SEMILINEAR STOCHASTIC EVOLUTION EQUATIONS

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Abstract

Let H be a separable Hilbert space. Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete stochastic basis with a right continuous filtration and $\{W_t, t \in \mathbf{R}\}$ is an H-valued cylindrical Brownian motion with respect to $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. $U(t,s)$ denotes an almost strong evolution operator generated by a family of unbounded closed linear operators on H. Consider the semilinear stochastic integral equation

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)f_s(X_s)ds + \int_0^t U(t,s)g_s(X)dW_s + V_t,$$

where

- f is of monotone type, i.e., $f_t(.) = f(t,\omega,.) : H \to H$ is semimonotone, demicontinuous, uniformly bounded, and for each $x \in H$, $f_t(x)$ is a stochastic process which satisfies certain measurability conditions.
- $g_s(.)$ is a uniformly-Lipschitz predictable functional with values in the space of Hilbert-Schmidt operators on H.
- V_t is a cadlag adapted process with values in H.
- X_0 is a random variable.

We obtain existence, uniqueness, boundedness of the solution of this equation. We show the solution of this equation changes continuously when one or all of X_0, f, g, and V are varied. We apply this result to find stationary solutions of certain equations, and to study the associated large deviation principles.

Let $\{Z_t, t \in \mathbf{R}\}$ be an H-valued semimartingale. We prove an Ito-type inequality and a Burkholder-type inequality for stochastic convolution $\int_0^t U(t,s)g_s(X)dZ_s$. These are the main tools for our study of the above stochastic integral equation.

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Acknowledgment

The building of the character of mankind involves many different factors. Included in this building process are teachers and the learning enviorment. Mathematics, as a beautiful and wonderful human activity, also is, included in this building process.

One of the most difficult problems a scholar can encounter is deciding which area they should choose to enter in terms of their ability and interest. This problem does not have a simple or easy solution. The solving, or partial solving of this dilemma took me several years. Of course, some of the factors which greatly influenced my final decision were the many teachers I had over the years. They helped and guided me over the course of my life and enabled me to eventually make a wise choice of field of study. I would like to speak a bit about them and to pay tribute to them.

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Chapter 1

INTRODUCTION

In recent years there has been increasing interest in the theory of stochastic evolution equations. This has been partly motivated by the needs in various applied fields such as control theory, mechanics, statistical hydromechanics, quantum mechanics, quantum field theory, population genetics, stochastic quantization, neurophysiology and random vibration. Several of those applications are presented in Curtain and Pritchard (1978), Dawson (1975), Krylov and Rozovskii (1981), Walsh (1981, 1984, 1986), Faris and Jona-Lasinio (1982), Crandall and Zhu (1983) and Biswas and Ahmed (1985).

Suppose one is given a dynamical system governed by a partial differential equation. Suppose that the system is then excited randomly by some sort of noise. Then the response of the system will be governed by a stochastic partial differential equation.

Example 1.1

Assume that an elastic string of length l is tightly stretched between two supports at the same horizontal level, so that the x-axis lies along the string. Let $u(t,x)$ denote its vertical displacement at the point x at time t. If damping effects such as air resistance are neglected, and if the amplitude of the motion is not too large, then $u(t,x)$ satisfies the P.D.E

$$u_{tt} = au_{xx} \tag{1.1}$$

in the domain $0 < x < l$, $t > 0$, where $a > 0$ is a constant.

If the string is excited randomly by a white noise $\dot{W}(t,x)$, then the dynamical response is governed by the stochastic partial differential equation (SPDE)

$$u_{tt} = a^2 u_{xx} + \dot{W}(t,x). \tag{1.2}$$

This equation is called the *stochastic wave equation* and has been well-studied in Cabaña (1970, 1972), Pardoux (1975), Walsh (1986), Biswas and Ahmed (1985), and Carmona and Nualart (preprint). The elastic string may be thought of as a violin string or a guitar string [see Walsh (1986), p. 0.1]

An important example of an elastic string is the electric power line which is excited by a white noise. According to Biswas and Ahmed (1985), " ...this distributed noise could be attributed to the random aerodynamic forces acting on the (transmission line) conductors, arising due to the randomness of wind velocity and the irregularity of ice formation on the conductor surface" (p. 1043). The vibrating transmission line is called *the galloping conductor* .

