In this thesis we study the orbifold Chow ring of smooth Deligne-Mumford stacks which are related to toric models. We give a new quotient construction of toric Deligne-Mumford stacks defined by Borisov-Chen-Smith such that toric Deligne-Mumford stacks have more representations as quotient stacks. We define toric stack bundles using this new construction and compute their orbifold Chow rings. As an interesting application, we compute the orbifold Chow ring of finite abelian gerbes over smooth schemes.

The extended stacky fans we introduced are used to give a new quotient construction of toric Deligne-Mumford stacks. These new combinatorial data have relations to stacky hyperplane arrangements, i.e. every stacky hyperplane arrangement determines an extended stacky fan. The hyperplane arrangement determines the topology of the associated hypertoric varieties. We define hypertoric Deligne-Mumford stacks using stacky hyperplane arrangements, generalizing the construction of Hausel and Sturmfels. Their orbifold Chow rings are computed as well.

Borisov, Chen and Smith computed the orbifold Chow ring of projective toric Deligne-Mumford stacks. We generalize their formula to semi-projective toric Deligne-Mumford stacks. The hypertoric Deligne-Mumford stack is a closed sub-stack of the Lawrence toric Deligne-Mumford stack associated to the stacky hyperplane arrangement which is semi-projective, but not projective. We prove that the orbifold Chow ring of a Lawrence toric Deligne-Mumford stack is isomorphic to the orbifold Chow ring of its associated hypertoric Deligne-Mumford stack. This is the orbifold Chow ring analogue of a result of Hausel and Sturmfels.
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The University of British Columbia
April 2007
Chapter 3 and Chapter 4 are joint works with Hsian-Hua Tseng. In Chapter 3, we found the research topic through discussions about the orbifold Chow rings. I had prepared most of the manuscripts including writing and typing.

The contents in Chapter 4 came from my idea to generalize the orbifold Chow ring formula to semi-projective toric Deligne-Mumford stacks. I had talked with Hsian-Hua Tseng about the proof details. We worked together about this project eventually.

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April 2007
Chapter 1

The Orbifold Chow Ring and
Toric Deligne-Mumford Stacks

1.1 Introduction and Motivation

In 1980's Dixon, Harvey, Vafa and Witten [DHVW1], [DHVW2] studied the string theory on orbifolds. Unlike the orbifolds which are generally singular, the string theory on orbifolds are smooth.

The physics results motivate the study the stringy properties on orbifolds in mathematics. In 1980's McKay studied the simple quotient singularities such as the quotients $X = \mathbb{C}^2/G$ for which $G$ is a finite group of $A, D, E$ type. For these quotient singularities, there exist unique crepant resolutions $Y$. McKay proved that the Euler and Hodge numbers of the crepant resolutions are equal to the orbifold Euler and Hodge numbers of the orbifold $X$ defined in physics, which is called McKay correspondence, see [Mckay].

In 2001 Ruan [R] gave a new conjecture called "Cohomological Hyperkahler Resolution Conjecture" (CHRC) based on the notion of orbifold Chow
ring (or Chen-Ruan orbifold cohomology) defined by Chen and Ruan [CR1], which states that the Chow ring of a hyperkahler resolution of an orbifold is isomorphic to the orbifold Chow ring of the orbifold. This conjecture generalized the McKay correspondence to the ring structure level. There are many works proving or trying to prove this conjecture, see [FG],[Uribe],[BCS].

The CHRC conjecture involved the computation of orbifold Chow ring which is also interesting in its own right. The orbifold Chow ring structure is the classical part of the orbifold Gromov-Witten theory developed by Chen and Ruan [CR2] for arbitrary orbifolds in symplectic category and Abramovich, Graber and Vistoli [AGV1],[AGV2] in algebraic category. The most interesting feature of this new cohomology theory is the obstruction bundle in the definition of the orbifold cup product. In the abelian orbifold case, the obstruction bundle is classified in [BCS],[CH] and [Jiang1]. In the general case, it is determined in [JKK]. In this thesis we do the abelian case.

Simplicial toric varieties provided good examples for the abelian orbifolds. Generalizing the quotient construction of Cox [Cox] for simplicial toric varieties, Borisov, Chen and Smith [BCS] defined toric Deligne-Mumford stacks using stacky fans and computed their orbifold Chow ring. However, their definition of stacky fan is a little restrictive. In this thesis we generalize their definition and define extended stacky fans. Using this new construction, in Chapter 2 we define toric Deligne-Mumford stacks using extended stacky fans so that toric Deligne-Mumford stacks have more representations as quotient stacks. We define toric stack bundles and compute their orbifold Chow ring.

The other outline of the thesis is as follows. In Chapter 3 we study the relation between stacky hyperplane arrangements and extended stacky fans. We
use stacky hyperplane arrangements to define hypertoric Deligne-Mumford stacks and study their orbifold Chow ring. In Chapter 4 we compute the orbifold Chow ring of semi-projective toric Deligne-Mumford stacks and use it to study the relation of orbifold Chow rings between Lawrence toric Deligne-Mumford stacks and their associated hypertoric Deligne-Mumford stacks. In Chapter 5 we talk about the relations among chapters and list some future studies. In the current Chapter we introduce the basic definition of orbifold Chow ring and review the construction of toric Deligne-Mumford stacks by Borisov-Chen-Smith.

This chapter is outlined as follow. In Section 1.2 we introduce the basic notion and properties of smooth Deligne-Mumford stacks. In Section 1.3 we define the orbifold Chow group of smooth Deligne-Mumford stacks and in Section 1.4 we define the orbifold cup product by studying the degree zero, genus zero orbifold Gromov-Witten invariants. In Section 1.5 we discuss the Gale duality of \( \beta : \mathbb{Z}^n \rightarrow N \) for any finitely generated abelian group \( N \). In Section 1.6 we define toric Deligne-Mumford stack and talk about some properties. Finally in Section 1.7 we study finite abelian gerbes over toric Deligne-Mumford stacks.

Convention

We consider rational coefficient in Chow rings and orbifold Chow rings in this thesis.

1.2 The Deligne-Mumford Stacks

Throughout this paper, let \( \mathcal{X} \) be a projective smooth Deligne-Mumford stack over the complex numbers \( \mathbb{C} \) with projective coarse moduli space \( X \).

In this section, we give the definition of smooth Deligne-Mumford stacks and discuss some general properties of \( \mathcal{X} \) and fix notations throughout. Let \( S \) be a
scheme, and let $\text{Sch}/S$ denote the category of schemes over $S$.

**Definition 1.2.1** A groupoid over $S$ is a category $\mathcal{X}$ together with a functor $p_\mathcal{X}: \mathcal{X} \to \text{Sch}/S$ such that:

(i) If $f: X \to Y$ is an arrow in $\text{Sch}/S$ and $\eta$ is an object of $\mathcal{X}$ with $p_\mathcal{X}(\eta) = Y$, then there exists an arrow $\varphi: \xi \to \eta$ in $\mathcal{X}$ such that $p_\mathcal{X}(\varphi) = f$;

(ii) If $\varphi: \xi \to \zeta$ and $\psi: \eta \to \zeta$ are arrows in $\mathcal{X}$, and $h: p_\mathcal{X}(\xi) \to p_\mathcal{X}(\eta)$ is such that $p_\mathcal{X}(\psi) \circ h = p_\mathcal{X}(\varphi)$, then there is a unique arrow $\chi: \xi \to \eta$ such that $\psi \circ \chi = \varphi$ and $p_\mathcal{X}(\chi) = h$.

Consider the groupoid $\mathcal{X}$, for any scheme $X$, let $\mathcal{X}(X)$ be the category of schemes over $X$.

**Definition 1.2.2** A groupoid $\mathcal{X}$ over $S$ is a stack if:

(i) For any $X$ in $\text{Sch}/S$ and any two objects $\xi_1$ and $\xi_2$ in $\mathcal{X}(X)$, the functor

$$Isom_\mathcal{X}(\xi_1, \xi_2): \text{Sch}/X \to \text{Sets},$$

which associates to a morphism $f: Y \to X$ the set of isomorphisms in $\mathcal{X}(Y)$ between $f^*\xi_1$ and $f^*\xi_2$ is a sheaf in the etale topology;

(ii) Let $\{X_i \to X\}$ be a covering of $X \in \text{Sch}/S$ in the etale topology. Let $\xi_i \in \mathcal{X}(X_i)$, and let

$$\varphi_{ij}: \xi_j|_{X_i \times X_j} \to \xi_i|_{X_i \times X_j}$$

be isomorphisms in $\mathcal{X}(X_i \times X_j)$ satisfying the cocycle condition. Then there is $\xi \in \mathcal{X}(X)$ with isomorphisms $\psi_i: \xi|_{X_i} \to \xi_i$, such that

$$\varphi_{ij} = (\psi_j|_{X_i \times X_j}) \circ (\psi_j|_{X_i \times X_j})^{-1}.$$  

**Example** If $G \to S$ is an affine group scheme, the groupoid $BG$ classifying isomorphism classes of principal $G$-bundles over schemes is a stack.
Example  Let $M$ be a scheme and $G$ an algebraic group acting on $M$. The category $\mathcal{X} = [M/G]$ which fibred isomorphism classes of principal $G$-bundles over scheme $X \in \text{Sch}/S$ together with a map to $M$ is a groupoid, see the following diagram

$$
\begin{array}{ccc}
E & \xrightarrow{f} & M \\
\downarrow \pi & & \downarrow \\
\mathcal{X} & & 
\end{array}
$$

We check from Definition 1.2.2 that $\mathcal{X}$ is a stack which we call a quotient stack.

The geometry of a stack of the form $[M/G]$ with $M$ a scheme and $G$ an algebraic group is essentially equivalent to the equivariant geometry of $M$ with respect to the $G$-action.

In this thesis all the stacks we consider are of this form for which $G$ is abelian, keeping this interpretation in mind may help the readers unfamiliar with stacks understand this notion. One may also think of a stack as some space whose points can have nontrivial automorphism groups. A point with nontrivial automorphism group is called a stacky point. This point of view helps one understand the notion of morphisms between stacks.

Remark  A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks can be thought of as a map between spaces, together with group homomorphisms $f_x : \text{Aut}(x) \rightarrow \text{Aut}(f(x)), \forall x \in \mathcal{X}$ between automorphism groups. A morphism $f$ is called representable if $f_x$ is injective for any $x \in \mathcal{X}$.

Definition 1.2.3 A stack $\mathcal{X}$ is a smooth Deligne-Mumford stack if:

(i) The diagonal $\Delta$ is representable, quasicompact and separated;

(ii) There is a scheme $U$ and an etale surjective morphism $U \rightarrow \mathcal{X}$. Such a morphism $U \rightarrow \mathcal{X}$ is called an atlas.
Remark In the above two examples, the stack $BG$ is Deligne-Mumford if $G$ is finite. And the stack $[M/G]$ is Deligne-Mumford if the $G$-action has only finite stabilizers.

To a Deligne-Mumford stack $X$ we can associate a coarse moduli space $X$ which is in general an algebraic space [L-MB]. We will often assume that $X$ is a projective scheme. This is related to the second interpretation above: a stack $X$ can be thought of as an additional structure on $X$ describing how $X$ locally looks like a quotient. Moreover, for a morphism $X' \rightarrow Y$ of stacks, there is an induced morphism $X \rightarrow Y$ between their coarse moduli spaces. This may be interpreted as forgetting the homomorphisms between automorphism groups. For comprehensive introductions to rigorous foundation of stacks the reader may consult [L-MB] and the Appendix of [V]. A very detailed treatment of the theory of algebraic stacks can be found in the forthcoming book [BEFFGK].

We now introduce the inertia stack associated to a stack $X'$, which plays a central role in the orbifold Chow ring of smooth Deligne-Mumford stacks.

Definition 1.2.4 Let $X$ be a smooth Deligne-Mumford stack. The inertia stack $IX$ associated to $X$ is defined to be the fiber product $IX$ in the following cartesian diagram:

$$
\begin{array}{ccc}
IX & \longrightarrow & X \\
\downarrow & & \downarrow \Delta \\
X' & \stackrel{\Delta}{\longrightarrow} & X \times X,
\end{array}
$$

where $\Delta: X \rightarrow X \times X$ is the diagonal morphism.

Remark 1. The objects in the category underlying $IX$ can be described as
follows:

\[ Ob(I\mathcal{X}) = \{(x,g) | x \in Ob(\mathcal{X}), g \in Aut(x)\} = \{(x,H,g) | x \in Ob(\mathcal{X}), H \subset Aut(x), g \text{ a generator of } H\}\]

2. For a stack \( \mathcal{X} \), \( I\mathcal{X} \) is isomorphic to the stack of representable morphisms from a constant cyclotomic gerbe to \( \mathcal{X} \),

\[ I\mathcal{X} \cong \coprod_{r \in \mathbb{N}} HomRep(B\mu_r, \mathcal{X}). \]

3. There is a natural projection \( q : I\mathcal{X} \to \mathcal{X} \). On objects we have \( q((x,g)) = x \).

The inertia stack \( I\mathcal{X} \) is in general not connected, but it is always smooth.

We write

\[ I\mathcal{X} = \coprod_{(g) \in T} \mathcal{X}_{(g)} \]

for the decomposition of \( I\mathcal{X} \) into a disjoint union of connected components. Here \( T \) is an index set which represents the conjugate classes of the local group of the stack \( \mathcal{X} \), i.e. \((g_1) \) and \((g_2) \) are equivalent if the local group \( H_2 \) is a subgroup of the local group \( H_1 \). Among all components there is a distinguished one (indexed by \( 0 \in T \)). \( \mathcal{X}_0 = \mathcal{X} \).

There is a natural involution \( I : I\mathcal{X} \to I\mathcal{X} \) defined by \( I((x,g)) = (x,g^{-1}) \).

**Example** Let \( \mathcal{X} \) be of the form \([M/G]\) with \( M \) a smooth variety and \( G \) a finite group. Then the index set \( T \) is the set \( \{(g) | g \in G\} \) of conjugacy classes of \( G \). In this case the centralizer \( C(g) \) acts on the locus \( M^g \) of \( g \)-fixed points and \( m \in M \) corresponds to \((m,g) \). We have \( \mathcal{X}_{(g)} = [M^g/C(g)] \) and the distinguished component is \([M/C(id)] = [M/G]\). The morphism \( I_{(g)} \) is an isomorphism between \( \mathcal{X}_{(g)} \) and \( \mathcal{X}_{(g^{-1})} \). If \( G \) is abelian, then

\[ I\mathcal{X} = \coprod_{g \in G} [M^g/G]. \]
1.3 The Orbifold Chow Groups

In this section we define the notion of orbifold cohomology group. From [CR1] (see also [AGV1], [AGV2]), for each component $X^g$ of $I\mathcal{X}$, the age $age(X^g)$ is defined as follows: Let $(x, g) \in X^g$. Let $r_g$ be the order the $g$ as an element in the local group. The tangent space $T_{x} \mathcal{X}$ is decomposed into a direct sum $\bigoplus_{0 \leq l < r_g} V_l$ of eigenspaces according to the $g$-action, where $V_l$ is the eigenspace with eigenvalue $\zeta^l_{r_g}$, $0 \leq l < r_g$, and $\zeta_{r_g} = \exp \left( \frac{2\pi i \sqrt{-1}}{r_g} \right)$. The age is defined to be

$$age(X^g) := \frac{1}{r_g} \sum_{0 \leq l < r_g} l \cdot dim \mathcal{C} V_l.$$

It is easy to see that this definition is independent of choices of $(x, g) \in X^g$.

We have the following Proposition from the definition of the age.

**Proposition 1.3.1** ([CR1], [AGV1])

$$age(X^g) + age(X^g_{-1}) = dim \mathcal{C} \mathcal{X} - dim \mathcal{C} X^g.$$

\[\square\]

**Definition 1.3.2** ([CR1],[AGV1]) The additive orbifold Chow group of the Deligne-Mumford stack $\mathcal{X}$ is defined by

$$A^*_{orb}(\mathcal{X}) := \bigoplus_{(g) \in T} A^{*-age(X^g)}(X^g).$$

**Remark** 1. In general, the Chow group of a stack can be defined as the Chow group of a geometric realization of the simplicial scheme associated to this stack. For our purpose we define the Chow group of a Deligne-Mumford stack as the Chow group of its coarse moduli space. These two definitions are equivalent if the coefficients are without torsion;
2. The ages on the components of the inertia stack give a grading on the orbifold Chow group. The notion of age comes from physics called "fermion number" or called "degree shifting number" in the sense of Chen and Ruan.

*Orbifold Poincare pairing*: According to [CR1], Section 3.3, the orbifold Poincaré pairing

\[(a, b)_{\text{orb}} : A^*_{\text{orb}}(\mathcal{X}) \times A^*_{\text{orb}}(\mathcal{X}) \rightarrow \mathbb{Q}\]

is defined as follows: For \(a \in A^*(\mathcal{X}(g))\), \(b \in A^*(\mathcal{X}(g^{-1}))\), define

\[(a, b)_{\text{orb}} := \int_{\mathcal{X}(g)} a \wedge I^* b.\]

The orbifold Poincaré pairing pairs cohomology classes from a component \(\mathcal{X}(g)\) with classes from the isomorphic component \(\mathcal{X}(g^{-1})\). The fact that it is a non-degenerate pairing follows from the fact that the usual Poincaré pairing on \(A^*(\mathcal{X}(g))\) is non-degenerate.

### 1.4 Orbifold Cup Product

The orbifold cup product is defined by the genus zero, degree zero orbifold Gromov-Witten invariant from which we can define the obstruction bundle. First we recall the basic degree zero orbifold Gromov-Witten theory.

**Definition 1.4.1** An \(n\)-pointed orbifold nodal curve \((\mathcal{C}, \Sigma_1, \ldots, \Sigma_n)\) is a diagram

\[
\bigcup_{i=1}^{n} \Sigma_i \longrightarrow \mathcal{C} \\
\downarrow \pi \\
\mathcal{C},
\]

where

1. \(\mathcal{C}\) is a proper Deligne-Mumford stack with coarse moduli scheme \(\mathcal{C}\);
(2) \((\mathcal{C}, \pi(\Sigma_1), \cdots, \pi(\Sigma_n))\) is an n-pointed nodal curve;

(3) Over the node \(xy = 0\) of \(\mathcal{C}\), \(\mathcal{C}\) has an etale chart

\[
[xy = 0/\mu_r]
\]

where the action is given by \((x, y) \mapsto (\xi x, \xi^{-1} y)\);

(4) Over a marked point \(\pi(\Sigma_i)\) of \(\mathcal{C}\), \(\mathcal{C}\) has etale chart

\[
[\mathcal{C}^1/\mu_r]
\]

where the action is given by \(u \mapsto \xi u\) and \(\Sigma_i\) is the substack defined by \(u = 0\).

After appropriately defining n-pointed twisted nodal curves (and morphisms of n-pointed twisted nodal curves) over an arbitrary base scheme, we have the following theorem.

**Theorem 1.4.2 ([Co]).** The category of n-pointed twisted nodal curves \(\mathcal{M}_{g,n}^{orb}\) is a smooth Artin stack of dimension \(3g - 3 + n\).

Let \(\mathcal{X}\) be a Deligne-Mumford stack with projective coarse moduli scheme \(X\).

**Definition 1.4.3** An n-pointed twisted stable map \((\mathcal{C}, \Sigma_1, \cdots, \Sigma_n)\) is a diagram

\[
\begin{array}{ccc}
\cup_{i=1}^n \Sigma_i & \xrightarrow{f} & \mathcal{C} \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{C} & \xrightarrow{\bar{f}} & \mathcal{X}
\end{array}
\]

where

(1) \((\mathcal{C}, \Sigma_1, \cdots, \Sigma_n)\) is an n-pointed orbifold nodal curve;

(2) \(f\) is representable and the induced map on coarse moduli spaces \(\bar{f}\) is stable.
Let $N_1(X)$ be the group of numerical equivalence classes of curves in $X$, and let $N^+(X) := N_1^+(X)$ be the monoid of effective classes. Then for $\beta \in N_1(X)$ and for $g$ a non-negative integer, one says that $(C \overset{f}{\to} X, \Sigma_1, \cdots, \Sigma_n)$ has degree $\beta$ and genus $g$ if the stable map $f$ does.

After appropriately defining n-pointed orbifold stable maps (and morphisms of n-pointed orbifold stable maps) over an arbitrary base scheme, we have the following theorem.

**Theorem 1.4.4 ([AGV1]).** The category of n-pointed orbifold stable maps of genus $g$ and degree $\beta$, $\overline{M}_{g,n}(X, \beta)$, is a proper Deligne-Mumford stack. The coarse moduli space $\overline{M}_{g,n}(X, \beta)$ of $\overline{M}_{g,n}(X, \beta)$ is projective.

To define the orbifold cup product, we are only interested in the moduli stack $\overline{M}_{0,3}(X, 0)$ of 3-pointed orbifold stable maps of genus zero and degree zero. There is a natural result in [CR1] and [AGV1].

**Proposition 1.4.5 ([CR1],[AGV1])** The moduli stack $\overline{M}_{0,3}(X, 0)$ is isomorphic to the double inertia stack $I_2X$ of $X$.

**Remark** The objects in the category underlying $I_2X$ can be described as follows:

$\text{Ob}(I_2X) = \{(x, (g_1, g_2)) \mid x \in \text{Ob}(X), g_1, g_2 \in \text{Aut}(x)\}$

$= \{(x, (g_1, g_2, g_3)) \mid x \in \text{Ob}(X), g_1, g_2 \in \text{Aut}(x), g_3 = (g_1g_2)^{-1}\}$,

where $(g_1, g_2)$ or $(g_1, g_2, g_3)$ represent the conjugacy classes of the 2 or 3 tuples.

Like the inertia stack, the double inertia stack $I_2X$ is in general not connected as well. We write

$$I_2X = \coprod_{(g_1, g_2, g_3) \in T^2} X_{(g_1, g_2, g_3)}$$
for the decomposition of $I_2 \mathcal{X}$ into a disjoint union of connected components. Here $T^2$ is an index set which represents the conjugate classes of the conjugate classes triples $(g_1, g_2, g_3)$ in the local group.

There are evaluation maps

$$e_i : I_2 \mathcal{X} \to I \mathcal{X}$$

defined by

$$(x, (g_1, g_2, g_3)) \mapsto (x, (g_i))$$

for $1 \leq i \leq 3$.

Consider the following diagram of universal maps

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{X} \\
\downarrow \pi & & \downarrow \\
\overline{M}_{0,3}(\mathcal{X}, 0),
\end{array}
$$

(1.1)

where $\mathcal{C}$ is the universal curve and $f$ is the universal map.

**Definition 1.4.6** ([CR1],[AGV1]) The obstruction bundle $E \to \overline{M}_{0,3}(\mathcal{X}, 0)$ over the double inertia stack is defined by

$$E := R^1 \pi_* f^* T_{\mathcal{X}}.$$ 

**Remark** If the universal curve $\mathcal{C} = [D/H]$ is a global quotient, we can describe $E$ in a more concrete way: write $\bar{\pi} : D \to \overline{M}_{0,3}(\mathcal{X}, 0)$ and $\bar{f} : D \to \mathcal{X}$ for the composite map, then $E = (R^1 \bar{\pi}_* \bar{f}^* T_{\mathcal{X}})^H$ is the $H$ invariant sub bundle of the usual obstruction bundle of $\bar{f}$. This is useful in figuring out some examples.

We classify the obstruction bundle for abelian Deligne-Mumford stacks. Let $\mathcal{X}$ be an abelian Deligne-Mumford stack, i.e. all the local groups are abelian.
$g = (g_1, g_2, g_3)$ and $X_{(g)}$ be a component in the double inertia stack $I_2\mathcal{X}$. Let $E_{(g)}$ be the obstruction over it defined in 1.4.6. Let $e : X_{(g)} \to \mathcal{X}$ be the embedding. Then $T_{\mathcal{X}|X_{(g)}} = T_{X_{(g)}} \oplus N(X_{(g)}/\mathcal{X})$, where $N(X_{(g)}/\mathcal{X})$ is the normal bundle of $X_{(g)}$ in $\mathcal{X}$. Let $H$ be the group generated by $g_1, g_2, g_3$, then $C = [D/H]$, where $D$ is smooth Riemann surface. The group $H$ acts on the normal bundle $N(X_{(g)}/\mathcal{X})$. Since $H$ is abelian, all the irreducible representations are one dimensional, so $N(X_{(g)}/\mathcal{X})$ can be represented as direct sum of line bundles based on the representations of $H$. We write $N(X_{(g)}/\mathcal{X}) = \oplus_{i=1}^{m} L_i$. So from the above Remark, we have

$$E_{(g)} = \left( R^1\pi_* j^* T_{\mathcal{X}} \right)^H = \left( H^1(D, \mathcal{O}_D) \otimes T_{\mathcal{X}|X_{(g)}} \right)^H = \left( H^1(D, \mathcal{O}_D) \otimes N(X_{(g)}/\mathcal{X}) \right)^H = \left( \bigoplus_{i=1}^{m} H^1(D, \mathcal{O}_D) \otimes L_i \right)^H.$$

Let $a(g_1) + a(g_2) + a(g_3) = \sum_{i=1}^{m} \alpha_i$, then $\alpha_i = 1$ or 2 since $g_1g_2g_3 = 1$. Then we have the following proposition due to [CH].

**Proposition 1.4.7** We have that $(H^1(D, \mathcal{O}_D) \otimes L_i)^H \cong L_i$ if $\alpha_i = 2$ and $(H^1(D, \mathcal{O}_D) \otimes L_i)^H = 0$ if $\alpha_i = 1$. So

$$E_{(g)} \cong \bigoplus_{\alpha_i=2} L_i.$$

\[\square\]

**Proposition 1.4.8** ([CR1]) Over the component $X_{(g_1, g_2, g_3)}$, the dimension of the obstruction bundle $E_{(g_1, g_2, g_3)}$ is given by

$$dim_{\mathbb{C}} E_{(g_1, g_2, g_3)} = dim_{\mathbb{C}} X_{(g_1, g_2, g_3)} + \sum_{i=1}^{3} age(X_{(g_i)}) - dim_{\mathbb{C}} \mathcal{X}. \quad (1.2)$$

\[\square\]
Definition 1.4.9 Let $\hat{e}_3 = I \circ e_3 : I_2X \to I\mathcal{X}$ be the composite map. For $\alpha, \beta \in A^*_{orb}(\mathcal{X})$, the orbifold cup product is defined by

$$\alpha \cup_{orb} \beta := \hat{e}_3^*(e_1^*\alpha \cup e_2^*\beta \cup e(E)),$$

where $e(E)$ is the Euler class of the obstruction bundle.

Theorem 1.4.10 ([CR1],[AGV1]) Under the cup product in Definition 1.4.9, the graded Chow group $A^*_{orb}(\mathcal{X})$ is a skew-symmetric associative ring. □

In the following special case, we can compare the orbifold cup product with the ordinary cup product of $A^*_{orb}(\mathcal{X})$. Let $q : I\mathcal{X} \to \mathcal{X}$ be the natural map defined by $(x, (g)) \mapsto x$.

Proposition 1.4.11 For $\alpha \in A^*(\mathcal{X})$ and $\beta \in A^*(\mathcal{X}_g)$, the orbifold cup product $\alpha \cup_{orb} \beta$ is equal to the ordinary product $q^*\alpha \cup \beta$ in $A^*_{orb}(\mathcal{X})$.

PROOF. Using the identification of $\overline{M}_{0,3}(\mathcal{X}, 0)$ with the double inertia stack, the component $\mathcal{X}_{(1, g, g^{-1})}$ is isomorphic to the component $\mathcal{X}_g$ and $\mathcal{X}_{(g^{-1})}$ in the inertia stack. So the evaluation map $e_1 = q$ and $e_2 = id$ on the component $\mathcal{X}_g$. From the definition of $\hat{e}_3$, we can take the map

$$\hat{e}_3 : \mathcal{X}_{(1, g, g^{-1})} \to \mathcal{X}_g$$

as the identity on $\mathcal{X}_g$. From the formula (1.2), the obstruction bundle $E_{(1, g, g^{-1})}$ has dimension zero. So from the definition of the orbifold cup product in Definition 1.4.9,

$$\alpha \cup_{orb} \beta = q^*\alpha \cup \beta.$$
**Example** We consider the weighted projective stack

\[ \mathcal{X} = \mathbb{P}(4,6) := [(A^2 \setminus \{0\})/\mathbb{G}_m] \]

with the action \((x; y) \mapsto (t^4 x; t^6 y)\). This is the moduli stack \(\overline{M}_{1,1}\) of 1-marked elliptic curves. We have a description of the inertia stack:

\[ I(\mathcal{X}) = \mathcal{X} \cup_{i=1}^2 B_{\mu_4} \cup_{i=1}^4 B_{\mu_6}. \]

The usual cohomology of \(\mathcal{X}\) is

\[ A^*(\mathcal{X}) = \mathbb{Q}[t]/(t^2). \]

Let \(A\) be the generator of the cohomology of \(B_{\mu_4}\) and \(B\) the generator of the cohomology of \(B_{\mu_6}\). The age of \(A\) is \(\frac{1}{2}\), and that of \(B\) is \(\frac{1}{3}\). We can present the orbifold Chow ring as

\[ A^*_{orb}(\mathcal{X}) = \mathbb{Q}[X, A, B, T]/(X^2 - 1, AT, BT, A^2 - XT, B^3 - XT, AB). \]

### 1.5 Gale Duality for Finitely Generated Abelian Groups

We review the basic form of Gale duality and its application to toric geometry as in [BCS]. Given \(n\) vectors \(b_1, \ldots, b_n\) which span \(\mathbb{Q}^d\), there is a dual configuration \([a_1, \ldots, a_n] \in \mathbb{Q}^{(n-d) \times n}\) such that

\[ 0 \longrightarrow \mathbb{Q}^d [b_1, \ldots, b_n]^T \longrightarrow \mathbb{Q}^n [a_1, \ldots, a_n] \longrightarrow \mathbb{Q}^{n-d} \longrightarrow 0 \quad (1.3) \]

is a short exact sequence; see Theorem 6.14 in [Zagier]. The set of vectors \(a_1, \ldots, a_n\) in \(\mathbb{Q}^{n-d}\) is uniquely determined up to a linear coordinates transformation in \(\mathbb{Q}^{n-d}\). This duality plays a role in study of smooth toric varieties. Let \(\Sigma\) be a fan with \(n\) rays such that the corresponding toric variety \(X(\Sigma)\) is smooth. Let \(N \cong \mathbb{Z}^d\) is the lattice

15
in $\Sigma$, then the minimal lattice points $b_1, \cdots, b_n$ generating the rays determine a map $\beta : \mathbb{Z}^n \to N$. By tensoring with $\mathbb{Q}$, we obtain a dual configuration $a_1, \cdots, a_n$. Since $X(\Sigma)$ is smooth, we have $a_i \in \mathbb{Z}^{n-d}$ and the set $a_1, \cdots, a_n$ is unique up to unimodular (determinant $\pm1$) coordinate transformations of $\mathbb{Z}^{n-d}$. Abbreviating $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$ by $(\cdot)^*$, it follows that the set $a_1, \cdots, a_n$ defines a map

$$\beta^* : (\mathbb{Z}^n)^* \to \mathbb{Z}^{n-d} = \text{Pic}(X)$$

and the short exact sequence (1.3) becomes

$$0 \to N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^*} \text{Pic}(X) \to 0;$$

see Section 3.4 in [F].

The idea of the notion of toric Deligne-Mumford stack is to generalize the above Gale duality of $\beta : \mathbb{Z}^n \to N$ for free abelian group $N$ to finitely generated abelian group $N$. The construction of toric Deligne-Mumford stack under this generalization gives rise to finite abelian gerbe structures over the underlying toric orbifolds coming from the torsion part of the finitely generated abelian group $N$.

Let $N$ be a finitely generated abelian group with rank $d$. Let

$$\beta : \mathbb{Z}^n \to N$$

be a map determined by $n$ integral vectors $\{b_1, \cdots, b_n\}$ in $N$. Taking $\mathbb{Z}^n$ and $N$ as $\mathbb{Z}$-modules, from the homological algebra, there exist projective resolutions $\hat{E}$ and $\hat{F}$ of $\mathbb{Z}^n$ and $N$ satisfying the following diagram

$$\begin{array}{ccc}
\hat{E} & \to & \mathbb{Z}^n \\
\downarrow & & \downarrow \beta \\
\hat{F} & \to & N.
\end{array}$$
Let $\text{Cone}(\beta)$ be the mapping cone of the map between $\hat{E}$ and $\hat{F}$. Then we have an exact sequence of the mapping cone:

$$0 \rightarrow \hat{F} \rightarrow \text{Cone}(\beta) \rightarrow \hat{E}[1] \rightarrow 0,$$

where $\hat{E}[1]$ is the shifting of $\hat{E}$ by 1. Since $\hat{E}$ is projective as $\mathbb{Z}$-modules, so we have the exact sequence by taking the $\text{Hom}(-, \mathbb{Z})$ functor

$$0 \rightarrow \hat{E}[1]^* \rightarrow \text{Cone}(\beta)^* \rightarrow \hat{F}^* \rightarrow 0.$$

Taking cohomology of the above sequence we get the exact sequence:

$$N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} H^1(\text{Cone}(\beta)^*) \xrightarrow{} \text{Ext}^1(\mathbb{Z}, \mathbb{Z}) \rightarrow 0. \quad (1.4)$$

**Definition 1.5.1** Let $DG(\beta) = H^1(\text{Cone}(\beta)^*)$. The map

$$\beta^\vee : (\mathbb{Z}^n)^* \rightarrow DG(\beta)$$

is called the Gale dual of the map $\beta$.

From [BCS], both $DG(\beta)$ and $\beta^\vee$ are well defined up to natural isomorphism.

Actually we can make this construction more clear. Since $N$ has rank $d$ and $\mathbb{Z}^n$ is a free $\mathbb{Z}$-module, the projection resolutions can be chosen as:

$$0 \rightarrow \mathbb{Z}^n \rightarrow 0 = \hat{E},$$

$$0 \rightarrow \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \rightarrow 0 = \hat{F},$$

where $Q$ is an integer matrix. Then there is a map from $\mathbb{Z}^n$ to $\mathbb{Z}^{d+r}$ defined by a matrix $B$ which gives the map between $\hat{E}$ and $\hat{F}$. The mapping cone $\text{Cone}(\beta)$ is given by the following complex:

$$0 \rightarrow \mathbb{Z}^{n+r} \xrightarrow{[B | Q]} \mathbb{Z}^{d+r} \rightarrow 0 = \text{Cone}(\beta).$$
Then we apply the snake lemma to the following diagram to get the sequence (1.4).

\[ \begin{array}{cccccc}
0 & \longrightarrow & 0 & \longrightarrow & (\mathbb{Z}^{d+r})^* & \longrightarrow & (\mathbb{Z}^{d+r})^* & \longrightarrow & 0 \\
& & \downarrow & & \downarrow[\mathbb{B},\mathbb{Q}]^* & & \downarrow & \\
0 & \longrightarrow & (\mathbb{Z}^n)^* & \longrightarrow & (\mathbb{Z}^{n+r})^* & \longrightarrow & (\mathbb{Z}^r)^*(\sigma) & \longrightarrow & 0.
\end{array} \] (1.5)

Then \( DG(\beta) = (\mathbb{Z}^{n+r})^*/\text{Im}([\mathbb{B},\mathbb{Q}]^*) \) and \( \beta^\vee \) is the composite map of the inclusion \( (\mathbb{Z}^n)^* \hookrightarrow (\mathbb{Z}^{n+r})^* \) and the quotient map \( (\mathbb{Z}^{n+r})^* \longrightarrow (\mathbb{Z}^{n+r})^*/\text{Im}([\mathbb{B},\mathbb{Q}]^*) \).

**Remark** If \( N \) is free, i.e. there is no torsion part in the group \( N \). Then from (1.5), the Gale dual \( \beta^\vee \) is the quotient map \( (\mathbb{Z}^n)^* \longrightarrow (\mathbb{Z}^n)^*/\text{Im}([\mathbb{B}]^*) \) and we have an exact sequence

\[ N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} H^1(\text{Cone}(\beta)^*) \longrightarrow 0. \]

Next we give two propositions in [BCS] for later use.

**Proposition 1.5.2** Let \( \beta : \mathbb{Z}^n \longrightarrow N \) be a map to a finitely generated abelian group \( N \). Then \( \beta^\vee \cong \beta \) is if and only if the cokernel of \( \beta \) is finite. Moreover in this case, \( \ker(\beta^\vee) = N^* \). □

**Proposition 1.5.3** Given a commutative diagram

\[ \begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^{n_1} & \longrightarrow & \mathbb{Z}^{n_2} & \longrightarrow & \mathbb{Z}^{n_3} & \longrightarrow & 0 \\
& & \downarrow\beta_1 & & \downarrow\beta_2 & & \downarrow\beta_3 & \\
0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0,
\end{array} \]

in which the rows are exact and the columns have finite cokernels, then there is a commutative diagram with exact rows:

\[ \begin{array}{cccccc}
0 & \longrightarrow & (\mathbb{Z}^{n_3})^* & \longrightarrow & (\mathbb{Z}^{n_2})^* & \longrightarrow & (\mathbb{Z}^{n_1})^* & \longrightarrow & 0 \\
& & \downarrow\beta_3^\vee & & \downarrow\beta_2^\vee & & \downarrow\beta_1^\vee & \\
0 & \longrightarrow & DG(\beta_3) & \longrightarrow & DG(\beta_2) & \longrightarrow & DG(\beta_1) & \longrightarrow & 0.
\end{array} \]
1.6 Toric Deligne-Mumford Stacks

In this section we introduce toric Deligne-Mumford stacks in the sense of Borisov-Chen-Smith. Let $N$ be a finitely generated abelian group. Let $N \longrightarrow \overline{N}$ be the natural map modulo torsion. Then $\overline{N}$ is a lattice. Let $\Sigma$ be a simplicial fan in the lattice $\overline{N}$ with $n$ rays $\{\rho_1, \cdots, \rho_n\}$. Choose $n$ integer vectors $\{b_1, \cdots, b_n\}$ such that $b_i$ generates the ray $\rho_i$ for $1 \leq i \leq n$. Then we have a map $\beta : \mathbb{Z}^n \longrightarrow N$ determined by the vectors $\{b_1, \cdots, b_n\}$. We require that $\beta$ has finite cokernel.

**Definition 1.6.1 ([BCS])** The triple $\Sigma := (N, \Sigma, \beta)$ is called a stacky fan.

We define toric Deligne-Mumford stack from a stacky fan $\sigma$. Since $\beta$ has finite cokernel, then from Proposition 1.5.2 and 1.5.3, we have the following exact sequences:

$$
0 \longrightarrow DG(\beta)^* \xrightarrow{\beta^*} \mathbb{Z}^n \xrightarrow{\beta} N \longrightarrow \text{Coker}(\beta) \longrightarrow 0, \quad (1.6)
$$

$$
0 \longrightarrow N^* \longrightarrow \mathbb{Z}^n \xrightarrow{\beta^*} DG(\beta) \longrightarrow \text{Coker}(\beta^*) \longrightarrow 0. \quad (1.7)
$$

Since $\mathbb{C}^\times$ is a divisible as a $\mathbb{Z}$-module, taking $\text{Hom}_\mathbb{Z}(\cdot, \mathbb{C}^\times)$ to the exact sequence (1.7) we get:

$$
1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (\mathbb{C}^\times)^n \longrightarrow T \longrightarrow 1, \quad (1.8)
$$

where $\mu = \text{Hom}_\mathbb{Z}(\text{Coker}(\beta^*), \mathbb{C}^\times)$ is finite, $G = \text{Hom}_\mathbb{Z}(DG(\beta), \mathbb{C}^\times)$ and $T$ is the $d$ dimensional torus $(\mathbb{C}^\times)^d$.

