Thin Sets and Strict-Two-Associatedness

By

Kathryn Elizabeth Hare

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Department of Mathematics

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

Date April 24, 1986
Abstract

Let $G$ be a compact, abelian group and let $E$ be a subset of its discrete, abelian, dual group $\hat{G}$.

$E$ is said to be a $A(p)$ set if for some $r < p$ there is a constant $c(r)$ so that

$$\|f\|_p \leq c(r)\|f\|_r$$

whenever the support of $\hat{f}$, the Fourier transform of $f$, is a finite subset of $E$.

The main result of this thesis, Theorem 3.5, is that if $E$ is a $A(p)$ set, $p > 2$, and $E$ satisfies a necessary technical condition, then for each $S \subset G$ of positive measure there is a constant $c(S, E) > 0$ so that

$$\|1_S f\|_2^2 \geq c(S, E)\|f\|_2^2$$

whenever $f \in L^2(G)$ and support $\hat{f} \subset E$. When such an inequality holds for all $f \in L^2(G)$ with support $\hat{f} \subset E$, then $E$ and $S$ are said to be strictly-2-associated. Actually we obtain the conclusion of strict-2-associatedness for a possibly larger class of sets than $A(p)$ sets, $p > 2$, so that our theorem improves upon previously known work even when $G$ is the circle group and $E \subset \mathbb{Z}$. Most of Chapter 3 is dedicated to proving this result and showing that it is almost best-possible. In the remainder of Chapter 3 we establish necessary and sufficient conditions for a
conclusion stronger than, but similar to strict-2-associatedness.

In Chapter 4 we prove that if $E$ is any $A(p)$ set, $p > 0$, (or any set with the same arithmetic structure as $A(p)$ sets) and if $E$ satisfies the same necessary technical condition as in Theorem 3.5, then $E$ is strictly-2-associated with all open subsets of $G$.

The proofs of these theorems depend on the arithmetic structure of $A(p)$ sets. This topic is discussed in detail in Chapter 2.

It has long been known that $A(p)$ sets in $\mathbb{Z}$ with $p > 2$, cannot contain arbitrarily long arithmetic progressions and have "uniformly large gaps". We prove that no $A(p)$ set, $p > 0$, can contain arbitrarily large parallelepipeds, a generalization of arithmetic progressions. This is new for $A(p)$ sets, $p < 1$, in groups other than the circle.

We introduce a definition which extends the notion of "uniformly large gaps" to the general setting. Combinatorial arguments are used to prove that sets which do not contain arbitrarily large parallelepipeds have this property.

Finally, parallelepipeds are used to show that $A(p)$ sets are built up from finite sets in a controlled way. This last fact and the notion of "uniformly large gaps" are central to the proofs we present of Theorems 3.5 and 4.1.
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Chapter 1

\( \Lambda(p) \) Sets - Introductory Results

In this chapter we will introduce the terminology and basic definitions used in the remainder of the paper. As well, we present background information on \( \Lambda(p) \) sets and uniformizable \( \Lambda(2) \) sets.

1.1 Notation and Terminology

Throughout this thesis \( G \) will denote a compact abelian group equipped with the normalized Haar measure \( m \), and \( \Gamma \) will denote its necessarily discrete abelian dual group. The most familiar example of such a group \( G \) is the circle group \( T \). Haar measure on \( T \) is normalized Lebesgue measure, and the dual of \( T \) is \( \mathbb{Z} \), the set of integers. For the group operation, our convention will be to use additive notation for \( \mathbb{Z} \) and multiplicative notation otherwise. A standard reference for harmonic analysis on such groups is Rudin [19].

\( Trig(G) \) will denote the space of complex-valued trigonometric polynomials on \( G \). For \( 0 < p < \infty \), \( L^p(G) \) will denote the space of equivalence classes of
complex-valued functions, measurable with respect to \( m \), and satisfying

\[
\|f\|_p = \left( \int_G |f|^p \, dm \right)^{1/p} < \infty.
\]

As usual we identify a function with the equivalence class to which it belongs.

\( L^\infty(G) \) will denote the Banach space (of equivalence classes) of complex-valued functions, measurable and essentially bounded with respect to the measure \( m \).

If \( E \) is a subset of \( \Gamma \), then a function \( f \) will be called an \( E \)-function provided its Fourier transform, \( \hat{f} \), vanishes off \( E \). A subscript \( E \) on a space of functions restricts that space to its \( E \)-functions. For example, \( Trig_E(G) \) consists of all \( E \)-polynomials.

1.2 Definition and Examples of \( A(p) \) Sets

**Definition 1.1** Let \( 0 < p < \infty \). A subset \( E \) of \( \Gamma \) is said to be an \( A(p) \) set if there is some \( r \in (0, p) \) and a constant \( c(p, r) \) so that

\[
\|f\|_p \leq c(p, r)\|f\|_r
\]

for all \( f \in Trig_E(G) \).

This notion was introduced by Rudin in [21] for subsets \( E \) of \( \mathbb{Z} \).

An application of Hölder's inequality as can be found in [21, 1.4] shows that if such a constant exists for some \( r \in (0, p) \), then for each \( s \in (0, p) \) there will be a
similar constant $c(p, s)$. Hölder's inequality also shows that if $E$ is a $A(p)$ set, then $E$ is a $A(t)$ set for all $0 < t < p$.

One could similarly define $A(\infty)$ sets, but the only $A(\infty)$ sets in any abelian group are finite sets.

Suppose $p > 1$ and that $E \subset \Gamma$ is a $A(p)$ set. Choose $\{h_\alpha\}$ an approximate unit for $L^1(G)$ consisting of trigonometric polynomials satisfying $\|h_\alpha\|_1 \leq 1$. If $f \in L^1_E(G)$, then $f * h_\alpha \in Trig_E(G)$ so that

$$\|f * h_\alpha\|_p \leq c(p, 1)\|f * h_\alpha\|_1 \leq c(p, 1)\|f\|_1\|h_\alpha\|_1 \leq c(p, 1)\|f\|_1.$$ 

By Alaoglu’s theorem the net $\{f * h_\alpha\}$ has a weak* cluster point $g \in L^p(G)$ satisfying

$$\|g\|_p \leq c(p, 1)\|f\|_1.$$ 

Since $\lim \hat{h_\alpha}(\chi) = 1$, $\hat{g}(\chi) = \hat{f}(\chi)$ for all $\chi \in \Gamma$, thus by the uniqueness theorem $g = f$.

Because the converse is obvious, this establishes

**Proposition 1.2** $E \subset \Gamma$ is a $A(p)$ set, for $1 < p < \infty$, if and only if there is a constant $c(p)$ so that

$$\|f\|_p \leq c(p)\|f\|_1.$$
for all \( f \in L_E^1(G) \).

As in [13, 5.3] an application of the Open Mapping Theorem proves

**Proposition 1.3** \( E \subseteq \Gamma \) is a \( \Lambda(p) \) set, for \( 1 < p < \infty \), if and only if there is some \( q < p \) so that whenever \( f \in L_E^q(G) \), then \( f \in L^p(G) \), and in this case the same statement holds for all \( q < p \).

**Proposition 1.4** For \( 2 < p < \infty \) the following are equivalent:

1. \( E \) is a \( \Lambda(p) \) set;
2. there is a constant \( c(p) \) so that \( \| f \|_p \leq c(p) \| f \|_2 \) for all \( f \in L_E^2(G) \); and
3. whenever \( f \in L_E^2(G) \), then \( f \in L^p(G) \).

The equivalence of (1) and (2) follow from Proposition 1.2 and Hölder's inequality, while the equivalence of (1) and (3) is a consequence of Proposition 1.3.

**Definition 1.5** \( E \subseteq \Gamma \) is called a **Sidon set** if there is a constant \( c \) so that for every \( f \in \text{Trig}_E(G) \)

\[
\sum_{x \in \Gamma} |\hat{f}(x)| \leq c \| f \|_\infty.
\]

A subset \( \{n_k\}_{k=1}^\infty \) of \( \mathbb{Z} \) is called a **lacunary** or **Hadamard set** if \( n_{k+1}/n_k \geq q > 1 \) for all \( k \). Lacunary sets are examples of Sidon sets. There are examples of Sidon sets which are not finite unions of lacunary sets (cf. [21, 2.5]).
Every Sidon set is a $A(q)$ set for all $q < \infty$. Indeed we have

**Theorem 1.6** [21, 3.1] \textit{If $E$ is a Sidon set with constant $c$ as in Definition 1.5, then for all $f \in \text{Trig}_E(G)$,}

\begin{enumerate}
\item $(1)$ $\|f\|_p \leq c\sqrt{p}\|f\|_2$ if $2 < p < \infty$; and
\item $(2)$ $\|f\|_2 \leq 2c\|f\|_1$.
\end{enumerate}

Condition (1) characterizes Sidon sets in the sense that if (1) holds for all $p > 2$ and all $E$-polynomials $f$, then indeed $E$ is a Sidon set [18].

In [3] Bonami constructed, for every infinite group $\Gamma$, examples of sets which she showed were $A(q)$ for all $q < \infty$. In [6] it is shown that these examples are non-Sidon sets. Rudin in [21, 4.8] constructed examples of subsets of $\mathbb{Z}$ which were $A(2s)$ but not $A(2s + \varepsilon)$ for any $\varepsilon > 0$, where $s$ is any integer greater than 1. In certain other discrete abelian groups examples have been constructed of sets which were $A(2s)$ but no better [7, 11]. Again $s$ must be an integer greater than 1.

There are no known examples of $A(p)$ sets for $p < 4$ which are not already $A(4)$ sets. It is known that if $E$ is a $A(p)$ set with $1 \leq p < 2$, then $E$ is a $A(p + \varepsilon)$ set for some $\varepsilon > 0$ [1], but many open questions remain. For example:

1. Is every $A(1)$ set a $A(2)$ set? a $A(4)$ set?

2. Is the union of two $A(p)$ sets, $p \leq 2$, a $A(r)$ set for some $r > 0$? For $p > 2$ this is obvious. Indeed the union of two $A(p)$ sets is another $A(p)$ set. In Chapter 2,
in the comments after Corollary 2.20, we give some evidence for an affirmative answer to this question.

1.3 Uniformizable $A(2)$ Sets

An alternate characterization of $A(p)$ sets, dual to Proposition 1.2, is

**Proposition 1.7** Let $1 < p < \infty$ and $1/p + 1/q = 1$. $E$ is an $A(p)$ set, if and only if there is a constant $c(p)$ so that corresponding to any function $g \in L^q(G)$ is a bounded function $h$ satisfying

(1) $\hat{h}(\chi) = \hat{g}(\chi)$ for all $\chi \in E$; and

(2) $\|h\|_\infty \leq c(p)\|g\|_q$.

**Remark** Proofs of this proposition may be found for subsets of $\mathbb{Z}$ in [21, 5.1] and for the general setting in [13, 5.3]. We give a proof here to illustrate the use of duality.

**Proof** Suppose $E$ is $A(p)$ and $\|f\|_p \leq c(p)\|f\|_1$ for all $f \in L^1_E(G)$. Fix $g \in L^q(G)$ and define the linear mapping $S : \text{Trig}_E(G) \to \mathbb{C}$ by

$$S(f) = \int_G f(x)\overline{g(x)} \, dm(x).$$
Then

\[ |S(f)| \leq \|f\|_p \|g\|_q \leq c(p)\|f\|_1 \|g\|_q \]

so by the Hahn-Banach Theorem $S$ may be extended to a linear functional on $L^1(G)$, also called $S$, with $\|S\| \leq c(p)\|g\|_q$. By the Riesz Representation Theorem there is a bounded function $h$ with

\[ S(f) = \int_G f(x)\overline{h(x)} \, dm(x) \]

and

\[ \|h\|_\infty = \|S\| \leq c(p)\|g\|_q . \]

Thus (2) holds for $h$.

If $\chi \in E$, the map $x \mapsto \chi(x)$ belongs to $\text{Trig}_E(G)$ hence

\[ S(\chi) = \int_G \chi(x)\overline{g(x)} \, dm(x) = \hat{g}(\chi) . \]

But also

\[ S(\chi) = \int_G \chi(x)\overline{h(x)} \, dm(x) = \hat{h}(\chi) \]

so $\hat{g}(\chi) = \hat{h}(\chi)$ for all $\chi \in E$ establishing (1).

For the converse, let $f \in \text{Trig}_E(G)$ and choose $g \in L^q(G)$ of $q$-norm equal to 1 so that

\[ \|f\|_p = \int_G f(x)\overline{g(x)} \, dm(x) . \]

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Choose the corresponding bounded function $h$ satisfying (1) and (2) for this choice of $g$. Using the equality above, Parseval's relation and (1) we obtain

$$\|f\|_p = \sum_{x \in E} \hat{f}(x) \hat{g}(x) = \sum_{x \in E} \hat{f}(x) \hat{h}(x) = \int_G f(x) \overline{h(x)} \, dm(x).$$

By Hölder's inequality and (2)

$$\|f\|_p \leq \|f\|_1 \|h\|_\infty \leq c(p) \|f\|_1.$$

Thus $E$ is $A(p)$.

---

Suppose $p = 2$. Then a choice of $g$ in $L^2(G)$ with $L^2$-norm equal to 1 and satisfying

$$\|f\|_2 = \int_G f(x) \overline{g(x)} \, dm(x)$$

is $g(x) = f(x)/\|f\|_2$, which belongs to $L^2_E(G)$. This observation together with the proof of the previous proposition yields

**Corollary 1.8** $E$ is $A(2)$ if and only if there exists a constant $c > 0$ so that whenever $g \in L^2_E(G)$, there is a bounded function $h$ with

1. $\hat{h}(x) = \hat{g}(x)$ for all $x \in E$; and

2. $\|h\|_\infty \leq c\|g\|_2$.
Following Blei [2], we define uniformizable $A(2)$ sets:

**Definition 1.9** $E \subseteq \Gamma$ is said to be a *uniformizable $A(2)$ set* if for each $\varepsilon > 0$ there is a constant $c(E, \varepsilon)$ so that whenever $g \in L^2_E(G)$ there is a bounded function $h$ with

1. $\hat{h}(x) = \hat{g}(x)$ for all $x \in E$;
2. $\|h\|_\infty \leq c(E, \varepsilon)\|g\|_2$; and
3. $\|\hat{h}|_{\Gamma \setminus E}\|_2 \leq \left(\sum_{x \in \Gamma \setminus E} |\hat{h}(x)|^2\right)^{1/2} \leq \varepsilon\|g\|_2$.

The least such constant $c(E, \varepsilon)$ will be called the *uniformizable $A(2)$ constant* for $E$ and $\varepsilon$.

**Remark** Observe that (1) and (3) imply that $\|h - g\|_2 \leq \varepsilon\|g\|_2$.

It is immediate from Corollary 1.8 that uniformizable $A(2)$ sets are $A(2)$ sets. Whether or not all $A(2)$ sets are indeed uniformizable $A(2)$ sets is an open problem. Since it is easy to see that the union of a uniformizable $A(2)$ set and a $A(2)$ set is another $A(2)$ set, a positive solution to this problem would answer the union question for $A(2)$ sets.

Some important properties of uniformizable $A(2)$ sets are outlined in the next theorem.
Theorem 1.10 [9] For \( E \subset \Gamma \) the following are equivalent:

(1) \( E \) is a uniformizable \( A(2) \) set.

(2) For all \( \varepsilon > 0 \) there is a constant \( c_1(E, \varepsilon) \) so that for each \( g \in L^2_E(G) \) there is a bounded function \( h \) satisfying

(a) \( \|h\|_\infty \leq c_1(E, \varepsilon)\|g\|_2 \), and

(b) \( \|h - g\|_2 \leq \varepsilon\|g\|_2 \).

(3) For all \( \varepsilon > 0 \) there is a constant \( c_2(E, \varepsilon) \) so that for each \( g \in L^2_E(G) \) there is a continuous function \( h \) satisfying (1), (2), and (3) of Definition 1.9 with this constant \( c_2(E, \varepsilon) \).

(4) For all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( S \) is a measurable subset of \( G \) with \( m(S) < \delta \), then

\[
\left( \int_S |f|^2 \right)^{1/2} \leq \varepsilon\|f\|_2
\]

for all \( f \in L^2_E(G) \). (The family \( \{f \in L^2_E(G) : \|f\|_2 = 1\} \) is said to be uniformly integrable.)

(5) There is a Young function \( \Phi \) with \( \frac{\Phi(x)}{x^2} \rightarrow \infty \) as \( x \rightarrow \infty \), for which we have

\[
L^2_E(G) \subset L^\Phi(G),
\]

the Orlicz space corresponding to \( \Phi \).

Proof For completeness we sketch the proofs as they will appear in [9].

