PATH PROPERTIES OF SUPERPROCESSES

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Abstract

Superprocesses are measure valued diffusions that arise as high density limits of particle systems undergoing spatial motion and critical branching.The most closely studied superprocess is super Brownian motion where the underlying spatial motion is Brownian. In chapter 1 we describe the approximating particle systems, the nonstandard model for a superprocess and some known path properties of super Brownian motion.

Super Brownian motion is effectively determined by its closed support. In chapter 2 we use the approximating particle systems to derive new path properties for the support process. We find the growth rate of the support for the process started at a point mass. We give a representation for the measure at a fixed time in terms of its support. We show that the support at a fixed time is nearly a totally disconnected set. Finally we calculate the Hausdorff dimension of the range of the process over random time sets.

A superprocess can be characterised as the solution to a martingale problem and in chapter 3 we use this characterisation to study the properties of general superprocesses. We investigate when the real valued process given by the measure of a half space under a super symmetric stable process is a semimartingale. We give a description of the behaviour of a general superprocess and its support near extinction. Finally we consider the problem of recovering the spatial motion from a path of the superprocess.

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Chapter 1

Construction

1.1 Introduction

Consider a population containing a large number of individuals. Each individual moves through space and produces offspring during its lifetime. The rules for the spatial motion and the number of offspring of any individual may depend on its location. To obtain a tractable mathematical model for the behaviour of such a population we make some assumptions. The future motion of each individual depends only on its present location. In particular the motion is independent of the behaviour of the rest of the population. At the end of its lifetime each individual produces a random (possibly zero) number of offspring independently of the other individuals. This model could be used to predict the dispersion of an asexually reproducing species. It also applies to the distribution of a rare gene in a diploid gene pool where the chance of two rare genes meeting and hence interacting can be ignored. Because of the assumption of no interaction between individuals many quantities of interest can be calculated. In Sawyer [21] the distribution and joint distribution of new individuals is calculated under a variety of initial distributions.

This thesis studies a continuous limit of such a model in which the mean number of offspring is one. To capture the positions of the whole population we consider the state as a measure consisting of small point masses at the locations of each individual. Then as the population size increases and the mean lifetime decreases we obtain a limiting measure valued process. The total mass represents the size of the population and the measure of a set A represents the number of individuals situated inside A. In some cases the limit process will have a density which can be thought of as a population density.

In passing to the limit some features of the original model are lost. For convenience we shall take the lifetimes of the individuals to be of fixed length but the same limiting process is obtained if the lifetimes are exponentially distributed. The exact distribution of the number of offspring is lost and only the variance is preserved, so we shall take critical binary branching where the number of offspring is 0 or 2 each with probability 1/2.

This thesis investigates the limiting measure valued process as a mathematical object and many of the theorems describe properties that are of purely mathematical interest. However the estimates needed to prove these properties might be interpreted to give information about the dispersion of a population. For example in section 2.3 we show that the support of the limiting process where the spatial motion is Brownian is nearly a totally disconnected set. To prove this we break the measure into groups of closely related individuals and estimate the fraction of the poulation that occurs in groups that are isolated from the remaining population.

1.2 Watanabe's Theorem

We first give an informal description for a construction of a branching superprocess. We will describe a particle system called a binary branching Feller process which will depend on a parameter μ . The idea is that as we let μ increase to infinity the particle system will converge in law to a superprocess.

Fix a large integer μ. At time zero we position $O(\mu)$ particles in space. On the time interval $[0, 1/\mu)$ these particles move independently according to the law of a fixed Feller process. At time $1/\mu$, for each particle independently we toss a fair coin. If the coin lands tails the particle dies and vanishes. If the coin land heads the particle splits into two. On the interval $[1/\mu, 2/\mu)$ the particles that are still alive move independently according to the Feller process. At time $2/\mu$ these particles again independently die or split into two. We continue this process for all time. Figure 1.2 shows the evolution for three generations (where we have drawn the particle motion as continuous paths for convenience).

One way to keep track of all the particles is to attach mass $1/\mu$ to each particle and consider the state at any fixed time as a measure. This measure will be a finite sum of point masses of size $1/\mu$. As the branches grow so the measure evolves in time. It is this measure valued process that will approximate a superprocess.

We wish to let $\mu \to \infty$. Notice that the parameter μ has several roles. There are $O(\mu)$ initial particles. The mass of each particle is $1/\mu$ so that taking μ large and choosing the initial positions carefully we can let the measure at time zero approximate any finite measure. However $1/\mu$ is also the time between each branching generation. We now investigate the approximating particle systems and in doing so we shall see that a branching rate of $1/\mu$ should lead to a nontrivial limit.

The number of particles descended from any one fixed particle is a Galton Watson process. We recall some results from branching processes (see Harris [11] p21-22.) Let $(X(n, i) : i, n \in N)$ be I.I.D. random

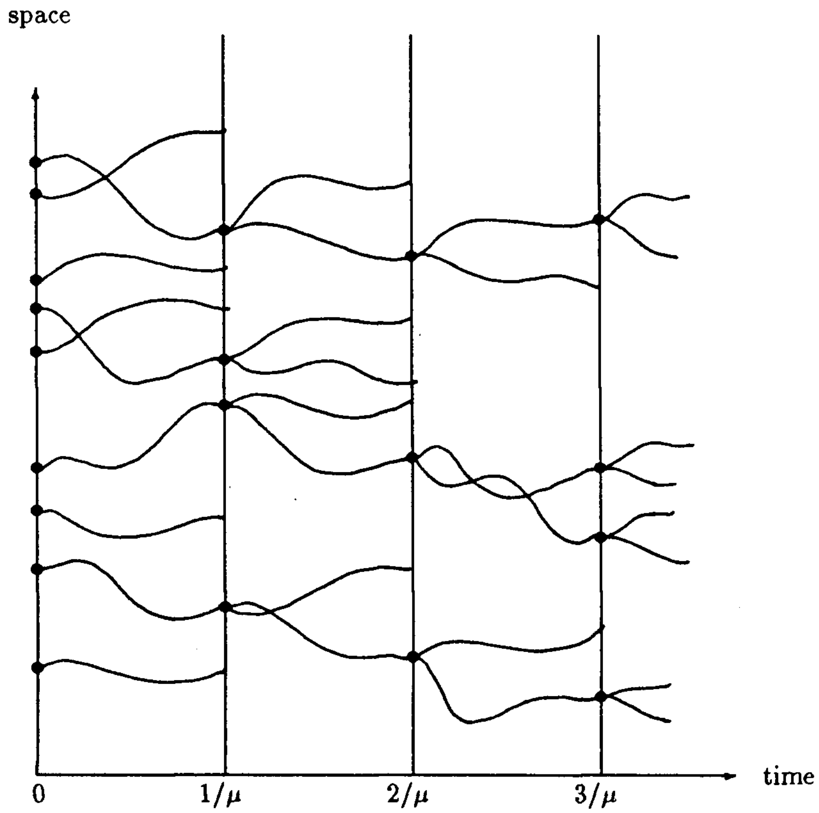

Figure 1.1: Binary Branching Feller Process.

variables on some probability space $(\Omega, \mathcal{F}, P)$ taking values 0 or 2 each with probability 1/2 . Let $Z_0 = 0$ and $Z_n = \sum_{i=1}^{Z_{n-1}} X(n,i)$. Then

$$\begin{aligned} \lim_{n\to\infty} nP(Z_n > 0) &= 2 && (1.1)\\ \lim_{n\to\infty} P(Z_n > nx | Z_n > 0) &= \exp(-2x) \text{ for all } x \geq 0. && (1.2) \end{aligned}$$

We use these results to analyse the measure at a fixed time $t > 0$. Look back in time a short distance $a > 0$. Let $I(t,a)$ be a list of those particles at time $t-a$ that have descendants alive at time t. We consider only those branches between times $t-a$ and t that end with a living particle at time t (see figure 1.2).

We see that the mass at time t comes in 'clusters' rooted at points in $I(t,a)$. Each particle at time $t-a$ has an equal and independent chance of having descendants alive at time t. So the number of clusters is a Binomial random variable with parameters (n,p) where

$$\begin{aligned} n &= \# \text{ of particles at time } (t-a) = \mu \times \text{ mass at time } (t-a)\\ p &= \text{Prob (one particle having descendants time a later)} = P(Z_{a\mu} > 0) \end{aligned}$$

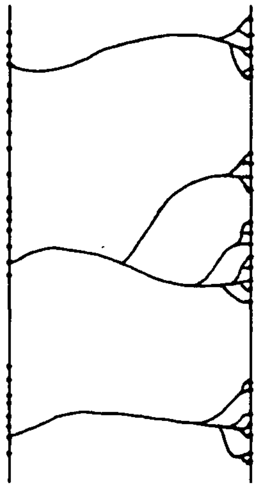

Figure 1.2: Decomposing the measure into clusters.

Now equation (1.1) shows that for large μ, conditional on the mass at time $t-a$

$$\#\text{ of clusters } \stackrel{D}{\approx} \text{ Poisson with mean equal to } (2/a)\times \text{ mass at } (t-a)\ .$$

The masses of particles in each cluster are independent random variables and equation (1.2) shows that for large μ

$$\text{mass of particles in a cluster } \stackrel{D}{\approx} \text{ Exponential with mean equal to (a/2)}.$$

Note that if the spatial motion is homogeneous then the exact shape of each cluster about its root is identically distributed.

Thus the measure at time t is the superposition of an approximately Poisson number of clusters whose masses are independent and approximately exponentially distributed and which are rooted at points uniformly chosen according to the measure at time $t-a$. This description becomes more and more accurate as $\mu \rightarrow \infty$ and is the basis for many properties of superprocesses.

As the parameter $\mu \rightarrow \infty$, if we pick the positions of the initial particles so that the initial measures converge, we hope that the approximating particle systems will converge in law to a finite measure valued process. That this is so and the exact way in which the spatial motion meshes with the branching is the content of Watanabe's Theorem. In the remainder of this section we will develop the notation to state this Theorem.This is essentially taken from Dawson,Iscoe and Perkins [5] section 2. We will descibe a labelling system (first used by Walsh [24]) which will allow us to point to any branch on a binary branching Feller process. This labelling system is used extensively throughout this thesis.

Notation.

$$\begin{aligned}
E &= \text{locally compact separable metric space.}\\
\mathcal{E} &= \text{Borel } \sigma\text{-algebra on } E.\\
b\mathcal{E} &= \text{bounded Borel measurable functions } f : E \to \mathrm{R}.\\
C_l &= \text{continuous functions } f : E \to \mathrm{R} \text{ with limits at infinity.}\\
C_0 &= \text{continuous functions } f : E \to \mathrm{R} \text{ vanishing at infinity.}\\
M(E) &= \text{all measures on } (E, \mathcal{E}).\\
M_F(E) &= \text{finite measures on } (E, \mathcal{E}).\\
M_1(E) &= \text{probability measures on } (E, \mathcal{E}).\\
M_F^\mu(E) &= ((1/\mu)\textstyle\sum_{i=1}^K \delta_{x_i} : x_i \in E, K \in \mathrm{N}_0).\\
m(f) &= \textstyle\int_E f(x)dm(x) \text{ for all } m \in M_F(E), f \in b\mathcal{E}.\\
f^+(x) &= f(x) \wedge 0.\\
E_\Delta &= E \cup \{\Delta\} \text{ where } \Delta \text{ is added as a discrete point.}\\
\mathrm{N} &= \{1, 2, \ldots\}\\
\mathrm{N}_0 &= \{0, 1, \ldots\}
\end{aligned}$$

For any metric space M we write $D(M)$ for the space of right continuous paths with left limits mapping $[0, \infty) \to M$ with the Skorohod topology and $C(M)$ for the space of continuous paths with the topology of uniform convergence on compacts. We give $M_F(E), M_1(E)$ the topology of weak convergence (which is metrizable).

Throughout this thesis C will denote a constant whose exact value is unimportant and may change from line to line. Distinguished constants will be denoted $c_1, c_2, \ldots$

There are two underlying sources of randomness for a branching superprocess, a spatial motion and a branching mechanism. As in the above description we shall take the spatial motion to be a Feller process and the branching mechanism to be critical binary branching. This will be sufficient for all the results of this thesis. We shall briefly describe more general branching superprocesses at the end of this section.

Let $((Y_t : t \geq 0), (P_0^y : y \in E))$ be a Feller process with state space E defined on some probability space $(\Omega_0, \mathcal{F}_0)$. Thus $(P_0^y : y \in E)$ is a strong Markov family and its associated semigroup satisfies

$$T_t : C_0(E) \to C_0(E) \text{ for all } t > 0$$

$$\|T_t f - f\| \to 0 \text{ as } t \to 0 \text{ for all } f \in C_0(E).$$

Define $A : D(A) \subset C_l(E) \to C_0(E)$ by

$$\begin{aligned} Af(x) &= \lim_{t\to 0}(T_t f(x) - f(x))/t \\ D(A) &= (f \in C_l(E) : \lim_{t\to 0}(T_t f(x) - f(x))/t \text{ exists uniformly in } x) \end{aligned}$$

It follows that $D(A)$ is dense in $C_l(E)$. We may extend the Feller process to E_Δ by setting $P_0^\Delta(Y_t = \Delta, \forall t > 0) = 1$.

Let e be a coin tossing random variable defined on $(\Omega_1, \mathcal{F}_1, P_1)$ taking the values 0 and 2 each with probability one half.

Let $I = \bigcup_{n\in \mathrm{N}}(\mathrm{N}_0 \times \{0,1\}^n)$. The elements of I will label the branches of the branching Feller process. If $\beta = (\beta_0, \beta_1, \ldots, \beta_j) \in I$ we write $|\beta| = j$ for the length of the label β . If β is of length j then it will label a branch upto time $(j+1)/\mu$. Write $\beta \sim t$ if $|\beta|/\mu \le t < (|\beta|+1)/\mu$ so that β labels a branch upto the first branching time after t. Let $\beta|_i = (\beta_0, \ldots, \beta_i)$ for $i \le j$ .Call β a descendant of γ and write $\beta \succ \gamma$ if $\gamma = \beta|_i$ for some $i \le |\beta|$. Let $\sigma(\beta, \gamma) = |\beta| - \inf(j : \beta|_j \ne \gamma|_j)$ be the number of generations back that β split from γ.

Let $\Omega_2 = (D(E_\Delta) \times \{0,1\})^I, \mathcal{F}_2 =$ product σ-field . Writing $\omega \in \Omega_2$ as $\omega = (Y^\alpha, e^\alpha)_{\alpha\in I}$ we define $G_n = \sigma((Y^\alpha, e^\alpha) : |\alpha| < n)$. Fix $\mu \in \mathrm{N}$ and $(x_i)_{i\in\mathrm{N}} \in E_\Delta^{\mathrm{N}}$. We wish to find a probability P on $(\Omega_2, \mathcal{F}_2)$ which satisfies for any measurable $A^\alpha \subseteq D(E_\Delta), B^\alpha \subseteq \{0,1\}$ and all $n > 0$

$$P(\omega : (Y^\alpha, e^\alpha)_{|\alpha|=0} \in \prod_{|\alpha|=0} A^\alpha \times B^\alpha) = \prod_{|\alpha|=0} P_0^{x_{\alpha_0}}(Y_{\cdot\wedge(1/\mu)} \in A^\alpha). \prod_{|\alpha|=0} P_1(e \in B^\alpha) \tag{1.3}$$

$$P(\omega : (Y^\alpha, e^\alpha)_{|\alpha|=n} \in \prod_{|\alpha|=0} A^\alpha \times B^\alpha | G_n)(\omega) = \prod_{|\alpha|=0} P_0^{x_{\alpha_0}}(Y_{\cdot\wedge((n+1)/\mu)} \in A^\alpha | Y_{\cdot\wedge(n/\mu)} = Y^{\alpha|_{n-1}}_{\cdot\wedge(n/\mu)})(\omega) \times \prod_{|\alpha|=n} P_1(e \in B^\alpha) \tag{1.4}$$

By an adaption of the Kolmogorov extension Theorem there exists a unique probability measure $P = P_2^{(x_i)_i,\mu}$ satisfying (1.3) and (1.4). It follows that

$$P(Y^\alpha \in A) = P_0^{x_{\alpha_0}}(Y_{\cdot\wedge(|\alpha|+1)/\mu} \in A)$$

so that each Y^α has the law of the Feller process upto time $(|\alpha|+1)/\mu$ when it is frozen. Also from (1.3),(1.4) $(e^\alpha : \alpha \in I)$ are I.I.D. copies of e and are independent of $(Y^\alpha : \alpha \in I)$. The e^α will indicate whether the particles split or die at the branching generations and this will be independent of the spatial motion. Finally from (1.3),(1.4) the random variables $(Y_t^\alpha : |\alpha| = n)$ are conditionally independent given G_n indicating that the particles move independently between branching times.

Let

$$\begin{aligned} \Omega &= E_\Delta^{\mathrm{N}} \times \Omega_2 \\ \mathcal{F} &= \text{product } \sigma\text{-field.} \\ P^{(x_i)_i,\mu} &= \delta_{(x_i)_i} \times P_2^{(x_i)_i,\mu} \end{aligned}$$

Then for $\omega = ((x_i)_i, (Y^\alpha, e^\alpha)_{\alpha \in I})$ particles will start at those x_i that are not equal to Δ. Define the death times for the branches as

$$\tau^\alpha = \begin{cases} 0 & \text{if } \alpha_0 = \Delta \\ \min((i+1)/\mu : e^{\alpha|_i} = 0) & \text{if this set is nonempty} \\ (|\alpha|+1)/\mu & \text{otherwise} \end{cases}$$

To each branch $\alpha \in I$ we associate a corresponding particle which moves along the branch until the death time. So the position of the particle on the branch α is given by

$$N_t^\alpha = \begin{cases} Y_t^\alpha & \text{for } t < \tau^\alpha \\ \Delta & \text{for } t \geq \tau^\alpha \end{cases}$$

Define a filtration where if $j/\mu \leq t < (j+1)/\mu$

$$\mathcal{A}_t^\mu = \sigma(Y^\alpha, e^\alpha : |\alpha| < j) \vee \bigcap_{u>t} \sigma(Y_s^\beta : |\beta| = j, s \leq u).$$

Also let $\mathcal{A}^\mu = \bigvee_{t \geq 0} \mathcal{A}_t^\mu$. For $\beta_1, \ldots, \beta_n \in I$ define the information in the branches $\beta_1, \ldots, \beta_n$ by

$$\mathcal{A}_{\beta_1,\ldots,\beta_n} = \sigma(Y^{\beta_i}, e^{\beta_i} : i = 1, \ldots, n).$$

Now we attach mass $1/\mu$ to each particle and define a measure valued process $N^\mu : [0,\infty) \to M_F(E)$ by

$$\begin{aligned} N_t^\mu(A) &= (1/\mu) \times \#(N_t^\alpha \in A : \alpha \sim t) \\ &= (1/\mu) \sum_{\alpha \sim t} \mathrm{I}(N_t^\alpha \in A). \end{aligned}$$

For $f \in b\mathcal{E}$ we write

$$N_t^\mu(f) = \int_E f(x) dN_t^\mu(x) = (1/\mu) \sum_{\alpha \sim t} f(N_t^\alpha)$$

where we shall always take $f(\Delta) = 0$. Then $N_t^\mu \in \mathcal{A}_t$ for all t and $N^\mu \in D(M_F(E))$ almost surely.

We shall sometimes need the total mass descended from one branch. Define

$$N_t(\beta) = (1/\mu) \sum_{\alpha \sim t, \alpha \succ \beta} \mathrm{I}(N_t^\alpha \neq \Delta).$$

If $m_\mu = (1/\mu)\sum_{i=1}^{K}\delta_{x_i} \in M_F^\mu(E)$ then we extend $(x_i)_{i\leq K}$ to $(x_i)_{i\in\mathbb{N}}$ by setting $x_i = \Delta$ for $i > K$. We write P^{m_μ} for $P^{(x_i)_i,\mu}$. This ignores the order of the $(x_i)_i$ but note that the order does not affect the measure on $\sigma(N_t^\mu : t \geq 0)$ in which we are mainly interested.

We shall need a strong Markov property . Let $T^\mu = (j/\mu : j = 1, 2, \ldots)$. In Perkins [16] Proposition 2.3. some shift operators are defined and a strong Markov property is proved for stopping times taking values in T^μ. (The construction in Perkins [16] for super stable processes is slightly different but the proposition applies here.)

Theorem 1.1 *(Watanabe [25].) Suppose $m_\mu \in M_F^\mu(E) \to m \in M_F(E)$ weakly as $\mu \to \infty$. Then*

$$P^{m_\mu}(N^\mu \in \cdot) \to Q^m(\cdot) \text{ on } D(M_F(E)) \text{ as } \mu \to \infty. \tag{1.5}$$

The law Q^m is supported on the subset of continuous paths. Writing Q^m again for the restriction to $C(M_F(E))$, X_t for the coordinate process and $\mathcal{F}_t^X$ for the canonical completed right continuous filtration then Q^m satisfies the following martingale problem

$$(M) \begin{cases} X_0 = m \\ X_t(f) = m(f) + \int_0^t X_s(Af)ds + Z_t(f) \text{ for all } f \in D(A) \\ Z_t(f) \text{ is a continuous } \mathcal{F}_t^X \text{ martingale s.t. } \langle Z(f)\rangle_t = \int_0^t X_s(f^2)ds \end{cases} \tag{1.6}$$

The family $(Q^m : m \in M_F(E))$ is a strong Markov family.

We reserve the symbol Q^m for the law of the branching super Feller process on path space starting at m.

In Watanabe [25] the convergence of the finite dimensional distributions is proved as well as the continuity of the superprocess under some conditions on the semigroup T_t. Also the Laplace functional of the superprocess is identified as follows. For fixed $f \in D(A)$ let u_t be the unique strong solution of the evolution equation

$$\begin{cases} du_t/dt = Au_t - u_t^2/2 \\ u_0 = f \end{cases}$$

Then

$$E^m(\exp(-X_t(f))) = \exp(-\int u_t dm). \tag{1.7}$$

For convergence as distributions on $D(M)$ and for the continuity of the paths of the superprocess in general see Roelly-Coppoletta [19] where it is shown that the law of a Markov process has Laplace functional given by (1.7) if and only if it satisfies the martingale problem (M).

From the construction we see that a superprocess inherits from the approximating particle systems the following 'branching' property. If $m_1, m_2 \in M_F(E)$ and X_t^1, X_t^2 are independent superprocesses started at m_1, m_2 then the process $X_t^1 + X_t^2$ has law $Q^{m_1+m_2}$.

One can build branching superprocesses with more general branching mechanisms. If the number of offspring has mean zero and a finite variance $\sigma(x)$ that depends continuously on the position of the parent $x \in E$ then the convergence in Theorem 1.1 still holds and only the variance σ of the offspring distribution enters into the limiting process. Fitzsimmons [10] considers infinite variance branching that depends measurably on position and also spatial motion given by a Borel right process. The correct Laplace functional is formally identified and a measure valued Borel right process that has this Laplace functional is constructed.

1.3 The nonstandard model

Why use nonstandard analysis to study a superprocess ? In taking the limit in Watanabe's Theorem we have lost the particle picture. The limiting process takes values in the space of measures. It no longer makes sense to talk of particles dying or having descendants. To do calculations using the intuition of the particle picture we must work with the approximating systems and use weak convergence arguments to obtain results about the superprocess. The idea is to work in the nonstandard universe and to construct a binary branching Feller process exactly as described in section 1.1 but with an infinite branching rate μ. This will give a process taking values in the nonstandard measures.Theorem 1.2 will show that we can derive from this nonstandard process a standard measure valued process in a very simple manner and that this standard process has the law of a superprocess. Now we can argue using particle calculations on the nonstandard model and transfer results to the superprocess . Many of the limiting arguments seem to be built into the model. Thus nonstandard analysis provides a tool to handle weak convergence arguments efficiently. The nonstandard model was introduced by Perkins and used successfully in Perkins [16],[17],[18] and Dawson, Iscoe and Perkins [5]. We now give an informal description of some definitions and results from nonstandard analysis that we hope motivate Theorem 1.2. Cutland [3] gives an introduction to nonstandard analysis for probabilists which is sufficient for our needs.

We start with a superstructure $V(S)$. S will be large enough to contain the basic spaces for constructing the binary branching processes i.e. it will contain the reals ,the metric space E, various measure spaces $(\Omega_0, \mathcal{F}_0)$ e.t.c. $V(S)$ is the superstructure obtained by repeated use of the power set operation and is large enough to do any calculations with the binary branching processes . The nonstandard model will live in an extended superstructure $V(^*S)$. We assume the existence of an embedding $* : V(S) \to V(^*S)$. Every object in $V(S)$ has an image under $*$ and the embedding satisfies three properties.

i. $^*\mathrm{R}$ is a proper extension of R. This will imply the existence of infinitesimal elements of $^*\mathrm{R}$. We will write elements of $^*\mathrm{R}$ as underscored characters $\underline{x}, \underline{t}, \ldots$ We identify the image of real numbers $r \in \mathrm{R}$ with their images $^*r$.

ii. The transfer principle. This allows us to transfer true statements about objects in $V(S)$ to true statements about their images under the embedding. It will imply, for instance, that $^*\mathrm{R}$ is an ordered field. $A \in V(^*S)$ is called internal if $A \in {^*B}$ for some $B \in V(S)$. These are precisely the sets in $V(^*S)$ that we can describe using the transfer principle. We give one example. Suppose the underlying Feller process was a Poisson process of rate one. Then

$$P_0(Y_t = 2) = e^{-t} t^2/2 \text{ for all } t \in \mathrm{R}_+$$

The transfer principle now implies that

$$^*P_0(^*Y_{\underline{t}} = 2) = e^{-\underline{t}}\, \underline{t}^2/2 \text{ for all } \underline{t} \in {^*\mathrm{R}_+}$$

where we have identified the reals $2, e$ with the nonstandard reals $^*2, {^*e}$.

iii. The saturation principle. This is needed, for instance, in the construction of Loeb measures but its statement would not be helpful here.

We can consider the construction of the binary branching Feller processes as a map $P : E_\Delta^{\mathrm{N}} \times \mathrm{N} \to M_1(\Omega)$ where $((x_i)_i, \mu) \to P^{(x_i)_i, \mu}$. Under the embedding we obtain a map $^*P : {^*(E_\Delta^{\mathrm{N}} \times \mathrm{N})} \to {^*M_1(\Omega)}$. So if $\mu \in {^*\mathrm{N}}, (x_i)_i \in {^*E_\Delta^{\mathrm{N}}}$ then $^*P^{(x_i)_i, \mu}$ is an internal probability on $(^*\Omega, {^*\mathcal{A}})$. We also have the embedding of all the particle structure e.g.

$$^*N_{\underline{t}}^{^*\beta} \in {^*E_\Delta} \text{ for all } {^*\beta} \in {^*I}, \underline{t} \in {^*\mathrm{R}_+}$$

To avoid a notational nightmare we drop the $*$ whenever the context makes clear we are talking about a nonstandard object. For instance we write

$$N_{\underline{t}}^{\beta} \in {^*E_\Delta} \text{ for all } \beta \in {^*I}, \underline{t} \in {^*\mathrm{R}_+}$$

The transfer principle will allow us to do calculations with the nonstandard branching processes as easily as with their standard equivalents.

Call $\underline{r} \in {}^*\mathrm{R}$ infinite if $|\underline{r}| > n, \ \forall n \in \mathrm{N}$. Otherwise it is called finite.

Call $\underline{r} \in {}^*\mathrm{R}$ infinitesimal if $|\underline{r}| < 1/n, \ \forall n \in \mathrm{N}$. For every finite $\underline{r} \in {}^*\mathrm{R}$ there is a unique $r \in \mathrm{R}$ such that $\underline{r} - r$ is infinitesimal. This unique r is called the standard part of $\underline{r}$. We write $\underline{r}_1 \approx \underline{r}_2$ if $\underline{r}_1 - \underline{r}_2$ is infinitesimal.

Similarly for any metric space M we call $\mathrm{mon}(y) = (x \in {}^*M : {}^*d(x, {}^*y) < 1/n, \ \forall n \in \mathrm{N})$ the monad about $y \in M$. If $x \in \mathrm{mon}(y)$ we call x nearstandard, y the standard part of x and write $y = \mathrm{st}_M(x)$. Let $ns({}^*M)$ be the set of nearstandard points in ${}^*M$. When the space we are working with is clear it is common to write ${}^\circ x$ for the standard part of x . Indeed for $\underline{r} \in {}^*\mathrm{R}$ we shall write r for the standard part .