Let $\mathbb{C}[z_1, \cdots, z_n]$ be the coordinate ring of the affine variety $\mathbb{A}^n$. Associated to the simplicial fan $\Sigma$, there is an irrelevant ideal $J_\Sigma$ generated by the elements:

$$
\left\langle \prod_{\rho_i \in \sigma} z_i : \sigma \in \Sigma \right\rangle. \quad (1.9)
$$

Let $Z := \mathbb{A}^n \setminus V(J_\Sigma)$. Then $Z$ is a quasi-affine variety. The torus $(\mathbb{C}^\times)^n$ acts on $Z$ naturally since $Z$ is a subvariety of $\mathbb{A}^n$. The algebraic group $G$ acts on the variety $Z$.
through the map $\alpha$ in the exact sequence (1.8). Then we have a translation groupoid $Z \times G \rightrightarrows Z$. The associated stack is the quotient stack $[Z/G]$ which is the fibre category of principle $G$-bundles.

**Definition 1.6.2** ([BCS]) The toric Deligne-Mumford stack $X(\Sigma)$ associated to the stacky fan $\Sigma$ is defined to be the quotient stack $[Z/G]$.

**Proposition 1.6.3** ([BCS]) The coarse moduli space of the toric Deligne-Mumford stack $X(\Sigma)$ is the toric variety $X(\Sigma)$ associated to the simplicial fan $\Sigma$. □

In [BCS], the authors proved that the morphism $Z \times G \rightarrow Z \times Z$ defined by the projection and action is a finite morphism so that the quotient stack $[Z/G]$ is Deligne-Mumford. We give a new explanation here. From the exact sequence (1.8), let $\overline{G} = \text{Im}(\alpha)$, then we have an exact sequence

$$1 \rightarrow \mu \rightarrow G \rightarrow \overline{G} \rightarrow 1.$$ 

Since all the groups are abelian, this sequence is a central extension. Then from [DP], the quotient stack $[Z/G]$ is the $\mu$-gerbe over the quotient stack $[Z/\overline{G}]$ determined by this central extension. Let $X_{\text{orb}}(\Sigma)$ be the underlying toric orbifold associated to the simplicial fan $\Sigma$ and $\{\bar{b}_1, \ldots, \bar{b}_n\}$ in $\overline{N}$.

**Remark** From Proposition 4.6 in [BN], any Deligne-Mumford stack is a $\mu$-gerbe over an orbifold for a finite group $\mu$. Our results are the toric case of that general result.

**Proposition 1.6.4** $X_{\text{orb}}(\Sigma)$ is isomorphic to the quotient stack $[Z/\overline{G}]$.

**Proof.** From the remark before Proposition 1.5.2, we have the following exact sequences

$$0 \rightarrow DG(\overline{\beta}) \rightarrow \mathbb{Z}^n \xrightarrow{\overline{\beta}} \overline{N} \rightarrow Cok(\overline{\beta}) \rightarrow 0;$$
0 \to N^* \to (\mathbb{Z}^n)^* \xrightarrow{\beta'} DG(\overline{\beta}) \to 0;

which are the two exact sequences on page 19 in [F]. So \( A_{d-1}(X(\Sigma)) = DG(\overline{\beta}) \) and from the construction of Cox [Cox], \( X_{\text{orb}}(\Sigma) = [\mathbb{Z}/\mathcal{G}] \). □

**Remark** Since the toric Deligne-Mumford stack \( X(\Sigma) \) is a \( \mu \)-gerbe over the toric orbifold \( X_{\text{orb}}(\Sigma) \) which is Deligne-Mumford and separated, from the standard stack theory (see [L-MB]), the stack \( X(\Sigma) \) is Deligne-Mumford.

### 1.7 Finite Abelian Gerbes over Toric Deligne-Mumford Stacks

In the last section we know that any toric Deligne-Mumford stack is a finite abelian gerbe over the toric orbifold. In this section we talk about when a finite abelian gerbe over toric Deligne-Mumford stack is again a toric Deligne-Mumford stack.

Given a toric Deligne-Mumford stack \( X(\Sigma) \) with stacky fan \( \Sigma \). Let \( \nu \) be a finite abelian group, and let \( \mathcal{G} \) be a \( \nu \)-gerbe over \( X(\Sigma) \). We give a sufficient condition so that \( \mathcal{G} \) is also a toric Deligne-Mumford stack. We have the following theorem:

**Theorem 1.7.1** Let \( X(\Sigma) \) be a toric Deligne-Mumford stack with stacky fan \( \Sigma \). If for any rays \( \rho_1, \rho_2, \rho_3 \), there exists a cone \( \sigma \in \Sigma \) such that \( \rho_1, \rho_2, \rho_3 \subseteq \sigma \), then every \( \nu \)-gerbe \( \mathcal{G} \) over \( X(\Sigma) \) is induced by a central extension

\[
1 \to \nu \to \widetilde{\mathcal{G}} \to \mathcal{G} \to 1,
\]
i.e., we have a Cartesian diagram:

\[
\begin{array}{c}
\mathcal{G} \\
\downarrow \\
\mathcal{X}(\Sigma) \\
\downarrow \\
BG.
\end{array}
\]

In general, the \( \nu \)-gerbe \( \mathcal{G} \) is not a toric Deligne-Mumford stack. But if the central extension is abelian, then we have:

\textbf{Corollary 1.7.2} If the \( \nu \)-gerbe \( \mathcal{G} \) is induced from an abelian central extension, it is a toric Deligne-Mumford stack.

\textbf{1.7.1 The proof of main results}

Consider the ideal in (1.9), let \( V = V(J_\Sigma) \). From [Cox], the codimension of \( V \) in \( \mathbb{C}^n \) is at least 2.

\textbf{Lemma 1.7.3} In the simplicial fan \( \Sigma \), the following two conditions are equivalent:

(a) \( \text{Codim}(V, \mathbb{C}^n) > 3 \);

(b) For any rays \( \rho_1, \rho_2, \rho_3 \), there exists a cone \( \sigma \in \Sigma \) such that \( \rho_1, \rho_2, \rho_3 \subseteq \sigma \).

\textbf{Proof.} (a) \( \Rightarrow \) (b): We prove the following claim: If there exist \( \rho_1, \rho_2, \rho_3 \) such that for any \( \sigma \in \Sigma \), there is some \( \rho_i \notin \sigma \), then \( \text{Codim}(V, \mathbb{C}^n) \leq 3 \). Since

\[ V = \left( \bigcup_{\rho_i \notin \sigma_1} V(z_1) \right) \cap \left( \bigcup_{\rho_i \notin \sigma_2} V(z_2) \right) \cap \cdots \]

It is easy to see that \( V(z_1) \cap V(z_2) \cap V(z_3) \subseteq V \). From the condition of the claim, \( \text{dim}(V(z_1) \cap V(z_2) \cap V(z_3)) \geq n - 3 \), so \( \text{dim}(V) \geq n - 3 \).

(b) \( \Rightarrow \) (a): Suppose not, then \( \text{dim}(V) \geq n - 3 \). So there exist \( \rho_1, \cdots, \rho_r \) such that \( r \leq 3 \), and \( \rho_i \notin \sigma_i \). This means that there exist \( \rho_1, \rho_2, \rho_3 \) such that for any \( \sigma \in \Sigma \), there is some \( \rho_i \notin \sigma \). A contradiction to (b). \( \square \)
Lemma 1.7.4 If \( \text{Codim}(V, \mathbb{C}^n) > 3 \), then \( H^1(Z, \nu) = H^2(Z, \nu) = 0 \), where \( \nu \) is the sheaf of abelian groups on \( \mathbb{C}^n \) with etale topology.

Proof. Consider the following exact sequence:

\[
0 \rightarrow H^0_V(\mathbb{C}^n, \nu) \rightarrow H^0(\mathbb{C}^n, \nu) \rightarrow H^0(Z, \nu) \rightarrow \]
\[
\rightarrow H^1_V(\mathbb{C}^n, \nu) \rightarrow H^1(\mathbb{C}^n, \nu) \rightarrow H^1(Z, \nu) \rightarrow \]
\[
\rightarrow H^2_V(\mathbb{C}^n, \nu) \rightarrow \cdots
\]

Since \( \text{Codim}(V, \mathbb{C}^n) > 3 \), \( H^i_V(\mathbb{C}^n, \nu) = 0 \) for \( i = 1, 2, 3 \), so from the exact sequence and \( H^i(\mathbb{C}^n, \nu) = 0 \) for all \( i > 0 \) we prove the lemma. \( \square \)

The proof of Theorem 1.7.1

Consider the following diagram

\[
\begin{array}{ccc}
Z & \rightarrow & \text{pt} \\
\downarrow & & \downarrow \\
[Z/G] & \rightarrow & BG
\end{array}
\]

which is Cartesian. Consider the Leray spectral sequence for the fibration \( \pi \):

\[
H^p(BG, R^q\pi_*\nu) \Rightarrow H^{p+q}([Z/G], \nu)
\]

So \( R^q\pi_*\nu = \text{pt} \times_G H^q(Z, \nu) = [H^q(Z, \nu)/G] \). When \( p = 2, q = 0 \), \( R^0\pi_*\nu = \nu \) because \( Z \) is connected, \( H^p(BG, R^q\pi_*\nu) = H^2(BG, \nu) \); When \( p = 1, q = 1 \), \( R^1\pi_*\nu = [H^1(Z, \nu)/G] \), so \( H^p(BG, R^q\pi_*\nu) = H^1(BG, H^1(Z, \nu)) \), but \( \text{Codim}(V, \mathbb{C}^n) > 3 \), from Lemma 2.2, \( H^1(Z, \nu) = 0 \), we have \( H^p(BG, R^q\pi_*\nu) = 0 \); When \( p = 0, q = 2 \), \( R^2\pi_*\nu = [H^2(Z, \nu)/G] \), so \( H^p(BG, R^q\pi_*\nu) = H^0(BG, H^2(Z, \nu)) \), also from Lemma 2.2, \( H^2(Z, \nu) = 0 \), we have \( H^p(BG, R^q\pi_*\nu) = 0 \). So we get

\[
H^2([Z/G], \nu) \cong H^2(BG, \nu)
\]
Since for the finite abelian group \( \nu \), the \( \nu \)-gerbes are classified by the second cohomology group with coefficient in the group \( \nu \). Theorem 2.4.1 is proved. 

The proof of Corollary 1.7.2

Let \( \mathcal{X}(\Sigma) = [Z/G] \). The \( \nu \)-gerbe \( \mathcal{G} \) over \( [Z/G] \) is induced from a \( \nu \)-gerbe \( B\tilde{G} \) over \( BG \) in the following central extension

\[
1 \to \nu \to \tilde{G} \to G \to 1,
\]

where \( \tilde{G} \) is an abelian group. So the pullback gerbe over \( Z \) under the map \( Z \to [Z/G] \) is trivial. Using the groupoid representation of the stack \( [Z/G] \). The \( \nu \)-gerbe over \( [Z/G] \) determines an extension

\[
\begin{array}{c}
Z \times \tilde{G} \\
\downarrow \alpha \downarrow \pi \downarrow \alpha \downarrow \pi
\end{array}
\]

where \( \tilde{G} \) is the central extension of \( G \) by \( \nu \). The stack \( [Z/\tilde{G}] \) is this \( \nu \)-gerbe \( \mathcal{G} \) over \( [Z/G] \). Consider the commutative diagram:

\[
\begin{array}{c}
\tilde{G} \\
\downarrow \alpha \downarrow \pi
\end{array}
\begin{array}{c}
G \\
\downarrow a
\end{array}
\]

where \( \alpha \) is the map in (2.3). From (1.10), we have \( ker(\alpha) \cong ker(\varphi) \otimes ker(\alpha) \). So we have the exact sequence like (2.3):

\[
1 \to \nu \otimes \mu \to \tilde{G} \xrightarrow{\tilde{\alpha}} (\mathbb{C}^\times)^n \to T \to 1
\]

where \( T \) is the torus of the simplicial toric variety \( X(\Sigma) \). Since the abelian groups \( \tilde{G}, G \) and \( (\mathbb{C}^\times)^n \) are all locally compact topological groups, taking Pontryagin duality
and Gale dual, we have the following diagrams:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & N^* & \longrightarrow & \mathbb{Z}^n & \overset{\beta^\vee}{\longrightarrow} & DG(\beta) & \longrightarrow & \text{Coker}(\beta^\vee) & \longrightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow p_\varphi & & & & \downarrow & \\
0 & \longrightarrow & \tilde{N}^* & \longrightarrow & \mathbb{Z}^n & \overset{(\tilde{\beta})^\vee}{\longrightarrow} & DG(\tilde{\beta}) & \longrightarrow & \text{Coker}((\tilde{\beta})^\vee) & \longrightarrow & 0,
\end{array}
\]

where \( p_\varphi \) is induced by \( \varphi \) in (1.10) under the Pontryagin duality. Suppose \( \tilde{\beta} : \mathbb{Z}^n \rightarrow \tilde{N} \) is given by \( \{\bar{b}_1, \cdots, \bar{b}_n\} \), then \( \tilde{\Sigma} := (\tilde{N}, \Sigma, \tilde{\beta}) \) is a new stacky fan. The toric Deligne-Mumford stack \( X(\tilde{\Sigma}) = [\mathbb{Z}/G] \) is the \( \nu \)-gerbe \( G \) over \( X(\Sigma) \).

**Remark** From the proof of Corollary 1.2, if a \( \nu \)-gerbe over \( X(\Sigma) \) comes from a gerbe over \( BG \) and the central extension is abelian, then we can construct a new toric Deligne-Mumford stack, see the example below.

### 1.7.2 An example

**Example** Let \( \Sigma \) be the complete fan of the projective line, \( N = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \), and \( \beta : \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) be given by the vectors \( \{b_1, b_2\} = \{(1,0), (-1,1)\} \). Then \( \Sigma = (N, \Sigma, \beta) \) is a stacky fan. We compute that \( (\beta)^\vee : \mathbb{Z}^2 \rightarrow DG(\beta) = \mathbb{Z} \) is given by the matrix \([3,3]\). So we get the following exact sequence:

\[
1 \longrightarrow \mu_3 \longrightarrow \mathbb{C}^\times \overset{[3,3]^\vee}{\longrightarrow} (\mathbb{C}^\times)^2 \longrightarrow \mathbb{C}^\times \longrightarrow 1 \tag{1.11}
\]

The toric Deligne-Mumford stack \( X(\Sigma) = [\mathbb{C}^2 - \{0\}/\mathbb{C}^\times] \), where the action is given by \( \lambda(x,y) = (\lambda^3 x, \lambda^3 y) \). So \( X(\Sigma) \) is the nontrivial \( \mu_3 \)-gerbe over \( \mathbb{P}^1 \) coming from
the canonical line bundle over $\mathbb{P}^1$. Let $\mathcal{G} \rightarrow \mathcal{X}(\Sigma)$ be a $\mu_2$-gerbe such that it comes from the $\mu_2$-gerbe over $\mathbb{C}^\times$ given by the central extension

$$1 \rightarrow \mu_2 \rightarrow \mathbb{C}^\times \xrightarrow{(\lambda^2)} \mathbb{C}^\times \rightarrow 1.$$  

(1.12)

From the sequence (1.11) and (1.12), we have:

$$1 \rightarrow \mu_3 \otimes \mu_2 \rightarrow \mathbb{C}^\times \xrightarrow{[6,6]^t} (\mathbb{C}^\times)^2 \rightarrow \mathbb{C}^\times \rightarrow 1.$$  

The Pontryagin dual of $\mathbb{C}^\times \xrightarrow{[6,6]^t} (\mathbb{C}^\times)^2$ is $(\beta)^\vee : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ which is given by the matrix $[6,6]$. Taking Gale dual we have:

$$\tilde{\beta} : \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_6,$$

which is given by the vectors $\{\tilde{b}_1 = (1,0), \tilde{b}_2 = (-1,1)\}$. Let $\tilde{\Sigma} = (\tilde{N}, \Sigma, \tilde{\beta})$ be a new stacky fan, then we have the toric Deligne-Mumford stack $\mathcal{X}(\tilde{\Sigma}) = [\mathbb{C}^2 - \{0\}/\mathbb{C}^\times]$, where the action is given by $\lambda(x, y) = (\lambda^6 x, \lambda^6 y)$. So $\mathcal{X}(\tilde{\Sigma})$ is the canonical $\mu_6$-gerbe over $\mathbb{P}^1$.

If the $\mu_2$-gerbe over $\mathbb{C}^\times$ is given by the central extension

$$1 \rightarrow \mu_2 \rightarrow \mathbb{C}^\times \times \mu_2 \xrightarrow{\alpha} \mathbb{C}^\times \rightarrow 1,$$

(1.13)

where $\alpha$ is given by the matrix $[3,0]$. Then from (1.11) and (1.13), we have

$$1 \rightarrow \mu_3 \otimes \mu_2 \rightarrow \mathbb{C}^\times \times \mu_2 \xrightarrow{\varphi} (\mathbb{C}^\times)^2 \rightarrow \mathbb{C}^\times \rightarrow 1,$$

where $\varphi$ is given by the matrix $\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$. The Pontryagin dual of $\varphi$ is: $(\beta)^\vee : \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_2$ which is given by the inverse of the above matrix. Taking Gale dual we get

$$\tilde{\beta} : \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2,$$
which is given by the vectors \( \tilde{b}_1 = (1, 0, 0), \tilde{b}_2 = (-1, 1, 0) \). So \( \tilde{\Sigma}' = (\tilde{N}', \Sigma, \tilde{\beta}') \) is a stacky fan. And \( \mathcal{X}(\tilde{\Sigma}') = [\mathbb{C}^2 - \{0\}/\mathbb{C}^* \times \mu_2] \), where the action is \((\lambda_1, \lambda_2) \cdot (x, y) = (\lambda_1^2 x, \lambda_2^2 y)\). So \( g' = \mathcal{X}(\tilde{\Sigma}') \) is the trivial \( \mu_2 \)-gerbe over \( \mathcal{X}(\Sigma) \). And \( \mathcal{X}(\tilde{\Sigma}) \not\cong \mathcal{X}(\tilde{\Sigma}') \).
Bibliography


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Chapter 2

Simplicial Toric Stack Bundles

2.1 Introduction

In this chapter we explicitly compute the orbifold Chow ring of toric stack bundles. These are bundles over a smooth base variety $B$ whose fibers are toric Deligne-Mumford stacks of Chapter 1.

From [Cox], to a simplicial fan $\Sigma$ with $n$ rays, one can associate a simplicial toric variety $X(\Sigma)$ expressed as a quotient $Z/G$, where $Z$ is an open subset of $\mathbb{C}^n$ and $G$ is a subgroup of $(\mathbb{C}^\times)^n$. Let $T := (\mathbb{C}^\times)^n/G$ be the torus acting on $X(\Sigma)$. Given a principle $T$-bundle $E \to B$, one can form a fibre bundle $^{E}X(\Sigma) := E \times_T X(\Sigma) \to B$ over $B$ with fibers isomorphic to $X(\Sigma)$. The cohomology ring of $^{E}X(\Sigma)$ was computed in [SU].

Generalizing Cox's construction, Borisov, Chen and Smith [BCS] constructed toric Deligne-Mumford stacks reviewed in Chapter 1. Let $P \to B$ be a principal

\footnote{The content of this chapter has been accepted by Illinois Journal of Mathematics for publication.}
\((\mathbb{C}^*)^n\)-bundle over a smooth variety \(B\). The group \(G\) acts on the fibre product \(P \times_{(\mathbb{C}^*)^n} Z\) via the map \(\alpha\). Define \(P X(\Sigma)\) to be the quotient stack \([(P \times_{(\mathbb{C}^*)^n} Z)/G]\) which we write as \(P \times_{(\mathbb{C}^*)^n} X(\Sigma)\). The fibre bundle \(P X(\Sigma) \to B\) is called a toric stack bundle over \(B\) whose fibre is the toric Deligne-Mumford stack \(X(\Sigma)\).

Both Cox and Borisov-Chen-Smith's construction used the minimal presentation of a toric variety (stack) as a quotient. One may expect that toric Deligne-Mumford stacks can be represented as a quotient stack of a larger space \(Z\) by a larger group \(G\). For example, the classifying stack \(B\mu_3 = [pt/\mu_3]\) is a toric Deligne-Mumford stack in the sense of Borisov, Chen and Smith, where the corresponding stacky fan is \((Z_3,0,0)\). The stack \(B\mu_3\) is isomorphic to the stack \([\mathbb{C}^*/\mathbb{C}^*]\), where \(\mathbb{C}^*\) acts on \(\mathbb{C}^*\) by \(\lambda x \to \lambda^3 x\). Given a line bundle \(L \to B\) over \(B\). Applying the construction above yields a \(\mu_3\)-gerbe \([\mu_3 \times_{\mathbb{C}^*} \mathbb{C}^*/\mathbb{C}^*]\) over \(B\) which is nontrivial if \(L\) is. The presentation \([pt/\mu_3]\) only produces trivial gerbes.

Motivated by the study of gerbes, the above discussion suggests that it is desirable to work with other presentations of toric Deligne-Mumford stacks. For this purpose, we introduce the notion of extended stacky fans. An extended stacky fan is a triple \(\Sigma^e := (N, \Sigma, \beta^e)\), where \(N\) and \(\Sigma\) are the same as in the stacky fan \(\Sigma\), but \(\beta^e : \mathbb{Z}^m \to N\) is determined by \(\{b_1, \ldots, b_n\}\) and additional elements \(\{b_{n+1}, \ldots, b_m\}\) in \(N\). An extended stacky fan \(\Sigma^e\) has an underlying stacky fan \(\Sigma\). Using \(\Sigma^e\) we define a quotient stack \(X(\Sigma^e) := [Z^e/G^e]\), where \(Z^e = Z \times (\mathbb{C}^*)^{m-n}\) and \(G^e\) acts on \(Z^e\) through the homomorphism \(\alpha^e : G^e \to (\mathbb{C}^*)^m\) determined by the extended stacky fan. We prove that \(X(\Sigma^e)\) is isomorphic to the toric Deligne-Mumford stack \(X(\Sigma)\). So enlarging the presentation from the minimal ones of Cox and Borisov-Chen-Smith is encoded in the extended stacky fan. For example, let \(N = \mathbb{Z}_3\), let \(\beta^e : \mathbb{Z} \to N\) be the map defined by \(b_1 = 1 \in \mathbb{Z}_3\), then \(\Sigma^e = (N, \Sigma, \beta^e)\) is
an extended stacky fan. (Note that this is not a stacky fan). The toric Deligne-Mumford stack is \( \mathcal{X}(\Sigma^e) = [\mathbb{C}^x/\mathbb{C}^x] \) which is isomorphic to \([pt/\mu_3]\).

Let \( P \to B \) be a principal \((\mathbb{C}^x)^m\)-bundle. The group \( G^e \) acts on the fibre product \( P \times_{(\mathbb{C}^x)^m} Z^e \) via the map \( \alpha^e \). The quotient stack \( P\mathcal{X}(\Sigma^e) := [(P \times_{(\mathbb{C}^x)^m} Z^e)/G^e] \) is called a toric stack bundle over \( B \) whose fibre is isomorphic to the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma^e) \).

In [BCS], Borisov, Chen and Smith computed the orbifold Chow ring of toric Deligne-Mumford stacks. The computation in the special case of weighted projective stack was pursued in [Jiang1]. In this paper we compute the orbifold cohomology ring of \( P\mathcal{X}(\Sigma^e) \). To describe the result, we introduce line bundles \( \xi_{\theta} \) for \( \theta \in M = N^* \). For \( \theta \in M \), let \( \chi^\theta : (\mathbb{C}^x)^m \to \mathbb{C}^x \) be the map induced by \( \theta \circ \beta^e : Z^m \to \mathbb{Z} \). The bundle \( \xi_{\theta} \to B \) is the line bundle \( P \times_{\chi^\theta} \mathbb{C} \). We introduce the deformed ring \( A^*(B)[N]^\Sigma^e = A^*(B) \otimes \mathbb{Q}[N]^\Sigma^e \), where \( \mathbb{Q}[N]^\Sigma^e := \bigoplus_{c \in N} \mathbb{Q}y^c \), \( y \) is a formal variable and \( A^*(B) \) is the Chow ring of \( B \). The multiplication of \( \mathbb{Q}[N]^\Sigma^e \) is given by:

\[
y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \bar{c}_1 \in \sigma, \bar{c}_2 \in \sigma, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}
\]

Let \( I(P\Sigma^e) \) be the ideal in \( A^*(B)[N]^\Sigma^e \) generated by the elements:

\[
\left( c_1(\xi_{\theta}) + \sum_{i=1}^n \theta(b_i)y^{b_i} \right)_{\theta \in M}, \tag{2.2}
\]

and \( A_{orb}^*(P\mathcal{X}(\Sigma^e)) \) the orbifold Chow ring of the toric stack bundle \( P\mathcal{X}(\Sigma^e) \).

**Theorem 2.1.1** If \( P\mathcal{X}(\Sigma^e) \to B \) is a toric stack bundle over a smooth variety \( B \) whose fibre is the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma^e) \) associated to an extended
stacky fan $\Sigma^e$, then we have an isomorphism of $\mathbb{Q}$-graded rings:

$$A^*_\text{orb}(P\mathcal{X}(\Sigma^e)) \cong \frac{A^*(B)[N]^{\Sigma^e}}{I(P\Sigma^e)}.$$ 

The extra data $\{b_{n+1}, \ldots, b_m\}$ in the extended stacky fan $\Sigma^e$ do affect the structure of $P\mathcal{X}(\Sigma^e)$, but do not affect its orbifold cohomology ring. When the variety $B$ is a point, this formula is the orbifold Chow ring formula of projective toric Deligne-Mumford stacks by Borisov-Chen-Smith.

To prove this theorem, we show that twisting by the $(\mathbb{C}^*)^m$-bundle $P$ does not "twist" the components of the inertia stack of the toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$. Thus we can describe the components of the inertia stack $I(P\mathcal{X}(\Sigma^e))$ of $P\mathcal{X}(\Sigma^e)$ using Box($\Sigma^e$), in a manner analogous to [BCS]. This makes it possible to use the similar methods as in [BCS] to determine 3-twisted sectors, obstruction bundles and compute the orbifold Chow ring of $P\mathcal{X}(\Sigma^e)$.

As an example, let $N$ be a finite abelian group and $\beta^e : \mathbb{Z} \rightarrow N$ any homomorphism, then $\Sigma^e = (N, 0, \beta^e)$ is an extended stacky fan. The toric Deligne-Mumford stack is $\mathcal{X}(\Sigma^e) = B\mu$, where $\mu = Hom(N, \mathbb{C}^*)$. Twisting this toric Deligne-Mumford stack by a line bundle $L$ over a smooth variety $B$ gives a $\mu$-gerbe $\mathcal{X}$ over $B$. So no matter the gerbe is trivial or not, our result gives that

$$H^*_\text{orb}(\mathcal{X}, \mathbb{Q}) = H^*(B, \mathbb{Q}) \otimes H^*_\text{orb}(B\mu, \mathbb{Q}).$$

This chapter is organized as follows. In Section 2.2 we introduce extended stacky fans and their associated toric Deligne-Mumford stacks. In Section 2.3 we define toric stack bundles and discuss their properties. In Section 2.4 we describe the orbifold Chow ring of toric stack bundles. In Section 2.5 we discuss the $\mu$-gerbe $\mathcal{X}$ mentioned above. Finally, in Section 2.6 we give some applications to crepant resolutions. We generalize a result of Borisov, Chen and Smith, showing that the orbifold Chow ring of a toric stack bundle and the Chow ring of its crepant resolution
can be put into a flat family.

2.2 A New Quotient Representation of Toric Deligne-Mumford stacks.

In this section we introduce extended stacky fans and construct a new representation of toric Deligne-Mumford stacks.

Let $\mathbb{N}$ be a finitely generated abelian group of rank $d$ and $\overline{\mathbb{N}}$ the lattice generated by $\mathbb{N}$ in the $d$-dimensional vector space $\mathbb{N}_Q := \mathbb{N} \otimes \mathbb{Q}$. Write $\overline{b}$ for the image of $b$ under the natural map $\mathbb{N} \rightarrow \overline{\mathbb{N}}$. Let $\Sigma$ be a rational simplicial fan in $\mathbb{N}_Q$. Suppose $\rho_1, \ldots, \rho_n$ are the rays in $\Sigma$. We fix $b_i \in \mathbb{N}$ for $1 \leq i \leq n$ such that $\overline{b_i}$ generates the cone $\rho_i$. Let $\{b_{n+1}, \ldots, b_m\} \subset \mathbb{N}$. We consider the homomorphism $\beta_e : \mathbb{Z}^m \rightarrow \mathbb{N}$ determined by the elements $\{b_1, \ldots, b_m\}$. We require that $\beta_e$ has finite cokernel.

**Definition 2.2.1** The triple $\Sigma^e := (\mathbb{N}, \Sigma, \beta_e)$ is called an extended stacky fan.

It is easy to see that any extended stacky fan $\Sigma^e = (\mathbb{N}, \Sigma, \beta_e)$ naturally determines a stacky fan $\Sigma := (\mathbb{N}, \Sigma, \beta)$, where $\beta : \mathbb{Z}^n \rightarrow \mathbb{N}$ is given by $\{b_1, \ldots, b_n\}$. Now since $\beta_e$ has finite cokernel, by Proposition 2.2 in [BCS], we have exact sequences:

$$0 \rightarrow DG(\beta_e)^* \rightarrow \mathbb{Z}^m \stackrel{\beta_e}{\rightarrow} \mathbb{N} \rightarrow \text{Coker}(\beta_e) \rightarrow 0,$$

$$0 \rightarrow N^* \rightarrow \mathbb{Z}^m(\beta_e)^\vee \rightarrow DG(\beta_e) \rightarrow \text{Coker}((\beta_e)^\vee) \rightarrow 0,$$

where $(\beta_e)^\vee$ is the Gale dual of $\beta_e$. As a $\mathbb{Z}$-module, $\mathbb{C}^x$ is divisible, so it is an injective $\mathbb{Z}$-module, and hence from [Lang], the functor $\text{Hom}_\mathbb{Z}(\cdot, \mathbb{C}^x)$ is exact. We get the exact sequence:

$$1 \rightarrow \text{Hom}_\mathbb{Z}(\text{Coker}((\beta_e)^\vee), \mathbb{C}^x) \rightarrow \text{Hom}_\mathbb{Z}(DG(\beta_e), \mathbb{C}^x) \rightarrow \text{Hom}_\mathbb{Z}(\mathbb{Z}^m, \mathbb{C}^x)$$
Let \( \mu := \text{Hom}_Z(\text{Coker}(((\beta^e)^{\mathbb{C}})), \mathbb{C}^\times) \), we have the exact sequence:

\[
1 \rightarrow \mu \rightarrow G^e \xrightarrow{\alpha^e} (\mathbb{C}^\times)^m \rightarrow T \rightarrow 1. \tag{2.3}
\]

From [BCS], the toric Deligne-Mumford stack \( X(\Sigma) = [Z/G] \) is a quotient stack, where they use the method of quotient construction of toric varieties [Cox]. Define \( Z^e := Z \times (\mathbb{C}^\times)^m \), then there exists a natural action of \((\mathbb{C}^\times)^m \) on \( Z^e \). The group \( G^e \) acts on \( Z^e \) through the map \( \alpha^e \) in (2.3). The quotient stack \([Z^e/G^e]\) is associated to the groupoid \( Z^e \times G^e \rightrightarrows Z^e \). Define the morphism \( \varphi : Z^e \times G^e \rightarrow Z^e \times Z^e \) to be \( \varphi(x, g) = (x, g \cdot x) \). Since \( Z^e = Z \times (\mathbb{C}^\times)^m \), we can mimic the proof the Lemma 3.1 in [BCS] to show that \( \varphi \) is finite which means that the stack \([Z^e/G^e]\) is a Deligne-Mumford stack.

**Lemma 2.2.2** The morphism \( \varphi : Z^e \times G^e \rightarrow Z^e \times Z^e \) is a finite morphism.

**Definition 2.2.3** For an extended stacky fan \( \Sigma^e = (N, \Sigma, \beta^e) \), we denote the quotient stack \([Z^e/G^e]\) by \( X(\Sigma^e) \).

**Proposition 2.2.4** For an extended stacky fan \( \Sigma^e = (N, \Sigma, \beta^e) \), the stack \( X(\Sigma^e) \) is isomorphic to the toric Deligne-Mumford stack \( X(\Sigma) \) associated to the underlying stacky fan \( \Sigma \).

**Proof.** From the definitions of extended stacky fan \( \Sigma^e \) and stacky fan \( \Sigma \), we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & Z^e & \longrightarrow & (\mathbb{C}^\times)^m & \longrightarrow & Z^{m-n} & \longrightarrow & 0 \\
& & \downarrow{\beta} & & \downarrow{\alpha^e} & & \downarrow{\beta^e} & & \\
0 & \longrightarrow & N & \xrightarrow{id} & N & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}
\]
From the definition of Gale dual, we compute that $DG(\tilde{\beta}) = \mathbb{Z}^{m-n}$ and $\tilde{\beta}^V$ is an isomorphism. So from Lemma 2.3 in [BCS], applying the Gale dual and the $\text{Hom}_{\mathbb{Z}}(-,\mathbb{C}^\times)$ functor to the above diagram we get:

\[
\begin{array}{cccccc}
1 & \longrightarrow & G & \xrightarrow{\varphi_1} & G^e & \longrightarrow & (\mathbb{C}^\times)^{m-n} & \longrightarrow & 1 \\
\alpha & \downarrow & \varphi & \downarrow & \alpha^e & \downarrow & \tilde{\alpha} & \\
1 & \longrightarrow & (\mathbb{C}^\times)^n & \longrightarrow & (\mathbb{C}^\times)^m & \longrightarrow & (\mathbb{C}^\times)^{m-n} & \longrightarrow & 1.
\end{array}
\tag{2.4}
\]

We define the morphism $\varphi_0 : Z \longrightarrow Z^e = Z \times (\mathbb{C}^\times)^{m-n}$ to be the inclusion defined by $z \mapsto (z,1)$. So $(\varphi_0 \times \varphi_1, \varphi_0) : (Z \times G \rightrightarrows Z) \longrightarrow (Z^e \times G^e \rightrightarrows Z^e)$ defines a morphism between groupoids. Let $\varphi : [Z/G] \longrightarrow [Z^e/G^e]$ be the morphism of stacks induced from $(\varphi_0 \times \varphi_1, \varphi_0)$. From the above commutative diagram we have the following commutative diagram:

\[
\begin{array}{ccc}
Z \times G & \xrightarrow{\varphi_0 \times \varphi_1} & Z^e \times G^e \\
\downarrow{(s,t)} & & \downarrow{(s,t)} \\
Z \times Z & \xrightarrow{\varphi_0 \times \varphi_0} & Z^e \times Z^e.
\end{array}
\]

In (2.4), $\tilde{\alpha}$ is an isomorphism, which implies that the left square in (2.4) is cartesian. So the above commutative diagram is cartesian and $\varphi : [Z/G] \longrightarrow [Z^e/G^e]$ is injective. Given an element $(z_1, \ldots, z_n, z_{n+1}, \ldots, z_m) \in Z^e$, there exists an element $g^e \in (\mathbb{C}^\times)^{m-n}$ such that $g^e \cdot (z_1, \ldots, z_n, z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n, 1, \ldots, 1)$. From (2.4), $g^e$ determines an element in $G^e$, so $\varphi$ is surjective. We conclude that the stacks $\mathcal{X}(\Sigma^e)$ and $\mathcal{X}(\Sigma)$ are isomorphic. □

**Remark** In view of Proposition 2.2.4, we call $\mathcal{X}(\Sigma^e)$ the toric Deligne-Mumford stack associated to the extended stacky fan $\Sigma^e$.

Let $X(\Sigma)$ be the simplicial toric variety associated to the simplicial fan $\Sigma$ in the extended stacky fan $\Sigma^e$. We have the following corollary:
Corollary 2.2.5 Given an extended stacky fan \( \Sigma^e \), the coarse moduli space of the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma^e) \) is also the simplicial toric variety \( X(\Sigma) \).

As in [BCS], for each top dimensional cone \( \sigma \) in \( \Sigma \), denote by \( \Box(\sigma) \) to be the set of elements \( v \in N \) such that \( v = \sum_{\rho \subseteq \tau} \alpha_i \bar{b}_i \) for some \( 0 \leq \alpha_i < 1 \). The elements in \( \Box(\sigma) \) are in one-to-one correspondence with the elements in the finite group \( N(\sigma) = N/N_\sigma \), where \( N(\sigma) \) is a local group of the stack \( \mathcal{X}(\Sigma^e) \). If \( \tau \subseteq \sigma \) is a low dimensional cone, we define \( Box(\tau) \) to be the set of elements in \( v \in N \) such that \( v = \sum_{\rho \subseteq \tau} \alpha_i \bar{b}_i \), where \( 0 \leq \alpha_i < 1 \). It is easy to see that \( Box(\tau) \subset Box(\sigma) \). In fact the elements in \( Box(\tau) \) generate a subgroup of the local group \( N(\sigma) \). Let \( Box(\Sigma^e) \) be the union of \( Box(\sigma) \) for all \( d \)-dimensional cones \( \sigma \in \Sigma \). For \( v_1, \ldots, v_n \in N \), let \( \sigma(\bar{v}_1, \ldots, \bar{v}_n) \) be the unique minimal cone in \( \Sigma \) containing \( \bar{v}_1, \ldots, \bar{v}_n \).

2.3 The Toric Stack Bundle \( P\mathcal{X}(\Sigma^e) \).

In this section we introduce the toric stack bundle \( P\mathcal{X}(\Sigma^e) \) and determine its twisted sectors. Let \( P \to B \) be a principal \((\mathbb{C}^\times)^m\)-bundle over a smooth variety \( B \). Through the map \( \alpha^e \) in (2.3), \( G^e \) acts on the fibre product \( P \times_{(\mathbb{C}^\times)^m} Z^e \).

Definition 2.3.1 We define the toric stack bundle \( P\mathcal{X}(\Sigma^e) \to B \) to be the quotient stack

\[
P\mathcal{X}(\Sigma^e) := [(P \times_{(\mathbb{C}^\times)^m} Z^e)/G^e].
\] (2.5)

Let \( \phi : Z^m \to Z^m \) be the map given by \( e_i \mapsto e_i \) for \( 1 \leq i \leq n \), and \( e_j \mapsto e_j + \sum_{i=1}^{n} a_i \bar{e}_i \) for \( n + 1 \leq j \leq m \), where \( a_i \in \mathbb{Z} \). Then from the following
we obtain a new extended stacky fan \( \tilde{\Sigma}^e = (N, \Sigma, \tilde{\beta}^e) \), where the extra data in \( \tilde{\Sigma}^e \) are \( b'_{n+1} = b_{n+1} + \sum_{i=1}^n a_i^{n+1} b_i, \ldots, b'_m = b_m + \sum_{i=1}^n a_i^m b_i \). The map \( \phi \) gives a map \( \mathbb{C}^n \times (\mathbb{C}^x)^{m-n} \to \mathbb{C}^n \times (\mathbb{C}^x)^{m-n} \) which is the identity on the first factor and given by \( \phi \) on the second factor. Since the map in the above diagram doesn't change the fan in the extended stacky fans, we have a map \( \varphi_0 : P \times (\mathbb{C}^x)^m Z^e \to P \times (\mathbb{C}^x)^m Z^e \). We may use the same argument as that in Proposition 2.3 to prove that \( P\mathcal{X}(\Sigma^e) \cong P\mathcal{X}(\tilde{\Sigma}^e) \). This means that we can always choose the extra data \( \{b_{n+1}, \ldots, b_m\} \) so that \( b_j = \sum_{i=1}^n a_i b_i \) for \( j = n + 1, \ldots, m \) and \( 0 \leq a_i < 1 \). These extra data are actually in the \( \text{Box}(\Sigma^e) \).

**Example** By above, the extra data can be chosen to lie in \( \text{Box}(\Sigma^e) \). In this example we prove that they can not be put into the torsion subgroup of \( N \). Let \( N = \mathbb{Z} \) and \( b_1 = 2, b_2 = -2 \). Then \( \Sigma = \{b_1, b_2\} \) is a simplicial fan in \( N \). Let \( \Sigma^e = (N, \Sigma, \beta^e) \), where \( \beta^e : \mathbb{Z}^3 \to \mathbb{Z} \) is determined by \( \{b_1, b_2, b_3 = 1\} \). We compute that \( DG(\beta^e) = \mathbb{Z}^2 \) and the Gale dual \( (\beta^e)^\vee : \mathbb{Z}^3 \to \mathbb{Z}^2 \) is given by the matrix \( \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \). The toric Deligne-Mumford stack is \( \mathcal{X}(\Sigma^e) = [(\mathbb{C}^2 - \{0\}) \times \mathbb{C}^x/(\mathbb{C}^x)^2] \), where the action is given by \( (\lambda_1, \lambda_2)(x, y, z) = (\lambda_1 \lambda_2^{-1} \cdot x, \lambda_1 \cdot y, \lambda_2 \cdot z) \).