(1 \( \Rightarrow \) 2) and (3 \( \Rightarrow \) 1) are clear.
(2 ⇒ 3) Here we use the fact that given \( h \in L^\infty(G) \) and \( \varepsilon > 0 \) we can find a polynomial \( p \) with \( \|p\|_\infty \leq \|h\|_\infty \) and \( \|p - h\|_2 < \varepsilon \). From this fact and (2) it follows that for each \( g \in L^2_E(G) \) there is a polynomial \( p_1 \) satisfying
\[
\|p_1\|_\infty \leq c(E, \varepsilon/4)\|g\|_2 \quad \text{and} \quad \|p_1 - g\|_2 \leq \frac{\varepsilon}{2}\|g\|_2.
\]

Set \( g_1 = g \) and inductively define a sequence \( \{g_n\}_{n=1}^\infty \subset L^2_E(G) \) by
\[
\hat{g}_n = \begin{cases} 
\hat{g}_{n-1} - \hat{p}_{n-1} & \text{on } E \\
0 & \text{otherwise}
\end{cases}
\]
and a sequence of polynomials \( \{p_n\} \) satisfying
\[
\|p_n - g_n\|_2 \leq \frac{\varepsilon}{2}\|g_n\|_2 \quad \text{and} \quad \|p_n\|_\infty \leq c(E, \varepsilon/4)\|g_n\|_2.
\]

The continuous function \( h \) satisfying (3) is \( \sum_{n=1}^\infty p_n \).

(2 ⇒ 4) Let \( \varepsilon > 0 \) and put \( \delta = \frac{\varepsilon^2}{c_1(E, \varepsilon)^2} \). Given \( f \in L^2_E(G) \) choose \( h \) as in (2). Writing \( f \) as \( h + (f - h) \) and applying the triangle inequality to \( \|1_S f\|_2 \), and then Hölder’s inequality, we see that if \( m(S) < \delta \) then \( \|1_S f\|_2 \leq 2\varepsilon\|f\|_2 \).

(4 ⇒ 2) Given \( \varepsilon > 0 \) and \( f \in L^2_E(G) \), let \( S = \{x \in G : |f(x)| > \|f\|_2/\sqrt{\delta}\} \), where \( \delta \) is chosen as in (4). By Chebyshev’s inequality \( m(S) < \delta \), thus if we let \( h = 1_S c f, h \) satisfies (2) with \( c_1(E, \varepsilon) = 1/\sqrt{\delta} \).

(4 ⇔ 5) This can essentially be found in [20, 3.1].
Remark In the proof of (5 ⇒ 4) the Closed Graph Theorem is used to ensure that the inclusion $L^2(E)(G) \hookrightarrow L^\Phi(G)$ is continuous. Thus there is a constant $M$ with $\int \Phi(|f|/M) \leq 1$ for all $f$ in the unit ball of $L^2(E)(G)$.

Corollary 1.11 Every $\Lambda(p)$ set, $p > 2$, is a uniformizable $\Lambda(2)$ set.

Proof Either verify (4) or use (5) with $\Phi(x) = x^p$.

This fact was first observed by Pisier. An alternate proof may be found in [2].
Chapter 2
Arithmetic Properties of $A(p)$ Sets

2.1 Introduction

$A(p)$ sets satisfy a number of arithmetic conditions which describe their "thinness". This chapter is devoted to a discussion of these conditions.

We begin by reviewing the well known relationship between $A(p)$ sets in $\mathbb{Z}$, $p > 2$; and arithmetic progressions; and extend these results to uniformizable $A(2)$ subsets of $\mathbb{Z}$. It easily follows that such sets have uniformly large gaps. We briefly discuss a generalization of this relationship for $A(p)$ sets, $p > 2$, in arbitrary discrete abelian groups.

Next, the notion of parallelepiped is introduced. We prove that no $A(p)$ set, $p > 0$, in any discrete abelian group, may contain parallelepipeds of arbitrarily large dimension. This fact was previously known for all $A(p)$ sets in $\mathbb{Z}$, but in the general setting only for $A(1)$ sets, and it can be used to obtain more results concerning arithmetic progressions.
We define a new arithmetic property, a generalization of the notion of uniformly large gaps, and prove that any set which does not contain parallelepipeds of arbitrarily large dimension has this property. This new property will be used in proving our main result in Chapter 3.

Finally, we extend a notion of Miheev's to the general setting and show that sets not containing parallelepipeds of arbitrarily large dimension are built from sets which have "large gaps" between any two members. This idea will also be used in Chapter 3.

2.2 Survey of Known Results

The first arithmetic conditions related to $A(p)$ sets are discussed in [21]. In that paper can be found the proofs of the next two propositions.

**Proposition 2.1** [21, 4.1] If $E \subset \mathbb{Z}$ is $A(1)$, then $E$ does not contain arbitrarily long arithmetic progressions.

**Proposition 2.2(a)** [21, 3.5] If $E \subset \mathbb{Z}$ is $A(p)$, $p > 2$, and $c = c(p, 2)$ as in Definition 1.1, then whenever $a, b \in \mathbb{Z}$, $b \neq 0$, and $N$ is a positive integer,

$$|\{a + b, a + 2b, \ldots, a + Nb\} \cap E| \leq 4c^2N^{2/p}.$$ 

*Here $| \cdot |$ denotes the cardinality of the set.*
Of course, the second proposition implies the first whenever $E$ is a $\Lambda(p)$ set for some $p > 2$.

If $E$ is only assumed to be a uniformizable $\Lambda(2)$ set in $\mathbb{Z}$ a result similar to Proposition 2.2(a) is true. Indeed we have

**Proposition 2.2(b)** If $E \subset \mathbb{Z}$ is a uniformizable $\Lambda(2)$ set with constant $c(E, \varepsilon)$, then

$$\left| \{a + b, a + 2b, \ldots, a + Nb\} \cap E \right| \leq 8(c(E, \varepsilon)^2 + \varepsilon^2 N).$$

**Proof** We appropriately modify [21, 3.5].

Let $K_N(t) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) e^{int}$ be the $N$'th Fejer kernel. It is well known that $\|K_N\|_1 = 1$ and $\|K_N\|_2^2 \leq N$.

Consider the arithmetic progression $A = \{a + b, a + 2b, \ldots, a + Nb\}$. Set $m = N/2$ or $\frac{N + 1}{2}$ depending on whether $N$ is even or odd, and define $Q(t) = e^{imbt} K_N(bt) e^{iat}$. The choice of $m$ ensures that $\dot{Q}(n) \geq 1/2$ for $n \in A$.

Suppose $A \cap E = \{n_1, \ldots, n_s\}$ and let $f$ be the $E$-polynomial $f(t) = \sum_{k=1}^{s} e^{int}$. Then $\|f\|_2 = \sqrt{s}$ and

$$s/2 \leq \sum_{k=1}^{s} \hat{Q}(n_k) \overline{f(n_k)} = \int Q(t) \overline{f(t)} \, dm(t).$$

Since $E$ is uniformizable $\Lambda(2)$ with constant $c(E, \varepsilon)$ there is an $h \in L^\infty(T)$
satisfying \( \|h - f\|_2 \leq \varepsilon \|f\|_2 \) and \( \|h\|_{\infty} \leq c(E, \varepsilon)\|f\|_2 \). Thus
\[
\frac{s}{2} \leq \int (f - h)Q + \int hQ \\
\leq \|f - h\|_2 \|Q\|_2 + \|h\|_{\infty} \|Q\|_1 \\
\leq \varepsilon \sqrt{sN} + c(E, \varepsilon)\sqrt{s}.
\]
Hence \( |A \cap E| = s \leq 8(\varepsilon^2 N + c(E, \varepsilon)^2) \).

An interesting consequence of Proposition 2.2(a) was observed by Miheev in [15], namely that \( A(p) \) sets in \( \mathbb{Z} \) with \( p > 2 \) have "uniformly large gaps". We make this notion precise and prove it for uniformizable \( A(2) \) sets.

**Corollary 2.3** Let \( E \) be a uniformizable \( A(2) \) set in \( \mathbb{Z} \) and let \( N \) be any positive integer. There is an integer \( M = M(E, N) \) such that any interval of length \( M \) in \( \mathbb{Z} \) contains a subinterval of length \( N \) free of points of \( E \), that is, \( E \) has uniformly large gaps.

**Proof** Given \( E \) and \( N \) set \( \varepsilon^2 = \frac{1}{16N} \). Let \( M = c(E, \varepsilon)^2 16N \).

By Proposition 2.2(b),
\[
\left| \{a, a + 1, \ldots, a + M - 1\} \cap E \right| \leq 8(c(E, \varepsilon)^2 + \varepsilon^2 M)
\]
for any \( a \in \mathbb{Z} \). Hence the interval \([a, a + M - 1]\) of length \( M \), must contain a subinterval of length at least \( \frac{M}{8(c(E, \varepsilon)^2 + \varepsilon^2 M)} = N \) free of points of \( E \).
In Corollary 2.23 we obtain the same conclusion for any $A(p)$ set, $p > 0$, in $\mathbb{Z}$.

In Definition 2.25 we generalize this notion to arbitrary discrete abelian groups.

**Definition 2.4** For positive integers $d$ and $N, X_1, \ldots, X_d \in \Gamma$ and $1 \leq r < \infty$, define as in [13, 6.2]

$$
A_r(N, X_1, \ldots, X_d) = \left\{ \sum_{j=1}^{d} n_j x_j^r : \sum_{j=1}^{d} |n_j|^r \leq N^r \right\}.
$$

These sets may be viewed as generalized arithmetic progressions. Indeed, if $\Gamma = \mathbb{Z}$ and $b \in \mathbb{Z}$ then

$$
A_r(N, b) = \{-Nb, \ldots, -b, 0, b, \ldots, Nb\}
$$

is an arithmetic progression of length $2N + 1$. Any arithmetic progression in $\mathbb{Z}$ of odd length is a translate of such a set.

The next theorem is a generalization of Proposition 2.2(a). For a proof see [13, 6.3–6.4].

**Theorem 2.5** Let $E \subseteq \Gamma$ be a $A(p)$ set with $p > 2$ and let $c = c(p, 2)$. Then

$$
|A_r(N, X_1, \ldots, X_d) \cap E| \leq e^2 c^2 (1 + \sqrt{\pi N})^{2d/p}
$$

for all $X_1, \ldots, X_d \in \Gamma$, $n \in \mathbb{Z}^+$ and $r = 1, 2$. 

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Again a slight modification, such as that given in the proof of Proposition 2.2(b), yields a corresponding conclusion whenever $E$ is a uniformizable $A(2)$ set. In Corollary 2.19 we present the proof of a similar statement for $A(p)$ sets, $p > 0$.

### 2.3 Parallelepipeds

**Definition 2.6** A subset $P$ of $\Gamma$ is called a **parallelepiped of dimension** $N$ if $P$ is the product of $N$ two element sets and $P$ has exactly $2^N$ elements.

Thus $P = \prod_{i=1}^{N} \{x_i, \psi_i\}$ with $x_i, \psi_i \in \Gamma$ and all $2^N$ terms are distinct.

If $A = \{a, a + b, \ldots, a + (2^N - 1)b\}$ is an arithmetic progression of length $2^N$ in $\mathbb{Z}$, then $A$ is a parallelepiped of dimension $N$, since $A = \{a, a + b\} + \{0, 2b\} + \ldots + \{0, 2^{N-1}b\}$. (In $\mathbb{Z}$ a parallelepiped of dimension $N$ is a $2^N$ element set which is the sum of $N$ two element sets.) Thus parallelepipeds are another generalization of arithmetic progressions.

**Proposition 2.7** $P$ is a parallelepiped of dimension $N$ if and only if

$$|P| = 2^N \quad \text{and} \quad P = \bigcup_{j=1}^{N} P_j, \quad \text{where} \quad |P_1| = 2,$$

(1)

and for each $j > 1$, $P_j = \left( \bigcup_{k=1}^{j-1} P_k \right) \gamma_j$, where $\gamma_j \in \Gamma$. 

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Proof If $P$ is as in (1) then $P = P_1 \prod_{i=2}^{N} \{1, \gamma_i\}$ so $P$ is a parallelepiped of dimension $N$.

For the converse, observe first that if $P$ is a parallelepiped of dimension 1 then $P$ clearly satisfies (1). Now proceed by induction. Assume any parallelepiped of dimension $N$ satisfies (1). If $P$ is a parallelepiped of dimension $N + 1$, say $P = \prod_{i=1}^{N+1} \{\chi_i, \psi_i\}$, then certainly $P' = \prod_{i=1}^{N} \{\chi_i, \psi_i\}$ is a parallelepiped of dimension $N$. Thus by the induction assumption $P' = \bigcup_{j=1}^{N} P'_j$ with $|P'_1| = 2$, and $P'_j = \left(\bigcup_{k=1}^{j-1} P'_k\right) \gamma_j$, $j = 2, 3, \ldots, N$.

Notice that every element of $P$ is either an element of $P' \chi_{N+1}$ or of $P' \psi_{N+1}$. If we let $P_1 = P'_1 \chi_{N+1}$, $\gamma_{N+1} = \chi_{N+1}^{-1} \psi_{N+1}$ and $P_j = \left(\bigcup_{i=1}^{j-1} P_i\right) \gamma_j$, $j = 2, \ldots, N + 1$, then we have $\bigcup_{j=1}^{N+1} P_j = P' \chi_{N+1}$ and $P_{N+1} = P' \psi_{N+1}$, hence $P = \bigcup_{j=1}^{N+1} P_j$. Thus $P$ satisfies (1).

In [16] Miheev makes the following definition:

Definition 2.8 If $\{m_i\}_{i=1}^{N} \subset \mathbb{Z}$ with $m_1 > 0$ and $m_k > m_1 + \ldots m_{k-1}$ for $k = 2, 3, \ldots, N$, and if $r \in \mathbb{Z}$, then the spectrum of the trigonometric polynomial

$$R(x) = e^{i r x} (1 + e^{i m_1 x}) (1 + e^{i m_2 x}) \cdots (1 + e^{i m_N x}),$$

i.e., the set

$$\{r, r + m_1\} + \{0, m_2\} + \{0, m_3\} + \ldots + \{0, m_N\},$$

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is called a *tracing segment of length* $2^N$.

In [15] this is called a *reflexive segment of length* $2^N$.

It is clear that any reflexive segment of length $2^N$ is a parallelepiped of dimension $N$ in $\mathbb{Z}$. Although not all parallelepipeds in $\mathbb{Z}$ are reflexive segments, any parallelepiped of dimension $(2^N - 1)$ contains a reflexive segment of length $2^N$.

To see this suppose $P = P_1 + P_2 + \cdots + P_{2^N - 1}$ is a parallelepiped of dimension $(2^N - 1)$. By translating and reordering if necessary we may assume $P_i = \{0, c_i\}$ with $c_{i+1} \geq c_i > 0$ for all $i = 1, \ldots, 2^N - 1$. Now set

$$m_1 = c_1, \quad m_2 = c_2 + c_3, \ldots, \quad m_N = \sum_{i=2^{N-1}}^{2^N-1} c_i.$$

Clearly $m_k > m_1 + \cdots + m_{k-1}$ for $k = 2, \ldots, N$ and the spectrum of

$$(1 + e^{im_1x})(1 + e^{im_2x}) \cdots (1 + e^{im_Nx})$$

is contained in $P$.

The main result of this section is:

**Theorem 2.9** If $E$ is a $\Lambda(p)$ set, $p > 0$, in any discrete abelian group, then there is an integer $N$ such that $E$ does not contain any parallelepipeds of dimension greater than $N$. 

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Remarks  Before beginning the proof we discuss previously known results of this type. For $p \geq 1$ a proof can be found in [10]. The proof there is given for $E \subset \mathbb{Z}$, but it directly generalizes to all discrete abelian groups. To obtain the conclusion for $A(1)$ sets this proof makes use of the fact that any $A(1)$ set is actually a $A(1 + \varepsilon)$ set for some $\varepsilon > 0$ [1]. Other proofs are given for $A(p)$ sets in $\mathbb{Z}$, in [15] ($p = 2$) and [16] ($p > 0$). The main idea in the proof of [16] is used in Lemma 2.13 below.

Proof of Theorem 2.9  Since any $A(p)$ set with $p \geq 1$ is a $A(s)$ set for any $s < 1$ we may without loss of generality assume $p < 1$.

We will show in fact that $N$ depends only on $c(p, p/2)$. Since a translate of a $A(p)$ set is a $A(p)$ set with the same constant $c(p, p/2)$, it suffices to show that $A(p)$ sets do not contain parallelepipeds of the form $P = \prod_{i=1}^{M} \{1, x_i\}$, $|P| = 2^M$, for $M > N$.

The proof will result by establishing a number of lemmas.

Let us say that $\{x_1, \ldots, x_N\} \subset \Gamma$ is quasi-dissociate if

$$\prod_{i=1}^{N} x_i^{e_i} = 1 \quad \text{for} \quad e_i = 0, \pm 1, i = 1, \ldots, N,$$

implies $e_i = 0$ for all $i = 1, \ldots, N$.