Let $(X, \mathcal{X}, \nu)$ be an internal measure space i.e. X is an internal set, $\mathcal{X}$ is an internal algebra of sets (closed under *-finite unions) and ν is a finitely additive internal measure on $\mathcal{X}$. (For example $({}^*E, {}^*\mathcal{E}, N_{\underline{t}})$ and $({}^*\Omega, {}^*\mathcal{A}, {}^*P^{m_\mu})$ are both internal measure spaces.) Define a real valued set function ${}^\circ\nu$ on $\mathcal{X}$ by

$${}^\circ\nu(A) = {}^\circ(\nu(A)) \text{ for all } A \in \mathcal{X}.$$

Loeb showed that the finitely additive measure ${}^\circ\nu$ has a σ-additive extension denoted by $L(\nu)$ on the σ-algebra $\sigma(\mathcal{X})$ generated by $\mathcal{X}$. Let $L(\mathcal{X})$ be the completion of $\sigma(\mathcal{X})$ under $L(\nu)$. Then $(X, L(\mathcal{X}), L(\nu))$ is a standard measure space called a Loeb space.

If E is a complete separable metric space and $\nu \in {}^*M_F(E)$ then there are two ways of obtaining a standard finite measure; we may take the Loeb measure $L(\nu)$ or if ν is nearstandard we may take its standard part $st_{M_F(E)}(\nu)$.These are connected by the following result (see Lemma 2 in Anderson and Rashid [2].)

$$\nu \in ns({}^*M_F(E)) \text{ if and only if } L(\nu)(ns({}^*E)^c) = 0$$

and in this case

$$st_{M_F(E)}(\nu)(A) = L(\nu)(st_E^{-1}(A)) \text{ for all } A \in \mathcal{E}. \tag{1.8}$$

Finally we have an elementary nonstandard criterion for convergence in a metric space M . Consider a sequence $a(n) \in M$ as a map $a : \mathrm{N} \to M$. We have an extension ${}^*a : {}^*\mathrm{N} \to {}^*M$. Then

$$a(n) \to a \in M \text{ if and only if } st_M({}^*a(\mu)) = a \text{ for all } \mu \in {}^*\mathrm{N} \setminus \mathrm{N}. \tag{1.9}$$

Now we state the main Theorem of this section. Fix $\eta \in {}^*\mathrm{N} \setminus \mathrm{N}$ and let $\mu = \eta!$ (this ensure that $\mathrm{Q} \subseteq T^\mu$).

Notation. We write $({}^*\Omega, \mathcal{F}, P^{m_\mu})$ for the Loeb space $({}^*\Omega, L({}^*\mathcal{A}), L({}^*P^{m_\mu}))$. Also when m_μ is fixed we shall often write $P, E, {}^*P, {}^*E$ for $P^{m_\mu}, E^{m_\mu}, {}^*P^{m_\mu}, {}^*E^{m_\mu}$ respectively.

Theorem 1.2 *Let $m \in M_F(E)$ and choose $m_\mu \in {}^*M_F^\mu(E)$ so that $st_{M_F(E)}(m_\mu) = m$. Then there is a unique (up to indistinguishability) continuous M_F valued process X_t on $({}^*\Omega, \mathcal{F}, P^{m_\mu})$ such that $P^{m_\mu} - a.s.$*

$$X_t(A) = L(N^\mu_{\underline{t}})(st^{-1}(A)) \text{ for all } \underline{t} \in ns({}^*[0,\infty)), A \in \mathcal{E}. \tag{1.10}$$

Moreover

$$P^{m_\mu}(X \in C) = Q^m(C) \text{ for all } C \in \mathcal{B}(C(M_F(E))).$$

This is nearly immediate from Theorem 1.1 and (1.8),(1.9). For the proof see Dawson,Iscoe and Perkins [5] Theorem 2.3.

We shall use the nonstandard model for super Brownian motion in R^d throughout chapter 2. In this case there are two very useful results connecting the nonstandard support of the process $(N_{\underline{t}} : \underline{t} > 0)$ to the support of the process $(X_t : t > 0)$. These are proved in Dawson, Iscoe and Perkins [5] Lemmas 4.8, 4.9.

Lemma 1.3 *a. For each nearstandard $\underline{t} \in {}^*[0,\infty)$ such that $t > 0$, with probability one*

$$S(X_t) = st_{\mathrm{R}^d}(S(N^\mu_{\underline{t}})). \tag{1.11}$$

b. With probability one, for all nearstandard $\underline{s}, \underline{t} \in {}^*[0,\infty)$ and $\gamma \sim \underline{t}$

$$\text{if } 0 < s < t \text{ and } N^\gamma_{\underline{t}} \neq \Delta \text{ then } {}^\circ N^\gamma_{\underline{s}} \in S(X_s). \tag{1.12}$$

1.4 Super Brownian motion

Super Brownian motion is the most intensively studied superprocess. We give a summary of those path properties that will be used in this thesis. We assume that the process is started at a finite measure.

In dimension one the measure X_t has a density $X(t,x)$ which is continuous in $(0,\infty) \times \mathrm{R}$. We shall not consider this case until section 3.1 and delay a careful statement of this result until Theorem 3.1 .

In dimension $d \geq 2$ the measures X_t are singular with respect to Lebesgue measure for all $t > 0$. Thus even if the process starts with a smooth density it instantly becomes singular. This result is proved in Dawson and Hochberg [4] for a fixed time and is extended for all times in Perkins [18] in a remarkable way which describes the exact nature of the measure X_t. We explain this result now.

For any continuous onto increasing function $\phi : [0,\infty) \to [0,\infty)$ define a function on the subsets of R^d by

$$\phi m(A) = \lim_{\delta \to 0+} \inf_{\substack{A \subseteq \cup D_i \\ \mathrm{diam}(D_i) \leq \delta}} \sum_{i=1}^{\infty} \phi(\mathrm{diam}(D_i)) \tag{1.13}$$

where the supremum is taken over all countable covers of $A \subseteq \mathrm{R}^d$ using sets of diameter less than δ. Then $\phi m(\cdot)$ is a Borel measure called Hausdorff ϕ-measure. If $\phi(x) = x^d$ then $\phi m(\cdot)$ is a multiple of d-dimensional Lebesgue measure. In general however, $\phi m(\cdot)$ is not a σ-finite measure. If $\phi(x)/x^d \to \infty$ as $x \to 0$ it gives a way of distinguishing between d-dimensional Lebesgue null sets. If $\phi(x) = x^r$ then $\phi m(\cdot)$ is called Hausdorff r-measure and will give positive measure to smoothly embedded subsets of R^r (e.g. for a smooth curve C , $x^1 m(C) = \mathrm{length}(C)$). Define for Borel $A \in R^d$

$$\dim(A) = \inf(r > 0 : x^r m(A) < \infty)$$

Then dim(A) , the Hausdorff dimension of A, takes values in $[0,d]$. Note that for A of dimension r we have

$$x^s m(A) = \begin{cases} 0 & \text{if } s > r \\ \in [0,\infty] & \text{if } s = r \\ \infty & \text{if } s < r \end{cases}$$

Notation. For any Borel measure m we write $S(m)$ for the closed support of m.

Theorem 1.4 *(Perkins [18]) Let $\phi(x) = x^2 \log^+ \log^+(1/x)$. Let X_t be super Brownian motion started at $m \in M_F(\mathrm{R}^d)$ in dimension $d \geq 3$. Then there exist constants $0 < c_1 \leq c_2 < \infty$ depending only on d such that with probability one*

$$c_1 \phi m(A \cap S(X_t)) \leq X_t(A) \leq c_2 \phi m(A \cap S(X_t)) \;\; \forall t > 0, \;\; \forall \textit{ Borel } A \tag{1.14}$$

So, upto a density bounded inside $[c_1, c_2]$, the measure X_t is a deterministic measure spread over a random closed set $S(X_t)$. This implies immediately that $S(X_t)$ has Hausdorff dimension 2 and hence is Lebesgue null for all $t > 0$. In dimension 2 there is a less precise result which still implies singularity.

Theorem 1.4 allows us to concentrate on the support process $(S(X_t) : t \geq 0)$ of super Brownian motion. The following two Theorems (proved in Dawson, Iscoe and Perkins [5] Theorems 1.1,4.5) show that the support moves with finite speed and gives a modulus of continuity for that speed.

Notation. For closed $A \subseteq R^d$ and $\varepsilon > 0$, let $A^\varepsilon = (x : d(x,A) \leq \varepsilon)$.

Theorem 1.5 *Let $h(t) = \sqrt{t(\log(t^{-1}) \vee 1)}$.*

a. For Q^m – a.a.ω and each $c > 2$, $\exists\ \delta(\omega,c)$ such that if $0 < t-s \le \delta$ then $S(X_t) \subseteq S(X_s)^{ch(t-s)}$.

b. For each $t \ge 0$, for Q^m – a.a.ω and each $c > \sqrt{2}$, $\exists\tilde{\delta}(\omega,c)$ such that if $0 < s \le \tilde{\delta}$ then $S(X_{t+s}) \subseteq S(X_t)^{ch(s)}$.

This Theorem can be derived from a global and local modulus of continuity for the motion of the particles in the nonstandard model. The key is that we can control the motion of all the particles simultaneously.

Theorem 1.6 *Let* $h(t) = \sqrt{t(\log(t^{-1}) \vee 1)}$.

a. For P^{m_μ} – a.a.ω and each $c > 2$, $\exists\ \delta(\omega,c)$ such that if $0 < t-s \le \delta$ for nearstandard $\underline{s},\underline{t} \in {}^*[0,\infty)$, $\beta \sim \underline{t}$ and $N^\beta_{\underline{t}} \ne \Delta$, then $|N^\beta_{\underline{t}} - N^\beta_{\underline{s}}| \le ch(\underline{t}-\underline{s})$.

b. For each nearstandard $\underline{t} \in {}^*[0,\infty]$, for P^{m_μ} – a.a.ω and each $c > \sqrt{2}$, $\exists\tilde{\delta}(\omega,c)$ such that if $0 < s \le \tilde{\delta}$, $\beta \sim \underline{t}+\underline{s}$, $N^\beta_{\underline{t}+\underline{s}} \ne \Delta$ then $|N^\beta_{\underline{t}+\underline{s}} - N^\beta_{\underline{t}}| \le ch(\underline{t}-\underline{s})$.

Theorem 1.5 follows from Theorem 1.6 and equations (1.11),(1.12) on the support of the nonstandard model.(The local modulus of continuity is not stated in Dawson, Iscoe and Perkins but the proof is entirely similar and simpler than the global modulus.) From the proofs of these results we also note that

$$Q(\delta(c) \le \rho) \le P(\delta(c) \le \rho) = O(\rho^{((c^2/2)-2)}) \text{ as } \rho \to 0 \tag{1.15}$$

Super Brownian motion has a space-time-mass scaling property. For $\beta > 0$ define $K_\beta : M_F(\mathrm{R}^d) \to M_F(\mathrm{R}^d)$ as follows

$$\int f(x)K_\beta m(dx) = \int f(\beta x)m(dx) \text{ for all measurable f.}$$

Proposition 1.7 *For* $m \in M_F(\mathrm{R}^d)$ *the law of the process* $(X_t : t \ge 0)$ *under* Q^m *equals the law of the process* $(\beta^{-1}K_{\beta^{-1/2}}X_{\beta t} : t \ge 0)$ *under* $Q^{\beta K_{\beta^{1/2}}m}$.

For a proof see Roelly-Coppoletta [19] Propositon 1.8.

Exact asymptotics for the probability of super Brownian motion giving mass to small balls were proved in Dawson,Iscoe and Perkins [5].

Theorem 1.8 a. *For* $d \ge 3$ *there exists a constant* $c_3 \in (0,\infty)$ *depending only on d such that for any* $\delta > 0$ *there exists* ε_0 *and*

$$|\varepsilon^{2-d}Q^m(X_t(B(x,\varepsilon)) > 0) - c_3\int_{\mathrm{R}^d} p_t(x,y)m(dy)| \le \delta m(1) + \varepsilon(c_3+\delta)^2 m(1)^2/2$$

for all $\varepsilon \le \varepsilon_0, t \ge \delta, x \in \mathrm{R}^d, m \in M_F(\mathrm{R}^d)$. *In particular*

$$\lim_{\varepsilon \to 0}(1/\varepsilon^{d-2})Q^m(X_t(B(x,\varepsilon)) > 0) = c_3 \int_{\mathrm{R}^d} p_t(x,y)m(dy)$$

and for any $\delta > 0, K < \infty$ *the convergence is uniform over* $t \ge \delta, x \in \mathrm{R}^d, m(1) \le K$.

b. For $d \ge 3$ *there exists a constant* $c_4 \in (0,\infty)$ *depending only on* d *such that for all* $x \in R^d$, $\varepsilon > 0$, $t \ge \varepsilon^2$, $m \in M_F(\mathrm{R}^d)$

$$Q^m(X_t(B(x,\varepsilon)) > 0) \le c_4\varepsilon^{d-2} \int_{\mathrm{R}^d} p_{t+\varepsilon^2}(x,y)dm(y).$$

c. There exists a constant $c_5 \in (0,\infty)$ *depending on* d *such that for all* ε, t, m, x

$$Q^m(X_t(B(x,\varepsilon)) > 0) \ge c_5 \left(\varepsilon^{d-2} \int_{\mathrm{R}^d} p_t(x,y)dm(y) \wedge 1\right).$$

The proofs of parts a,b follow from the proof of Dawson,Iscoe and Perkins [5] Theorem 3.1 and part c follows from Evans and Perkins [8] Lemma 1.3 .

Finally we state a result on the effect of changing the initial measure on the law of of the process (Evans and Perkins [8] Corollary 2.4).

Theorem 1.9 *For any* $m_1, m_2 \in M_F(\mathrm{R}^d)$ *and* $s,t > 0$ *the laws* $Q^{m_1}(X_t \in \cdot)$ *and* $Q^{m_2}(X_s \in \cdot)$ *are mutually absolutely continuous.*

When trying to prove almost sure results about $Q^m(X_t \in \cdot)$ this will allow us to choose m and $t > 0$ at our convenience.

Chapter 2

The support process of super Brownian motion

2.1 The support process started at a point

For super Brownian motion started at a point mass the growth of the support is controlled by the local modulus of continuity Theorem 1.5b. We show that in this case there is a limit result for the rate of growth.

Theorem 2.1 *Let* $g(t) = \sqrt{2t\log(t^{-1})}$ *and* $\rho(t) = \inf\{r : S(X_t) \subseteq B(0,r)\}$. *Then*

$$\text{For } Q^{\delta_0} - a.a.\omega : \lim_{t\to 0}\frac{\rho(t)}{g(t)} = 1$$

First we reinterpret the classical results on Galton Watson processes in (1.1),(1.2) in terms of the nonstandard model.

Lemma 2.2 *For nearstandard* $\underline{t}, \underline{x} \geq 0$ *such that* $x, t > 0$ *and* $\gamma \sim \underline{t}$

a. $\mu^{\bullet}P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) > 0) \approx 2t^{-1}$

b. $P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) > \underline{x}|N^{\mu}_{\underline{t}}(1) > 0) = e^{-(2x/t)}$

c. $P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) \geq \underline{x}|N^{\gamma}_{\underline{t}} \neq \Delta) \geq 2xt^{-1}\exp(-2x/t)$

PROOF OF LEMMA 2.2. Parts a,b follow from (1.1),(1.2) and the transfer principle. For part c, fix $j \in \{0, 1, \ldots, 2^{\mu t}\}$. Then a counting argument shows

$${}^{\bullet}P^{\mu^{-1}\delta_0}(\mu N^{\mu}_{\underline{t}}(1) = j, N^{\gamma}_{\underline{t}} \neq \Delta) = j2^{-\mu t \bullet}P^{\mu^{-1}\delta_0}(\mu N^{\mu}_{\underline{t}} = j).$$

So for $\underline{x} \in \{0, 1/\mu, 2/\mu, \ldots, 2^{\mu t}/\mu\}$

$$\begin{aligned}
{}^{\bullet}P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) \geq \underline{x}|N^{\gamma}_{\underline{t}} \neq \Delta) &= 2^{\mu t \bullet}P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) \geq \underline{x}, N^{\gamma}_{\underline{t}} \neq \Delta) \\
&= \sum_{j=\mu\underline{x}}^{2^{\mu t}} j^{\bullet}P^{\mu^{-1}\delta_0}(\mu N^{\mu}_{\underline{t}}(1) = j) \\
&\geq \mu\underline{x}^{\bullet}P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) \geq \underline{x}) \\
&= \underline{x}(\mu^{\bullet}P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) > 0))^{\bullet}P^{\mu^{-1}\delta_0}(N^{\mu}_{\underline{t}}(1) \geq \underline{x}|N^{\mu}_{\underline{t}}(1) > 0)
\end{aligned}$$

and the result follows from parts a,b.□

PROOF OF THEOREM 2.1. The local modulus of continuity (Theorem 1.5b) implies it is enough to show

$$\text{For } Q^{\delta_0} - a.a.\omega: \ \liminf_{t\to 0} \frac{\rho(t)}{g(t)} \geq 1.$$

We use the nonstandard model for super Brownian motion taking $x_i = 0$, $i = 1, \ldots, \mu$ so that the initial mass m_μ equals δ_0. Fix $\theta \in \mathrm{Q}$, $\varepsilon \in \mathrm{R}$ such that $\varepsilon, \theta \in (0,1)$.Define

$$A_n = \left\{\omega : \ \sup(|N^\beta_{\theta^n}| : \beta \sim \theta^n) \leq (1-\varepsilon)g(\theta^n)\right\}$$

Recall that $I(\theta^n, \theta^n)$ lists the particles at time zero that have descendants alive at time θ^n. For each $\gamma \in I(\theta^n, \theta^n)$ pick $\tilde{\gamma} \sim \theta^n$ such that $\gamma \prec \tilde{\gamma}$ and $N^{\tilde{\gamma}}_{\theta^n} \neq \Delta$.

$$\begin{aligned} P(A_n) &\leq P\left(\sup(|N^{\tilde{\gamma}}_{\theta^n}| : \gamma \in I(\theta^n, \theta^n)) \leq (1-\varepsilon)g(\theta^n)\right) \\ &= E\left[\prod_{\gamma \in I(\theta^n,\theta^n)} {}^{*}P(|N^{\tilde{\gamma}}_{\theta^n}| \leq (1-\varepsilon)g(\theta^n)\right] \\ &= E\left[(1-I_n)^{Z(\theta^n,\theta^n)}\right] \end{aligned}$$

where $I_n = P_0(|B_{\theta^n}| > (1-\varepsilon)g(\theta^n))$ and $Z(\theta^n, \theta^n)$ is the cardinality of $I(\theta^n, \theta^n)$. Lemma 2.2 shows that $Z(\theta^n, \theta^n)$ has a *-binomial distribution $B(n,p)$ with $n = \mu$ and $\mu p \approx 2\theta^n$. Using the bound $P_0(|B_1| \geq x) \geq C(d)x^{d-2}e^{-x^2/2}$ for $x \geq 1$ we have

$$\begin{aligned} P(A_n) &\leq {}^{\circ}((1 - I_n p)^{\mu}) \\ &= \exp(-2\theta^{-n} I_n) \\ &\leq \exp\left(-Cn^{d-2}\theta^{-\varepsilon n}\right) \end{aligned}$$

Borel Cantelli implies there exists $N(\omega) < \infty$ almost surely such that

$$\text{For all } n \geq N(\omega),\ \exists \beta \sim \theta^n \text{ such that } |N^\beta_{\theta^n}| > (1-\varepsilon)g(\theta^n)$$

Now fix ω such that $N(\omega) < \infty$ and outside a null set so that the support relations (1.11),(1.12) and the modulus of continuity for particles Theorem 1.6a holds. Choose $n \geq N(\omega)$ such that $\theta^n < \delta(\omega, 3)$. Find $\beta \sim \theta^n$ such that $|N^\beta_{\theta^n}| > (1-\varepsilon)g(\theta^n)$ .The modulus of continuity implies

$$\begin{aligned} |N^\beta_{\underline{t}}| &\geq (1-\varepsilon)g(\theta^n) - 3\sqrt{\theta^n(1-\theta)}\sqrt{\log((\theta^n(1-\theta))^{-1})} \\ &\qquad\qquad \text{for } \theta^{n+1} \leq \underline{t} \leq \theta^n \end{aligned}$$

$$\geq \; g(\theta^n)\left(1-\varepsilon-(3/\sqrt{2})\left((1-\theta)(1+\log(1-\theta)/n\log(\theta)\right)^{1/2}\right)$$
$$\geq \; g(\underline{t})\left(1-\varepsilon-(3/\sqrt{2})\left((1-\theta)(1+\log(1-\theta)/n\log(\theta)\right)^{1/2}\right)$$

But ${}^\circ N_{\underline{t}}^{\beta} \in S(X_t)$ by (1.11) so

$$\liminf_{t\to 0}\frac{\rho(t)}{g(t)} \geq 1-\varepsilon-3\sqrt{1-\theta}$$

Now take sequences $\varepsilon_n \downarrow 0, \theta_n \uparrow 1$ to show that with probability one

$$\liminf_{t\to 0}\frac{\rho(t)}{g(t)} \geq 1$$

Finally we note that the set $\{\omega : \liminf_{t\to 0}\rho(t)/g(t) \geq 1\}$ is Borel in $C([0,\infty), M_F)$ and so we may transfer the result to path space.□

Thus for small t the support is approximately contained in a ball of radius $g(t)$. We may normalise the support so that it has radius one. We show in Corollary 2.5 that as $t \to 0$ this normalised support 'fills' the whole unit ball. The following Lemmas examine how fast holes appear in the support.

Lemma 2.3 *For $d \geq 3, 0 < r < 1$*

a. If $k < (1-r^2)/(d-2)$ then with probability one

$$\lim_{t\to 0+} t^{-k} \sup_{x\in B(0,r)} d(x, S(X_t)/g(t)) = 0.$$

b. If $k > (1-r^2)/(d-2)$ then with probability one

$$\limsup_{t\to 0+} t^{-k} \sup_{x\in B(0,r)} d(x, S(X_t)/g(t)) = \infty.$$

PROOF. a. Fix $r \in (0,1), k < (1-r^2)/(d-2)$. We look for holes inside $S(X_t) \cap B(0, rg(t))$ of size $t^{k+(1/2)}$. Define

$$\text{Grid} = \left((t,x) : t = n^{-1/3}\;, n = 1, \ldots\;,\; x \in (t^{k+(1/2)}/2\sqrt{d})Z^d, |x| \leq rg(t)\right)$$

We first show that for small t there are no holes centered at a point of the Grid. For $(n^{-1/3}, x) \in$ Grid we have

$$\begin{aligned} Q^{\delta_0}\left(X_{n^{-1/3}}(B(x,(1/4)n^{-(2k+1)/6}))=0\right) &= Q^{n^{1/3}\delta_0}\left(X_1(B(n^{1/6}x,(1/4)n^{-k/3}))=0\right) \\ &= \left[1-Q^{\delta_0}\left(X_1(B(n^{1/6}x,(1/4)n^{-k/3}))>0\right)\right]^{n^{1/3}} \end{aligned}$$

by first the space-time-mass scaling and then the branching property of super Brownian motion. From Theorem 1.8c

$$Q^{\delta_0}\left(X_1(B(n^{1/6}x,(1/4)n^{-k/3}))>0\right)\geq C(n^{-k(d-2)/3}p_1(n^{1/6}x)\wedge 1).$$

Pick ε such that $k<(1-(1-\varepsilon)^{-1}r^2)/(d-2)$ and choose n_0 such that $(1/4)n_0^{-k/3}<\varepsilon$. Noting that $|n^{1/6}x|\leq r\sqrt{(2/3)\log n}$ we have for $n\geq n_0$

$$\begin{aligned} Q^{\delta_0}\left(X_{n^{-1/3}}(B(x,(1/4)n^{-(2k+1)/6}))=0\right) &\leq \left[1-Cn^{-k(d-2)/3}\exp(-r^2\log(n)/3(1-\varepsilon))\right]^{n^{1/3}} \\ &\leq \exp(-Cn^{(1/3)-k(d-2)/3-r^2/3(1-\varepsilon)}) \end{aligned}$$

Therefore

$$\sum_{(t,x)\in \mathrm{Grid}:t\leq n_0^{-1/3}} Q^{\delta_0}(X_t(B(x,(1/4)t^{k+(1/2)}))=0)\leq \sum_{n\geq n_0} Cn^{kd/3}(\log n)^d\exp(-Cn^{(1-k(d-2)-r^2/(1-\varepsilon))/3})$$

This sums over n and Borel Cantelli implies $\exists N(\omega)<\infty$ almost surely such that

$$\text{if } (t,x)\in \text{Grid and } t\leq N^{-1/3} \text{ then } X_t(B(x,(1/4)t^{k+(1/2)}))>0 \tag{2.16}$$

Fix ω such that $N(\omega)<\infty$ and off a null set so that the global modulus of continuity (Theorem 1.5a) holds.Now argue by contradiction. Suppose $\exists t\leq \min(\delta(\omega,3),N^{-1/3},2^{-20})$ and an $x\in B(0,rg(t))$ such that $X_t(B(x,t^{k+(1/2)}))=0$ .Pick $n\geq N$ such that $(n+1)^{-1/3}<t\leq n^{-1/3}$. We use the global modulus of continuity for the support to show there must be a hole (of smaller radius) centered at a grid point.The modulus of continuity implies

$$X_{n^{-1/3}}\left(B(x,t^{k+(1/2)}-3h((n)^{-1/3}-(n+1)^{-1/3}))\right)=0$$

But

$$\begin{aligned} t^{k+(1/2)}-3h((n)^{-1/3}-(n+1)^{-1/3}) &\geq n^{-(2k+1)/6}-(3n^{-4/3}\log(3n^{4/3}))^{1/2} \\ &\geq (1/2)n^{-(2k+1)/6} \end{aligned}$$

so that

$$X_{n^{-1/3}}(B(x,(1/2)n^{-(2k+1)/6}))=0$$

But we can find x_0 such that $(n^{-1/3},x_0)\in$ Grid and $B(x_0,(1/4)n^{-(2k+1)/6})\subseteq B(x,(1/2)n^{-(2k+1)/6})$. Thus

$$X_{n^{-1/3}}(B(x_0,(1/4)n^{-(2k+1)/6}))=0$$

which contradicts equation 2.16 . So for small t , for all $x \in B(0, rg(t))$ we have $t^{-k-(1/2)}d(x, S(X_t)) \leq 1$. Thus

$$\limsup_{t\to 0+} t^{-k}(2\log(t^{-1}))^{1/2} \sup_{x\in B(0,r)} d(x, S(X_t)/g(t)) \leq 1.$$

But $k < (1-r^2)/(d-2)$ was arbitrary and the result follows.

b. Let x_t have coordinates $(rg(t), 0, \ldots, 0)$ for $t < 1$. Then using Theorem 1.8b

$$\begin{aligned} Q^{\delta_0}(X_t(B(x_t, t^l)) > 0) &\leq Ct^{l(d-2)}p(t + t^{2l}, rg(t)) \\ &\leq Ct^{l(d-2)-(d/2)+r^2(1+t^{2l-1})^{-1}} \end{aligned}$$

If $l > (1/2) + (1-r^2)/(d-2)$ this probability tends to zero as $t \to 0$ so that along a fast enough sequence $t_n \to 0$ Borel Cantelli guarantees $X_{t_n}(B(x_{t_n}, (t_n)^l)) = 0$ for large n. So if $k > (1-r^2)/(d-2)$

$$\limsup_{t\to 0+} t^{-k}(2\log(t^{-1}))^{1/2} \sup_{x\in B(0,r)} d(x, S(X_t)/g(t)) \geq 1.$$

But $k > (1-r^2)/(d-2)$ was arbitrary and the result follows. □

In dimension 1 and 2 we do not have estimates on the probability of charging small balls given by Theorem 1.8.The following Lemma gives such an estimate and hence an upper bound for the rate at which holes appear in the support in these dimensions .While this bound is certainly not best possible it will be sufficient to prove Corollary 2.5.