In random vibration, structural elements such as beams, cables, arches, membranes, and shells can be excited by some sort of random loadings. According to Crandall (1979), "Random loadings on such structural elements can arise from earthquakes ..., or windstorms ..., acting on onshore structures, from storm winds and waves ...on offshore structures, from turbulent boundary layers and jet noise on high-speed aircraft ..., or from turbulent flow in and around the tubes in heat exchangers " (p. 1). See also Crandall and Zhu (1983) for a survey of the recent developments of random vibration.

Stochastic Partial Differential Equations (SPDEs)

As in the classical theory of PDE, there are two methods to study SPDEs. First, one can study the multiparameter processes $u(t,x)$ as solutions of the SPDE. This method emphasizes sample path properties of real-valued multiparameter processes $u(t,x)$. Walsh

(1986) contains a systematic treatment of this approach. See also Walsh (1981, 1983), Dozzi (1989) and the references given therein for further information on this approach, though we shall not follow it closely here.

The second method, which we will follow, is to consider one-parameter Banach-valued processes, $u = \{u_t : t \in \mathbf{R}\}$ as solutions of the SPDEs. Here one envisages the SPDEs as *stochastic evolution equations* in an appropriate Banach space, where functional analytic techniques are applied. Dawson (1975) contains a rigorous treatment of the theory of stochastic evolution equations and reviews the subject up to 1975 and has extensive references. See also Curtain and Pritchard (1978).

Though these methods are nearly equivalent, some problems are more natural to pose from one point of view than the other.

Notations and Definitions

Let H be a real separable Hilbert space with norm $\|\ \|$ and inner product $<\ ,\ >$. Let $L_2(H)$ be the space of Hilbert-Schmidt operators on H with norm $\|\ \|_2$.

Let g be an H-valued function defined on a set $D(G) \subset H$. Recall that g is *monotone* if for each pair

$$x, y \in D(g), \quad < g(x) - g(y), x - y > \geq 0,$$

and g is *semi-monotone* with *parameter* M if, for each pair $x, y \in D(g)$,

$$< g(x) - g(y), x - y > \geq -M\|x - y\|^2.$$

On the real line we can represent any semimonotone function with parameter M, by $f(x) - Mx$; where f is a non-decreasing function on $\mathbf{R}$.

We say g is *bounded* if there exists an increasing continuous function ψ on $[0, \infty)$ such that $\|g(x)\| \leq \psi(\|x\|), \forall x \in D(g)$. g is *demi-continuous* if, whenever (x_n) is a sequence in $D(g)$ which converges strongly to a point $x \in D(g)$, then $g(x_n)$ converges weakly to $g(x)$.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete stochastic basis with a right continuous filtration.

We follow Yor (1974) and define cylindrical Brownian motion as

Definition 1.1 *A family of random linear functionals $\{W_t, t \geq 0\}$ on H is called a cylindrical Brownian motion on H if it satisfies the following conditions:*
(i) *$W_0 = 0$ and $W_t(x)$ is $\mathcal{F}_t$-adapted for every $x \in H$.*
(ii) *For every $x \in H$ such that $x \neq 0$, $W_t(x)/\|x\|$ is a one-dimensional Brownian motion.*

Note that cylindrical Brownian motion is not H-valued because its covariance is not nuclear. For the properties of cylindrical Brownian motion and the definition of stochastic integrals with respect to the cylindrical Brownian motion see Yor (1974).