Lemma 2.10  Fix a positive integer $N_0$ and let $N_1 = 3^{N_0} + 1$. Any subset of $\Gamma$ of cardinality $N_1$ contains a quasi-dissociate subset of cardinality $N_0$.

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Proof  This is essentially an application of the Pigeon Hole Principle.

Consider the subset \( \{X_i\}_{i=1}^{N_1} \subset T \). Choose \( \psi_1 \in \{X_1, X_2\} \) so that \( \psi_1 \neq 1 \). If \( A_1 = \{\psi_1^{e_1} : e_1 = 0, \pm 1\} \) then \( |A_1| \leq 3 \) so it is possible to choose \( \psi_2 \in \{X_i\}_{i=1}^{N_1} \) with \( \psi_2 \notin A_1 \).

Now proceed inductively. Assume \( \psi_1, \ldots, \psi_n \) have been chosen. Let

\[
A_n = \{\psi_1^{e_1} \psi_2^{e_2} \cdots \psi_n^{e_n} : e_i = 0, \pm 1, i = 1, \ldots n\}.
\]

Since \( |A_n| \leq 3^n \) we may choose \( \psi_{n+1} \in \{X_i\}_{i=1}^{N_1} \) with \( \psi_{n+1} \notin A_n \).

We may choose \( \{\psi_i\}_{i=1}^{N_0} \subset \{X_i\}_{i=1}^{N_1} \) in this way since \( N_1 = 3^{N_0} + 1 \).

Now suppose \( \prod_{i=1}^{N_0} \psi_i^{e_i} = 1 \) with \( e_i = 0, \pm 1, i = 1, \ldots, N_0 \). Let \( k \) be the largest integer with \( e_k \neq 0 \). We cannot have \( k = 1 \) for then \( \psi_1^{e_1} = 1 \) and hence \( \psi_1 = 1 \). If \( k > 1 \) then without loss of generality, \( e_k = 1 \), so \( \psi_k = \prod_{i=1}^{k-1} \psi_i^{-e_i} \). But this implies \( \psi_k \in A_{k-1} \) contradicting its selection. Thus \( e_i = 0 \) for all \( i = 1, 2, \ldots, N_0 \) and hence \( \{\psi_i\}_{i=1}^{N_0} \) is a quasi-dissociate set.  /////

Let us say that the parallelepiped \( P_N = \prod_{i=1}^{N} \{1, X_i\} \) is

1. **of order 2** if \( X_i^2 = 1 \) for \( i = 1, \ldots, N \);

2. **dissociate** if \( \prod_{i=1}^{N} X_i^{e_i} = 1 \) with \( e_i = 0, \pm 1, \pm 2 \), implies \( e_i = 0 \) for all \( i = 1, \ldots, N \); and
(3) quasi-dissociate if \( \prod_{i=1}^{N} x_i^{\varepsilon_i} = 1 \) with \( \varepsilon_i = 0, \pm 1 \) implies \( \varepsilon_i = 0 \) for all \( i = 1, \ldots, N \).

With this notation an immediate corollary of the previous lemma is

**Corollary 2.11** If \( E \) contains \( P = \prod_{i=1}^{N_1} \{1, x_i\} \), a parallelepiped of dimension \( N_1 = 3^{N_0} + 1 \), then \( E \) contains a quasi-dissociate, \( N_0 \)-dimensional parallelepiped.

Next we will prove

**Lemma 2.12** Let \( E \) be a \( A(p) \) set, \( 0 < p < 1 \), with constant \( c(p, p/2) \). There is an integer \( N_1 \) depending on \( c(p, p/2) \) such that \( E \) does not contain any parallelepipeds of order 2 with dimension greater than \( N_1 \).

**Proof** Choose an integer \( N_0 \) so that \( 2^{N_0}/p = 2^{(1-1/p)N_0} > c(p, p/2) \) and set \( N_1 = 3^{N_0} + 1 \). By Corollary 2.11 if \( E \) contains a parallelepiped of order 2 with dimension \( N_1 \) then \( E \) contains a quasi-dissociate parallelepiped of order 2 with dimension \( N_0 \), say \( \prod_{i=1}^{N_0} \{1, x_i\} \). Being quasi-dissociate and of order 2 the set \( \{x_i\}_{i=1}^{N_0} \) is probabilistically independent. Hence

\[
\left( \int \prod_{i=1}^{N_0} |1 + x_i|^p \right)^{1/p} = \left( \prod_{i=1}^{N_0} \int |1 + x_i|^p \right)^{1/p} = 2^{(1-1/p)N_0}.
\]
Similarly
\[
\left( \int \prod_{i=1}^{N_0} |1 + X_i|^{p/2} \right)^{2/p} = 2^{(1-2/p)N_0}.
\]

Thus if \( f(x) = \prod_{i=1}^{N_0} (1 + X_i(x)) \), then \( f \in \text{Trig}_E(G) \) and
\[
\|f\|_p = 2^{(1-1/p)N_0} > c(p, p/2)2^{(1-2/p)N_0} = c(p, p/2)\|f\|_{p/2}
\]
contradicting the fact that \( E \) is a \( A(p) \) set with constant \( c(p, p/2) \).

\[
\text{Lemma 2.13} \quad \text{Let } E \text{ be a } A(p) \text{ set, } 0 < p < 1, \text{ with constant } c(p, p/2). \text{ There is an integer } N \text{ depending on } c(p, p/2) \text{ such that } E \text{ does not contain any dissociate parallelepipeds of dimension } N.
\]

\[
\text{Proof} \quad \text{It is shown in [16] that for any fixed } r \in (0, 1) \text{ with } \frac{r}{(1-r)^3} < \frac{p^2}{256},\]
\[
A = \left( 1 - \frac{\frac{p}{4}(1 - \frac{p}{4})r^2}{4} - \left( \frac{r}{1-r} \right)^3 \right)^{1/p} > \left( 1 - \frac{\frac{p}{4}(1 - \frac{p}{4})r^2}{4} + \left( \frac{r}{1-r} \right)^3 \right)^{2/p} = B.
\]

Choose \( N \) so that \( A^N > c(p, p/2)B^N \), and suppose \( E \) contains the dissociate parallelepiped \( \prod_{i=1}^{N} \{1, X_i\} \). Let \( R \) be the least solution of \( r = \frac{2R}{1 + R^2} \).

Let \( f = \prod_{i=1}^{N} (1 + RX_i) \). Then \( f \in \text{Trig}_E(G) \), and
\[
\|f\|_p = \left( \int \prod_{i=1}^{N} (|1 + RX_i|^{p/2}) \right)^{1/p}
\]
\[
= (1 + R^2)^{N/2} \left( \int \prod_{i=1}^{N} \left( 1 + r \left( \frac{X_i + X_i}{2} \right) \right)^{p/2} \right)^{1/p}.
\]

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An application of MacLaurin's formula shows that for any \( \alpha \in (0, 1) \)

\[
(1 + x)^\alpha = 1 + \alpha x - \frac{\alpha(1 - \alpha)x^2}{2} + \text{Rem}(x)
\]

where \( |\text{Rem}(x)| \leq \left( \frac{r}{1 - r} \right)^3 \) provided \( x \in [-r, r] \) and \( r \in (0, 1) \).

Now \(-r \leq r\left( \frac{X_i(x) + \overline{X_i}(x)}{2} \right) \leq r \) so applying MacLaurin's formula to (1) with \( \alpha = p/2 \) we obtain

\[
\|f\|_p \geq (1 + R^2)^{N/2} \left( \prod_{i=1}^{N} \left( 1 + \frac{p}{2}r(\frac{X_i + \overline{X_i}}{2}) - \frac{p}{2}(1 - \frac{p}{2})r^2(\frac{X_i + \overline{X_i}}{2}) \right) \right)^{1/p}
\]

\[
= (1 + R^2)^{N/2} \left( \prod_{i=1}^{N} \left( 1 - \frac{p}{2}(1 - \frac{p}{2})r^2 - \left( \frac{r}{1 - r} \right)^3 + \frac{p}{2}r(\frac{X_i + \overline{X_i}}{2}) \right) \right)^{1/p}
\]

\[
= (1 + R^2)^{N/2} \prod_{i=1}^{N} \left( 1 - \frac{p}{2}(1 - \frac{p}{2})r^2 - \left( \frac{r}{1 - r} \right)^3 \right)^{1/p}
\]

because of the dissociateness assumption.

Similarly

\[
\|f\|_{p/2} \leq (1 + R^2)^{N/2} \prod_{i=1}^{N} \left( 1 - \frac{p}{4}(1 - \frac{p}{4})r^2 + \left( \frac{r}{1 - r} \right)^3 \right)^{2/p}
\]

Thus

\[
\|f\|_p \geq (1 + R^2)^{N/2} A^N > (1 + R^2)^{N/2} c(p, p/2) B^N
\]

\[
= c(p, p/2)\|f\|_{p/2}
\]

contradicting the fact that \( E \) is a \( A(p) \) set with constant \( c(p, p/2) \).
Lemma 2.14 For each positive integer $N_0$ there is an integer $N_2 = N_2(N_0)$ so that if $P = \prod_{i=1}^{N_2} \{1, x_i\}$ is a parallelepiped of dimension $N_2$ with the property that for each $i = 1, 2, \ldots, N_2$ the set $\{j \neq i : x_j^2 = x_i^2\}$ is empty, then $P$ contains a dissociate parallelepiped of dimension $N_0$.

Proof This is another application of the Pigeon Hole Principle similar to Lemma 2.10. //

Lemma 2.15 For each positive integer $N_0$ there is an integer $N = N(N_0)$ so that if $E$ contains a parallelepiped of dimension $N$, then a translate of $E$ contains either a dissociate parallelepiped or a parallelepiped of order 2, with dimension $N_0$.

Proof Fix $N_0$. Put $N = 2N_0N_2$ with $N_2 = N_2(N_0)$ as in Lemma 2.14. Assume that a translate of $E$ contains the $N$-dimensional parallelepiped $P = \prod_{i=1}^{N} \{1, x_i\}$.

We will say that $x_i \sim x_j$ if $x_i^2 = x_j^2$. Let $S_i$ be the equivalence class containing $x_i$. We consider two cases.

Case 1: For some $i \in \{1, 2, \ldots, N\}$, $|S_i| \geq 2N_0$. Without loss of generality $i = 1$ and $\{1, 2, \ldots, 2N_0\} \subset S_1$, i.e., $x_k^2 = x_1^2$ for $k = 1, 2, \ldots, 2N_0$. Then $x_1x_k^{-1} \equiv \varphi_k$ satisfies $\varphi_k^2 = 1$ for $k = 1, \ldots, 2N_0$.

Certainly $\prod_{j=1}^{N_0} \{x_1 \varphi_{2j-1}, x_1 \varphi_{2j}\} \subset P$ and hence is a parallelepiped of dimension $N_0$ contained in $E$. A further translate of $E$ contains the $N_0$-dimensional parallelepiped $\prod_{j=1}^{N_0} \{1, \varphi_{2j} \varphi_{2j-1}^{-1}\}$ of order two.
Case 2: Otherwise $|S_i| \leq 2N_0$ for all $i = 1, 2, \ldots, N$. In this case there must be at least $N_2$ distinct equivalence classes, say $S_1, \ldots, S_{N_2}$. Lemma 2.14 may be applied to $\prod_{i=1}^{N_2} \{1, X_i\}$ to obtain a dissociate parallelepiped of dimension $N_0$ in the original translate of $E$.

Completion of the Proof of Theorem 2.9

Put together Lemmas 2.12, 2.13 and 2.15.

In the remainder of this chapter we discuss properties of sets which do not contain parallelepipeds of arbitrarily large dimension and thus obtain new results for $A(p)$ sets. The results of Sections 2.4 and 2.5 will be used in Chapter 3.

Definition 2.16 $P \subset \Gamma$ will be called a \textit{pseudo-parallelepiped} of dimension $N$ if

$$P = \prod_{i=1}^{N} \{x_i, \psi_i\}, \text{ where } x_i, \psi_i \in \Gamma.$$ 

Note that a parallelepiped of dimension $N$ is a pseudo-parallelepiped of dimension $N$ with cardinality $2^N$.

Proposition 2.17 There are constants $c(n)$ and $0 < c(n) < 1$ so that if $E \subset \Gamma$ does not contain any parallelepipeds of dimension greater than $n$, then

$$|E \cap P_d| \leq c(n)2^{dc(n)}$$
whenever $P_d$ is a pseudo-parallelepiped of dimension $d$.

This proposition is proved in [16] for $E \subseteq \mathbb{Z}$ and $P_d$ a parallelepiped of dimension $d$. Proposition 2.17 can be established by making appropriate modifications to this proof. We carry out similar modifications later, in the proof of Theorem 2.31.

We will use this proposition to prove a series of results. Corollary 2.21 may be found in [16], as can Corollaries 2.18 and 2.20 for subsets of $\mathbb{Z}$.

**Corollary 2.18** If $E$ does not contain any parallelepipeds of dimension $n$ and if $A_N$ is any arithmetic progression of length $N$, then

$$|E \cap A_N| \leq 2^{e(n)} c(n) N^{e(n)}$$

where $c(n)$ and $0 < e(n) < 1$ are as in Proposition 2.17.

In particular, if $E \subseteq \Gamma$ is a $\Lambda(p)$ set for any $p > 0$ then there are constants $c(E)$ and $0 < e(E) < 1$ so that

$$|E \cap A_N| \leq c(E) N^{e(E)}.$$

(Compare with Proposition 2.2.)

**Proof** Choose $M$ such that $2^{M-1} < N \leq 2^M$. Observe that the arithmetic progression $A_N = \{x, x\psi, \ldots, x\psi^{N-1}\}$ is contained in the pseudo-parallelepiped $\{x, x\psi\} \cdot \prod_{i=2}^{M} \{1, \psi^{2^{i-1}}\}$ of dimension $M$. 

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Hence

$$|E \cap A_N| \leq c(n)2^{Me(n)} \leq 2^{e(n)}c(n)N^{e(n)}.$$ 

Recalling Definition 2.4, it is natural to define

$$A_\infty(N, X_1, \ldots, X_d) \equiv \left\{ \prod_{i=1}^{d} X_i^{n_i} : \sup_{1 \leq i \leq d} |n_i| \leq N \right\}.$$ 

**Corollary 2.19**  
If $E$ does not contain parallelepipeds of dimension $n$ then

$$|E \cap A_r(N, X_1, \ldots, X_d)| \leq 2^{de(n)}c(n)(2N + 1)^{de(n)}$$

for $1 \leq r \leq \infty$, where $c(n)$ and $0 < e(n) < 1$ are as in Proposition 2.17.

(Compare with Theorem 2.5.)

**Proof**  
Observe that

$$A_r(N, X_1, \ldots, X_d) \subset A_\infty(N, X_1, \ldots, X_d) = \prod_{i=1}^{d} A_\infty(N, X_i).$$

But $A_\infty(N, X_i)$ is an arithmetic progression of length $(2N + 1)$, so $\prod_{i=1}^{d} A_\infty(N, X_i)$ is contained in a pseudo-parallelepiped of dimension at most $Md$, where $M$ is chosen so that $2^{M-1} < 2N + 1 \leq 2^M$. 

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Hence

$$|E \cap A_r(N, X_1, \ldots, X_d)| \leq c(n)2^{dM\varepsilon(n)} \leq 2^{dc(n)}c(n)(2N + 1)^{dc(n)}.$$ 

\\

**Corollary 2.20**  If $E_1$ and $E_2$ do not contain parallelepipeds of arbitrarily large dimension then neither does $E_1 \cup E_2$.

**Proof**  Assume $E_1$ and $E_2$ do not contain parallelepipeds of dimension $n$ and let $P_N$ be any parallelepiped of dimension $N$.

Then

$$|(E_1 \cup E_2) \cap P_N| \leq |E_1 \cap P_N| + |E_2 \cap P_N|$$

$$\leq 2c(n)2^{N\varepsilon(n)}$$

which is less than $|P_N|$ if $N$ is sufficiently large. 

\\

Although this corollary by no means settles the union question for $A(p)$ sets it does provide some evidence to support the belief in an affirmative answer.

**Corollary 2.21**  The set of primes in $\mathbb{Z}$ is not a $A(p)$ set for any $p > 0$. Indeed, if $E = \{n_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ does not contain parallelepipeds of arbitrarily large dimension, then

$$\sum_{n_k \neq 0} \frac{1}{|n_k|} < \infty.$$
Proof If $E$ does not contain parallelepipeds of arbitrarily large dimension then there are constants $c$ and $0 < \varepsilon < 1$ so that $|E \cap [2^j, 2^{j+1})| \leq c2^{j\varepsilon}$ since $[2^j, 2^{j+1})$ is an arithmetic progression of length $2^j$.