Lemma 2.4 *a. For all $x \in \mathbb{R}^d, t > 0$*

$$Q^{\delta_0}(X_t(B(x,\varepsilon)) = 0) \leq \exp\left(-2(2\pi)^{d/2}\varepsilon^d t^{-(d+2)/2}\exp(-t^{-1}(\|x\| + \varepsilon)^2)\right)$$

b. For $d = 1, 2$ and $0 < r < 1$ if $k < (1-r^2)/d$ then with probability one

$$\lim_{t\to 0+} t^{-k} \sup_{x\in B(0,r)} d(x, S(X_t)/g(t)) = 0.$$

PROOF. a. It will be enough to prove this for the nonstandard model with $x_i = 0,\ i = 1, \ldots, \mu$. $I(t,t)$ lists the particles alive at time zero that have descendents alive at time t. For each $\gamma \in I(t,t)$ pick $\tilde{\gamma} \sim t$ such that $\gamma \prec \tilde{\gamma}$ and $N_t^{\tilde{\gamma}} \neq \Delta$. Using (1.11) we have

$$\begin{aligned} P(X_t(\overline{B(x,\varepsilon)}) = 0) &\leq P\left(\bigcap_{\gamma\in I(t,t)} (N_t^{\tilde{\gamma}} \notin B(x,\varepsilon))\right) \\ &= E\left[(1 - I_t)^{Z(t,t)}\right] \end{aligned}$$

where $I_t = P_0(B_t \in B(x,\varepsilon))$. $Z(t,t)$ has a $*$-binomial distribution $B(\mu,p)$ where $\mu p \approx 2t^{-1}$. So

$$\begin{aligned}
P(X_t(\overline{B(x,\varepsilon)}) = 0) &\leq {}^{\circ}((1-I_t p)^{\mu}) \\
&= \exp(-2t^{-1}I_t) \\
&\leq \exp\left(-2(2\pi)^{d/2}\varepsilon^d t^{-(d+2)/2}\exp(-t^{-1}(\|x\|+\varepsilon)^2)\right)
\end{aligned}$$

The bound is continuous in ε so we may replace $\overline{B(x,\varepsilon)}$ by $B(x,\varepsilon)$.

b. We follow the proof of Theorem 2.3a. replacing the equation 2.16 using the bound above by

$$Q^{\delta_0}\left(X_{n^{-1/3}}(B(x,(1/4)n^{-(2k+1)/6})) = 0\right) \leq \exp(-Cn^{-((2k+1)d-(d+2)+2r^2)/6})$$

The remainder of the proof carries through.□

The Hausdorff metric on compact subsets of R^d is defined as follows. For K_1, K_2 nonempty compact sets

$$\begin{aligned}
d_H(K_1,K_2) &= \max(\sup_{x\in K_1} d(x,K_2)\wedge 1, \sup_{x\in K_2} d(x,K_1)\wedge 1) \\
d_H(K_1,\emptyset) &= 1
\end{aligned}$$

Combining Theorem 2.1 and Lemmas 2.3,2.4 we obtain

Corollary 2.5 *If $d \geq 1$ then with probability one*

$$d_H(S(X_t)/g(t), \overline{B(0,1)}) \to 0 \ \textit{as}\ t \to 0+.$$

2.2 Recovering the measure from the support

For a fixed time $t > 0$, Theorem 1.4 can be improved as follows. For $d \geq 3$ there is a constant c_6 depending only on the dimension so that with probability one

$$X_t(A) = c_6\phi m(A \cap S(X_t)) \text{ for all Borel } A.$$

The proof, due to Perkins (private communication), uses the $0-1$ law explained in Proposition 2.11. Thus for $d \geq 3$ and fixed $t > 0$ the measure X_t can be completely recovered from its support.

We now give an alternate method for recovering X_t from its support. It is an analogue of a Theorem of Kingman on Brownian local time.Let $l(t,x)$ be the local time of a Brownian motion B_t. Let $Z(t,x) = \{s \leq t : B_s = x\}$. Recall that for a closed set A we write A^ε for the set $\{x : d(x,A) \leq \varepsilon\}$. We also write

Leb(A) for the Lebesgue measure of a set A. In Kingman [13] it is shown that there exists a constant c_7 such that for fixed x, t with probability one

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/2} \operatorname{Leb}(Z(t,x)^\varepsilon) = c_7 l(t,x).$$

Theorem 2.6 *For $d \geq 3, t > 0$ and Borel A of finite Lebesgue measure, with probability one*

$$\varepsilon^{2-d} \operatorname{Leb}(S(X_t)^\varepsilon \cap A) \xrightarrow{L^2} c_3 X_t(A) \ as\ \varepsilon \to 0$$

where c_3 is the universal constant occuring in Theorem 1.8 a.

PROOF. Fix $t > 0$ and Borel A. Let $K_t^\varepsilon = \varepsilon^{2-d}\operatorname{Leb}(S(X_t) \cap A)$. We shall show

$$\begin{aligned} \limsup_{\varepsilon \to o} E((K_t^\varepsilon)^2) &\leq c_3^2 E((X_t(A))^2) & (2.17) \\ \lim_{\varepsilon \to 0} E(K_t^\varepsilon Y) &= c_3 E(X_t(A)Y) \text{ for all } Y \in L^2(\bigvee_{\delta > 0} \mathcal{F}_{t-\delta}). & (2.18) \end{aligned}$$

The result then follows for

$$\begin{aligned} & \limsup_{\varepsilon \to 0} E((K_t^\varepsilon - c_3 X_t(A))^2) \\ = \ & \limsup_{\varepsilon \to 0} (E((K_t^\varepsilon)^2) - 2c_3 E(K_t^\varepsilon X_t(A)) + c_3^2 E((X_t(A))^2)) \\ \leq \ & 0 \end{aligned}$$

since $t \to X_t(A)$ is continuous and hence $X_t(A) \in L^2(\bigvee_{\delta>0} \mathcal{F}_{t-\delta})$.

Proof of (2.17). We shall prove this for the nonstandard model.

$$E((K_t^\varepsilon)^2) = \varepsilon^{4-2d} \int_{A \times A} P(X_t(B(x,\varepsilon)) > 0, X_t(B(y,\varepsilon)) > 0) dx dy. \tag{2.19}$$

To calculate the probability that occurs as the integrand in (2.19) we use the following idea. Recall that the support moves with a modulus of continuity given by $ch(t)$ where $c > 2$. We shall choose a suitable value of c later. Using the notation of Theorem 1.6 let $G_a = (\omega : \delta(\omega, c) \geq a)$ be the set where the global modulus of continuity for particles holds for time intervals less than a . For a path in G_δ , the only particles that can enter $B(x,\varepsilon)$ at time t must lie in $B(x, \varepsilon + ch(\delta))$ at time $t - \delta$. So if the distance between $B(x,\varepsilon)$ and $B(y,\varepsilon)$ is at least $2ch(\delta)$ then on the set G_δ , if we condition on the measure $X_{t-\delta}$, the events $(X_t(B(x,\varepsilon)) > 0)$ and $(X_t(B(y,\varepsilon)) > 0)$ are independent.

Fix $a \geq \varepsilon^2, \delta \in (\varepsilon^2, a)$ and x, y such that $|x-y| > 3ch(\delta)$.We write $m|_B$ for the measure m restricted to B. Using Theorem 1.8a we can find ε_0 such that for all $\varepsilon \leq \varepsilon_0$

$$\begin{aligned}
&\varepsilon^{4-2d} P(X_t(B(x,\varepsilon)) > 0, X_t(B(y,\varepsilon)) > 0, G_a | \sigma(\mathcal{A}_{t-\delta})) \\
\leq\; & \varepsilon^{4-2d} P^{X_{t-\delta}|_{B(x,\varepsilon+ch(\delta))}}(X_\delta(B(x,\varepsilon)) > 0) P^{X_{t-\delta}|_{B(y,\varepsilon+ch(\delta))}}(X_\delta(B(y,\varepsilon)) > 0) \\
\leq\; & \varepsilon^{4-2d} P^{X_{t-\delta}}(X_\delta(B(x,\varepsilon)) > 0) P^{X_{t-\delta}}(X_\delta(B(y,\varepsilon)) > 0) \\
\leq\; & \left(c_3 \int p_\delta(x,z) X_{t-\delta}(dz) + \delta X_{t-\delta}(1) + \varepsilon(c_3+\delta)^2 X_{t-\delta}^2(1)/2\right) \\
& \times \left(c_3 \int p_\delta(y,z) X_{t-\delta}(dz) + \delta X_{t-\delta}(1) + \varepsilon(c_3+\delta)^2 X_{t-\delta}^2(1)/2\right) \\
\leq\; & c_3^2 \int p_\delta(x,z) X_{t-\delta}(dz) \int p_\delta(y,z) X_{t-\delta}(dz) + C(\delta+\varepsilon)(1 + X_{t-\delta}^4(1))
\end{aligned}$$

So for $\varepsilon \leq \varepsilon_0$

$$\begin{aligned}
&\varepsilon^{4-2d} \int\limits_{A \times A \cap (|x-y| \geq 3ch(\delta))} \int P(X_t(B(x,\varepsilon)) > 0, X_t(B(y,\varepsilon)) > 0, G_a) dx dy \\
\leq\; & c_3^2 E(X_t^2(A)) + C(\delta+\varepsilon)|A|^2(1 + E(X_{t-\delta}^4(1)))
\end{aligned} \tag{2.20}$$

Similarly using Theorem 1.8b and conditioning on $\sigma(\mathcal{F}_{t-\varepsilon^2})$

$$\begin{aligned}
&\varepsilon^{4-2d} \int\limits_{A \times A \cap (|x-y| < 3ch(\delta))} \int P(X_t(B(x,\varepsilon)) > 0, X_t(B(y,\varepsilon)) > 0, G_a) dx dy \\
\leq\; & \varepsilon^{4-2d} \int\limits_{A \times A \cap (|x-y| < 3ch(\varepsilon^2))} \int P(X_t(B(x,\varepsilon)) > 0, G_a) dx dy \\
& + \varepsilon^{4-2d} E\Big(\int\limits_{A \times A \cap (|x-y| < 3ch(\delta))} \int P^{X_{t-\varepsilon^2}}(X_{\varepsilon^2}(B(x,\varepsilon)) > 0) P^{X_{t-\varepsilon^2}}(X_{\varepsilon^2}(B(y,\varepsilon)) > 0) dx dy\Big) \\
\leq\; & Cm(1)|A|\varepsilon^{2-d}(\varepsilon \log(1/\varepsilon))^d + C|A|(h(\delta))^d E(X_{t-\varepsilon^2}^2(1)).
\end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21) we see that

$$\limsup_{\varepsilon \to 0} E((K_t^\varepsilon)^2 \mathrm{I}_{G_a}) \leq c_3^2 E(X_t^2(A)) + C\delta$$

where C depends only on m, d, t, A. However $\delta \leq a$ was arbitrary and so $\limsup E((K_t^\varepsilon)^2 \mathrm{I}_{G_a}) \leq c_3^2 E(X_t^2(A))$. To remove the restriction to G_a we note that $P(G_a) \to 1$ as $a \downarrow 0$ and so it is certainly enough to show $\sup_{\varepsilon > 0} E((K_t^\varepsilon)^4) < \infty$.

$$\varepsilon^{8-4d} \int\limits_{\substack{A \times A \times A \times A \\ (|x_i - x_j| \geq 3ch(\varepsilon^2) : i \neq j)}} \int\int\int P(\prod_{i=1}^4 X_t(B(x_i,\varepsilon)) > 0) dx_1 \ldots dx_4)$$

$$\le \ \varepsilon^{8-4d} \int \underset{\substack{A\times A\times A\times A \\ (|x_i - x_j| \ge 3ch(\varepsilon^2)\,:\, i\neq j)}}{\iiint} E\left(\prod_{i=1}^{4} P^{X_{t-\varepsilon^2}}(X_{\varepsilon^2}(B(x_i,\varepsilon)) > 0) + P(G_{\varepsilon^2})\right)$$

$$\le \ c_1^4 E\left(\left(\int p_{\varepsilon^2}(a,z) X_{t-\varepsilon^2}(dz)\right)^4\right) + C\varepsilon^{8-4d}\varepsilon^{c^2-4}$$

using equation (1.15) . Fixing c so that $c^2 - 4 \ge 4d - 8$ this expression is uniformly bounded in ε. The other regions of $A \times A \times A \times A$ give smaller contributions as in the derivation of (2.21).

Proof of (2.18). Fix $\delta > 0$ and $C \in \mathcal{F}_{t-\delta}$. Using Theorem 1.8a

$$\begin{aligned}
\lim_{\varepsilon\to 0} E(\mathrm{I}_C K_t^\varepsilon | \mathcal{F}_{t-\delta}) &= \lim_{\varepsilon\to 0} \varepsilon^{2-d}\mathrm{I}_C \int_A Q^{X_{t-\delta}}(X_\delta(B(x,\varepsilon)) > 0)dx \\
&= c_3 \mathrm{I}_C \int_A \int_{\mathrm{R}^d} p_\delta(x,y) X_{t-\delta}(dy)dx
\end{aligned}$$

By the uniform integrability of the K_t^ε

$$\begin{aligned}
\lim_{\varepsilon\to 0} E(\mathrm{I}_C K_t^\varepsilon) &= E(c_3 \mathrm{I}_C \int_{\mathrm{R}^d} p_\delta(y,A) X_{t-\delta}(dy)) \\
&= c_3 E(\mathrm{I}_C X_t(A)).
\end{aligned}$$

Since $\sup_{\varepsilon>0} E((K_t^\varepsilon)^4) < \infty$ we may extend to all $Y \in L^2(\bigvee_{\delta>0} \mathcal{F}_{t-\delta})$. □

2.3 The connected components of the support

The arguments that lead to the upper bound on the Hausdorff measure of the support use covers of the support that have a Cantor set like structure. Don Dawson asked the following question:

For fixed $t > 0$, is $S(X_t)$ a totally disconnected set ?

We now prove the following partial answer.

Theorem 2.7 *Let Comp(x) denote the connected component of $S(X_t)$ containing x. If $d \ge 3$ then for all $m \in M_F(\mathrm{R}^d)$ and $t > 0$, with probability one*

$$\mathrm{Comp(x)} = \{x\} \text{ for } X_t - a.a.x.$$

Notation. For $\underline{t}, \underline{a}, \underline{\theta} \in T^\mu$, $\beta \sim \underline{t}$ let

$$Z^\beta(\underline{a}) = \mu^{-1} \sum_{\gamma\sim \underline{t}+\underline{a}^2, \gamma \succ \beta} \mathrm{I}(N^\gamma_{\underline{t}+\underline{a}^2} \neq \Delta)$$

$$W^\beta(\underline{a},\underline{\theta}) = \mu^{-1} \sum_{\gamma\sim\underline{t}+\underline{a}^2,\gamma\succ\beta} \mathrm{I}(|N^\gamma_{\underline{t}+\underline{a}^2} - N^\beta_{\underline{t}}| > \underline{a}\,\underline{\theta})$$

$Z^\beta(\underline{a})$ is the mass of the 'cluster' of particles descended from $N^\beta_{\underline{t}}$ that are alive at time $\underline{t}+\underline{a}^2$. The following lemma shows that there is a good chance (independent of $\underline{a}$) that these particles have not spread more than a distance $O(\underline{a})$ from their common root. For $m \in M_F(\mathrm{R}^d)$ choose $m_\mu \in {}^*M^\mu_F(\mathrm{R}^d)$ such that $st_{M_F}(m_\mu) = m$.

Lemma 2.8 *For nearstandard $\underline{a},\underline{\theta} \in {}^*[0,\infty)$ such that $a = {}^\circ\underline{a} > 0, \theta = {}^\circ\underline{\theta} > 0$ we have*

$$P^{m_\mu}(W^\beta(\underline{a},\underline{\theta}) = 0|Z^\beta(\underline{a}) > 0) \;=\; p(\theta) \tag{2.22}$$

$$E^{m_\mu}\left(Z^\beta(\underline{a})\mathrm{I}(W^\beta(\underline{a},\underline{\theta}) = 0)|Z^\beta(\underline{a}) > 0\right) \;=\; r(\theta)a^2 \tag{2.23}$$

where if $\theta \in \mathrm{R}, \theta > 0$ then $p(\theta) > 0, r(\theta) > 0$ and

$$Q^{(1/2)\delta_0}(X_1(B(0,\theta)^c) = 0) \;=\; \exp(p(\theta) - 1) \tag{2.24}$$

$$E^{(1/2)\delta_0}(X_1(B(0,\theta))\mathrm{I}(X_1(B(0,\theta)^c) = 0)) \;=\; r(\theta)\exp(p(\theta) - 1) \tag{2.25}$$

PROOF. This is essentially due to scaling (Proposition 1.7.) Fix nearstandard $\underline{a},\underline{\theta}$ such that $a,\theta > 0$. For β such that $\beta|_0 \neq \Delta$ define

$$\begin{aligned} p(\underline{a},\underline{\theta}) &= {}^*P^{m_\mu}(W^\beta(\underline{a},\underline{\theta}) = 0|Z^\beta(\underline{a}) > 0) \\ r(\underline{a},\underline{\theta}) &= {}^*E^{m_\mu}(Z^\beta(\underline{a})\mathrm{I}(W^\beta(\underline{a},\underline{\theta}) = 0)|Z^\beta(\underline{a}) > 0) \end{aligned}$$

The values of $p(\underline{a},\underline{\theta}), r(\underline{a},\underline{\theta})$ do not depend on the choice of m_μ or β.Take $x_i = 0$ for $i = 1,\ldots,[\mu\underline{a}^2/2]$ and $x_i = \Delta$ otherwise , so that $st_{m_F}(m_\mu) = (a^2/2)\delta_0$. Then

$$\begin{aligned} & Q^{(1/2)\delta_0}(X_1(B(0,\theta)^c) = 0) \\ = \;& Q^{(a^2/2)\delta_0}(X_{a^2}(B(0,a\theta)^c = 0) \\ = \;& P^{m_\mu}(N_{\underline{a}^2}(\mathrm{st}^{-1}(B(0,a\theta)^c)) = 0) \\ = \;& P^{m_\mu}(N_{\underline{a}^2}(B(0,\underline{a}\,\underline{\theta})^c) = 0) \end{aligned}$$

using scaling and the fact that $X_1(\partial B(0,r)) = 0,$ almost surely for any r. So

$$\begin{aligned} & Q^{(1/2)\delta_0}(X_1(B(0,\theta)^c) = 0) \\ = \;& P^{m_\mu}\Big(\bigcap_{i=1}^{[\mu\underline{a}^2/2]} (W^{x_i}(\underline{a},\underline{\theta}) = 0)\Big) \end{aligned}$$

$$
\begin{aligned}
&= {}^\circ \prod_{i=1}^{[\mu \underline{a}^2/2]} {}^\bullet P^{m_\mu}(W^{x_i}(\underline{a},\underline{\theta}) = 0) \\
&= {}^\circ \prod_{i=1}^{[\mu \underline{a}^2/2]} ({}^\bullet P^{m_\mu}(W^{x_i}(\underline{a},\underline{\theta}) = 0 | Z^{x_i}(\underline{a}) > 0) {}^\bullet P^{m_\mu}(Z^{x_i}(\underline{a}) > 0) + {}^\bullet P^{m_\mu}(Z^{x_i}(\underline{a}) = 0)) \\
&= {}^\circ \left[\left(1 + (p(\underline{a},\underline{\theta}) - 1) {}^\bullet P^{m_\mu}(Z^{x^1}(\underline{a}) > 0)\right)^{[\mu \underline{a}^2/2]} \right] \\
&= \exp({}^\circ p(\underline{a},\underline{\theta}) - 1)
\end{aligned}
$$

since $[\mu \underline{a}^2/2] {}^\bullet P^{m_\mu}(Z^{x_1}(\underline{a}) > 0) \sim 1$ from Lemma 2.2a. So ${}^\circ p(\underline{a},\underline{\theta})$ is constant in $\underline{a}$ and (2.22) and (2.24) follow taking $p(\theta) = {}^\circ p(\underline{a},\underline{\theta})$.

Similarly

$$
\begin{aligned}
& E^{\delta_0/2}(X_1(B(0,\theta)) \mathrm{I}(X_1(B(0,\theta)^c) = 0)) \\
&= E^{a^2\delta_0/2}(a^{-2} X_{a^2}(B(0,a\theta)) \mathrm{I}(X_{a^2}(B(0,a\theta)^c) = 0)) \\
&= (a^{-2}) \circ \left({}^\bullet E\Big(\sum_{i=1}^{[\mu \underline{a}^2/2]} Z^{x_i}(\underline{a}) \mathrm{I}\big(\bigcap_{j=1}^{[\mu \underline{a}^2/2]} W^{x_j}(\underline{a},\underline{\theta}) = 0\big)\Big) \right) \\
&= (a^{-2}) \circ \left(\sum_{i=1}^{[\mu \underline{a}^2/2]} {}^\bullet E(Z^{x_i}(\underline{a}) | W^{x_i}(\underline{a},\underline{\theta}) = 0) {}^\bullet P\big(\bigcap_{j=1}^{[\mu \underline{a}^2/2]} W^{x_j}(\underline{a},\underline{\theta}) = 0\big) \right) \\
&= (a^{-2} \exp(p(\theta)) - 1))^\circ \Big[\sum_{i=1}^{[\mu \underline{a}^2/2]} {}^\bullet E(Z^{x_i}(\underline{a}) \mathrm{I}(W^{x_i}(\underline{a},\underline{\theta}) = 0) | Z^{x_i}(\underline{a}) > 0) \\
&\qquad \times {}^\bullet P(Z^{x_i}(\underline{a}) > 0) / {}^\bullet P(W^{x_i}(\underline{a},\underline{\theta}) = 0) \Big] \\
&= (a^{-2} \exp(p(\theta)) - 1))^\circ \left([\mu \underline{a}^2/2] {}^\bullet P(Z^{x_1}(\underline{a}) > 0) r(\underline{a},\underline{\theta}) / (1 - (1 - p(\underline{a},\underline{\theta})) {}^\bullet P(Z^{x_1}(\underline{a}) > 0)) \right) \\
&= (a^{-2} \exp(p(\theta)) - 1))^\circ r(\underline{a},\underline{\theta})
\end{aligned}
$$

So $(a^{-2})\,{}^\circ r(\underline{a},\underline{\theta})$ is independent of $\underline{a}$ and (2.23) , (2.25) follow taking $r(\theta) = (a^{-2})\,{}^\circ r(\underline{a},\underline{\theta})$.

Finally $Q^{\delta_0/2}(X_1(\mathrm{R}^d) = 0) = \exp(-1)$ so that $p(\theta), r(\theta) > 0$ will follow if we can show

$$
Q^{\delta_0/2}(X_1(B(0,\theta)^c) = 0, X_1(\mathrm{R}^d) \neq 0) > 0 \tag{2.26}
$$

But since the support of the process moves with finite speed , for small enough s we have

$$
Q^{\delta_0/2}(X_s(B(0,\theta)^c) = 0, X_s(\mathrm{R}^d) \neq 0) > 0
$$

and Theorem 1.9 implies (2.26) holds.□

Notation.

For $a \in (0,\infty)$ define

$$Q_a = \left\{(y,r,\delta) \in \mathrm{Q}^d \times \mathrm{Q} \times \mathrm{Q} \;:\; r,\delta > 0\,,\ |y| < r-\delta\,,\ |y|+r+\delta < a\right\}$$

For $a \in [0,\infty)$, $m \in M_F(\mathrm{R}^d)$ define $\mathrm{Ann}(m,a) \subseteq \mathrm{R}^d$ by

$$\mathrm{Ann}(m,a) = \bigcup_{(y,r,\delta)\in Q_a} \left\{x \in \mathrm{R}^d : m(z:\ r-\delta < |z-x-y| < \ r+\delta)\right\}$$

For $\underline{a} \in {}^*[0,\infty)$, $m \in {}^*M_F(\mathrm{R}^d)$ define $\mathrm{Ann}(m,\underline{a}) \subseteq {}^*\mathrm{R}^d$ by

$$\mathrm{Ann}(m,\underline{a}) = \bigcup_{(y,r,\delta)\in Q_a} \left\{\underline{x} \in {}^*\mathrm{R}^d : m(\underline{z}:\ r-\delta < |\underline{z}-\underline{x}-y| < \ r+\delta)\right\}$$

$\mathrm{Ann}(m,\underline{a})$ is defined so that for $\underline{x} \in \mathrm{Ann}(m,\underline{a})$ there is a mass free annulus of positive standard rational thickness that disconnects $\underline{x}$ from $B(\underline{x},\underline{a})^c$.

Note that if $x \in \mathrm{Ann}(X_t,a)$ then $\mathrm{Comp}(x) \subseteq B(x,a)$. Thus

$$x \in \bigcap_{n=1}^{\infty} \mathrm{Ann}(X_t,n^{-1}) \implies \mathrm{Comp}(x) \subseteq \{x\}$$

For $\underline{t},\underline{a},\underline{\theta} \in T^\mu$, $\beta \sim \underline{t}$, $N^\beta_{\underline{t}} \neq \Delta$ define

$$V^\beta(\underline{a},\underline{\theta}) = \mu^{-1} \sum_{\gamma\sim\underline{t}+\underline{a}^2,\gamma\not\sim\beta} \mathrm{I}(|N^\gamma_{\underline{t}+\underline{a}^2} - N^\beta_{\underline{t}}| < 2\underline{a}\,\underline{\theta})$$

Note that if $\underline{a},\underline{\theta},\underline{t} \in T^\mu$, $a > 0$, $0 < \theta < 1/2$, $\beta \sim \underline{t}$ are such that $N^\beta_{\underline{t}-\underline{a}^2} \neq \Delta$, $V^{\beta|\underline{t}-\underline{a}^2}(\underline{a},\underline{\theta}) = W^{\beta|\underline{t}-\underline{a}^2}(\underline{a},\underline{\theta}) = 0$ then there is a particle free annulus surrounding $N^\beta_{\underline{t}}$

$$N^\mu_{\underline{t}}(\underline{z} : 5\underline{a}\,\underline{\theta}/4 < |\underline{z} - N^\beta_{\underline{t}-\underline{a}^2}| < 7\underline{a}\,\underline{\theta}/4) = 0$$

$$|N^\beta_{\underline{t}} - N^\beta_{\underline{t}-\underline{a}^2}| \leq a\theta$$

We may shift the annulus slightly to be centered at a rational and have positive rational thickness so that $N^\beta_{\underline{t}} \in \mathrm{Ann}(N^\mu_{\underline{t}},\underline{a})$.

The following lemma shows that on average a positive fraction of the initial particles will lie inside $\mathrm{Ann}(N^\mu_{\underline{t}},\underline{a})$.