We get \( \mathcal{X}(\Sigma^e) = \mathbb{P}^1 \times [\mathbb{C}^x/\mathbb{C}^x] = \mathbb{P}^1 \times \mathbb{B}_{\mu_2} \). Now let \( \tilde{\Sigma}^e = (N, \Sigma, \tilde{\beta}^e) \), where \( \tilde{\beta}^e : \mathbb{Z}^3 \to \mathbb{Z} \) is determined by \( \{b_1, b_2, \tilde{b}_3 = 0\} \), then we compute that \( DG(\beta^e) = \mathbb{Z}^2 \oplus \mathbb{Z}_2 \).
and the Gale dual $\tilde{\beta}^\vee : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_2$ is given by the matrix
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

The toric Deligne-Mumford stack is $\mathcal{X}(\tilde{\Sigma}^e) = [(\mathbb{C}^2 - \{0\}) \times \mathbb{C}^\times / (\mathbb{C}^\times)^2 \times \mu_2]$, where the action is given by $(\lambda_1, \lambda_2, \lambda_3)(x, y, z) = (\lambda_1 \cdot x, \lambda_1 \cdot y, \lambda_2 \cdot z)$. We get $\mathcal{X}(\tilde{\Sigma}^e) = [\mathbb{P}^1/\mu_2] = \mathbb{P}^1 \times B\mu_2$. Let $B = \mathbb{P}^1$ and $P = \mathbb{C}^\times \oplus \mathbb{C}^\times \oplus \mathcal{O}(1)^\times$, then $^P\mathcal{X}(\Sigma^e)$ is a nontrivial $\mu_2$-gerbe over $\mathbb{P}^1 \times \mathbb{P}^1$ coming from the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$. Let $Q = \mathcal{O}(n_1)^\times \oplus \mathcal{O}(n_2)^\times \oplus \mathcal{O}(n_3)^\times$, then $^Q\mathcal{X}(\tilde{\Sigma}^e)$ is the trivial $\mu_2$-gerbe over the $\mathbb{P}^1$-bundle $E$ over $\mathbb{P}^1$. So $^P\mathcal{X}(\Sigma^e)$ is not isomorphic to $^Q\mathcal{X}(\tilde{\Sigma}^e)$ for any $Q$.

From Corollary 2.2.5, $\mathcal{X}(\Sigma^e)$ has the coarse moduli space $\mathcal{X}(\Sigma)$ which is the simplicial toric variety associated to the simplicial fan $\Sigma$. From the exact sequence in (2.3), a $((\mathbb{C}^\times)^m)$-bundle over $B$ naturally determines a $T$-bundle over $B$. Let $E \rightarrow B$ be the principal $T$-bundle induced by $P$, then we have the twists $^P\mathcal{X}_{\text{red}}(\Sigma^e) \rightarrow B$ with fibre the toric orbifold $\mathcal{X}_{\text{red}}(\Sigma^e)$ and $^E\mathcal{X}(\Sigma) \rightarrow B$ with fibre the simplicial toric variety $\mathcal{X}(\Sigma)$, where $^P\mathcal{X}_{\text{red}}(\Sigma^e) := [(P \times (\mathbb{C}^\times)^m Z^e) / G^e]$, $^E\mathcal{X}(\Sigma) := E \times_T \mathcal{X}(\Sigma)$, and $G^e = \text{Im}(\alpha^e)$ in (2.3). We obtain the exact sequence:
\[
1 \rightarrow \mu \rightarrow G^e \rightarrow ^e\mathcal{G} \rightarrow 1. \tag{2.6}
\]

From [DP], we have:

**Proposition 2.3.2** $^P\mathcal{X}(\Sigma^e)$ is a $\mu$-gerbe over $^P\mathcal{X}_{\text{red}}(\Sigma^e)$ for a finite abelian group $\mu$.

Because any toric stack bundle is a $\mu$-gerbe over the corresponding toric orbifold bundle and can be represented as a quotient stack, we have the following propositions:
Proposition 2.3.3 The simplicial toric bundle $E X(\Sigma)$ is the coarse moduli space of the toric stack bundle $P X(\Sigma^e)$ and the toric orbifold bundle $P X_{\text{red}}(\Sigma^e)$.

**Proof.** The toric stack bundle $P X(\Sigma^e)$ is a $\mu$-gerbe over the simplicial toric orbifold bundle $P X_{\text{red}}(\Sigma^e)$ for a finite abelian group $\mu$. The stacks $P X(\Sigma^e) = [(P \times_{(C^\times)^m} Z^e)/G^e]$ and $P X_{\text{red}}(\Sigma^e) = [(P \times_{(C^\times)^m} Z^e)/G^e]$ are quotient stacks. Taking geometric quotient, we have the coarse moduli space $(P \times_{(C^\times)^m} Z^e)//G^e = (P \times Z^e)//((C^\times)^m \times G^e)$. From Corollary 2.4 $X(\Sigma) = Z//G = Z^e//G^e$, so

$$E \times_T (Z^e//G^e) = (P \times_{(C^\times)^m} T) \times_T (Z^e//G^e) = (P \times Z^e)//((C^\times)^m \times G^e).$$

From the universal geometric quotients in [KM], $E X(\Sigma)$ is the coarse moduli space of $P X(\Sigma^e)$ and $P X_{\text{red}}(\Sigma^e)$. □

Proposition 2.3.4 The toric stack bundle $P X(\Sigma^e)$ is a Deligne-Mumford stack.

**Proof.** From (2.5), $P X(\Sigma^e) = [(P \times_{(C^\times)^m} Z^e)/G^e]$ is a quotient stack, where $G^e$ acts trivially on $P$. The action of $G^e$ on $Z^e$ has finite, reduced stabilizers because the stack $[Z^e/G^e]$ is a Deligne-Mumford stack, so the action of $G^e$ on $P \times_{(C^\times)^m} Z^e$ also has finite, reduced stabilizers. From Corollary 2.2 of [Ed], $P X(\Sigma^e)$ is a Deligne-Mumford stack. □

For an extended stacky fan $\Sigma^e$, let $\sigma \in \Sigma$ be a cone, we define

$$\text{link}(\sigma) := \{\tau: \sigma + \tau \in \Sigma, \sigma \cap \tau = \emptyset\}.$$  

Let $\{\tilde{b}_1, \ldots, \tilde{b}_l\}$ be the rays in $\text{link}(\sigma)$ and $s := |\sigma|$. Then $\Sigma^e/\sigma = (N(\sigma) = N/N_{\sigma}, \Sigma/\sigma, \beta^e(\sigma))$ is an extended stacky fan, where $\beta^e(\sigma): Z^{m-s} \rightarrow N(\sigma)$ is given by the images of $b_i$ for $\rho_i$ not in $\sigma$ under $N \rightarrow N(\sigma)$, here $\{\tilde{b}_1, \ldots, \tilde{b}_i\}$ generate the quotient fan $\Sigma/\sigma$, all others are extra data. From the construction of toric
Deligne-Mumford stacks, we have $X(T, e/a) := [Z_e(a)/G_e(a)]$, where $Z_e(a) = (A_1 - V(J_{S/a}) \times (C^x)^{m-s}) = Z(a) \times (C^x)^{m-s}$. We have an action of $(C^x)^m$ on $Z_e(a)$ induced by the natural action of $(C^x)^{m-s}$ on $Z_e(a)$ and the projection $(C^x)^m \rightarrow (C^x)^{m-s}$. We consider

$$PX(S/a) = [(P \times (C^x)^{m-s} \times (C^x)^{m-s})/G_e(a)].$$

Then we have:

**Proposition 2.3.5** Let $\sigma$ be a cone in the extended stacky fan $\Sigma^e$, then $PX(\Sigma^e/\sigma)$ defines a closed substack of $PX(\Sigma^e)$.

**Proof.** Let $W_e(\sigma)$ be the closed subvariety of $Z_e$ defined by $J(\sigma) := < z_i : \rho_i \subseteq \sigma >$ in $C[z_1, \ldots, z_n, z_{n+1}^{\pm 1}, \ldots, z_m^{\pm 1}]$, then we see that $W_e(\sigma) = W(\sigma) \times (C^x)^{m-n}$, where $W(\sigma)$ is the closed subvariety of $Z$ defined by $J(\sigma) := < z_i : \rho_i \subseteq \sigma >$ in $C[z_1, \ldots, z_n]$. From [BCS], there is a map $\varphi_0 : W(\sigma) \rightarrow Z(\sigma)$ which is $(C^x)^n$-equivariant. We define the map $W_e(\sigma) \rightarrow Z_e(\sigma)$ by $\varphi_0 \times 1$. From the following diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & Z^s & \rightarrow & Z^m & \rightarrow & Z^{m-s} & \rightarrow & 0 \\
& \downarrow&\beta & & \downarrow & \beta_e & & \downarrow & \\
0 & \rightarrow & N_e & \rightarrow & N & \rightarrow & N(\sigma) & \rightarrow & 0,
\end{array}
$$

applying Gale dual yields

$$
\begin{array}{ccccccccc}
0 & \rightarrow & Z^{m-s} & \rightarrow & Z^m & \rightarrow & Z^s & \rightarrow & 0 \\
& \downarrow & (\beta_e)^\vee & & \downarrow & (\beta_e)^\vee & & \downarrow & \\
0 & \rightarrow & DG(\beta_e(\sigma)) & \rightarrow & DG(\beta_e) & \rightarrow & DG(\tilde{\beta}) & \rightarrow & 0.
\end{array}
$$

The dimension of the cone $\text{dim}(\sigma)$ is $s$ since the cone $\sigma$ is simplicial, then $N_\sigma \cong Z^s$ and $DG(\tilde{\beta}) \cong 0$. Applying $Hom_Z(-, C^x)$ functor we get:

$$
\begin{array}{cccccc}
1 & \rightarrow & 1 & \rightarrow & G^e & \xrightarrow{\varphi_1} & G^e(\sigma) & \rightarrow & 1 \\
& \downarrow & & \downarrow & \alpha_e & & \downarrow & \alpha_e(\sigma) \\
1 & \rightarrow & (C^x)^s & \rightarrow & (C^x)^m & \rightarrow & (C^x)^{m-s} & \rightarrow & 1.
\end{array}
$$

(2.7)
From (2.7), \( \varphi_1 \) is an isomorphism. From the definition of \( W(\sigma) \), we have that \( W(\sigma) \cong Z(\sigma) \times (\mathbb{C}^\times)^{n-l-s} \). So \( W^e(\sigma) \cong Z(\sigma) \times (\mathbb{C}^\times)^{n-l-s} \times (\mathbb{C}^\times)^{m-n} = Z^e(\sigma) \) and \([W^e(\sigma)/G^e] \cong [Z^e(\sigma)/G^e(\sigma)]\). Since \( W^e(\sigma) \) is a closed subvariety of \( Z \times (\mathbb{C}^\times)^{m-n} \), so the stack \( \mathcal{X}(\Sigma^e/\sigma) \) is a closed substack of \( \mathcal{X}(\Sigma^e) \). Twisting it by the bundle \( P \), we have a map \( \varphi_0 : P \times (\mathbb{C}^\times)_m W^e(\sigma) \to P \times (\mathbb{C}^\times)_m Z^e(\sigma) \). So we get a map of groupoids: \( \varphi_0 \times \varphi_1 : P \times (\mathbb{C}^\times)_m W^e(\sigma) \times G^e \to P \times (\mathbb{C}^\times)_m Z^e(\sigma) \times G^e(\sigma) \) which is Morita equivalent. So we have an isomorphism of stacks \([P \times (\mathbb{C}^\times)_m W(\sigma))/G^e] \cong [(P \times (\mathbb{C}^\times)_m Z^e(\sigma))/G^e(\sigma)]\). Since \( W^e(\sigma) \) is a subvariety of \( Z^e \), and \( P \times (\mathbb{C}^\times)_m W^e(\sigma) \) is a subvariety of \( P \times (\mathbb{C}^\times)_m Z^e, \) so \( [(P \times (\mathbb{C}^\times)_m W^e(\sigma))/G^e] \) is a substack of \([P \times (\mathbb{C}^\times)_m Z^e)/G^e] = P \mathcal{X}(\Sigma^e) \). So \( P \mathcal{X}(\Sigma^e/\sigma) \) is a closed substack of \( P \mathcal{X}(\Sigma^e) \). □

**Remark** The proof is similar to Proposition 4.2 in [BCS] where the authors used the notion of quotient stacky fan. We found that the exact sequences they wrote down there are wrong. We write the correct sequences here and found that the quotient stacky fan is naturally an extended stacky fan. Note that the underlying stacky fan in the quotient extended stacky fan \( \Sigma^e/\sigma \) is the quotient stacky fan in the sense of [BCS].

**Remark** From [BCS], \( W(\sigma) = Z^{<g_1, \ldots, g_r>} \) for some group elements \( g_1, \ldots, g_r \in G \).

From Proposition 2.2.4, the toric Deligne-Mumford stack \([Z^e(\sigma)/G^e(\sigma)]\) is isomorphic to the stack \([Z(\sigma)/G(\sigma)]\). Let \( g_1, \ldots, g_r \) still represent the elements in \( G^e \) through the map \( \varphi_1 \) in (2.4). Then \( W^e(\sigma) = (Z^e)^{<g_1, \ldots, g_r>} \).

**Proposition 2.3.6** Let \( P \mathcal{X}(\Sigma^e) \to B \) be a toric stack bundle over a smooth variety \( B \) whose fibre \( \mathcal{X}(\Sigma^e) \) is the toric Deligne-Mumford stack associated to the
extended stacky fan $\Sigma^e$, then the $r$-th inertia stack of this toric stack bundle is

$$I_r(\mathcal{X}(\Sigma^e)) = \prod_{(v_1, \ldots, v_r) \in \Box(\Sigma^e)^r} \mathcal{X}(\Sigma^e / \sigma(v_1, \ldots, v_r)).$$

PROOF. From (2.5), $\mathcal{X}(\Sigma^e) = [(P \times (\mathbb{C}^*)^m Z^e) / G^e]$ is a quotient stack. Because $G^e$ is an abelian group and the the action has finite, reduced stabilizers, we have the $r$-th inertia stack:

$$I_r(\mathcal{X}(\Sigma^e)) = \left[\left(\prod_{(g_1, \ldots, g_r) \in (G^e)^r} (P \times (\mathbb{C}^*)^m Z^e)^H\right) / G^e\right],$$

where $H$ is the subgroup in $G^e$ generated by the elements $g_1, \ldots, g_r$. From Lemma 4.6 in [BCS], there is a map from $\Box(\Sigma^e)$ to $G$. So from the map $\varphi_1$ in (2.4), we have a map $\rho : \Box(\Sigma^e) \to G^e$ such that $\rho(v) = g(v)$. For a $r$-tuple $(v_1, \ldots, v_r)$ in the $\Box(\Sigma^e)$, from Proposition 3.5 and the above Remark, we have:

$$\mathcal{X}(\Sigma^e / \sigma(v_1, \ldots, v_r)) \cong [P \times (\mathbb{C}^*)^m (Z^e)^H / G^e].$$

Taking the disjoint union over all $r$-tuples in $\Box(\Sigma^e)$ we get a map:

$$\psi : \prod_{(v_1, \ldots, v_r) \in \Box(\Sigma^e)^r} \mathcal{X}(\Sigma^e / \sigma(v_1, \ldots, v_r)) \to I_r(\mathcal{X}(\Sigma^e)).$$

The toric stack bundle $\mathcal{X}(\Sigma^e)$ locally is the product of a smooth variety with the toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$. From [BCS], the map $\psi$ is an isomorphism locally in the Zariski topology of the base $B$, so $\psi$ is an isomorphism globally. We complete the proof of the Proposition. □

Remark For any pair $(v_1, v_2) \in \Box(\Sigma^e)^2$, there exists a unique element $v_3 \in \Box(\Sigma^e)$ such that $v_1 + v_2 + v_3 \in N$. This means that in the local group $N/N_{\sigma(\bar{v}_1, \bar{v}_2)}$, the corresponding group elements $g_1, g_2, g_3$ satisfy $g_1g_2g_3 = 1$. So this implies that $\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3) = \sigma(\bar{v}_1, \bar{v}_2)$. In fact, Proposition 2.3.6 determines all 3-twisted sectors of the toric stack bundle $\mathcal{X}(\Sigma^e)$. See also [Jiang] for the case of toric varieties.
2.4 The Orbifold Cohomology Ring.

In this section we describe the ring structure of the orbifold cohomology of toric stack bundles.

2.4.1 The module structure on $A^*_{\text{orb}}(P\mathcal{X}(\Sigma^e))$.

Let $\Sigma^e$ be an extended stacky fan, $P \to B$ a $(\mathbb{C}^\times)^m$-bundle and $P\mathcal{X}(\Sigma^e) \to B$ the associated toric stack bundle. Let $M = N^*$ be the dual of $N$. For $\theta \in M$, let $\chi^0 : (\mathbb{C}^\times)^m \to \mathbb{C}^\times$ be the map induced by $\theta \circ \beta^e : \mathbb{Z}^m \to \mathbb{Z}$. Let $\xi_\theta : B \to B$ be the line bundle $P \times_{\chi_\theta} \mathbb{C}$. We give several definitions.

**Definition 2.4.1** Let $A^*(B)$ denote the Chow ring over $\mathbb{Q}$ of the smooth variety $B$. Define the deformed ring $A^*(B)[N]_{\Sigma^e}$ as follows: $A^*(B)[N]_{\Sigma^e} = A^*(B) \otimes_\mathbb{Q} \mathbb{Q}[N]_{\Sigma^e}$, where $\mathbb{Q}[N]_{\Sigma^e} = \bigoplus_{c \in N} \mathbb{Q} y^c$, and $y$ is a formal variable. Multiplication is given by (2.1).

The deformed ring $A^*(B)[N]_{\Sigma^e}$ has a $\mathbb{Q}$-grading defined as follows: if $\bar{c} = \sum_{\rho \subseteq \sigma(c)} a_i \bar{b}_i$, $\deg(y^c) = \sum a_i \in \mathbb{Q}$.

**Definition 2.4.2** Let $\Sigma^e = (N, \Sigma, \beta^e)$ be an extended stacky fan. Let $S_{\Sigma}$ be the ring $A^*(B)[x_1, \ldots, x_n]/I_{\Sigma}$, where the ideal $I_{\Sigma}$ is generated by the square-free monomials $\{x_{i_1} \cdots x_{i_n} : \rho_{i_1} + \cdots + \rho_{i_n} \notin \Sigma\}$.

Note that $S_{\Sigma}$ is a subring of $A^*(B)[N]_{\Sigma^e}$ given by the map $x_i \mapsto y^{b_i}$ for $1 \leq i \leq n$. Let $\{\rho_1, \ldots, \rho_n\}$ be the rays of $\Sigma^e$, then each $\rho_i$ corresponds to a line bundle $L_i$ over the toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$. This line bundle can be defined as follow. The line bundle $L_i$ on the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is given by the trivial line bundle $\mathbb{C} \times Z$ over $Z$ with the $G$ action on $\mathbb{C}$ given by the
i-th component $\alpha_i$ of $\alpha : G \to (\mathbb{C}^\times)^n$ in (2.3) when $\Sigma^e = \Sigma$. From (2.4), we have:

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi_1} & G^e \\
\downarrow \alpha & & \downarrow \alpha^e \\
(\mathbb{C}^\times)^n & \xrightarrow{i} & (\mathbb{C}^\times)^m.
\end{array}
$$

(2.8)

**Definition 2.4.3** For each $\rho_i$, define the line bundle $L_i$ over $\mathcal{X}(\Sigma^e)$ to be the quotient of the trivial line bundle $Z^e \times \mathbb{C}$ over $Z^e$ under the action of $G^e$ on $\mathbb{C}$ through the component of $\alpha^e$ such that the pullback component in $\alpha$ through (2.8) is $\alpha_i$. Twisting it by the principal $(\mathbb{C}^\times)^m$-bundle $P$, we get the line bundle $L_i$ over the toric stack bundle $P\mathcal{X}(\Sigma^e)$.

Let $\mathcal{I}(P\Sigma^e)$ be the ideal in (2.2). We first describe the ordinary Chow ring of the toric stack bundle $P\mathcal{X}(\Sigma^e)$.

**Lemma 2.4.4** Let $P\mathcal{X}(\Sigma^e) \to B$ be a toric stack bundle over a smooth variety $B$ whose fibre $\mathcal{X}(\Sigma^e)$ is the toric Deligne-Mumford stack associated to the extended stacky fan $\Sigma^e$, then there is an isomorphism of $\mathbb{Q}$-graded rings:

$$
\frac{S^*_{\Sigma^e}}{\mathcal{I}(P\Sigma^e)} \cong A^*(P\mathcal{X}(\Sigma^e))
$$

given by $x_i \mapsto c_1(L_i)$.

**Proof.** From Corollary 2.4, let $X(\Sigma)$ be the coarse moduli space of the toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$. Let $E \to B$ be the principal $T$-bundle induced from the $(\mathbb{C}^\times)^m$-bundle $P$. Then from Proposition 3.3, $E X(\Sigma)$ is the coarse moduli space of the toric stack bundle $P\mathcal{X}(\Sigma^e)$. Let $a_i$ be the first lattice vector in the ray generated by $\bar{b}_i$, then $\bar{b}_i = l_i a_i$ for some positive integer $l_i$. The ideal $\mathcal{I}(P\Sigma^e)$ in (2.2) also defines an ideal in $S^*_{\Sigma^e}$. From [SU], we have

$$
\frac{S^*_{\Sigma^e}}{\mathcal{I}(P\Sigma^e)} \cong A^*(E X(\Sigma))
$$
which is given by \( x_i \mapsto E(V(\rho_i)) \), where \( E(V(\rho_i)) \) is the associated bundle over \( B \) corresponding to the \( T \)-invariant divisor \( V(\rho_i) \). From [V], the Chow ring of the stack \( \mathcal{P}(\Sigma^n) \) is isomorphic to the Chow ring of its coarse moduli space \( \mathcal{E}(\Sigma) \) given by \( c_1(L_i) \mapsto l_i^{-1} \cdot E(V(\rho_i)) \). Then we conclude by \( c_1(\xi) + \sum_{i=1}^{n} \theta(a_i) l_i y^{b_i} = c_1(\xi) + \sum_{i=1}^{n} \theta(b_i) y^{b_i} \). □

Now we discuss the module structure of \( A_{\text{orb}}^{\star}(\mathcal{P}(\Sigma^n)) \). Because \( \Sigma \) is a simplicial fan, we have:

**Lemma 2.4.5** For any \( c \in N \), let \( \sigma \) be the minimal cone in \( \Sigma \) containing \( \bar{c} \), then there exists a unique expression

\[
\begin{align*}
c = v + \sum_{\rho_i \subseteq \sigma} m_i b_i
\end{align*}
\]

where \( m_i \in \mathbb{Z}_{\geq 0} \), and \( v \in \text{Box}(\sigma) \). □

**Lemma 2.4.6** Let \( \tau \) is a cone in the complete simplicial fan \( \Sigma \) and \( \{\rho_1, \ldots, \rho_s\} \subset \text{link}(\tau) \). Suppose \( \rho_1, \ldots, \rho_s \) are contained in a cone \( \sigma \subset \Sigma \). Then \( \sigma \cup \tau \) is contained in a cone of \( \Sigma \).

**Proof.** Using the following result, see [F]. Let \( \rho_1, \ldots, \rho_s \) be rays in the complete simplicial fan \( \Sigma \). If for any \( i, j \), \( \rho_i, \rho_j \) generate a cone, then \( \rho_1, \ldots, \rho_s \) generate a cone. □

**Proposition 2.4.7** Let \( \mathcal{P}(\Sigma^n) \to B \) be a toric stack bundle over a smooth variety \( B \) whose fibre \( \mathcal{X}(\Sigma^n) \) is the toric Deligne-Mumford stack associated to the extended stacky fan \( \Sigma^n \), then we have an isomorphism of \( A^\star(\mathcal{P}(\Sigma^n)) \)-modules:

\[
\bigoplus_{v \in \text{Box}(\Sigma^n)} A^\star(\mathcal{P}(\Sigma^n/\sigma(\bar{v}))) [\deg(y^n)] \cong \frac{A^\star(B)[N]_{\Sigma^n}}{I(\mathcal{P}(\Sigma^n))}.
\]
PROOF. From the definition of $A^*(B)[N]^{\Sigma_v}$ and Lemma 2.4.5, we see that $A^*(B)[N]^{\Sigma_v} = \bigoplus_{v \in Box(\Sigma^e)} y^v \cdot S_\Sigma$. Since $I(P \Sigma^e)$ is the ideal in $A^*(B)[N]^{\Sigma_v}$ defined in (2.2). Then $\bigoplus_{v \in Box(\Sigma^e)} y^v \cdot I(P \Sigma^e)$ is the ideal $I(P \Sigma^e)$ in $\bigoplus_{v \in Box(\Sigma^e)} y^v \cdot S_\Sigma = A^*(B)[N]^{\Sigma_v}$. So we obtain the isomorphism of $A^*(P, X(\Sigma^e))$-modules:

$$\frac{A^*(B)[N]^{\Sigma_v}}{I(P \Sigma^e)} \cong \bigoplus_{v \in Box(\Sigma^e)} y^v \cdot S_\Sigma / y^v \cdot I(P \Sigma^e). \quad (2.9)$$

For any $v \in Box(\Sigma^e)$, let $\sigma(\overline{v})$ be the minimal cone in $\Sigma$ containing $\overline{v}$. Let $\rho_1, \ldots, \rho_l \in link(\sigma(\overline{v}))$, and $\tilde{\rho}_i$ be the image of $\rho_i$ under the natural map $N \to N(\sigma(\overline{v})) = N/N_\sigma(\overline{v})$. Then $S_{\Sigma(\sigma(\overline{v}))} \subset A^*(B)[N(\sigma(\overline{v}))]^{\Sigma_v/\sigma(\overline{v})}$ is the subring given by: $\tilde{x}_i \mapsto y^{\tilde{\rho}_i}$, for $\rho_i \in link(\sigma(\overline{v}))$. Consider the morphism: $i : A^*(B)[\tilde{x}_1, \ldots, \tilde{x}_l] \to A^*(B)[x_1, \ldots, x_n]$ given by $\tilde{x}_i \mapsto x_i$. From Lemma 2.4.6, it is easy to check that the ideal $I_{\Sigma(\sigma(\overline{v}))}$ goes to the ideal $I_{\Sigma}$, so we have a morphism $S_{\Sigma(\sigma(\overline{v}))} \to S_\Sigma$. Since $S_\Sigma$ is a subring of $A^*(B)[N]^{\Sigma_v}$ given by $x_i \mapsto y^{h_i}$, we use the notations $y^{h_i}$. Let $\tilde{\Psi}_v : S_{\Sigma(\sigma(\overline{v}))}[deg(y^v)] \to y^v \cdot S_\Sigma$ be the morphism given by: $y^{\tilde{\rho}_i} \mapsto y^v \cdot y^{h_i}$. If $\sum_{i=1}^l \tilde{\theta}(b_i)y^{\tilde{\rho}_i} + c_1(\xi_0)$ belongs to the ideal $I(P \Sigma^e/\sigma(\overline{v}))$, then

$$\tilde{\Psi}_v \left( \sum_{i=1}^l \tilde{\theta}(b_i)y^{\tilde{\rho}_i} + c_1(\xi_0) \right) = y^v \cdot \left( \sum_{i=1}^l \tilde{\theta}(b_i)y^{\tilde{\rho}_i} + c_1(\xi_0) \right)$$

$$= y^v \cdot \left( \sum_{i=1}^n \theta(b_i)y^{h_i} + c_1(\xi_0) \right),$$

where $\theta$ is determined by the diagram:

$$\begin{array}{ccc}
N & \xrightarrow{\pi} & \mathbb{Z} \\
\downarrow \phi & & \downarrow \theta \\
N(\sigma(\overline{v})) & \xrightarrow{\overline{\phi}} & \mathbb{Z}. \\
\end{array} \quad (2.10)$$

So $\theta(b_i) = \overline{\theta}(\tilde{b}_i)$. From the definition of the line bundle $\xi_\theta$, we have $\xi_\theta \cong \xi_{\tilde{\theta}}$. We obtain that $\tilde{\Psi}_v(\sum_{i=1}^l \tilde{\theta}(b_i)y^{\tilde{\rho}_i} + c_1(\xi_0)) \in y^v \cdot I(P \Sigma^e)$. So $\tilde{\Psi}_v$ induce a morphism $\Psi_v : \frac{S_{\Sigma(\sigma(\overline{v}))}}{I(P \Sigma^e/\sigma(\overline{v}))}[deg(y^v)] \to \frac{y^v \cdot S_{\Sigma}}{y^v \cdot I(P \Sigma^e)}$ such that $\Psi_v([y^{\tilde{\rho}_i}]) = [y^v \cdot y^{h_i}]$. 

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Conversely, for such \( v \in Box(\Sigma^e) \) and \( \rho_i \subset \sigma(\overline{v}) \), choose \( \theta_i \in Hom(N, \mathbb{Q}) \) such that \( \theta_i(b_i) = 1 \) and \( \theta_i(b'_i) = 0 \) for \( b'_i \neq b_i \in \sigma(\overline{v}) \). We consider the following morphism \( p : A^*(B)[x_1, \ldots, x_n] \rightarrow A^*(B)[\overline{x}_1, \ldots, \overline{x}_l] \), where \( p \) is given by:

\[
x_i \mapsto \begin{cases} 
\overline{x}_i & \text{if } \rho_i \subseteq \text{link}(\sigma(\overline{v})), \\
-\sum_{j=1}^l \theta_i(b_j)\overline{x}_j & \text{if } \rho_i \subseteq \sigma(\overline{v}), \\
0 & \text{if } \rho_i \not\subseteq \sigma(\overline{v}) \cup \text{link}(\sigma(\overline{v})).
\end{cases}
\]

For any \( x_{i_1} \cdots x_{i_8} \) in \( I_{\Sigma} \), also from Lemma 2.4.6 we prove that \( p(x_{i_1} \cdots x_{i_8}) \in I_{\Sigma/\sigma(\overline{v})} \).

We also use the notations \( y^h \) to replace \( x_i \). The map \( p \) induces a surjective map:

\( S_{\Sigma} \rightarrow S_{\Sigma/\sigma(\overline{v})} \) and a surjective map: \( \tilde{\Phi}_v : y^v \cdot S_{\Sigma} \rightarrow S_{\Sigma/\sigma(\overline{v})}[\text{deg}(y^v)] \). Let \( y^v \cdot (\sum_{i=1}^n \theta_i(b_i)y^{\overline{b}_i} + c_1(\xi_0)) \) belong to the ideal \( y^v \cdot I(\Sigma^e) \). For \( \theta \in M \), we have \( \theta = \theta_v + \theta'_v \), where \( \theta_v \in N(\sigma(\overline{v}))^* = M \cap \sigma(\overline{v})^\perp \) and \( \theta'_v \) belongs to the orthogonal complement of the subspace \( \sigma(\overline{v})^\perp \) in \( M \). From (2.10), we have:

\[
\tilde{\Phi}_v \left( y^v \cdot \left( \sum_{i=1}^n \theta(b_i)y^{\overline{b}_i} + c_1(\xi_0) \right) \right) = \sum_{i=1}^l \theta_v(\overline{b}_i)y^{\overline{b}_i} + c_1(\xi_{\theta_v}) + \sum_{\rho_i \subset \sigma(\overline{v})} \theta'_v(b_i) \left( -\sum_{j=1}^l \theta_i(b_j)y^{\overline{b}_j} \right) + c_1(\xi_{\theta'_v}) + \sum_{i=1}^l \theta'_v(b_i)y^{\overline{b}_i}.
\]

Note that \( \left( \sum_{i=1}^l \theta_v(\overline{b}_i)y^{\overline{b}_i} + c_1(\xi_{\theta_v}) \right) \in I(\Sigma^e/\sigma(\overline{v})) \). From the definition of \( \xi_{\theta_v} \) over \( \mathcal{X}(\Sigma^e/\sigma(\overline{v})) \), \( \xi_{\theta'_v} = 0 \). Now let \( \theta'_v = \sum_{\rho_i \subset \sigma(\overline{v})} a_i \theta_i \), where \( a_i \in \mathbb{Q} \), then \( \sum_{\rho_i \subset \sigma(\overline{v})} \theta'_v(b_i) = \sum_{\rho_i \subset \sigma(\overline{v})} a_i \theta_i(b_i) \). We have:

\[
\sum_{\rho_i \subset \sigma(\overline{v})} a_i \theta_i(b_i) \left( -\sum_{j=1}^l \theta_i(b_j)y^{\overline{b}_j} \right) + \sum_{\rho_i \subset \sigma(\overline{v})} a_i \theta_i(b_i)y^{\overline{b}_i} = 0,
\]

so we have \( \tilde{\Phi}_v \left( y^v \cdot (\sum_{i=1}^n \theta(b_i)y^{\overline{b}_i} + c_1(\xi_0)) \right) \in I(\Sigma^e/\sigma(\overline{v})) \).

So \( \tilde{\Phi}_v \) induces a morphism

\[
\Phi : y^v \cdot S_{\Sigma} \rightarrow \frac{S_{\Sigma/\sigma(\overline{v})}}{I(\Sigma^e/\sigma(\overline{v})[\text{deg}(y^v)]},
\]

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Note that $\Phi_v \Phi_v = 1$ is easy to check. For any $[y^v \cdot y^{h_i}] \in \frac{y^v}{y^v \mathbb{Z}(P \Sigma^e)}$, since $y^v \cdot \left(- \sum_{j=1}^l \theta_j (b_j) y^{b_j} + \sum_{j=1}^n \theta_j (b_j) y^{b_j}\right) = y^v \cdot y^{h_i}$, we have $[y^v \cdot \left(- \sum_{j=1}^l \theta_j (b_j) y^{b_j}\right)] = [y^v \cdot y^{h_i}]$, we check that $\Phi_v \Phi_v = 1$. So $\Phi_v$ is an isomorphism. From Lemma 2.4.4, for any $v \in Box(\Sigma^e)$, we have an isomorphism of Chow rings: $\frac{S_{\Sigma/\mathbb{Z}(P \Sigma^e)}(v)}{\mathbb{Z}(P \Sigma^e)} \cong A^{*}(P \mathcal{X}(\Sigma^e/\sigma(v)))$. Taking into account all the $v$ in $Box(\Sigma^e)$ and (2.9) we have the isomorphism: $\bigoplus_{v \in Box(\Sigma^e)} A^{*}(P \mathcal{X}(\Sigma^e/\sigma(v))) [deg(y^v)] \cong A^{*}(\mathbb{Z}(P \Sigma^e))$. Note that both sides of (2.9) are $S_{\Sigma/\mathbb{Z}(P \Sigma^e)}$-modules, we complete the proof. □

Remark In Proposition 5.2 of [BCS], the authors give a proof of Proposition 2.4.7 for toric Deligne-Mumford stacks. We give a more explicit proof of this isomorphism for the toric stack bundle.

2.4.2 The orbifold cup product.

In this section we consider the orbifold cup product on $A^{*}_{orb}(P \mathcal{X}(\Sigma^e))$. First we determine the 3-twisted sectors of $P \mathcal{X}(\Sigma^e)$ which are the components of the double inertia stack $I_2(P \mathcal{X}(\Sigma^e))$ of $P \mathcal{X}(\Sigma^e)$, see [CR2]. It follows that all 3-twisted sectors of $P \mathcal{X}(\Sigma^e)$ are:

$$\bigcap_{(g_1,g_2,g_3) \in Box(\Sigma)^3, g_1 g_2 g_3 = 1} P \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)), \quad (2.11)$$

where $\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)$ is the minimal cone in $\Sigma$ containing $\bar{g}_1, \bar{g}_2, \bar{g}_3$. For any 3-twisted sector $P \mathcal{X}(\Sigma^e)_{(g_1, g_2, g_3)} = P \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))$, we have an inclusion

$$e : P \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \rightarrow P \mathcal{X}(\Sigma^e)$$

because $P \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))$ is a substack of $P \mathcal{X}(\Sigma^e)$. Let $H$ be the subgroup generated by $g_1, g_2, g_3$, then the genus zero, degree zero orbifold stable map to $P \mathcal{X}(\Sigma^e)$ determines a Galois covering $\pi : C \rightarrow \mathbb{P}^1$ branching over three marked
points 0, 1, \infty such that the transformation group of this covering is \( H \). We have the definition:

**Definition 2.4.8** ([CR1]) The obstruction bundle \( O_{(g_1, g_2, g_3)} \) over \( P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \) is defined as the \( H \)-invariant bundle:

\[
(e^*T(P\mathcal{X}(\Sigma^e)) \otimes H^1(C, \mathcal{O}_C))^H.
\]

**Proposition 2.4.9** Let \( P\mathcal{X}(\Sigma^e)_{(g_1, g_2, g_3)} = P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \) be a 3-twisted sector of the stack \( P\mathcal{X}(\Sigma^e) \). Let \( g_1 + g_2 + g_3 = \sum_{\mu \in \sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)} a_i b_i \), \( a_i = 1, 2 \), then the Euler class of the obstruction bundle \( O_{(g_1, g_2, g_3)} \) over \( P\mathcal{X}(\Sigma^e)_{(g_1, g_2, g_3)} \) is:

\[
\prod_{a_i = 2} c_1(L_i)|P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))|
\]

where \( L_i \) is the line bundle over \( P\mathcal{X}(\Sigma^e) \) in definition 2.4.3.

**Proof.** Let \( \mathcal{X}(\Sigma^e) \) be the toric Deligne-Mumford stack corresponding to the extended stacky fan \( \Sigma^e \). Let \( \sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3) \) be the minimal cone in \( \Sigma \) containing \( \bar{g}_1, \bar{g}_2, \bar{g}_3 \).

From (2.11) we have the 3-twisted sector \( \mathcal{X}(\Sigma^e)_{(g_1, g_2, g_3)} = \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \) and \( P\mathcal{X}(\Sigma^e)_{(g_1, g_2, g_3)} = P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \). Since \( e : \mathcal{X}(\Sigma^e)_{(g_1, g_2, g_3)} \to \mathcal{X}(\Sigma^e) \) is an inclusion, we have an exact sequence:

\[
0 \to T\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \to e^*T\mathcal{X}(\Sigma^e) \to \mathcal{N}(\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))/\mathcal{X}(\Sigma^e)) \to 0,
\]

where \( \mathcal{N}(\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))/\mathcal{X}(\Sigma^e)) \) is the normal bundle of \( \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \) in \( \mathcal{X}(\Sigma^e) \).