Hence

$$\sum_{n_k \neq 0} \frac{1}{|n_k|} = \sum_{j=0}^{\infty} \sum_{2^j \leq |n_k| < 2^{j+1}} \frac{1}{|n_k|} \leq \sum_{j=0}^{\infty} \frac{c2^{j\varepsilon}}{2^j} < \infty.$$  

For $E$ a subset of the positive integers and $n \in \mathbb{Z}$ let $r_2(E, n)$ be, as in [21], the number of ordered pairs $\{m_1, m_2\}$ with $m_1, m_2 \in E$ and $m_1 + m_2 = n$.

In [17] Neugebauer showed that if $E$ was a $A(p)$ set in $\mathbb{Z}^+$ with $p = 2q' > 2$, then if $1/q + 1/q' = 1$, 

$$\left( \sum_{n=1}^{N} r_2(E, n)^q \right)^{1/q} \leq cN^{2/p}.$$ 

For $p = 4$, $q' = 2$ this result may be found in [21, 4.5].

Corollary 2.22 If $E \subset \mathbb{Z}^+$ does not contain parallelepipeds of arbitrarily large dimension, then there is some $p = 2q' > 2$ and a constant $c$ so that $E$ satisfies the inequality

$$\left( \sum_{n=1}^{N} r_2(E, n)^q \right)^{1/q} \leq cN^{2/p}$$
for all positive integers \( N \).

**Proof** Observe that if \( \{a_i, b_i\} \) is the set of pairs in \( E \) with \( a_i + b_i = n \), then \( a_i, b_i \in (0, n] \) and if \( \{a_i, b_i\} \) and \( \{a_j, b_j\} \) are distinct pairs as above, then \( a_i \neq a_j \).

Thus

\[
\frac{r_2(E, n)}{2} \leq |(0, n] \cap E| \leq 2n^c c
\]

where constants \( c \) and \( 0 < \varepsilon < 1 \) are chosen as in Proposition 2.17.

Hence

\[
\left( \sum_{n=1}^{N} r_2(E, n)^q \right)^{1/q} \leq \left( \sum_{n=1}^{N} (2n^c c)^q \right)^{1/q} \leq N^{1/q} 2N^c c.
\]

Now \( N^{2/p} = N^{1-1/q} \) so that if \( \frac{2}{1 - \varepsilon} \leq q < \infty \) then \( p = 2q' \geq \frac{4}{1 + \varepsilon} > 2 \), hence

\[
\left( \sum_{n=1}^{N} r_2(E, n)^q \right)^{1/q} \leq 2c N^{2/p}.
\]

Corollary 2.23 If \( E \subset \mathbb{Z} \) does not contain parallelepipeds of arbitrarily large dimension then

(a) \( E \) has uniformly large gaps, and

(b) \( E \) has zero uniform density i.e., \( \lim_{N \to \infty} \sup_{a \in \mathbb{Z}} \frac{|[a, a + N] \cap E|}{N} = 0 \).

(Compare with Corollary 2.3.)
Proof  (a) Imitate the proof of Corollary 2.3 replacing Proposition 2.2 by Corollary 2.18. Alternatively, (a) follows from (b); see for example the proof of Corollary 2.33.

(b) Apply Corollary 2.18 to the arithmetic progression \([a, a + N]\) of length \(N + 1\).

Previously the zero uniform density of \(A(p)\) sets was readily seen for \(p > 2\) as a result of Proposition 2.2. It was only known that \(A(1)\) sets had zero density, and this as a result of a difficult theorem of Szemerédi [22] on sets not containing arbitrarily long arithmetic progressions. The proof based on parallelepipeds not only gives a stronger conclusion but seems much easier.

These corollaries show that sets which do not contain parallelepipeds of arbitrarily large dimension satisfy the previously known necessary arithmetic properties of \(A(2)\) sets.

2.4 The Uniformly Large Gap Property

Corollaries 2.18, 2.19, 2.22 and 2.23 use the notion of parallelepipeds to prove extensions in the spirit of previously known results. Our next goal will be to generalize the notion of uniformly large gaps to the setting of arbitrary discrete abelian
groups and show that sets which do not contain parallelepipeds of arbitrarily large
dimension have this property.

Toward this end we make the following definitions:

**Definition 2.24** Let $F$ be a finite subset of $\Gamma$ and $\chi, \psi \in \Gamma$. We say that $\chi$ is
$F$-equivalent to $\psi$ if for some positive integer $m$ there is a sequence

$$\chi = \chi_1, \ldots, \chi_m = \psi$$

with $\chi_{i+1} \chi_i^{-1} \in F$ for $i = 1, 2, \ldots, m - 1$. Such a sequence will be called an $F$-chain
joining $\chi$ to $\psi$.

If $\chi_i \in E$ for $i = 1, 2, \ldots, m$, then $\chi_1, \ldots, \chi_m$ will be said to be an $F$-chain in $E$
joining $\chi$ to $\psi$, and in this case $\chi$ will be said to be $(E, F)$-equivalent to $\psi$.

When $F$ is a symmetric subset of $\Gamma$ containing the identity this relation is an
equivalence relation.

The terminology is suggested by that of [13, 8.9].

If $E \subset \mathbb{Z}$ and $F = [-N, N]$ then $n < m \in E$ are $(E, F)$-equivalent provided
the interval $[n, m]$ does not contain any subinterval of length greater than $N$ free
of points of $E$. $E$ has uniformly large gaps provided for every integer $N$ there is an
integer $s > 0$ with the property that if $n - m \notin [-sN, sN]$ then $n$ and $m$ are not
$(E \cup \{n, m\}, [-N, N])$-equivalent. This has the obvious generalization stated below.
Definition 2.25  

$E \subseteq \Gamma$ has the **uniformly large gap property** provided for each finite, symmetric set $F \subseteq \Gamma$ containing the identity, there is an integer $s > 0$ such that if $\chi\psi^{-1} \notin F^s$, then $\chi$ and $\psi$ are not $(E \cup \{\chi, \psi\}, F)$-equivalent.

Our next theorem extends Corollary 2.23(a) to the general setting with this interpretation of uniformly large gaps.

**Theorem 2.26**  

Suppose $E \subseteq \Gamma$ does not contain any parallelepipeds of dimension $n$ and $F$ is a finite symmetric subset of $\Gamma$ containing the identity. There is a constant $s = s(n, F)$ so that whenever $\{X_i\}_{i \in I} \subseteq \Gamma$ satisfy $X_iX_j^{-1} \notin F^s$ for all $i \neq j$, then $X_i$ and $X_j$ belong to distinct $(E \cup \{X_i\}_{i \in I}, F)$-equivalence classes when $i \neq j$.

Thus $E$ has the uniformly large gap property whenever $E$ does not contain parallelepipeds of arbitrarily large dimension. In particular, all $A(p)$ sets, $p > 0$, have the uniformly large gap property.

Before proceeding with the proof we establish two lemmas. The second is motivated in part by [23].

**Lemma 2.27**  

Let $1 \neq \varphi \in \Gamma$. Suppose $P' = \{X_i\}_{i=1}^{2^N} \cup \{\psi_i\}_{i=1}^{2^N}$ consists of $2^{N+1}$ distinct elements of $\Gamma$ with $X_i\psi_i^{-1} = \varphi$ for $i = 1, 2, \ldots, 2^N$. If one of $\{X_i\}_{i=1}^{2^N}$ or $\{\psi_i\}_{i=1}^{2^N}$ is a parallelepiped of dimension $N$, then $P'$ is a parallelepiped of dimension $N + 1$.  

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Proof. Without loss of generality assume \( \{X_i\}_{i=1}^{2N} = P_1 \cdots P_N = P \) with \(|P_1| = 2\) and \(|P| = 2^N\). Then \( P^{-1} = \{\psi_i\}_{i=1}^{2N} \) so \( P' = P_1 \cdots P_N \cdot \{1, \psi^{-1}\} \).

Lemma 2.28. Let \( F \) be any finite subset of \( \Gamma \). For each positive integer \( n \) there is a constant \( k(n) = k(n, F) \) so that if \( r \geq k(n) \) and \( \{X_i\}_{i=1}^r \) is an \( F \)-chain joining \( X_1 \) and \( X_r \), with \( X_i \neq X_j \) if \( i \neq j \), then \( \{X_i\}_{i=1}^r \) contains a parallelepiped of dimension \( n \).

Proof. Since any two element set is a parallelepiped of dimension one we may set \( k(1) = 2 \).

Now proceed inductively assuming the result for \( n \). We consider the \( F \)-chain of distinct terms \( \{X_i\}_{i=1}^r \) with \( r \geq 2k(n)|F|^{k(n)-1} = k(n+1) \).

Notice that each of the sets

\[ B_1 = \{X_i\}_{i=1}^{k(n)} \quad B_2 = \{X_i\}_{i=k(n)+1}^{2k(n)} \quad \cdots \quad B_N = \{X_i\}_{i=(N-1)k(n)+1}^{Nk(n)} \]

where \( N = k(n+1)/k(n) \), form an \( F \)-chain of \( k(n) \) distinct terms, so by the induction hypothesis each contains a parallelepiped of dimension \( n \).

Observe that any two subsets of \( \Gamma \), say \( A = \{\alpha_i\}_{i=1}^m \) and \( B = \{\beta_i\}_{i=1}^m \), are translates of one another, i.e., \( A = BX \) for some \( X \in \Gamma \), if \( \alpha_{i+1} \alpha_i^{-1} = \beta_{i+1} \beta_i^{-1} \) for all \( i = 1, \ldots, m-1 \). If in addition \( A \cap B = \emptyset \) and \( A \) contains a parallelepiped of
dimension \( n \), then by Lemma 2.27 \( A \cup B \) contains a parallelepiped of dimension \( n + 1 \).

Since the set \( \{X_i\}_{i=1}^r \) is an \( F \)-chain, there are only \( |F| \) choices for each of the characters \( X_{i+1}X_i^{-1} \). Thus there can be at most \( |F|^{k(n)-1} \) different sets of the form

\[
\{X_{i+1}X_i^{-1}\}_{i=(j-1)k(n)+1}^{jk(n)-1}, \ j = 1, \ldots, N.
\]

(Here we count different orderings as different sets.) But \( N \) was chosen to be twice this number, so at least two of the sets \( B_1, \ldots, B_N \) must be translates. Their union, and hence \( \{X_i\}_{i=1}^r \), must contain a parallelepiped of dimension \( n + 1 \). This completes the induction step.

We now prove the theorem. Recall that by assumption \( E \) does not contain parallelepipeds of dimension \( n \).

**Proof of Theorem 2.26** We will show that if \( s = k(n, F) \) and \( X_iX_j^{-1} \notin F^g \) for all \( i \neq j \), then \( X_i \) and \( X_j \) belong to distinct \((E \cup \{X_i\}_{i \in I}, F)\)-equivalence classes when \( i \neq j \).

Suppose not. Then there is an \( F \)-chain in \( E \cup \{X_i\}_{i \in I}, \psi_1, \ldots, \psi_m \), joining some pair \( X_i, X_j \). Without loss of generality we may assume \( \psi_2, \ldots, \psi_{m-1} \in E \), i.e., \( X_j \) is the first occurrence in the chain after \( X_i \) from possibly outside of \( E \). If two of the characters \( \psi_k \) and \( \psi_l \) were equal, then the sequence \( \psi_1, \ldots, \psi_k, \psi_{l+1}, \ldots, \psi_m \),
upon renumbering, would still be an $F$-chain joining $X_i$ and $X_j$, so we may assume $\psi_2, \ldots, \psi_{m-1}$ are distinct.

Because $\psi_{i+1}\psi_i^{-1} \in F$ for $i = 1, \ldots, m - 1$, it follows that $X_j \in F^{m-1}X_i$ and since $F^{m-1} \subseteq F^s$ if $m - 1 \leq s$, we must have $m - 2 \geq s = k(n, F)$.

Thus the $F$-chain in $E$, $\{\psi_i\}_{i=2}^{m-1}$, consists of at least $k(n, F)$ distinct terms and hence by the lemma $E$ must contain a parallelepiped of dimension $n$. This contradiction establishes the theorem. \\

Arguments similar to those above, but not given here, can be used to show that if $E$ contains no parallelepipeds of dimension $n$, then the cardinality of any $F$-equivalence class in $E$ is at most $\frac{|F|^{k(n, F)} - 1}{|F| - 1}$.

\section{M\textsubscript{n} Sets}

\textbf{Definition 2.29} A subset $E$ of $\Gamma$ is said to \textit{tend to infinity} if for each finite subset $\Delta$ of $\Gamma$, there is a finite subset $F$ of $E$ such that if $\chi, \psi \in E \setminus F$ and $\chi \neq \psi$ then $\chi\psi^{-1} \not\in \Delta$.

A subset $E$ of $\mathbb{Z}$ tends to infinity if for each positive integer $N$ only finitely many points of $E$ differ in absolute value by at most $N$. Thus any lacunary set, for example, tends to infinity.
Any Sidon set is known to be a finite union of sets which tend to infinity. (See [13, 9.1] for countable Sidon sets and [4] for the general case.)

Sets which tend to infinity can readily be shown to have special analytic properties. For example:

(1) Deschamps-Gondim [5, 5.1]:

If $E$ is a symmetric Sidon set which tends to infinity, then for every compact set $K$ in $G$ with non-void interior there is a finite set $F \subset E$ and a constant $c > 0$ such that

$$\sum_{x \in F} |\hat{f}(x)| \leq c \|\max(f, 0) 1_K\|_{\infty}$$

for all real-valued $E \setminus F$ polynomials.

(2) Bonami [3, chap. IV]:

If $E$ is a $A(4)$ set which tends to infinity and $S$ is a subset of a connected group $G$ with the Haar measure of $S$ positive, then there is a constant $c(S) > 0$ such that

$$\|f\|_2^2 \leq c(S) \|f 1_S\|_2^2$$

for all $E$-polynomials $f$.

The hypothesis of tending to infinity was shown by Deschamps-Gondim to be unnecessary in example 1. The conclusion of example 1 is in fact true for all Sidon...
sets. For a presentation of this see [13, chap. 9]. We will show in Chapter 3 that it is unnecessary in example 2 as well.

One of the ideas we use to prove this is to show that sets which tend to infinity are a central feature in the structure of all $A(p)$ sets. In a sense all $A(p)$ sets are built out of sets which tend to infinity and sets with large gaps between members. These ideas were introduced by Miheev in [15] for subsets of $\mathbb{Z}$. We extend them to all discrete abelian groups.

We define sets of class $M_n$ inductively in the following manner:

**Definition 2.30** $M_0$ is the class of subsets of $\mathbb{I}$ which tend to infinity.

$M_n$ is the class of subsets $E$ of $\mathbb{I}$ which have the following property: for each finite set $\Delta \subset \mathbb{I}$, $E$ can be expressed as $E_1 \cup E_2$, where $\chi, \psi \in E_2$, $\chi \neq \psi$, implies $\chi \psi^{-1} \notin \Delta$, and $E_1$ is a finite union of sets in class $M_{n-1}$.

In [15] Miheev shows that each class $M_n$ contains a $A(4)$ set in $\mathbb{Z}$ which is not a finite union of sets in class $M_{n-1}$. He shows that any subset of $\mathbb{Z}$ which does not contain parallelepipeds of arbitrarily large dimension belongs to class $M_n$ for some $n$. He also establishes that any sequence of integers which belongs to some class $M_n$ has zero uniform density. A consequence of this fact, as is shown in Corollary 2.33, is that all subsets of $\mathbb{Z}$ which belong to some class $M_n$ have uniformly large gaps.
This gives another proof that all $A(p)$ sets in $\mathbb{Z}$, $p > 0$, have the uniformly large gap property.

First however, we illustrate how [15, Thm. 3] may be adapted to the general setting.

**Theorem 2.31** Every subset $E$ of $\Gamma$ which does not contain parallelepipeds of dimension $n$ belongs to class $M_{n-2}$. In particular all $A(p)$ sets, $p > 0$, belong to class $M_n$ for some $n \geq 0$.

**Proof** If $E$ does not tend to infinity then for some finite subset $\Delta$ of $\Gamma$ and each finite subset $F$ of $E$, there are distinct members of $E \setminus F$, say $\chi$ and $\psi$, with $\chi\psi^{-1} \in \Delta$. Since $\Delta$ is a finite set infinitely many of these pairs satisfy $\chi\psi^{-1} = \varphi$ for some $\varphi \in \Delta$, $\varphi \neq 1$. By Lemma 2.27 $E$ contains a parallelepiped of dimension two proving the theorem for $n = 2$.

Now proceed inductively assuming the result for $n (\geq 2)$. Suppose that $E$ contains no parallelepipeds of dimension $n + 1$. Let $\Delta$ be any finite subset of $\Gamma$. If only finitely many pairs $\{\chi, \psi\} \subset E$, $\chi \neq \psi$, satisfy $\chi\psi^{-1} \in \Delta$, then $E$ satisfies the definition of class $M_{n-2}$, (even $M_0$), with regards to $\Delta$; so assume otherwise. As before $E$ contains infinitely many two element sets $P_i = \{\chi_i, \psi_i\}$ with $\chi_i\psi_i^{-1} = \varphi_i \in \Delta$, $\varphi_i \neq 1$. Choose a maximal collection of these sets, say $\{P_i\}_{i \in I}$, subject to $P_i \cap P_j = \emptyset$ for $i \neq j$. Then whenever $\chi\psi^{-1} = \varphi_1$, $\chi, \psi \in E$,
at least one of $X$ or $\psi$ belongs to a set in the maximal collection. By Lemma 2.27 neither of the sets $\{X_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ may contain any parallelepipeds of dimension $n$, since $E$ was assumed to contain no parallelepipeds of dimension $n + 1$. Applying the induction hypothesis we conclude that both sets belong to class $M_{n-2}$.