Lemma 2.9 *For nearstandard $\underline{a},\underline{t} \in T^\mu$ with $0 < a < 1$, $2a^{1/3} < t < \infty$, $d \geq 3$ if $\mathrm{st}_M(m_\mu) = m \in M_F(\mathrm{R}^d)$ then there exists a constant $\rho > 0$ depending only on d such that*

$$E^{m_\mu}\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(N^\gamma_{\underline{t}} \in \mathrm{Ann}(N^\mu_{\underline{t}},\underline{a}), N^\gamma_{\underline{t}} \neq \Delta)\right) \geq \rho m(\mathrm{R}^d)$$

PROOF. The remark following the definition of $V^\gamma(\underline{a},\underline{\theta})$ shows that

$$
\begin{aligned}
& E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(N^\gamma_{\underline{t}} \in \mathrm{Ann}(N^\mu_{\underline{t}},\underline{a}), N^\gamma_{\underline{t}} \neq \Delta)\right) \\
\geq\ & E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(N^\gamma_{\underline{t}} \neq \Delta, W^{\gamma|\underline{t}-\underline{a}^2}(\underline{a},\underline{\theta}) = 0, V^{\gamma|\underline{t}-\underline{a}^2}(\underline{a},\underline{\theta}) = 0)\right) \\
=\ & E\left(\sum_{\gamma\sim\underline{t}-\underline{a}^2} Z^\gamma(\underline{a})\mathrm{I}(V^\gamma(\underline{a},\underline{\theta}) = W^\gamma(\underline{a},\underline{\theta}) = 0)\right) \\
=\ & E\left(\sum_{\gamma\sim\underline{t}-\underline{a}^2} {}^*E(Z^\gamma(\underline{a})\mathrm{I}(W^\gamma(\underline{a},\underline{\theta}) = 0)|\mathcal{A}_{\underline{t}-\underline{a}^2})\,{}^*P(V^\gamma(\underline{a},\underline{\theta}) = 0|\mathcal{A}_{\underline{t}-\underline{a}^2})\right)
\end{aligned}
\tag{2.27}
$$

since conditional on $\mathcal{A}_{\underline{t}-\underline{a}^2}$ the random variables $V^\gamma(\underline{a},\underline{\theta})$ and $Z^\gamma(\underline{a})\mathrm{I}(W^\gamma(\underline{a},\underline{\theta}) = 0)$ are *-independent. Now using the *-Markov property (see Perkins [16] Proposition 2.3.)

$$
\begin{aligned}
& {}^{\circ *}P\left(V^\gamma(\underline{a},\underline{\theta}) = 0)|\mathcal{A}_{\underline{t}-\underline{a}^2}\right) \\
=\ & {}^{\circ *}P^{N^\mu_{\underline{t}-\underline{a}^2}-\mu^{-1}\delta_{N^\gamma_{\underline{t}-\underline{a}^2}}}\left(N_{\underline{a}^2}(B(\cdot,2\underline{a}\underline{\theta})) = 0\right)\Big|_{N^\gamma_{\underline{t}-\underline{a}^2}} \\
=\ & Q^{X_{t-a^2}}\left(X_{a^2}(B(\cdot,2a\theta)) = 0\right)\Big|_{{}^\circ N^\gamma_{\underline{t}-\underline{a}^2}}
\end{aligned}
$$

where in the second step we used the continuity in θ of $Q^m(X_s(B(x,2a\theta)) = 0)$ and the fact that $\mathrm{st}_M(N^\mu_{\underline{t}-\underline{a}^2} - \mu^{-1}\delta_{N^\gamma_{\underline{t}-\underline{a}^2}}) = X_{t-a^2}$. Also

$$
\begin{aligned}
& {}^{\circ *}E\left(\mu Z^\gamma(\underline{a})\mathrm{I}(W^\gamma(\underline{a},\underline{\theta}) = 0)|\mathcal{A}_{\underline{t}-\underline{a}^2}\right) \\
=\ & {}^{\circ *}E\left(\mu Z^\gamma(\underline{a})\mathrm{I}(W^\gamma(\underline{a},\underline{\theta}) = 0)|N^\gamma_{\underline{t}-\underline{a}^2} \neq \Delta\right)\mathrm{I}(N^\gamma_{\underline{t}-\underline{a}^2} \neq \Delta) \\
=\ & {}^\circ\left[{}^*E\left(Z^\gamma(\underline{a})\mathrm{I}(W^\gamma(\underline{a},\underline{\theta}) = 0|Z^\gamma(\underline{a}) > 0\right)\mu\,{}^*P\left(Z^\gamma(\underline{a}) > 0|N^\gamma_{\underline{t}-\underline{a}^2} \neq \Delta\right)\right]\mathrm{I}(N^\gamma_{\underline{t}-\underline{a}^2} \neq \Delta) \\
=\ & 2r(\theta)\mathrm{I}(N^\gamma_{\underline{t}-\underline{a}^2} \neq \Delta)
\end{aligned}
$$

using Lemma 2.8 and Lemma 2.2. Substituting into equation 2.27 we get

$$
\begin{aligned}
& E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(N^\gamma_{\underline{t}} \in \mathrm{Ann}(N^\mu_{\underline{t}},\underline{a}), N^\gamma_{\underline{t}} \neq \Delta)\right) \\
\geq\ & 2r(\theta)E\left[\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(N^\gamma_{\underline{t}-\underline{a}^2} \neq \Delta)Q^{X_{t-a^2}}\left(X_{a^2}(B(\cdot,2a\theta)) = 0\right)\Big|_{{}^\circ N^\gamma_{\underline{t}-\underline{a}^2}}\right] \\
=\ & 2r(\theta)E\left[\int_{\mathbb{R}^d} Q^{X_{t-a^2}}(X_{a^2}(B(x,2a\theta)) = 0)dX_{t-a^2}(x)\right]
\end{aligned}
$$

$$\geq \; 2r(\theta)E\left[\int_{\mathbf{R}^d}\left(1-c_4(2a\theta)^{d-2}(2\pi a^2(1+4\theta^2))^{-d/2}\right.\right.$$

$$\left.\left.\cdot\int_{\mathbf{R}^d}\exp(-(x-y)^2(2a^2(1+4\theta^2))^{-1})dX_{t-a^2}(y)\right)dX_{t-a^2}(x)\right]$$

using the estimate on hitting balls in Theorem 1.8b. Lemma 2.10 gives an estimate on the expectation of this double integral and leads directly to

$$E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}}\mathrm{I}(N_{\underline{t}}^{\gamma}\in\mathrm{Ann}(N_{\underline{t}}^{\mu},\underline{a}),N_{\underline{t}}^{\gamma}\neq\Delta)\right)$$

$$\geq \; 2r(\theta)m(\mathbf{R}^d)\left(1-C\theta^{d-2}(1+4\theta^2)^{1-d/2}(2^{d/2}+m(\mathbf{R}^d))\right)$$

Now take $\theta>0$ small enough so that the right hand side is strictly positive. □

Lemma 2.10 *If* $0<a\leq 1, t\geq a^{1/3}, d\geq 3, m\in M_F(\mathbf{R}^d)$ *then*

$$E^m\left[\int\int\exp(-(x-y)^2/2a)dX_t\,dX_t\right]\leq am(\mathbf{R}^d)(2^{d/2}+m(\mathbf{R}^d))$$

PROOF. Let $p_t(x)$ be the Brownian transition density with associated semigroup P_t. We have for positive measurable f,g (see Dynkin [6] Theorem 1.1)

$$E^m\left[X_t(f)X_t(g)\right]=\int P_tf(x)dm(x)\int P_tg(x)dm(x)+\int dm(x)\int_0^t P_{t-s}(P_sfP_sg)(x)ds$$

By approximating positive measurable $h(x,y)$ by functions of the form $\sum_{i=1}^n f_i(x)g_i(x)$ we have

$$\begin{aligned}E^m[\int\int h(x,y)dX_t(x)\,dX_t(y)] &= \int dm(z_1)\int dm(z_2)\int dx\int dy\,h(x,y)p_t(x-z_1)p_t(y-z_2)\\ &\quad+\int dm(x)\int_0^t ds\int dy\,p_s(x-y)\int dz_1\int dz_2\,p_s(y-z_1)p_s(y-z_2)h(z_1,z_2)\\ &\leq (m(\mathbf{R}^d))^2\sup_{z_1,z_2}E\left[h(B_t^1,B_t^2)\right]+m(\mathbf{R}^d)\int_0^t\sup_{z_1,z_2}E\left[h(B_s^1,B_s^2)\right] \qquad (2.28)\end{aligned}$$

where B_t^1,B_t^2 are independent Brownian motions satrting at z_1,z_2.For $h(x,y)=\exp(-(x-y)^2/2a)$ we have

$$\sup_{z_1,z_2}E\left[h(B_s^1,B_s^2)\right]=\int\exp(-x^2/2a)p_{2s}(x)dx\leq(2a/a+2s)^{d/2}.$$

Substituting into equation (2.28) and using the bounds on a and t gives the result. □

Proof of Theorem 2.7 We prove the result first for the nonstandard model with $x_i=0$ for $i=1,\ldots\mu$, $x_i=\Delta$ for $i>\mu$ so that if $m_\mu=\mu^{-1}\sum_i\delta_{x_i}\mathrm{I}(x_i\neq\Delta)$ then $P^{m_\mu}(X\in\cdot)=Q^{\delta_0}(X\in\cdot)$. From

Lemma 2.9 we have for nearstandard $\underline{a}, \underline{t} \in T^{\mu}$ such that $0 < a < 1, 2a^{1/3} < t < \infty$

$$\begin{aligned} 0 < \rho &\le E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(N^{\gamma}_{\underline{t}} \in \mathrm{Ann}(N^{\mu}_{\underline{t}}, \underline{a}), N^{\gamma}_{\underline{t}} \ne \Delta)\right) \\ &= E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} {}^{*}P(N^{\gamma}_{\underline{t}} \in \mathrm{Ann}(N^{\mu}_{\underline{t}}, \underline{a}) | N^{\gamma}_{\underline{t}} \ne \Delta)\mathrm{I}(N^{\gamma}_{\underline{t}} \ne \Delta)\right) \\ &= P\left(N^{\gamma}_{\underline{t}} \in \mathrm{Ann}(N^{\mu}_{\underline{t}}, \underline{a}) | N^{\gamma}_{\underline{t}} \ne \Delta\right) \end{aligned}$$

where $P(\cdot|N^{\gamma}_{\underline{t}} \ne \Delta)$ is the Loeb measure induced by ${}^{*}P(\cdot|N^{\gamma}_{\underline{t}} \ne \Delta)$. Since $\mathrm{Ann}(N^{\mu}_{\underline{t}}, \underline{a})$ decreases as $\underline{a}$ decreases we have for any $\gamma \sim \underline{t}$

$$P(N^{\gamma}_{\underline{t}} \in \bigcap_{n=1}^{\infty} \mathrm{Ann}(N^{\mu}_{\underline{t}}, n^{-1}) | N^{\gamma}_{\underline{t}} \ne \Delta) \ge \rho > 0 \tag{2.29}$$

We now use a zero-one law to show this probability is in fact 1.

<u>Notation.</u> Fix $\gamma \sim \underline{t}$. For $\underline{u}, \underline{v} \in T^{\mu}, \underline{u} \le \underline{v}$ define

$$\begin{aligned} \overline{\mathcal{H}}_{\underline{u},\underline{v}} &= {}^{*}\sigma(N^{\beta}_{\underline{t}} - N^{\beta}_{\underline{t}-\mu^{-1}\sigma(\beta,\gamma)} : \underline{u} < \mu^{-1}\sigma(\beta,\gamma) \le \underline{v}) \bigvee {}^{*}\sigma(N^{\gamma}_{\underline{t}} - N^{\gamma}_{\underline{t}-\underline{s}} : \underline{u} < \underline{s} \le \underline{v}) \\ \mathcal{H}_{\underline{u},\underline{v}} &= \sigma(\overline{\mathcal{H}_{\underline{u},\underline{v}}}) \\ \mathcal{H}_{\underline{v}} &= \bigvee_{n} \mathcal{H}_{n^{-1},\underline{v}} \\ \mathcal{H}_{0+} &= \bigcap_{n} \mathcal{H}_{n^{-1}} \end{aligned}$$

The following two results are due to Ed Perkins (personal communication.)

Proposition 2.11 *For $A \in \mathcal{H}_{0+}$,* $P(A|N^{\gamma}_{\underline{t}} \ne \Delta) = 0$ or 1.

Proposition 2.12 *If $0 < \alpha < 2^{-4/d}$ then $P(\cdot|N^{\gamma}_{\underline{t}} \ne \Delta)$ almost surely , for r small enough*

$$d\left(\{N^{\beta}_{\underline{t}} : \mu^{-1}\sigma(\beta,\gamma) \ge (2r)^{\alpha}\}, N^{\gamma}_{\underline{t}}\right) > r$$

For $(y, r, \delta) \in Q_a$, $\underline{u}, \underline{v} \in T^{\mu}$, $\underline{u} \le \underline{v}$ define

$$\begin{aligned} \Gamma_{y,r,\delta,\underline{u},\underline{v}}(\gamma) &= \left\{\omega : |N^{\beta}_{\underline{t}} - N^{\gamma}_{\underline{t}} - y| \notin (r-\delta, r+\delta) \text{ for all } \beta \text{ s.t. } \underline{u} < \mu^{-1}\sigma(\beta,\gamma) \le \underline{v}\right\} \\ \Gamma(\gamma) &= \bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{(y,r,\delta)\in Q_{n^{-1}}}\bigcap_{j=k+1}^{\infty} \Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma) \in \mathcal{H}_{0+} \end{aligned}$$

If $\omega \in \bigcap_{n=1}^{\infty}\{N^{\gamma}_{\underline{t}} \in \mathrm{Ann}(N_{\underline{t}}, n^{-1})$ then $\omega \in \Gamma(\gamma)$ and so equation 2.29 and the zero-one law imply $P(\Gamma(\gamma)|N^{\gamma}_{\underline{t}} \ne \Delta) = 1$. Let

$$\Lambda(\gamma) = \left\{\omega : \text{for small } r\ ,\ d(\{N^{\beta}_{\underline{t}} : \mu^{-1}\sigma(\beta,\gamma) \ge (2r)^{\alpha}\}, N^{\gamma}_{\underline{t}}) > r\right\}$$

so that Proposition 2.12 says $P(\Lambda(\gamma)|N_{\underline{t}}^{\gamma} \neq \Delta) = 1$. Then

$$\begin{aligned}
& E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(\omega \notin \Gamma(\gamma)\cap\Lambda(\gamma), N_{\underline{t}}^{\gamma} \neq \Delta)\right) \\
= \; & E\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} {}^{*}P(\omega \notin \Gamma(\gamma)\cap\Lambda(\gamma)|N_{\underline{t}}^{\gamma} \neq \Delta)\mathrm{I}(N_{\underline{t}}^{\gamma} \neq \Delta)\right) \\
= \; & P\left(\omega \notin \Gamma(\gamma)\cap\Lambda(\gamma)|N_{\underline{t}}^{\gamma} \neq \Delta\right) \\
= \; & 0
\end{aligned}$$

From the global modulus of continuity for particles, with probability one all the particles move only an infinitessimal distance in an infinitessimal time. So equation (1.11) and the above imply we can pick a single P null set N such that if $\omega \notin N$ we have simultaneously

$$^{\circ}\left(\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}(\omega \notin \Gamma(\gamma)\cap\Lambda(\gamma), N_{\underline{t}}^{\gamma} \neq \Delta)\right) = 0 \tag{2.30}$$

For all nearstandard $\underline{s} < \underline{t}$ and $\beta \sim \underline{t}$, $N_{\underline{t}}^{\beta} \neq \Delta$

we have $N_{\underline{s}}^{\beta} \approx N_{\underline{t}}^{\beta}$ (2.31)

$$\mathrm{st}(S(N_{\underline{t}})) = S(X_t) \tag{2.32}$$

Now fix $\omega \notin N$, $\gamma \sim \underline{t}$ such that $N_{\underline{t}}^{\gamma} \neq \Delta$, $\omega \in \Gamma(\gamma)\bigcap\Lambda(\gamma)$. We claim

$$^{\circ}N_{\underline{t}}^{\gamma} \in \bigcap_{n=1}^{\infty} \mathrm{Ann}(X_t, n^{-1})$$

To show this find k so that

$$\omega \in \bigcap_{n=1}^{\infty} \bigcup_{(y,r,\delta)\in Q_{n-1}} \bigcap_{j=k+1}^{\infty} \Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma)$$

Find r_0 such that $(2r_0)^{1/2} \leq k^{-1}$ and

$$d\left(\{N_{\underline{t}}^{\beta} : \mu^{-1}\sigma(\beta,\gamma) \geq (2r_0)^{1/2}\}, N_{\underline{t}}^{\gamma}\right) > r_0 \tag{2.33}$$

Pick n so that $n^{-1} \leq r_0$ and find $(y,r,\delta) \in Q_{n^{-1}}$ so that

$$\omega \in \bigcap_{j=k+1}^{\infty} \Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma) \tag{2.34}$$

For $\beta \sim \underline{t}$ such that $\mu^{-1}\sigma(\beta,\gamma) \geq (2r_0)^{1/2}$ equation (2.33) ensures that $|N_{\underline{t}}^{\beta} - N_{\underline{t}}^{\gamma}| > n^{-1}$ and so $|N_{\underline{t}}^{\beta} - N_{\underline{t}}^{\gamma} - y| \notin (r-\delta, r+\delta)$.

For $\beta \sim \underline{t}$ such that $0 < {}^\circ(\mu^{-1}\sigma(\beta,\gamma)) \leq k^{-1}$ equation (2.34) and the definition of $\Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma)$ ensure $|N_{\underline{t}}^{\beta} - N_{\underline{t}}^{\gamma} - y| \notin (r-\delta, r+\delta)$.

For $\beta \sim \underline{t}$ such that $\mu^{-1}\sigma(\beta,\gamma) \approx 0$ equation (2.31) ensures $N_{\underline{t}}^{\beta} \approx N_{\underline{t}}^{\gamma}$. So

$$\left\{z \in {}^*\mathrm{R}^d : r-\delta < |z - N_{\underline{t}}^{\gamma} - y| < r+\delta\right\} \cap \left\{N_{\underline{t}}^{\beta} : \beta \sim \underline{t}\right\} = \emptyset$$

Equation (2.32) now gives

$$\left\{z \in \mathrm{R}^d : r-\delta < |z - {}^\circ N_{\underline{t}}^{\gamma} - y| < r+\delta\right\} \cap S(X_t) = \emptyset$$

Since we were free to pick n arbitrarily large this proves the claim. Now equation (2.30) and the claim give

$$^\circ\left[\mu^{-1}\sum_{\gamma\sim\underline{t}} \mathrm{I}({}^\circ N_{\underline{t}}^{\gamma} \notin \bigcap_{n=1}^{\infty} \mathrm{Ann}(X_t, n^{-1}), N_{\underline{t}}^{\gamma} \neq \Delta)\right] = 0$$

so that for $P^{\delta_0} - a.a.\omega$

$$X_t(\mathrm{R}^d \setminus \bigcap_{n=1}^{\infty} \mathrm{Ann}(X_t, n^{-1})) = 0 \tag{2.35}$$

It is possible to show that the map $m \longrightarrow m(\mathrm{R}^d \setminus \bigcap_{n=1}^{\infty} \mathrm{Ann}(m, n^{-1}))$ is Borel measurable. So we can apply Theorem 1.9 and conclude that for any $m \in M_F(\mathrm{R}^d)$ equation (2.35) holds for $Q^m - a.a.\omega$.□

2.4 The range of the process over random time sets

In Perkins [17] it is shown that with probability one the support process $(S(X_t) : t \geq 0)$ has right continuous paths with left limits in the Hausdorff topology on compact sets. If we write $S(X_t)_-$ for $\lim_{s\uparrow t} S(X_s)$ then $S(X_t)_- \supseteq S(X_t)$ and $\bigcup_{t>0} S(X_t)_- \setminus S(X_t)$ is shown to be almost surely a countable set of points. We deduce that if A is a Borel subset of $(0,\infty)$ then $\bigcup_{t\in A} S(X_t)$ is almost surely a Borel subset of R^d since for any $t_0 > 0$

$$\overline{\bigcup_{t\in A\cap[t_0,\infty)} S(X_t)} \setminus \bigcup_{t\in A\cap[t_0,\infty)} S(X_t) \subseteq \bigcup_{t>0} S(X_t)_- \setminus S(X_t).$$

Note also that Hausdorff dimensions are unaffected by the addition of countably many points .

In Dawson,Iscoe and Perkins [5] , if $d \geq 4$, suitable Hausdorff measure functions are found for the set $\overline{\cup_{a<t<b} S(X_t)}$ from which it follows that the Hausdorff dimension is 4. We now find the dimension

of $\bigcup_{t \in A} S(X_t)$ for possibly random time sets A. This is an analogue of a Theorem of Kauffman for Brownian motion. In Kauffman [12] it is shown that if B_t is a two dimensional Brownian motion then with probability one

$$\dim(B_t : t \in A) = 2\dim(A) \text{ for all Borel } A \subseteq [0, \infty)$$

and the result is true for Brownian motion in dimensions $d \geq 2$.

Notation. For $R, K \in [0, \infty)$ define

$$\tau_k = \inf\{t \geq 0 : X_t(\mathrm{R}^d) = k\}$$

$$\sigma_k = \inf\{t \geq 0 : X_t(B(0, k)^c) > 0\}$$

Theorem 2.13 *For $d \geq 4$ and any initial measure $m \in M_F(\mathrm{R}^d)$, with probability one*

$$\dim\left(\bigcup_{t \in A} S(X_t)\right) = 2 + 2\dim(A) \text{ for all nonempty Borel } A \subseteq (0, \tau_0) \tag{2.36}$$

PROOF. Lower bound. Let $T_k = k \wedge \tau_k \wedge \tau_{k^{-1}} \wedge \sigma_k$. We will show that if $m \in M_F(\mathrm{R}^d)$ is of compact support, $k \in \mathrm{N}$ satisfies $m(\mathrm{R}^d) \in (k^{-1}, k)$ and $\varepsilon > 0$ then for $Q^m - a.a.\omega$

$$\dim\left(\bigcup_{t \in A} S(X_t)\right) \geq 2 + 2\dim(A) - 2\varepsilon \text{ for all nonempty Borel } A \subseteq (k^{-1}, T_k) \tag{2.37}$$

If $m \in M_F(\mathrm{R}^d)$ is of compact support then almost surely the total mass remains bounded and the support remains bounded so that $T_k \uparrow \tau_0 < \infty$ as $k \to \infty$ and $\dim(A \cap (k^{-1}, T_k)) \uparrow \dim(A \cap (0, \tau_0))$. So we may take sequences $k_n \uparrow \infty, \varepsilon_n \downarrow 0$ to conclude that if $m \in M_F(\mathrm{R}^d)$ is of compact support then the lower bound on the dimension of $\cup_{t \in A} S(X_t)$ in equation (2.36) holds Q^m almost surely. For initial measures $m \in M_F(\mathrm{R}^d)$ we can argue by decomposing m into countably many finite measures with compact support and use the branching property.

Fix $m \in M_F(\mathrm{R}^d)$ of compact support, $k \in \{2^n : n = 1, 2, \ldots\}$ satisfying $m(\mathrm{R}^d) \in (k^{-1}, k)$ and $\varepsilon \in (0, 1)$.Define a time grid as follows. Let $t_j^n = j2^{-n}$, $T^n = \{t_j^n : j = 1, 2 \ldots\}$, $I_j^n = [t_j^n, t_{j+1}^n)$. Along any fixed sample path we say I_j^n charges $B(x, a)$ if there exists $s \in I_j^n$ such that $X_s(B(x, a)) > 0$. For large n we do not expect many balls $B(x, 2^{-n})$ to be charged repeatedly by many of the I_j^{2n}'s. The following Lemma shows that there is not much mass in such balls .

Lemma 2.14 *For $Q^m - a.a.\omega$ and sufficiently large n*

$$X_t\left\{x : \sum_{I_j^{2n} \subseteq [k^{-1}, k]} \mathrm{I}(I_j^{2n} charges B(x, 2^{-n})) \geq 2^{n\varepsilon})\right\} \leq 2^{-(d+5)n} \tag{2.38}$$

$$\text{for all } t \in T^{3n+6} \cap [0,k]$$

For any fixed ball of radius a , if it is charged by the measure X_t we do not expect it to be charged much more than a^2 . In Perkins [16] (proof of Theorem 4.5) the following very precise result is shown. There exist constants $c_a, c_b > 0$ such that for $Q^m - a.a.\omega$ and sufficiently large n

$$X_t\left\{x : X_t(B(x, 2^{-n})) \geq c_a \log(n) 2^{-2n}\right\} \leq c_b/n^2 \text{ for all } t \in [k^{-1}, k] \tag{2.39}$$

Define a space grid as follows . Let $G_k^n = d^{-1/2} 2^{-n} \mathbb{Z}^d \cap B(0, k+1)$. Note that any point is in at most 3^d of the balls $\{B(x_i, 2^{-n}) : x_i \in G_k^n)$.

Lemma 2.15 *For $Q^m - a.a.\omega$ and sufficiently large n*

$$\sup_{t \in I_j^{3n}} X_t(B(x, 2^{-n})) \leq 2^{(d+4)n} X_{t_j^{3n}}(B(x, 2^{-(n-1)})) \tag{2.40}$$
$$\text{for all } t \in T^{3n} \cap [0,k], x \in G_k^n.$$

We delay the proofs of Lemmas 2.14 , 2.15 . The strategy of the proof is as follows . Given a cover $\mathcal{C}$ of $\bigcup_{t \in A} S(X_t)$ we will construct a cover $\mathcal{C}_A$ of $A \times [0, 1/2k]$. Since $A \times [0, 1/2k]$ has dimension at least $1 + \dim(A)$ (see Falconer [9] Corollary 5.10) this will lead to a lower bound on how efficiently we can cover $\bigcup_{t \in A} S(X_t)$.

Fix ω and $n_0 < \infty$ so that equations (2.38), (2.39) ,2.40 hold for $n \geq n_0$ and so that $3^d 2^{-n_0} 2^{2d+10} \leq 1/4k$ and $\sum_{n \geq n_0} c_b/n^2 \leq 1/4k$. Fix Borel $A \subseteq (k^{-1}, T_k)$ of dimension α. Since $A \subseteq [0, \sigma_k)$ we have $\bigcup_{t \in A} S(X_t) \subseteq \overline{B(0,k)}$. Choose a cover of $\bigcup_{t \in A} S(X_t)$

$$\mathcal{C} = \left\{B_i = B(x_i, 2^{-n_i}) : i = 1, 2 \ldots\right\} \text{ with } x_i \in G_k^{n_i}, n_i \geq n_0 + 2. \tag{2.41}$$

We assume all the balls in $\mathcal{C}$ are distinct . Split the cover into two parts $\mathcal{C}_1 = \{B_i : i \in I_1\}, \mathcal{C}_2 = \{B_i : i \in I_2\}$ where

$$S_{(n_i-2)} = \left\{ x : \sum_{I_j^{2(n_i-1)} \subseteq [k^{-1}, k]} I(I_j^{2(n_i-2)} \text{ charges } B(x, 2^{-(n_i-2)})) \geq 2^{(n_i-2)\epsilon} \right\}$$

$$i \in I_1 \quad \text{if} \quad B(x_i, 2^{-(n_i-1)}) \subseteq S_{(n_i-2)}$$
$$i \in I_2 \quad \text{if} \quad B(x_i, 2^{-(n_i-1)}) \cap S^c_{(n_i-2)} \neq \emptyset$$

For each $i \in I_2$ find z_i such that $B(x_i, 2^{-(n_i-1)}) \subseteq B(z_i, 2^{-(n_i-2)})$ and less than $2^{(n_i-2)\epsilon}$ of the intervals $I_j^{2(n_i-2)} \subseteq [k^{-1}, k]$ charge $B(z_i, 2^{-(n_i-2)})$.

Now we form the cover $\mathcal{C}_A$ of $A \times [0, 1/2k]$. Consider the balls in $\mathcal{C}_2$ one by one in order of decreasing radius. For a ball $B(x_i, 2^{-n_i}) \in \mathcal{C}_2$ there are less than $4.2^{(n_i-2)\varepsilon}$ of the intervals $I_j^{2n_i} \subseteq [k^{-1}, k]$ that charge $B(x_i, 2^{-n_i})$. For each $I_j^{2n_i}$ that does charge $B(x_i, 2^{-n_i})$ choose a rectangle in $[k^{-1}, k] \times \mathrm{R}$ based above $I_j^{2n_i}$ and of height $c_a \log(n_i - 1)2^{-2(n_i-1)}$ so that the base of the rectangle lies on top of any previous rectangle above $I_j^{2n_i}$ (or the x-axis if there are none). Repeating this procedure for each ball $B_i \in \mathcal{C}_2$ gives a collection of rectangles which we call $\mathcal{C}_A$. Note that each $B(x_i, 2^{-n_i})$ gave rise to at most $4.2^{(n_i-2)\varepsilon}$ rectangles of diameter less than $(1 + c_a)\log(n_i - 1)2^{-2(n_i-1)}$.