Since \( \mathcal{X}(\Sigma^e) = [Z^e/G^e] \), the tangent bundle \( T(\mathcal{X}(\Sigma^e)) = [T(Z^e)/T(G^e)] \) is a quotient stack. \( Z^e \) is an open subvariety of \( \mathbb{A}^n \times (\mathbb{C}^\times)^{m-n} \), so \( T(Z^e) = \mathcal{O}_{Z^e}^2 \).
Now from the construction of the line bundle $L_k$ over $\mathcal{X}(\Sigma^n)$, we have a canonical map: $\bigoplus_{k=1}^n L_k \to T(\mathcal{X}(\Sigma^n))$. Since we have a natural map $T(\mathcal{X}(\Sigma^n)) \to N(\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))/\mathcal{X}(\Sigma^n))$, we obtain a map of vector bundles over $\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))$:

$$\varphi: \bigoplus_{\rho_k \in \sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)} L_k \to N(\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))/\mathcal{X}(\Sigma^n)).$$

Then from the definition of the line bundle $L_k$ over $P\mathcal{X}(\Sigma^n)$, we have the map:

$$\tilde{\varphi}: \bigoplus_{\rho_k \in \sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)} L_k \to N(P\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))/P\mathcal{X}(\Sigma^n)),$$

where $N(P\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))/P\mathcal{X}(\Sigma^n))$ is the normal bundle of $P\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))$ in $P\mathcal{X}(\Sigma^n)$. For any point map:

$$x : Spec \mathbb{C} \to \mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)) \to P\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)),$$

note that $x^*\tilde{\varphi}$ is an isomorphism, so $\tilde{\varphi}$ is an isomorphism. We have the exact sequence:

$$0 \to T(P\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))) \to e^*T(P\mathcal{X}(\Sigma^n)) \to \bigoplus_{\rho_k \in \sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3)} L_k \to 0.$$

Now using the result in the proof of Proposition 6.3 in [BCS], we have

$$\dim_{\mathbb{C}}(L_k \otimes H^1(C, \mathcal{O}_C))^H = \begin{cases} 
0 & \text{if } a_k = 1, \\
1 & \text{if } a_k = 2.
\end{cases}$$

So from the Definition 2.4.8, we have:

$$e(O_{(g_1, g_2, g_3)}) \cong \prod_{a_i = 2} e_1(L_i)|_{P\mathcal{X}(\Sigma^n/\sigma(\bar{g}_1, \bar{g}_2, \bar{g}_3))}.$$ 

\(\square\)
2.4.3 Proof of Theorem 2.1.1

From the definition of the orbifold cohomology in [CR1], we have that
\[ A^*_{\text{orb}}(P\mathcal{X}(\Sigma^e)) = \bigoplus_{g \in \text{Box}(\Sigma^e)} A^*(P\mathcal{X}(\Sigma^e/\sigma(g))) [\text{deg}(y^g)]. \]
From Proposition 2.4.7, we have an isomorphism between \( A^*(P\mathcal{X}(\Sigma^e)) \)-modules:
\[ \bigoplus_{g \in \text{Box}(\Sigma^e)} A^*(P\mathcal{X}(\Sigma^e/\sigma(g))) [\text{deg}(y^g)] \cong \frac{A^*(B)[N]\Sigma^e}{\mathcal{I}(P\Sigma^e)}. \]
So we have an isomorphism of \( A^*(P\mathcal{X}(\Sigma^e)) \)-modules:
\[ A^*_{\text{orb}}(P\mathcal{X}(\Sigma^e)) \cong \frac{A^*(B)[N]\Sigma^e}{\mathcal{I}(P\Sigma^e)}. \]

Next we show that the orbifold cup product defined in [CR1] coincides with the product in ring \( A^*(B)[N]\Sigma^e/\mathcal{I}(P\Sigma^e) \). From the above isomorphisms, it suffices to consider the canonical generators \( y^{b_i}, y^\gamma \) where \( g \in \text{Box}(\Sigma^e) \) and \( \gamma \in A^*(B) \). Since \( b_i \in N \), the twisted sector determined by \( b_i \) is the whole toric stack bundle \( P\mathcal{X}(\Sigma^e) \), \( y^{b_i} \cup_{\text{orb}} \gamma \) is the usual product \( y^{b_i} \cdot \gamma \) in the deformed ring because \( y^{b_i} \) and \( \gamma \) belong to the ordinary Chow ring of \( P\mathcal{X}(\Sigma^e) \).

For \( y^g \cup_{\text{orb}} y^{b_i} \) and \( y^g \cup_{\text{orb}} \gamma \), where \( g \in \text{Box}(\Sigma^e) \). \( g \) determines a twisted sector \( P\mathcal{X}(\Sigma^e/\sigma(g)) \). The corresponding twisted sectors to \( b_i \) and \( \gamma \) are the whole toric stack bundle \( P\mathcal{X}(\Sigma^e) \). It is easy to see that the 3-twisted sector corresponding to \( (g, b_i) \) and \( (g, \gamma) \) are \( P\mathcal{X}(\Sigma^e)_{(g, 1, g^{-1})} \cong P\mathcal{X}(\Sigma^e/\sigma(g)) \), where \( g^{-1} \) is the inverse of \( g \) in the local group. From the dimension formula in [CR1], the obstruction bundle over \( P\mathcal{X}(\Sigma^e)_{(g, 1, g^{-1})} \) has rank zero. So from the definition of orbifold cup product in [CR1] it is easy to check that \( y^g \cup_{\text{orb}} y^{b_i} = y^g \cdot y^{b_i} \), \( y^g \cup_{\text{orb}} \gamma = y^g \cdot \gamma \).

For the orbifold product \( y^{g_1} \cup_{\text{orb}} y^{g_2} \), where \( g_1, g_2 \in \text{Box}(\Sigma^e) \). From (2.11), we see that if there is no cone in \( \Sigma \) containing \( \bar{g}_1, \bar{g}_2 \), then there is no 3-twisted sector corresponding to the elements \( g_1, g_2 \), so the orbifold cup product is zero from the definition. On the other hand from the definition of the group ring \( A^*(B)[N]\Sigma^e \), \( y^{g_1} \cdot y^{g_2} = 0 \), so \( y^{g_1} \cup_{\text{orb}} y^{g_2} = y^{g_1} \cdot y^{g_2} \). If there is a cone \( \sigma \in \Sigma \) such that \( \bar{g}_1, \bar{g}_2 \in \sigma \), let
$g_3 \in Box(\Sigma^e)$ such that $\bar{g}_3 \in \sigma(\bar{g}_1, \bar{g}_2)$ and $g_1g_2g_3 = 1$ in the local group. Using the same method in the proof of main Theorem in [BCS], we get: $y^{g_1 \cup_{orb} y^{g_2}} = y^{g_1} \cdot y^{g_2}$. The theorem is proved. □

2.5 The $\mu$-Gerbe.

In this section we study the degenerate case of toric Deligne-Mumford stacks. In this case $N$ is a finite abelian group, the simplicial fan $\Sigma$ is 0. The toric stack bundle is a $\mu$-gerbe $\mathcal{X}$ over $B$ for a finite abelian group $\mu$.

Let $N = \mathbb{Z}_{p_1}^{n_1} \oplus \cdots \oplus \mathbb{Z}_{p_s}^{n_s}$ be a finite abelian group, where $p_1, \cdots, p_s$ are prime numbers and $n_1, \cdots, n_s > 1$. Let $\beta^e : \mathbb{Z} \to N$ be given by the vector $(1, 1, \cdots, 1)$. $N_0 = 0$ implies that $\Sigma = 0$, then $\Sigma^e = (N, \Sigma^e, \beta^e)$ is an extended stacky fan from Section 2.1. Let $n = lcm(p_1^{n_1}, \cdots, p_s^{n_s})$, then $n = p_{i_1}^{n_{i_1}} \cdots p_{i_t}^{n_{i_t}}$, where $p_{i_1}, \cdots, p_{i_t}$ are the distinct prime numbers which have the highest powers $n_{i_1}, \cdots, n_{i_t}$. Note that the vector $(1, 1, \cdots, 1)$ generates an order $n$ cyclic subgroup of $N$. We calculate the Gale dual $(\beta^e)^\vee : \mathbb{Z} \to \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i}$, where $DG(\beta^e) = \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i}$.

We have the following exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{\beta^e} \to N \to \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \to 0,$$

$$0 \to 0 \to \mathbb{Z}^{(\beta^e)^\vee} \to \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \to \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \to 0.$$  

So we obtain

$$1 \to \mu \to \mathbb{C}^\times \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i} \xrightarrow{\alpha^e} \mathbb{C}^\times \to 1,$$

(2.12)
where the map $\alpha^e$ in (2.12) is given by the matrix
\[
\begin{bmatrix}
  n \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
\]
and $\mu = \mu_n \times \prod_{i \in \{ i_1, \ldots, i_t \}} \mu_{p_i}^n \cong N$. The toric Deligne-Mumford stack is $X(\Sigma^e) = [\mathbb{C}^\times / \mathbb{C}^\times \times \prod_{i \in \{ i_1, \ldots, i_t \}} \mu_{p_i}^n] = B\mu$, the classifying stack of the group $\mu$. Let $L$ be a line bundle over a smooth variety $B$ and $L^\times$ the principal $\mathbb{C}^\times$-bundle induced from $L$ removing the zero section. From our twist we have $L^\times X(\Sigma^e) = L^\times \times_{\mathbb{C}^\times} [\mathbb{C}^\times / \mathbb{C}^\times \times \prod_{i \in \{ i_1, \ldots, i_t \}} \mu_{p_i}^n] = [L^\times / \mathbb{C}^\times \times \prod_{i \in \{ i_1, \ldots, i_t \}} \mu_{p_i}^n]$, which is exactly a $\mu$-gerbe $X$ over $B$. The structure of this gerbe is a $\mu_n$-gerbe coming from the line bundle $L$ plus a trivial $\prod_{i \in \{ i_1, \ldots, i_t \}} \mu_{p_i}^n$-gerbe over $B$. For this toric stack bundle, $Box(\Sigma^e) = N$, so we have the following Proposition for the inertia stack.

**Proposition 2.5.1** The inertia stack of this toric stack bundle $X$ is $p_1^{n_1} \cdots p_s^{n_s}$ copies of the $\mu$-gerbe $X$.

From our main Theorem, we have:

**Proposition 2.5.2** The orbifold cohomology ring of the toric stack bundle $X$ is given by:
\[
H^*_\text{orb}(X, \mathbb{Q}) \cong H^*(B, \mathbb{Q}) \otimes H^*_\text{orb}(B\mu, \mathbb{Q}),
\]
where $H^*_\text{orb}(B\mu; \mathbb{Q}) = \mathbb{Q}[t_1, \cdots, t_s] / (t_1^{n_1} - 1, \cdots, t_s^{n_s} - 1)$.

Let $N = \mathbb{Z}_r$, and $\beta : \mathbb{Z} \rightarrow \mathbb{Z}_r$ be the natural projection. The toric Deligne-Mumford stack $X(\Sigma^e) = B\mu_r$. Let $L \rightarrow B$ be a line bundle, then the toric stack bundle $X = B_{(i,r)}$ is the $\mu_r$-gerbe over $B$ determined by the line bundle $L$. We have:
**Corollary 2.5.3** The orbifold cohomology ring of $B_{(t,r)}$ is isomorphic to $H^*(B)[t]/(t^r - 1)$.

If the variety $B$ is not a toric variety, then the toric stack bundle over $B$ is not a toric Deligne-Mumford stack. But suppose $B$ is a smooth toric variety, then a $\mu$-gerbe $\mathcal{X}$ can give a toric Deligne-Mumford stack in the sense of [BCS].

**Example** Let $B = \mathbb{P}^d$ be the $d$-dimensional projective space. We give stacky fan $\Sigma = (N, \Sigma, \beta)$ as follows. Let $N = \mathbb{Z}^d \oplus \mathbb{Z}_r$ and $\beta : \mathbb{Z}^{d+1} \rightarrow N$ be the map determined by the vectors: $\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1,0),(-1,-1,\ldots,-1,1)\}$. Then $DG(\beta) = \mathbb{Z}$, and the Gale dual $\beta^\vee$ is given by the matrix $[r,r,\ldots,r]$. So we have the following exact sequences:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^{d+1} \overset{\beta}{\rightarrow} \mathbb{Z}^d \oplus \mathbb{Z}_r \rightarrow 0 \rightarrow 0,$$

$$0 \rightarrow \mathbb{Z}^d \rightarrow \mathbb{Z}^{d+1} \overset{\beta^\vee}{\rightarrow} \mathbb{Z} \rightarrow \mathbb{Z}_r \rightarrow 0.$$

Then we obtain the exact sequence:

$$1 \rightarrow \mu_r \rightarrow \mathbb{C}^\times \overset{\alpha}{\rightarrow} \left(\mathbb{C}^\times\right)^{d+1} \rightarrow \left(\mathbb{C}^\times\right)^d \rightarrow 1.$$

The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) := [\mathbb{C}^{d+1} - \{0\}/\mathbb{C}^\times]$ is the canonical $\mu_r$-gerbe over the projective space $\mathbb{P}^d$ coming from the canonical line bundle, where the $\mathbb{C}^\times$ action is given by $\lambda \cdot (z_1, \ldots, z_{d+1}) = (\lambda^{r} \cdot z_1, \ldots, \lambda^{r} \cdot z_{d+1})$. Denote this toric Deligne-Mumford stack by $\mathcal{G}_r = \mathbb{P}(r,\ldots,r)$. If the homomorphism $\beta : \mathbb{Z}^{d+1} \rightarrow N$ is determined by the vectors: $\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1,0),(-1,-1,\ldots,-1,0)\}$, then $DG(\beta) = \mathbb{Z} \oplus \mathbb{Z}_r$. Comparing to the former exact sequence, we have the exact sequence:

$$1 \rightarrow \mu_r \rightarrow \mathbb{C}^\times \times \mu_r \overset{\alpha}{\rightarrow} \left(\mathbb{C}^\times\right)^{d+1} \rightarrow \left(\mathbb{C}^\times\right)^d \rightarrow 1.$$
The corresponding toric Deligne-Mumford stack is the trivial $\mu_r$-gerbe $\mathbb{P}^d \times \mathcal{B}\mu_r$ coming from the trivial line bundle over $\mathbb{P}^d$. The coarse moduli spaces of these two stacks are both projective space $\mathbb{P}^d$. From the Theorem of this paper or the main Theorem in [BCS], the orbifold cohomology rings of these two stacks are isomorphic, although as stacks they are different.

**Remark**  Let $H$ represent the hyperplane class of $\mathbb{P}^d$, then $H_{orb}^*(G_r, \mathbb{Q}) \cong \mathbb{Q}[H]/(H^{d+1}) \otimes \mathbb{Q}[t]/(t^r - 1)$. We conjecture that the orbifold quantum cohomology ring of $G_r$ defined in [CR2] is isomorphic to $\mathbb{Q}[H]/(H^{d+1} - f(H, q)) \otimes \mathbb{Q}[t]/(t^r - 1 - g(t, q))$, where $f, g$ are two relations and $q$ is the quantum parameter. The orbifold quantum cohomology of trivial gerbe case is easy to compute, where $f(H, q) = q$ and $g(t, q) = 0$.

**Remark** We conjecture that the small orbifold quantum cohomology ring of the nontrivial $\mu_r$-gerbe and trivial $\mu_r$-gerbe over the projective space $\mathbb{P}^d$ should be different. This means that the orbifold quantum cohomology can classify these two different stacks.

### 2.6 Application.

In this section we generalize a result of Borisov, Chen and Smith [BCS] to the toric stack bundle case.

Let $X(\Sigma)$ be a simplicial toric variety, and let $X(\Sigma)$ be the associated toric Deligne-Mumford stack, where $\Sigma = (N, \Sigma, \beta)$ is the stacky fan associated to $\Sigma$. Let $\Sigma'$ be a subdivision of $\Sigma$ such that $X(\Sigma')$ is a crepant resolution of $X(\Sigma)$. Suppose there are $m$ rays in $\Sigma'$, let $i : (\mathbb{C}^*)^m \rightarrow (\mathbb{C}^*)^m$ be the inclusion. From the following
commutative diagram:

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathbb{Z}^{n-d} & \longrightarrow & \mathbb{Z}^n \\
& & \downarrow & & \downarrow \beta \\
0 & \longrightarrow & \mathbb{Z}^{m-d} & \longrightarrow & \mathbb{Z}^m \\
& & \downarrow & & \downarrow i \\
& & \downarrow id & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}
\]

taking Gale dual we get:

\[
\begin{array}{cccc}
0 & \longrightarrow & N^* & \longrightarrow & (\mathbb{Z}^m)^* \\
& & \downarrow id & & \downarrow (\beta')^\vee \\
0 & \longrightarrow & N^* & \longrightarrow & (\mathbb{Z}^n)^* \\
& & \downarrow & & \downarrow \beta^\vee \\
0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}
\]
So applying the \(\text{Hom}\) functor we have the following diagram:

\[
\begin{array}{ccc}
(C^\times)^n & \xrightarrow{i} & (C^\times)^m \\
\downarrow & & \downarrow \\
T & \xrightarrow{id} & T.
\end{array}
\]

Let \(\text{P} \to \text{B}\) be a principal \((C^\times)^n\)-bundle, we still use \(\text{P}\) to represent the principal \((C^\times)^m\)-bundle induced by \(i\), then they induce the same principal \(T\) bundle \(\text{E}\) over \(\text{B}\). So \(\text{E}(\Sigma') \to \text{E}(\Sigma)\) is a crepant resolution. And \(\text{E}(\Sigma)\) is the coarse moduli space of the toric stack bundle \(\text{P}(\text{X}(\Sigma))\) from Proposition 3.3. We have the following result.

**Proposition 2.6.1** If the Chow ring of the smooth variety \(\text{B}\) is a Cohen-Macaulay ring. Then there is a flat family \(\text{S} \to \mathbb{P}^1\) of schemes such that \(\text{S}_0 \cong \text{Spec}(A^{\text{Ord}}(\text{P}(\text{X}(\Sigma))))\) and \(\text{S}_\infty \cong \text{Spec}(A^*(\text{E}(\Sigma') \to \Sigma))\).

**Proof.** We also construct a family of algebras over \(\mathbb{P}^1\) such that the fiber over 0 and \(\infty\) are \(A^{\text{Ord}}(\text{P}(\text{X}(\Sigma)))\) and \(A^*(\text{E}(\Sigma'))\) respectively. \(\text{X}(\Sigma')\) is a smooth variety, and \(\{b_1, \cdots, b_n, b_{n+1}, \cdots, b_m\}\) generate the whole lattice \(N\), then \(A^*(\text{B})[N]^{\Sigma}\) is the quotient ring of the ring \(S := A^*(\text{B})[y^{b_1}, \cdots, y^{b_m}]\) by the binomial ideal determined
by (2.1). Let $I_2$ denote this ideal. Let $I_1$ denote the ideal generated by $c_1(\xi_{\sigma_1}) + \sum_{i=1}^{m} \theta_j(b_i)y^{h_i}$ for $1 \leq j \leq d$, where $\theta_1, \ldots, \theta_d$ is a basis of $N^*$. Since $\Sigma'$ is a regular subdivision of $\Sigma$, then there is a $\Sigma'$-linear support function $h : N \to \mathbb{Z}$ such that $h(b_i) = 0$ for $1 \leq i \leq n$, $h(b_i) > 0$ for $n + 1 \leq i \leq m$. For any lattice points $c_1, c_2$ lying in the same cone of $\Sigma$, $h(c_1 + c_2) \geq h(c_1) + h(c_2)$, and the inequality is strict unless $c_1, c_2$ lies in the same cone of $\Sigma'$.

We describe the family over $\mathbb{P}^1 - \{\infty\}$. Let $\tilde{I}_1$ be the ideal in $S[t_1]$ generated by $c_1(\xi_{\sigma_1})^i_1(b_i) + \sum_{i=1}^{m} \theta_j(b_i)y^{h_i}$ for $1 \leq j \leq d$. So the choice of $h$ implies that

$$\frac{S[t_1]}{\tilde{I}_1 + I_2 + < t_1 >} \cong \frac{S}{c_1(\xi_{\sigma_1}) + \sum_{i=1}^{m} \theta_j(b_i)y^{h_i} : 1 \leq j \leq d > + I_2} \cong A^*_{\text{orb}}(\mathbb{P}X(\Sigma))$$

The sequence $c_1(\xi_{\sigma_1}) + \sum_{i=1}^{m} \theta_j(b_i)y^{h_i}$ for $1 \leq j \leq d$ is also a homogeneous system of parameters on $S/I_2$. The Chow ring $A^*(B)$ is a Cohen-Macaulay ring, so $S/I_2$ is also Cohen-Macaulay. So the sequence is a regular sequence. Therefore, the Hilbert function of the family $S[t_1/(\tilde{I}_1 + I_2)]$ is constant outside a finite set in $\mathbb{Q}^*$.

On the other hand, for the family over $\mathbb{P}^1 - \{0\}$, let $\tilde{I}_2$ be the binomial ideal in $S[t_2]$ given by

$$y^{c_1 + c_2 + h(c_1 + c_2) - h(c_1) - h(c_2)} \quad \text{if } \exists \sigma \in \Sigma \text{ such that } \overline{c}_1 \in \sigma, \overline{c}_2 \in \sigma,$$

$$0 \quad \text{otherwise}.$$ 

From the property of the function $h$. This product becomes (2.1) for the fan $\Sigma'$ over $t_2 = 0$. Hence $S[t_2]/(I_1 + \tilde{I}_2 + < t_2 >) \cong A^*(\mathbb{P}X(\Sigma'))$. The sequence $c_1(\xi_{\sigma_1}) + \sum_{i=1}^{m} \theta_j(b_i)y^{h_i}$ for $1 \leq j \leq d$ is a regular sequence on $S/I_2$ and $S/I_{\Sigma'}$, and we have the same Hilbert function for $S/(I_1 + I_2)$ and $S/(I_1 + I_{\Sigma'})$.

There exists an automorphism $\varphi$ between these two families so that we construct such a family over $\mathbb{P}^1$. The rest of the proof is the same as in [BCS]. We omit the details. \qed
Remark  Ruan [R] conjectured that the cohomology ring of crepant resolution is isomorphic to the orbifold Chow ring of the orbifold if we add some quantum corrections on the ordinary cohomology ring of the crepant resolution coming from the exceptional divisors. Let $\mathbb{P}(1,1,2)$ be the weighted projective plane with one orbifold point whose local group is $\mathbb{Z}_2$. The Hirzburch surface $\mathbb{F}_2$ is the crepant resolution of $\mathbb{P}(1,1,2)$. We can compute the quantum correction of the cohomology ring of the Hirzburch surface and check Ruan’s conjecture. This case has been done recently in [Per].
Bibliography


Chapter 3

Hypertoric Deligne-Mumford Stacks

3.1 Introduction

Hypertoric varieties (cf. [BD], [P]) are the hyperkähler analogue of Kähler toric varieties. The algebraic construction of hypertoric varieties was given by Hausel and Sturmfels [HS]. Modelling on their construction, in this chapter we construct hypertoric Deligne-Mumford stacks and study their orbifold Chow rings.

According to [BD], the topology of hypertoric varieties is determined by hyperplane arrangements. In this chapter we define stacky hyperplane arrangements from which we define hypertoric Deligne-Mumford stacks.

Let $N$ be a finitely generated abelian group of rank $d$ and $N \to \overline{N}$ the natural projection modulo torsion. Let $\beta : \mathbb{Z}^m \to N$ be a homomorphism determined by $^1$

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$^1$The content of this chapter has been accepted by Journal für die reine und angewandte Mathematik for publication.
a collection of nontorsion integral vectors \( \{b_1, \ldots, b_m\} \subseteq N \). We require that \( \beta \) has finite cokernel. The Gale dual of \( \beta \) is denoted by \( \beta^\vee : (\mathbb{Z}^m)^* \rightarrow DG(\beta) \). A generic element \( \theta \) in \( DG(\beta) \) and the vectors \( \{\overline{b}_1, \ldots, \overline{b}_m\} \) determine a hyperplane arrangement \( \mathcal{H} = (H_1, \ldots, H_m) \) in \( N^*_\mathbb{R} \). We call \( A := (N, \beta, \theta) \) a stacky hyperplane arrangement.

For \( \beta : \mathbb{Z}^m \rightarrow N \) in \( A \), we consider the Lawrence lifting \( \beta_L : \mathbb{Z}^m \oplus \mathbb{Z}^m \rightarrow N_L \) of \( \beta \) where \( N_L \) is a finitely generated abelian group with rank \( m+d \). The map \( \beta_L \) is given by vectors \( \{b_{L,1}, \ldots, b_{L,m}, b'_{L,1}, \ldots, b'_{L,m}\} \subseteq N_L \). The generic element \( \theta \) determines a Lawrence simplicial fan \( \Sigma_\theta \) in \( \overline{N}_L \). We call \( \Sigma_\theta = (N_L, \Sigma_\theta, \beta_L) \) a Lawrence stacky fan and \( \mathcal{X}(\Sigma_\theta) \) the Lawrence toric Deligne-Mumford stack. The hypertoric Deligne-Mumford stack \( \mathcal{M}(A) \) associated to \( A \) is defined as a quotient stack which is a closed substack of the Lawrence toric Deligne-Mumford stack \( \mathcal{X}(\Sigma_\theta) \), generalizing the construction of [HS].

The stacky hyperplane arrangement \( A \) also determines an extended stacky fan \( \Sigma = (N, \Sigma, \beta) \) introduced in [Jiang2]. Here \( \Sigma \) is the normal fan of the bounded polytope \( \Gamma \) of the hyperplane arrangement \( \mathcal{H} \). The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \) defined in [Jiang2] is the associated toric Deligne-Mumford stack of \( \mathcal{M}(A) \).

To the map \( \beta \) we associate a multi-fan \( \Delta_\beta \) in the sense of [HM], which consists of cones generated by linearly independent subsets \( \{\overline{b}_{i_1}, \cdots, \overline{b}_{i_k}\} \) in \( \overline{N} \) for \( \{i_1, \cdots, i_k\} \subset \{1, \cdots, m\} \), see Section 3.4. We assume that the \( \text{supp}(\Delta_\beta) = \overline{N} \). We prove that each top dimensional cone in \( \Delta_\beta \) gives a local chart for the hypertoric Deligne-Mumford stack \( \mathcal{M}(A) \). We define a set \( \text{Box}(\Delta_\beta) \) consisting of all pairs \((v, \sigma)\), where \( \sigma \) is a cone in the multi-fan \( \Delta_\beta \), \( v \in N \) such that \( \overline{v} = \sum_{i \in \sigma} \alpha_i \overline{b}_i \) for \( 0 < \alpha_i < 1 \). For \((v, \sigma) \in \text{Box}(\Delta_\beta)\) we consider a closed substack of \( \mathcal{M}(A) \) given by the quotient stacky hyperplane arrangement \( A(\sigma) \). The inertia stack of \( \mathcal{M}(A) \)
is the disjoint union of all such closed substacks, see Section 3.4.

We now describe the orbifold Chow ring of $\mathcal{M}(A)$. The multi-fan $\Delta_\beta$ naturally gives a “matroid” $M_\beta$. The vertex set is $\{1, \ldots, m\}$, and the faces are the subsets $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ such that $\{\bar{b}_{i_1}, \ldots, \bar{b}_{i_k}\}$ are linearly independent in $\bar{N}$. Note that the faces of $M_\beta$ are the cones in $\Delta_\beta$. According to [HS], the ordinary cohomology ring of the hypertoric variety corresponding to the hyperplane arrangement $\mathcal{H}$ is isomorphic to the “Stanley-Reisner” ring of the matroid $M_\beta$. Our result shows that the orbifold Chow ring of hypertoric Deligne-Mumford stacks is a generalization of the Stanley-Reisner ring of the matroid $M_\beta$ to the multi-fan $\Delta_\beta$. Let $N^\Delta_\beta$ denote all the pairs $(c, \sigma)$, where $c \in N$, $\sigma$ is a cone in $\Delta_\beta$ such that $c = \sum_{\rho_i \subseteq \sigma} a_i \bar{b}_i$ and $a_i > 0$ are rational numbers. Then $N^\Delta_\beta$ gives rise to a group ring

$$\mathbb{Q}[\Delta_{\beta}] = \bigoplus_{(c, \sigma) \in N^\Delta_\beta} \mathbb{Q} \cdot y^{(c, \sigma)},$$

where $y$ is a formal variable. For any $(c, \sigma) \in N^\Delta_\beta$, there exists a unique element $(v, \tau) \in \text{Box}(\Delta_{\beta})$ such that $\tau \subseteq \sigma$ and $c = v + \sum_{\rho_i \subseteq \sigma} m_i \bar{b}_i$, where $m_i$ are non-negative integers. We call $(v, \tau)$ the fractional part of $(c, \sigma)$. For $(c, \sigma)$ we define the ceiling function $[c]_{\sigma}$ by $[c]_{\sigma} = \sum_{\rho_i \subseteq \sigma} b_i + \sum_{\rho_i \subseteq \sigma} m_i b_i$. Note that if $v = 0$, $[c]_{\sigma} = \sum_{\rho_i \subseteq \sigma} m_i b_i$. For two pairs $(c_1, \sigma_1)$, $(c_2, \sigma_2)$, if $\sigma_1 \cup \sigma_2$ is a cone in $\Delta_{\beta}$, define $\epsilon(c_1, c_2) := [c_1]_{\sigma_1} + [c_2]_{\sigma_2} - [c_1 + c_2]_{\sigma_1 \cup \sigma_2}$. Let $\sigma_c \subseteq \sigma_1 \cup \sigma_2$ be the minimal cone in $\Delta_{\beta}$ containing $\epsilon(c_1, c_2)$ so that $(\epsilon(c_1, c_2), \sigma_c) \in N^\Delta_{\beta}$. We define the grading on $\mathbb{Q}[\Delta_{\beta}]$ as follows. For any $(c, \sigma)$, write $c = v + \sum_{\rho_i \subseteq \sigma} m_i \bar{b}_i$, then

$$\deg(y^{(c, \sigma)}) := |\tau| + \sum_{\rho_i \subseteq \sigma} m_i,$$

where $|\tau|$ is the dimension of $\tau$. By abuse of notation, we write $y^{(h_i, \rho_i)}$ as $y^{h_i}$. The
multiplication is defined by

\[
y^{(c_1, \sigma_1)} \cdot y^{(c_2, \sigma_2)} := \begin{cases} 
(-1)^{|\sigma_1|} y^{(c_1 + c_2 + e(c_1, c_2), \sigma_1 \cup \sigma_2)} & \text{if } \sigma_1 \cup \sigma_2 \text{ is a cone in } \Delta_\beta, \\
0 & \text{otherwise}.
\end{cases}
\]

(3.1)

Using the property of ceiling functions we check that the multiplication is commutative and associative. So \(\mathbb{Q}[\Delta_\beta]\) is a unital associative commutative ring. Let \(\text{Cir}(\Delta_\beta)\) be the ideal in \(\mathbb{Q}[\Delta_\beta]\) generated by the elements:

\[
\sum_{i=1}^{m} e(b_i) y^{b_i}, \quad e \in N^*.
\]

(3.2)

Let \(A_{\text{orb}}^*(M(A))\) be the orbifold Chow ring of the hypertoric Deligne-Mumford stack \(M(A)\). We have the following Theorem:

**Theorem 3.1.1** Let \(M(A)\) be the hypertoric Deligne-Mumford stack associated to the stacky hyperplane arrangement \(A\). Then there is an isomorphism of graded \(\mathbb{Q}\)-algebras:

\[
A_{\text{orb}}^*(M(A)) \cong \frac{\mathbb{Q}[\Delta_\beta]}{\text{Cir}(\Delta_\beta)}.
\]

The orbifold Chow ring of the hypertoric Deligne-Mumford stack \(M(A)\) is independent of the generic element \(\theta\). It only depends on the map \(\beta\).

Theorem 3.1.1 is proven by a direct approach. The inertia stack of a hypertoric Deligne-Mumford stack \(M(A)\) is the disjoint union of closed substacks \(M(A(\sigma))\) for all \((v, \sigma) \in \text{Box}(\Delta_\beta)\). To determine the ring structure, we identify the 3-twisted sectors as closed substacks of \(M(A)\) indexed by triples \(((v_1, \sigma_1), (v_2, \sigma_2), (v_3, \sigma_3))\) in \(\text{Box}(\Delta_\beta)^3\) such that \(v_1 + v_2 + v_3 \in N\) is an integral linear combination of \(b_i\)'s. We then determine the obstruction bundle over any 3-twisted sector and prove that the orbifold cup product is the same as the product of the ring \(\mathbb{Q}[\Delta_\beta]\) described above.
The multi-fan $\Delta_\beta$ is equal to the simplicial fan $\Sigma$ in $\Sigma$ induced from the stacky hyperplane arrangement $\mathcal{A}$ if and only if $\mathcal{H}$ has $n$ hyperplanes $\{H_1, \cdots, H_n\}$ whose normal polytope is a product of simplices. So in this case $\Sigma$ is a stacky fan and the simplicial fan $\Sigma$ is a product of normal fans of simplices, the toric variety $X(\Sigma)$ is a product of weighted projective spaces. Then by [BD] the associated hypertoric variety is the cotangent bundle of the toric variety $X(\Sigma)$. So $\mathcal{M}(\mathcal{A}) \simeq T^*X(\Sigma)$, the cotangent bundle of the toric Deligne-Mumford stack $X(\Sigma)$. The ring $\mathbb{Q}[\Delta_\beta]$ coincides (as vector spaces) with the deformed ring $\mathbb{Q}[N|^2$ as defined in [BCS].

**Corollary 3.1.2** Let $\Sigma$ be as above. Then there is an isomorphism of $\mathbb{Q}$-vector spaces

$$A^*_{orb}(\mathcal{M}(\mathcal{A})) \simeq A^*_{orb}(X(\Sigma)).$$

Here is an example which shows that the orbifold Chow ring of $\mathcal{M}(\mathcal{A})$ is not isomorphic as a ring to the orbifold Chow ring of the associated toric Deligne-Mumford stack $X(\Sigma)$. Consider the weighted projective stack $\mathbb{P}(1,2)$ which is a toric Deligne-Mumford stack with stacky fan $\Sigma = (N, \Sigma, \beta)$, where $N = \mathbb{Z}$, $\beta : \mathbb{Z}^2 \to N$ is given by the vectors $b_1 = (1), b_2 = (-2)$ and $\Sigma$ is the simplicial fan in the lattice $N$ consisting cones $\rho_1$ and $\rho_2$ generated by $b_1 = (1)$ and $b_2 = (-2)$ respectively. The Gale dual map $\beta' : \mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(2)$. Choosing generic element $\theta = (1) \in \mathbb{Z}$, we get a stacky hyperplane arrangement $\mathcal{A} = (N, \beta, \theta)$. The hypertoric Deligne-Mumford stack $\mathcal{M}(\mathcal{A})$ is the cotangent bundle $T^*\mathbb{P}(1,2)$ whose core is the toric Deligne-Mumford stack $\mathbb{P}(1,2)$. Both $\mathbb{Q}[\Delta_\beta]$ and $\mathbb{Q}[N]^\Sigma$ are generated by $y^{b_1}$, $y^{b_2}$, and $y^{(\frac{1}{2} b_2, \rho_2)}$. According to Theorem 1.1 and the main theorem in [BCS], their orbifold Chow rings are given as follows:

$$A^*_{orb}(X(\Sigma); \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, x_2, v]}{(x_1 - 2x_2, v^2 - x_2, vx_1, x_1x_2)} \cong \frac{\mathbb{Q}[v]}{(v^3)},$$

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It is easy to see that these two rings are not isomorphic. Thus the orbifold Chow ring of a hypertoric Deligne-Mumford stack is not necessarily isomorphic to the orbifold Chow ring of its core. (However, their Chow rings are isomorphic, see Theorem 1.1 of [HS].) This also proves that the orbifold Chow ring has no homotopy invariance property. We remark that the core of a general hypertoric Deligne-Mumford stack can be singular, it is not clear how to define orbifold Chow ring. But in the case of a cotangent bundle over weighted projective space, the core is the weighted projective space and the orbifold Chow ring is well-defined. On the other hand, the orbifold Chow ring of a Lawrence toric Deligne-Mumford stack is isomorphic to its associated hypertoric Deligne-Mumford stack, see [JT2].

Computations of orbifold cohomology rings of hypertoric orbifolds in symplectic geometry have been pursued in [GH]. They used the method in [GHK] and symplectic method to compute the orbifold cohomology. Our method is purely algebraic. It would be interesting to compare these two cohomology rings combinatorially.

This chapter is organized as follows. In Section 3.2 we discuss the relation between stacky hyperplane arrangements and extended stacky fans. We define hypertoric Deligne-Mumford stack $\mathcal{M}(A)$ associated to the stacky hyperplane arrangement $A$. In Section 3.3 we discuss the properties of hypertoric Deligne-Mumford stacks. In Section 3.4 we determine closed substacks of a hypertoric Deligne-Mumford stack. This yields a description of its inertia stacks. We prove Theorem 3.1.1 in Section 3.5, and in Section 3.6 we give some examples.
Conventions

For cones $\sigma_1, \sigma_2$ in $\mathbb{R}^d$, we use $\sigma_1 \cup \sigma_2$ to represent the set of union of the generators of $\sigma_1$ and $\sigma_2$.

3.2 The Hypertoric Deligne-Mumford Stacks

In this section we define hypertoric Deligne-Mumford stacks, mimicking the construction of hypertoric varieties in [HS].

3.2.1 Stacky hyperplane arrangements

We introduce stacky hyperplane arrangements. We explain how a stacky hyperplane arrangement gives extended stacky fans.

Let $N$ be a finitely generated abelian group and $\beta : \mathbb{Z}^m \rightarrow N$ a map given by nontorsion integral vectors $\{b_1, \ldots, b_m\}$. We have the following exact sequences:

\[
0 \rightarrow DG(B)^* \xrightarrow{(\beta^\vee)^*} \mathbb{Z}^m \xrightarrow{\beta} N \rightarrow \text{Coker}(\beta) \rightarrow 0, \tag{3.3}
\]

\[
0 \rightarrow N^* \rightarrow \mathbb{Z}^m \xrightarrow{\beta^\vee} DG(\beta) \rightarrow \text{Coker}(\beta^\vee) \rightarrow 0, \tag{3.4}
\]

where $\beta^\vee$ is the Gale dual of $\beta$ (see [BCS]). The map $\beta^\vee$ is given by the integral vectors $\{a_1, \ldots, a_m\} \subseteq DG(\beta)$. Choose a generic element $\theta \in DG(\beta)$ which lies in the image of $\beta^\vee$ and let $\psi := (r_1, \ldots, r_m)$ be a lifting of $\theta$ in $\mathbb{Z}^m$ such that $\theta = -\beta^\vee \psi$.

Note that $\theta$ is generic if and only if it is not in any hyperplane of the configuration determined by $\beta^\vee$ in $DG(\beta)$. Let $M = N^*$ be the dual of $N$ and $M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}$, then $M_\mathbb{R}$ is a $d$-dimensional $\mathbb{R}$-vector space. Associated to $\theta$ there is a hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_m\}$ in $M_\mathbb{R}$ defined by $H_i$ the hyperplane

\[
H_i := \{v \in M_\mathbb{R} | <b_i, v> + r_i = 0\} \subseteq M_\mathbb{R}. \tag{3.5}
\]
This determines hyperplane arrangements in $M_\mathbb{R}$, up to translation.

**Definition 3.2.1** We call $A := (N, \beta, \theta)$ a **stacky hyperplane arrangement**.

It is well-known that hyperplane arrangements determine the topology of hypertoric varieties [BD]. Let

$$\Gamma = \bigcap_{i=1}^{m} F_i, \text{ where } F_i = \{ v \in M_\mathbb{R} \mid <b_i, v> + \tau_i \geq 0 \}.$$

Let $\Sigma$ be the normal fan of $\Gamma$ in $M_\mathbb{R} = \mathbb{R}^d$ with one dimensional rays generated by $\overline{b}_1, \ldots, \overline{b}_n$. By reordering, we may assume that $H_1, \ldots, H_n$ are the hyperplanes that bound the polytope $\Gamma$, and $H_{n+1}, \ldots, H_m$ are the other hyperplanes. Then we have an extended stacky fan $\Sigma = (N, \Sigma, \beta)$ defined in [Jiang2], where $\beta : \mathbb{Z}^m \to N$ is given by $\{b_1, \ldots, b_n, b_{n+1}, \ldots, b_m\} \subset N$, and $\{b_{n+1}, \ldots, b_m\}$ are the extra data.

By [Jiang2], the extended stacky fan $\Sigma$ determines a toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. It is the same stack as in [BCS]. Its coarse moduli space is the toric variety corresponding to the normal fan of $\Gamma$. According to [BD], a hyperplane arrangement $\mathcal{H}$ is **simple** if the codimension of the nonempty intersection of any $l$ hyperplanes is $l$. A hypertoric variety is the coarse moduli space of an **orbifold** if the corresponding hyperplane arrangement is simple.

**Example** Let $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$, see Figure 1. The polytope $\Gamma$ of the hyperplane arrangement is the shaded triangle whose toric variety is the projective plane. The extended stacky fan is given by the fan of the projective plane $\mathbb{P}^2$ and an extra ray $(0, 1)$. 