1.1 Linear Stochastic Evolution Equation

Linear stochastic evolution equations have been extensively studied in recent years. They occur in Dawson (1975), Miyahara (1981), Ito (1978, 1982), Holley and Strook (1978), Kallianpur and Wolpert (1984), Ichikawa (1978), Walsh (1981, 1984, 1986), Ustunel (1982), Kallianpur and Perez-Abreu (1987), Da Prato et al. (1982a, 1982b), Da Prato (1983) , and Leon (1989). Consider the linear stochastic evolution equation

$$dX_t = A(t)X_t\,dt + dM_t, \tag{1.3}$$

where M_t is an H-valued martingale and for each $t \in \mathbf{R}, A(t)$ is a closed unbounded linear operator on H which satisfies certain conditions.

The mild solution of (1.3) with initial condition $X(0) = 0$ can be represented as a *stochastic convolution integral* $\int_0^t U(t,s)dM_s$ [see Kotelenez (1982)], where $U(t,s)$ is the evolution operator generated by $A(t)$.

Kotelenez (1982, 1984) proved a submartingale-type and stopped-Doob inequality for stochastic convolution integrals. An Ito-type inequality and a Burkholder-type inequality for this object will be proved in Chapter 3.

1.2 Non-linear Stochastic Evolution Equations

Most of the work on strong solutions of non-linear stochastic evolution equations in recent years has concentrated on two types of equations. The first has the form

$$dX_t = A(t)X_t dt + f(X_t)dt + g(X_t)dW_t, \tag{1.4}$$

where $f : H \to H$ and $g : H \to L_2(H)$ are uniformly Lipschitz mappings. In solving this equation one usually uses semigroup techniques.

The other non-linear stochastic evolution equation is based on a Gelfand triple, $B \subset H \subset B^\star$ (where B is a Banach space and $B^\star$ is its dual) and has the form

$$dX_t = F(X_t)dt + G(X_t)dW_t, \tag{1.5}$$

where $F : B \to B^\star$ and $G : B \to L_2(H)$ satisfy certain monotonocity and coercivity conditions. In this setting, one often uses a variational approach.

Let us briefly discuss the strong solutions in each of these two classes.

1.2.1 First Type

If $A(t) \equiv 0$, (1.4) is called a *stochastic differential equation* in H and has been well-studied by several authors [see Dawson (1975) and Métivier (1982) for references].

When $A(t)$ is an unbounded closed linear operator, equation (1.4) is called a *semilinear stochastic evolution equation* and since f and g are Lipschitz, we say it is of *Lipschitz type.* Here we usually look for *mild solutions* of (1.4), which are strong solutions of the following integral equation:

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)f(X_s)ds + \int_0^t U(t,s)g(X_s)dW_s, \tag{1.6}$$

where $U(t,s)$ is an evolution operator generated by $A(t)$.

When $A(t) \equiv A$ is a negative, self-adjoint operator such that A^{-1} is nuclear, the existence and uniqueness of the mild solution of (1.4) has been proved by Dawson (1972, 1975). The existence and uniqueness still apply when g takes values in the space of bounded linear operators on H, instead of the Hilbert-Schmidt operators.

If $A(t)$ is a generator of an evolution operator $U(t,s)$, the existence and uniqueness of the solution of (1.4) have been studied by several authors [see for example Ichikawa (1982), Kotelenez (1988)]. Kotelenez (1984) has studied a more general case.

Ahmed (1985) and Da Prato and Zabczyk (1988) have proved the existence and uniqueness of (1.3) in the context of Banach spaces.

There are several results in this theory of a qualitative character. We shall briefly mention a few.

Marcus (1974) considered problems of the asymptotic stationarity of the solution of (1.3) (in the case $g \equiv I$). Funaki (1983) applied equation (1.4) to examine the random vibration of strings. Ichikawa (1982, 1983, 1984) has results on the stability, boundedness and invariant measures of (1.4). Maslowski (1989) has results on the uniqueness and stability of invariant measures. Da Prato and Zabczyk (1988) obtained the Wentzel-Freidlin large-deviations estimate for the solution in the case $g \equiv I$.