Remove $\{X_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ from $E$. Notice that if $X$ and $\psi$ belong to this new set, then $X \psi^{-1} \neq \varphi_1$, thus after repeating this process at most $|\Delta|$ times and deleting from $E$ the (at most) $2|\Delta|$ sets of class $M_{n-2}$ so obtained, there will be only finitely many pairs $\{X, \psi\}, X \neq \psi$, remaining in $E$, with $X \psi^{-1} \in \Delta$. Since the addition of finitely many terms does not affect the class we may write $E$ as $E_1 \cup E_2$, where $E_1$ consists of finitely many sets of class $M_{n-2}$, and whenever $X, \psi \in E_2$, $X \neq \psi, X \psi^{-1} \notin \Delta$. Since $\Delta$ was arbitrary this implies that $E$ belongs to class $M_{n-1}$.

Of course, the converse to Theorem 2.31 is not true. There are subsets of $\mathbb{Z}$ which tend to infinity and contain parallelepipeds of arbitrarily large dimension, even arbitrarily long arithmetic progressions; for example

$$\{1, 2, 4, 7, 10, 13, 17, 21, 25, 29, 34, 39, 44, 49, 54, \ldots \}.$$ 

**Proposition 2.32** If $E \subset \mathbb{Z}$ belongs to class $M_n$ for some $n$, then $E$ has zero uniform density.
A proof by induction on $n$ can be found in [15, Thm. 2].

**Corollary 2.33** If $E \subseteq \mathbb{Z}$ belongs to class $M_n$ for some $n$, then $E$ has the uniformly large gap property.

**Proof** Actually we will prove that if a subset of $\mathbb{Z}$ has zero uniform density then it has the uniformly large gap property.

Suppose this is not the case. Then there is a positive integer $N$ with the property that for every $M$ there is an interval of length $M$ which does not contain a subinterval of length $N$ free of points of $E$. These intervals must contain at least $M/N$ terms from $E$. Hence

$$\limsup_{M \to \infty} \sup_{a \in \mathbb{Z}} \frac{|E \cap [a, a + M]|}{M} \geq 1/N$$

contradicting the fact that $E$ has zero uniform density.

As there are sets in class $M_n$ which contain parallelepipeds of arbitrarily large dimension this result is more general than Corollary 2.23. Indeed the proof shows that any set with zero uniform density, such as the primes in $\mathbb{Z}$, has uniformly large gaps.
Chapter 3
Strict-Two-Associatedness for Uniformizable $A(2)$ Sets

3.1 Preliminaries

**Definition 3.1** [13, 9.3] $E \subset \Gamma$ is said to be *strictly-2-associated* with a measurable subset $S$ of $G$ provided there is a constant $c = c(S, E) > 0$ so that for all $E$-polynomials $f$,

$$\|1_S f\|_2^2 \geq c(S, E)\|f\|_2^2.$$  \hfill (1)

In this case, inequality (1) holds for all $f \in L^2_E(G)$. To see this choose a net $\{P_\alpha\}$ of $E$-polynomials converging in $L^2(G)$ to $f \in L^2_E(G)$. For example, we could take for $P_\alpha$, $f \ast h_\alpha$, where $\{h_\alpha\}$ is a bounded approximate unit for $L^1(G)$ consisting of trigonometric polynomials. Certainly

$$\|P_\alpha 1_S\|_2^2 \to \|f1_S\|_2^2 \quad \text{and} \quad \|P_\alpha 1_S\|_2^2 \geq c\|P_\alpha\|_2^2 \to c\|f\|_2^2,$$

hence (1) holds for $f$. 44
Any $c > 0$ which satisfies (1) for all $E$-polynomials $f$ will be called a constant of strict-2-associatedness for $E$ and $S$.

Our main result may be stated for connected groups, such as $T$, as follows:

**Theorem 3.2**  
Let $G$ be a connected group. If $E$ is a uniformizable $A(2)$ set in $T$ then $E$ is strictly-2-associated with all measurable subsets of $G$ with positive measure.

Theorem 3.2 is a special case of Theorem 3.5, which will be stated and proved later in this chapter.

In [15] Miheev obtained the conclusion of Theorem 3.2 for $A(p)$ sets in $Z$ with $p > 2$. As all $A(p)$ sets with $p > 2$ are uniformizable $A(2)$ sets, our work is a theoretical improvement even for subsets of $Z$. In [3] Bonami showed that if $G$ was a connected group and $E$ was a $A(4)$ set which tended to infinity, then $E$ was strictly-2-associated with all measurable subsets of $G$ with positive measure. As there are many known examples of $A(4)$ sets which do not tend to infinity, our work improves upon this result as well.

We follow the general outline used by Miheev but give completely different proofs of most of the intermediate steps, as his proofs do not seem to adapt to uniformizable $A(2)$ sets and/or to the general setting. On $Z$ our proofs are simpler.
than Miheev's. For the general setting we will use the uniformly large gap property discussed in Chapter 2.

As is proven in Proposition 3.4 below, when \( G \) is not connected one can construct polynomials which vanish on certain subsets of \( G \) having positive measure. Thus in the general setting a further hypothesis is necessary. This hypothesis is related to the next definition which is given in [13, 8.2].

**Definition 3.3** \( E \subseteq \Gamma \) is said to be \( X_0 \)-subtransversal, for \( X_0 \) a subgroup of \( \Gamma \), if whenever \( X, \psi \in E, X \neq \psi \), we have \( X\psi^{-1} \notin X_0 \).

This says that each coset of \( X_0 \) intersects \( E \) in at most one point.

**Proposition 3.4** A subset \( E \) of \( \Gamma \) is \( X_0 \)-subtransversal for all finite subgroups \( X_0 \) of \( \Gamma \) if and only if the only \( E \)-polynomial which can vanish on an open, non-empty subset of \( G \) is the identically zero function.

**Proof** \((\Leftarrow)\) If \( X_0 \) is a finite subgroup of \( \Gamma \) then its annihilator \( G_0 \) is open and non-empty, hence has positive measure. Suppose \( X, \psi \in E \) and \( X\psi^{-1} \in X_0 \). The \( E \)-polynomial \( \chi - \psi \) vanishes on \( G_0 \) and thus must be identically zero. Hence \( \chi = \psi \) and \( E \) is \( X_0 \)-subtransversal.
A proof is given in [13, 8.12]. We prove a stronger result in Proposition 3.9.

Thus in order for $E$ to be strictly-2-associated with all subsets of $G$ it is necessary that $E$ be $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$. This is also the sufficient additional hypothesis we need to make the conclusion of Theorem 3.2 true in the general setting.

### 3.2 The Main Theorem

**Theorem 3.5** Let $E$ be a uniformizable $A(2)$ set in $\Gamma$. If $E$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$ then $E$ is strictly-2-associated with all measurable subsets of $G$ with positive measure.

Before beginning the proof we make a few observations.

**Remarks** If $G$ is a connected group then $\Gamma$ is a torsion free group, and $\{1\}$ is the only finite subgroup of $\Gamma$. Thus any subset $E$ of $\Gamma$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$, and so Theorem 3.5 reduces to Theorem 3.2 when $G$ is connected.

In [12] López proves that all $A(4)$ sets which tend to infinity and are $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$ are strictly-2-associated with all mea-
surable subsets of $G$ with positive measure. A presentation of this may be found in [13, chap. 9].

The proof of Theorem 3.5 will comprise most of the rest of this chapter. Its organization is roughly as follows:

(1) Show that whenever subsets of a uniformizable $A(2)$ set are all strictly-2-associated with $S \subset G$ with a common constant of strict-2-associatedness, and these subsets have sufficiently large "gaps" between them, then their union is again strictly-2-associated with $S$. This will be established in Lemma 3.6.

(2) Use (1) together with the uniformly large gap property established in Chapter 2, to show that the union of a uniformizable $A(2)$ set strictly-2-associated with $S$ and a uniformizable $A(2)$ set with large "distances" between members is strictly-2-associated with $S$. This step will be accomplished in 3.7 - 3.11.

(3) Use the definition of class $M_n$ as the union of a set with large "distances" between members and finitely many sets in class $M_{n-1}$ to present an induction argument completing the proof of the theorem.

Early in this process (Corollary 3.10) we will be able to extend the results of Bonami and López to uniformizable $A(2)$ sets which tend to infinity.

Unless specified otherwise, $E$ will denote a uniformizable $A(2)$ set in $\Gamma$, and $S$
will be a subset of $G$ of positive Haar measure.

**Lemma 3.6** For each $c > 0$, there is a finite symmetric set $F = F(E, S, c) \subset \Gamma$, containing the identity, so that if the subsets $\{E_i\}_{i \in I}$ of $E$ are strictly-2-associated with $S$ with $c$ a constant of strict-2-associatedness for each, and

$$E_i E_j^{-1} \cap F = \emptyset \text{ for } i \neq j,$$

then $\bigcup_{i \in I} E_i$ is also strictly-2-associated with $S$.

**Proof** Since $E$ is a uniformizable $A(2)$ set there is a Young function $\Phi$ and constant $K$ so that $\|f\|_{\Phi}^2 \leq K \|f\|_2^2$ whenever $f \in L_2^2(G)$, with $\Phi(x) = \phi(x^2)$, $\phi$ a "strongly convex" function (Theorem 1.10(5)). Without loss of generality suppose $\phi$ is itself a Young function and hence has conjugate $\psi$. Define a Young function $\beta(x) = \psi(x^2)$. Let $\varepsilon = \frac{c}{24K}$ and choose a trigonometric polynomial $P$ satisfying $\|P - 1_S\|_{\beta}^2 < \varepsilon$.

Let $F = F(E, S, c) = (\text{supp} \hat{P})(\text{supp} \hat{P})^{-1}$. Clearly $F$ is a finite symmetric subset of $\Gamma$ containing the identity.

Suppose $\{E_i\}_{i \in I} \subset E$ are strictly-2-associated with $S$ with constant of strict-2-associatedness $c$, and satisfy $E_i E_j^{-1} \cap F = \emptyset$ for all $i \neq j$. We estimate $\|1_S f\|_2^2$ for any $\bigcup_{i \in I} E_i$-polynomial $f$. Write $f$ as $\sum_{i \in I} f_i$ with $f_i$ an $E_i$-polynomial for each
\[ f_i \mathcal{P}(x) \neq 0 \quad \text{and} \quad f_j \mathcal{P}(x) \neq 0, \]

then

\[(\text{supp} f_i)(\text{supp} f_j)^{-1} \cap F \neq \emptyset.\]

In particular \(E_i E_j^{-1} \cap F \neq \emptyset; \) so \(i = j.\) Hence

\[\|f P\|_2^2 = \sum_{i \in I} \|f_i P\|_2^2.\]

Since the sets \(\{E_i\}_{i \in I}\) are disjoint and strictly-2-associated with \(S\) with constant of strict-2-associatedness \(c,\) we have

\[ c\|f\|_2^2 = \sum_{i \in I} c\|f_i\|_2^2 \leq \sum_{i \in I} \|f_i 1s\|_2^2. \]

Hence

\[ c\|f\|_2^2 \leq 2 \sum_{i \in I} \left(\|1s - P\| f_i\|_2^2 + \|P f_i\|_2^2\right) \]
\[ \leq 2 \sum_{i \in I} 2\|1s - P\|_2^2 \|f_i\|_2^2 + 2\|P f\|_2^2 \]
\[ \leq 2 \sum_{i \in I} 2e\|f_i\|_2^2 + 2\|P f\|_2^2, \]

with the last inequality following from the choice of \(P.\)

Now the functions \(f_i\) are \(E\)-polynomials, so \(\|f_i\|_2^2 \leq K\|f_i\|_2^2.\) Thus we obtain

\[ c\|f\|_2^2 \leq 4eK \sum_{i \in I} \|f_i\|_2^2 + 2\|P f\|_2^2 \]
\[ \leq 4eK\|f\|_2^2 + 2\|P f\|_2^2. \]
Similarly
\[
\| Pf \|_2^2 \leq 2\| (1 - P) f \|_2^2 + 2\| 1_s f \|_2^2
\]
\[
\leq 4\varepsilon K \| f \|_2^2 + 2\| 1_s f \|_2^2.
\]
Thus
\[
c\| f \|_2^2 \leq 12\varepsilon K \| f \|_2^2 + 4\| 1_s f \|_2^2.
\]

Our choice of \( \varepsilon \) implies
\[
\| 1_s f \|_2^2 \geq \frac{c}{8} \| f \|_2^2
\]
for all \( \bigcup_{i \in I} E_i \)-polynomials \( f \), which completes the proof. ////

For \( E_i, E_j \subset \mathbb{Z} \) let
\[
d(E_i, E_j) = \min\{|n_i - n_j| : n_i \in E_i, n_j \in E_j\},
\]
the size of the smallest gap between \( E_i \) and \( E_j \).

Lemma 3.6 says that if the sets \( \{E_i\} \subset E \subset \mathbb{Z} \) are strictly-2-associated with \( S \) with a common constant of strict-2-associatedness \( c \), then there is an integer \( N = N(E, S, c) \) so that if \( d(E_i, E_j) \geq N \) for \( i \neq j \), then \( \bigcup_{i \in I} E_i \) is strictly-2-associated with \( S \). Hence if the gaps between the sets \( \{E_i\}_{i \in I} \) are sufficiently large, their union is also strictly-2-associated with \( S \).

For the second step indicated in the outline of the proof of Theorem 3.5 we first establish
Proposition 3.7  Suppose $E$ is strictly-2-associated with $S$. There is a finite subgroup $X$ of $\Gamma$, depending on $E$ and $S$, so that whenever the set $E \cup \{X\}$, $X \in \Gamma$, is $X$-subtransversal, then $E \cup \{X\}$ is strictly-2-associated with $S$, with constant of strict-2-associatedness independent of $X$.

Indeed, if $c$ is a constant of strict-2-associatedness for $E$ and $S$, then $E \cup \{X\}$ and $S$ have as a constant of strict-2-associatedness

$$
\left( \frac{c \varepsilon}{m(V)} + m(S) + 2 \right)^{-1} \frac{c \varepsilon m(S)}{4m(V)}
$$

where $V$ is an open subset of $G$ whose choice depends on $E, S$ and $c$, and $\varepsilon$ is a constant determined by $V$.

First we state and prove a preliminary lemma.

Lemma 3.8  Let $g$ be a real-valued non-negative integrable function on $G$ which is not identically zero. There is a finite subgroup $X_0$ of $\Gamma$ and a constant $\varepsilon(g) > 0$ so that if $X, \psi \in \Gamma$ and $X\psi^{-1} \notin X_0$, then

$$
\int_G g|X - \psi|^2 \geq 2\varepsilon(g).
$$

Proof  Since $g$ is not identically zero there is a $\delta > 0$ so that the measure of the set $A = \{x \in G : g \geq \delta\}$ is positive.

Let $X, \psi \in \Gamma$. Observe that the integral $\int_A (X\psi^{-1}) \leq m(A)$ with equality if
and only if $X\psi^{-1} = 1$ on $A$, and hence on the smallest open subgroup containing $A$. Let $X_0$ be the annihilator of this subgroup.

If $G$ is connected the only open subgroup of $G$ is $G$ itself and thus $X_0$ would be trivial. In general, $X_0$ is a finite subgroup of $\Gamma$ and $\text{Re} \hat{\Delta}(X\psi^{-1}) = m(A)$ if and only if $X\psi^{-1} \in X_0$. An application of the Riemann-Lebesgue Lemma yields an $\epsilon > 0$ so that $\text{Re} \hat{\Delta}(X\psi^{-1}) < (1 - \epsilon)m(A)$ whenever $X\psi^{-1} \notin X_0$.