We now check that $\mathcal{C}_A$ covers $A \times [0, 1/2k]$. Fix $t \in A$ and let $\hat{t}^n = \sup\{t_j^n : t_j^n \leq t\}$. Let $J_n = \{i \in I_1 : n_i = n\}$ so that $I_1 = \bigcup_{n \geq n_0+2} J_n$.

$$
\begin{aligned}
& \sum_{j \in J_n} X_t(B(x_j, 2^{-n})) \\
\leq\ & 2^{(d+4)n} \sum_{j \in J_n} X_{\hat{t}^{3n}}(B(x_j, 2^{-(n-1)})) \text{ (by equation (2.40))} \\
\leq\ & 2^{(d+4)n} 3^d X_{\hat{t}^{3n}}\Big(\bigcup_{j \in J_n} B(x_j, 2^{-(n-1)})\Big) \\
\leq\ & 2^{(d+4)n} 3^d X_{\hat{t}^{3n}} \left\{ x : \sum_{I_j^{2(n-2)} \subseteq [k^{-1}, k]} \mathrm{I}(I_j^{2(n-2)} \text{ charges } B(x, 2^{-(n-2)})) \geq 2^{(n-2)\varepsilon} \right\} \\
\leq\ & 2^{(d+4)n} 3^d 2^{-(d+5)(n-2)} \text{ (by equation 2.38)}
\end{aligned}
$$

So

$$
\begin{aligned}
\sum_{i \in I_1} X_t(B_i) &= \sum_{n \geq n_0+2} \sum_{j \in J_n} X_t(B(x_j, 2^{-n})) \\
&\leq 3^d 2^{2d+10} 2^{-n_0} \leq 1/4k.
\end{aligned}
$$

Since $t \leq \tau_{k^{-1}}$ we know that $X_t(\mathrm{R}^d) \geq 1/k$. Let $C_t = \bigcup_{(n \geq n_0+2)} \{x : X_t(B(x, 2^{-n})) \geq c_a \log(n) 2^{-2n}\}$. Equation (2.39) now gives

$$
\begin{aligned}
3/4k &\leq X_t(S(X_t) \setminus C_t) \\
&\leq \sum_{i \in I_1 \cup I_2} X_t(B_i)\mathrm{I}(B_i \not\subseteq C_t, X_t(B_i) > 0) \\
&\leq 1/4k + \sum_{i \in I_2} X_t(B_i)\mathrm{I}(B_i \not\subseteq C_t, X_t(B_i) > 0)
\end{aligned}
$$

If $B(x_i, 2^{-n_i}) \not\subseteq C_t$ choose $y_i \in B(x_i, 2^{-n_i}) \cap C_t^c$. Then

$$\begin{aligned} 1/2k &\leq \sum_{i \in I_2} X_t(B(y_i, 2^{-(n_i-1)}))\mathrm{I}(X_t(B_i) > 0, B_i \not\subseteq C_t) \\ &\leq \sum_{i \in I_2} c_b(n_i - 1)2^{2(n_i-1)}\mathrm{I}([\hat{t}^{2n_i}, \hat{t}^{2n_i} + 2^{-2n_i}) \text{ charges } B_i) \end{aligned} \tag{2.42}$$

The right hand side of equation (2.42) is the total height of the rectangles in $\mathcal{C}_A$ that lie above t. So indeed $\mathcal{C}_A$ covers $A \times [0, 1/2k]$.

Now suppose that $\dim(\bigcup_{t \in A} S(X_t)) < 2\gamma \leq 4$. Then we can find a sequence of covers $\{\mathcal{C}^m\}_m$ of the form 2.41 satisfying $\mathrm{diam}(\mathcal{C}^m) \to 0$ and

$$\sum_{B_i \in \mathcal{C}^m} |B_i|^{2\gamma} = \sum_i 2^{-2\gamma n_i} \leq 1 \text{ for all } m$$

Then $\mathrm{diam}(\mathcal{C}_A^m) \to 0$ and

$$\begin{aligned} \sum_{R \in \mathcal{C}_A^m} |R|^{\gamma+\varepsilon} &\leq \sum_i 4.2^{(n_i-2)\varepsilon}((1+c_a)(n_i - 1)2^{-2(n_i-1)})^{\gamma+\varepsilon} \\ &\leq \text{Constant for all } m. \end{aligned}$$

So

$$1 + \alpha \leq \dim(A \times [0, 1/2k]) \leq \gamma + \varepsilon$$

Therefore

$$2\gamma \geq 2 + 2\alpha - 2\varepsilon$$

and

$$\dim(\bigcup_{t \in A} S(X_t)) \geq 2 + 2\alpha - 2\varepsilon$$

and the lower bound is finished.

Upper bound. The upper bound is a straightforward application of the global modulus of continuity for the motion of the particles in the nonstandard model for super Brownian motion.

It will be enough to show that if $m \in M_F(\mathrm{R}^d)$ and $m(\mathrm{R}^d) \leq k$ then for $Q^m - a.a.\omega$.

$$\dim(\bigcup_{t \in A} S(X_t)) \leq 2 + 2\dim(A) \text{ for all Borel } A \subseteq (0, \tau_k \wedge k).$$

Fix $m_\mu \in {}^*M_F^\mu({}^*\mathrm{R}^d)$ so that $\mathrm{st}_{M_F}(m_\mu) = m$ and fix $k \in \mathrm{N}$ so that $m(1) \leq k$. Define for $\underline{t}, \underline{\varepsilon} \in \{j/\mu : j \in {}^*N\}$

$$I(\underline{t}, \underline{\varepsilon}) = \{\gamma \sim \underline{t} : \exists \beta \sim \underline{t} + \underline{\varepsilon}, \beta \succ \gamma, N^\beta_{\underline{t}+\underline{\varepsilon}} \neq \Delta\}$$

$$Z(\underline{t},\underline{\varepsilon}) = |I(\underline{t},\underline{\varepsilon})|$$

Conditional on $\mathcal{A}_{\underline{t}}$, $Z(\underline{t},\underline{\varepsilon})$ has a *-Binomial distribution under ${}^*P$ with parameters $\mu N_{\underline{t}}^{\mu}(1)$ and $p_{\underline{\varepsilon}} = {}^*P^{\mu^{-1}\delta_0}(N_{\underline{\varepsilon}}^{\mu}(1) > 0)$. From Lemma 2.2 ${}^{\circ}(\mu p_{\underline{\varepsilon}}) = 2/\underline{\varepsilon}$ whenever $\underline{\varepsilon}$ is nearstandard . Let Poisson(n) be a random variable having the Poisson distribution with mean n under P_0.

$$\begin{aligned}
& P\left(\exists t_j^n \in [0, k \wedge \tau_k] \text{ such that } Z(t_j^n, 2^{-n}) \geq k2^{(n+2)}\right) \\
\leq & \sum_{t_j^n \in [0,k]} P\left(Z(t_j^n, 2^{-n}) \geq k2^{(n+2)}, \tau_k \geq t_j^n\right) \\
\leq & \; k2^n P_0(\text{Poisson}(k2^{n+1}) \geq k2^{(n+2)}) \\
\leq & \; k2^n \exp(-k2^{n+1}(2 - e^{1/2}))
\end{aligned}$$

which sums over n. Thus for ω off a single null set we may find $n_0(\omega) < \infty$ so that $Z(t_j^n, 2^{-n}) \leq k2^{(n+2)}$ for all $t_j^n \in [0, k \wedge \tau_k]$ and $n \geq n_0$, so that the global modulus of continuity for particles (Theorem 1.6a) holds with $2^{-n_0} \leq \delta(\omega, 3)$ and so that equation (1.12) holds.

Fix ω so that $n_0(\omega) < \infty$. Fix $A \subseteq (0, \tau_k \wedge k)$ of dimension α. Given $\varepsilon > 0$ we can find a cover $\mathcal{C} = \{I_{j_i}^{n_i} : i = 1, 2 \ldots\}$ of A so that $\text{diam}(\mathcal{C}) \leq \varepsilon$ and

$$\sum_i 2^{n_i(\alpha+\varepsilon)} \leq 1$$

Using the modulus of continuity we have

$$\begin{aligned}
\bigcup_{t \in A} S(X_t) & \subseteq \bigcup_i \bigcup_{t \in I_{j_i}^{n_i}} S(X_t) \\
& \subseteq \bigcup_i \bigcup_{\gamma \in I(t_{j_i-1}^{n_i}, 2^{-n_i})} B(N_{t_{j_i-1}^{n_i}}^{\gamma}, 3h(2^{-(n_i-1)}))
\end{aligned} \tag{2.43}$$

The right hand side of (2.43) forms a cover $\mathcal{C}_S$ of $\cup_{t \in A} S(X_t)$ of diameter less than $6h(2\varepsilon)$ satisfying

$$\begin{aligned}
\sum_{B \in \mathcal{C}_S} |B|^{2\alpha+2+3\varepsilon} & \leq \sum_i k2^{n_i+2} |5\sqrt{n_i} 2^{-n_i/2}|^{2\alpha+2+3\varepsilon} \\
& \leq \text{Constant}
\end{aligned}$$

Choosing a sequence of covers of decreasing diameter gives

$$\dim(\bigcup_{t \in A} S(X_t)) \leq 2 + 2\dim(A) \text{ for } P - a.a.\omega.$$

It is possible to show that the set

$$\Big\{\omega : \text{for sufficiently large n, if } I_j^n \subseteq [0, k \wedge \tau_k] \text{ then } \cup_{t \in I_j^n} S(\omega_t) \text{ can be covered}$$

$$\text{by } k2^{n+1} \text{ balls with rational centers and radius } 3h(2^{-(n-1)})\Big\}$$

is Borel in $C([0,\infty), M_F(\mathrm{R}^d))$. So by transfering to path space at the correct point in the proof we can show the lower bound holds Q^m almost surely.□

Notation.

Define

$$GMC_n = \Big\{\omega : |N_{\underline{t}}^\beta - N_{\underline{t}}^\beta| \le 3h(\underline{t} - \underline{s}) \text{ for all nearstandard } \underline{s}, \underline{t}, \beta \sim \underline{t}$$

$$\text{satisfying } 0 < t - s \le 2^{-n}, N_{\underline{t}}^\beta \ne \Delta\Big\}$$

Then from (1.15) $P(GMC_n^c) \to 0$ geometrically fast.

Proof of lemma 2.15. It is enough to prove the lemma for the nonstandard model.

$$P\left(\exists t \in T^{3n} \cap [0,k], x \in G_k^n \ s.t. \sup_{s\in[t,t+2^{-3n}]} X_s(B(x,2^{-n})) > 2^{(d+4)n} X_t(B(x, 2^{-(n-1)}))\right)$$

$$\le \sum_{t\in T^{3n}\cap[0,k]} \sum_{x\in G_k^n} P\left(GMC_{3n} \cap \{ \sup_{s\in[t,t+2^{-3n}]} X_s(B(x,2^{-n})) > 2^{(d+4)n} X_t(B(x,2^{-(n-1)}))\}\right)$$

$$+ P(GMC_{3n}^c) \tag{2.44}$$

Now

$$P\left(GMC_{3n} \cap (\sup_{s\in[t,t+2^{-3n}]} X_s(B(x,2^{-n})) > 2^{(d+4)n} X_t(B(x,2^{-(n-1)})))\right)$$

$$\le P\left(\sup_{\underline{s}\in[0,2^{-3n}]} \mu^{-1} \sum_{\gamma\sim t+\underline{s}} \mathrm{I}(N_{t+\underline{s}}^\gamma \ne \Delta, N_t^\gamma \in B(x, 2^{-(n-1)})) \right.$$

$$\left. > 2^{(d+4)n} \mu^{-1} \sum_{\gamma\sim t} \mathrm{I}(N_t^\gamma \in B(x,2^{-(n-1)}))\right)$$

Let $\nu = \mu^{-1}\sum_{\gamma\sim t} \mathrm{I}(N_t^\gamma \in B(x,2^{-(n-1)}))$. Then using the *-Markov property at time t the process $\{\mu^{-1}\sum_{\gamma\sim t+\underline{s}} \mathrm{I}(N_{t+\underline{s}}^\gamma \ne \Delta, N_t^\gamma \in B(x,2^{-(n-1)})) : \underline{s} \ge 0\}$ has the same law as $\{N_{\underline{s}}^\mu(1) : \underline{s} \ge 0\}$ under P^ν. But

$$\begin{aligned} & P^\nu(\sup_{\underline{s}\in[0,2^{-3n}]} N_{\underline{s}}^\mu(1) > 2^{(d+4)n} N_0^\mu(1)) \\ \le\ & P^\nu(\sup_{s\ge 0} X_s(\mathrm{R}^d) \ge 2^{(d+4)n} X_0(\mathrm{R}^d)) \\ =\ & 2^{-(d+4)n} \end{aligned}$$

since $X_s(\mathrm{R}^d)$ is a continuous martingale. Substituting into (2.44) and noting the summation is over less than $(k+1)^{d+1}2^{(d+3)n}$ terms , Borel Cantelli gives the result. □

Lemma 2.16 *For $s \in T^{3n+6} \cap [0,k], p \in \mathrm{N}$ there exists a constant $C = C(p,k,m,d)$ such that for all $n \ge (5+\log_2(p))/\varepsilon$*

$$E^m\left[X_s\Big(x : \sum_{t_j^{2n}\in[k^{-1},k]} \mathrm{I}(X_{t_j^{2n}}(B(x,cn2^{-n})) \ge 2^{-2n}) \ge 2^{n\varepsilon}\Big)\right] \le Cn^{p(d+1)}2^{-n\varepsilon p}.$$

PROOF. Fix n and write B_x for $B(x,cn2^{-n})$. C will be a constant depending on p,k,m,d but independent of n whose value may change from line to line.

$$\begin{aligned}
& E\left[X_s\Big(x : \sum_{t_j^{2n}\in[k^{-1},k]} \mathrm{I}(X_{t_j^{2n}}(B_x) \ge 2^{-2n}) \ge 2^{n\varepsilon}\Big)\right] \\
\le\ & E\left[X_s\Big(x : \sum_{t_j^{2n}\in[s+2^{-2n+1},k]} \mathrm{I}(X_{t_j^{2n}}(B_x) \ge 2^{-2n}) \ge 2^{n\varepsilon-2}\Big)\right] \\
& +E\left[X_s\Big(x : \sum_{t_j^{2n}\in[k^{-1},s-2^{-2n+1}]} \mathrm{I}(X_{t_j^{2n}}(B_x) \ge 2^{-2n}) \ge 2^{n\varepsilon-2}\Big)\right] \qquad (2.45)
\end{aligned}$$

We will bound the first term on the right hand side of (2.45). It is similar and slightly easier to show the second term has the same bound.

$$\begin{aligned}
& E\left[X_s\Big(x : \sum_{t_j^{2n}\in[s+2^{-2n+1},k]} \mathrm{I}(X_{t_j^{2n}}(B_x) \ge 2^{-2n}) \ge 2^{n\varepsilon-2}\Big)\right] \\
\le\ & E\left[X_s\left(x : \sum_{\substack{t_{j_1}^{2n},\dots,t_{j_p}^{2n}\in[s+2^{-2n+1},k] \\ \min_{k,l}|t_{j_l}^{2n}-t_{j_k}^{2n}|\ge 2^{-2n+1}}} \prod_{i=1}^{p} \mathrm{I}(X_{t_{j_i}^{2n}}(B_x) \ge 2^{-2n}) \ge 2^{(n\varepsilon-3)p}\right)\right] \\
\le\ & C2^{-n\varepsilon p} \sum_{\substack{t_{j_1}^{2n}\le\dots t_{j_p}^{2n}\in[s+2^{-2n+1},k] \\ \min_k|t_{j_{k+1}}^{2n}-t_{j_k}^{2n}|\ge 2^{-2n+1}}} E\left[\int \prod_{i=1}^{p} \mathrm{I}(X_{t_{j_i}^{2n}}(B_x) \ge 2^{-2n})dX_s(x)\right] \\
\le\ & C2^{-n\varepsilon p}\,2^{2np} \sum_{\substack{t_{j_1}^{2n}\le\dots t_{j_p}^{2n}\in[s+2^{-2n+1},k] \\ \min_k|t_{j_{k+1}}^{2n}-t_{j_k}^{2n}|\ge 2^{-2n+1}}} E\left[\int X_{t_{j_1}^{2n}}(B_x)\dots X_{t_{j_p}^{2n}}(B_x)dX_s(x)\right] \qquad (2.46)
\end{aligned}$$

The following lemma gives an upper bound for such expectations.

Lemma 2.17 *If $k^{-1} \le s < t_1 < t_2 < \ldots < t_p \le k$ then*

$$E\left[\int X_{t_1}(B(x,a))\dots X_{t_p}(B(x,a))dX_s(x)\right] \le Ca^{pd}[(t_p-t_{p-1})\dots(t_2-t_1)(t_1-s)]^{1-d/2}$$

where C depends on k,d,m,p but not a or the t_i's.

Proof of Lemma 2.17 . The required expectation is the standard part of

$$
\begin{aligned}
& {}^*E\left[\mu^{-(p+1)}\sum_{\beta\sim s}\sum_{\alpha_1\sim t_1}\cdots\sum_{\alpha_p\sim t_p}\prod_{i=1}^{p}\mathrm{I}(|N_s^\beta - N_{t_i}^{\alpha_i}|\le a)\right] \\
= {} & \mu^{-p}\sum_{\beta\sim s}\sum_{\alpha_1\sim t_1}\cdots\sum_{\alpha_{p-1}\sim t_{p-1}} {}^*E\left[\prod_{i=1}^{p-1}\mathrm{I}(|N_s^\beta - N_{t_i}^{\alpha_i}|\le a)\right. \\
& \left.\times\,\mu^{-1}\sum_{i=\mu(t_p-t_{p-1})}^{\mu t_p}\ \sum_{\substack{\alpha_p\sim t_p \\ \sigma(\alpha_p:\beta,\alpha_1,\ldots,\alpha_{p-1})=i}} {}^*E[\mathrm{I}(|N_{t_p}^{\alpha_p} - N_s^\beta|\le a)|\mathcal{A}_{(\beta,\alpha_1,\ldots,\alpha_{p-1})}]\right]
\end{aligned}
\tag{2.47}
$$

where $\sigma(\alpha:\gamma_1,\ldots,\gamma_n) = |\alpha| - \inf(j : \alpha|_j \neq \gamma_i|_j\ \forall i = 1,\ldots,n)$ is the number of generations back that α branched off from any of the branches $\gamma_1,\ldots,\gamma_n$.

Recall $Y(t)$ is a *-Brownian motion under ${}^*P_0$. If $i \in \{\mu(t_p - t_{p-1}),\ldots,\mu t_p\}$ then

$$
\begin{aligned}
& \sum_{\substack{\alpha_p\sim t_p \\ \sigma(\alpha_p:\beta,\alpha_1,\ldots,\alpha_{p-1})=i}} {}^*E[\mathrm{I}(|N_{t_p}^{\alpha_p} - N_s^\beta|\le a)|\mathcal{A}_{(\beta,\alpha_1,\ldots,\alpha_{p-1})}] \\
\le {} & \begin{cases} p\,{}^*P_0(|Y(i\mu^{-1})|\le a) & \text{if } i\neq \mu t_p \\ (\mu m^\mu({}^*\mathrm{R}^d)-1)\,{}^*P_0(|Y(s+t_p)|\le a) & \text{if } i = \mu t_p \end{cases}
\end{aligned}
$$

So

$$
\begin{aligned}
& \mu^{-1}\sum_{i=\mu(t_p-t_{p-1})}^{\mu t_p}\ \sum_{\substack{\alpha_p\sim t_p \\ \sigma(\alpha_p:\beta,\alpha_1,\ldots,\alpha_{p-1})=i}} {}^*E\left[\mathrm{I}(|N_{t_p}^{\alpha_p} - N_s^\beta|\le a)|\mathcal{A}_{(\beta,\alpha_1,\ldots,\alpha_{p-1})}\right] \\
\le {} & C\left(m(\mathrm{R}^d)a^d(s+t_p)^{-d/2} + a^d\mu^{-1}\sum_{i=\mu(t_p-t_{p-1})}^{\mu t_p - 1}(i\mu^{-1})^{-d/2}\right) \\
\le {} & Ca^d(t_p - t_{p-1})^{1-d/2}.
\end{aligned}
$$

Substituting into (2.47) and using induction over p gives the result .□

Completion of proof of Lemma 2.16. Using Lemma 2.17 and the bound in equation (2.46) we have

$$
\begin{aligned}
& E\left[X_s(x : \sum_{t_j^{2n}\in[s+2^{-2n+1},k]}\mathrm{I}(X_{t_j^{2n}}(B_x)\ge 2^{-2n})\ge 2^{n\varepsilon-2})\right] \\
\le {} & Cn^{pd}2^{(2-d-\varepsilon)np}\sum_{\substack{t_{j_1}^{2n}\le\ldots t_{j_p}^{2n}\in[s+2^{-2n+1},k] \\ \min_k |t_{j_{k+1}}^{2n} - t_{j_k}^{2n}|\ge 2^{-2n+1}}}[(t_{j_p}^{2n} - t_{j_{p-1}}^{2n})\ldots(t_{j_2}^{2n} - t_{j_1}^{2n})(t_{j_1}^{2n} - s)]^{1-d/2} \\
\le {} & Cn^{pd}2^{(4-d-\varepsilon)np}\int_{s+2^{-2n}}^{k}dt_1\int_{t_1+2^{-2n}}^{k}dt_2\ldots\int_{t_{p-1}+2^{-2n}}^{k}dt_p\,[(t_p - t_{p-1})\ldots(t_2-t_1)(t_1-s)]^{1-d/2} \\
\le {} & Cn^{(p+1)d}2^{-n\varepsilon p}.
\end{aligned}
$$

$\square$

Notation. Define for $j, n \in \mathrm{N}, \underline{x} \in {}^*\mathrm{R}$ the events

$$\begin{aligned}
A_j^n(\underline{x}) &= \{\exists \gamma \sim (j-1)2^{-2n} \text{ s.t. } N^\gamma_{(j-1)2^{-2n}} \in B(\underline{x}, 9h(2^{-2n})), N_{j2^{-2n}}(\gamma) > 0\} \\
B_j^n(\underline{x}) &= \{\exists \gamma \sim (j-1)2^{-2n} \text{ s.t. } N^\gamma_{(j-1)2^{-2n}} \in B(\underline{x}, 9h(2^{-2n})), N_{j2^{-2n}}(\gamma) > 2^{-2n}\} \\
\tilde{A}_j^n(x) &= \{\exists s \in I_j^{2n}, \gamma \sim s \text{ such that } {}^\circ N_s^\gamma \in B(x, 12h(2^{-2n}))\} \\
\tilde{B}_j^n(x) &= \{X_{j2^{-2n}}(B(x, 12h(2^{-2n}))) \geq 2^{-2n}\}
\end{aligned}$$

Note that up to a null set

$$\tilde{A}_j^n(x) \cap GMC_{2n} \subseteq A_j^n(\underline{x}) \tag{2.48}$$

$$B_j^n(\underline{x}) \cap GMC_{2n} \subseteq \tilde{B}_j^n(x) \tag{2.49}$$

Lemma 2.16 gives a bound on

$$E\left[\int \mathrm{I}\Big(\sum_{i=k-1 2^{2n}}^{k2^{2n}} \mathrm{I}\tilde{B}_i^n(x) \geq 2^{n\epsilon}\Big) dX_s\right] \tag{2.50}$$

whereas to prove Lemma 2.14 we wish to bound

$$E\left[\int \mathrm{I}\Big(\sum_{i=k-1 2^{2n}}^{k2^{2n}} \mathrm{I}\tilde{A}_i^n(x) \geq 2^{n\epsilon}\Big) dX_s\right]. \tag{2.51}$$

By restricting to GMC_{2n} equations (2.48), (2.49) will allow us to replace $\tilde{A}_j^n(x), \tilde{B}_j^n(x)$ by $A_j^n(\underline{x}), B_j^n(\underline{x})$. We will show that each time $A_j^n(\underline{x})$ occurs there is a good chance that $B_j^n(\underline{x})$ occurs and use the following Lemma to convert our bound on (2.50) into a bound on (2.51).

Lemma 2.18 *On a probability space $(\Omega, (\mathcal{F}_j)_{j\in \mathrm{Z}_+}, P)$ let $A_n, B_n, n = 1, \ldots, N$ be events satisfying for some $q \in [0,1]$*

i. $A_j, B_j \in \mathcal{F}_j$.

ii. $B_j \subseteq A_j$.

iii. $P(B_j|\mathcal{F}_{j-1}) \geq qP(A_j|\mathcal{F}_{j-1})$.

Then

$$P\left(\sum_{i=1}^N \mathrm{I}_{A_i} \geq n, \sum_{i=1}^N \mathrm{I}_{B_i} \leq a|\mathcal{F}_0\right) \leq P_0(B(n,q) \leq a) \quad P-a.s.$$

where $B(n,q)$ has a Binomial distribution under P_0 with parameters n, q.

PROOF. Define $\tau_0 = 0, \tau_j = \inf(m \geq \tau_{j-1} : \omega \in A_m)$ for $j = 1, \ldots, N$. Let $X_j = \mathrm{I}_{A_j}, Y_j = \mathrm{I}_{B_j}$.We claim

$$E[Y_{\tau_j}\mathrm{I}(\tau_j < \infty)|\mathcal{F}_{\tau_{j-1}}] \geq qE[\mathrm{I}(\tau_j < \infty)|\mathcal{F}_{\tau_{j-1}}] \text{ for } j = 1, \ldots, N. \tag{2.52}$$

To prove this pick $C \in \mathcal{F}_{\tau_{j-1}}$ .Note that

$$\{\tau_j = n\} = \bigcup_{\substack{x\in\{0,1\}^{n-1} \\ \sum x_i = j-1}} A_n \cap \{(X_1, \ldots, X_{n-1}) = x)\}$$

and this union is disjoint. If $x \in \{0,1\}^{n-1}$ satisfies $\sum x_i = j-1$ then $\{(X_1, \ldots, X_{n-1}) = x\} \subseteq \{\tau_{j-1} = k\}$ for some $k = 1, \ldots, n-1$, so that

$$C \cap \{(X_1, \ldots, X_{n-1}) = x\} = C \cap \{(X_1, \ldots, X_{n-1}) = x\} \cap \{\tau_{j-i} = k\} \in \mathcal{F}_{n-1}.$$

Then

$$\begin{aligned}
\int_C Y_{\tau_j}\mathrm{I}(\tau_j < \infty)dP &= \sum_{n=1}^{N} \int_{C\cap\{\tau_j = n\}} Y_n dP \\
&= \sum_{n=1}^{N} \sum_{\substack{x\in\{0,1\}^{n-1} \\ \sum x_i = j-1}} \int_{C\cap A_n \cap \{(X_1,\ldots,X_{n-1})=x\}} Y_n dP \\
&= \sum_{n=1}^{N} \sum_{\substack{x\in\{0,1\}^{n-1} \\ \sum x_i = j-1}} \int_{C\cap\{(X_1,\ldots,X_{n-1})=x\}} Y_n dP \\
&\geq q\sum_{n=1}^{N} \sum_{\substack{x\in\{0,1\}^{n-1} \\ \sum x_i = j-1}} \int_{C\cap\{(X_1,\ldots,X_{n-1})=x\}} X_n dP \\
&= q\int_C X_{\tau_j}\mathrm{I}(\tau_j < \infty)dP \\
&= q\int_C \mathrm{I}(\tau_j < \infty)dP
\end{aligned}$$

which proves the claim.