1.2.2 Second Type

In equation (1.5), assume the functions F and G satisfy the following conditions: there exist $C > 0$, $\epsilon > 0$ and $p \geq 1$ such that

- **Coercivity** of (F, G):

$$2 < X, F(X) >_{B \times B^*} + \|G(X)\|_2^2 + \epsilon \|X\|_B^p \leq C(1 + \|X\|^2).$$

- **Monotonicity** of (F, G):

$$2 < X_2 - X_1, F(X_2) - F(X_1) >_{B \times B^*} + \|G(X_2) - G(X_1)\|_2^2 + \leq C\|X_2 - X_1\|^2.$$

- **Boundedness** of the growth of F:

$$\|F(X)\|_{B\star} \leq C(1 + \|X\|_B^{p-1}).$$

- **Semicontinuity** of F: the function $< X, F(X_1 + \lambda X_2) >_{B \times B^*}$ is continuous in λ on $\mathbf{R}^1$.

In this approach one is usually interested in the strong solution of the integral equation

$$X_t = X_0 + \int_0^t F(X_s)ds + \int_0^t G(X_s)dW_s. \tag{1.7}$$

When $G \equiv 1$, the existence and the uniqueness of the solution of (1.7) is proved in Bensoussan and Temam (1972).

In the general case, the existence and uniqueness of the solution was first proved by Pardoux (1975) under stronger assumptions on (F, G). This was proved in connection with the theory developed in Pardoux (1975). A direct proof was later given by Krylov and Rozovskii (1981). This has been generalized by Gyongy and Krylov (1982a).

This was one of the most active areas in the theory of the stochastic evolution equations in the last decade. There have been extensive works by Pardoux, Krylov, Rozovskii, Gyongy, such as Gyongy (1982, 1988, 1989a, 1989b), and Gyongy and Krylov (1982b).

1.2.3 Comparing the Two Types of Equations

Each one of these approaches has some advantages. For example, if the differential operator is non-linear as in the Navier-Stokes equation, we have to pose the problem in the second setting, and if the differential operator is linear but does not have the coercivity property, as in the wave equation or in the symmetric hyperbolic system, then it is more natural to pose it in the first setting.

One advantage of the semigroup approach is that it gives a unified treatment of a wide class of parabolic, hyperbolic and functional differential equations.

In the case of parabolic equations, one can employ the variational method, which applies to non-Lipschitz (F, G) [see Krylov and Rozovskii (1981)].

Consider the second order semilinear stochastic evolution equation on H, written formally as

$$\frac{\partial^2 u(t)}{\partial t^2} + Au(t) = \tilde{f}\left(u(t), \frac{\partial u}{\partial t}\right) + \tilde{g}\left(u(t), \frac{\partial u}{\partial t}\right)\dot{W}_t, \tag{1.8}$$

- where $\tilde{f} : H \times H \to H$ and $\tilde{g} : H \times H \to L_2(H)$ satisfy certain conditions;
- W_t is a cylindrical Brownian motion on H;
- A is strictly positive definite, self-adjoint unbounded operator on H. When f and g satisfy the Lipschitz condition, equation (1.8) falls in the first category.

Pardoux (1975) has studied this equation when f satisfies certain monotonicity and coercivity conditions. He has constructed the solution of (1.8) in connection with the theory developed in part three of Pardoux (1975). His approach is a stochastic version of Lions and Strauss (1965).

In Chapters 4 and 7 of this thesis, we will study Equation (1.8) as a corollary of our existence and uniqueness Theorem.

1.2.4 The Semilinear Stochastic Evolution Equation of Monotone Type

The preceding two approaches are both stochastic versions of well-known methods in the theory of deterministic evolution equations. In the latter theory, there is yet another approach in which a large class of problems could be studied. This approach is a generalization of the first one above and can be used to study *semilinear evolution equations of monotone type.*

Consider

$$\dot{X}_t = A(t)X_t + f(X_t), \tag{1.9}$$

where f is of monotone type, i.e., $-f$ is semimonotone, demicontinuous and bounded by φ [see for example Browder (1964), Kato (1964), Vainberg (1973), Tanabe (1979) and Carroll (1969)].