Thus, if $X\psi^{-1} \notin X_0$ we have

$$\int g|X - \psi|^2 \geq \delta \int 1_A|X - \psi|^2$$

$$= \delta \left( 2m(A) - 2\text{Re} \hat{\Delta}(X\psi^{-1}) \right)$$

$$\geq 2\epsilon \delta m(A).$$

Setting $\epsilon(g) = \epsilon \delta m(A)$, the lemma is established. ////

**Proof of Proposition 3.7** Suppose $c$ is a constant of strict-2-associatedness for $E$ and $S$.

Since $E$ is a uniformizable $A(2)$ set we can find by Theorem 1.10(4) a $\delta > 0$ so that whenever $m(A) < \delta$ and $f \in L^2_B(G)$,

$$\|1_A f\|_2^2 \leq \frac{c}{2} \|f\|_2^2.$$

The function

$$v \mapsto m(S) - 1_S \ast 1_{S^{-1}}(v) = m(S \setminus (SV^{-1}))$$
is continuous; so there is a neighbourhood $V$ of the identity in $G$ with

$$m(S \setminus (Sv^{-1})) < \delta$$

whenever $v \in V$. Hence if $S_v = (Sv^{-1}) \cap S$ and $f$ is an $E$-polynomial then

$$\|1_{S_v}f\|_2^2 \geq \|1_Sf\|_2^2 - \|1_{S \setminus S_v}f\|_2^2 \geq \frac{c}{2}\|f\|_2^2$$

whenever $v \in V$.

Given any $E \cup \{x\}$-polynomial $f$ and $v \in V$, we follow Bonami [3] and let

$$f_v(x) = f(xv) - \chi(v)f(x).$$

Observe that

$$\hat{f}_v(\psi) = \hat{f}(\psi)(\psi(v) - \chi(v))$$

so that $f_v$ is an $E$-polynomial.

The choice of $S_v$ ensures that

$$\|1_Sf\|_2^2 \geq \frac{1}{m(V)} \int_V \int_{S_v} \frac{|f(xv)|^2 + |f(x)|^2}{2} dm(x)dm(v)$$

so that by applying the basic inequality

$$2(|a|^2 + |b|^2) \geq |a + b|^2$$
to the line above and then using (1) and (2) we obtain

$$\|1_s f\|_2^2 \geq \frac{1}{4m(V)} \int_V \int_{S_0} |f_\nu(x)|^2 \, dm(x) \, dm(\nu)$$

$$\geq \frac{1}{4m(V)} \int_V \frac{c}{2} \|f_\nu\|_2^2 \, dm(\nu)$$

$$= \frac{1}{4m(V)} \frac{c}{2} \sum_{\psi \neq \chi} |\hat{f}(\psi)|^2 \int 1_V |\psi(x) - \chi(x)|^2 \, dm(\nu).$$

We assume that $E \cup \{\chi\}$ is $X$-subtransversal where $X$ is the finite subgroup $X_0$ chosen in Lemma 3.8 for the integrable function $1_V$. Then for all $\psi \in E$, $\chi \psi^{-1} \notin X_0$, hence by Lemma 3.8

$$\|1_s f\|_2^2 \geq \frac{1}{4m(V)} \frac{c}{2} \sum_{\psi \neq \chi} |\hat{f}(\psi)|^2 2\varepsilon(1_V).$$

(3)

The basic inequality used before also shows that

$$\|1_s f\|_2^2 \geq \frac{1}{2} \int_S |\hat{f}(x) \psi(x)|^2 \, dm(x) - \int_S \left| \sum_{\psi \neq \chi} \hat{f}(\psi) \psi(x) \right|^2 \, dm(x)$$

$$\geq \frac{1}{2} |\hat{f}(x)|^2 m(S) - \sum_{\psi \neq \chi} |\hat{f}(\psi)|^2.$$

(4)

Thus by considering the two cases:

(i) $\sum_{\psi \neq \chi} |\hat{f}(\psi)|^2 \geq \delta \|f\|_2^2$; or

(ii) $\sum_{\psi \neq \chi} |\hat{f}(\psi)|^2 \leq \delta \|f\|_2^2$ in which case $|\hat{f}(x)|^2 \geq (1 - \delta) \|f\|_2^2$;

for

$$\delta = m(S) \left( \frac{c\varepsilon(1_V)}{2m(V)} + m(S) + 2 \right)^{-1}$$

and substituting into (3) or (4), respectively, we obtain the constant of strict-2-
associatedness given in the proposition with $\epsilon = \epsilon(1_Y)$.

Remarks Miheev in [15, Thm. 6] proved a similar result for subsets of $\mathbb{Z}$, without obtaining a specific constant of strict-2-associatedness. His proof relied on special properties of $\mathbb{Z}$.

A weaker version of Proposition 3.7, without the requirement that the constant of strict-2-associatedness be independent of $X$, was proved for $A(p)$ sets, $p > 2$, by Bonami for connected groups and López for the general case. A presentation of this may be found in [13, chap. 9]. The same proof may be applied to uniformizable $A(2)$ sets by making use of the uniform integrability property, Theorem 1.10(4). It is possible to prove Proposition 3.7 without obtaining a specific value for the constant of strict-2-associatedness by soft methods based on Bonami's result. We do not give the details here.

Lemma 3.8 is a special case of the next proposition. For subsets of $\mathbb{Z}$ this proposition was known by Zygmund [24], and for the general setting a proof can be found in [13, 8.14]. We present here a new proof which is constructive. The technique is similar to the proof of Proposition 3.7.

Proposition 3.9 Let $\sigma$ be a positive integer and $g$ be a real-valued, non-negative integrable function on $G$ which is not identically zero. There is a finite
subgroup \( X \) of \( \Gamma \) and a constant \( \varepsilon_1(\sigma, g) > 0 \) so that whenever \( f \) is a polynomial with \( \text{supp} \hat{f} \) \( X \)-subtransversal and \( |\text{supp} \hat{f}| \leq \sigma \), then

\[
\int_G g |f|^2 \, dm \geq \varepsilon_1(\sigma, g) \|f\|_2^2.
\]

**Proof** As with the proof of Lemma 3.8 we may assume without loss of generality that \( g = 1_S \) for some measurable set \( S \) with \( m(S) > 0 \).

If \( \sigma = 1 \) the result holds trivially with \( \varepsilon_1 = m(S) \). We proceed by induction, assuming the result for all polynomials \( f \) with \( |\text{supp} \hat{f}| \leq \sigma - 1 \) and \( \text{supp} \hat{f} \) \( X \)-subtransversal for the appropriate subgroup \( X \).

As is seen in the proof of Proposition 3.7, it is possible to choose a neighbourhood \( V \) of the identity in \( G \) with

\[
m(S \setminus S v^{-1}) < \frac{\varepsilon_1(\sigma - 1, 1_S)}{2(\sigma - 1)}
\]

for all \( v \in V \). Let \( S_v = S v^{-1} \cap S \). Take for \( X \) the finite subgroup generated by the union of the finite subgroup given by the induction assumption for \( 1_S \) and \( \sigma - 1 \), and the finite subgroup \( X_0 \) determined in Lemma 3.8 for \( 1_V \).

Choose any \( x \in \text{supp} \hat{f} \) and let \( f_v(x) = f(xv) - \chi(v)f(x) \) for each \( v \in V \).

As in the proof of Proposition 3.7 we have

\[
\int_S |f|^2 \, dm \geq \frac{1}{4m(V)} \int_V \int_{S_v} |f_v(x)|^2 \, dm(x) \, dm(v).
\]
Again \( \hat{f}_v(\psi) = \hat{f}(\psi)(\psi(v) - \chi(v)) \) so \( \text{supp} \hat{f}_v \subset \text{supp} \hat{f} \). Since \( |\text{supp} \hat{f}_v| \leq \sigma - 1 \), from the induction assumption (and choice of \( X \)) we have

\[
\int_{S_v} |f_v(x)|^2 \, dm(x) = \int_S |f_v(x)|^2 \, dm(x) - \int_{S \setminus S_v^{-1}} |f_v(x)|^2 \, dm(x) \\
\geq \varepsilon_1(\sigma - 1, 1_s) \|f_v\|_2^2 - \|f_v\|_\infty^2 m(S \setminus S_v^{-1}).
\]

But

\[
\|f_v\|_\infty \leq \sum_\psi |\hat{f}_v(\psi)| \leq (\sigma - 1) \max_\psi |\hat{f}_v(\psi)| \leq (\sigma - 1) \|f_v\|_2.
\]

Thus

\[
\int_{S_v} |f_v(x)|^2 \, dm(x) \geq \varepsilon_1(\sigma - 1, 1_s) \|f_v\|_2^2 - (\sigma - 1) \|f_v\|_2^2 \left( \frac{\varepsilon_1(\sigma - 1, 1_s)}{2(\sigma - 1)} \right) \\
\geq \frac{\varepsilon_1(\sigma - 1, 1_s)}{2} \|f_v\|_2^2.
\]

Hence

\[
\int_S |f|^2 \, dm \geq \frac{\varepsilon_1(\sigma - 1, 1_s)}{8m(V)} \int_V \|f_v\|_2^2 \, dm(v) \\
= \frac{\varepsilon_1(\sigma - 1, 1_s)}{8m(V)} \sum_{\psi \neq X} |\hat{f}(\psi)|^2 \int_V |\psi(v) - \chi(v)|^2 \, dm(v) \\
\geq \frac{\varepsilon_1(\sigma - 1, 1_s)}{8m(V)} \sum_{\psi \neq X} |\hat{f}(\psi)|^2 2\varepsilon(1_V),
\]

the last inequality from Lemma 3.8 (again using the choice of \( X \)). But also

\[
\int_S |f|^2 \, dm \geq \frac{m(S)}{2} |\hat{f}(\chi)|^2 - \sum_{\psi \neq X} |\hat{f}(\psi)|^2.
\]

Again consider the two cases:

(i) \( \sum_{\psi \neq X} |\hat{f}(\psi)|^2 \geq \delta \|f\|_2^2 \), or
(ii) \( \sum_{\psi \neq \chi} |\hat{f}(\psi)|^2 \leq \delta \|f\|_2^2 \); for

\[
\delta = m(S)\left( \frac{\varepsilon_1(\sigma - 1,1_s)\varepsilon(1_V)}{2m(V)} + m(S) + 2 \right)^{-1}.
\]

Substituting into (5) or (6), respectively, we obtain the conclusion of the proposition with

\[
\varepsilon_1(\sigma,1_s) = \frac{\delta \varepsilon_1(\sigma - 1, 1_s)\varepsilon(1_V)}{4m(V)}.
\]

A consequence of Lemma 3.6 and Proposition 3.7 is

**Corollary 3.10** If \( E \), in addition to being a uniformizable \( A(2) \) set, is \( X_0 \)-subtransversal for all finite subgroups \( X_0 \) of \( \Gamma \) and tends to infinity, then \( E \) is strictly-2-associated with all measurable subsets of \( G \) with positive measure.

**Proof** Let \( S \subset G \) have positive measure. Let \( F = F(E,S,m(S)) \) be the finite set from Lemma 3.6. Since \( E \) is assumed to tend to infinity there is a finite set \( \Delta \) so that if \( \chi, \psi \in E \setminus \Delta, \chi \neq \psi \), then \( \chi \psi^{-1} \notin F \).

Now apply Lemma 3.6, taking as the sets \( E_i \) the singleton sets whose union is \( E \setminus \Delta \). Since the singleton sets are strictly-2-associated with \( S \) with constant of strict-2-associatedness equal to \( m(S) \), the choice of \( F \) ensures that \( E \setminus \Delta \) is also strictly-2-associated with \( S \).
Applying Proposition 3.7 $|\Delta|$ times we conclude that $E$ is strictly-2-associated with $S$.

The uniformly large gap property is used now to complete step 2 of the outline of the proof of Theorem 3.5.

**Lemma 3.11** Suppose $E$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$ and suppose $E' \subset E$ is strictly-2-associated with $S$. Then there is a finite set $F_1$ depending on $E$, $E'$ and $S$, so that whenever $E'' = \{X_i\}_{i \in I} \subset E$ satisfies $X_iX_j^{-1} \notin F_1$ if $i \neq j$, then $E' \cup E''$ is also strictly-2-associated with $S$.

**Proof** Let $c > 0$ be a constant of strict-2-associatedness for all of the sets $E' \cup \{X\}$, $X \in E$, and choose the finite symmetric set $F = F(E, S, c)$ containing the identity as in Lemma 3.6.

Being a uniformizable $A(2)$ set $E$ does not contain parallelepipeds of arbitrarily large dimension, and hence $E$ has the uniformly large gap property (Theorem 2.26). Choose the constant $s$ so that if $X_iX_j^{-1} \notin F^s$ for $i \neq j$, and $E'' = \{X_i\}_{i \in I}$, then the $(E, F)$-equivalence class containing $X_i$ does not contain any other $X_j \in E''$. Denote by $E_i$ the elements of this class which belong to $E' \cup E''$. Set $E_0 = E' \cup E'' \setminus \cup_{i \in I} E_i$.

We take for $F_1$ the finite set $F^s$.

For $i \in I$, $E_i \subset E' \cup \{X_i\}$ while $E_0 \subset E'$, hence $\{E_i\}_{i \in I \cup \{0\}}$ are strictly-2-associated with $S$ with constant of strict-2-associatedness $c$. By construction of the
equivalence relation, \( E_i E_j^{-1} \cap F = \emptyset \) for \( i, j \in I \cup \{0\}, \ i \neq j \) so by Lemma 3.6
\[ \bigcup_{i \in I \cup \{0\}} E_i = E' \cup E'' \] is strictly-2-associated with \( S \).

Remark  For the case when \( E \) is a subset of \( Z \), Lemma 3.11 says that if \( E' \subset E \) is strictly-2-associated with \( S \) and \( \{n_k\} \subset E \) satisfies \( |n_k - n_j| \geq M \) for all \( k \neq j \) and for some sufficiently large \( M \), then \( E' \cup \{n_k\} \) is also strictly-2-associated with \( S \). Thus if we adjoin to a uniformizable \( A(2) \) subset of \( Z \) which is strictly-2-associated with \( S \) a uniformizable \( A(2) \) set with the property that the distance between any two members is sufficiently large, then this new set is still strictly-2-associated with \( S \). As Theorem 2.31 says that \( E \) is built up inductively from sets which tend to infinity by sets with large distances between members, Lemma 3.11 is the tool we need to complete the proof of Theorem 3.5.

Proof of Theorem 3.5  Recall from Theorem 2.31 that \( E \) belongs to class \( M_k \) for some \( k \); thus it suffices to show that any subset of \( E \) which belongs to \( M_k \) is strictly-2-associated with each set \( S \) of positive measure, for all positive integers \( k \).

We proceed by induction on \( k \).

Fix \( S \) and let \( E' \) be any subset of \( E \) which is strictly-2-associated with \( S \). Suppose \( E'' \subset E \) belongs to class \( M_0 \), i.e., \( E'' \) tends to infinity. Choose the finite set \( F_1 = F_1(E, E', S) \) by Lemma 3.11 so that whenever \( \{\chi_i\}_{i \in I} \subset E \) satisfies \( \chi_i \chi_j^{-1} \notin F_1 \) for \( i, j \in I, i \neq j \), then \( E' \cup \{\chi_i\}_{i \in I} \) is strictly-2-associated with \( S \).
Since $E''$ tends to infinity there is a finite set $F$ so that if $x, \psi \in E'' \setminus F$ and $x \neq \psi$ then $x\psi^{-1} \notin F_1$. Thus $E' \cup (E'' \setminus F)$ is strictly-2-associated with $S$. Applying Proposition 3.7 $|F|$ times we conclude that $E' \cup E''$ is strictly-2-associated with $S$.

Now suppose we have established that whenever $E' \subset E$ is strictly-2-associated with $S$ and $E_{k}' \subset E$ belongs to class $M_k$, then $E' \cup E_{k}'$ is also strictly-2-associated with $S$.

Let $E' \subset E$ be any set strictly-2-associated with $S$ and assume that $E_{k+1}' \subset E$ belongs to class $M_{k+1}$. Again choose the finite set $F_1 = F_1(E,E',S)$ as in Lemma 3.11. Since $E_{k+1}'$ belongs to class $M_{k+1}$, it is the union of two sets $E_1$ and $E_2$, where if $x, \psi \in E_1, x \neq \psi$, then $x\psi^{-1} \notin F_1$, and $E_2$ is a finite union of sets in class $M_k$. From Lemma 3.11 we may conclude that $E' \cup E_1$ is strictly-2-associated with $S$. By the induction hypothesis $(E' \cup E_1) \cup E_2 = E' \cup E_{k+1}'$ is as well.

This completes the induction step and hence the proof of the theorem. \///

**Corollary 3.12** Let $E$ be a uniformizable $\Lambda(2)$ set which is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$. Let $g \in L^2(G)$, $g \neq 0$. Then there is a constant $c(g, E) > 0$ so that for all $f \in L^2_E(G)$,

$$\|gf\|_2^2 \geq c(g, E)\|f\|_2^2.$$
Proof: Choose $\varepsilon > 0$ and a subset $S$ of $G$ with positive measure, such that $|g| \geq \varepsilon$ on $S$. Then
\[ \|gf\|^2 \geq \varepsilon^2 \|1_S f\|^2 \geq \varepsilon^2 c(S, E)\|f\|^2 \]
for all $f \in L^2(E, G)$.

Corollary 3.13: Let $E$ be a uniformizable $A(2)$ set which is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$. If for some $f \in L^2(E, G)$, $m\{x : f(x) = 0\} > 0$, then $f \equiv 0$.