We now check by induction on n that for $n = 1, \ldots, N$, $a = 0, \ldots, n$

$$P\left(\sum_{j=1}^{n} Y_{\tau_j} \leq a, \tau_n < \infty|\mathcal{F}_0\right) \leq P_0(B(n,q) \leq a) \tag{2.53}$$

The case $n = 1$ is immediate from (2.52). Assume equation (2.53) for $n = 1, \ldots, k$.

$$P(\sum_{j=1}^{k+1} Y_{\tau_j} \leq a, \tau_{k+1} < \infty|\mathcal{F}_{\tau_k})$$

$$
\begin{aligned}
&= \mathrm{I}(\sum_{j=1}^{k} Y_{\tau_j} = a) P(Y_{\tau_{k+1}} = 0, \tau_{k+1} < \infty | \mathcal{F}_{\tau_k}) + \mathrm{I}(\sum_{j+1}^{k} Y_{\tau_j} \le a-1) P(\tau_{k+1} < \infty | \mathcal{F}_{\tau_k}) \\
&\le \left((1-q)\mathrm{I}(\sum_{j=1}^{k} Y_{\tau_j} = a) + \mathrm{I}(\sum_{j=1}^{k} Y_{\tau_j} \le a-1) \right) P(\tau_{k+1} < \infty | \mathcal{F}_{\tau_k}) \\
&\le \left((1-q)\mathrm{I}(\sum_{j=1}^{k} Y_{\tau_j} \le a) + q\mathrm{I}(\sum_{j=1}^{k} Y_{\tau_j} \le a-1) \right) \mathrm{I}(\tau_k < \infty)
\end{aligned}
$$

since $\{\tau_k < \infty\} \subseteq \{\tau_{k+1} < \infty\}$. So taking conditional expectations given $\mathcal{F}_0$ and using the induction hypothesis

$$
\begin{aligned}
P(\sum_{j=1}^{k+1} Y_{\tau_j} \le a, \tau_{k+1} < \infty | \mathcal{F}_0) &\le (1-q) P_0(B(k,q) \le a) + q P_0(B(k,q) \le a-1) \\
&= P_0(B(k+1,q) \le a)
\end{aligned}
$$

completing the induction. Finally

$$
\begin{aligned}
P(\sum_{j=1}^{N} \mathrm{I}_{A_j} \ge n, \sum_{j=1}^{N} \mathrm{I}_{B_j} \le a | \mathcal{F}_0) &= P(\tau_n < \infty, \sum_{j=1}^{N} Y_j \le a | \mathcal{F}_0) \\
&\le P(\tau_n < \infty, \sum_{j=1}^{n} Y_{\tau_j} \le a | \mathcal{F}_0) \\
&\le P_0(B(n,q) \le a).
\end{aligned}
$$

□

<u>Proof of Lemma 2.14</u>. Fix $n \in \mathrm{N}$ and nearstandard $\underline{t} \in \{j/\mu : j \in {}^*N\}, \beta \sim \underline{t}$ such that $N_{\underline{t}}^{\beta} \ne \Delta$. Let $\hat{j}^n(\underline{t}) = \sup\{j : j2^{-n} \le \underline{t}\}$. We will apply Lemma 2.18 to the events $A_j^n(N_{\underline{t}}^{\beta}), B_j^n(N_{\underline{t}}^{\beta}),\ j = k^{-1}2^{2n}, \ldots, k2^{2n},\ j \ne \hat{j}^{2n}(\underline{t}) + 1$ and the internal filtration

$$
\mathcal{G}_j^n = \mathcal{A}_{j2^{-2n}} \vee {}^*\sigma(N_{\underline{s}}^{\beta} : \underline{s} \le \underline{t}) \ \ j = k^{-1}2^{2n}, \ldots, k2^{2n},\ j \ne \hat{j}^{2n}(\underline{t}) + 1
$$

under the internal probability ${}^*P$. Conditions i. and ii. of the Lemma is immediate and we claim that condition iii. holds , namely that there exists $\underline{q} \in {}^*\mathrm{R}^d$ satisfying $q = \exp^{-2}$ such that for $j \ne \hat{j^{2n}}(\underline{t}) + 1$

$$
{}^*P(B_j^n(N_{\underline{t}}^{\beta}) | \mathcal{G}_{j-1}^n) \ge \underline{q}\, {}^*P(A_j^n(N_{\underline{t}}^{\beta}) | \mathcal{G}_{j-1}^n) \tag{2.54}
$$

Before proving (2.54) we complete the proof of Lemma 2.14. For $t \in T^{3n+6} \cap [0,k]$

$$
E\left[X_t\{x : \sum_{j=k^{-1}2^{2n}}^{k2^{2n}} \mathrm{I}(\tilde{A}_j^n(x)) \ge 2^{n\epsilon}\} \mathrm{I}(GMC_{2n}) \right]
$$

$$\leq \quad E\left[X_t\{x : \sum_{j=k^{-1}2^{2n}}^{k2^{2n}} \mathrm{I}(\tilde{B}_j^n(x)) \geq 2^{n\varepsilon/2}\}\right] \tag{2.55}$$
$$+E\left[X_t\{x : \sum_{j=k^{-1}2^{2n}}^{k2^{2n}} \mathrm{I}(\tilde{A}_j^n(x)) \geq 2^{n\varepsilon}, \sum_{j=k^{-1}2^{2n}}^{k2^{2n}} \mathrm{I}(\tilde{B}_j^n(x)) \leq 2^{n\varepsilon/2}\}\mathrm{I}(GMC_{2n})\right]$$

Lemma 2.16 gives the upper bound $Cn^{p(d+1)}2^{-n\varepsilon p/2}$ for the first term in (2.55). The value of the second term in (2.55) is less than the standard part of

$$\mu^{-1}\sum_{\beta\sim t} {}^*P\left[\sum_{\substack{j=k^{-1}2^{2n} \\ j\neq j^{2n}(\underline{t})+1}}^{k2^{2n}} \mathrm{I}(\tilde{A}_j^n(N_t^\beta)) \geq 2^{n\varepsilon-1}, \sum_{\substack{j=k^{-1}2^{2n} \\ j\neq j^{2n}(\underline{t})+1}}^{k2^{2n}} \mathrm{I}(\tilde{B}_j^n(N_t^\beta)) \leq 2^{n\varepsilon/2}, GMC_{2n}, N_t^\beta \neq \Delta\right]$$

But by Lemma 2.18

$$\begin{aligned}
&{}^*P\left(\sum_{\substack{j=k^{-1}2^{2n} \\ j\neq j^{2n}(\underline{t})+1}}^{k2^{2n}} \mathrm{I}(A_j^n(N_t^\beta)) \geq 2^{n\varepsilon-1}, \sum_{\substack{j=k^{-1}2^{2n} \\ j\neq j^{2n}(\underline{t})+1}}^{k2^{2n}} \mathrm{I}(B_j^n(N_t^\beta)) \leq 2^{n\varepsilon/2}, GMC_{2n} \;\middle|\; \mathcal{G}_{k^{-1}2^{2n}}^n\right) \\
\leq\;& {}^*P_0(Y(2^{n\varepsilon-1},q) \leq 2^{n\varepsilon/2})
\end{aligned}$$

So (using (2.48)(2.49)) the second term of (2.55) is bounded by

$$\begin{aligned}
&\mu^{-1}\sum_{\beta\sim t} {}^*P(N_t^\beta \neq \Delta){}^*P_0(Y(2^{n\varepsilon-1},q) \leq 2^{n\varepsilon/2}) \\
\leq\;& m(\mathrm{R}^d)(1-e^{-2})^{2^{n\varepsilon/3}} \quad \text{for large } n
\end{aligned}$$

So

$$P\left[X_t\{x : \sum_{I_j^{2n}\subseteq[k^{-1},k]} \mathrm{I}(I_j^{2n} \text{ charges } B(x,2^{-n})) \geq 2^{n\varepsilon}\} > 2^{-(d+5)n} \text{ for some } t \in T^{3n+6}\cap[0,k]\}\right]$$

$$\begin{aligned}
\leq\;& \sum_{t\in T^{3n+6}\cap[0,k]} 2^{(d+5)n}E\left[X_t\{x : \sum_{j=k^{-1}2^{2n}}^{k2^{2n}} \mathrm{I}(\tilde{A}_j^n(x)) \geq 2^{n\varepsilon}\}\mathrm{I}(GMC_{2n})\right] + P(GMC_{2n}^c) \\
\leq\;& C2^{3n+6}2^{(d+5)n}\left[n^{(d+1)p}2^{-n\varepsilon p/2} + (1-e^{-2})^{2^{n\varepsilon/3}}\right] + P(GMC_{2n}^c)
\end{aligned}$$

for large n.Taking $p = 2(d+9)/\varepsilon$, Borel Cantelli gives the desired result.

It remains to prove (2.54).Fix $n, j, \underline{t}, \beta \sim \underline{t}$ such that $\underline{t} \notin [(j-1)2^{-2n}, j2^{-2n}]$. For $\gamma \sim (j-1)n2^{-2n}$ define

$$\begin{aligned}
A_j^n(\gamma,\underline{x}) &= \{N_{(j-1)2^{-2n}}^\gamma \in B(\underline{x}, 9h(2^{-2n})), N_{j2^{-2n}}(\gamma) > 0\} \\
B_j^n(\gamma,\underline{x}) &= \{N_{(j-1)2^{-2n}}^\gamma \in B(\underline{x}, 9h(2^{-2n})), N_{j2^{-2n}}(\gamma) > 2^{-2n}\}
\end{aligned}$$

Then

$$\begin{aligned} A_j^n(\underline{x}) &= \bigcup_{\gamma\sim(j-i)2^{-2n}} A_j^n(\gamma,\underline{x}) \\ B_j^n(\underline{x}) &= \bigcup_{\gamma\sim(j-i)2^{-2n}} B_j^n(\gamma,\underline{x}) \end{aligned}$$

Fix $S=(\gamma_1,\ldots,\gamma_r)$ where $\gamma_i \sim (j-1)2^{-2n}, r \geq 1$. Consider the set

$$A_S(\underline{x}) = \bigcap_{i=1}^{r} A_j^n(\gamma_i,\underline{x}) \cap \bigcap_{\substack{\gamma\sim(j-1)2^{-2n} \\ \gamma\notin S}} (A_j^n(\gamma,\underline{x}))^c \tag{2.56}$$

As S ranges over nonempty subsets of $\{\gamma : \gamma \sim (j-1)2^{-2n}\}$ the sets $A_S(\underline{x})$ form a disjoint partition of $A_j^n(\underline{x})$.

$$\begin{aligned} & {}^{\bullet}P\left(B_j^n(N_{\underline{t}}^\beta)|\mathcal{G}_{j-1}^n, A_S(N_{\underline{t}}^\beta)\right) \\ \geq\ & {}^{\bullet}P\left(N_{j2^{-2n}}(\gamma_1) \geq 2^{-2n}|\mathcal{G}_{j-1}^n, A_S(N_{\underline{t}}^\beta)\right) \\ \geq\ & {}^{\bullet}P\left(N_{j2^{-2n}}(\gamma_1) \geq 2^{-2n}|N_{j2^{-2n}}(\gamma_1) > 0\right) \mathrm{I}(A_S(N_{\underline{t}}^\beta))\mathrm{I}(\beta|_{(j-1)2^{-2n}} \neq \gamma_1) \\ & + {}^{\bullet}P(N_{j2^{-2n}}(\beta|_{(j-1)2^{-2n}}) \geq 2^{-2n}|N_{j2^{-2n}}(\beta|_{(j-1)2^{-2n}}) > 0, N_{\underline{t}}^\beta \neq \Delta) \\ & \qquad \times \mathrm{I}(A_S(N_{\underline{t}}^\beta))\mathrm{I}(\beta|_{(j-1)2^{-2n}} = \gamma_1) \end{aligned}$$

Lemma 2.2 b. c. gives

$$\begin{aligned} & {}^{\bullet}P(N_{j2^{-2n}}(\gamma_1) \geq 2^{-2n}|N_{j2^{-2n}}(\gamma_1) > 0) \approx e^{-2} \\ & {}^{\bullet}P(N_{j2^{-2n}}(\beta|_{(j-1)2^{-2n}}) \geq 2^{-2n}|N_{j2^{-2n}}(\beta|_{(j-1)2^{-2n}}) > 0, N_{\underline{t}}^\beta \neq \Delta) \\ =\ & {}^{\bullet}P(N_{j2^{-2n}}(\beta|_{(j-1)2^{-2n}}) \geq 2^{-2n}|N_{j2^{-2n}}^\beta \neq \Delta) \approx 2e^{-2} \end{aligned}$$

So with $\underline{q} \approx e^{-2}$

$${}^{\bullet}P(B_j^n(N_{\underline{t}}^\beta)|\mathcal{G}_{j-1}^n, A_S(N_{\underline{t}}^\beta)) \geq \underline{q}\ \mathrm{I}(A_S(N_{\underline{t}}^\beta)).$$

Since this is true for all sets $A_S(N_{\underline{t}}^\beta)$ of the form (2.56) we have

$${}^{\bullet}P(B_j^n(N_{\underline{t}}^\beta)|\mathcal{G}_{j-1}^n) \geq \underline{q}\ {}^{\bullet}P(A_j^n(N_{\underline{t}}^\beta)|\mathcal{G}_{j-1}^n)$$

and the proof is complete.□

Chapter 3

The martingale problem characterisation

3.1 The measure of a half space

The martingale problem satisfied by a superprocess gives a semimartingale decomposition for $X_t(f)$ where f is in the domain of the generator A of the underlying spatial motion, namely

$$\begin{aligned} X_t(f) &= m(f) + \int_0^t X_s(Af)ds + M_t(f) \\ \langle M(f)\rangle_t &= \int_0^t X_s(f^2)ds \end{aligned} \tag{3.57}$$

We look for a similar decomposition of $X_t(f)$ for general bounded measurable f. Perkins has shown (private communication) that if the semigroup of the underlying process satisfies a continuity condition, for instance

$$\begin{aligned} &\exists C, \beta_1, \beta_2 > 0 \text{ such that for all } 0 \le \delta \le \nu, f \in b\mathcal{E} \\ &\|T_{\nu+\delta} f - T_\nu f\| \le C\|f\|\,\delta^{\beta_1}(\nu^{-\beta_2} \vee 1) \end{aligned} \tag{3.58}$$

then with probability one the processes $t \longrightarrow X_t(f)$ for bounded measurable f are all continuous on $(0, \infty)$. The proof shows that if $f_n \in D(A)$ are uniformly bounded and converge pointwise to f then almost surely the paths $X_t(f_n)$ converge to $X_t(f)$ uniformly on compact subintervals of $(0, \infty)$. Also $m(f_n) \longrightarrow m(f)$ by dominated convergence and

$$E(\sup_{t\le T}(M_t(f_n) - M_t(f_m))^2) \le 4E(\int_0^T X_s((f_n - f_m)^2)ds) \longrightarrow 0 \tag{3.59}$$

as $m, n \longrightarrow \infty$ again by dominated convergence. So along a subsequence n' the martingales $M_t(f_{n'})$ converge almost surely and uniformly on compacts to a continuous martingale $M_t(f)$. So under the hypothesis (3.58) ,with probability one the processes $\int_0^t X_s(Af_n)ds$ have a subsequence which converges uniformly on compacts in $(0, \infty)$ to a continuous limit.

We examine the case where X_t is a super symmetric $\alpha-$ stable process (so that hypothesis (3.58) is satisfied with $\beta_1 = \beta_2 = 1$) and f is the indicator of a halfspace . Theorem 3.2 shows that $X_t(f)$ fails to be a semimartingale if $1 < \alpha \leq 2$. We will need the existence of a density for X_t in dimension 1 when $1 < \alpha \leq 2$. We state the necessary results as a Theorem.

Theorem 3.1 *Let $m \in M_F(\mathrm{R})$ have a continuous density $u(x)$. Let $\alpha \in (1,2]$ and X_t be a one dimensional super symmetric $\alpha-$stable process starting at m defined on a probabilty space $(\Omega, \mathcal{F}, P)$. Then X_t has a density $X(t,x)$ which is continuous on $[0,\infty) \times \mathrm{R}$. There is a space-time white noise $W_{t,x}$ defined on an enlargement of $(\Omega, \mathcal{F}, P)$ such that for all $f \in C^\infty(\mathrm{R})$ of compact support*

$$X_t(f) = m(f) + \int_0^t X_s(Af)ds + \int_0^t \int_{\mathrm{R}} \sqrt{X(s,x)} f(x) dW_{s,x} \quad \forall t \geq 0 \tag{3.60}$$

For fixed $x \in \mathrm{R}, t \geq 0$

$$X(t,x) = T_t u(x) + \int_0^t \int_{\mathrm{R}} p_{t-s}(y,x) \sqrt{X(s,y)} dW_{s,y} \tag{3.61}$$

where $p_t(x,y)$ is the $\alpha-$stable transition density.

If $u(x)$ is bounded and uniformly Hölder continuous then there exist $\gamma > 0$ and C depending only on m and α such that

$$E((X(t,x) - X(s,x))^2) \leq C(t-s)^\gamma \textit{ for all } t,s \geq 0 \tag{3.62}$$

The existence of a jointly continuous density satisfying (3.60) is proved in Konno and Shiga [14] Theorem 1.4. Equation (3.61) is established during the proof in Konno and Shiga (although they consider more general initial measures and thus work on $[t_0, \infty)$ for $t_0 > 0$, it is easy to extend (3.61) to $[0,\infty)$ for initial measures that have a continuous density.) The proof uses moment estimates of the type in (3.62) but since we can't point to exactly what we need we give a proof.

PROOF OF (3.62) . From (3.61)

$$\begin{aligned} X(t,x) - X(s,x) &= (T_t - T_s)u(x) \\ &+ \int_0^s \int (p_{t-r}(x,y) - p_{s-r}(x,y)) \sqrt{X(r,y)} dW_{r,y} \\ &+ \int_s^t \int p_{t-r}(x,y) \sqrt{X(r,y)} dW_{r,y} \end{aligned}$$

Find $C, \beta \in (0,1]$ such that $|u(x) - u(y)| \leq C|x-y|^\beta$ for all $x,y \in \mathrm{R}$. Then

$$\begin{aligned} \|(T_t - T_s)u\| &\leq \|(T_{t-s} - I)u\| \\ &\leq C E_0(|Y_{t-s}|^\beta) \\ &\leq C(t-s)^{\beta/\alpha} \end{aligned}$$

The stable density satisfies $p_1(x) \leq C(|x|^{-(1+\alpha)} \wedge 1)$ and the scaling equation $p_t(x) = t^{-1/\alpha} p_1(t^{-1/\alpha} x)$.

$$\begin{aligned} E((\int_s^t p_{t-r}(x,y)\sqrt{X(r,y)}dW_{s,y})^2) &= \int_s^t mT_r(p_{t-r}^2(x-\cdot))dr \\ &\leq \|u\| \int_0^{t-s} \int_{\infty}^{\infty} p_r^2(x)dx\,dr \\ &\leq C \int_0^{t-s} r^{-1/\alpha} dr \\ &= C(t-s)^{(\alpha-1)/\alpha} \end{aligned}$$

Similarly

$$E((\int_{s-(t-s)^{1/2}}^{s} (p_{t-r} - p_{s-r})\sqrt{X(r,y)}dW_{r,y})^2) \leq C(t-s)^{(\alpha-1)/2\alpha}.$$

Finally $\|p_t - p_s\| \leq C(t-s)s^{-(\alpha+1)/\alpha}$ for $0 < s < t$ so

$$\begin{aligned} & E((\int_0^{s-(t-s)^{1/2}} (p_{t-r}(x,y) - p_{s-r}(x,y))\sqrt{X(r,y)}dW_{r,y})^2) \\ \leq\ & Cm(1) \int_{(t-s)^{1/2}}^{s} (t-s)^2 r^{-2(\alpha+1)/\alpha} dr \\ \leq\ & C(t-s)^{(3/2)-(1/\alpha)}. \end{aligned}$$

□

Theorem 3.2 *Let $m \in M_F$ and X_t be a super symmetric stable process of index α. Let H be the indicator of a halfspace .Define*

$$\phi(\alpha) = 2\alpha/(\alpha+1) \text{ if } \alpha > 1$$

Then for any $T > 0$ we have the following decomposition.

$$X_t(H) = m(H) + V_t + M_t \text{ for } 0 \leq t \leq T$$

where M_t is a continuous L^2 martingale satisfying $\langle M \rangle_t = \int^t X_s(H)ds$ and V_t is continuous on $(0,T]$.

If $0 < \alpha < 1$ and m has a bounded density then $X_t(H)$ is a semimartingale and V_t has integrable variation on $[0,T]$.

If $1 < \alpha \leq 2$ and m has a bounded density then V_t has integrable $\phi(\alpha)$ variation on $[0,T]$. If in addition the density is uniformly Hölder continuous and satisfies $u(0,x) > 0$ for some x on the boundary of the halfspace then with probability one V_t has strictly positive $\phi(\alpha)$ variation on $[0,T]$ and hence X_t fails to be a semimartingale.

The proof uses the following well known Green's function representation for $X_t(\phi)$, $\phi \in b\mathcal{E}$.

$$X_t(\phi) = m(T_t\phi) + \int_0^t \int_E T_{t-s}\phi(x)dZ_{s,x} \tag{3.63}$$

where T_t is the semigroup of the underlying motion and $Z_{s,x}$ is an orthogonal martingale measure satisfying

$$\langle \int_0^t \int_E f(s,x)dZ_{s,x} \rangle = \int_0^t X_s(f^2(s,\cdot))ds \tag{3.64}$$

for any measurable $f(s,x)$ such that $E(\int_0^t X_s(f^2(s,\cdot))ds) < \infty$, $\forall t$. For the theory of stochastic integration with respect to martingale measures see Walsh [24].Briefly, equation (3.63) may be derived from the martingale problem as follows. Rewrite (M) as

$$X_t(f) = m(f) + \int_0^t X_s(Af)ds + \int_0^t \int_E f(x)dZ_{s,x}.$$

Considering functions of the form $f_t(x) = \sum g_i(x)h_i(t)$ and then passing to the limit we have

$$X_t(f_t) = m(f_0) + \int_0^t X_s(Af_s + df_s/ds)ds + \int_0^t \int_E f_s(x)dZ_{s,x} \tag{3.65}$$

for sufficiently smooth $f_s(x)$. Fixing $t > 0$ and checking that $f_s(x) = T_{t-s}\phi(x)$ is smooth enough to apply (3.65) we immediately obtain (3.63) for nice ϕ. Extension to all $\phi \in b\mathcal{E}$ is straightforward.

PROOF OF PROPOSITION 3.2. It is enough to consider the case $d = 1$ and $H = \mathrm{I}(x \geq 0)$. We start with the Green's function representation.

$$X_t(H) = m(T_tH) + \int_0^t \int_{\mathbb{R}} T_{t-s}H\,dZ_{s,x}$$

If we set $M_t = \int_0^t \int H\,dZ_{s,x}$ then M_t is a continuous L^2 martingale satisfying $\langle M\rangle_t = \int_0^t X_s(H)ds$. The decomposition follows by setting

$$V_t = m((T_t - I)H) + \int_0^t \int_{\mathbb{R}} (T_{t-s} - I)H\,dZ_{s,x}.$$

Now

$$m((T_t - I)H) = m((T_t - I)(H)\mathrm{I}(x \geq 0)) - m((I - T_t)(H)\mathrm{I}(x < 0))$$

is the difference of two decreasing processes and so of bounded variation. It remains to check the variation of

$$W_t := \int_0^t \int_{\mathbb{R}} (T_{t-s} - I)H\,dZ_{s,x}. \tag{3.66}$$

An upper bound for the expected value of the size of an increment of W_t can be obtained using the isometry for $Z_{s,x}$ (equation 3.64) . We delay the calculations and state the result as a Lemma.

Lemma 3.3 *If m has a bounded density then there is a constant C depending only on T, α, m such that for $0 \le s \le t \le T$*

$$E((W_t - W_s)^2) \le C \begin{cases} (t-s)^{(\alpha+1)/\alpha} & \text{if } \alpha > 1 \\ (t-s)^2 & \text{if } \alpha < 1 \end{cases}$$

Since we are interested in a continuous version of W_t it is enough to check the variation over one sequence of decreasing nested partitions .Let $\Delta = T/n$ and $s_j = j\Delta$. If $1 < \alpha$ then

$$\begin{aligned} E\left(\sum_{j=1}^{n} |W_{s_j} - W_{s_{j-1}}|^{\phi(\alpha)}\right) &\le \sum_{j=1}^{n} (E((W_{s_j} - W_{s_{j-1}})^2))^{\phi(\alpha)/2} \\ &\le CT. \end{aligned}$$

So W_t and hence V_t has integrable $\phi(\alpha)$ variation on $[0,T]$. Similarly if $\alpha < 1$ then V_t has integrable variation on $[0,T]$.

We now assume that $u(0,x)$ is bounded, uniformly Hölder continuous and satisfies $u(0,0) > 0$. If $1 < \alpha \le 2$ then X_t has a jointly continuous density $u(t,x)$ and

$$\int_0^t \int_{\mathrm{R}} f(s,x) dZ_{s,x} = \int_0^t \int_{\mathrm{R}} f(s,x)\sqrt{u(s,x)}dW_{s,x}$$

where $W_{s,x}$ is a space-time white noise (see Theorem 3.1).

We split an increment of W_t into three parts as follows. Fix n and let $t_j = j/n$.

$$\begin{aligned} W_{t_{j+1}} - W_{t_j} &= \int_0^{t_j} \int_{\mathrm{R}} (T_{t_{j+1}-s} - T_{t_j-s}) H dZ_{s,x} \\ &+ \int_{t_j}^{t_{j+1}} \int_{\mathrm{R}} (T_{t_{j+1}-s} - I)(H)\sqrt{u(t_j,x)}dW_{s,x} \\ &+ \int_{t_j}^{t_{j+1}} \int_{\mathrm{R}} (T_{t_{j+1}-s} - I)(H)(\sqrt{u(s,x)} - \sqrt{u(t_j,x)})dW_{s,x} \\ &=: \zeta_j + \epsilon_j + \eta_j \end{aligned}$$

We wish to show that W_t has strictly positive $\phi(\alpha)$ variation . We will first show that $|\eta_j|^{\phi(\alpha)}$ is small and does not contribute to the variation. Then noting that ζ_j is $\mathcal{F}_{t_j}$ measurable , we will show that conditional on $\mathcal{F}_{t_j}$, ϵ_j has a mean zero Normal distribution with variance more than $Cn^{-1}X_{t_j}(B(0,n^{-1/\alpha}))$. Since X_t has a density $u(t,x)$ bounded away from zero at $t = x = 0$, this variance will be of the order of $n^{-(\alpha+1)/\alpha}$ and the increment $|\zeta_j + \epsilon_j|^{\phi(\alpha)}$ will be of the order of n^{-1}.

$$E(|\eta|^2) \le E(\int_{t_j}^{t_{j+1}} \int_{\mathrm{R}} (T_{t_{j+1}-s} - I)^2(H)(\sqrt{u(s,x)} - \sqrt{u(t_j,x)})^2 ds\, dx)$$

$$
\begin{aligned}
&\leq \int_{t_j}^{t_{j+1}} \int_{\mathrm{R}} (T_{t_{j+1}-s} - I)^2(H)[E(u(s,x) - u(t_j,x))^2)]^{1/2} ds\, dx \\
&\leq Cn^{-\gamma} \int_{t_j}^{t_{j+1}} \int_{\mathrm{R}} (T_{t_{j+1}-s} - I)^2(H) ds\, dx
\end{aligned}
$$

where $\gamma > 0$ from Lemma 3.1 which uses the Hölder continuity of $u(0,x)$. Using the bound $(T_r - I)H(x) \leq Cr|x|^{-\alpha} \wedge 1$ (see equation 3.71) we have

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} \int_{\mathrm{R}} (T_{t_{j+1}-s} - I)^2(H) ds\, dx &\leq Cn^{-1} \left(\int_0^{n^{-1/\alpha}} dx + \int_{n^{-1/\alpha}}^{\infty} n^{-2}|x|^{-2\alpha} dx \right) \\
&\leq Cn^{-(\alpha+1)/\alpha}.
\end{aligned}
$$

So

$$
\begin{aligned}
E\left[\sum_{j=0}^{[nT]-1} |\eta_j|^{\phi(\alpha)} \right] &\leq \sum_{j=0}^{[nT]-1} (E(\eta_j^2))^{\phi(\alpha)/2} \\
&\leq C \sum_{j=0}^{n[T]-1} (n^{-(\alpha+1)/\alpha} n^{-\gamma})^{\phi(\alpha)/2} \\
&= Cn^{-\gamma\alpha/(\alpha+1)}. \qquad (3.67)
\end{aligned}
$$

Conditional on $\mathcal{F}_{t_j}$, ϵ_j has a Normal mean zero distribution with variance

$$
X_{t_j} \left(\int_{t_j}^{t_{j+1}} (T_{t_{j+1}-s} - I)^2 H\, ds \right).
$$

Let Y_t be a symmetric α-stable process under P_0.

$$
\begin{aligned}
|(T_r - I)H(x)| &= P_0(Y_1 \geq |x|/r^{1/\alpha}) \\
&\geq P_0(Y_1 \geq 2^{1/\alpha})\, \mathrm{I}(|x| \leq (2r)^{1/\alpha})
\end{aligned}
$$

So

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} (T_{t_{j+1}-s} - I)^2 H\, ds &\geq \int_{(2n)^{-1}}^{n^{-1}} (P_0(Y_1 \geq 2^{1/\alpha}))^2\, \mathrm{I}(|x| \leq (n)^{-1/\alpha}) ds \\
&= C_2 n^{-1} \mathrm{I}(|x| \leq n^{-1/\alpha})
\end{aligned}
$$

where $C_2 = (P_0(Y_1 \geq 2^{1/\alpha})^2/2$.