The study of the stochastic version of the above equation, i.e., the study of the equation (1.4) when f is of monotone type and $g \neq 0$ is uniformly Lipschitz, is not in the literature. In this dissertation, we shall see that we may use semigroup theory to extend the above deterministic method to equation (1.9) and then use this to study the stochastic equation (1.4). We will obtain the existence, uniqueness, boundedness and the continuity with respect to a parameter. We will also apply it to find stationary solutions of certain equations, and to study the associated large deviation principle.

Now we shall briefly outline our main results and the contents of this thesis.

1.3 The Main Results

Let us consider the following generalization of the integral equation (1.4):

$$X_t = u(t,0)X_0 + \int_0^t U(t,s)f_s(X_s)ds + \int_0^t U(t,s)g_s(X)dW_s + V_t, \tag{1.10}$$

where

- $f_t(.) = f(t,\omega,.) : H \to H$ is of monotone type, and for each $x \in H$ $f_t(x)$ is a stochastic process which satisfies certain measurability conditions;
- $g_s(.)$ is a uniformly Lipschitz predictable functional with values in $L_2(H)$;

- W_s is an H-valued cylindrical Brownian motion;
- V_t is a cadlag, adapted process with values in H.

1.3.1 The Method of Study

We construct the solution (1.10) by first constructing its solution when $g \equiv 0$. This latter will be shown to be a weak limit of solutions of (1.10) in the case when $g \equiv 0$ and $A \equiv 0$, which in turn have been constructed by the Galerkin approximation of the finite-dimensional equation. Pardoux, Krylov, Rozovskii, and Gyongy built their analysis of (1.7) on a generalized Ito's Lemma decomposition of $||u(t)||^2$. Since we use semigroup techniques in this dissertation, our object is to solve the equation by means of convolution integrals. Our proof is based upon a different version of an Ito-type inequality (Theorem 3.1) and on a Burkholder-type inequality (Theorem 3.2).

1.3.2 Existence, Uniqueness, and Boundedness of the Solution

- *Equation (1.10) in case* $g \equiv 0$:

The main problem in this case is to show the measurability of the solution. The proof is in chapters two and four. In Chapter 4, we show that several important stochastic semilinear equations fall in this setting.

- *Extending equation (1.10) to the general case*:

In Chapters 7 and 8 we prove that if V_t *is a continuous adapted process*, and if f is bounded by a polynomial, then the integral equation (1.10) has a continuous adapted solution. We also give a bound for the moments of this solution in Chapter 8.

In Chapter 7 several examples are studied including the second order equation and the semilinear hyperbolic system.

1.3.3 The Semilinear Integral Equation on the Whole Real Line and the Stationarity of its Solutions

Consider the stochastic semilinear equation

$$dX_t = AX_t dt + f_t(X_t)dt + dW_t, \tag{1.11}$$

where A is a closed, self-adjoint, negative definite, unbounded operator such that A^{-1} is nuclear. A mild solution of (1.11) with initial condition $X(0) = X_0$ is the solution of the integral equation

$$X_t = U(t,0)X_0 + \int_0^t U(t-s)f_s(X_s)ds + \int_0^t U(t-s)dW_s, \tag{1.12}$$

where $U(t)$ is the semigroup generated by A.

Marcus (1974) has proved that when f is independent of t and ω, and f is uniformly Lipschitz, then the solution of (1.12) is asymptotically stationary. To prove this, he studied the following integral equation:

$$X_t = \int_{-\infty}^t U(t-s)f_s(X_s)ds + \int_{-\infty}^t U(t-s)dW_s, \tag{1.13}$$

where the parameter set of the processes is extended to the whole real line. This motivated us to study the existence of the solution of the slightly more general equation

$$X_t = \int_{-\infty}^t U(t-s)f_s(X_s)ds + V_t, \tag{1.14}$$

where f is of monotone type and is bounded by a polynomial, and V_t is a cadlag adapted process. In Chapter 4, we prove the existence the uniqueness of the solution to (1.12). In Chapter 5, we will prove that finite dimensional Galerkin approximations converge strongly to the solution of (1.14). In Chapter 6 we prove under certain conditions that if V_t is a stationary process, then X_t is also a stationary process.