Corollary 3.14: Suppose $E$ is a uniformizable $A(2)$ set and that either $E$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$ or $E$ tends to infinity. If $f = \sum_{x \in E} a_x x$ converges pointwise on some set of positive measure, then $f \in L^2(G)$.

Proof: By Egoroff's theorem there is a set $S$ of positive measure on which $\sum_{x \in E} a_x x$ converges uniformly.

If $E$ tends to infinity then the proof of Corollary 3.10 shows that there is a finite set $\Delta \subset E$ so that $E \setminus \Delta$ and $S$ are strictly-2-associated. If $E$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$, then $E$ itself is strictly-2-associated with $S$ so let $\Delta = \emptyset$.

For any finite set $F$ let $S_F = \sum_{x \in (E \setminus \Delta) \cap F} a_x x$. Then
\[ \|1_S(S_F - S_{F'})\|^2 \geq c(E, S)\|S_F - S_{F'}\|^2 \]

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whenever $F'$ is a finite subset of $\Gamma$, since $S_F - S'_F \in L^2_{E \setminus \Delta}(G)$.

Since $\sum_{x \in E} a_x x$ converges uniformly on $S$, and $\Delta$ is a finite set, $\{1_S S_F\}$ is a Cauchy net (indexed by $F$) in $L^2(G)$. Thus $\{S_F\}$ is Cauchy in $L^2(G)$ and hence $\sum_{x \in E \setminus \Delta} a_x x \in L^2(G)$. Since $\Delta$ is a finite set, $f \in L^2(G)$.

Remark If $\Gamma$ is a torsion group then the assumption that $E$ tends to infinity is necessary to obtain the conclusion of the previous corollary. To see this suppose that $E$ does not tend to infinity. Then for some finite set $\Delta \subset \Gamma$ there is an infinite set $\{(x_i, \psi_i)\} \subset E$ of distinct pairs with $x_i \psi_i^{-1} \in \Delta \setminus \{1\}$. Without loss of generality we may assume $\Delta$ is a finite subgroup. Let $S = \Delta^\perp$, the annihilator of $\Delta$. Then $S$ is an open set, hence a set of positive measure, but $\sum_i (x_i - \psi_i) = 0$ on $S$, while $\sum_i (x_i - \psi_i) \notin L^2(G)$.

Whether it is necessary for $E$ to be a uniformizable $A(2)$ set for Theorem 3.5 or these corollaries to be true is unknown. In 3.18–3.20 we discuss very strongly-2-associatedness, a conclusion stronger than strict-2-associatedness, which does imply that $E$ is a uniformizable $A(2)$ set. We can show however that if $E$ is strictly-2-associated with all subsets of $G$ of sufficiently large measure then $E$ must be a $A(2)$ set. Thus Theorem 3.5 is almost best-possible. Indeed, if all $A(2)$ sets are uniformizable $A(2)$ sets then the hypotheses of Theorem 3.5 would be both necessary and sufficient for strict-2-associatedness.
Proposition 3.15  Let $A$ be a subset of $G$ with positive measure and let $0 < \varepsilon < m(A)$. If $E$ is strictly-2-associated with all subsets of $A$ with measure at least $m(A) - \varepsilon$, then there is a constant $c(A)$ such that

$$\|f 1_A\|_1 \geq c(A) \|f\|_2$$

for all $f \in \text{Trig}_E(G)$.

When $A = G$ the conclusion is that $E$ is a $\Lambda(2)$ set. An alternate proof of this case may be found in [8].

**Proof**  If no such constant $c(A)$ exists then there is a sequence $\{f_n\} \subset \text{Trig}_E(G)$ such that $\|f_n\|_2 = 1$ and $\|f_n 1_A\|_1 < 1/2^n$. Let $B_n = \{x \in A : |f_n(x)| > 1/\varepsilon\}$. By Chebyshev’s inequality $m(B_n) < \varepsilon \|f_n 1_A\|_1 \leq \varepsilon / 2^n$. Thus if we let $B = A \setminus \bigcup_n B_n$, then $m(B) > m(A) - \varepsilon > 0$.

By hypothesis $E$ is strictly-2-associated with $B$, contradicting the fact that

$$\int_B |f_n|^2 \leq \|f_n 1_B\|_\infty \|f_n 1_B\|_1$$

$$\leq 1/\varepsilon \|f_n 1_A\|_1$$

$$\to 0 \text{ as } n \to \infty.$$  

///

By duality we obtain the following corollary for uniformizable $\Lambda(2)$ sets which
are $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$.

**Corollary 3.16** If any set $E \subset \Gamma$ is strictly-2-associated with all sufficiently large subsets of a subset $A$ of $G$, then for such a set $A$ there is a constant $c_1(A)$ so that corresponding to each $\phi \in l^2(E)$ there is a bounded function $h$ supported on $A$ with

$$\|h\|_\infty \leq c_1(A)\|\phi\|_2 \quad \text{and} \quad \hat{h}(\chi) = \phi(\chi) \quad \text{for all} \quad \chi \in E.$$  

**Remark** The novelty here is that $h$ is zero off $A$.

**Proof** For fixed $\phi \in l^2(E)$ consider the map

$$S : \{f1_A : f \in Trig_B(G)\} \subset L^1(A) \to \mathbb{C}$$

given by

$$S(f1_A) = \sum_{x \in \Gamma} \hat{f}(x)\overline{\phi(x)}.$$  

Since $E$ is strictly-2-associated with $A$, the map $S$ is well defined. By Proposition 3.15

$$|S(f1_A)| \leq \|f\|_2\|\phi\|_2 \leq \frac{1}{c(A)}\|f1_A\|_1\|\phi\|_2,$$

so that $S$ is bounded.

Now extend $S$ to $L^1(A)$, preserving the norm, by the Hahn-Banach Theorem. The proof is completed in the same manner as Proposition 1.7 by applying the Riesz...
Before turning to the notion of very strong-2-associatedness we state a partial converse to Proposition 3.15. Another proof of it can be found in [8].

**Proposition 3.17** If $E \subset \Gamma$ is a $A(2)$ set, then $E$ is strictly-2-associated with all subsets of $G$ of sufficiently large measure.

**Proof** Choose $c$ so that $||f||_2 \leq c||f||_1$ for all $f \in L^2_E(G)$ and suppose $S \subset G$ has measure greater than $1 - 1/c^2$.

If $f \in L^2_E(G)$ then
\[
||f||_2 \leq c||f||_1 \leq c||f1_S||_1 + c||f1_{S^c}||_1 \\
\leq c||f1_S||_1 + c||f||_2||1_{S^c}||_2 ,
\]
the last inequality by Cauchy-Schwarz.

Transposing we obtain
\[
||f||_2(1 - cm(S^c)^{1/2}) \leq c||f1_S||_1 .
\]
Since $m(S^c)^{1/2} < 1/c$ we obtain the result.

### 3.3 Very Strong-Two-Associatedness

We discuss in this section a property similar to, although stronger than, strict-2-
associatedness with all subsets of positive measure. This property is only possessed by uniformizable $A(2)$ sets.

**Definition 3.18** Let $E$ be a subset of $\Gamma$ and $S$ a subset of $G$. $E$ is said to be **very strongly-2-associated** with $S$ if for each $\lambda > 1$ there is a finite set $F \subset E$ so that if $f \in \text{Trig}_E \setminus F(G)$ then

$$\lambda^{-1} \|f\|_2^2 \leq \frac{1}{m(S)} \|1_S f\|_2^2 \leq \lambda \|f\|_2^2.$$ 

In other words, the average of $|f|^2$ over $S$ is nearly its average over all of $G$.

**Proposition 3.19** $E$ is very strongly-2-associated with all sets $S$ of positive measure provided for each such set $S$ and each $\lambda > 1$ there is a finite set $F \subset E$ so that if $f \in \text{Trig}_E \setminus F(G)$ then

$$\lambda^{-1} \|f\|_2^2 \leq \frac{1}{m(S)} \|1_S f\|_2^2 \leq \lambda \|f\|_2^2.$$ (1)

**Proof** Let $S$ be a subset of $G$ and $\lambda > 1$. Certainly we may assume $m(S) < 1$. Notice that if (1) holds for some $\lambda > 1$ then it also holds for all larger $\lambda$, thus we may assume $\lambda m(S) < 1$.

If we set $\delta \equiv \frac{m(S^c)}{1 - m(S)\lambda}$ then $\delta > 1$ and there is a finite set $F$ so that

$$\frac{1}{m(S^c)} \|1_{S^c} f\|_2^2 \geq \delta^{-1} \|f\|_2^2.$$
for all $f \in \text{Trig}_{E \setminus F}(G)$. For all such $f$ we have

$$\|1_s f\|_2^2 = \|f\|_2^2 - \|1_{S^c} f\|_2^2 \leq (1 - \delta^{-1} m(S^c)) \|f\|_2^2 = m(S) \lambda \|f\|_2^2$$

as required.

Remark Similar arguments may be used to show that $E$ is very strongly-2-associated with all sets $S$ of positive measure provided for each such set $S$ and each $\lambda > 1$ there is a finite set $F \subset E$ so that if $f \in \text{Trig}_{E \setminus F}(G)$ then

$$\frac{1}{m(S)} \|1_s f\|_2^2 \leq \lambda \|f\|_2^2.$$

It was shown by Zygmund in [25] that the lacunary sets in $\mathbb{Z}$ are very strongly-2-associated with all subsets of $T$ with positive measure. Bonami in [3] showed that every $A(4)$ set in $\Gamma$ which tended to infinity has this property. In [9] Fournier obtained the same conclusion for uniformizable $A(2)$ sets which tend to infinity. We will prove that these hypotheses are both necessary and sufficient.

Theorem 3.20 $E \subset \Gamma$ is a uniformizable $A(2)$ set which tends to infinity if and only if $E$ is very strongly-2-associated with all subsets of $G$ of positive measure.
Proof Our proof of the sufficiency of the hypotheses will be proved by techniques similar to those used in Lemma 3.6.

Fix $S$ and $\lambda > 1$. With the notation as in the proof of Lemma 3.6 choose a polynomial $P$ with

$$\|P - 1_S\|_2 \leq \varepsilon \equiv \frac{(1 - \lambda^{-1})m(S)}{3 + \sqrt{2K}},$$

$$\|P\|_\infty \leq 2 \quad \text{and} \quad \|P^2 - 1_S\|_\beta \leq \varepsilon.$$

Let $\Delta = (\text{supp } \hat{P})(\text{supp } \hat{P})^{-1}$. Since $E$ is assumed to tend to infinity there is a finite set $F$ so that if $X, \psi \in E \setminus F, X \neq \psi$, then $X\psi^{-1} \notin \Delta$. Suppose $f \in \text{Trig}_{E \setminus F}(G)$, say $f = \sum_{X, \psi \in E \setminus F} a_i X_i$.

As in the proof of Lemma 3.6 we have

$$\|fP\|_2^2 = \sum_i \|a_i X_i P\|_2^2 = \sum_i \|a_i\|^2 \|P\|_2^2 = \|f\|_2^2 \|P\|_2^2.$$

Now

$$\|fP\|_2^2 \leq \int |f|^2 1_S + \int |f|^2 |P^2 - 1_S|$$

$$\leq \|f 1_S\|_2^2 + \|f\|_2 \|f(P^2 - 1_S)\|_2.$$

But

$$\|f(P^2 - 1_S)\|_2 \leq \sqrt{2}\|f\|_\Phi \|P^2 - 1_S\|_\beta$$

$$\leq \sqrt{2}K\varepsilon\|f\|_2$$

since $L^2_E(G) \subset L^\Phi(G)$.

Hence

$$\|f 1_S\|_2^2 \geq \|fP\|_2^2 - \sqrt{2}K\varepsilon\|f\|_2^2$$

$$\geq \|f\|_2^2(\|P\|_2^2 - \sqrt{2}K\varepsilon).$$
Standard arguments show that $m(S) \leq 3\varepsilon + \|P\|_2^2$. Thus

$$\|f1_S\|_2^2 \geq \|f\|_2^2 (m(S) - \varepsilon (3 + \sqrt{2}K)) \geq \|f\|_2^2 m(S) \lambda^{-1}.$$ 

This inequality combined with Proposition 3.19 shows the sufficiency of the hypotheses.

Now we will establish the necessity of the hypotheses.

First, suppose $E$ does not tend to infinity. Then there is an infinite family $\Omega$ of pairwise disjoint two-element sets $\{X, \psi\}$, with $X, \psi \in E$ and $X\psi^{-1} = \gamma \neq 1$.

Let $S = \{z \in G : |\gamma(z) - 1| < 1/2\}$. $S$ is an open, non-empty set so $m(S) \neq 0$.

For each finite set $F$ there is a pair $(X, \psi) \in \Omega$, with $X, \psi \in E \setminus F$, $\|X - \psi\|_2^2 = 2$ and

$$\frac{1}{m(S)} \int_S |X - \psi|^2 \leq \|(X - \psi)1_S\|_\infty^2 \leq \|(X\psi^{-1} - 1)1_S\|_\infty^2 \leq 1/4.$$

Hence $S$ does not satisfy the definition of very strongly-2-associated for any $\lambda < 8$.

To complete the proof we show that $E$ must be a uniformizable $A(2)$ set. This requires the following lemma whose proof we defer to the end of the proof of this theorem.
Lemma 3.21 Suppose \( E \) is not a uniformizable \( A(2) \) set. Then there is an \( \varepsilon > 0 \) so that for each finite set \( F \) and each positive integer \( n \), there is a subset \( A = A(n, F) \) of \( G \) and an \( E \setminus F \)-polynomial \( g_n \) satisfying

\[
m(A) \leq \frac{1}{n} \quad \text{and} \quad \int_A |g_n|^2 > \varepsilon \|g_n\|_2^2.
\]

Proof of Theorem 3.20 (continued)

Suppose \( E \) is not a uniformizable \( A(2) \) set. Choose \( \varepsilon \) as in the lemma. Apply the lemma first with \( F \) equal to the empty set to obtain \( A_1 \subset G \) and \( f_1 \in Trig_E(G) \) with \( m(A_1) \leq \varepsilon/4 \) and

\[
\int_{A_1} |f_1|^2 > \varepsilon \|f_1\|_2^2.
\]

Apply the lemma again with \( F = \text{supp} f_1 \) to obtain \( A_2 \subset G \) with \( m(A_2) \leq \varepsilon/8 \), and \( f_2 \in Trig_E(G) \) with \( \text{supp} f_2 \cap \text{supp} f_1 = \emptyset \) and

\[
\int_{A_2} |f_2|^2 > \varepsilon \|f_2\|_2^2.
\]

Repeat this process to obtain sets \( A_n \subset G \) with \( m(A_n) \leq \varepsilon/2^{n+1} \) and \( f_n \in Trig_E(G) \) with

\[
\text{supp} f_n \cap \left( \bigcup_{i=1}^{n-1} \text{supp} f_i \right) = \emptyset \quad \text{for} \quad n = 2, 3, \ldots
\]

and

\[
\int_{A_n} |f_n|^2 > \varepsilon \|f_n\|_2^2.
\]

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Let $S = \bigcap_{n=1}^{\infty} A_n^c$. Then

$$m(S) = 1 - m\left(\bigcup_{n=1}^{\infty} A_n\right) \geq 1 - \varepsilon/2 > 0.$$ 

$E$ and $S$ are very strongly-2-associated; so given any fixed $\lambda$ with $1 < \lambda < \frac{1 - \varepsilon/2}{1 - \varepsilon}$ there is a finite set $F \subset E$ with the property that whenever $f \in Trig_{E \setminus F}(G)$,

$$\int_S |f|^2 \geq \lambda^{-1} m(S)\|f\|_2^2.$$ 

Our construction of the functions $f_n$ ensures the existence of an integer $N$ so that whenever $n \geq N$, $f_n \in Trig_{E \setminus F}(G)$. Hence for all $n \geq N$,

$$\int_S |f_n|^2 \geq \lambda^{-1} m(S)\|f_n\|_2^2 \geq (1 - \varepsilon)\|f_n\|_2^2.$$ 

However

$$\int_S |f_n|^2 \leq \int_{A_n^c} |f_n|^2$$

$$= \int_G |f_n|^2 - \int_{A_n} |f_n|^2$$

$$< (1 - \varepsilon)\|f_n\|_2^2.$$ 

This contradiction shows that $E$ must indeed be a uniformizable $A(2)$ set. /////<

We turn now to proving Lemma 3.21. First we will prove a weaker version.

**Lemma 3.21'** If $E$ is not a uniformizable $A(2)$ set there is an $\varepsilon' > 0$ with the property that for each positive integer $n$ there is a subset $A_n$ of $G$ with $m(A_n) \leq 1/n$.
and an $E$-polynomial $f_n$ with

$$\int_{A_n} |f_n|^2 > \epsilon' \|f_n\|_2^2.$$ 

**Proof** This is the failure of the uniform integrability property for non-uniformizable $A(2)$ sets (Theorem 1.10(4)). We remark that the $E$-functions $g_n$ in $L^2(G)$ which satisfy

$$\int_{A_n} |g_n|^2 > \epsilon' \|g_n\|_2^2$$

for some set $A_n$, $m(A_n) \leq 1/n$, may be assumed to be $E$-polynomials, since $\text{Trig}_E(G)$ is dense in $L^2_E(G)$.