Let N have a Normal mean zero variance one distribution under P_0.

$$
\begin{aligned}
Q(|\epsilon_j|^{\phi(\alpha)} \geq \kappa n^{-1} | \mathcal{F}_{t_j}) &\geq P_0(N^2 \geq C_2^{-1} \kappa^{2/\phi(\alpha)} n^{(1-2\phi(\alpha)^{-1})} / X_{t_j}(B(0, n^{-1/\alpha}))) \\
&\geq (1/5)\mathrm{I}\left[X_{t_j}(B(0, n^{-1/\alpha})) \geq C_2^{-1} \kappa^{2/\phi(\alpha)} n^{-1/\alpha} \right]
\end{aligned}
$$

using $P_0(N^2 \geq 1) \geq 1/5$. Since ζ_j is $\mathcal{F}_{t_j}$ measurable

$$Q(|\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa n^{-1} | \mathcal{F}_{t_j}) \geq (1/10) \mathrm{I}\left[X_{t_j}(B(0, n^{-1/\alpha})) \geq C_2^{-1} \kappa^{2/\phi(\alpha)} n^{-1/\alpha}\right].$$

The density $u(t,x)$ is jointly continuous and $u(0,0) > 0$ so given $\varepsilon > 0$ we may find $n_0 \geq 1, \kappa_0 > 0, t_0 > 0$ so that for all $n \geq n_0$

$$Q(X_t(B(0, n^{-1/\alpha}) < C_2^{-1} \kappa_0^{2/\phi(\alpha)} n^{-1/\alpha} \text{ for some } 0 \leq t \leq t_0) \leq \varepsilon.$$

Then for $n \geq n_0$

$$\begin{aligned}
& Q(\sum_{j=0}^{[nT]-1} |\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0 t_0/20) \\
\geq\ & Q(\sum_{j=0}^{[nT]-1} \mathrm{I}(|\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0/n) \geq n t_0/20) \\
\geq\ & Q\left(\sum_{j=0}^{[nT]-1} \mathrm{I}(X_{t_j}(B(0, n^{-1/\alpha})) \geq C_2 \kappa_0^{2/\phi(\alpha)} n^{-1/\alpha}) \geq n t_0\right) \\
& -Q\left(\sum_{j=0}^{[nT]-1} \mathrm{I}(X_{t_j}(B(0, n^{-1/\alpha})) \geq C_2 \kappa_0^{2/\phi(\alpha)} n^{-1/\alpha}) \geq n t_0, \right. \\
& \qquad\qquad \left. \sum_{j=0}^{[nT]-1} \mathrm{I}(|\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0/n) < n t_0/20\right) \\
\geq\ & (1 - \varepsilon) - P_0(B(n t_0, 1/10) < n t_0/20)
\end{aligned}$$

where B has a Binomial distribution under P_0, using Lemma 2.18.

So for large n

$$Q(\sum_{j=0}^{n[T]-1} |\epsilon_j + \zeta_j|^{\phi(\alpha)} \geq \kappa_0 t_0/20) \geq 1 - 2\varepsilon.$$

But from (3.67) for large n

$$Q(\sum_{j=0}^{n[T]-1} |\eta_j|^{\phi(\alpha)} \geq \kappa_0 t_0/40) \leq \varepsilon.$$

Now Minkowski's inequality and Fatou's Lemma give

$$Q(\sum_{j=0}^{n[T]-1} |W_{t_{j+1}} - W_{t_j}|^{\phi(\alpha)} \geq \kappa_0 t_0/80 \text{ infinitely often }) \geq 1 - 3\varepsilon.$$

Since ε was arbitrary it follows that the $\phi(\alpha)$ variation of W_t over $[0,T]$ is strictly positive. □

PROOF OF LEMMA 3.3 . From (3.66)

$$
\begin{aligned}
&E[(W_t - W_s)^2] \\
=\ & E\left[\left(\int_s^t (T_{t-r} - I) H\, dZ_{r,x} + \int_0^s (T_{t-r} - T_{s-r}) H\, dZ_{r,x}\right)^2\right] \\
=\ & \int_s^t mT_r(((T_{t-r} - I)H)^2)dr + \int_0^s mT_r(((T_{t-r} - T_{s-r})H)^2)dr.
\end{aligned}
\tag{3.68}
$$

For fixed $x \geq 0$

$$
\begin{aligned}
(T_{r+\delta} - T_r)H(x) &= P_0(Y_{r+\delta} \geq -x) - P_0(Y_r \geq -x) \\
&= P_0(Y_1 \in [x/(r+\delta)^{1/\alpha}, x/r^{1/\alpha}]) \\
&\leq |(x/r^{1/\alpha}) - (x/(r+\delta)^{1/\alpha})| p_1(x/(r+\delta)^{1/\alpha}).
\end{aligned}
$$

C will be a constant depending only on T, α, m whose value may change from line to line.Using the bound $p_1(x) \leq C(|x|^{-(1+\alpha)} \wedge 1)$ we have for $r \geq \delta$

$$
\begin{aligned}
|(T_{r+\delta} - T_r)H(x)| &\leq C(\delta|x|^{-\alpha} \wedge \delta|x|r^{-(\alpha+1)/\alpha}) \\
&\leq C\delta(|x|^{-\alpha} \wedge r^{-1})
\end{aligned}
\tag{3.69}
$$

for $r \leq \delta$

$$
\begin{aligned}
|(T_{r+\delta} - T_r)H(x)| &\leq P_0(Y_1 \in [|x|/\delta^{1/\alpha}, \infty)) \\
&\leq C\delta|x|^{-\alpha} \wedge 1
\end{aligned}
\tag{3.70}
$$

and for $r > 0$

$$
\begin{aligned}
|(T_r - I)H(x)| &\leq P_0(Y_1 \in [|x|/r^{1/\alpha}, \infty)) \\
&\leq Cr|x|^{-\alpha} \wedge 1
\end{aligned}
\tag{3.71}
$$

Find a constant K so that the densities of the measures mT_r are bounded by K for all $r \geq 0$. From (3.69), for $0 \leq r \leq s - (t - s)$

$$
\begin{aligned}
&mT_r((T_{t-r} - T_{s-r})^2)H \\
\leq\ & 2(t-s)^2 \left(K(s-r)^{-2} \int_0^{(s-r)^{1/\alpha}} dx + K \int_{(s-r)^{1/\alpha}}^1 |x|^{-2\alpha} dx + mT_r(|x| \geq 1)\right) \\
\leq\ & C(t-s)^2 \begin{cases} 1 + (s-r)^{(1/\alpha)-2} & \text{if } \alpha \neq 1/2 \\ 1 + \log^+((s-r)^{-1}) & \text{if } \alpha = 1/2 \end{cases}
\end{aligned}
$$

From (3.70), for $s-(t-s) \le r \le s$

$$
\begin{aligned}
& mT_r((T_{t-r}-T_{s-r})^2)H \\
\le\ & 2CK\int_0^{(t-s)^{1/\alpha}} dx + 2CK(t-s)^2\int_{(t-s)^{1/\alpha}}^1 |x|^{-2\alpha}dx + 2C(t-s)^2 mT_r(|x|\ge 1) \\
\le\ & C\begin{cases} (t-s)^{(2\wedge(1/\alpha))} & \text{if } \alpha \ne 1/2 \\ (t-s)^2(1+\log^+((t-s)^{-1}) & \text{if } \alpha = 1/2 \end{cases}
\end{aligned}
$$

So

$$
\begin{aligned}
& \int_0^s mT_r((T_{t-r}-T_{s-r})^2)H\,dr \\
\le\ & C(t-s)^2\int_0^{(t-s)} dr + C(t-s)^2\int_{(t-s)}^{s-(t-s)} \left\{ \begin{array}{l} 1+(s-r)^{(1/\alpha)-2} \\ 1+\log^+((s-r)^{-1}) \end{array} \right\} dr \\
& +C\int_{s-(t-s)}^{s} \left\{ \begin{array}{l} (t-s)^{(2\wedge(1/\alpha))} \\ (t-s)^2(1+\log^+((t-s)^{-1})) \end{array} \right\} dr \\
\le\ & C\begin{cases} (t-s)^2 & \text{if } \alpha < 1 \\ (t-s)^{(\alpha+1)/\alpha} & \text{if } \alpha > 1 \end{cases}
\end{aligned}
$$

Similar arguments give an upper bound of no larger order for the first term in (3.68). □

Remarks.

i.For any $m \in M_F(R^d)$ similar arguments show that if $0 < S < T$, $0 < \alpha < 1$ then V_t has integrable variation on $[S,T]$.

ii. If $1 < \alpha < 2$ then the instantaneous propagation of the support (see (3.78)) implies that V_t will have strictly positive $\phi(\alpha)$ variation on $[0,T]$ for any $T > 0$. If $\alpha = 2$ and $m \ne 0$ then there is positive probability that for some $s > 0$ the measure X_s will have a uniformly Hölder continuous bounded density that is strictly positive at some point on the boundary of the half space. Thus for any $X_0 \ne 0$ the process X_t fails to be a semimartingale.

iii.Sugitani [23] shows that for super Brownian motion in dimension one the local time process $Y(t,x) = \int_0^t X(s,x)ds$ is differentiable in x and that if m is atomless the derivative $D_xY(t,x)$ is jointly continuous in t, x almost surely. We can easily identify the drift term V_t in the decomposition of $X_t(H)$ as $(1/2)D_xY(t,x)$.

Take $m \in M_F(\mathrm{R})$ atomless and of compact support. Define $f_a(x) = ((x-a)\vee 0)^2$. We may find $f_n \in D(A)$ so that $f_n \uparrow f_a(x)$ and $Af_n \to \mathrm{I}(x \ge a)$ bounded pointwise. We have enough domination (e.g. $E(\sup_{t \le T} X_t(f_a^2)) < \infty$) to take limits in the martingale problem and obtain

$$\begin{aligned} X_t(f_a) &= m(f_a) + \int_0^t X_s(\mathrm{I}(x \le a))ds + M_t(f_a) \\ &= m(f_a) + \int_a^\infty Y(t,x)dx + M_t(f_a) \end{aligned} \tag{3.72}$$

We wish to differentiate (3.72) twice with respect to a and again we have enough domination. Thus for a fixed t

$$2X_t((x-a) \vee 0) = 2m((x-a) \vee 0) + Y(t,a) + M_t(2(x-a) \vee 0) \tag{3.73}$$

Now continuity of both sides in t gives (3.73) for all t. Repeating the argument and using the continuity of $D_x Y(t,x)$ gives

$$X_t(\mathrm{I}(x \le a)) = m(\mathrm{I}(x \le a)) + (1/2)D_x Y(t,a) + M_t(\mathrm{I}(x \le a)).$$

3.2 The death point

We use the characterisation of a Superprocess X_t as a solution to a martingale problem (equation 3.57) to study the sample path behaviour near the time of death. Set $\xi = \inf\{t \ge 0 : X_t(1) = 0\}$ where we write 1 for the constant function with value one. If $m \in M_F$ then $\xi < \infty$ almost surely. Define

$$C_t = \int_0^t 1/X_s(1)\, ds.$$

In Konno-Shiga [14] Theorem 2.1 it is shown that with probability one C_t is a homeomorphism between $[0,\xi)$ and $[0,\infty)$. Let $D_t : [0,\infty) \to [0,\xi)$ be the continuous strictly increasing inverse to C_t. Shiga [22] uses D_t as a time change together with a renormalisation to convert a class of measure valued processes into a class of probability valued processes. The Superprocesses studied here do not seem to fall directly into his context. However the time change will still be useful. By stretching out the interval $[0,\xi)$ into $[0,\infty)$ we can use the behaviour at infinity of the time changed process to give infomation about X_t before death.

For $t \in [0,\infty)$ define

$$\begin{aligned} \tilde{Y}_t &= X_{D_t} \\ Y_t &= \tilde{Y}_t / \tilde{Y}_t(1) \\ \mathcal{G}_t &= \mathcal{F}_{D_t} \end{aligned}$$

Note that $\{Y_t : t \geq 0\}$ is a probability valued process.We derive the martingale problem for Y_t. For $f \in D(A)$

$$\begin{aligned}
\tilde{Y}_t(f) &= m(f) + \int_0^{D_t} X_s(Af)ds + M_{D_t}(f) \\
&= m(f) + \int_0^t \tilde{Y}_s(Af)\tilde{Y}_s(1)ds + \tilde{N}_t(f)
\end{aligned}$$

where, since D_t is a continuous time change , $\tilde{N}_t(f)$ is a continuous $\mathcal{G}_t$ local martingale satisfying

$$\begin{aligned}
\langle \tilde{N}(f) \rangle_t &= \int_0^{D_t} X_s(f^2)ds \\
&= \int_0^t \tilde{Y}_s(f^2)\tilde{Y}_s(1)ds.
\end{aligned}$$

In particular

$$\begin{aligned}
\tilde{Y}_t(1) &= m(1) + \tilde{N}_t(1) \\
\langle \tilde{N}(1) \rangle_t &= \int_0^t (\tilde{Y}_s(1))^2 ds \\
\langle \tilde{N}(f), \tilde{N}(1) \rangle_t &= \int_0^t \tilde{Y}_s(f)\tilde{Y}_s(1)ds
\end{aligned}$$

Applying Ito's formula and noting that $\tilde{Y}_t(1) > 0$ for all $t > 0$ we have

$$Y_t(f) = m(f) + \int_0^t \tilde{Y}_s(Af)ds + N_t(f) \tag{3.74}$$

where $N_t(f)$ is a continuous $\mathcal{G}_t$ local martingale satisfying

$$\langle N(f) \rangle_t = \int_0^t Y_s(f^2) - (Y_s(f))^2 ds$$

The martingale problem for Y_t is frustratingly close to that for the probability valued diffusion known as the Fleming-Viot process (where the drift term in (3.74) would be replaced by $\int Y_s(Af)ds$). In Konno-Shiga [14] this 'connection' between the martingale problems is used to derive the existence of a continuous density for the Fleming-Viot process in dimension 1 from that for super Brownian motion .

The following result shows that as $t \rightarrow \xi$, what mass that remains is concentrated near a single point.

Theorem 3.4 *For $m \in M_F$ there exists an E valued random variable F such that with probability one*

$$X_t/X_t(1) \rightarrow \delta_F \text{ as t} \rightarrow \xi \tag{3.75}$$

where the convergence is weak convergence of measures.The law of F given the history of the total mass process $\mathcal{H} = \sigma(X_t(1) : t \geq 0)$ satisfies

$$E^m(f(F)|\mathcal{H}) = 1/m(1) \int_E T_\xi f\, dm. \tag{3.76}$$

Remarks.

i. Equation (3.76) implies that the law of F can be constructed as follows. Position a particle in E at random according to the measure $m(\cdot)/m(1)$. Let the particle move according to the underlying spatial motion but independently to the process. Stop the particle at time ξ. The final position of the particle will have law F.

ii. The law of ξ is given by $P(\xi \leq t) = \exp(-2m(1)/t)$

PROOF. First assume E is compact. Take $f \in D(A)$.

$$\begin{aligned} \int_0^t \tilde{Y}_s(Af)ds &\leq \|Af\| \int_0^t \tilde{Y}_s(1)ds \\ &= \|Af\| \int_0^t X_{D_s}(1)ds \\ &= \|Af\| D_t < \|Af\|\xi. \end{aligned}$$

So

$$N_t(f) \geq -m(f) - \|Af\|\xi.$$

For any continuous local martingale $(M_t; t \geq 0)$, with probability one either M_t converges to a finite limit or $\limsup M_t = -\liminf M_t = \infty$ (see Rogers Williams [20] Corollary IV.34.13). So $N_t(f)$ converges as $t \to \infty$ to a finite limit. Also

$$\left| \int_s^t \tilde{Y}_r(Af)dr \right| \leq \|Af\|(D_t - D_s) \to 0 \text{ as } s, t \to \infty.$$

So $\int_0^t \tilde{Y}_s(Af)ds$ converges as $t \to \infty$.Thus $Y_t(f)$ converges a.s. to a finite limit which we call $Y_\infty(f)$.

Since $C(E)$ is separable and $D(A)$ is dense in $C(E)$ we may pick $\{\phi_n\}_n \subseteq D(A)$ dense in $C(E)$. Off a null set N we have $Y_t(\phi_n) \to Y_\infty(\phi_n), \forall n$. Fix $\omega \notin N$. Then by approximation $Y_t(f)$ converges to a finite limit $Y_\infty(f)$ for all $f \in C(E)$. Also $f \to Y_\infty(f)$ is a positive linear functional with $Y_\infty(1) = 1$ and thus arises from a probability which we call Y_∞. For $f \in D(A)$

$$N_t^2(f) - \int_0^t Y_s(f^2) - (Y_s(f))^2 ds$$

is a continuous local martingale. Since $N_t(f), Y_s(f^2), Y_s(f)$ all converge to finite limits this local martingale must converge requiring $Y_\infty(f^2) = (Y_\infty(f))^2$ a.s. So the probability Y_∞ is concentrated on a level set of f. But E is a metric space so that $C(E)$ and hence $\{\phi_n\}_n$ separate points and this forces $Y_\infty = \delta_F$ a.s. for some F.

We have been unable to deduce the law of F directly from the martingale problem but it comes immediately from the particle picture. Take the nonstandard model with $m_\mu = \mu^{-1}\sum_i \delta_{x_i}$ satisfying $st_{M_F}(m_\mu) = m$. Let $\mathcal{G}$ be the internal algebra generated by the total mass process $\{N_{\underline{t}}(1) : \underline{t} \geq 0\}$ and $\sigma(\mathcal{G})$ the standard σ-algebra generated by $\mathcal{G}$. Note that $N_{\underline{t}}({}^*E)$ is $\mathcal{G}$ measurable for all $\underline{t}$ so that $\mathcal{H}$ is a sub σ - algebra of $\sigma(\mathcal{G})$. Let $\xi_n = inf(\underline{t} \geq 0 : N^\mu_{\underline{t}}(1) \leq 1/n)$. For any $f \in C(E), n \geq m(1)^{-1}$

$$
\begin{aligned}
& {}^*E(N^\mu_{\xi_n}({}^*f)/N^\mu_{\xi_n}(1)|\mathcal{G}) \\
= \; & n\,{}^*E(\mu^{-1}\sum_{\gamma\sim\xi_n} {}^*f(N^\gamma_{\xi_n})|\mathcal{G}) \\
= \; & n\mu^{-1}\sum_{\gamma\sim\xi_n} {}^*P(N^\gamma_{\xi_n} \neq \Delta|\mathcal{G})T_{\xi_n}f(\gamma|_0)
\end{aligned}
$$

Now $\underline{p} = {}^*P(N^\gamma_{\xi_n} \neq \Delta|\mathcal{G})$ is independent of γ, so

$$
\begin{aligned}
& {}^*E(N^\mu_{\xi_n}({}^*f)/N^\mu_{\xi_n}(1)|\mathcal{G}) \\
= \; & n\underline{p}2^{\mu\xi_n}(\mu^{-1}\sum_i T_{\xi_n}f(x_i)) \\
= \; & n/m(1)\int_{{}^*E} T_{\xi_n}f(\underline{x})dm_\mu(\underline{x})(\mu^{-1}\sum_{\gamma\sim\xi_n} {}^*P(N^\gamma_{\xi_n} \neq \Delta|\mathcal{G})) \\
= \; & 1/m(1)\int_{{}^*E} T_{\xi_n}f dm_\mu.
\end{aligned}
$$

So

$$E(X_{{}^\circ\xi_n}(f)/X_{{}^\circ\xi_n}(1)|\sigma(\mathcal{E})) = 1/m(1)\int_E T_{{}^\circ\xi_n} f\, dm$$

using Albevario et al. [1] Proposition 3.2.12.Now ${}^\circ\xi_n \uparrow \xi$ as $n \to \infty$ so that

$$E(f(F)|\mathcal{H}) = 1/m(1)\int_E T_\xi f\, dm. \tag{3.77}$$

When E is only locally compact we can extend the semigroup T_t to $E \cup \{\infty\}$ the one point compactification of E by taking $T_t(\infty, \{\infty\}) = 1, T_t(x, \{\infty\}) = 0$ for all $x \in E, t > 0$. Working with this new Feller process on $E \cup \{\infty\}$ the above argument gives the existence of a death point F taking values in $E \cup \{\infty\}$ and satisfying (3.75) and (3.76). Since $P(\xi < \infty) = 1$ the characterisation of the law of F (3.77) ensures $P(F \in E) = 1$. □

Example. Let X_t be a super Poisson process. Define $T_k = \inf(t \geq 0 : X_t(\{0, \ldots, k-1\}) = 0)$. In

Perkins [17] Corollory 3.1 it is shown that $T_k \uparrow \xi$ and

$$S_t = \{k, k+1, \ldots\} \text{ for Lebesgue a.a.t in } [T_k, T_{k+1}), k \in \mathrm{Z}_+, P^m - a.s.$$

Theorem 3.4 shows that only finitely many of the T_k's are distinct. Indeed

$$0 = T_0 \le T_1 \le \ldots \le T_F \le T_{F+1} = T_{F+2} = \ldots = \xi \;\; P^m - a.s.$$

There is positive probability for any combination of equalities amoung $T_0, T_1, \ldots, T_F$.

3.3 The support near extinction

The closed support of a superprocess X_t at a fixed time has been studied in Perkins [17] and Evans and Perkins [8] . If the spatial motion is a Lévy process on R^d with Lévy measure μ then in Evans and Perkins [8] Theorem 5.1 it is shown that for all $t > 0$

$$\bigcup_{k=1}^{\infty} S(\mu^{*k} * X_t) \subseteq S(X_t) \;\; P^m - a.s.$$

where μ^{*k} is the k'th fold convolution of μ with itself.For a super symmetric stable process this implies

$$S(X_t) = \emptyset \text{ or } \mathrm{R}^d \;\; P^m - a.s. \, \forall t > 0. \tag{3.78}$$

Similar results for certain Feller processes are obtained . Consider a Markov jump process with bounded generator A so that

$$Af(x) = \rho \int_E \mu(x, dy)(f(y) - f(x)) \tag{3.79}$$

with $\rho > 0$ and μ a probability kernel such that $x \to \int \mu(x, dy) f(y) \in C_0(E)$ for all $f \in C_0(E)$. Then for $t > 0$

$$\bigcup_{k=1}^{\infty} S\left(\int \ldots \int X_t(dx_1)\mu(x_1, dx_2)\ldots\mu(x_k, \cdot)\right) \subseteq S(X_t) \; P^m - a.s. \tag{3.80}$$

We shall show that (3.78),(3.80) are far from being sample path properties and that near the time of death there will be exceptional times at which the support is concentrated arbitrarily close to the death point.

We start by examining the case where the spatial motion is a Markov jump process as described above. Note that Af is well defined by (3.79) for any bounded measurable f . A monotone class argument shows that for any bounded measurable f the process $X_t(f)$ is continuous and satisfies the usual semimartingale decomposition.

Theorem 3.5 *For all $\varepsilon > 0$, with probability one there exist distinct $t_n \uparrow \xi$ such that*

$$S(X_{t_n}) \subseteq B(F, \varepsilon).$$

PROOF. Take $A \subseteq E$ and let $f = \mathrm{I}(x \in A)$. We shall use the time changed process $Y_t(f)$ as in section 3.2. Let $\tilde{B}_t$ be an independent Brownian motion defined if necessary on an extension of the original probability space. Define

$$\bar{B}_t = \int_0^t (Y_s(f)(1 - Y_s(f)))^{-1/2}\mathrm{I}(Y_s(f) \neq 0)dN_s(f) + \int_0^t \mathrm{I}(Y_s(f) = 0)d\tilde{B}_s$$

so that $\bar{B}_t$ is a Brownian motion and

$$Y_t(f) = m(f) + \int_0^t \tilde{Y}_s(Af)ds + \int_0^t (Y_s(f)(1 - Y_s(f)))^{1/2}d\bar{B}_s.$$

If $Y_t(f) = 0$ or 1 then Y_t is supported on A^c or A respectively. So we look for times at which $Y_t(f)(1 - Y_t(f))$ becomes zero. Fix $N \in \mathrm{N}$ and define $Z_t(f) = Y_{N+t}(f)(1 - Y_{N+t}(f))$. By Itô's formula we have

$$\begin{aligned} Z_t(f) &= Z_0(f) + \int_0^t (1 - 2Y_{N+s}(f))(Y_{N+s}(f)(1 - Y_{N+s}(f)))^{1/2}d\bar{B}_s \\ &\quad + \int_0^t (1 - 2Y_{N+s}(f))\tilde{Y}_{N+s}(Af)ds - \int_0^t Y_{N+s}(f)(1 - Y_{N+s}(f))ds \\ &= Z_0(f) + \int_0^t (\beta_s - Z_s(f))ds + \int_0^t (Z_s(f)(1 - 4Z_s(f)))^{1/2}dB_s \end{aligned}$$

where

$$\begin{aligned} \beta_s &= (1 - 2Y_{N+s}(f))\tilde{Y}_s(Af) \\ B_t &= \int_N^{N+t} \mathrm{sgn}(1 - 2Y_s(f))d\bar{B}_s \end{aligned}$$

so that B_t is another Brownian motion. Since the function $(x(1-4x))^{1/2}$ satisfies the Yamada-Watanabe criterion (see Rogers and Williams [20] Theorem V.40.1) we have a unique solution on the same probability space to the stochastic differential equation

$$X_t = Z_0(f) + \int_0^t ((1/8) - X_s)ds + \int_0^t |X_s(1 - 4X_s)|^{1/2}dB_s.$$

Lemma 3.6 shows that X_t takes values in $[0, 1/4]$ and zero is a recurrent point . Define

$$T_N = \inf(t \geq 0 : \tilde{Y}_{N+t}(1) \geq (1/8||Af||))$$

which is a $\mathcal{G}_{N+t}$ stopping time. For $s \leq T_N$

$$|\beta_s| = |(1 - 2Y_{N+s}(f))\tilde{Y}_{N+s}(Af)| \leq 1/8.$$

So by a comparison Theorem for one dimensional diffusions (see Rogers and Williams [20] Theorem V.43.1) we have

$$Z_t(f) \leq X_t \text{ for } t \leq T_N \;\; P^m - a.s.$$

(We have applied the comparison Theorem up to a stopping time .The changes needed in the proof of Theorem V.43.1 are easy.)

Since zero is recurrent for X_t , on the set $\{T_N = \infty\}$, $Z_t(f)$ must hit zero infinitely often as $t \to \infty$. Since $\tilde{Y}_s(1) \to 0$ as $s \to \infty$, $P(T_N = \infty) \uparrow 1$ as $N \to \infty$. So with probability one

$$\text{there exist } t_n \uparrow \infty \text{ so that } Y_{t_n}(f) = 0 \text{ or } 1. \tag{3.81}$$

Given $\varepsilon > 0$ let $(A_m)_m$ be a countable collection of open balls of radius $\varepsilon/2$ that cover E . Fix ω so that (3.81) holds simultaneously for all $f_m = \mathrm{I}(x \in A_m)$. Find $m_0(\omega)$ so that $F(\omega) \in A_{m_0(\omega)}$. Since $Y_t \to \delta_F$ then $Y_t(A_{m_0}) \to 1$. So there exist $t_n \uparrow \infty$ so that $Y_{t_n}(\mathrm{I}(x \notin A_{m_0})) = 0$ and

$$S(X_{D_{t_n}}) = S(Y_{t_n}) \subseteq B(F, \varepsilon) \text{ for all } n.$$

□

Lemma 3.6 *Let B_t be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Let X_t^x be the unique solution to the stochastic differential equation*

$$\begin{aligned} dX_t &= ((1/8) - X_t)dt + |X_t(1 - 4X_t)|^{1/2} dB_t \\ X_0 &= x \in [0, 1/4] \end{aligned} \tag{3.82}$$

Then $P(\exists n$ such that $X_t > 0$ for all $t \geq n) = 0$.

PROOF. Equation (3.82) is pathwise exact so we may find a pathwise unique solution on any space and any two solutions have the same law. Let P^x be the law of X^x on path space. Then the laws $(P^x)_x$ form a strong Markov family. We write x_t for the coordinate function on path space.

Let Y_t be the unique solution on $(\Omega, \mathcal{F}, P)$ to the S.D.E.

$$\begin{aligned} dX_t &= ((1/8) - X_t) \wedge 0 \, dt + |X_t(1 - 4X_t)|^{1/2} dB_t \\ X_0 &= 0 \end{aligned}$$

By uniqueness $Y_t = 0$, $\forall t \geq 0$. So by a comparison Theorem (see Rogers and Williams [20] Theorem V.43.1) $X_t^x \geq 0$, $\forall t \; P^m - a.s.$ We may treat the boundary $x = 1/4$ similarly and conclude

$$P^x(x_t \in [0, 1/4], \; \forall t \geq 0) = 1$$

It is enough to show that there exist t_0 so that

$$P^{1/4}(\exists 0 < t \le t_0,\ x_t = 0) = c > 0 \tag{3.83}$$

for then by the strong Markov property $P^x(\exists 0 < t \le t_0,\ x_t = 0) \ge c$ for all $x \in [0, 1/4]$ and setting $A_n = (\exists t \in (nt_0, (n+1)t_0],\ x_t = 0)$ we have $P^x(A_n | A_1, \ldots, A_{n-1}) \ge c$ and $P^x(A_n\ i.o.) = 1$ which implies the result.