1.3.4 Continuity with Respect to a Parameter

Faris and Jona-Lasinio (1982) have studied the equation (1.10) in the case when $g \equiv 0$, the generator of U is $\frac{d^2}{dx^2}$, and $f(x) = -\lambda x^3 - \mu x$. They showed that the solution X is a continuous function of V in this case.

Da Prato and Zabczyk (1988) generalized this to the case where U is a general analytic semigroup and f is a locally Lipschitz function on a Banach space. In Chapter 5 we generalize these results by proving that [in case $g \equiv 0$] the solution of (1.9) changes continuously when any or all of V, f, A, and X_0 are varied. As a corollary, we prove a generalization of Faris and Jona-Lasino's theorem for semimonotone f and more general U; this was open after Faris and Jona-Lasinio (1982) [see for example Smolenski et al.(1986), p. 230]. We also prove the strong convergence of the finite dimensional Galerkin approximations to the solution of (1.10) (in case $g \equiv 0$). Métivier (1980) has proved that when $A \equiv 0$, $V \equiv 0$, and f is Lipschitz, then the solution of equation (1.10) changes continuously as f, g and X_0 are varied.

In Chapter 8 we generalize this by proving that the solution of (1.10) in the general case changes continuously when one or all of X_0, f, g, and V are varied.

Chapter 2

THE MEASURABILITY OF THE SOLUTION

2.1 The Main Theorem

Let H be a real separable Hilbert space with an inner product and a norm denoted by $<,>$ and $\| \|$, respectively. Let $(G, \mathcal{G})$ be a measurable space, i.e., G is a set and $\mathcal{G}$ is a σ-field of subsets of G. Let $T > 0$ and let $S = [0, T]$. Let β be the Borel field of S. Let $L^2(S, H)$ be the set of all H-valued square integrable functions on S.

Consider the initial value problem, formally written as

$$\begin{cases} \frac{du}{dt} &= f(t, u(t)), \quad t \in S; \\ u(0) &= u_0, \end{cases} \tag{2.1}$$

where $f : S \times H \to H$ and $u_0 \in H$. We say u is a solution of (2.1) if it is a solution of the integral equation

$$u(t) = u_0 + \int_0^t f(s, u(s))ds. \tag{2.2}$$

We will actually be interested in slightly more general equations. Consider the integral equation

$$u(t, y) = \int_0^t f(s, y, u(s, y))ds + V(t, y), \quad t \in S, \ y \in G. \tag{2.3}$$

In this case $f : S \times G \times H \to H$ and $V : S \times G \to H$. The variable y is a parameter, which in practice will be an element ω of a probability space.

Our aim in this chapter is to show that under proper hypotheses on f and V there exists a unique solution u to (2.3), and that this solution is a $\beta \times \mathcal{G}$-measurable function of t and the parameter y.

In this chapter we say $X(.,.)$ is *measurable* if it is $\beta \times \mathcal{G}$-measurable.

We will study (2.3) in the case where $-f$ is demi-continuous and semi-monotone on H and V is right continuous and has left limits in t *(cadlag).*

This has been well-studied in the case in which V is continuous and f is bounded by a polynomial and does not depend on the parameter y. See for example Bensoussan and Temam (1972).

Let $\mathcal{H}$ be the Borel field of H. Consider functions f and V

$$f : S \times G \times H \to H$$

$$V : S \times G \to H.$$

We impose the following conditions on f and V:

Hypothesis 2.1 **(a)** *f is $\beta \times \mathcal{G} \times \mathcal{H}$-measurable and V is $\mathcal{G} \times \mathcal{H}$-measurable.*
(b) *For each $t \in S$ and $y \in G$, $x \to f(t,y,x)$ is demicontinuous and uniformly bounded in t. (That is, there is a function $\varphi = \varphi(x,y)$ on $\mathbf{R}_+ \times G$ which is continuous and increasing in x and such that for all $t \in S$, $x \in H$, and $y \in G$, $\|f(t,y,x)\| \leq \varphi(y, \|x\|)$.)*
(c) *There exists a non-negative $\mathcal{G}$-measurable function $M(y)$ such that for each $t \in S$ and $y \in G$, $x \to -f(t,y,x)$ is semimonotone with parameter $M(y)$.*
(d) *For each $y \in G$, $t \to V(t,y)$ is cadlag.*