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Remark If for a generic element $\theta \in DG(\beta)$ the hyperplane arrangement $\mathcal{H}$ bounds a polytope whose normal fan is $\Sigma$, then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan defined in [BCS].

3.2.2 Lawrence toric Deligne-Mumford stacks

Consider the Gale dual map $\beta^\vee : \mathbb{Z}^m \rightarrow DG(\beta)$ in (3.4). We denote the Gale dual map of

\[ \mathbb{Z}^m \oplus \mathbb{Z}^m (\beta^\vee, -\beta^\vee) \rightarrow DG(\beta) \]

by

\[ \beta_L : \mathbb{Z}^{2m} \rightarrow N_L, \tag{3.6} \]

where $N_L$ is a lattice of dimension $2m - (m - d)$. The map $\beta_L$ is given by the integral vectors $\{b_{L,1}, \ldots, b_{L,m}, b'_{L,1}, \ldots, b'_{L,m}\}$ and $\beta_L$ is called the Lawrence lifting of $\beta$.

Given the generic element $\theta$, let $\overline{\theta}$ be the natural image of $\theta$ under the projection $DG(\beta) \rightarrow \overline{DG(\beta)}$. Then the map $\overline{\beta^\vee} : \mathbb{Z}^m \rightarrow \overline{DG(\beta)}$ is given by $\overline{\beta^\vee} = (\overline{a_1}, \ldots, \overline{a_m})$. For any basis of $\overline{DG(\beta)}$ of the form $C = \{\overline{a_i}, \ldots, \overline{a}_{m-d}\}$,
there exist unique $\lambda_1, \ldots, \lambda_{m-d}$ such that

$$a_{i_0} \lambda_1 + \cdots + a_{i_{m-d}} \lambda_{m-d} = \bar{\theta}.$$  

Let $\mathbb{C}[z_1, \ldots, z_m, w_1, \ldots, w_m]$ be the coordinate ring of $\mathbb{C}^{2m}$. Let

$$\sigma(C, \theta) = \{b_{L,i} \mid \lambda_j > 0\} \cup \{b'_{L,i} \mid \lambda_j < 0\} \quad \text{and} \quad C(\theta) = \{z_j \mid \lambda_j > 0\} \cup \{w_j \mid \lambda_j < 0\}.$$  

We put

$$I_\theta := \left\{ \prod C(\theta) \mid C \text{ is a basis of } DG(\beta) \right\}, \quad (3.7)$$

and

$$\Sigma_\theta := \{\sigma(C, \theta) : C \text{ is a basis of } DG(\beta)\}, \quad (3.8)$$

where $\sigma(C, \theta) = \{b_{L,i}, \ldots, b_{L,m}, b'_{L,i}, \ldots, b'_{L,m}\} \setminus \sigma(C, \theta)$ is the complement of $\sigma(C, \theta)$ and corresponds to a maximal cone in $\Sigma_\theta$. From [HS], $\Sigma_\theta$ is the fan of a Lawrence toric variety $X(\Sigma_\theta)$ corresponding to $\theta$ in the lattice $N_L$, and $I_\theta$ is the irrelevant ideal. The construction above establishes the following:

**Proposition 3.2.2** A stacky hyperplane arrangement $A = (N, \beta, \theta)$ also gives a stacky fan $\Sigma_\theta = (N_L, \Sigma_\theta, \beta_L)$ which is called a Lawrence stacky fan.

**Proof.** From Proposition 4.3 in [HS], $\Sigma_\theta$ is a simplicial fan in $N_L$. The rays $\rho_{L,i}, \rho'_{L,i}$ are generated by $b_{L,i}, b'_{L,i}$. The map $\beta_L$ is the map (3.6) given by $\{b_{L,1}, \ldots, b_{L,m}, b'_{L,1}, \ldots, b'_{L,m}\}$. So by [BCS], $\Sigma_\theta = (N_L, \Sigma_\theta, \beta_L)$ is a stacky fan. \qed

**Definition 3.2.3** The toric Deligne-Mumford stack $\mathcal{X}(\Sigma_\theta)$ is called the Lawrence toric Deligne-Mumford stack.

For the map $\beta^\vee_L : \mathbb{Z}^m \oplus \mathbb{Z}^m \to DG(\beta)$ given by $(\beta^\vee, -\beta^\vee)$, there is an exact sequence

$$0 \to N_L^* \to \mathbb{Z}^{2m} \to DG(\beta) \to \text{Coker}(\beta^\vee_L) \to 0. \quad (3.9)$$
Applying $\text{Hom}_\mathbb{C}(\mu, \mathbb{C})$ to (3.9) yields

$$1 \rightarrow \mu \rightarrow G \rightarrow \mathbb{C} \rightarrow T_L \rightarrow 1,$$

where $\mu := \text{Hom}_\mathbb{C}(\text{Coker}(\beta L^0), \mathbb{C})$ and $T_L$ is the torus of dimension $m + d$. From [BCS] and Proposition 3.2.2, the toric Deligne-Mumford stack $\mathcal{X}(\Sigma_0)$ is the quotient stack $[(\mathbb{C}^{2m} \setminus V(I_0))/G]$, where $G$ acts on $\mathbb{C}^{2m} \setminus V(I_0)$ through the map $\alpha L$.

### 3.2.3 Hypertoric Deligne-Mumford stacks

Define an ideal in $\mathbb{C}[z, w]$ by:

$$I_{\beta v} := \left\langle \sum_{i=1}^{m} (\beta')^*(x) z_i w_i \mid x \in DG(\beta)^* \right\rangle,$$

where $(\beta')^*$ is the map in (3.3) and $(\beta')^*(x)_i$ is the $i$-th component of the vector $(\beta')^*(x)$.

According to Section 6 in [HS], $I_{\beta v}$ is a prime ideal. Let $Y$ be the closed subvariety of $\mathbb{C}^{2m} \setminus V(I_0)$ determined by the ideal (3.11). Since $(\mathbb{C}^*)^{2m}$ acts on $Y$ naturally and the group $G$ acts on $Y$ through the map $\alpha L$, we have the quotient stack $[Y/G]$. Since $Y \subseteq \mathbb{C}^{2m} \setminus V(I_0)$ is a closed subvariety, the quotient stack $[Y/G]$ is a closed substack of $\mathcal{X}(\Sigma_0)$, and is Deligne-Mumford.

**Definition 3.2.4** The hypertoric Deligne-Mumford stack $\mathcal{M}(\mathcal{A})$ associated to the stacky hyperplane arrangement $\mathcal{A}$ is defined to be the quotient stack $[Y/G]$.  

**Example** Let $N = \mathbb{Z} \oplus \mathbb{Z}_2$, $\Sigma$ the fan of projective line $\mathbb{P}^1$, and $\beta : \mathbb{Z}^3 \rightarrow N$ given by $\{b_1 = (1,0), b_2 = (-1,1), b_3 = (1,0)\}$. Then the Gale dual $\beta' : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ is given by the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$. Choose a generic element $\theta = (1,1)$ in $\mathbb{Z}^2$ which determines the fan $\Sigma$. The stacky hyperplane arrangement is $\mathcal{A} = (N, \beta, \theta)$.
\( G = (\mathbb{C}^x)^2 \) and \( Y \) is the subvariety of \( \text{Spec}(\mathbb{C}[z_1, z_2, z_3, w_1, w_2, w_3]) \) determined by the ideal \( I_{\beta^\vee} = (z_1w_1 + z_3w_3, 2z_1w_1 + 2z_2w_2) \). Then by [HS], the coarse moduli space is the \textit{crepant resolution} of the Gorenstein orbifold \( [\mathbb{C}^2/\mathbb{Z}_3] \), see Figure 3. The corresponding hyperplane arrangement \( \mathcal{H} \) consists of three distinct points on the real line \( \mathbb{R}^1 \), and the bounded polyhedron is two segments intersecting at one point. So the core of the hypertoric variety is two \( \mathbb{P}^1 \) intersecting at one point. The hypertoric Deligne-Mumford stack \( \mathcal{M}(A) \) is a nontrivial \( \mu_2 \)-gerbe over the \textit{crepant resolution} according to the action given by the inverse of the above matrix. If we change \( b_2 \) to \( (-1, 0) \), we will see an example in Section 3.4 that the hypertoric Deligne-Mumford stack is a trivial \( \mu_2 \)-gerbe over the crepant resolution.

### 3.3 Properties of Hypertoric Deligne-Mumford Stacks

The coarse moduli space

Each Deligne-Mumford stack has an underlying coarse moduli space. In this section we prove that the coarse moduli space of \( \mathcal{M}(A) \) is the underlying hypertoric variety.

Consider again the map \( \beta^\vee : \mathbb{Z}^m \to \overline{DG}(\beta) \) in (3.4), which is given by the vectors \( (a_1, \ldots, a_m) \). As in section 2, let \( \overline{\theta} \) be the natural image of \( \theta \) under the projection \( \overline{DG}(\beta) \to \overline{DG}(\beta) \). Then the map \( \overline{\beta^\vee} : \mathbb{Z}^m \to \overline{DG}(\beta) \) is given by \( \overline{\beta^\vee} = (\overline{a_1}, \ldots, \overline{a_m}) \). From the map \( \overline{\beta^\vee} \) we get the simplicial fan \( \Sigma_{\theta} \) in (3.8). By [BCS], the toric variety \( X(\Sigma_{\theta}) \), which is the geometric quotient \( (\mathbb{C}^{2m} - V(\mathcal{I}_{\theta}))/G \), is the coarse moduli space of the Lawrence toric Deligne-Mumford stack \( X(\Sigma_{\theta}) \). The toric variety \( X(\Sigma_{\theta}) \) is semi-projective, but not projective. In [HS], from \( \beta^\vee \) and \( \theta \), the authors define the hypertoric variety \( Y(\beta^\vee, \theta) \) as the complete intersection of
the toric variety $X(\Sigma_\theta)$ by the ideal (3.11), which is the geometric quotient $Y/G$.

We have the following Proposition.

**Proposition 3.3.1** The coarse moduli space of $\mathcal{M}(A)$ is $Y(\beta^\vee, \theta)$.

**Proof.** Let $X = (\mathbb{C}^2 - V(I_\theta))$. By the universal property of geometric quotients ([KM]), we have that $X \times_{X(\Sigma_\theta)} Y(\beta^\vee, \theta) = Y$. From Lemma 3.3 in [JT2], the stabilizers of points in $X$ are the same as the stabilizers of the points in $Y$, which are determined by the box elements in the Lawrence simplicial fan and extended stacky fan. So we have the following diagram

$$
\begin{array}{ccc}
\mathcal{M}(A) & \xleftarrow{\Delta} & X(\Sigma_\theta) \\
\downarrow & & \downarrow \\
Y(\beta^\vee, \theta) & \xleftarrow{\Delta} & X(\Sigma_\theta),
\end{array}
$$

which is cartesian. The Lawrence toric variety $X(\Sigma_\theta)$ is the coarse moduli space of the Lawrence toric Deligne-Mumford stack $X(\Sigma_\theta)$. So $\mathcal{M}(A)$ has coarse moduli space $Y(\beta^\vee, \theta)$. □

**Remark** In [HS], the authors began with the map $\beta^\vee$, and assumed that $N$ and $DG(\beta)$ are free. In our case $DG(\beta)$ is a finitely generated abelian group, the toric variety $X(\Sigma_\theta)$ is again semi-projective since $\Sigma_\theta$ is a semi-projective fan. The hypertoric variety $Y(\beta^\vee, \theta)$ is the complete intersection determined by the ideal (3.11). This reduces to the case in [HS] when $N$ and $DG(\beta)$ are free.

**Independence of coorientations of hyperplanes**

From (3.5), a hyperplane $H_i$ is naturally oriented. Changing the orientation of $H_i$ means changing the map $\beta$ by replacing $b_i$ by $-b_i$. 

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Proposition 3.3.2 \( \mathcal{M}(A) \) is independent to the coorientations of the hyperplanes in the hyperplane arrangement \( \mathcal{H} = (H_1, \ldots, H_m) \) corresponding to the stacky hyperplane arrangement \( A \).

Remark Note that changing coorientations does change the corresponding normal fan of the weighted polytope \( \Gamma \).

Proof. It suffices to prove the Proposition when we change the coorientation of one hyperplane, say \( H_j \) for some \( j \). Let \( \mathcal{H}' = (H_1, \ldots, H'_j, \ldots, H_m) \). Then we have a new stacky hyperplane arrangement \( A' = (N, \beta', \theta) \), where \( \beta' : \mathbb{Z}^m \to N \) is given by \( \{b_1, \ldots, -b_j, \ldots, b_m\} \). Using the technique of Gale dual in [BCS], it is easy to check that if the Gale dual \( \beta^\vee \) is given by the integral vectors \( \beta^\vee = (a_1, \ldots, a_m) \), then the Gale dual \( (\beta')^\vee \) is given by the integral vectors \( (\beta')^\vee = (a_1, \ldots, -a_j, \ldots, a_m) \). Let \( \psi : \mathbb{Z}^m \to \mathbb{Z}^m \) be the map given by \( e_i \to e_i \) if \( i \neq j \) and \( e_j \to -e_j \), then we have the following commutative diagrams:

\[
\begin{array}{ccc}
\mathbb{Z}^m & \xrightarrow{\psi} & \mathbb{Z}^m \\
\beta & \downarrow & \beta' \\
N & \xrightarrow{id} & N,
\end{array}
\quad
\begin{array}{ccc}
(\mathbb{Z}^m)^* & \xrightarrow{(\mathbb{Z}^m)^*} & (\mathbb{Z}^m)^* \\
\beta^\vee & \downarrow & \beta'^\vee \\
DG(\beta') & \xrightarrow{DG(\beta')} & DG(\beta).
\end{array}
\]

Consider the diagram

\[
(\mathbb{Z}^{2m})^* \xrightarrow{[\beta'^\vee, -\beta'^\vee]} DG(\beta') \xrightarrow{DG(\beta)} (\mathbb{Z}^{2m})^*.
\]

Applying \( \text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times) \) yields the following diagram of abelian groups

\[
G \xrightarrow{\varphi^l} G' \xrightarrow{(\alpha^l)^\vee} (\mathbb{C}^\times)^{2m} \xrightarrow{(\alpha^l)^\vee} (\mathbb{C}^\times)^{2m}.
\]
Recall that $Y$ is a subvariety of $\mathbb{C}^{2m} \setminus V(I_\theta)$ defined by the ideal $I_\theta$ in (3.11). When we change the coorientation of $H_j$, the ideals do not change, so $Y' = Y$. By (3.12), the following diagram is Cartesian:

\[
\begin{array}{ccc}
Y \times G & \xrightarrow{\phi_0 \times \phi_1} & Y' \times G' \\
(s,t) \downarrow & & \downarrow (s,t) \\
Y \times Y & \xrightarrow{\phi_0 \times \phi_0} & Y' \times Y',
\end{array}
\]

(3.13)

where $\phi_0$ is determined by the map $\psi$. So the groupoid $Y \times G \rightrightarrows Y$ is Morita equivalent to the groupoid $Y' \times G' \rightrightarrows Y'$. The stack $[Y/G]$ is isomorphic to the stack $[Y'/G']$, and $\mathcal{M}(A) \cong \mathcal{M}(A')$. □

**Remark** Let $\Sigma = (N, \Sigma, \beta)$ be the extended stacky fan induced by $\mathcal{A}$. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is the quotient stack $[Z/G]$, where $Z = (\mathbb{C}^n \setminus V(J_\Sigma)) \times (\mathbb{C}^\times)^{m-n}$ as in [Jiang2], and $J_\Sigma$ is the square-free ideal of the fan $\Sigma$. So every hypertoric Deligne-Mumford stack $\mathcal{M}(A)$ has an associated toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ whose simplicial fan is the normal fan of the bounded polytope $\Gamma$ in the hyperplane arrangement $\mathcal{H}$ determined by the stacky hyperplane arrangement $\mathcal{A}$. But by Proposition 3.3.2, $\mathcal{M}(A)$ does not determine $\mathcal{X}(\Sigma)$.

**Example** Consider Figure 1 again. The corresponding toric variety is $\mathbb{P}^2$. If we change the co-orintation of the hyperplane 2, then the corresponding normal fan $\Sigma$ of $\Gamma$ changes. The resulting toric variety is a Hirzebruch surface. So the associated toric Deligne-Mumford stacks are different. But the hypertoric Deligne-Mumford stacks are the same.
3.4 Substacks of Hypertoric Deligne-Mumford Stacks

In this section we consider substacks of hypertoric Deligne-Mumford stacks. In particular, we determine the inertia stack of a hypertoric Deligne-Mumford stack.

Let $A = (N, \beta, \theta)$ be a stacky hyperplane arrangement and $\Sigma = (N, \Sigma, \beta)$ the extended stacky fan induced from $A$. Let $\mathcal{M}(A)$ denote the corresponding hypertoric Deligne-Mumford stack. Consider the map $\beta : \mathbb{Z}^m \rightarrow N$ given by $\{b_1, \ldots, b_m\}$. Let $Cone(\beta)$ be a partially ordered finite set of cones generated by $\vec{b}_1, \ldots, \vec{b}_m$. The partial ordering is defined by requiring that $\sigma < \tau$ if $\sigma$ is a face of $\tau$. We have the minimum element $\hat{0}$ which is the cone consisting of the origin. Let $Cone(N)$ be the set of all convex polyhedral cones in the lattice $N$. Then we have a map

$$ C : Cone(\beta) \rightarrow Cone(N), $$

such that for any $\sigma \in Cone(\beta)$, $C(\sigma)$ is the cone in $N$. Then $\Delta_\beta := (C, Cone(\beta))$ is a simplicial multi-fan in the sense of [HM].

Closed substacks

Recall that in Section 3.2 we have the fan $\Sigma_\theta$ for the Lawrence toric variety corresponding to $\pm \beta^\vee$. Let $\Lambda(B) = \{\vec{b}_{L,1}, \ldots, \vec{b}_{L,m}, \vec{b}_{L,1}', \ldots, \vec{b}_{L,m}'\} \subset N_L$ be the Lawrence lifting of $B = \{\vec{b}_1, \ldots, \vec{b}_m\} \subset N$. We have the following lemma.

**Lemma 3.4.1** If $\sigma_\theta = (\vec{b}_{L,i_1}, \ldots, \vec{b}_{L,i_k}, \vec{b}_{L,i_1}', \ldots, \vec{b}_{L,i_k}')$ forms a cone in $\Sigma_\theta$, then $\sigma = (\vec{b}_{i_1}, \ldots, \vec{b}_{i_k})$ forms a cone in $\Delta_\beta$.

**Proof.** This can be easily proved from the definition of fan $\Sigma_\theta$ in (3.8). □

For a cone $\sigma$ in the multi-fan $\Delta_\beta$, let $link(\sigma) = \{b_i : \rho_i + \sigma$ is a cone in $\Delta_\beta\}$. Then we have a quotient extended stacky fan $\Sigma/\sigma = (N(\sigma), \Sigma/\sigma, \beta(\sigma))$, where
\( \beta(\sigma) : \mathbb{Z}^l \to N(\sigma) \) is given by the images of \( \{b_i\} \)'s in \( \text{link}(\sigma) \). Let \( s := |\sigma| \), then \( \dim(N_{\sigma}) = s \) since \( \sigma \) is simplicial. Consider the commutative diagrams

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^{l+s} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{m-l-s} & \longrightarrow & 0 \\
\downarrow \bar{\beta} & & \downarrow \beta & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \xrightarrow{\alpha} & N & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}
\]

and

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \mathbb{Z}^{l+s} & \longrightarrow & \mathbb{Z}^l & \longrightarrow & 0 \\
\downarrow \bar{\beta}_s & & \downarrow \bar{\beta} & & \downarrow \beta(\sigma) & & \downarrow & & \\
0 & \longrightarrow & N_{\sigma} & \longrightarrow & N & \longrightarrow & N(\sigma) & \longrightarrow & 0.
\end{array}
\]

Applying the Gale dual yields

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^{m-l-s} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{l+s} & \longrightarrow & 0 \\
\downarrow \bar{\alpha} & & \downarrow \beta^\vee & & \downarrow \bar{\beta}^\vee & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^{m-l-s} & \longrightarrow & DG(\beta) & \xrightarrow{\phi_1} & DG(\bar{\beta}) & \longrightarrow & 0,
\end{array}
\]

and

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^l & \longrightarrow & \mathbb{Z}^{l+s} & \longrightarrow & \mathbb{Z}^s & \longrightarrow & 0 \\
\downarrow \beta(\sigma)^\vee & & \downarrow \bar{\beta}^\vee & & \downarrow \beta^\vee_s & & \downarrow & & \\
0 & \longrightarrow & DG(\beta(\sigma)) & \xrightarrow{\phi_2} & DG(\bar{\beta}) & \longrightarrow & DG(\beta_{\sigma}) & \longrightarrow & 0.
\end{array}
\]

Since \( \mathbb{Z}^s \cong N_{\sigma} \), the Gale dual \( DG(\beta_{\sigma}) = 0 \). And again applying the \( \text{Hom}_{\mathbb{Z}}(-,\mathbb{C}^\times) \) functor to the above two diagrams (3.14), (3.15) yields

\[
\begin{array}{cccccc}
1 & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & (\mathbb{C}^\times)^{m-l-s} & \longrightarrow & 1 \\
\downarrow \tilde{\alpha} & & \downarrow \alpha & & \downarrow \alpha & & \downarrow & & \\
1 & \longrightarrow & (\mathbb{C}^\times)^{l+s} & \longrightarrow & (\mathbb{C}^\times)^m & \longrightarrow & (\mathbb{C}^\times)^{m-l-s} & \longrightarrow & 1,
\end{array}
\]

and

\[
\begin{array}{cccccc}
1 & \longrightarrow & 1 & \longrightarrow & \tilde{G} & \xrightarrow{\alpha} & G(\sigma) & \longrightarrow & 1 \\
\downarrow \tilde{\alpha} & & \downarrow \alpha & & \downarrow \alpha(\sigma) & & \downarrow & & \\
1 & \longrightarrow & (\mathbb{C}^\times)^s & \longrightarrow & (\mathbb{C}^\times)^{l+s} & \longrightarrow & (\mathbb{C}^\times)^l & \longrightarrow & 1.
\end{array}
\]

Since \( \theta \in DG(\beta) \), from the map \( \phi_1 \) in (3.14) \( \theta \) induces \( \bar{\theta} \) in \( DG(\bar{\beta}) \). From the isomorphism \( \phi_2 \) in (3.15), we get \( \theta(\sigma) \in DG(\beta(\sigma)) \). Then \( A = (N, \beta, \theta) \) gives \( A(\sigma) = (N(\sigma), \beta(\sigma), \theta(\sigma)) \) whose induced extended stacky fan is \( \Sigma / \sigma \).
From (3.14), (3.15) we have the following diagrams

\[
\begin{array}{c}
\mathbb{Z}^{2m} \longrightarrow \mathbb{Z}^{2(l+s)} \\
\downarrow{\frac{\partial^{\nu}}{\partial^{\nu}} - \frac{\partial^{\nu}}{\partial^{\nu}}} & \downarrow{\frac{\partial^{\nu}}{\partial^{\nu}} - \frac{\partial^{\nu}}{\partial^{\nu}}} \\
DG(\beta) \longrightarrow DG(\bar{\beta}), & DG(\beta(\sigma)) \longrightarrow DG(\bar{\beta}).
\end{array}
\]

(3.18)

Taking \(\text{Hom}_G(-, \mathbb{C}^\times)\) gives

\[
\begin{array}{c}
\tilde{G} \xrightarrow{\varphi_1} G \\
\tilde{G} \xrightarrow{\sigma} G(\sigma)
\end{array}
\]

\[
\begin{array}{c}
(C^\times)^{2(l+s)} \longrightarrow (C^\times)^{2m}, \\
(C^\times)^{2(l+s)} \longrightarrow (C^\times)^{2l}.
\end{array}
\]

(3.19)

Let \(X(\sigma) := (\mathbb{C}^{2l} \setminus V(I_{\theta}(\sigma)))\) and \(Y(\sigma)\) the closed subvariety of \(X(\sigma)\) defined by the ideal

\[
I_{\beta(\sigma)^G} := \left\{ \sum_{i=1}^l (\beta(\sigma)^G)(x)z_iw_i : \forall x \in DG(\beta(\sigma))^G \right\},
\]

(3.20)

where \((\beta(\sigma)^G)^* : DG(\beta(\sigma))^* \rightarrow \mathbb{Z}^l\) is the dual map of \(\beta(\sigma)^G\) and \((\beta(\sigma)^G)^*(x)\) the \(i\)-th component of the vector \((\beta(\sigma)^G)^*(x)\). Then from the definition of hypertoric Deligne-Mumford stacks, we have \(\mathcal{M}(\mathcal{A}(\sigma)) = [Y(\sigma)/G(\sigma)]\). We have the following result:

**Proposition 3.4.2** If \(\sigma\) is a cone in the multi-fan \(\Delta_\beta\), then \(\mathcal{M}(\mathcal{A}(\sigma))\) is a closed substack of \(\mathcal{M}(\mathcal{A})\).

**PROOF.** Let \(I_{\theta}\) be the irrelevant ideal in (3.7). The hypertoric stack \(\mathcal{M}(\mathcal{A})\) is the quotient stack \([Y/G]\), where \(Y \subset X := (\mathbb{C}^{2m} \setminus V(I_{\theta}))\) is the subvariety determined by the ideal \(I_{\beta(\sigma)^G}\) in (3.11).

Taking duals to (3.14), (3.15) we get:

\[
\begin{array}{c}
0 \longrightarrow DG(\bar{\beta})^* \longrightarrow DG(\beta)^* \longrightarrow \mathbb{Z}^{m-l-s} \longrightarrow 0 \\
\downarrow{(\bar{\beta}^{\nu})^*} & \downarrow{(\beta^{\nu})^*} & \downarrow{\alpha} \\
0 \longrightarrow \mathbb{Z}^{l+s} \longrightarrow \mathbb{Z}^{m} \longrightarrow \mathbb{Z}^{m-l-s} \longrightarrow 0,
\end{array}
\]

(3.21)
Let $W(\sigma)$ be the subvariety of $X$ defined by the ideal $J(\sigma) := \langle z_i, w_i : \rho_i \subseteq \sigma \rangle$. Then $W(\sigma)$ contains the $C$-points $(z, w) \in C^{2m}$ such that the cone spanned by $\{\rho_i : z_i = w_i = 0\}$ containing $\sigma$ belongs to $\Delta_\beta$. From Lemma 3.4.1, the $C$-point $(z, w)$ in $W(\sigma)$ such that $\rho_i \notin \sigma \cup \text{link}(\sigma)$ implies that $z_i \neq 0$ or $w_i \neq 0$. It is clear that $W(\sigma)$ is invariant under the $G$-action defined by (3.10). Let $V(\sigma) := Y \cap W(\sigma)$. Then from (3.11),(3.20) and (3.21),(3.22), $V(\sigma) \cong Y(\sigma) \times (C^\times)^{m-s-t} \times (0)^{m-s-t}$ and the components $0$ are determined by the choice of the generic element $\theta$.

Let $\varphi_0 : Y(\sigma) \rightarrow V(\sigma)$ be the inclusion given by $(z, w) \mapsto (z, w, 1, 0)$. From the map $\varphi_1$ in (3.19), we have a morphism of groupoids $\varphi_0 \times \varphi_1 : Y(\sigma) \times G(\sigma) \rightarrow V(\sigma) \times G$ which induces a morphism of stacks $\varphi : [Y(\sigma)/G(\sigma)] \rightarrow [V(\sigma)/G]$. To prove that it is an isomorphism, we first prove that the following diagram is cartesian:

$$
\begin{array}{ccc}
Y(\sigma) \times G(\sigma) & \xrightarrow{\varphi_0 \times \varphi_1} & V(\sigma) \times G \\
(s, t) \downarrow & & \downarrow (s, t) \\
Y(\sigma) \times Y(\sigma) & \xrightarrow{\varphi_0 \times \varphi_0} & V(\sigma) \times V(\sigma).
\end{array}
$$

This is easy to prove. Given an element $((z_1, w_1), (z_2, w_2)) \in Y(\sigma) \times Y(\sigma)$, under the map $\varphi_0 \times \varphi_0$, we get $((z_1, w_1, 1, 0), (z_2, w_2, 1, 0)) \in V(\sigma) \times V(\sigma)$. If there is an element $g \in G$ such that $g(z_1, w_1, 1, 0) = (z_2, w_2, 1, 0)$, then from the exact sequence in the first row of (3.16), there is an element $g(\sigma) \in G(\sigma)$ such that $g(\sigma)(z_1, w_1) = (z_2, w_2)$. Thus we have an element $((z_1, w_1), g(\sigma)) \in Y(\sigma) \times G(\sigma)$. So the morphism $\varphi : [Y(\sigma)/G(\sigma)] \rightarrow [V(\sigma)/G]$ is injective. Let $(z, w, s, 0)$ be an element in $V(\sigma)$, then there exists an element $g \in (C^\times)^{m-l-s}$ such that $g(z, w, s, 0) = \ldots$
(z, w, 1, 0). From (3.16), \( g \) determines an element in \( G \), so \( \varphi \) is surjective and \( \varphi \) is an isomorphism. Clearly the stack \([V(\sigma)/G]\) is a closed substack of \( \mathcal{M}(A) \), so the stack \( \mathcal{M}(A(\sigma)) = [Y(\sigma)/G(\sigma)] \) is also a closed substack of \( \mathcal{M}(A) \). □

**Open substacks**

We now study open substacks of \( \mathcal{M}(A) \). Let \( \sigma \) be a top dimensional cone in \( \Delta_\beta \). Then \( \sigma = (Z^d, \sigma, \beta_\sigma) \) is a stacky fan, where \( \beta_\sigma : Z^d \to N \) is given by \( b_i \) for \( \rho_i \subseteq \sigma \). Since \( N \) has rank \( d \), we find that \( DG(\beta_\sigma) \) is a finite abelian group. So in this case the generic element \( \theta \) induces zero in \( DG(\beta_\sigma) \). This is the degenerate case, which means that the corresponding ideal (3.11) is zero. Thus

\[ Y_\sigma = \mathbb{C}^{2d}. \]

Note that \( G_\sigma = \text{Hom}_\mathbb{Z}(DG(\beta_\sigma), \mathbb{C}^\times) \) is a finite abelian group. According to the construction of hypertoric Deligne-Mumford stack in Section 3.2.3, the hypertoric Deligne-Mumford stack \( \mathcal{M}(\sigma) \) associated to \( \sigma \) is the quotient stack \([Y_\sigma/G_\sigma] \) which can be regarded as a local chart of the hypertoric orbifold \([Y/G]\).

**Proposition 3.4.3** If \( \sigma \) is a top-dimensional cone in the multi-fan \( \Delta_\beta \), then \( \mathcal{M}(\sigma) \) is an open substack of \( \mathcal{M}(A) \).

**Proof.** Since \( \sigma \) is a top dimensional cone in \( \Delta_\beta \), from (3.4) we get a basis \( C \) of \( DG(\beta) \). Let \( U_\sigma \) be the open subvariety of \( \mathbb{C}^{2m} \setminus V(I_\theta) \) defined by the monomials \( \prod C(\theta) \) in (3.7). Let \( V_\sigma = U_\sigma \cap Y \), i.e. the points in \( U_\sigma \) staisfying (3.11). Then we have the groupoid \( V_\sigma \times G \rightrightarrows V_\sigma \) associated to the action of \( G \) on \( V_\sigma \). It is clear that this groupoid defines an open substack of \( \mathcal{M}(A) \). Next we show that this substack is isomorphic to \( \mathcal{M}(\sigma) \).
Consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^d & \rightarrow & \mathbb{Z}^m & \rightarrow & \mathbb{Z}^{m-d} & \rightarrow & 0 \\
& & \downarrow{\beta} & & \downarrow{\alpha} & & \downarrow{\beta'} & \\
0 & \rightarrow & N & \xrightarrow{id} & N & \rightarrow & 0 & \rightarrow & 0.
\end{array}
\]

Applying Gale dual and \( \text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^\times) \), we obtain

\[
\begin{array}{cccccc}
1 & \rightarrow & G_\sigma & \xrightarrow{\varphi_1} & G & \rightarrow & (\mathbb{C}^\times)^{m-d} & \rightarrow & 1 \\
& & \downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{id} & \\
1 & \rightarrow & (\mathbb{C}^\times)^d & \rightarrow & (\mathbb{C}^\times)^m & \rightarrow & (\mathbb{C}^\times)^{m-d} & \rightarrow & 1.
\end{array}
\]

We construct a morphism \( \varphi_0 : Y_\sigma \rightarrow V_\sigma \). For \( \rho_j \not\in \sigma \), we set \( z_j = 1 \) if \( z_j \) is component of a monomial of \( C(\theta) \) in (3.7) or \( w_j = 1 \) if \( w_j \) is component of a monomial of \( C(\theta) \) in (3.7), then from (3.11), the corresponding \( w_j \) or \( z_j \) can be represented as linear component of \( \{z_iw_i\} \) for \( \rho_i \subseteq \sigma \). Let \( \tilde{\varphi}_0 : \mathbb{C}^d \rightarrow U_\sigma \) be the morphism given by \( z_i, w_i \mapsto z_i, w_i \) for \( \rho_i \subseteq \sigma \), and \( z_j, w_j \) to the corresponding 1 or linear combination of \( \{z_iw_i\} \) for \( \rho_i \subseteq \sigma \) in the above analysis. Then \( \tilde{\varphi}_0 \) induces a morphism \( \varphi_0 : Y_\sigma \rightarrow V_\sigma \).

Hence we have a morphism of groupoids

\[
\Phi := (\varphi_0 \times \varphi_0, \varphi_0 \times \varphi_1) : [Y_\sigma \times G_\sigma \Rightarrow Y_\sigma] \rightarrow [V_\sigma \times G \Rightarrow V_\sigma],
\]

where \( \varphi_1 \) is the morphism in (3.23). This morphism determines a morphism of the associated stacks. The isomorphism of these two stacks comes from the following Cartesian diagram:

\[
\begin{array}{cccccc}
Y_\sigma \times G_\sigma & \xrightarrow{\varphi_0 \times \varphi_1} & V_\sigma \times G \\
\downarrow{(s,t)} & & \downarrow{(s,t)} \\
Y_\sigma \times Y_\sigma & \xrightarrow{\varphi_0 \times \varphi_0} & V_\sigma \times V_\sigma.
\end{array}
\]

\[\square\]

**Inertia stacks**

Let \( N_\sigma \) be the sublattice generated by \( \sigma \), and \( N(\sigma) := N/N_\sigma \). Note that when \( \sigma \) is a top dimensional cone, \( N(\sigma) \) is the local orbifold group in the local chart of
the coarse moduli space of the hypertoric toric Deligne-Mumford stack. Namely:

**Lemma 3.4.4** Let $\sigma$ be a top-dimensional cone in the multi-fan $\Delta_\beta$. Then $G_{\sigma} \cong N(\sigma)$.

**Proof.** The proof is the same as the proof for a top dimensional cone in a simplicial fan in Proposition 4.3 in [BCS]. \( \square \)

Recall that $G$ acts on $(\mathbb{C}^\times)^{2m}$ via the map $\alpha^L : G \to (\mathbb{C}^\times)^{2m}$ in (3.10). We write

$$\alpha^L(g) = (\alpha^L_1(g), \ldots, \alpha^L_m(g), \alpha^L_{1+m}(g), \ldots, \alpha^L_{2m}(g)).$$

**Lemma 3.4.5** Let $(z,w) \in Y$ be a point fixed by $g \in G$. If $\alpha^L_1(g) \neq 1$, then $z_i = w_i = 0$.

**Proof.** Since $G$ acts on $\mathbb{C}^{2m}$ through the matrix $\beta^y_L = [\beta^y, -\beta^y]$ in (3.9), we have that $\alpha^L_{i+m}(g) = \alpha^L_i(g)^{-1}$. The Lemma follows immediately. \( \square \)

Given the multi-fan $\Delta_\beta$, we consider the pairs $(v, \sigma)$, where $\sigma$ is a cone in $\Delta_\beta$, $v \in N$ such that $v = \sum_{\alpha_i \leq \sigma} \alpha_i b_i$ for $0 < \alpha_i < 1$. Note that $\sigma$ is the minimal cone in $\Delta_\beta$ satisfying the above condition. Let $Box(\Delta_\beta)$ be the set of all such pairs $(v, \sigma)$.

**Proposition 3.4.6** There is an one-to-one correspondence between $g \in G$ with nonempty fixed point set and $(v, \sigma) \in Box(\Delta_\beta)$. Moreover, for such $g$ and $(v, \sigma)$ we have $[Y^g/G] \cong M(\mathcal{A}(\sigma))$.

**Proof.** Let $(v, \sigma) \in Box(\Delta_\beta)$. Since $\sigma$ is contained in a top dimensional cone $\tau$ in $\Delta_\beta$, we have $v \in N(\tau)$. By Lemma 3.4.4, $N(\tau) \cong G_{\tau}$. Hence $v$ determines an element in $G_{\tau}$. Using the morphism $\varphi_1$ in (3.23), we see that $g$ fixes a point in $Y$. 85
Conversely, suppose $g \in G$ fixes a point $(z, w)$ in $Y$, where $(z, w) \in \mathbb{C}^{2m}$. By Lemma 3.4.5, the point $(z, w)$ satisfies the condition that if $\alpha^T_k(g) \neq 1$ then $z_i = w_i = 0$. From the definition of $\mathbb{C}^{2m} \setminus V(I_\theta)$, there is a cone in $\Sigma_\theta$ containing the rays for which $z_i = w_i = 0$. By Lemma 3.4.1, the rays $\rho_i$ for which $z_i = 0$ is a cone in $\Delta_\beta$ which we call $\sigma$. So $g$ stabilizes $Y_\tau = \mathbb{C}^{2d}$ in $V_\tau$ through $\varphi_0$ in (3.24) for any top dimensional cone $\tau$ containing $\sigma$, and $g$ corresponds to an element $(v, \sigma) \in Box(\Delta_\beta)$.

From the definition of $W(\sigma)$ and $V(\sigma)$ in Proposition 3.4.2, we have $W(\sigma) \cong Y^g$ and $[V(\sigma)/G] \cong [Y^g/G]$ which is $\mathcal{M}(A(\sigma))$. □

We determine the inertia stack of a hypertoric Deligne-Mumford stack.

**Proposition 3.4.7** The inertia stack of $\mathcal{M}(A)$ is given by

$$I(\mathcal{M}(A)) = \prod_{(v, \sigma) \in Box(\Delta_\beta)} \mathcal{M}(A(\sigma)).$$

**Proof.** The hypertoric Deligne-Mumford stack $\mathcal{M}(A) = [Y/G]$ is a quotient stack. Its inertia stack is determined as

$$I(\mathcal{M}(A)) = \left[ \left( \prod_{g \in G} Y^g \right) / G \right].$$

By Proposition 3.4.6, the stack $[Y^g/G]$ is isomorphic to the stack $\mathcal{M}(A(\sigma))$ for some $(v, \sigma) \in Box(\Delta_\beta)$. □

**Example** Let $\Sigma = (N, \Sigma, \beta)$ be an extended stacky fan, where $N = \mathbb{Z}^2$, the simplicial
fan $\Sigma$ is the fan of weighted projective plane $\mathbb{P}(1,2,2)$, and $\beta : \mathbb{Z}^4 \to N$ is given by the vectors \{$(b_1 = (1,0), b_2 = (0,1), b_3 = (-2,-2), b_4 = (0,-1))$, where $b_1, b_2, b_3$ are the generators of the rays in $\Sigma$. Choose generic element $\theta = (1,1) \in DG(\beta) \cong \mathbb{Z}^2$. Then $A = (N, \beta, \theta)$ is the stacky hyperplane arrangement whose induced extended stacky fan is $\Sigma$. A lifting of $\theta$ in $\mathbb{Z}^4$ through the Gale dual map $\beta^\vee$ is $r = (1,1,-3,0)$.