On the interval $[\delta, (1/4) - \delta]$ where $0 < \delta < 1/8$ will be chosen later , we can construct a weak solution to (3.82) using a scale and time change of Brownian motion in a standard manner (see Rogers and Williams V.44 .) We shall then examine the behaviour near the endpoints seperately. Set

$$s(x) = \int_{\delta/2}^{x} (u(1-4u))^{-1/4} du \text{ for } x \in [\delta, (1/4) - \delta]$$

so that $s(x)$ is a strictly increasing C^2 function taking $[\delta, (1/4) - \delta] \to [a, b]$. Set $h(x) = s'(x)x(1-4x)$ and $g(x) = h(s^{-1}(x))$. Then g is a continuous function on $[a, b]$ bounded away from zero by a constant K.

Let $\tilde{B}_t$ be a Brownian motion on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ started at $s(x) \in [a, b]$. Set $\tilde{T}_y = \inf(t \ge 0 : \tilde{B}_t = y)$ and

$$A_t = \int_0^t g(\tilde{B}_u)^{-2} du \text{ for } t \le \tilde{T}_a \wedge \tilde{T}_b.$$

Let γ_t be the continuous strictly increasing inverse to A_t. Then $Y_t = \tilde{B}_{\gamma_t}$ solves the S.D.E. $dY_t = g(Y_t)dB_t$ for some Brownian motion B_t and $Z_t = s^{-1}(Y_t)$ is a weak solution to (3.82) up till the time $\inf(t \ge 0 : Z_t \in \{\delta, (1/4) - \delta\})$. Now for $t_0 > 0$

$$\tilde{P}(\tilde{T}_a < \tilde{T}_b \le t_0) > 0$$

so if $T_y = \inf(t \ge 0 : x_t = y)$ then

$$P^x(T_\delta < T_{(1/4)-\delta} \le K^2 t_0) > 0 \ \ \forall x \in (\delta, (1/4) - \delta). \tag{3.84}$$

For the behaviour near $x = 1/4$ we need only that we can find $t_0, \delta > 0$ so that

$$P^{1/4}(x_{t_0} \le (1/4) - 2\delta) > 0 \tag{3.85}$$

and this follows since $X_t \equiv 1/4$ is not a solution to (3.82).

For the behaviour near $x = 0$ we use another comparison . Let $Y_t = (1 + \cos(2B_t))/8$ where B_t is a Brownian motion started at x_0 and $\delta = (1 + \cos(2x_0))/8$. Itô's formula gives

$$Y_t = \delta + 2\int_0^t ((1/8) - Y_s) ds + \int_0^t (Y_s(1 - 4Y_s))^{1/2} dW_s$$

where W_t is another Brownian motion. Let X_t be a solution on the same space to the equation (3.82) with $x = 1/4$ and W_t the Brownian motion. Then the comparison Theorem shows $X_t \leq Y_t$ up till the time

$$T = \inf(t \geq 0 : X_t \wedge Y_t = 1/8) = \inf(t \geq 0 : Y_t = 1/8).$$

But the construction of Y_t implies that there exists t_0 such that with positive probability , $Y_t = 0$ for some $t \leq t_0 \leq T$. So

$$P^\delta(T_0 < T_{1/8} \leq t_0) > 0. \tag{3.86}$$

Equation (3.84),(3.85),(3.86) together imply (3.83). □

Example. We examine the simplest nontrivial superprocess. Let $E = \{a, b\}$ and the underlying spatial motion be a Markov chain leaving each state at rate one. Then if we write $X_t(a), X_t(b)$ for $X_t(\{a\}), X_t(\{b\})$ the martingale problem reduces to a pair of linked stochastic differential equations.

$$X_t(a) = X_0(a) + \int_0^t (X_s(b) - X_s(a))ds + \int_0^t (X_s(a))^{1/2} dB_s^a \tag{3.87}$$

$$X_t(b) = X_0(b) + \int_0^t (X_s(a) - X_s(b))ds + \int_0^t (X_s(b))^{1/2} dB_s^b \tag{3.88}$$

where B_t^a, B_t^b are independent Brownian motions. So we consider the superprocess as a diffusion on R_+^2.

Let $D = ((x,0) : x > 1/2) \cup ((0,y) : y > 1/2)$. We will show that with probability one $(X_t(a), X_t(b))$ never hits D.Define

$$D_r = ((x,0) : x > (1/2) + r)$$

$$R_r = ((x,y) : y > x - (1/2) - r)$$

It will be enough to show $P((X_t(a), X_t(b)) \in D_r$ for some $t > 0) = 0$ for all $r > 0$.

The properties of the α-dimensional Bessel process (see Rogers and Williams [20] V.48) show that if Z_t satisfies

$$dZ_t = \alpha dt + (Z_t)^{1/2} dB_t \tag{3.89}$$

then for $\alpha \geq 1/2$, $P(Z_t > 0, \forall t > 0) = 1$ and $\alpha = 1/2$ is critical. By comparing (3.88) to the S.D.E. (3.89) solved on the same space with respect to B_t^b we see that $X_t(b) > 0$ up till the time $S_0 = \inf(t \geq 0 : (X_t(a), X_t(b)) \in \overline{R}_0)$. Define

$$T_n = \inf(t \geq S_{n-1} : (X_t(a), X_t(b)) \in R_{r/2}^c)$$

$$S_n = \inf(t \geq T_n : (X_t(a), X_t(b)) \in \overline{R}_0)$$

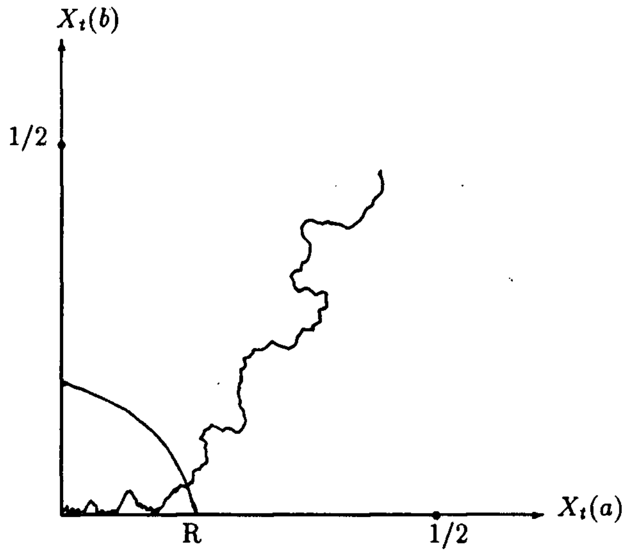

Figure 3.3: Typical sample path of $(X_t(a), X_t(b))$.

Then by the strong Markov property and the same comparison argument $X_t(b) > 0$ for $T_n \le t \le S_n$. By continuity of paths $T_n \uparrow \infty$ and the result is proved.

The finite lifetime of the process implies that the diffusion converges to $(0,0)$ and Theorem 3.5 shows that it approaches the origin in a particular manner . There exists $R(\omega) > 0$ such that inside $B((0,0), R)$ the diffusion will not hit one axis and will hit the other axis at an infinite number of points that accumulate at $(0,0)$.

We would like to extend Theorem 3.5 to superprocesses with more general spatial motion. We take one step in this direction by showing that the semimartingale decomposition for $X_t(H)$ in section 3.1 allows us to extend Theorem 3.5 to super symmetric stable processes of index $\alpha < 1/2$. Note also that the result is true for super Brownian motion (see Liu [15] where it is shown that the diameter of the support of super Brownian motion converges to zero at extinction).

Lemma 3.7 *Let X_t be a one dimensional super symmetric stable process of index α started at $m \in M_F(\mathrm{R})$. If $0 < \alpha, \beta < 1/2$ then the path $t \to X_t(|x|^{-\beta})$ is continuous on $(0,\infty)$.*

PROOF. We use a stopping time argument similar to that in Perkins [16] Proposition 4.4. Take the nonstandard model with $m_\mu \approx m$. Find $\tilde{\alpha}, \tilde{\beta}$ so that $0 < \max(\alpha,\beta) < \tilde{\beta} < \tilde{\alpha} < 1/2$. Write B_n for $B(0, 2^{-n})$ and rB for $(rx : x \in B)$. Fix integer $M \ge 1$ and define

$$T_n = \inf(\underline{t} \in T^\mu \cap [1/M, \infty) : N_{\underline{t}}(B_n) \ge 2.2^{-n\tilde{\beta}}).$$

Let $t_j^n = j2^{-n\tilde{\alpha}}, I_j^n = [t_j^n, t_{j+1}^n)$. Then

$$\begin{aligned}
\overline{P}(T_n \in I_j^n) &= \overline{P}(T_n \in I_j^n, N_{t_{j+1}^n}(2B_n) \geq 2^{-n\tilde{\beta}}) \\
&+ \overline{P}(T_n \in I_j^n, \mu^{-1} \sum_{\gamma \sim t_{j+1}^n} \mathrm{I}(N_{T_n}^\gamma \in B_n, N_{t_{j+1}^n}^\gamma \neq \Delta) \leq 2^{-n\tilde{\beta}}) \\
&+ \overline{P}(T_n \in I_j^n, \mu^{-1} \sum_{\gamma \sim t_{j+1}^n} \mathrm{I}(N_{T_n}^\gamma \in B_n, |N_{t_{j+1}^n}^\gamma - N_{T_n}^\gamma| > 2^{-n}) \geq N_{T_n}(B_n) 2^{-n\tilde{\beta}})
\end{aligned} \tag{3.90}$$

Denote the terms on the right hand side of (3.90) as I,II and III.For $t_j^n \geq 1/m$

$$\begin{aligned}
I &\leq \overline{P}(N_{t_{j+1}^n}(2B_n) \geq 2^{-n\tilde{\beta}}) \\
&\leq 2^{n\tilde{\beta}} E(X_{t_{j+1}^n}(2B_n)) \\
&\leq 2m(1) M^{-1/\alpha} 2^{-n(1-\tilde{\beta})}.
\end{aligned}$$

T_n is a $\mathcal{A}_t$ stopping time so by the strong Markov property

$$\begin{aligned}
II &= \overline{E}\left(\mathrm{I}(T_n \in I_j^n) \overline{P}^{N_{T_n(\omega)}}(\mu^{-1} \sum_{\gamma \sim t_{j+1}^n - T_n(\omega)} \mathrm{I}(N_0^\gamma \in B_n, N_{t_{j+1}^n - T_n(\omega)}^\gamma \neq \Delta) \leq N_0(B_n)/2) \right) \\
&\leq \overline{P}(T_n \in I_j^n) \exp(-2^{n(\tilde{\alpha}-\tilde{\beta})}/4)
\end{aligned}$$

using Perkins [16] Lemma 4.1.a. Similarly

$$\begin{aligned}
& III \\
&\leq \overline{E}\left(\mathrm{I}(T_n \in I_j^n) \overline{P}^{N_{T_n(\omega)}}(\mu^{-1} \sum_{\gamma \sim t_{j+1}^n - T_n(\omega)} \mathrm{I}(|N_{t_{j+1}^n - T_n(\omega)}^\gamma - N_0^\gamma| > 2^{-n}) \geq N_{T_n(\omega)}(B_n) - 2^{-n\tilde{\beta}}) \right) \\
&\leq \overline{P}(T - n \in I_j^n) 2. P_0(|Y_{2^{-n\tilde{\alpha}}}| > 2^{-n}) \\
&\leq C\overline{P}(T_n \in I_j^n) 2^{-n(\tilde{\alpha}-\alpha)}
\end{aligned}$$

using Perkins [16] Lemma 4.2.a. Summing (3.90) over $I_j^n \subseteq [1/M, M]$ we have

$$\overline{P}(T_n \in [1/M, M]) \leq C(2^{-n(1-\tilde{\alpha}-\tilde{\beta})} + \exp(-2^{n(\tilde{\alpha}-\tilde{\beta})}) + 2^{-n(\tilde{\alpha}-\alpha)})$$

which sums over n. So for large n , for all $t \in [1/M, M]$

$$X_t((-2^{-n}, 2^{-n})) \leq 2.2^{-n\tilde{\beta}}.$$

Thus $X_t(|x|^{-\beta})$ is uniformly bounded for $t \in [1/M, M]$ and $(X_t(|x|^{-\beta} \wedge n) : n = 1, 2, \ldots)$ is a Cauchy sequence in $C[1/M, M]$, $P^m - a.s.$ So $X_t(|x|^{-\beta})$ is continuous on $[1/M, M]$ for any M. □

Remark. In Perkins [16] Theorem 6.5 it shown that if $\alpha < 1$ then there exist constants $0 < c_8 \leq c_9 < \infty$ such that for any $m \in M_F(R)$, setting

$$\begin{aligned}\phi_\alpha(x) &= x^\alpha \log^+ \log^+ 1/x \\ \Lambda_t &= (x : \limsup_{a \downarrow 0} X_t(B(x,a))\phi_\alpha(a)^{-1} \in [c_8, c_9])\end{aligned}$$

then

$$X_t(\Lambda_t^c) = 0\ , \forall t > 0\ , P^m - a.s.$$

We call a point in Λ_t a point of density for X_t. For $\alpha < 1$ it follows from the facts that

i. Λ_t is Lebesgue null for all $t > 0$ (Perkins [16])

ii. The laws of super symmetric stable processes on $\sigma(X_s : s \geq t_0 > 0)$ are equivalent under translation of initial measures (Evans and Perkins [8])

that for any fixed x , $P(x$ is a point of density at some $t > 0) = 0$. This also follows (for $\alpha < 1/2$) from Lemma 3.7 for if $0 \in \Lambda_t$ then $X_t(|x|^{-\alpha}) = \infty$. Contrast this with the fact that equation (3.78) implies that for a fixed $t > 0$ the points of density are dense in R.

Proposition 3.8 *If X_t is a super symmetric stable process of index $\alpha < 1/2$ in dimension one started at a finite measure then the conclusions of Theorem 3.5 still hold.*

PROOF. Fix an open ball $B = (a, b)$ of finite radius in R. From Theorem 3.2 and the following remark we have the decomposition

$$X_t(B) = X_0(B) + V_t + M_t(B)$$

where $\langle M(B) \rangle_t = \int^t X_s(B)ds$ and V_t has finite variation on $[S, T]$ for any $0 < S < T < \infty$. Define

$$v_t = \begin{cases} \lim_{h \to 0+}(V_{t+h} - V_t)/h & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}$$

We will find an upper bound on $|v_s|$. Note that for any $\delta > 0$, $T_\delta I_B(x)$ is a C^∞ function vanishing at infinity. Let $g(x) = \sup_{\delta > 0} |AT_\delta I_B(x)|$. Scaling arguments show there exists C such that $g(x) \leq C(|x-a|^{-\alpha} + |x-b|^{-\alpha})$. For fixed $0 < s < t$

$$\begin{aligned}&|X_t(T_\delta I_B) - X_s(T_\delta I_B) - M_t(T_\delta I_B) + M_s(T_\delta I_B)| \\ =\ & |\int_s^t X_r(AT_\delta I_B)dr| \leq \int_s^t X_r(g)dr.\end{aligned}$$

Letting $\delta \downarrow 0$

$$|V_t - V_s| = |X_t(B) - X_s(B) - M_t(B) + M_s(B)| \le \int_s^t X_r(g)dr.$$

$X_r(g)$ is continuous and bounded by Lemma 3.7 so V_t is absolutely continuous and $|v_r| \le X_r(g)$ for $a.a.r$ $Q^m-a.s.$ Now we follow the proof of Theorem 3.5. Recall that

$$t = \int_0^{D_t} 1/X_r(1)dr \tag{3.91}$$

and we set $Y_t(B) = X_{D_t}(B)/X_{D_t}(1)$ and $Z_t(B) = Y_t(B)(1 - Y_t(B))$. Then $Z_t(B)$ satisfies

$$Z_t = Z_0 + \int_0^t ((1 - 2Y_{N+s}(B))v_{D_{N+s}} - Z_s)ds + \int_0^t (Z_s(B)(1 - 4Z_s(B)))^{1/2}dB_s.$$

Now the comparison argument of Theorem 3.5 will work provided we can show

$$P(|v_{D_{N+s}}| \le 1/8 \text{ for } a.a.s) \to 1 \text{ as } N \to \infty \tag{3.92}$$

But from (3.91) we have $|t-s| \le |D_t - D_s| \max_{(D_s \le r \le D_t)}(1/X_r(1))$ so that $|v_{D_r}| \le X_{D_r}(g)$ for $a.a.r.$ $Q^m-a.s.$ From Lemma 3.7 $X_{D_r}(g) \to 0$ as $r \to \infty$ and (3.92) follows. □

3.4 Recovering the spatial motion

How much can you tell about the underlying spatial motion from a single path of a superprocess? In the following result we use an arbitrarily short piece of the path but recover only partial information.

Lemma 3.9 *Let X_t be a superprocess started at $m \in M_F(E)$ with spatial motion a Feller process with generator A . For $f \in D(A)$ satisfying $m(|f|) = 0$ and*

$$Var(X_t(f)) = O(t^2) \text{ as t} \to 0 \tag{3.93}$$

there is a sequence $t_j \downarrow 0$ such that

$$(1/n)\sum_{j=1}^{n} t_j^{-1} X_{t_j}(f) \xrightarrow{L^2} m(Af). \tag{3.94}$$

PROOF. Set $Z_t = (1/t)X_t(f)$.Then

$$\begin{aligned} E(Z_t) &= (1/t)(m(f) + \int_0^t E(X_s(Af))ds \\ &= (1/t)\int_0^t mT_s(Af)ds \to m(Af) \text{ as } t \to 0 \end{aligned} \tag{3.95}$$

Hypothesis (3.93) ensures that Var(Z_t) remains bounded so it remains only to show that we can pick $t_j \downarrow 0$ fast enough that the Z_{t_j}'s are nearly uncorrelated. Using a product moment formula (Dynkin [6] Theorem 1.1) we have for $s < t$

$$\begin{aligned} Cov(Z_s, Z_t) &= (1/st) \int_E dm \int_0^s T_r(T_{t-r}(f) T_{s-r}(f)) dr \\ &\leq (1/t) \|f\| m T_s(|f|). \end{aligned}$$

Note that $mT_s(|f|) \to m(|f|) = 0$ as $s \to 0$. Now pick $t_0 > 0$ arbitrarily small and t_n inductively so that

$$Cov(Z_{t_n}, Z_{t_m}) \leq 2^{-|m-n|}. \tag{3.96}$$

Then

$$\begin{aligned} & E(((1/n)\sum_{j=1}^{n} Z_{t_j} - m(Af))^2) \\ = \; & (1/n^2) \sum_{j=1}^{n} E((Z_{t_j} - m(Af))^2) + (1/n^2) \sum_{i \neq j}^{n} Cov(Z_{t_i}, Z_{t_j}) \\ & + (1/n^2) \sum_{i \neq j}^{n} E(Z_{t_i} - m(Af)) E(Z_{t_j} - m(Af)) \end{aligned}$$

(3.93),(3.95),(3.96) ensure that all three terms go to zero. □

Remark. Since

$$(1/t^2) Var(X_t(f)) = (1/t^2) \int_0^t m T_r(T_{t-r}^2(f)) dr \leq (1/t) m T_t(f^2)$$

a sufficient condition for hypothesis (3.93) to hold is that $f^2 \in D(A)$ for then

$$(1/t) m T_t(f^2) = (1/t) \int_0^t m T_r(A(f^2)) dr \leq \|A(f^2)\| m(1).$$

If this condition holds then we may take $t_n = 2^{-n}$ in (3.94).

As an example we take the underlying motion to be a pure jump Levy process on the line $(Y_t : t \geq 0)$. Hence

$$E(\exp(-i\theta Y_t)) = \exp(t \int_{\mathrm{R}} (e^{i\theta x} - 1 - (i\theta x/(1+x^2))) \mu(dx))$$

where μ, the Levy measure , gives finite mass to $(-a, a)^c$ for any $a > 0$. We show that from any initial segment of a path of the super Levy process X_t started at δ_0 we can recover the Levy measure μ. Fix $a > 0$ and let $f = \mathrm{I}(a, \infty)$ Then although $f \notin D(A)$ it can be shown that

$$(T_t f(0) - f(0))/t = P(Y_t \in (a, \infty))/t \to \mu(a, \infty) \text{ as } t \to 0. \tag{3.97}$$

Find t_0 so that $P(|Y_s| > a/2) \leq 2s\mu([-a/2,a/2]^c)$ for all $s \leq t_0$.

$$\begin{aligned}
Var(X_t(f)) &= \int_0^t T_r(T_{t-r}^2 f)(0)dr \\
&\leq \int_0^t P(|Y_r| \geq a/2) + (P(Y_{t-r} \in (a/2,\infty))^2 dr \\
&\leq 2\mu([-a/2,a/2]^c)\int_0^t (r + (t-r)^2)dr = O(t^2) \qquad (3.98)
\end{aligned}$$

The proof of Lemma 3.9 shows that the bounds (3.97),(3.98) together will imply

$$S_n := (1/n)\sum^n 2^{-j}X_{2^{-j}}(f) \xrightarrow{L^2} \mu(a,\infty)$$

and along a subsequence $(n_k)_k$ therefore $S_{n_k} \to \mu(a,\infty)$ almost surely . From a countable number of intervals all bounded away from zero we can recover the entire measure μ in this way.

Corollary 3.10 *Suppose A_1, A_2 are generators of two conservative Feller processes on E and that there exists $f \in C(E)$ satisfying $f, f^2 \in D(A_1) \cap D(A_2)$ and $A_1 f(x) \neq A_2 f(x)$ for some $x \in E$. Let P_i be the law of the superprocess with spatial motion generated by A_i and started at δ_x .Then P_1 and P_2 are singular.*

PROOF. For conservative Feller processes we have $A_i 1 = 0$ so that replacing f by $f - f(x) \in D(A_i)$ we may assume that $f(x) = 0$. The proof of Lemma 3.9 and the following remark shows that we can find an explicit subsequence $(n_k)_k$ such that P_1 lives on the measurable subset

$$\Omega_1 = \{\omega : (1/n_k)\sum_{j=1}^{n_k} 2^j \omega_{2^{-j}}(f) \to A_1 f(x) \text{ as k} \to \infty\}$$

Since $(1/n_k)\sum^{n_k} 2^j\omega_{2^{-j}}(f)$ will have a subsequence that converges almost surely to $A_2 f(x)$, under P_2 we have $P_2(\Omega_1) = 0.\Box$

We return to the example of section 3.3 to show that it may not be possible to recover the entire underlying motion.

Let $E = \{a,b\}, \Omega = D([0,\infty), M_F(E)), X_t(\omega) = \omega(t), \mathcal{F} = \sigma(X_s : s \geq 0)$. Let $P_r^{(a_0,b_0)}$ be the law on Ω of the superprocess with spatial motion a Markov chain on E with generator $Af(a) = -Af(b) = rf(b) - f(a)$ and started at $a_0\delta_a + b_0\delta_b$ .Thus the 'particles' jump from a to b at rate one and from b to a at rate r.

Proposition 3.11 *For $r_1, r_2 > 0$ there exist $a_0, b_0 > 0$ such that $P_{r_1}^{(a_0,b_0)}$ and $P_{r_2}^{(a_0,b_0)}$ are not singular measures.*

PROOF. We write $X_t(a), X_t(b)$ for $X_t(\{a\}), X_t(\{b\})$. Define

$$T = \begin{cases} \inf(0 \leq t < \xi : X_t(a)/X_t(1) \leq 1/2) & \text{if this set is non-empty} \\ +\infty & \text{otherwise} \end{cases}$$

We will find $M > 0$ so that if $\Omega_0 = \{T = +\infty, \xi \leq M, \sup_{t \leq \xi} X_t(1) \leq M\}$ then $P_{r_1}^{(a_0,b_0)}(\Omega_0) > 0$ and then give an explicit Radon-Nikodym derivative for

$$dP_{r_2}^{(a_0,b_0)}|_{\Omega_0} / dP_{r_1}^{(a_0,b_0)}|_{\Omega_0}$$

From the characterisation of the law of the death point (equation (3.76)) we have $P_{r_1}^{(1,1)}(F = a) > 0$. So

$$P_{r_1}^{(1,1)}(X_t(a)/X_t(1) \to 1) > 0.$$

Pick n so that if $\xi_n = \inf(t \geq 0 : X_t(1) \leq 1/n)$ then

$$P_{r_1}^{(1,1)}(X_t(a)/X_t(1) > 1/2 \text{ for all } \xi_n \leq t \leq \xi) > 0.$$

Then by the strong markov property if $F(x,y)$ is the distribution function of the pair $(X_{\xi_n}(a), X_{\xi_n}(b))$ under $P_{r_1}^{(1,1)}$ then

$$0 < \int_0^\infty \int_0^\infty P_{r_1}^{(x,y)}(T = +\infty) dFx, y.$$

So we may pick a_0, b_0 so that $P_{r_1}^{(a_0,b_0)}(T = +\infty) > 0$ and since $\xi < \infty$ and $\sup_{t \leq \xi} X_t(1) < \infty$ almost surely we may find M so that $P_{r_1}^{(a_0,b_0)}(\Omega_0) > 0$.

Let $R = T \wedge \inf(t \geq 0 : X_t(1) > M) \wedge \inf(t \geq M : X_t(1) > 0)$ where we let $\inf(\emptyset) = \infty$. Under $P_{r_1}^{(a_0,b_0)}$

$$\begin{aligned} M_t^R(a) &= X_{t\wedge R}(a) - a_0 - \int_0^{t\wedge R} (r_1 X_s(b) - X_s(a)) ds \\ M_t^R(b) &= X_{t\wedge R}(b) - b_0 - \int_0^{t\wedge R} (X_s(a) - r_1 X_s(b)) ds \end{aligned}$$

are martingales satisfying $\langle M^R(a)\rangle_t = \int_0^{t\wedge R} X_s(a) ds$, $\langle X^R(b)\rangle_t = \int_0^{t\wedge R} X_s(b) ds$ and $\langle M^R(a), M^R(b)\rangle_t = 0$. Define

$$\begin{aligned} Z_t &= \int_0^t (r_2 - r_1) X_s(b)/X_s(a) dM_s^R(a) + \int_0^t (r_1 - r_2) dM_s^R(b) \\ \eta_t &= \mathcal{E}(Z_t) = \exp(Z_t - (1/2)\langle Z\rangle_t) \end{aligned}$$

Then

$$\langle Z\rangle_t = \int_0^{t\wedge R} (r_1 - r_2)^2 ((X_s^2(b)/X_s(a)) + X_s(b)) ds.$$

For $t \leq T$, $X_s(b)/X_s(a) \leq 1$ so that $\langle Z \rangle_t \leq M^2(r_2 - r_1)^2$ for all t . This ensures that η_t is a uniformly integrable martingale (see Elliot [7] Theorem 13.27) . Define Q by $dQ/dP_{r_1}^{(a_0,b_0)} = \eta_\infty$ Then (see Rogers and Williams [20] Theorem IV.38.4) under Q

$$\begin{aligned}
M_t^R(a) - \langle M^R(a), Z \rangle_t &= X_{t\wedge R}(a) - a_0 - \int_0^{t\wedge R} (r_2 X_s(b) - X_s(a))ds \\
M_t^R(b) - \langle M^R(b), Z \rangle_t &= X_{t\wedge R}(b) - b_0 - \int_0^{t\wedge R} (X_s(a) - r_2 X_s(b))ds
\end{aligned}$$

are martingales with the same brackets processes as $M_t^R(a), M_t^R(b)$. This characterises the law of Q on $\mathcal{F}_R$ which must agree with $P_{r_2}^{(a_0,b_0)}$. But $\Omega_0 = \{R = \infty\}$ so if $A \subseteq \Omega_0$ then $A = A \cap \{R = \infty\} \in \mathcal{F}_R$. Thus $Q|_{\Omega_0} = P_{r_2}^{(a_0,b_0)}|_{\Omega_0}$.□

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