Theorem 2.1 *Suppose f and V satisfy the Hypothesis 2.1. Then for each $y \in G$, (2.3) has a unique cadlag solution $u(\cdot,y)$, and $u(\cdot,\cdot)$ is $\beta \times \mathcal{G}$-measurable. Furthermore*

$$\|u(t,y)\| \leq \|V(t,y)\| + 2\int_0^t e^{M(y)(t-s)}\|f(s,y,V(s,y))\|ds; \tag{2.4}$$

$$\|u(.,y)\|_\infty \leq \|V(.,y)\|_\infty + 2C_T\varphi(y, \|V(.,y)\|_\infty), \tag{2.5}$$

where $\|u\|_\infty = \sup_{0\le t\le (T)} \|u(t)\|$, *and*

$$C_T = \begin{cases} \frac{1}{M(y)} e^{M(y)T} & \text{if } M(y) \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

.

Let us reduce this theorem to the case when $M = 0$ and $V = 0$. Define the transformation

$$X(t, y) = e^{M(y)t}(u(t, y) - V(t, y)) \tag{2.6}$$

and set

$$g(t, y, x) = e^{M(y)t} f(t, y, V(t, y) + xe^{-M(y)t}) + M(y)x. \tag{2.7}$$

Lemma 2.1 *Suppose f and V satisfy Hypothesis 2.1 . Let X and g be defined by (2.6) and (2.7). Then g is $\beta \times \mathcal{G} \times \mathcal{H}$-measurable and $-g$ is monotone, demicontinuous, and uniformly bounded in t. Moreover u satisfies (2.3) if and only if X satisfies*

$$X(t, y) = \int_0^t g(s, y, X(s, y))ds, \quad \forall t \in S,\ y \in G. \tag{2.8}$$

Proof: The verification of this is straightforward. Suppose that V and f satisfy Hypothesis 2.1. We claim g satisfies the above conditions.

- g is $\beta \times \mathcal{G} \times \mathcal{H}$-measurable.

Indeed, if $h \in H$ then $< f(t, y, .), h >$ is continuous and $V(t, y) + xe^{-M(y)t}$ is $\beta \times \mathcal{G} \times \mathcal{H}$-measurable, so $< f(t, y, V(t, y) + xe^{-M(y)t}), h >$ is $\beta \times \mathcal{G} \times \mathcal{H}$-measurable. Since H is separable then $f\left(t, y, V(t, y) + xe^{-M(y)t}\right)$ is also $\beta \times \mathcal{G} \times \mathcal{H}$-measurable, and since $e^{M(y)t}$ and $M(y)x$ are $\beta \times \mathcal{G} \times \mathcal{H}$-measurable, then g is $\beta \times \mathcal{G} \times \mathcal{H}$-measurable.

- g is bounded, since $\sup_t \|V_t(y)\| < \infty$ and $\|g(t, y, x)\| \le \phi(y, \|x\|)$, where

$$\phi(y, \xi) = e^{M(y)T} \phi(y, \xi + \sup_t \|V_t\|) + M(y)\xi.$$

- g is demicontinous.

- $-g$ is monotone.

Furthermore, one can check directly that if X is measurable, so is u. Since X is continuous in t and V is cadlag, u must be cadlag. It is easy to see that different solutions of (2.7) correspond to different solutions of (2.3). Q.E.D

By Lemma 2.1, Theorem 2.1 is a direct consequence of the following.

Theorem 2.2 *Let $g = g(t, y, x)$ be a $\beta \times \mathcal{G} \times \mathcal{H}$- measurable function on $S \times G \times H$ such that for each $t \in S$ and $y \in G$, $x \rightarrow -g(t, y, x)$ is demicontinous, monotone and bounded by φ. Then for each $y \in G$ the equation (2.8) has a unique continuous solution $X(., y)$, and $(t, y) \rightarrow X(t, y)$ is $\beta \times \mathcal{G}$-measurable.*

Furthermore X satisfies (2.5) with $M = 0$ and $V = 0$.