The corresponding hyperplane arrangement $\mathcal{H} = (H_1, H_2, H_3, H_4)$ consists of 4 lines, see Figure 2. Take $v = \frac{1}{2}b_3$, then $(v, \sigma) \in Box(\Delta_\beta)$, where $\sigma$ is the ray generated by $b_3$. Consider the following diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & Z & \longrightarrow & Z^4 & \longrightarrow & Z^3 & \longrightarrow & 0 \\
\downarrow{\beta_{\sigma}} & & \downarrow{\beta} & & \downarrow{\beta(\sigma)} & & \\
0 & \longrightarrow & N_{\sigma} & \longrightarrow & N & \longrightarrow & Z \oplus Z_2 & \longrightarrow & 0.
\end{array}
$$

We have the quotient extended stacky fan $\Sigma/\sigma = (N(\sigma), \Sigma/\sigma, \beta(\sigma))$, where $\beta(\sigma) : Z^3 \to N(\sigma)$ is given by the vectors $\{(1,0), (-1,0), (1,0)\}$, and $(1,0)$ is the extra data in the quotient extended stacky fan. Taking Gale dual, we get

$$
\begin{array}{cccccc}
0 & \longrightarrow & Z^3 & \longrightarrow & Z^4 & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow{\beta(\sigma)^{\vee}} & & \downarrow{\beta^{\vee}} & & \downarrow{\beta_{\sigma}^{\vee}} & & \\
0 & \longrightarrow & Z^2 \oplus Z_2 & \longrightarrow & Z^2 & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}
$$

where $\beta^{\vee}$ is given by the matrix $\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $\beta(\sigma)^{\vee}$ is given by $\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

The associated generic element $\theta(\sigma) = (1,1,0)$ and the lifting of $\theta(\sigma)$ in $Z^3$ is $r(\sigma) = \frac{1}{2}b_3$. 

Figure 2: The correspondence of the hyperplane arrangement and an extended stacky fan
(1, 1, -3). So the quotient hyperplane arrangement $A(\sigma) = (N(\sigma), \beta(\sigma), \theta(\sigma))$ is a line with three distinct points \{-1, 1, 3\}. The bounded polyhedron of this hyperplane arrangement is two segments intersecting at one point, see Figure 3.

Figure 3: The bounded polyhedron

The core of $M(A(\sigma))$ corresponds to these two segments, hence is two $\mathbb{P}^1$'s meeting at one point. Adding the stacky structure the twisted sector $M(A(\sigma))$ corresponding to the element $v$ is the trivial $\mu_2$-gerbe over the crepant resolution of the stack $[\mathbb{C}^2/\mathbb{Z}_3]$.

### 3.5 Orbifold Chow Ring of $M(A)$

In this section we discuss the orbifold Chow ring of hypertoric Deligne-Mumford stacks. We determine its module structure, then compute the orbifold cup product.

#### 3.5.1 The module structure

We first consider the ordinary Chow ring for hypertoric Deligne-Mumford stacks. According to [K], the cohomology ring of $M(A)$ is generated by the Chern classes of some line bundles defined as follows. Applying $\text{Hom}_\mathbb{Z}(-, \mathbb{C}^\times)$ to (3.4), we have

\[ 1 \rightarrow \mu \rightarrow G \overset{\alpha}{\rightarrow} (\mathbb{C}^\times)^m \rightarrow T \rightarrow 1. \]

**Definition 3.5.1** For every $b_i$ in the stacky hyperplane arrangement, define the line bundle $L_i$ over $M(A)$ to be the trivial line bundle $Y \times \mathbb{C}$ with the $G$-action on $\mathbb{C}$ defined via the $i$-th component of the morphism $\alpha : G \rightarrow (\mathbb{C}^\times)^m$ in the above exact sequence.
For any \( c \in N \), there is a cone \( \sigma \in \Delta_\beta \) such that \( c = \sum_{\rho_i \subseteq \sigma} \alpha_i \bar{b}_i \) where \( \alpha_i > 0 \) are rational numbers. Let \( N^\Delta_\beta \) denote all the pairs \((c, \sigma)\). Then \( N^\Delta_\beta \) gives rise a group ring

\[
\mathbb{Q}[\Delta_\beta] = \bigoplus_{(c, \sigma) \in N^\Delta_\beta} \mathbb{Q} \cdot y^{(c, \sigma)},
\]

where \( y \) is a formal variable. By abuse of notation, we write \( y^{(b_i, \rho_i)} \) as \( y^{b_i} \). The multiplication is given in terms of the ceiling function for fans which we define below. Since the multi-fan \( \Delta_\beta \) is simplicial, we have the following Lemma.

**Lemma 3.5.2** For any \( c \in N \), there exists a unique cone \( \sigma \in \Delta_\beta \) and \((v, \tau) \in Box(\Delta_\beta)\) such that \( \tau \subseteq \sigma \) and

\[
c = v + \sum_{\rho_i \subseteq \sigma} m_i b_i
\]

where \( m_i \in \mathbb{Z}_{\geq 0}. \)

**Definition 3.5.3** \((v, \tau)\) is called the fractional part of \((c, \sigma)\).

Now for \((c, \sigma) \in N^\Delta_\beta\), from Lemma 3.5.2, we write \( c = v + \sum_{\rho_i \subseteq \sigma} m_i b_i \), where \( m_i \)'s are nonnegative integers. We define the ceiling function \([c]_\sigma\) by

\[
[c]_\sigma = \sum_{\rho_i \subseteq \tau} b_i + \sum_{\rho_i \subseteq \sigma} m_i b_i.
\]

Note that if \( \nu = 0 \), \([c]_\sigma = \sum_{\rho_i \subseteq \sigma} m_i b_i \). For two pairs \((c_1, \sigma_1), (c_2, \sigma_2)\), if \( \sigma_1 \cup \sigma_2 \) is a cone in \( \Delta_\beta \), define \( \epsilon(c_1, c_2) := [c_1]_{\sigma_1} + [c_2]_{\sigma_2} - [c_1 + c_2]_{\sigma_1 \cup \sigma_2} \). Let \( \sigma_\epsilon \subseteq \sigma_1 \cup \sigma_2 \) be the minimal cone in \( \Delta_\beta \) containing \( \epsilon(c_1, c_2) \) so that \((\epsilon(c_1, c_2), \sigma_\epsilon) \in N^\Delta_\beta \). The ceiling function \([c]_\sigma\) is an integral linear combination of \( b_i \)'s for \( \rho_i \subseteq \sigma \). We define the grading on \( \mathbb{Q}[\Delta_\beta] \) as follows. For any \((c, \sigma)\), write \( c = v + \sum_{\rho_i \subseteq \sigma} m_i b_i \), then

\[
deg(y^{(c, \sigma)}) := |\tau| + \sum_{\rho_i \subseteq \sigma} m_i.
\]
where $|\tau|$ is the dimension of $\tau$. Let $Cir(\Delta_\beta)$ be the ideal in $Q[\Delta_\beta]$ generated by the elements in (3.2). The multiplication $y^{(c_1, \sigma_1)} \cdot y^{(c_2, \sigma_2)}$ is defined by (3.1).

**Lemma 3.5.4** The multiplication (3.1) is associative.

**Proof.** For any three pairs $(c_1, \sigma_1), (c_2, \sigma_2), (c_3, \sigma_3)$, if $\sigma_1 \cup \sigma_2 \cup \sigma_3$ is a cone in $\Delta_\beta$, let $\sigma \subseteq \sigma_1 \cup \sigma_2 \cup \sigma_3$ be the minimal cone in $\Delta_\beta$ containing

$$
\epsilon(c_1, c_2, c_3) := [c_1]_{\sigma_1} + [c_2]_{\sigma_2} + [c_3]_{\sigma_3} - [c_1 + c_2 + c_3]_{\sigma_1 \cup \sigma_2 \cup \sigma_3},
$$

such that $(\epsilon(c_1, c_2, c_3), \sigma) \in N^{\Delta_\beta}$. Then we check from the properties of ceiling function that $(y^{(c_1, \sigma_1)} \cdot y^{(c_2, \sigma_2)}) \cdot y^{(c_3, \sigma_3)}$ and $y^{(c_1, \sigma_1)} \cdot (y^{(c_2, \sigma_2)} \cdot y^{(c_3, \sigma_3)})$ are both equal to

$$
\begin{cases} 
(-1)^{|\sigma|}y^{(c_1 + c_2 + c_3 + \epsilon(c_1, c_2, c_3), \sigma_1 \cup \sigma_2 \cup \sigma_3)} & \text{if } \sigma_1 \cup \sigma_2 \cup \sigma_3 \text{ is a cone in } \Delta_\beta, \\
0 & \text{otherwise}.
\end{cases}
$$

It is easy to check that the product preserves the grading, and the proof is left to readers. $\square$

Consider the map $\beta : \mathbb{Z}^m \to N$ which is given by $\{b_1, \cdots, b_m\}$. We take $\{1, \cdots, m\}$ as the vertex set. The matroid complex $M_\beta$ is defined using $\beta$ by requiring that $F \in M_\beta$ iff the normal vectors $\{\bar{b}_i\}_{i \in F}$ are linearly independent in $N$. The Stanley-Reisner ring of the matroid $M_\beta$ is

$$
Q[M_\beta] = \frac{Q[y^{b_1}, \cdots, y^{b_m}]}{I_{M_\beta}},
$$

where $I_{M_\beta}$ is the matroid ideal generated by the set of square-free monomials

$$
\{y^{b_{i_1} \cdots b_{i_k}} | \bar{b}_{i_1}, \cdots, \bar{b}_{i_k} \text{ linearly dependent in } N\}.
$$

It is clear that $Q[M_\beta]$ is a subring of $Q[\Delta_\beta]$ under the injection $y^{b_i} \mapsto y^{(b_i, \sigma_i)}$. 

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Lemma 3.5.5 Let $\mathcal{A} = (N, \beta, \theta)$ be a stacky hyperplane arrangement and $\mathcal{M}(\mathcal{A})$ the corresponding hypertoric Deligne-Mumford stack, then we have an isomorphism of graded rings

$$A^*(\mathcal{M}(\mathcal{A})) \cong \frac{\mathbb{Q}[M_{\beta}]}{Cir(\Delta_\beta)},$$

given by $c_1(L_i) \rightarrow y^{b_i}$, where $Cir(\Delta_\beta)$ is the ideal generated by elements in (3.2).

**Proof.** Let $Y(\beta^\vee, \theta)$ be the coarse moduli space of the hypertoric Deligne-Mumford stack $\mathcal{M}(\mathcal{A})$. By [HS], we have

$$A^*(Y(\beta^\vee, \theta)) \cong \frac{\mathbb{Q}[M_{\beta}]}{Cir(\Delta_\beta)},$$

given by $D_i \rightarrow y^{b_i}$, where $D_i$ is the $T$-equivariant Weil divisor on $Y(\beta^\vee, \theta)$. Let $a_i$ be the first lattice vector in the ray generated by $b_i$, then $\bar{b}_i = l_i a_i$ for some positive integer $l_i$. By [V], the Chow ring of the stack $\mathcal{M}(\mathcal{A})$ is isomorphic to the Chow ring of its coarse moduli space $Y(A, \theta)$ via $c_1(L_i) \rightarrow l^{-1}_i \cdot D_i$, and $\sum_{i=1}^{m} e(a_i) l_i y^{b_i} = \sum_{i=1}^{m} e(b_i) y^{b_i}$ for $e \in N^*$. □

Let $A^*_{orb}(\mathcal{M}(\mathcal{A}))$ denote the orbifold Chow ring of $\mathcal{M}(\mathcal{A})$, which by definition is $A^*(I(\mathcal{M}(\mathcal{A})))$ as a group. By Proposition 3.4.7, we have

$$A^*(I(\mathcal{M}(\mathcal{A}))) \cong \bigoplus_{(v, \sigma) \in Box(\Delta_\beta)} A^*(\mathcal{M}(\mathcal{A}(\sigma))).$$

For $(v, \sigma) \in Box(\Delta_\beta)$, there is an exact sequence of vector bundles,

$$0 \rightarrow TM(\mathcal{A}(\sigma)) \rightarrow TM(\mathcal{A})|_{\mathcal{M}(\mathcal{A}(\sigma))} \rightarrow N_v \rightarrow 0,$$

where $N_v$ denotes the normal bundle of $\mathcal{M}(\mathcal{A}(\sigma))$ in $\mathcal{M}(\mathcal{A})$. On the other hand, there is a surjective morphism

$$\bigoplus_{i=1}^{m} (L_i \oplus L_i^{-1}) \rightarrow TM(\mathcal{A}).$$

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Restricting this to $\mathcal{M}(A(\sigma))$ yields a surjective map

$$\bigoplus_{\rho_i \in \sigma(v)} (L_i \oplus L_i^{-1}) \to N_v.$$  

Moreover, the element in the local group represented by $v$ acts trivially on the kernel. Let $v$ act on $L_i$ with eigenvalue $e^{2\pi \sqrt{-1} w_i}$, where $w_i \in [0,1) \cap \mathbb{Q}$. It follows that the age function on the component $\mathcal{M}(A(\sigma))$ assumes the value

$$\sum_{\rho_i \subset \sigma} (w_i + (1 - w_i)) = |\sigma|.$$  

Hence $A^*_{orb}(\mathcal{M}(A))$ as a graded module coincides with

$$\bigoplus_{(v,\alpha) \in Box(A^0)} A^*(\mathcal{M}(A(\sigma)))[[\sigma]].$$

Note that $A^*_{orb}(\mathcal{M}(A))$ is $\mathbb{Z}$-graded, due to the fact that $\mathcal{M}(A)$ is hyperkähler.

Again since the multi-fan $\Delta_\beta$ is simplicial, we have the following result, similar to Lemma 4.6 in [Jiang2].

**Lemma 3.5.6** Let $\tau$ be a cone in the multi-fan $\Delta_\beta$. If $\{\rho_1, \cdots, \rho_t\} \subseteq \text{link}(\tau)$, and suppose $\rho_1, \cdots, \rho_t$ are contained in a cone $\sigma \in \Delta_\beta$. Then $\sigma \cup \tau$ is contained in a cone of $\Delta_\beta$. □

**Proposition 3.5.7** Let $\mathcal{M}(A)$ be the hypertoric Deligne-Mumford stack associated to the stacky hyperplane arrangement $A$, then we have an isomorphism of graded $A^*(\mathcal{M}(A))$-modules:

$$\frac{\mathbb{Q}[\Delta_\beta]}{\text{Cir}(\Delta_\beta)} \cong \bigoplus_{(v,\alpha) \in Box(\Delta_\beta)} A^*(\mathcal{M}(A(\sigma)))[\text{deg}(y^{(v,\alpha)})].$$

**Proof.** We use arguments similar to those in Proposition 4.7 of [Jiang2]. From Lemma 3.5.2 it is easy to see that

$$\mathbb{Q}[\Delta_\beta] \cong \bigoplus_{(v,\alpha) \in Box(\Delta_\beta)} y^{(v,\alpha)} \cdot \mathbb{Q}[M_\beta].$$

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Consider $\bigoplus_{(v,\sigma) \in Box(\Delta_\beta)} y^{(v,\sigma)} \cdot Cir(\Delta_\beta)$. It is an ideal of the ring $\bigoplus_{(v,\sigma) \in Box(\Delta_\beta)} y^{(v,\sigma)}$. 

$\mathbb{Q}[M_\beta]$, so

$$\frac{\mathbb{Q}[\Delta_\beta]}{Cir(\Delta_\beta)} \cong \bigoplus_{(v,\sigma) \in Box(\Delta_\beta)} \frac{y^{(v,\sigma)} \cdot \mathbb{Q}[M_\beta]}{y^{(v,\sigma)} \cdot Cir(\Delta_\beta)}.$$ 

For an element $(v,\sigma) \in Box(\Delta_\beta)$, let $\rho_1, \ldots, \rho_t \in link(\sigma)$. Then we have an induced matroid complex $M_\beta(\sigma)$, where $\beta(\sigma)$ is the map in the quotient stacky hyperplane arrangement $A(\sigma)$ and the quotient extended stacky fan $\Sigma/\sigma$. Similarly from $\beta(\sigma)$, we have multi-fan $\Delta_\beta(\sigma)$ in $N(\sigma)$. By Lemma 3.5.5, $A^*(M(A(\sigma))) \cong \mathbb{Q}[M_\beta(\sigma)]/Cir(\Delta_\beta(\sigma))$. For any element $(v,\sigma) \in Box(\Delta_\beta)$, we construct an isomorphism

$$\Psi_v : \frac{\mathbb{Q}[\Delta_\beta]}{Cir(\Delta_\beta)}[deg(y^{(v,\sigma)})] \longrightarrow \frac{y^{(v,\sigma)} \cdot \mathbb{Q}[M_\beta]}{y^{(v,\sigma)} \cdot Cir(\Delta_\beta)}.$$ 

as follows. Consider the quotient stacky hyperplane arrangement $A(\sigma) = (N(\sigma), \beta(\sigma), \theta(\sigma))$. The hypertoric Deligne-Mumford stack $M(A(\sigma))$ is a closed substack of $M(A)$. Consider the morphism $i : \mathbb{Q}[y^{b_1}, \ldots, y^{b_t}] \rightarrow \mathbb{Q}[y^{b_1}, \ldots, y^{b_m}]$ given by $y^{b_i} \rightarrow y^{b_i}$. By Lemma 3.5.6, it is easy to check that the ideal $I_{M_\beta(\sigma)}$ is mapped to the ideal $I_{M_\beta}$, so we have a morphism $\mathbb{Q}[M_\beta(\sigma)] \rightarrow \mathbb{Q}[M_\beta]$. Since $\mathbb{Q}[M_\beta]$ is a subring of $\mathbb{Q}[\Delta_\beta]$. Let $\bar{\Psi}_v : \mathbb{Q}[M_\beta(\sigma)][deg(y^{(v,\sigma)})] \rightarrow \frac{y^{(v,\sigma)} \cdot \mathbb{Q}[M_\beta]}{y^{(v,\sigma)} \cdot Cir(\Delta_\beta)}$ be the morphism given by $y^{b_i} \rightarrow y^{(v,\sigma)} \cdot y^{b_i}$. Using similar arguments as in Proposition 4.7 of [Jiang2], we find that the ideal $Cir(\Delta_\beta(\sigma))$ goes to the ideal $y^{(v,\sigma)} \cdot Cir(\Delta_\beta)$, so we have the morphism $\Psi_v$ such that $\Psi_v(y^{b_i}) = [y^{(v,\sigma)} \cdot y^{b_i}]$.

Conversely, for $(v,\sigma) \in Box(\Delta_\beta)$, since $\sigma$ is simplicial, for $\rho_i \subset \sigma$ we can choose $\theta_i \in Hom_\mathbb{Z}(N, \mathbb{Q})$ such that $\theta_i(b_i) = 1$ and $\theta_i(b_{i'}) = 0$ for $b_{i'} \neq b_i \in \sigma$. We
consider the following morphism \( p : \mathbb{Q}[y^b_1, \ldots, y^b_m] \to \mathbb{Q}[y^b_1, \ldots, y^b_n] \) given by:

\[
y^b_i \mapsto \begin{cases} 
y^b_i & \text{if } \rho_i \subseteq \text{link}(\sigma), \\
- \sum_{j=1}^{l} \theta_i(b_j)y^b_j & \text{if } \rho_i \subseteq \sigma, \\
0 & \text{if } \rho_i \not\subseteq \sigma \cup \text{link}(\sigma).
\end{cases}
\]

Again by Lemma 3.5.6, the ideal \( I_{M^0} \) is mapped by \( p \) to the ideal \( I_{M^0} \). Then \( p \) induces a surjective map \( \mathbb{Q}[M_\beta] \to \mathbb{Q}[M_\beta] \) and a surjective map \( \Phi_v : y^{(v, \sigma)} \cdot \mathbb{Q}[M_\beta] \to \mathbb{Q}[M_\beta] \). Using the same computation as in Proposition 4.7 in [Jiang2], the relations \( y^{(v, \sigma)} \cdot \text{Cir}(\Delta_\beta) \) is seen to go to \( \text{Cir}(\Delta_\beta) \). This yields another morphism

\[
\Phi_v : y^{(v, \sigma)} \cdot \mathbb{Q}[M_\beta] \to \mathbb{Q}[M_\beta][\text{deg}(y^{(v, \sigma)})]
\]

so that \( \Phi_v \Psi_v = 1, \Psi_v \Phi_v = 1 \). So \( \Psi_v \) is an isomorphism. We conclude by Lemma 3.5.5. \( \square \)

### 3.5.2 The orbifold product

In this section we compute the orbifold cup product. First for any \((v_1, \sigma_1), (v_2, \sigma_2) \in \text{Box}(\Delta_\beta)\), we have the following lemma:

**Lemma 3.5.8** If \( \sigma_1 \cup \sigma_2 \) is a cone in the multi-fan \( \Delta_\beta \), there exists a unique \( (v_3, \sigma_3) \in \text{Box}(\Delta_\beta) \) such that \( \sigma_1 \cup \sigma_2 \cup \sigma_3 \) is a cone in the multi-fan \( \Delta_\beta \) and \( v_1 + v_2 + v_3 \) has no fractional part.

**Proof.** Let \( v_3 = [v_1 + v_2]_{\sigma_1 \cup \sigma_2} - v_1 - v_2 \) and \( \sigma_3 \) the minimal cone in \( \sigma_1 \cup \sigma_2 \) containing \( v_3 \). Then \( (v_3, \sigma_3) \) satisfies the conditions of the Lemma. \( \square \)

The notation \((v_1, \sigma_1) + (v_2, \sigma_2) + (v_3, \sigma_3) \equiv 0\) means that \((v_1, \sigma_1), (v_2, \sigma_2), (v_3, \sigma_3)\) satisfies the conditions in Lemma 3.5.8.
By [CR2], the 3-twisted sector $M(A)_{(g_1, g_2, g_3)}$ is the moduli space of 3-pointed genus 0 degree 0 orbifold stable maps to $M(A)$. Let $\mathbb{P}^1(0, 1, \infty)$ be a genus 0 twisted curve with stacky structures possibly at $0, 1, \infty$. Consider a constant map $f : \mathbb{P}^1(0, 1, \infty) \to M(A)$ with image $x \in M(A)$. This induces a morphism
\[
\rho : \pi_1^{orb}(\mathbb{P}^1(0, 1, \infty)) \to G_x,
\]
where $G_x$ is the local group of the point $x$. Let $\gamma_i$ be generators of $\pi_1^{orb}(\mathbb{P}^1(0, 1, \infty))$ and $g_i := \rho(\gamma_i)$. The $g_i$ fixes the point $x$. By Proposition 3.4.6, $g_i$ corresponds to an element $(v_i, \sigma_i) \in Box(\Delta_\beta)$. An argument similar to that in Proposition 6.1 in [BCS] shows that 3-twisted sectors of the hypertoric Deligne-Mumford stack $M(A)$ are given by
\[
\prod_{(v_1, \sigma_1), (v_2, \sigma_2), (v_3, \sigma_3) \in Box(\Delta_\beta), (v_1, \sigma_1) + (v_2, \sigma_2) + (v_3, \sigma_3) \equiv 0} M(A(\sigma_{123})),
\]
where $\sigma_{123}$ is the cone in $\Delta_\beta$ satisfying $v_1 + v_2 + v_3 = \sum_{\rho_i \subseteq \sigma_{123}} a_i b_i$, $a_i = 1, 2$. Let $ev_i : M(A(\sigma_{123})) \to M(A(\sigma_i))$ be the evaluation map. We have the obstruction bundle (see [CR1]) $Ob(v_1, v_2, v_3)$ over the 3-twisted sector $M(A(\sigma_{123}))$,
\[
Ob(v_1, v_2, v_3) = (e^*T(M(A)) \otimes H^1(C, O_C))^H
\]
where $e : M(A(\sigma_{123})) \to M(A)$ is the embedding, $C \to \mathbb{P}^1$ is the $H$-covering branching over three marked points $\{0, 1, \infty\} \subset \mathbb{P}^1$, and $H$ is the group generated by $v_1, v_2, v_3$. Let $(v, \sigma) \in Box(\Delta_\beta)$, say $v \in N(\tau)$ for some top dimensional cone $\tau$. Let $(\bar{v}, \sigma) \in Box(\Delta_\beta)$ be the inverse of $v$ as an element in the group $N(\tau)$. Equivalently, if $\bar{v} = \sum_{\rho_i \supseteq \sigma} a_i \bar{b}_i$ for $0 < a_i < 1$, then $\bar{v} = \sum_{\rho_i \supseteq \sigma} (1 - a_i) \bar{b}_i$.

**Lemma 3.5.9** Let $(v_1, \sigma_1), (v_2, \sigma_2), (v_3, \sigma_3) \in Box(\Delta_\beta)$ such that $v_1 + v_2 + v_3 \equiv 0$. Then if $\bar{v}_1 + \bar{v}_2 + \bar{v}_3 = \sum_{\rho_i \subseteq \sigma_{123}} a_i \bar{b}_i$, $\bar{v}_1 + \bar{v}_2 + \bar{v}_3 = \sum_{\rho_i \subseteq \sigma_{123}} c_i \bar{b}_i$, where $a_i, c_i = 1$ or 2, then $a_i + c_i = 2$ or 3.
PROOF. Let $\bar{v}_i = \sum_{\rho_j \subseteq \sigma_i} \alpha_j^i \bar{b}_j$, with $0 < \alpha_j^i < 1$ and $i = 1, 2, 3$. Then $\bar{v}_i = \sum_{\rho_j \subseteq \sigma_i} (1 - \alpha_j^i) \bar{b}_j$. From the condition we have $\alpha_j^i + \alpha_j^2 + \alpha_j^3 = a_j = 1$ or $2$ and $(1 - \alpha_j^i) + (1 - \alpha_j^2) + (1 - \alpha_j^3) = c_j = 2$ or $1$. So if $\rho_j$ belongs to $\sigma_1, \sigma_2$ and $\sigma_3$, then $\alpha_j^1, \alpha_j^2, \alpha_j^3$ exist and if $a_j = 1$ or $2$, then $c_j = 2$ or $1$. If $\rho_j$ belongs to $\sigma_1, \sigma_2$, but not $\sigma_3$, then $\alpha_j^3$ doesn't exist and $\alpha_j^1 + \alpha_j^2 = a_j = 1$, $(1 - \alpha_j^1) + (1 - \alpha_j^2) = c_j = 1$. The cases that $\rho_j$ belongs to $\sigma_1, \sigma_3$ but not $\sigma_2$, to $\sigma_2, \sigma_3$ but not $\sigma_1$ are similar. We omit them. □

The stack $\mathcal{M}(A)$ is an abelian Deligne-Mumford stack, i.e. the local groups are all abelian groups. For any 3-twisted sector $\mathcal{M}(\sigma_{123})$, the normal bundle $N(\mathcal{M}(\sigma_{123}))/\mathcal{M}(A)$ can be split into the direct sum of some line bundles under the group action. It follows from the definition that if $\bar{v} = \sum_{\rho_i \subseteq \sigma_{123}} \alpha_i \bar{b}_i$, then the action of $\nu$ on the normal bundle $N(\mathcal{M}(\sigma_{123}))/\mathcal{M}(A)$ is given by the diagonal matrix $diag(\alpha_i, 1 - \alpha_i)$. A general result in [CH] and [JKK] about the obstruction bundle and Lemma 3.5.9 imply the following Proposition.

**Proposition 3.5.10** Let $\mathcal{M}(A)(v_1, v_2, v_3) = \mathcal{M}(\sigma_{123})$ be a 3-twisted sector of the stack $\mathcal{M}(A)$ such that $v_1, v_2, v_3 \neq 0$. Then the Euler class of the obstruction bundle $Ob(v_1, v_2, v_3)$ on $\mathcal{M}(\sigma_{123})$ is

$$\prod_{\alpha_i = 2} c_1(L_i)|_{\mathcal{M}(\sigma_{123})} \cdot \prod_{\alpha_i = 1, \alpha_j^1, \alpha_j^2, \alpha_j^3 \text{ exist}} c_1(L_i^{-1})|_{\mathcal{M}(\sigma_{123})},$$

where $L_i$ is the line bundle over $\mathcal{M}(A)$ defined in Definition 3.5.1.

To prove the main theorem, we introduce two Lemmas. For any two pairs $(c_1, \sigma_1), (c_2, \sigma_2) \in N^{\Delta_{\beta}}$, there exist two unique elements $(v_1, \tau_1), (v_2, \tau_2) \in Box(\Delta_{\beta})$ such that $\tau_1 \subseteq \sigma_1, \tau_2 \subseteq \sigma_2$ and $c_1 = v_1 + \sum_{\rho_i \subseteq \sigma_1} m_i b_i$, $c_2 = v_2 + \sum_{\rho_i \subseteq \sigma_2} n_i b_i$, where $m_i, n_i$ are nonnegative integers. Let $(v_3, \sigma_3)$ be the unique element in $Box(\Delta_{\beta})$ such
that \( v_1 + v_2 + v_3 \equiv 0 \) in the local group given by \( \sigma_1 \cup \sigma_2 \). Let \( \bar{v}_i = \sum_{\rho_j \subseteq \sigma_i} \alpha_j^i \bar{b}_j \), with \( 0 < \alpha_j^i < 1 \) and \( i = 1, 2, 3 \). The existence of \( \alpha_1^i, \alpha_2^i, \alpha_3^i \) means that \( \rho_j \) belongs to \( \sigma_1, \sigma_2, \sigma_3 \). Let \( \sigma_{123} \) be the cone in \( \Delta_\beta \) such that \( \bar{v}_1 + \bar{v}_2 + \bar{v}_3 = \sum_{\rho_i \subseteq \sigma_{123}} a_i \bar{b}_i \), with \( a_i = 1 \) or \( 2 \). Let \( I \) be the set of \( i \) such that \( a_i = 1 \) and \( \alpha_1^i, \alpha_2^i, \alpha_3^i \) exist, \( J \) the set of \( j \) such that \( \rho_j \) belongs to \( \sigma_{123} \) but not to \( \sigma_3 \). We have the following Lemma for the ceiling functions:

**Lemma 3.5.11** \( [c_1]_{\sigma_1} + [c_2]_{\sigma_2} - [c_1 + c_2]_{\sigma_1 \cup \sigma_2} = [v_1]_{\tau_1} + [v_2]_{\tau_2} - [v_1 + v_2]_{\tau_1 \cup \tau_2} \).

**Proof.** By the definition of ceiling functions, we have \( [c_1]_{\sigma_1} = [v_1]_{\tau_1} + \sum_{\rho_i \subseteq \sigma_1} m_i b_i \) and \( [c_2]_{\sigma_2} = [v_2]_{\tau_2} + \sum_{\rho_i \subseteq \sigma_2} n_i b_i \). The Lemma follows. \( \square \)

**Lemma 3.5.12** If \( \sigma_1 \cup \sigma_2 \) is a cone in \( \Delta_\beta \) for the two pairs \( (c_1, \sigma_1), (c_2, \sigma_2) \), then the product \( y^{(c_1, \sigma_1)} \cdot y^{(c_2, \sigma_2)} \) in (3.1) can be given by

\[
\begin{cases}
(-1)|I| \cdot y^{(c_1+c_2+\sum_{i \in I} b_i + \sum_{j \in J} b_j, \sigma_1 \cup \sigma_2)} & \text{if} \ \bar{v}_1, \bar{v}_2 \neq 0 \ \text{and} \ \bar{v}_1 \neq \bar{v}_2, \\
(-1)|J| \cdot y^{(c_1+c_2+\sum_{i \in I} b_i, \sigma_1 \cup \sigma_2)} & \text{if} \ \bar{v}_1, \bar{v}_2 \neq 0 \ \text{and} \ \bar{v}_1 = \bar{v}_2, \\
y^{(c_1+c_2, \sigma_1 \cup \sigma_2)} & \text{if} \ \bar{v}_1 \ \text{or} \ \bar{v}_2 = 0.
\end{cases}
\] (3.27)

**Proof.** First for a fixed ray \( \rho_i \) and \( 0 < \alpha_1, \alpha_2 < 1 \), by the definition of ceiling functions, we find that

\[
[\alpha_1 b_i]_{\rho_i} + [\alpha_2 b_i]_{\rho_i} - [\alpha_1 + \alpha_2 b_i]_{\rho_i} = \begin{cases} 
0 & \text{if} \ \alpha_1 + \alpha_2 > 1, \\
b_i & \text{if} \ \alpha_1 + \alpha_2 \leq 1.
\end{cases}
\] (3.28)

Since \( e(c_1, c_2) = [c_1]_{\sigma_1} + [c_2]_{\sigma_2} - [c_1 + c_2]_{\sigma_1 \cup \sigma_2} \), by Lemma 3.5.11, we need to check that

\[
[v_1]_{\tau_1} + [v_2]_{\tau_2} - [v_1 + v_2]_{\tau_1 \cup \tau_2} = \begin{cases} 
\sum_{\rho_i \subseteq \sigma_1} b_i + \sum_{\rho_j \subseteq \sigma_2} b_j & \text{if} \ \bar{v}_1, \bar{v}_2 \neq 0 \ \text{and} \ \bar{v}_1 \neq \bar{v}_2, \\
\sum_{\rho_i \subseteq \sigma_1} b_i & \text{if} \ \bar{v}_1, \bar{v}_2 \neq 0 \ \text{and} \ \bar{v}_1 = \bar{v}_2, \\
0 & \text{if} \ \bar{v}_1 \ \text{or} \ \bar{v}_2 = 0.
\end{cases}
\]

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3.5.3 Proof of Theorem 3.1.1

By Proposition 3.5.17, it remains to prove that the orbifold cup product is the same as the product in the ring $\mathbb{Q}[\Delta_\beta]$. By Lemma 3.5.12, we need to prove that the orbifold cup product is the same as the product in (3.27). It suffices to consider the canonical generators $y^{b_i}$, $y^{(v,\sigma)}$ for $(v, \sigma) \in Box(\Delta_\beta)$.

Consider $y^{(v,\sigma)} \cup_{orb} y^{b_i}$ with $(v, \sigma) \in Box(\Delta_\beta)$. The element $(v, \sigma)$ determines a twisted sector $M(A(\sigma))$. The corresponding twisted sector to $b_i$ is the whole hypertoric stack $M(A)$. It is easy to see that the 3-twisted sector relevant to this product is $M(A)_{\{v, v^{-1}\}} \cong M(A(\sigma))$, where $v^{-1}$ denotes the inverse of $v$ in the local group. It follows from the dimension formula in [CR1] that the obstruction bundle over $M(A)_{\{v, v^{-1}\}}$ has rank zero. It is immediate from definition that $y^{(v_1,\sigma_1)} \cup_{orb} y^{b_i} = y^{(v+b_i,\sigma \cup \rho)}$ if there is a cone in $\Delta_\beta$ containing $v, b_i$. This is the third case in (3.27).

Now consider $y^{(v_1,\sigma_1)} \cup_{orb} y^{(v_2,\sigma_2)}$, where $(v_1, \sigma_1), (v_2, \sigma_2) \in Box(\Delta_\beta)$. By (3.25), we see that if $\sigma_1 \cup \sigma_2$ is not a cone in $\Delta_\beta$, then there is no 3-twisted sector corresponding to the elements $v_1, v_2$. Thus the product is zero by definition. On the other hand, by definition of the ring $\mathbb{Q}[\Delta_\beta]$, we have $y^{(v_1,\sigma_1)} \cdot y^{(v_2,\sigma_2)} = 0$. So $y^{(v_1,\sigma_1)} \cup_{orb} y^{(v_2,\sigma_2)} = y^{(v_1,\sigma_1)} \cdot y^{(v_2,\sigma_2)}$. If $\sigma_1 \cup \sigma_2$ is a cone in $\Delta_\beta$, let $(v_3, \sigma_3) \in Box(\Delta_\beta)$ such that $v_3 \in \sigma_{123}$ and $v_1 v_2 v_3 = 1$ in the local group. Then we have the 3-twisted sector $M(A(\sigma_{123}))$. Let $ev_i : M(A(\sigma_{123})) \to M(A(\sigma_i))$ be the evaluation maps. The element $y^{(v,\sigma)}$ is the class 1 in the cohomology of the twisted sector $M(A(\sigma))$.

From the definition of orbifold cup product [CR1], [AGV2], we have:

$$y^{(v_1,\sigma_1)} \cup_{orb} y^{(v_2,\sigma_2)} = (ev_3)_* (ev_1^* y^{(v_1,\sigma_1)} \cdot ev_2^* y^{(v_2,\sigma_2)} \cdot e(Ob_{(v_1, v_2, v_3)})),$$

where $ev_3 = I \circ ev_3 : M(A(\sigma_{123})) \to M(A(\sigma_3))$ is the composite of $ev_3$ and the
natural involution \( I : M(A)_{(v_3)} \to M(A)_{(v_3)} \). Let \( \overline{v}_i = \sum_{\rho_j \subseteq \sigma_i} \alpha_j^i \overline{b}_j \), with \( 0 < \alpha_j^i < 1 \) and \( i = 1, 2, 3 \). Let \( I \) denote the set of \( i \) such that \( a_i = 1 \) and \( \alpha_j^i, \alpha_j^2, \alpha_j^3 \) exist, \( J \) the set of \( j \) such that \( \rho_i \) belongs to \( \sigma_{123} \), but not belong to \( \sigma_3 \).

If some \( \overline{v}_i = 0 \), for example, \( \overline{v}_1 = 0 \), then \( v_1 \) is a torsion element in \( N \) which means that the action of \( v_1 \) is trivial on the hypertoric Deligne-Mumford stack. Then the 3-twisted sector corresponding to \( v_1, v_2 \) is isomorphic to the twisted sector \( M(A(\sigma_2)) \) and the obstruction bundle over \( M(A(\sigma_2)) \) is zero by [CR1]. In this case the set \( I \) and \( J \) are all empty. So \( y^{(v_1, \sigma_1)} \cup_{\text{orb}} y^{(v_2, \sigma_2)} = y^{(v_1 + v_2, \sigma_1 \cup \sigma_2)} \). This is again the third case in (3.27).

Now we assume that \( \overline{v}_1, \overline{v}_2 \neq 0 \). If \( \overline{v}_1 = \overline{v}_2 \), then \( \overline{v}_3 = 0 \), \( \sigma_{123} = \sigma_1 \) and \( v_1 + v_2 = \sum_{\rho_j \subseteq \sigma_1} b_j \). So the 3-twisted sector corresponding to \( v_1, v_2 \) is isomorphic to the twisted sector \( M(A(\sigma_1)) \) and the obstruction bundle over \( M(A(\sigma_1)) \) is zero by [CR1]. The set \( J \) is the set \( j \) such that \( \rho_j \subseteq \sigma_1 \). So we have

\[
y^{(v_1, \sigma_1)} \cup_{\text{orb}} y^{(v_2, \sigma_2)} = y^0 \cdot \prod_{i \in J} y^{b_i} \cdot \prod_{i \in J} (-y^{b_i}) = (-1)^{|J|} \cdot y^{(v_1 + v_2 + \sum_{i \in J} b_i, \sigma_1 \cup \sigma_2)},
\]

which is the second case in (3.27).

If \( \overline{v}_1 \neq \overline{v}_2 \), then \( \overline{v}_3 \neq 0 \) and the obstruction bundle over the 3-twisted sector \( M(A(\sigma_{123})) \) is given by Proposition 3.5.10. So we have:

\[
y^{(v_1, \sigma_1)} \cup_{\text{orb}} y^{(v_2, \sigma_2)} = y^{(v_3, \sigma_3)} \cdot \prod_{a_i = 2} y^{b_i} \cdot \prod_{i \in J} y^{b_i} \cdot \prod_{i \in I} (-y^{b_i}) \cdot \prod_{i \in J} (-y^{b_i}).
\]

Since \( \overline{v}_3 + \sum_{a_i = 2} b_i + \sum_{i \in J} b_i = v_1 + v_2 \), we have

\[
y^{(v_1, \sigma_1)} \cup_{\text{orb}} y^{(v_2, \sigma_2)} = (-1)^{|J| + |J|} \cdot y^{(v_1 + v_2, \sigma_1 \cup \sigma_2)} \cdot \prod_{i \in I} y^{b_i} \cdot \prod_{i \in J} y^{b_i} = (-1)^{|I| + |J|} \cdot y^{(v_1 + v_2 + \sum_{i \in I} b_i + \sum_{i \in J} b_i, \sigma_1 \cup \sigma_2)},
\]

which is the first case in (3.27).

\[\Box\]
3.6 Applications

In this section we compute some examples of the orbifold Chow rings of hypertoric Deligne-Mumford stacks. In particular, we relate the hypertoric stack to crepant resolutions.