Proof of Lemma 3.21 If $E$ is not a uniformizable $A(2)$ set Lemma 3.21' holds for some $\epsilon' > 0$. Set $\epsilon = \epsilon'/4$.

Assume the lemma is false. Then there is a finite set $F$, and a positive integer $n$ so that whenever $A \subset G$ satisfies $m(A) \leq 1/n$ and $f \in \text{Trig}_{E \setminus F}(G)$, then

$$\int_A |f|^2 \leq \epsilon \|f\|_2^2 = \frac{\epsilon'}{4} \|f\|_2^2. \quad (1)$$

Choose a positive integer $N$ with

$$\frac{1}{N} \leq \min\left(\frac{1}{n}, \frac{\epsilon'}{4|F|^2}\right).$$

Let $A$ be any subset of $G$ with measure less than $1/N$ and let $f$ be any $E$-polynomial. Denote by $f_1$ the $E \setminus F$-polynomial $\sum_{x \in E \setminus F} f(x)x$. 74
By (1),

\[ \int_A |f_1|^2 \leq \frac{\varepsilon'}{4} \|f_1\|_2^2 \leq \frac{\varepsilon'}{4} \|f\|_2^2. \]

Now

\[ \|f - f_1\|_\infty \leq |F| \|\tilde{f}\|_\infty \leq |F| \|f\|_2, \]

hence

\[
\int_A |f|^2 \leq 2 \int_A |f_1|^2 + 2 \int_A |f - f_1|^2 \\
\leq \frac{\varepsilon'}{2} \|f\|_2^2 + 2 \|f - f_1\|_\infty^2 m(A) \\
\leq \left(\frac{\varepsilon'}{2} + 2|F|^2 m(A)\right) \|f\|_2^2 \\
\leq \varepsilon' \|f\|_2^2
\]

since \( m(A) \leq \frac{\varepsilon'}{4|F|^2} \).

Since \( A \) and \( f \) were arbitrary this contradicts the weaker version of the lemma.

/////
Chapter 4

Strict-Two-Associatedness With Open Sets

The structural arguments used in Chapter 3 to prove Theorem 3.5 can also be used to prove

**Theorem 4.1** Suppose $E \subset \Gamma$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$. If $E$ belongs to class $M_n$ for some $n$ and $E$ has the uniformly large gap property, then $E$ is strictly-2-associated with all open, non-empty subsets of $G$.

Before presenting the proof we mention some applications of this theorem.

**Corollary 4.2** Suppose $E$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$ and

1. $E$ tends to infinity; or

2. $E$ does not contain parallelepipeds of arbitrarily large dimension (in particular, if $E$ is a $A(p)$ set for some $p > 0$); or

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(3) $E$ is a subset of $\mathbb{Z}$ and belongs to class $M_n$ for some $n$.

Then $E$ is strictly-2-associated with all open, non-empty subsets of $G$.

**Proof**  
(1) If $E$ tends to infinity, then $E$ belongs to class $M_0$ and clearly has the uniformly large gap property.

(2) From Theorems 2.26 and 2.31 we know that $E$ satisfies the hypotheses of the theorem.

(3) All subsets of $\mathbb{Z}$ which belong to class $M_n$ have the uniformly large gap property by Corollary 2.33.  

The following four lemmas will be used to prove Theorem 4.1. The first is a standard topological argument.

**Lemma 4.2**  
Let $U \subset G$. If $S$ is any open set which contains the closure of $U$ then there is a neighbourhood $V$ of the identity such that $U \cdot V \subset S$.

**Proof**  
For each $u \in \overline{U}$, choose an open set $N_u$ containing $u$ such that $N_u \subset S$.

Choose a neighbourhood of the identity $V_u$ so that $V_u \cdot V_u \subset N_u u^{-1}$. The open sets $\{V_u u\}_{u \in \overline{U}}$ cover $\overline{U}$ hence we may choose a finite subcover $\{V_{u_i} u_i\}_{i=1}^m$. Let $V = \bigcap_{i=1}^m V_{u_i}$.

If $u \in U$ and $v \in V$ then $u \in V_{u_i} u_i$ for some $i = 1, 2, \ldots, m$, and $v \in V_{u_i}$, thus $uv \in V_{u_i} u_i V_{u_i} \subset N_{u_i} u_i u_i^{-1} \subset S$.  

///
Lemma 4.3 Suppose $E \subset \Gamma$ and $U$ is an open, non-empty subset of $G$ such that $E$ and $U$ are strictly-2-associated. Let $S$ be any open subset of $G$ containing $\overline{U}$. There is a finite subgroup $X_0$ of $\Gamma$, depending on $S$ and $E$, so that whenever the set $E \cup \{\chi\}, \chi \in \Gamma$, is $X_0$-subtransversal, then $E \cup \{\chi\}$ is strictly-2-associated with $S$, with constant of strict-2-associatedness independent of $\chi$.

(Compare with Proposition 3.7.)

Proof Choose a neighbourhood $V$ of the identity so that $UV \subset S$. Given $v \in V$ and $f \in L^2_{E \cup \{x\}}(G)$, let $f_v(x) = f(xv) - \chi(v)f(x)$. Observe that

$$\|1_s f\|_2^2 = \frac{1}{m(V)} \int_V \int_S |f(x)|^2 \, dm(x) \, dm(v) \geq \frac{1}{4 m(V)} \int_V \int_U |f_v(x)|^2 \, dm(x) \, dm(v).$$

If we assume $\|1_U f\|_2^2 \geq c_1 \|f\|_2^2$ whenever $f \in Trig_E(G)$, then since $f_v \in Trig_E(G)$ for all $v \in V$ we have

$$\|1_s f\|_2^2 \geq \frac{c_1}{4 m(V)} \int_V \|f_v\|_2^2 \, dm(v).$$

The proof of the lemma is completed in the same manner as was the proof of Proposition 3.7. //://

Corollary 4.4 Suppose $E \subset \Gamma$ is $X_0$-subtransversal for all finite subgroups $X_0$ of $\Gamma$. If there is a finite set $F$ and an open, non-empty set $U$ so that $E \setminus F$
and $U$ are strictly-2-associated, then $E$ is strictly-2-associated with any open set $S$ containing $\bar{U}$.

**Proof** Assume $F = \{X_1, \ldots, X_N\}$. Being a compact Hausdorff space, $G$ is normal. Thus it is possible to choose open sets $S_1, \ldots, S_{N-1}$ satisfying

$$\bar{U} \subset S_1 \subset S_1 \subset S_2 \subset \ldots \subset S_{N-1} \subset S.$$ 

By the previous lemma $(E \setminus F) \cup \{X_1\}$ is strictly-2-associated with $S_1$, hence

$$(E \setminus F) \cup \{X_1, X_2\}$$

is strictly-2-associated with $S_2$, and so by induction $E$ is strictly-2-associated with $S$. 

///

**Lemma 4.5** Let $S$ and $S_1$ be open, non-empty subsets of $G$ with $\overline{S_1} \subset S$. Given $c > 0$ there is a finite, symmetric set $F = F(c, S_1, S) \subset \Gamma$, containing the identity, so that if the sets $\{E_i\}_{i \in I} \subset \Gamma$ are strictly-2-associated with $S_1$, with constant of strict-2-associatedness $c$, and $E_i E_j^{-1} \cap F = \emptyset$ for all $i, j \in I, i \neq j$, then

$\bigcup_{i \in I} E_i$ is strictly-2-associated with $S$.

(Compare with Lemma 3.6.)

**Proof** Since $G$ is normal there is a continuous function $g : G \to [0, 1]$ with $g(S_1) = 1$ and $g(S^c) = 0$. Let $P$ be a polynomial with

$$\|P - g\|_\infty^2 < \varepsilon \equiv c/12.$$  

(1)
Set $F = (\text{supp} \hat{P})(\text{supp} \hat{P})^{-1}$. As in Lemma 3.6, if $f \in \text{Trig}_{uE_i}(G)$, $f = \sum_{i \in I} f_i$ with $f_i \in \text{Trig}_{E_i}(G)$, then

$$\| Pf \|^2 = \sum_{i \in I} \| Pf_i \|^2.$$ 

Thus

$$\| 1_s f \|^2 \geq \| g f \|^2$$

$$\geq \frac{1}{2} \| Pf \|^2 - \| (P - g) f \|^2$$

$$\geq \frac{1}{2} \sum_{i \in I} \| Pf_i \|^2 - \varepsilon \| f \|^2$$

the last step by an application of Hölder’s inequality and (1).

But also

$$\| Pf_i \|^2 \geq \frac{1}{2} \| g f_i \|^2 - \varepsilon \| f_i \|^2$$

$$\geq \frac{1}{2} \| 1_s f_i \|^2 - \varepsilon \| f_i \|^2$$

$$\geq \frac{1}{2} c \| f_i \|^2 - \varepsilon \| f_i \|^2$$

since the sets $\{E_i\}_{i \in I}$ are strictly-2-associated with $S_1$ with constant of strict-2-associatedness $c$.

Hence

$$\| 1_s f \|^2 \geq \sum_{i \in I} \left( \frac{1}{4} c \| f_i \|^2 - \frac{1}{2} \varepsilon \| f_i \|^2 \right) - \varepsilon \| f \|^2$$

$$\geq \frac{1}{4} c \| f \|^2 - \frac{3}{2} \varepsilon \| f \|^2$$

$$\geq \frac{1}{8} c \| f \|^2.$$
Lemma 4.6  Suppose $E$ satisfies the hypotheses of the theorem and $S$ is an open subset of $G$. Assume $E' \subset E$ is strictly-2-associated with the open set $S_2$ whose closure is contained in $S$. There is a finite set $F_1 = F_1(E, E', S, S_2)$ so that whenever $E'' = \{X_i\}_{i \in I} \subset E$ satisfies $X_iX_j^{-1} \notin F_1$ if $i \neq j$, then $E' \cup E''$ is strictly-2-associated with $S$.

(Compare with Lemma 3.11.)

Proof  Choose $S_1$ open with $\overline{S_2} \subset S_1 \subset \overline{S_1} \subset S$. By Lemma 4.3 there is a constant $c > 0$ which is a constant of strict-2-associatedness for $S_1$ and each of the sets $E' \cup \{X\}, X \in \Gamma$. By Lemma 4.5 obtain the finite set $F = F(c, S_1, S)$ with the property that if $\{E_i\}_{i \in I}$ are strictly-2-associated with $S_1$ with constant of strict-2-associatedness $c$, and $E_iE_j^{-1} \cap F = \emptyset$, then $\bigcup_{i \in I} E_i$ is strictly-2-associated with $S$.

Since $E$ has the uniformly large gap property the proof may be completed as was Lemma 3.11. ///

Proof of Theorem 4.1  Since $E$ is assumed to belong to $M_n$ for some $n$ we carry out an induction proof similar to that given for Theorem 3.5.

The induction assumption will be as follows: Assume that for each non-empty, open subset $S$ of $G$, whenever $E' \subset E$ is strictly-2-associated with the open set $S_1$
whose closure is contained in $S$, and $E'' \subset E$ belongs to class $M_k$, then $E' \cup E''$ is strictly-2-associated with $S$.

Since the proof is very similar to the proof of Theorem 3.5 we present the details only for $k = 0$.

Let $S$ be any open, non-empty subset of $G$. Let $E' \subset E$ be strictly-2-associated with the open set $S_1$, $S_1 \subset S$, and let $E'' \subset E$ belong to class $M_0$. Choose $S_2$ open with $S_1 \subset S_2 \subset S$.

Choose the finite set $F_1 = F_1(E, E', S_2, S_1)$ from Lemma 4.6. Since $E''$ tends to infinity there is a finite set $F$ so that if $X, \psi \in E'' \setminus F$ and $X \neq \psi$, then $X\psi^{-1} \notin F_1$.

By Lemma 4.6 $E' \cup (E'' \setminus F)$ is strictly-2-associated with $S_2$. By Corollary 4.4 $E' \cup E''$ is strictly-2-associated with $S$.

**Corollary 4.7** If $E$ satisfies the hypotheses of Theorem 4.1 and $f \in L^2_B(G)$ vanishes on an open set, then $f \equiv 0$.

As with Theorem 3.5 the $X_0$-subtransversality condition is necessary to obtain the corollary above.

In [14] Mandelbrojt proved

**Theorem 4.8** Let $E = \{n_k\} \subset \mathbb{Z}$ and let $\beta = \sup\{\alpha : \sum_{|n_k|}^{1/\alpha} \text{diverges}\}$. If

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\[ \beta < 1, \, f \in L^2_E(T) \text{ and } \]
\[ \limsup_{\alpha \to 0^+} \frac{\ln \left( -\ln \int_0^\alpha |f(x)| \right)}{-\ln \alpha} > \frac{\beta}{1 - \beta}, \]

then \( f \equiv 0. \)

In particular, if \( f \) vanishes on an open set, then \( f \equiv 0. \)

If \( E \) does not contain parallelepipeds of arbitrarily large dimension it follows from Corollary 2.18 that \( \beta < 1. \) Thus the conclusion of Corollary 4.7 can be obtained for such sets by Mandelbrojt's theorem.

If however \( E \) is only assumed to satisfy the assumption of Mandelbrojt's theorem, then \( E \) need not be strictly-2-associated with all open subsets of \( T. \) To prove this, observe that there are sets which satisfy Mandelbrojt's hypothesis and yet contain arbitrarily long sequences of consecutive integers. Such sets cannot be strictly-2-associated with all open sets, as a result of the next proposition.

**Proposition 4.9** Suppose \( E \subset \mathbb{Z} \) contains arbitrarily long arithmetic progressions \( \{a_N + b, \ldots, a_N + Nb\} \), for some fixed \( b. \) Then \( E \) is not strictly-2-associated with any open set that does not have full measure.

**Proof** Let \( S \) be any open subset of \( T \) with measure less than one. Let \( f_N \) be the \( N \)'th partial sum of the function \( x \mapsto 1_{S^c}(bx). \)
Suppose \( 1_{S^c}(x) = \sum_n c_ne^{inx} \). Then
\[
 f_N(x) = \sum_{|n| \leq N} c_ne^{inx} = e^{-i\alpha_Nz}e^{-i(N+1)bz}g_N(x)
\]
where \( g_N \in L^2_E(G) \) if we assume that the arithmetic progression
\[
 \{a_N + b, \ldots, a_N + (2N + 1)b\} \subset E.
\]

Now
\[
 \|g_N(x)1_S(bx)\|_2 = \|f_N(x)1_S(bx)\|_2
\]
and since \( \{f_N\} \) converges to \( 1_{S^c}(bx) \) in \( L^2(T) \), \( \|g_N(x)1_S(bx)\|_2 \to 0 \). But
\[
 \|g_N\|_2 = \|f_N\|_2 \to \|1_{S^c}(bx)\|_2 = m(S^c)^{1/2} > 0.
\]
Thus \( E \) cannot be strictly-2-associated with \( S \).

It is possible to prove a similar statement for \( E \subset \Gamma \) although we do not give the details here. We do not know if the hypotheses that \( E \) belong to class \( M_n \) or have the uniformly large gap property are necessary for the strict-2-associatedness conclusion. As mentioned in Chapter two there are sets \( E \) in \( \mathbb{Z} \) which tend to infinity but have arbitrarily long arithmetic progressions, thus \( E \) may be strictly-2-associated with all open subsets of \( G \) without being a \( \mathcal{A}(p) \) set for any \( p > 0 \).

Good necessary conditions for the conclusion of Theorem 4.1 are unknown at this time.
Open Problems

We conclude by presenting a short list of open problems suggested by topics discussed in the thesis.

1. Are there sets which do not contain parallelepipeds of arbitrarily large dimension and which are not $A(4)$ sets? which are not $A(p)$ sets for any $p > 0$? Lemma 2.27 together with [21, 4.5] can be used to show that sets which do not contain parallelepipeds of dimension 2 are indeed $A(4)$ sets.

If the answer to the first question posed above is no, then all $A(p)$ sets, $p > 0$, are $A(4)$ sets. If just the second part has a negative answer the union problem for $A(p)$ sets would be solved.

2. If $E \subset \mathbb{Z}$ is a $A(2)$ set, is $E$ strictly-2-associated with all subsets of $T$ with positive measure?

3. What conditions must $E \subset \Gamma$ satisfy if

(i) $E$ is strictly-2-associated with all open, non-empty subsets of $G$? or

(ii) no $L^2_E(G)$ function may vanish on an open, non-empty subset of $G$ without being identically zero?

4. Are the results of this thesis true in the setting of $A(p)$ sets in discrete, non-abelian groups?
Bibliography