Let \( N = \mathbb{Z} \) and \( \Sigma \) the fan of projective line \( \mathbb{P}^1 \) generated by \{ (1), (-1) \}. Let \( \beta : \mathbb{Z}^n \to N \) be the map given by \( b_1 = (1), b_2 = (-1) \) and \( b_i = (1) \) for \( i \geq 2 \).

Consider the following exact sequences

\[
0 \to \mathbb{Z}^{n-1} \to \mathbb{Z}^n \xrightarrow{\beta} N \to 0 \to 0,
\]

\[
0 \to \mathbb{Z} \to \mathbb{Z}^n \xrightarrow{\beta^\vee} \mathbb{Z}^{n-1} \to 0 \to 0,
\]

where the Gale dual \( \beta^\vee \) is given by the column vectors of the matrix

\[
A = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & -1 & 0 & \cdots \\
1 & 0 & 0 & -1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -1
\end{bmatrix}.
\]

Note that \( A \) is unimodular in the sense of [HS]. Taking \( \text{Hom}_\mathbb{Z}(-, \mathbb{C}^\times) \) yields

\[
1 \to (\mathbb{C}^\times)^{n-1} \xrightarrow{a} (\mathbb{C}^\times)^n \to \mathbb{C}^\times \to 1.
\]

So \( G = (\mathbb{C}^\times)^{n-1} \). Choose \( \theta = (1, 1, \cdots, 1) \) in \( \mathbb{Z}^{n-1} \), then it is a generic element. The extended stacky fan \( \Sigma = (N, \Sigma, \beta) \) is induced from the stacky hyperplane arrangement \( \mathcal{A} = (N, \beta, \theta) \), where \( \mathcal{H} \) is the hyperplane arrangement whose normal fan is \( \Sigma \). It is easy to see that the toric Deligne-Mumford stack is the projective line \( \mathbb{P}^1 \).

The hypertoric Deligne-Mumford stack is the crepant resolution of the Gorenstein
orbifold $[\mathbb{C}^2/\mathbb{Z}_n]$. To see this, from the construction of hypertoric Deligne-Mumford stack, we have:

$$1 \rightarrow (\mathbb{C}^\times)^{n-1} \xrightarrow{\alpha^L} (\mathbb{C}^\times)^{2n} \rightarrow (\mathbb{C}^\times)^{n+1} \rightarrow 1,$$

(3.29)

where $\alpha^L$ is given by the matrix $[\beta^\vee, -\beta^\vee]$. Let $\mathbb{C}[z_1, ..., z_n, w_1, ..., w_n]$ be the coordinate ring of $\mathbb{C}^{2n}$. So the ideal $I_{\beta^\vee}$ in (3.11) is generated by the following equations:

$$
\begin{align*}
z_1 w_1 + z_2 w_2 &= 0,
z_1 w_1 - z_3 w_3 &= 0,
\vdots & \vdots 
z_1 w_1 - z_n w_n &= 0.
\end{align*}
$$

Hence $Y$ is the subvariety of $\mathbb{C}^{2n} - V(I_\delta)$ determined by the above ideal. The action of $G$ on $Y$ is through the map $\alpha^L$ in (3.29). The hypertoric Deligne-Mumford stack associated to $A$ is $\mathcal{M}(A) = [Y/G]$. From Proposition 3.3.2, the hypertoric Deligne-Mumford stack is independent to the coorientations of the hyperplanes. This means that we can give the stacky hyperplane arrangement $A$ as follows. Let $b_i = 1$ for $1 \leq i \leq n$. Then the Gale dual map $\beta^\vee : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-1}$ is given by the matrix

$$
A = \begin{bmatrix}
1 & -1 & 0 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & \cdots & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{bmatrix},
$$

which is exactly the matrix in Lemma 10.2 in [HS], from which it follows that the coarse moduli space $Y(\beta^\vee, \theta)$ of $\mathcal{M}(A) = [Y/G]$ is the crepant resolution of the Gorenstein orbifold $[\mathbb{C}^2/\mathbb{Z}_n]$. The core of the hypertoric Deligne-Mumford stack
$\mathcal{M}(A)$ is a chain of $n - 1$ copies of $\mathbb{P}^1$ with normal crossing divisors corresponding to the multi-fan $\Delta_\beta$.

**Remark** This is an example of [Kro], in which it is shown that the minimal resolution of a surface singularity of ADE type can be constructed as a hyperkähler quotient.

The $\mathbb{Z}_n$-action defining the Gorenstein orbifold $[\mathbb{C}^2/\mathbb{Z}_n]$ is given by $\lambda(x, y) = (\lambda x, \lambda^{-1} y)$ for $\lambda \in \mathbb{Z}_n$. There are $n - 1$ twisted sectors each of which is isomorphic to $B\mathbb{Z}_n$ with age 1. There are only dimension zero cohomology on the untwisted sector and twisted sectors. So we prove the following Proposition:

**Proposition 3.6.1** The orbifold Chow ring $A^*_{orb}([\mathbb{C}^2/\mathbb{Z}_n])$ of $[\mathbb{C}^2/\mathbb{Z}_n]$ is isomorphic to the ring

$$\mathbb{C}[x_1, \ldots, x_{n-1}] / \{x_i x_j : 1 \leq i, j \leq n - 1\}.$$ 

Since the crepant resolution is a manifold, the orbifold Chow ring is the ordinary Chow ring. By Theorem 3.1.1, we have

**Proposition 3.6.2** The Chow ring of $\mathcal{M}(A)$ is isomorphic to the ring

$$\mathbb{C}[y_1, \ldots, y_{n-1}] / \{y_i y_j : 1 \leq i, j \leq n - 1\},$$

which is isomorphic to the orbifold cohomology ring of the Gorenstein orbifold $[\mathbb{C}^2/\mathbb{Z}_n]$.

**Proof.** By Theorem 3.1.1, the Chow ring of $\mathcal{M}(A)$ is isomorphic to the ring:

$$\mathbb{C}[y_1, \ldots, y_n] / \{y_1 - y_n + y_3 + \cdots + y_{n-1}, y_i y_j : 1 \leq i, j \leq n - 1\},$$

which we can easily check that this ring is isomorphic to the orbifold cohomology ring of $[\mathbb{C}^2/\mathbb{Z}_n]$ in Proposition 3.6.1. $\Box$
Y. Ruan [R] conjectured that, among other things, the orbifold cohomology ring of a hyperkähler orbifold is isomorphic to the ordinary cohomology ring of a hyperkähler resolution (which is crepant). For the orbifold \([\mathbb{C}^2/Z_n]\), the crepant resolution \(Y(\beta^\nu, \theta)\) is smooth, we have that \(M(A) \cong Y(\beta^\nu, \theta)\). From Proposition 3.6.2, the conjecture is true.

A conjecture equating Gromov-Witten theories of an orbifold and its crepant resolutions, as proposed in [BG], is recently proven in genus 0 for \([\mathbb{C}^2/Z_3]\), see [BGP]. The comparison of two Gromov-Witten theories requires certain change of variables. For \([\mathbb{C}^2/Z_4]\) case, see [BGP]. For \([\mathbb{C}^2/Z_4]\) case the following change of variables is found in [BJ]:

\[
\begin{align*}
  y_1 &= \frac{1}{4}(\sqrt{2}x_1 + 2ix_2 - \sqrt{2}x_3), \\
  y_2 &= \frac{1}{4}(\sqrt{2}ix_1 - 2ix_2 + \sqrt{2}ix_3), \\
  y_3 &= \frac{1}{4}(\sqrt{2}x_1 + 2ix_2 + \sqrt{2}x_3).
\end{align*}
\]

Under this change of variables, the genus zero Gromov-Witten potential of the crepant resolution is seen to coincide with the genus zero orbifold Gromov-Witten potential of the orbifold \([\mathbb{C}^2/Z_4]\), see [BJ].

For a toric orbifold, it is known that adding rays in the simplicial fan can give a crepant resolution. In the end of the paper we compute an example and explain that adding rays in the stacky hyperplane arrangement doesn’t give a smooth hypertoric variety which means that it is not easy in general to give a crepant resolution in hyperkähler geometry.

**Example** Let \(\Sigma = (N, \Sigma, \beta)\) be an extended stacky fan, where \(N = \mathbb{Z}^2\), the simplicial fan \(\Sigma\) is the fan of weighted projective plane \(\mathbb{P}(1,1,2)\), and \(\beta : \mathbb{Z}^3 \to N\) is given by the vectors \(\{b_1 = (1,0), b_2 = (0,1), b_3 = (-1,-2)\}\), where \(b_1, b_2, b_3\)
are the generators of the rays in $\Sigma$. The generic element $\theta = (1) \in DG(\beta) \cong \mathbb{Z}$ determines the fan $\Sigma$. The stacky hyperplane arrangement $A = (N, \beta, \theta)$ induces $\Sigma$. The hypertoric DM stack is $\mathcal{M}(A) = T^*(\mathbb{P}(1,1,2))$. From Theorem 3.1.1,

$$A^*_{\text{orb}}(\mathcal{M}(A)) \cong \frac{\mathbb{Q}[x_1, x_2, x_3, x_4]}{(x_1 - x_3, x_2 - 2x_3, x_4^2, x_1x_2x_3, x_4x_2, x_4x_1x_3)} \cong \frac{\mathbb{Q}[x_3, x_4]}{(x_4^2, x_3^2, x_3x_4)}.$$  

Let $b_4 = (0, -1)$ and consider the new map $\beta' : \mathbb{Z}^4 \to N$ which is given by the vectors $\{b_1, b_2, b_3, b_4\}$. Choose generic element $\theta' = (1, 1) \in \mathbb{Z}^2 = DG(\beta')$ and we get a new stacky hyperplane arrangement $A' = (N, \beta', \theta')$ which induces the extended stacky fan $\Sigma' = (N, \Sigma, \beta')$. The hypertoric DM stack $\mathcal{M}(A')$ is the stack corresponding to $A'$. From the definition of Box, $(\frac{1}{2}b_1 + \frac{1}{2}b_3, \rho_1 + \rho_3)$ is again a box element which determines a twisted sector. We compute that $A^*_{\text{orb}}(\mathcal{M}(A'))$ is isomorphic to

$$\frac{\mathbb{Q}[x_1, x_2, x_3, x_4, v]}{(x_1 - x_3, x_2 - 2x_3 - x_4, x_2x_4, x_1x_2x_3, x_1x_3x_4, v^2, vx_2, vx_4)} \cong \frac{\mathbb{Q}[x_3, x_4, v]}{(x_3x_4 + x_4^2, x_3^2x_4, v^2, vx_3, vx_4)}.$$  

We check that $A^*_{\text{orb}}(\mathcal{M}(A))$ is not isomorphic to the ring $A^*_{\text{orb}}(\mathcal{M}(A'))$.

We give two comments here. First, if the crepant resolution conjecture is true, then $\mathcal{M}(A')$ is not a hyperkähler resolution. On the other hand, the map $\beta$ is given by the matrix $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$ and the map $\beta'$ given by $B' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix}$.

It is easy to see that $B'$ is not unimodular which means that $\mathcal{M}(A')$ is not smooth, see [HS]. So adding columns in $B$ can't make it unimodular which means that adding rays in the stacky hyperplane arrangement can not give a hyperkähler resolution. Second, since $\mathcal{M}(A')$ is still a hypertoric orbifold, we wish that $\mathcal{M}(A')$ is a partial resolution and there is a morphism $\mathcal{M}(A') \to \mathcal{M}(A)$. But this is impossible because the two rings have different dimensions and it would violate the McKay correspondence statement if there exists a morphism $\mathcal{M}(A') \to \mathcal{M}(A)$, see [Ya1], [Ya2].
Question: Is there a combinatorial description of a hyperkähler resolution of hypertoric orbifolds?
Bibliography


[CH] B. Chen and S. Hu, A deRham model for Chen-Ruan cohomology ring of abelian orbifolds, math.SG/0408265.


Chapter 4

Semi-projective Toric Deligne-Mumford Stacks

4.1 Introduction

The main goal of this chapter is to generalize the orbifold Chow ring formula of Borisov-Chen-Smith for projective toric Deligne-Mumford stacks to the case of semi-projective toric Deligne-Mumford stacks.

In Chapter 1 we reviewed the construction of toric Deligne-Mumford stacks. The construction of toric Deligne-Mumford stacks was slightly generalized in Chapter 2, in which the notion of extended stacky fans was introduced. This new notion is based on that of stacky fans plus some extra data. Extended stacky fans yield toric Deligne-Mumford stacks in the same way as stacky fans do. The main point is that extended stacky fans provide presentations of toric Deligne-Mumford stacks.

1The content of this chapter has been submitted for publication.
not available from stacky fans.

When \( X(\Sigma) \) is projective, it is found in [BCS] that the orbifold Chow ring (or Chen-Ruan cohomology ring) of \( \mathcal{X}(\Sigma) \) is isomorphic to a deformed ring of the group ring of \( N \). We call a toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \) semi-projective if its coarse moduli space \( X(\Sigma) \) is semi-projective. Hausel and Sturmfels [HS] computed the Chow ring of semi-projective toric varieties. Their answer is also known as the "Stanley-Reisner" ring of a fan. Using their result, we prove a formula of the orbifold Chow ring of semi-projective toric Deligne-Mumford stacks.

Consider an extended stacky fan \( \Sigma = (N, \Sigma, \beta) \), where \( \Sigma \) is the simplicial fan of the semi-projective toric variety \( X(\Sigma) \). Let \( N_{\text{tor}} \) be the torsion subgroup of \( N \), then \( N = \bar{N} \oplus N_{\text{tor}} \). Let \( N_{\Sigma} := |\Sigma| \oplus N_{\text{tor}} \). Note that \( |\Sigma| \) is convex, so \( |\Sigma| \oplus N_{\text{tor}} \) is a subgroup of \( N \). Define the deformed ring \( \mathbb{Q}[N_{\Sigma}] := \bigoplus_{c \in N_{\Sigma}} \mathbb{Q}y^c \) with the product structure given by

\[
y^{c_1} \cdot y^{c_2} := \begin{cases} 
y^{c_1 + c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \bar{c}_1 \in \sigma, \bar{c}_2 \in \sigma; \\
0 & \text{otherwise.} \end{cases} \tag{4.1}
\]

Note that if \( \mathcal{X}(\Sigma) \) is projective, then \( N_{\Sigma} = N \) and \( \mathbb{Q}[N_{\Sigma}] \) is the deformed ring \( \mathbb{Q}[N]^{\Sigma} \) in [BCS]. Let \( A^{*\text{orb}}(\mathcal{X}(\Sigma)) \) denote the orbifold Chow ring of the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \).

**Theorem 4.1.1** Assume that \( \mathcal{X}(\Sigma) \) is semi-projective. There is an isomorphism of rings

\[
A^{*\text{orb}}(\mathcal{X}(\Sigma)) \cong \frac{\mathbb{Q}[N_{\Sigma}]}{\left\{ \sum_{i=1}^{n} e(h_i)y^{h_i} : e \in N^{*\Sigma} \right\}}.
\]

The strategy of proving Theorem 4.1.1 is as follows. We use a formula in [HS] for the ordinary Chow ring of semi-projective toric varieties. We prove that each twisted sector is also a semi-projective toric Deligne-Mumford stack. With this, we
use a method similar to that in [BCS] and [Jiang2] to prove the isomorphism as modules. The argument to show the isomorphism as rings is the same as that in [BCS], except that we only take elements in the support of the fan.

An interesting class of examples of semi-projective toric Deligne-Mumford stack is the Lawrence toric Deligne-Mumford stacks. We discuss the properties of such stacks. We prove that each 3-twisted sector or twisted sector is again a Lawrence toric Deligne-Mumford stack. This allows us to draw connections to hypertoric Deligne-Mumford stacks studied in [JT1]. We prove that the orbifold Chow ring of a Lawrence toric Deligne-Mumford stack is isomorphic to the orbifold Chow ring of its associated hypertoric Deligne-Mumford stack. This is an analog of Theorem 1.1 in [HS] for orbifold Chow rings.

The rest of this text is organized as follows. In Section 4.2 we define semi-projective toric Deligne-Mumford stacks and prove Theorem 4.1.1. Results on Lawrence toric Deligne-Mumford stacks are discussed in Section 4.3.

**Convention**

In this chapter we use $N^*$ to represent the dual of $N$ and $\mathbb{C}^*$ the multiplicative group $\mathbb{C} - \{0\}$.

### 4.2 Semi-projective Toric Deligne-Mumford Stacks and Their Orbifold Chow Rings

In this section we define semi-projective toric Deligne-Mumford stacks and discuss their properties.
4.2.1 Semi-projective toric Deligne-Mumford stacks

**Definition 4.2.1 ([HS])** A toric variety $X$ is called semi-projective if the natural map

$$\pi : X \to X_0 = \text{Spec}(H^0(X, \mathcal{O}_X)),$$

is projective and $X$ has at least one torus-fixed point.

**Definition 4.2.2 ([Jiang2])** An extended stacky fan $\Sigma$ is a triple $(N, \Sigma, \beta)$, where $N$ is a finitely generated abelian group, $\Sigma$ is a simplicial fan in $N_\mathbb{R}$ and $\beta : \mathbb{Z}^m \to N$ is the map determined by the elements $\{b_1, \ldots, b_m\}$ in $N$ such that $\{\tilde{b}_1, \ldots, \tilde{b}_n\}$ generate the simplicial fan $\Sigma$ (here $m \geq n$).

Given an extended stacky fan $\Sigma = (N, \Sigma, \beta)$, we have the following exact sequences:

$$0 \to DG(\beta)^* \to \mathbb{Z}^m \overset{\beta}{\to} N \overset{Coker(\beta)}{\to} 0, \quad (4.2)$$

$$0 \to N^* \to \mathbb{Z}^m \overset{\beta^\vee}{\to} DG(\beta) \overset{Coker(\beta^\vee)}{\to} 0, \quad (4.3)$$

where $\beta^\vee$ is the Gale dual of $\beta$ (see Chapter 2). Applying $\text{Hom}_\mathbb{Z}(\_ , \mathbb{C}^*)$ to (4.3) yields

$$1 \to \mu \to G \overset{\alpha}{\to} (\mathbb{C}^*)^m \to (\mathbb{C}^*)^d \to 1. \quad (4.4)$$

The toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is the quotient stack $[Z/G]$, where $Z := (\mathbb{C}^n \setminus V(J_\Sigma)) \times (\mathbb{C}^*)^{m-n}$, $J_\Sigma$ is the irrelevant ideal of the fan $\Sigma$ and $G$ acts on $Z$ through the map $\alpha$ in (4.4). The coarse moduli space of $\mathcal{X}(\Sigma)$ is the simplicial toric variety $X(\Sigma)$ corresponding to the simplicial fan $\Sigma$, see [BCS] and [Jiang2].

**Definition 4.2.3** A toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is *semi-projective* if the coarse moduli space $X(\Sigma)$ is semi-projective.

**Theorem 4.2.4** The following notions are equivalent:
1. A semi-projective toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$;

2. A toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ such that the simplicial fan $\Sigma$ is a regular triangulation of $B = \{\bar{b}_1, \cdots, \bar{b}_n\}$ which spans the lattice $\overline{N}$.

**Proof.** Since the toric Deligne-Mumford stack is semi-projective if its coarse moduli space is semi-projective, the theorem follows from results in [HS]. $\square$

4.2.2 The inertia stack

Let $\Sigma$ be an extended stacky fan and $\sigma \in \Sigma$ a cone. Define $\text{link}(\sigma) := \{\tau : \sigma + \tau \in \Sigma, \sigma \cap \tau = 0\}$. Let $\{\tilde{\rho}_1, \ldots, \tilde{\rho}_l\}$ be the rays in $\text{link}(\sigma)$. Consider the quotient extended stacky fan $\Sigma/\sigma = (N(\sigma), \Sigma/\sigma, \beta(\sigma))$, with $\beta(\sigma) : \mathbb{Z}^{l+m-n} \to N(\sigma)$ given by the images of $b_1, \ldots, b_l$ and $b_{n+1}, \ldots, b_m$ under $N \to N(\sigma)$. By the construction of toric Deligne-Mumford stacks, if $\sigma$ is contained in a top dimensional cone in $\Sigma$, we have $\mathcal{X}(\Sigma/\sigma) := [Z(\sigma)/G(\sigma)]$, where $Z(\sigma) = (\mathbb{A}^l \setminus \cup(J_{\Sigma/\sigma})) \times (\mathbb{C}^*)^{m-n}$ and $G(\sigma) = \text{Hom}_{\mathbb{Z}}(DG(\beta(\sigma)), \mathbb{C}^*)$.

**Lemma 4.2.5** If $\mathcal{X}(\Sigma)$ is semi-projective, so is $\mathcal{X}(\Sigma/\sigma)$.

**Proof.** Semi-projectivity of the stack $\mathcal{X}(\Sigma)$ means the simplicial fan $\Sigma$ is a fan coming from a regular triangulation of $B = \{\bar{b}_1, \cdots, \bar{b}_n\}$ which spans the lattice $\overline{N}$. Let $\text{pos}(B)$ be the convex polyhedral cone generated by $B$. Then from [HS], the triangulation is supported on $\text{pos}(B)$ and is determined by a simple polyhedron whose normal fan is $\Sigma$. So $\sigma$ is contained in a top-dimensional cone $\tau$ in $\Sigma$. The image $\tilde{\tau}$ of $\tau$ under quotient by $\sigma$ is a top-dimensional cone in the quotient fan $\Sigma/\sigma$. So the toric variety $\mathcal{X}(\Sigma/\sigma)$ is semi-projective by Theorem 4.2.4, and the stack $\mathcal{X}(\Sigma/\sigma)$ is semi-projective by definition. $\square$

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Recall in [BCS] that for each top-dimensional cone $\sigma$ in $\Sigma$, define $Box(\sigma)$ to be the set of elements $v \in N$ such that $v = \sum_{\rho_i \leq \sigma} a_i \tilde{b}_i$ for some $0 \leq a_i < 1$. Elements in $Box(\sigma)$ are in one-to-one correspondence with elements in the finite group $N(\sigma) = N/N'_{\sigma}$, where $N(\sigma)$ is a local group of the stack $X(\Sigma)$. In fact, we write $v = \sum_{\rho_i \leq \sigma(\bar{v})} a_i \tilde{b}_i$ for some $0 < a_i < 1$, where $\sigma(\bar{v})$ is the minimal cone containing $\bar{v}$. Denoted by $Box(\Sigma)$ the union of $Box(\sigma)$ for all top-dimensional cones $\sigma$.

**Proposition 4.2.6** The $r$-inertia stack is given by

$$I_r(X(\Sigma)) = \prod_{(v_1, \ldots, v_r) \in Box(\Sigma)^r} X(\Sigma/\sigma(v_1, \ldots, v_r)), \quad (4.5)$$

where $\sigma(v_1, \ldots, v_r)$ is the minimal cone in $\Sigma$ containing $v_1, \ldots, v_r$.

**Proof.** Since $G$ is an abelian group, we have

$$I_r(X(\Sigma)) = \left( \prod_{(v_1, \ldots, v_r) \in (G)^r} Z(v_1, \ldots, v_r)/G \right),$$

where $Z(v_1, \ldots, v_r) \subset Z$ is the subvariety fixed by $v_1, \ldots, v_r$. Since $\sigma(v_1, \ldots, v_r)$ is contained in a top-dimensional cone in $\Sigma$. We use the same method as in Lemma 4.6 and Proposition 4.7 of [BCS] to prove that $[Z(v_1, \ldots, v_r)/G] \cong X(\Sigma/\sigma(v_1, \ldots, v_r))$. □

Note that in (4.5) each component is semi-projective.

### 4.2.3 The orbifold Chow ring

In this section we compute the orbifold Chow ring of semi-projective toric Deligne-Mumford stacks and prove Theorem 4.1.1.
The module structure

Let $\Sigma = (N, \Sigma, \beta)$ be an extended stacky fan such that the toric Deligne-Mumford stack $X(\Sigma)$ is semi-projective. Since the fan $\Sigma$ is convex, $|\Sigma|$ is an abelian subgroup of $N$. We put $N_\Sigma := |\Sigma| \oplus N_{tor}$, where $N_{tor}$ is the torsion subgroup of $N$. Define the deformed ring $\mathbb{Q}[N_\Sigma] := \bigoplus_{c \in N_\Sigma} \mathbb{Q}y^c$ with the product structure given by (4.1).

Let $\{\rho_1, \ldots, \rho_n\}$ be the rays of $\Sigma$, then each $\rho_i$ corresponds to a line bundle $L_i$ over the toric Deligne-Mumford stack $X(\Sigma)$ given by the trivial line bundle $\mathbb{C} \times \mathbb{Z}$ over $\mathbb{Z}$ with the $G$ action on $\mathbb{C}$ given by the $i$-th component $\alpha_i$ of $\alpha : G \to (\mathbb{C}^*)^m$ in (4.4). The first Chern classes of the line bundles $L_i$, which we identify with $y^{b_i}$, generate the cohomology ring of the simplicial toric variety $X(\Sigma)$.

Let $S_\Sigma$ be the quotient ring $\mathbb{Q}[y^{b_1}, \ldots, y^{b_n}]_{J_\Sigma}$, where $J_\Sigma$ is the square-free ideal of the fan $\Sigma$ generated by the monomials

$$\{y^{b_{i_1}} \cdots y^{b_{i_k}} : \overline{b}_{i_1}, \ldots, \overline{b}_{i_k} \text{ do not generate a cone in } \Sigma\}.$$  

It is clear that $S_\Sigma$ is a subring of the deformed ring $\mathbb{Q}[N_\Sigma]$.

**Lemma 4.2.7** Let $A^*(X(\Sigma))$ be the ordinary Chow ring of a semi-projective toric Deligne-Mumford stack $X(\Sigma)$. Then there is a ring isomorphism:

$$A^*(X(\Sigma)) \cong \frac{S_\Sigma}{\{e(h_i)y^{b_i} : e \in N^*\}}.$$  

**Proof.** The Lemma is easily proven from the fact that the Chow ring of a Deligne-Mumford stack is isomorphic to the Chow ring of its coarse moduli space (IV) and Proposition 2.11 in [HS].

Now we study the module structure on $A^*_{orb}(X(\Sigma))$. Because $\Sigma$ is a simplicial fan, we have:
Lemma 4.2.8 For any $c \in N_S$, let $\sigma$ be the minimal cone in $\Sigma$ containing $c$. Then there is a unique expression $c = v + \sum_{\rho \in \sigma} m_i b_i$ where $m_i \in \mathbb{Z}_{\geq 0}$, and $v \in Box(\sigma)$.

Proposition 4.2.9 Let $\mathcal{X}(\Sigma)$ be a semi-projective toric Deligne-Mumford stack associated to an extended stacky fan $\Sigma$. We have an isomorphism of $A^*(\mathcal{X}(\Sigma))$-modules:

$$\bigoplus_{v \in Box(\Sigma)} A^*(\mathcal{X}(\Sigma/\sigma(v))) [\deg(y^v)] \cong \frac{\mathbb{Q}[N_S]}{\{\sum_{i=1}^n e(b_i)y^{b_i} : e \in N^*\}}.$$

Proof. From the definition of $\mathbb{Q}[N_S]$ and Lemma 4.2.8, we see that $\mathbb{Q}[N_S] = \bigoplus_{v \in Box(\Sigma)} y^v \cdot S_\Sigma$. The rest is similar to the proof of Proposition 4.7 in [Jiang2], we leave it to the readers. □

The Chen-Ruan product structure

The orbifold cup product on a Deligne-Mumford stack $\mathcal{X}$ is defined using genus zero, degree zero 3-pointed orbifold Gromov-Witten invariants on $\mathcal{X}$. The relevant moduli space is the disjoint union of all 3-twisted sectors (i.e. the double inertia stack). By (4.5), the 3-twisted sectors of a semi-projective toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ are

$$\prod_{(v_1,v_2,v_3) \in Box(\Sigma)^3, v_1v_2v_3=1} \mathcal{X}(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3)). \tag{4.6}$$

Let $ev_i : \mathcal{X}(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3)) \to \mathcal{X}(\Sigma/\sigma(\overline{v}_i))$ be the evaluation maps. The obstruction bundle (see [CR2]) $Ob_{(v_1,v_2,v_3)}$ over the 3-twisted sector $\mathcal{X}(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3))$ are defined by

$$Ob_{(v_1,v_2,v_3)} := (e^*T(\mathcal{X}(\Sigma)) \otimes H^1(C, \mathcal{O}_C))^H, \tag{4.7}$$

where $e : \mathcal{X}(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3)) \to \mathcal{X}(\Sigma)$ is the embedding, $C \to \mathbb{P}^1$ is the $H$-covering branched over three marked points $\{0, 1, \infty\} \subset \mathbb{P}^1$, and $H$ is the group generated
by $v_1, v_2, v_3$.

A general result in [CH] and [JKK] about the obstruction bundle implies the following.

**Proposition 4.2.10** Let $\mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3))$ be a 3-twisted sector of the stack $\mathcal{X}(\Sigma)$.

Suppose $v_1 + v_2 + v_3 = \sum_{\rho_i \subseteq \sigma(v_1, v_2, v_3)} a_i b_i$, $a_i = 1$ or 2. Then the Euler class of the obstruction bundle $Ob((v_1, v_2, v_3))$ on $\mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3))$ is

$$\prod_{a_i=2} c_1(L_i))_{|\mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3))},$$

where $L_i$ is the line bundle over $\mathcal{X}(\Sigma)$ corresponding to the ray $\rho_i$.

Let $v \in Box(\Sigma)$, say $v \in N(\sigma)$ for some top-dimensional cone $\sigma$. Let $\tilde{v} \in Box(\Sigma)$ be the inverse of $v$ as an element in the group $N(\sigma)$. Equivalently, if $v = \sum_{\rho_i \subseteq \sigma(v)} \alpha_i b_i$ for $0 < \alpha_i < 1$, then $\tilde{v} = \sum_{\rho_i \subseteq \sigma(v)} (1 - \alpha_i) b_i$. Then for $\alpha_1, \alpha_2 \in A^*_{orb}(\mathcal{X}(\Sigma))$, the orbifold cup product is defined by

$$\alpha_1 \cup_{orb} \alpha_2 = \tilde{e}v_3^* (ev_1^* \alpha_1 \cup ev_2^* \alpha_2 \cup e(Ob((v_1, v_2, v_3)))),$$

(4.8)

where $\tilde{e}v_3 = I \circ ev_3$, and $I : \mathcal{I}\mathcal{X}(\Sigma) \rightarrow \mathcal{I}\mathcal{X}(\Sigma)$ is the natural map given by $(x, g) \rightarrow (x, g^{-1})$.

**Proof of Theorem 4.1.1**

By Proposition 4.2.9, it remains to consider the cup product. In this case, for any $v_1, v_2 \in Box(\Sigma)$, we also have

$$v_1 + v_2 = \tilde{v}_3 + \sum_{a_i=2} b_i + \sum_{i \in J} b_i,$$

where $J$ represents the set of $j$ such that $\rho_j$ belongs to $\sigma(\bar{v}_1, \bar{v}_2)$, but not belong to $\sigma(\bar{v}_3)$. Then the proof is the same as the proof in [BCS]. We omit the details.
4.3 Lawrence Toric Deligne-Mumford Stacks

In this section we study a special type of semi-projective toric Deligne-Mumford stacks called the Lawrence toric Deligne-Mumford stacks. Their orbifold Chow rings are shown to be isomorphic to the orbifold Chow rings of their associated hypertoric Deligne-Mumford stacks studied in Chapter 3. We refer to Section 3.2 about the construction of Lawrence toric Deligne-Mumford stacks and hypertoric Deligne-Mumford stacks.

4.3.1 Comparison of inertia stacks

Next we compare the orbifold Chow ring of the hypertoric Deligne-Mumford stack and the orbifold Chow ring of the Lawrence toric Deligne-Mumford stack. First we compare the inertia stacks. From the map $\beta : \mathbb{Z}^m \to N$ which is given by vectors $\{b_1, \cdots, b_m\}$. Let $\text{Cone}(\beta)$ be a partially ordered finite set of cones generated by $\vec{b}_1, \cdots, \vec{b}_m$. The partial order is defined by: $\sigma \prec \tau$ if $\sigma$ is a face of $\tau$, and we have the minimum element $\hat{0}$ which is the cone consisting of the origin. Let $\text{Cone}(\mathcal{N})$ be the set of all convex polyhedral cones in the lattice $\mathcal{N}$. Then we have a map

$$C : \text{Cone}(\beta) \to \text{Cone}(\mathcal{N}),$$

such that for any $\sigma \in \text{Cone}(\beta)$, $C(\sigma)$ is the cone in $\mathcal{N}$. Then $\Delta_{\beta} := (C, \text{Cone}(\beta))$ is a simplicial multi-fan in the sense of [HM].

For the multi-fan $\Delta_{\beta}$, let $\text{Box}(\Delta_{\beta})$ be the set of pairs $(v, \sigma)$, where $\sigma$ is a cone in $\Delta_{\beta}$, $v \in N$ such that $v = \sum_{\rho_i \subseteq \sigma} \alpha_i b_i$ for $0 < \alpha_i < 1$. (Note that $\sigma$ is the minimal cone in $\Delta_{\beta}$ satisfying the above condition.) From Chapter 3, an element $(v, \sigma) \in \text{Box}(\Delta_{\beta})$ gives a component of the inertia stack $\mathcal{I}(\mathcal{M}(\mathcal{A}))$. Also consider the set $\text{Box}(\Sigma_{\theta})$ associated to the stacky fan $\Sigma_{\theta}$, see Section 4.2.2 for its definition.
An element \( v \in Box(\Sigma_\theta) \) gives a component of the inertia stack \( \mathcal{I}(\mathcal{X}(\Sigma_\theta)) \).

By the Lawrence lifting property, a vector \( \overline{b}_i \) in \( \mathcal{N} \) lifts to two vectors \( \overline{b}_{L,i}, \overline{b}'_{L,i} \) in \( \mathcal{N}_L \). Let \( \{ \overline{b}_{L,i_1}, \ldots, \overline{b}_{L,i_k}, \overline{b}'_{L,i_1}, \ldots, \overline{b}'_{L,i_k} \} \) be the Lawrence lifting of \( \{ \overline{b}_{i_1}, \ldots, \overline{b}_{i_k} \} \).

**Lemma 4.3.1** \( \{ \overline{b}_{i_1}, \ldots, \overline{b}_{i_k} \} \) generate a cone \( \sigma \) in \( \Delta_\beta \) if and only if \( \{ \overline{b}_{L,i_1}, \ldots, \overline{b}_{L,i_k}, \overline{b}'_{L,i_1}, \ldots, \overline{b}'_{L,i_k} \} \) generate a cone \( \sigma_\theta \) in \( \Sigma_\theta \).

**Proof.** Suppose \( \sigma \) is a cone in \( \Delta_\beta \) generated by \( \{ \overline{b}_{i_1}, \ldots, \overline{b}_{i_k} \} \), it is contained in a top-dimensional cone \( \tau \). Assume that \( \tau \) is generated by \( \{ \overline{b}_{i_1}, \ldots, \overline{b}_{i_k}, \overline{b}_{i_{k+1}}, \ldots, \overline{b}_{i_d} \} \).

Let \( C \) be the complement \( \{ \overline{b}_1, \ldots, \overline{b}_m \} \setminus \tau \). Then \( C \) corresponds to a column basis of \( \overline{\beta}^\vee \) in the map \( \overline{\beta}^\vee : \mathbb{Z}^m \to \text{DG}(\beta) \). By the definition of \( \Sigma_\theta \) in (3.8), \( C \) corresponds to a maximal cone \( \tau_\theta \) in \( \Sigma_\theta \) which contains the rays generated by \( \{ \overline{b}_{L,i_1}, \ldots, \overline{b}_{L,i_k}, \overline{b}'_{L,i_1}, \ldots, \overline{b}'_{L,i_k} \} \). Thus these rays generate a cone \( \sigma_\theta \) in \( \Sigma_\theta \).

Conversely, suppose \( \sigma_\theta \) is a cone in \( \Sigma_\theta \) generated by \( \{ \overline{b}_{L,i_1}, \ldots, \overline{b}_{L,i_k}, \overline{b}'_{L,i_1}, \ldots, \overline{b}'_{L,i_k} \} \). Using the similar method above we prove that \( \{ \overline{b}_{i_1}, \ldots, \overline{b}_{i_k} \} \) must be contained in a top-dimensional cone of \( \Delta_\beta \). So \( \{ \overline{b}_{i_1}, \ldots, \overline{b}_{i_k} \} \) generate a cone \( \sigma \) in \( \Delta_\beta \). \( \square \)

**Lemma 4.3.2** There is an one-to-one correspondence between the elements in \( Box(\Sigma_\theta) \) and the elements in \( Box(\Delta_\beta) \). Moreover, their degree shifting numbers coincide.

**Proof.** First we rewrite two exact sequences in Chapter 3 here:

\[
1 \rightarrow \mu \rightarrow G \xrightarrow{\alpha} \left( \mathbb{C}^* \right)^m \rightarrow T \rightarrow 1,
\]

and

\[
1 \rightarrow \mu \rightarrow G \xrightarrow{\alpha_L} \left( \mathbb{C}^* \right)^{2m} \rightarrow T_L \rightarrow 1.
\]

The torsion elements in \( Box(\Sigma_\theta) \) and \( Box(\Delta_\beta) \) are both isomorphic to \( \mu = \ker(\alpha) = \ker(\alpha^L) \) in the above two exact sequences. Let \( (v, \sigma) \in Box(\Delta_\beta) \) with \( \overline{v} = \sum_{i=1}^k \alpha_i \overline{b}_i \).
Then $v$ may be identified with an element (which we ambiguously denote by) $v \in G := \text{Hom}_\mathbb{Z}(DG(\beta), \mathbb{C}^\ast)$. Certainly $v$ fixes a point in $\mathbb{C}^m$. Consider the map $\alpha$ in (3.9), put $\alpha(v) = (\alpha^1(v), \cdots, \alpha^m(v))$. Then $\alpha^i(v) \neq 1$ if $\rho_i \subseteq \sigma$, and $\alpha^i(v) = 1$ otherwise. By Lemma 4.3.1, let $\{b_{L,i}, \bar{b}_{L,i} : i = 1, \cdots, |\sigma|\}$ be the Lawrence lifting of $\{b_i\}_{\rho_i \subseteq \sigma}$. Since the action of $v$ on $\mathbb{C}^m$ is given by $(v, v^{-1})$, $v$ fixes a point in $\mathbb{C}^m$ and yields an element $v_0$ in $Box(\Sigma_\theta)$. From the map $\alpha^L$, let

$$\alpha^L(v_0) = (\alpha^L_1(v_0), \cdots, \alpha^L_m(v_0), \alpha^L_{m+1}(v_0), \cdots, \alpha^L_{2m}(v_0)). \quad (4.9)$$

Then $\alpha^L_1(v_0) \neq 1$ and $\alpha^L_{i+m}(v_0) \neq 1$ if $\rho_i \subseteq \sigma$; $\alpha^L_1(v_0) = \alpha^L_{i+m}(v_0) = 1$ otherwise. So $\sigma_\theta(\bar{v}_\theta) = \{b_{L,i}, \bar{b}_{L,i} : i = 1, \cdots, |\sigma|\}$ is the minimal cone in $\Sigma_\theta$ containing $\bar{v}_\theta$. Furthermore, $\bar{v}_\theta = \sum_{\rho_i \subseteq \sigma} \alpha_i b_{L,i} + \sum_{\rho_i \subseteq \sigma} (1 - \alpha_i)\bar{b}_{L,i}$.

Moreover, given an element $v_0 \in Box(\Sigma_\theta)$, let $\sigma_\theta(\bar{v}_\theta)$ be the minimal cone in $\Sigma_\theta$ containing $\bar{v}_\theta$. Then from the action of $G$ on $\mathbb{C}^{2m}$ and (4.9), we have $\alpha^L_1(v_0) = (\alpha^L_{i+m}(v_0))^{-1}$. If $\alpha^L_1(v_0) \neq 1$, then $\alpha^L_{i+m}(v_0) \neq 1$, which means that $b_{L,i}, \bar{b}_{L,i+m} \in \sigma_\theta(\bar{v}_\theta)$. The cone $\sigma_\theta(\bar{v}_\theta)$ is the one in $\Sigma_\theta$ containing $b_{L,i}, \bar{b}_{L,i+m}$'s satisfying this condition. Then $\bar{v}_\theta = \sum_i (\alpha_i b_{L,i} + (1 - \alpha_i)\bar{b}_{L,i})$. By Lemma 4.3.1, $\sigma_\theta(\bar{v}_\theta)$ is the Lawrence lifting of a cone $\sigma$ generated by the $\{b_i\}$'s in $\Delta_\beta$. Let $v = \sum_{\rho_i \subseteq \sigma} \alpha_i b_i$. So it also determines an element $(v, \sigma) \in Box(\Delta_\beta)$.

For $(v_1, \sigma_1), (v_2, \sigma_2), (v_3, \sigma_3) \in Box(\Delta_\beta)$, let $\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ be the minimal cone containing $\bar{v}_1, \bar{v}_2, \bar{v}_3$ in $\Delta_\beta$ such that $\bar{v}_1 + \bar{v}_2 + \bar{v}_3 = \sum_{\rho_i \subseteq \sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)} a_i b_i$ and $a_i = 1, 2$. Let $v_{\theta,1}, v_{\theta,2}, v_{\theta,3}$ be the corresponding elements in $Box(\Sigma_\theta)$ and $\sigma(\bar{v}_{\theta,1}, \bar{v}_{\theta,2}, \bar{v}_{\theta,3})$ the minimal cone containing $\bar{v}_{\theta,1}, \bar{v}_{\theta,2}, \bar{v}_{\theta,3}$ in $\Sigma_\theta$. Then by Lemmas 4.3.1 and 4.3.2, $\sigma(\bar{v}_{\theta,1}, \bar{v}_{\theta,2}, \bar{v}_{\theta,3})$ is the Lawrence lifting of $\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. Suppose that $\sigma$ is generated by $\{b_{i_1}, \cdots, b_{i_s}\}$, then $\sigma(\bar{v}_{\theta,1}, \bar{v}_{\theta,2}, \bar{v}_{\theta,3})$ is generated by $\{\bar{b}_{L,i_1}, \cdots, \bar{b}_{L,i_s}, \bar{b}_{L,i_1}, \cdots, \bar{b}_{L,i_s}\}$, the Lawrence lifting of $\{b_{i_1}, \cdots, b_{i_s}\}$. Let $\{\bar{b}_{j_1}, \cdots, \bar{b}_{j_{m-1-2}}\}$ be the rays not in
σ ∪ link(σ), we have the Lawrence lifting \( \{ b_{L,j_1}, \ldots, b_{L,j_{m-l-s}}, b'_{L,j_1}, \ldots, b'_{L,j_{m-l-s}} \} \).

Then from the definition of Lawrence fan \( Σ_θ \) in (3.8), we have the following lemma:

**Lemma 4.3.3** There exist \( m-l-s \) vectors in \( \{ b_{L,j_1}, \ldots, b_{L,j_{m-l-s}}, b'_{L,j_1}, \ldots, b'_{L,j_{m-l-s}} \} \) such that the rays they generate plus the rays in \( σ(\overline{v}_{\theta,1}, \overline{v}_{\theta,2}, \overline{v}_{\theta,3}) \) generate a cone \( σ_θ \) in \( Σ_θ \). □

**Proposition 4.3.4** The stack \( X(Σ_θ/σ_θ) \) is also a Lawrence toric DM stack.

**Proof.** For simplicity, put \( σ := σ(\overline{v}_1, \overline{v}_2, \overline{v}_3) \). Suppose there are \( l \) rays in the link(σ). Then by Lemma 4.3.1 there are \( 2l \) rays in link(σ_θ), the Lawrence lifting of link(σ). Let \( s := |σ| \), then \( 2s + m - l - s = |σ_θ| \). Applying Gale dual to the diagrams

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^{s} & \rightarrow & \mathbb{Z}^{l+s} & \rightarrow & \mathbb{Z}^{l} & \rightarrow & 0 \\
& & \downarrow{β_σ} & & \downarrow{β} & & \downarrow{β(σ)} & \\
0 & \rightarrow & N_σ & \rightarrow & N & \rightarrow & N(σ) & \rightarrow & 0,
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^{l+s} & \rightarrow & \mathbb{Z}^{m} & \rightarrow & \mathbb{Z}^{m-l-s} & \rightarrow & 0 \\
& & \downarrow{β} & & \downarrow{β} & & & \\
0 & \rightarrow & N & \xrightarrow{σ} & N & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

yields

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^{l} & \rightarrow & \mathbb{Z}^{l+s} & \rightarrow & \mathbb{Z}^{s} & \rightarrow & 0 \\
& & \downarrow{β(σ)^{\vee}} & & \downarrow{β^{\vee}} & & \downarrow{β_σ^{\vee}} & & (4.10) \\
0 & \rightarrow & DG(β(σ)) & \xrightarrow{φ_1} & DG(β) & \rightarrow & DG(β_σ) & \rightarrow & 0,
\end{array}
\]

and

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^{m-l-s} & \rightarrow & \mathbb{Z}^{m} & \rightarrow & \mathbb{Z}^{l+s} & \rightarrow & 0 \\
& & \downarrow{σ} & & \downarrow{β^{\vee}} & & \downarrow{β^{\vee}} & & (4.11) \\
0 & \rightarrow & \mathbb{Z}^{m-l-s} & \rightarrow & DG(β) & \xrightarrow{φ_2} & DG(β) & \rightarrow & 0.
\end{array}
\]

Since \( \mathbb{Z}^{s} \cong N_σ \), we have that \( DG(β_σ) = 0 \). We add two exact sequences

\[
0 \rightarrow \mathbb{Z}^{l} \rightarrow \mathbb{Z}^{m} \rightarrow \mathbb{Z}^{m-l} \rightarrow 0,
\]

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and
\[
0 \to 0 \to \mathbb{Z}^{m} \to \mathbb{Z}^{m} \to 0,
\]
on the rows of the diagrams (4.10),(4.11) and make suitable maps to the Gale duals we get
\[
\begin{array}{c}
0 \to \mathbb{Z}^{2l} \to \mathbb{Z}^{l+s+m} \to \mathbb{Z}^{s+m-l} \to 0 \\
\downarrow (\beta(\sigma)^{\vee}, -\beta(\sigma)^{\vee}) \quad \downarrow (\tilde{\beta}^{\vee}, -\beta^{\vee}) \quad \downarrow 0
\end{array}
\]
\[
0 \to DG(\beta(\sigma)) \xrightarrow{\cong} DG(\tilde{\beta}) \to 0 \to 0,
\]
and
\[
\begin{array}{c}
0 \to \mathbb{Z}^{m-l-s} \to \mathbb{Z}^{2m} \to \mathbb{Z}^{l+s+m} \to 0 \\
\downarrow \cong \quad \downarrow (\beta^{\vee}, -\beta^{\vee}) \quad \downarrow (\tilde{\beta}^{\vee}, -\beta^{\vee})
\end{array}
\]
\[
0 \to \mathbb{Z}^{m-l-s} \to DG(\beta) \to DG(\tilde{\beta}) \to 0.
\]
Applying Gale dual to (4.12), (4.13) we get
\[
\begin{array}{c}
0 \to \mathbb{Z}^{s+m-l} \to \mathbb{Z}^{l+s+m} \to \mathbb{Z}^{2l} \to 0 \\
\downarrow \cong \quad \downarrow \tilde{\beta}_{L} \quad \downarrow \beta_{L}(\sigma_{\theta})
\end{array}
\]
\[
0 \to \mathbb{Z}^{s+m-l} \to \tilde{N}_{L} \to N_{L}(\sigma_{\theta}) \to 0,
\]
and
\[
\begin{array}{c}
0 \to \mathbb{Z}^{l+s+m} \to \mathbb{Z}^{2m} \to \mathbb{Z}^{m-l-s} \to 0 \\
\downarrow \tilde{\beta}_{L} \quad \downarrow \beta_{L} \quad \downarrow 0
\end{array}
\]
\[
0 \to \tilde{N}_{L} \xrightarrow{\cong} N_{L} \to 0 \to 0.
\]
For the generic element $\theta$, from them map $\varphi_{2}$ in (4.11), $\theta$ induces $\tilde{\theta} \in DG(\tilde{\beta})$, and from the isomorphism $\varphi_{1}$ in (4.10), $\tilde{\theta} = \theta(\sigma) \in DG(\beta(\sigma))$. So we a quotient stacky hyperplane arrangement $A(\sigma) = (N(\sigma), \beta(\sigma), \theta(\sigma))$. From the above diagrams we see that the quotient fan $\Sigma_{\theta_{0}}/\sigma_{\theta}$ in $N_{L}(\sigma_{\theta})$ also comes from a Lawrence construction of the map $\beta(\sigma)^{\vee} : \mathbb{Z}^{l} \to DG(\beta(\sigma))$. Let $X(\sigma) = \mathbb{C}^{2l} \setminus V(I_{\theta(\sigma)})$, where $I_{\theta(\sigma)}$ is the irrelevant ideal of the quotient fan $\Sigma_{\theta}/\sigma_{\theta}$. Let $G(\sigma) = Hom_{\mathbb{Z}}(DG(\beta(\sigma)), \mathbb{C}^{*})$. The stack $\mathcal{X}(\Sigma_{\theta}/\sigma_{\theta}) = [X(\sigma)/G(\sigma)]$ is a Lawrence toric Deligne-Mumford stack. □

**Corollary 4.3.5** $\mathcal{M}(A(\sigma(\overline{v}_{1}, \overline{v}_{2}, \overline{v}_{3})))$ is the hypertoric DM stack associated to the quotient Lawrence toric DM stack $\mathcal{X}(\Sigma_{\theta}/\sigma_{\theta})$. 123
PROOF. $\mathcal{M}(A(\varphi(\overline{v}_1, \overline{v}_2, \overline{v}_3)))$ is constructed in [JT1] as a quotient stack $[Y(\sigma)/G(\sigma)]$, where $Y(\sigma) \subset X(\sigma)$ is defined by $I_{\beta(\sigma)^\vee}$, which is the ideal in (3.11) corresponding to the map $\beta(\sigma)^\vee$ in (4.10). So the stack $\mathcal{M}(A(\varphi(\overline{v}_1, \overline{v}_2, \overline{v}_3)))$ is the associated hypertoric Deligne-Mumford stack in the Lawrence toric Deligne-Mumford stack $\mathcal{X}(\Sigma_\theta/\sigma_\theta)$. □

**Remark** For any $v_0 \in Box(\Sigma_\theta)$, let $v_0^{-1}$ be its inverse. We have the quotient Lawrence toric stack $\mathcal{X}(\Sigma_\theta/\sigma_\theta)$. Let $(v, \sigma)$ be the corresponding element in $Box(\Delta_\sigma)$, then

$$\mathcal{M}(A(\varphi(v, v^{-1}, 1))) \cong \mathcal{M}(A(\sigma)).$$

By Proposition 4.3.4 and Corollary 4.3.5, the twisted sector $\mathcal{M}(A(\sigma))$ is the associated hypertoric Deligne-Mumford stack of the Lawrence toric Deligne-Mumford stack $\mathcal{X}(\Sigma_\theta/\sigma_\theta)$.

**Remark** From Lemma 4.3.3, the cone $\sigma_\theta$ is not the minimal cone $\varphi(\overline{v}_{\theta,1}, \overline{v}_{\theta,2}, \overline{v}_{\theta,3})$ containing $\overline{v}_{\theta,1}, \overline{v}_{\theta,2}, \overline{v}_{\theta,3}$ in $\Sigma_\theta$. So $\mathcal{X}(\Sigma_\theta/\sigma(\overline{v}_{\theta,1}, \overline{v}_{\theta,2}, \overline{v}_{\theta,3}))$ is not a Lawrence toric Deligne-Mumford stack. But from the construction of Lawrence toric Deligne-Mumford stack, the quotient stack $\mathcal{X}(\Sigma_\theta/\sigma(\overline{v}_{\theta,1}, \overline{v}_{\theta,2}, \overline{v}_{\theta,3}))$ is homotopy equivalent to the quotient stack $\mathcal{X}(\Sigma_\theta/\sigma_\theta)$. Since we do not need this to compare the orbifold Chow ring, we omit the details.

### 4.3.2 Comparison of orbifold Chow rings

Recall that $N_L = N_L + N_{L,\text{tor}}$, where $N_{L,\text{tor}}$ is the torsion subgroup of $N_L$. Let $N_{\Sigma_\theta} = N_{L,\text{tor}} + |\Sigma_\theta|$. By Theorem 4.1.1, we have
Proposition 4.3.6 The orbifold Chow ring $A^*_\orb(X(Y,o))$ of the Lawrence toric Deligne-Mumford stack $X(Y,o)$ is isomorphic to the ring

$$\mathbb{Q}[N_{\Sigma}] / \left( \sum_{i=1}^{m} e(b_{L,i})y^{b_{L,i}} + \sum_{i=1}^{m} e(b'_{L,i})y^{b'_{L,i}} : e \in N_{\Sigma}^1 \right).$$ (4.14)

Recall in Chapter 4 that for any $c \in N$, there is a cone $\sigma \in \Delta_\beta$ such that $ar{c} = \sum_{\rho_i \subseteq \sigma} \alpha_i \bar{b}_i$ where $\alpha_i > 0$ are rational numbers. Let $N^{\Delta_\beta}$ denote all the pairs $(c, \sigma)$. Then $N^{\Delta_\beta}$ gives rise a group ring $\mathbb{Q}[\Delta_\beta] = \bigoplus_{(c, \sigma) \in N^{\Delta_\beta}} \mathbb{Q} \cdot y^{(c, \sigma)}$, where $y$ is a formal variable. For any $(c, \sigma) \in N^{\Delta_\beta}$, there exists a unique element $(v, \tau) \in Box(\Delta_\beta)$ such that $\tau \subseteq \sigma$ and $c = v + \sum_{\rho_i \subseteq \sigma} m_i b_i$, where $m_i$ are nonnegative integers. We call $(v, \tau)$ the fractional part of $(c, \sigma)$. We define the ceiling function for fans. For $(c, \sigma)$ define the ceiling function $[c]_\sigma$ by $[c]_\sigma = \sum_{\rho_i \subseteq \tau} b_i + \sum_{\rho_i \subseteq \sigma} m_i b_i$. Note that if $v = 0$, $[c]_\sigma = \sum_{\rho_i \subseteq \sigma} m_i b_i$. For two pairs $(c_1, \sigma_1)$, $(c_2, \sigma_2)$, if $\sigma_1 \cup \sigma_2$ is a cone in $\Delta_\beta$, define $\epsilon(c_1, c_2) := [c_1]_{\sigma_1} + [c_2]_{\sigma_2} - [c_1 + c_2]_{\sigma_1 \cup \sigma_2}$. Let $\sigma_e \subseteq \sigma_1 \cup \sigma_2$ be the minimal cone in $\Delta_\beta$ containing $\epsilon(c_1, c_2)$ so that $(\epsilon(c_1, c_2), \sigma_e) \in N^{\Delta_\beta}$. We define the grading on $\mathbb{Q}[\Delta_\beta]$ as follows. For any $(c, \sigma)$, write $c = v + \sum_{\rho_i \subseteq \sigma} m_i b_i$, then $\text{deg}(y^{(c, \sigma)}) = |\tau| + \sum_{\rho_i \subseteq \sigma} m_i$, where $|\tau|$ is the dimension of $\tau$. By abuse of notation, we write $y^{(b_i, \rho_i)}$ as $y^{b_i}$. The multiplication is defined by

$$y^{(c_1, \sigma_1)} \cdot y^{(c_2, \sigma_2)} := \begin{cases} (-1)^{|\sigma_1|} y^{(c_1 + c_2 + \epsilon(c_1, c_2), \sigma_1 \cup \sigma_2)} & \text{if } \sigma_1 \cup \sigma_2 \text{ is a cone in } \Delta_\beta, \\ 0 & \text{otherwise.} \end{cases}$$ (4.15)

From the property of ceiling function we check that the multiplication is commutative and associative. So $\mathbb{Q}[\Delta_\beta]$ is a unital associative commutative ring. In Chapter
Consider the map $\beta : \mathbb{Z}^m \to N$ which is given by the vectors $\{b_1, \ldots, b_m\}$. We take $\{1, \ldots, m\}$ as the vertex set of the matroid complex $M_\beta$, defined from $\beta$ by requiring that $F \in M_\beta$ iff the vectors $\{\bar{b}_i\}_{i \in F}$ are linearly independent in $N$.

A face $F \in M_\beta$ corresponds to a cone in $\Delta_\beta$ generated by $\{\bar{b}_i\}_{i \in F}$. By [S], the "Stanley-Reisner" ring of the matroid $M_\beta$ is

$$Q[M_\beta] = \frac{Q[y^{b_1}, \ldots, y^{b_m}]}{I_{M_\beta}},$$

where $I_{M_\beta}$ is the matroid ideal generated by the set of square-free monomials

$$\{y^{b_{i_1} \cdot \cdot \cdot y^{b_{i_k}}} | \bar{b}_{i_1}, \ldots, \bar{b}_{i_k} \text{ linearly dependent in } N\}.$$

It is proved in Chapter 3 that

$$Q[\Delta_\beta] \cong \bigoplus_{(v, \sigma) \in Box(\Delta_\beta)} y^{(v, \sigma)} \cdot Q[M_\beta].$$

For any $(v_1, \sigma_1), (v_2, \sigma_2) \in Box(\Delta_\beta)$, let $(v_3, \sigma_3)$ be the unique element in $Box(\Delta_\beta)$ such that $v_1 + v_2 + v_3 \equiv 0$ in the local group given by $\sigma_1 \cup \sigma_2$, where $\equiv 0$ means that there exists a cone $\sigma(v_1, v_2, v_3)$ in $\Delta_\beta$ such that $v_1 + v_2 + v_3 = \sum_{\rho_j \subseteq \sigma} a_j \bar{b}_j$, where $a_1 = 1$ or 2. Let $\bar{v}_1 = \sum_{\rho_j \subseteq \sigma} a_j^1 \bar{b}_j, \bar{v}_2 = \sum_{\rho_j \subseteq \sigma} a_j^2 \bar{b}_j, \bar{v}_3 = \sum_{\rho_j \subseteq \sigma} a_j^3 \bar{b}_j$ with $0 < a_j^1, a_j^2, a_j^3 < 1$. Let $I$ be the set of $i$ such that $a_i = 1$ and $a_j^1, a_j^2, a_j^3$ exist, $J$ the set of $j$ such that $\rho_j$ belongs to $\sigma(v_1, v_2, v_3)$ but not $\sigma_3$. If $(v, \sigma) \in Box(\Delta_\beta)$, let $(\bar{v}, \sigma)$ be the inverse of $(v, \sigma)$. Except torsion elements, equivalently, if $\bar{v} = \sum_{\rho_i \subseteq \sigma} \alpha_i \bar{b}_i$ for $0 < \alpha_i < 1$, then $\bar{v} = \sum_{\rho_i \subseteq \sigma} (1 - \alpha_i) \bar{b}_i$. By abuse of notation, we write $y^{(b_i, \alpha)}$ as $y^{b_i}$.

We have that $v_1 + v_2 = \bar{v}_3 + \sum_{a_i = 2} b_i + \sum_{j \in J} b_j$. From (4.15), Lemma 3.5.11 and
Lemma 3.5.12 in Chapter 3, if \( \overline{v}_1, \overline{v}_2 \neq 0 \), we have

\[
[v_1]_{\sigma_1} + [v_2]_{\sigma_2} - [v_1 + v_2]_{\sigma_1 \cup \sigma_2} = \begin{cases} 
\sum_{i \in I} b_i + \sum_{j \in J} b_j & \text{if } \overline{v}_1 \neq \overline{v}_2, \\
\sum_{j \in J} b_j & \text{if } \overline{v}_1 = \overline{v}_2.
\end{cases}
\]

So it is easy to check that the multiplication \( y^{(v_1, \sigma_1)} \cdot y^{(v_2, \sigma_2)} \) can be written as

\[
\begin{cases} 
(-1)^{|I|+|J|} y^{(\beta, \sigma_3)} \cdot \prod_{i=2} y^{b_i} \cdot \prod_{i \in I} y^{b_{L,i}} \cdot \prod_{j \in J} y^{2b_j} & \text{if } \overline{v}_1, \overline{v}_2 \in \sigma \text{ for } \sigma \in \Delta_\beta \text{ and } \overline{v}_1 \neq \overline{v}_2,
\end{cases}
\]

\[
(-1)^{|J|} \prod_{j \in J} y^{2b_j}
\]

\[
0
\]

if \( \overline{v}_1, \overline{v}_2 \in \sigma \text{ for } \sigma \in \Delta_\beta \text{ and } \overline{v}_1 = \overline{v}_2,
\]

otherwise.

(4.17)

The following is the main result of this Section.

**Theorem 4.3.7** There is an isomorphism of orbifold Chow rings \( A^*_\text{orb}(\mathcal{X}(\Sigma_\theta)) \cong A^*_\text{orb}(\mathcal{M}(A)) \).

**Proof.** The ring \( \mathbb{Q}[N_{\Sigma_\theta}] \) is generated by \( \{y^{b_{L,i}}, y^{b_{L,i}} : i = 1, \ldots, m\} \) and \( y^w \) for \( v_0 \in \text{Box}(\Sigma_\theta) \) by the definition. By Lemma 4.3.2, define a morphism

\[
\phi : \mathbb{Q}[N_{\Sigma_\theta}] \to \mathbb{Q}[\Delta_\beta]
\]

by \( y^{b_{L,i}} \to y^{b_{L,i}}, y^{b_{L,i}} \to -y^{b_{L,i}} \) and \( y^w \to y^{(v,o)} \). By [HS], the ideal \( I_\theta \) goes to the ideal \( I_{M_\beta} \) and the relation \( \{\sum_{i=1}^m e(b_{L,i}) y^{b_{L,i}} + \sum_{i=1}^m e(b_{L,i}) y^{b_{L,i}} : e \in N_1^*\} \) goes to the relation \( \{\sum_{i=1}^m e(b_{L,i}) y^{b_{L,i}} : e \in N^*\} \). Thus the two rings are isomorphic as modules.

It remains to check the multiplications. For any \( y^w \) and \( y^{b_{L,i}} \) or \( y^{b_{L,i}} \), let \( y^{(v,o)} \) be the corresponding element in \( \mathbb{Q}[\Delta_\beta] \). By the property of \( v_0 \) and Lemma 4.3.2, the minimal cone in \( \Sigma_\theta \) containing \( \overline{v}_0, \overline{b}_{L,i} \) must contains \( \overline{b}_{L,i} \). By Lemma 4.3.1, there is a cone in \( \Delta_\beta \) containing \( \overline{v}, \overline{b}_{L,i} \). In this way, \( y^{v_0} \cdot y^{b_{L,i}} \) goes to \( y^{(v,o)} \cdot y^{b_{L,i}} \) and \( y^{v_0} \cdot y^{b_{L,i}} \) goes to \( -y^{(v,o)} \cdot y^{b_{L,i}} \). If there is no cone in \( \Sigma_\theta \) containing \( \overline{v}_0, \overline{b}_{L,i}, \overline{b}_{L,i} \).
then by Lemma 4.3.1 there is no cone in $\Delta_\beta$ containing $\vec{v}, \vec{b}_i$. So $y^{v_0} \cdot y^{b_i,i} = 0$ goes to $y^{(v,\sigma)} \cdot y^{b_i} = 0$ and $y^{v_0} \cdot y'^{b_i,i} = 0$ goes to $-y^{(v,\sigma)} \cdot y^{b_i} = 0$.

For any $y^{v_0,i}, y^{v_0,2}$, let $y^{(v_1,\sigma_1)}, y^{(v_2,\sigma_2)}$ be the corresponding elements in $Q[\Delta_\beta]$. If there is no cone in $\Sigma_\theta$ containing $\vec{v}_{\theta,1}, \vec{v}_{\theta,2}$, then by Lemmas 4.3.1 and 4.3.2, there is no cone in $\Delta_\beta$ containing $\vec{v}_1, \vec{v}_2$. So $y^{v_0,i} \cdot y^{v_0,3} = 0$ goes to $y^{(v_1,\sigma_1)} \cdot y^{(v_2,\sigma_2)} = 0$. Suppose there is a cone containing $\vec{v}_{\theta,1}, \vec{v}_{\theta,2}$, let $\vec{v}_{\theta,3} \in Box(\Sigma_\theta)$ such that $v_{\theta,1} + v_{\theta,2} + v_{\theta,3} = 0$. Let $\sigma(\vec{v}_{\theta,1}, \vec{v}_{\theta,2}, \vec{v}_{\theta,3})$ be the minimal cone containing $\vec{v}_{\theta,1}, \vec{v}_{\theta,2}, \vec{v}_{\theta,3}$ in $\Sigma_\theta$. Then by Lemmas 4.3.1 and 4.3.2, $\sigma(\vec{v}_{\theta,1}, \vec{v}_{\theta,2}, \vec{v}_{\theta,3})$ is the Lawrence lifting of $\sigma(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ for $(v_1, \sigma_1), (v_2, \sigma_2), (v_3, \sigma_3) \in Box(\Delta_\beta)$. So we may write $\vec{v}_{\theta,1} + \vec{v}_{\theta,2} + \vec{v}_{\theta,3} = \sum_{\rho_i \subseteq \sigma(\vec{v}_1, \vec{v}_2, \vec{v}_3)} \alpha_i \vec{b}_{L,i} + \sum_{\rho_i \subseteq \sigma(\vec{v}_1, \vec{v}_2, \vec{v}_3)} \alpha'_i \vec{b}'_{L,i}$. The corresponding $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \sum_{\rho_i \subseteq \sigma(\vec{v}_1, \vec{v}_2, \vec{v}_3)} \alpha_i \vec{b}_i$. Let $(\vec{\sigma}, \sigma)$ be the inverse of $(v, \sigma)$ in $Box(\Delta_\beta)$, i.e. if $v$ is nontorsion and $\vec{v} = \sum_{\rho_i \subseteq \sigma} \alpha_i \vec{b}_i$ for $0 < \alpha_i < 1$, then $\vec{v} = \sum_{\rho_i \subseteq \sigma} (1 - \alpha_i) \vec{b}_i$. The $\vec{\sigma}$ is defined similarly in $Box(\Sigma_\theta)$. The notation $J$ represents the set of $j$ such that $\rho_j$ belongs to $\sigma(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ but not $\sigma_3$, the corresponding $\rho_{L,j}, \rho'_{L,j}$ belong to $\sigma(\vec{v}_{\theta,1}, \vec{v}_{\theta,2}, \vec{v}_{\theta,3})$ but not $\sigma(\vec{v}_{\theta,3})$.

If some $\vec{v}_{\theta,1} = 0$ which means that $v_{\theta,1}$ is a torsion. Then from Lemma (4.3.2) the corresponding $v$ is also a torsion element. In this case we know that the orbifold cup product $y^{v_0,i} \cdot y^{v_0,2}$ is the usual product, and under the map $\phi$, is equal to $y^{(v_1,\sigma_1)} \cdot y^{(v_2,\sigma_2)}$.

If $\vec{v}_{\theta,1} = \vec{v}_{\theta,2}, \vec{v}_{\theta,3} = 0$ and the obstruction bundle over the corresponding 3-twisted sector is zero. The set $\mathcal{J}$ is the set $j$ such that $\rho_j$ belongs to $\sigma(\vec{v}_{\theta,1})$. So from [BCS], we have

$$y^{v_0,i} \cdot y^{v_0,2} = \prod_{j \in \mathcal{J}} y^{b_{L,j}} \cdot y'^{b_{L,j}}.$$

Under the map $\phi$ we see that $y^{(v_1,\sigma_1)} \cdot y^{(v_2,\sigma_2)}$ is equal to the second line in the product (4.17).
If \( \overline{v}_{\theta,1} \neq \overline{v}_{\theta,2} \), then \( \overline{v}_{\theta,3} \neq 0 \) and the obstruction bundle is given by Proposition 3.5.10. If all \( \alpha_1^1, \alpha_2^2, \alpha_3^3 \) exist, the coefficients \( a_i \) and \( a_i' \) satisfy that if \( a_i = 1 \) then \( a_i' = 2 \), and if \( a_i = 2 \) then \( a_i' = 1 \). So from [BCS],

\[
y^{v_{\theta,1}} \cdot y^{v_{\theta,2}} = y^{v_{\theta,3}} \cdot \prod_{a_i = 2} y^{b_{L,i}} \cdot \prod_{i \in I} y^{\nu_{L,i}} \cdot \prod_{j \in J} y^{b_{L,j}} \cdot y^{\nu_{L,j}}.
\]

Under the map \( \phi \) we see that \( y^{(v_1, \sigma_1)} \cdot y^{(v_2, \sigma_2)} \) is equal to the first line in the product (4.17). By Lemma 4.3.2, the box elements have the same orbifold degrees. By Corollary 4.3.5 and the definition of orbifold cup product in (4.8), the products \( y^{v_{\theta,1}} \cdot y^{v_{\theta,3}} \) and \( y^{(v_1, \sigma_1)} \cdot y^{(v_2, \sigma_2)} \) have the same degrees in both Chow rings. So \( \phi \) induces a ring isomorphism \( A^\ast_{orb}(\mathcal{A}(\Sigma_\theta)) \cong A^\ast_{orb}(\mathcal{M}(\mathcal{A})). \)

**Remark** The presentation (4.16) of orbifold Chow ring only depends on the matroid complex corresponding to the map \( \beta : \mathbb{Z}^m \to N \), not \( \theta \). Note that the presentation (4.14) depends on the fan \( \Sigma_\theta \). We couldn’t see explicitly from this presentation that the ring is independent to the choice of generic elements \( \theta \).
Bibliography


5.1 Relations Among Chapters.

Chapter 1 is an introduction chapter. We introduced orbifold Chow ring of smooth Deligne-Mumford stacks. We reviewed the definition of toric Deligne-Mumford stacks by Borisov, Chen and Smith. In Chapter 2 we defined extended stacky fans and constructed toric Deligne-Mumford stacks associated to extended stacky fans. Every extended stacky fan has an underlying stacky fan and they gave isomorphic toric Deligne-Mumford stacks. The main point of extended stacky fan is that it can give different representations of toric Deligne-Mumford stacks.

Any extended stacky fan $\Sigma$ can be constructed from a stacky hyperplane arrangement $\mathcal{A} = (N, \beta, \theta)$ defined in Chapter 3. The hyperplane arrangement determines the topology of hypertoric varieties [BD]. Using stacky hyperplane arrangement $\mathcal{A}$ we constructed hypertoric Deligne-Mumford stack $\mathcal{M}(\mathcal{A})$ so that the toric Deligne-Mumford stack $\mathcal{K}(\Sigma)$ is the associated toric Deligne-Mumford stack of $\mathcal{M}(\mathcal{A})$. We computed the orbifold Chow ring of hypertoric Deligne-Mumford stacks.

For any stacky hyperplane arrangement $\mathcal{A} = (N, \beta, \theta)$, we associate a
Lawrence stacky fan $\Sigma_\theta$ constructed from the Lawrence lifting. The hypertoric Deligne-Mumford stack $\mathcal{M}(A)$ is defined as a closed substack of the Lawrence toric Deligne-Mumford stack $\mathcal{X}(\Sigma_\theta)$. The Lawrence toric Deligne-Mumford stack $\mathcal{X}(\Sigma_\theta)$ is semi-projective, but not projective. In Chapter 4 we generalized the orbifold Chow ring formula of projective toric Deligne-Mumford stacks to semi-projective toric Deligne-Mumford stacks. We prove that the orbifold Chow ring of a Lawrence toric Deligne-Mumford stack $\mathcal{X}(\Sigma_\theta)$ is isomorphic to the orbifold Chow ring of its associated hypertoric Deligne-Mumford stack $\mathcal{M}(A)$ which is the orbifold Chow ring analogue of a theorem of Hausel and Sturmfels.

5.2 Importance of the Thesis.

5.2.1 Good reading materials for students.

In this thesis we reviewed the basic definition of smooth Deligne-Mumford stacks and the definition of orbifold Chow ring. We gave the construction of toric Deligne-Mumford stacks by Borisov-Chen-Smith and generalized it in Chapter 2. We defined hypertoric Deligne-Mumford stacks in Chapter 3, semi-projective toric Deligne-Mumford stacks in Chapter 4 and computed their orbifold Chow rings.

The thesis is a good introductory reading materials for graduate students who want to study orbifold Chow ring of Deligne-Mumford stacks and want to learn about the computations. Through reading students will learn about the techniques for computing the orbifold Chow rings, for example, what's the technical part and how to deal with the obstruction bundles in the definition of orbifold cup product. And through reading students will know some nice abelian Deligne-Mumford stacks.
5.2.2 Importance of the definitions and results.

In the definition of toric Deligne-Mumford stacks by Borisov, Chen and Smith in Chapter 1, the authors used the notion of stacky fans. We introduced a new notion of extended stacky fan and used this notion to define toric Deligne-Mumford stacks. The importance of this new definition mainly lies in two cases:

The first case is that it gave more representations of toric Deligne-Mumford stacks as quotients, see the examples in Chapter 2.

The second case is that the extended stacky fan $\Sigma$ is natural when we consider closed substacks corresponding to cones in the fan $\Sigma$. In Proposition 4.2 of [BCS], the authors used quotient stacky fan $\Sigma/\sigma$ to study the closed substack $\mathcal{X}(\Sigma/\sigma)$ of $\mathcal{X}(\Sigma)$. We found that the exact sequences in that proof are wrong and the quotient stacky fan is naturally an extended stacky fan. We give a new proof in Proposition 2.3.5 of Chapter 2.

An interesting application of the toric stack bundle discussed in Chapter 2 is that any finite abelian gerbes over a smooth scheme is a toric stack bundle. Thus we compute the orbifold Chow ring of any $\mu$-gerbes over smooth scheme $B$ for a finite abelian group $\mu$.

We first define the notion of hypertoric Deligne-Mumford stacks in Chapter 3 from stacky hyperplane arrangements and studied their orbifold Chow rings. We generalize the orbifold Chow ring formula of projective toric Deligne-Mumford stacks to semi-projective toric Deligne-Mumford stacks so that we compute more examples of orbifold Chow rings. All these results are positive contributions to the research field of stringy orbifold theory.
5.2.3 Application of the results.

The main results of this thesis are the Chow ring formula for toric stack bundles, hypertoric Deligne-Mumford stacks and semi-projective toric Deligne-Mumford stacks. These results will have many applications.

One interesting case is the special case of toric stack bundles, i.e. the $\mu$-gerbe $\mathcal{G} \to B$ over a smooth scheme $B$ for a finite abelian group $\mu$. We compute the orbifold cohomology of $\mathcal{G}$ and show that the $\mu$-gerbes over $B$, no matter they are trivial or not, have the same orbifold cohomology. In a physics paper [PS], the authors predicted that the quantum cohomology of $\mathcal{G}$ should be $|\mu|$ copies of the quantum cohomology of $B$. Since quantum cohomology is a deformation of the orbifold cohomology, based on our computation we can study the quantum cohomology of such gerbes.

Hypertoric Deligne-Mumford stacks are good examples of hyperkahler Deligne-Mumford stacks. Ruan’s CHRC conjecture says that the cohomology of the hyperkahler resolution of an orbifold is isomorphic to its orbifold cohomology. So our results of the orbifold Chow ring of hypertoric Deligne-Mumford stacks will be used to check the CHRC.

Quantum cohomology of Deligne-Mumford stacks will be an interesting topic in the future. We already compute the orbifold cohomology of toric Deligne-Mumford stacks and hypertoric Deligne-Mumford stacks. What’s the quantum cohomology of these two stacks? Our results will have applications in studying the quantum cohomology of Deligne-Mumford stacks.
5.3 Future Research.

5.3.1 Twisted orbifold Chow ring

Ruan [Ruan2] defined twisted orbifold cohomology using inner local systems on the inertia orbifold. Later on people discovered that a $\mathbb{C}^*$-gerbe over an orbifold naturally gives an inner local system. I will study the $\mathbb{C}^*$-gerbes over toric Deligne-Mumford stacks. Since a toric Deligne-Mumford stack is a quotient stack $[Z/G]$, a class of discrete torsion $\alpha \in H^2(G, \mathbb{C}^*)$ defines a $\mathbb{C}^*$-gerbe over the toric Deligne-Mumford stack. But not every $\mathbb{C}^*$-gerbe comes in this way. Is there a combinatorial description of the gerbes over toric Deligne-Mumford stacks that comes from discrete torsion elements? I will study this question and compute the twisted orbifold Chow rings of toric Deligne-Mumford stacks.

5.3.2 Quantum cohomology of gerbes

Orbifold cohomology of finite abelian gerbes over smooth schemes were computed in [Jiang1]. Let $\mu$ be a finite abelian group and $X$ a smooth scheme. Then trivial and nontrivial $\mu$-gerbes over $X$ have the same orbifold cohomology [Jiang1]. For the quantum cohomology, Pantev and Sharpe [PS] predicted $k$ copies of the quantum cohomology of $X$ for the quantum cohomology of a $\mu_k$-gerbe over $X$. For example, for the nontrivial $\mu_2$-gerbe $\mathcal{X}$ over $\mathbb{P}^1$, they predicted that the quantum cohomology should be $QH^*_{orb}(\mathcal{X}) \cong \mathbb{Q}[x, t, q]/(x^2 - qt, t^2 - 1)$. For the trivial gerbe, we have $QH^*_{orb}([\mathbb{P}^1/\mu_2]) \cong \mathbb{Q}[x, t, q]/(x^2 - q, t^2 - 1)$. These two gerbes have the same orbifold cohomology by letting $q$ be zero, while they have different quantum cohomology which are both 2 copies of the quantum cohomology of $\mathbb{P}^1$. I will study the quantum cohomology of such gerbes and compute the quantum cohomology of...
5.3.3 The Crepant resolution conjecture.

One of the motivation to compute orbifold Chow ring is to check the Cohomological Hyperkahler Resolution Conjecture of Ruan [Ruan]. For toric Deligne-Mumford stacks, Borisov, Chen and Smith gave an example [BCS] to show that the CHRC conjecture is not true in general, or maybe true when we add some quantum corrections. I will study this question in the future studies. In Chapter 4 we computed the orbifold Chow ring of any hypertoric Deligne-Mumford stack. We don’t know if there exists a crepant resolution for a hypertoric orbifold, or suppose it exists, if it is a smooth hypertoric variety. Since hypertoric varieties are hyperkahler, maybe we can check the conjecture if we know the crepant resolution.

Bryan and Graber [BG] generalized the CHRC conjecture to the whole Gromov-Witten theory which we call the ”crepant resolution conjecture”. In [BG], Bryan and Graber solved the genus zero conjecture for the symmetric product orbifold $[(C^2)^n/S_n]$. Let $G$ be a finite group and act on $C^2$. We take the semi-product $G^n \times S_n = G_n$ and the group $G_n$ acts on $(C^2)^n$. The quotient $[(C^2)^n/G_n]$ is called the wreath product stack. Let $Y$ be the crepant resolution of $C^2/G$, then the Hilbert scheme $Y^{[n]}$ of $n$-points on $Y$ is the crepant resolution of the wreath product $(C^2)^n/G_n$, see [W], [QW]. In [QW], the authors proved that the ordinary cohomology ring of the Hilbert scheme $Y^{[n]}$ is isomorphic to the orbifold cohomology ring of $[(C^2)^n/G_n]$ when $G$ is a cyclic group. A natural question is to ask if the CRC is true for wreath product stacks. For $A_n$ resolutions $Y$, Maulik [Maulik] already computed the Gromov-Witten theory of $Y$ and proved the Gromov-Witten/Hilbert-Scheme correspondence. I will study the orbifold Gromov-Witten theory of the
wreath product to see if the CRC is true.
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