Remark that the transformation (2.6) $u \rightarrow X$ is bicontinuous and in particular, implies if X satisfies (2.4) and (2.5) for $M = 0$ and $V = 0$, then u satisfies (2.4) and (2.5)

Note that y serves only as a nuisance parameter in this theorem. It only enters in the measurability part of the conclusion. In fact, one could restate the theorem somewhat informally as: if f and u_0 depend measurably on a parameter y in (2.2), so does the solution.

The proof of Theorem 2.2 in the case in which f is independent of y is a well-known theorem of Browder (1964) and Kato (1964). One proof of this theorem can be found in Vainberg(1973), Th (26.1), page 322. The proof of the uniqueness and existence are in Vainberg (1973). In this section we will prove the uniqueness of the solution and inequalities (2.4) and (2.5). In Section 2.3 we will prove the measurability and outline the proof of the existence of the solution of equation (2.8)

Since y is a nuisance parameter, which serves mainly to clutter up our formulas, we will only indicate it explicitly in our notation when we need to do so.

Let us first prove a lemma which we will need for proof of the uniqueness and for the proof of inequalities (2.4) and (2.5).

Lemma 2.2 *If $a(.)$ is an H-valued integrable function on S and if $X(t) := X_0 + \int_0^t a(s)ds$, then*

$$\|X(t)\|^2 = \|X_0\|^2 + 2\int_0^t < X(s), a(s) > ds.$$

Proof: Since $a(s)$ is integrable, then $X(t)$ is absolutely continuous and $X'(t) = a(t)$ a.e. on S. Then $\|X(t)\|$ is also absolutely continuous and

$$\frac{d}{dt}\|X(t)\|^2 = 2 < \frac{dX(t)}{dt}, X(t) >= 2 < a(t), X(t) > \quad \text{a.e.}$$

so that

$$\int_0^t \frac{d}{ds}\|X(s)\|^2 ds = \|X(t)\|^2 - \|X_0\|^2.$$

Thus

$$\|X(t)\|^2 - \|X_0\|^2 = 2\int_0^t < X(s), a(s) > ds.$$

Q.E.D.

Now we can prove inequalities (2.4) and (2.5) in case $M = 0$ and $V = 0$.

Lemma 2.3 *If $M = V = 0$, the solution of the integral equation (2.8) satisfies the inequality*

$$\|X(t)\| \leq 2\int_0^t \|g(s,0)\|ds \leq 2T\varphi(0).$$

Proof: Since $X(t)$ is a solution of the integral equation (2.8), then by Lemma 2.2 we have

$$\begin{aligned}\|X(t)\|^2 &= 2\int_0^t < g(s,X(s)),\ X(s) > ds \\ &= 2\int_0^t < g(s,X(s)) - g(s,0), X(s) > ds \\ &\quad + 2\int_0^t < g(s,0), X(s) > ds \\ &\leq 2\int_0^t < g(s,X(s)) - g(s,0), X(s) > ds \\ &\quad + 2\int_0^2 \|g(s,0)\|\ \|X(s)\| ds.\end{aligned}$$

Since $-g$ is monotone, the first integral is negative. We can bound the second integral and rewrite the above inequality as

$$\begin{aligned}\|X(t)\|^2 &\leq 2\int_0^t \|g(s,0)\|\|X(s)\|ds \\ &\leq 2\sup_{0\leq s\leq t}\|X(s)\| \int_0^t \|g(s,0)\|ds.\end{aligned}$$

Thus $\sup_{0\leq s\leq t}\|X(s)\| \leq 2\int_0^t \|g(s,0)\|ds$. Since $\sup_{0\leq s\leq t}\|g(s,x)\| \leq \varphi(\|x\|)$, the proof is complete. Q.E.D

Proof of Uniqueness

Let X and Y be two solutions of (2.8). Then we have

$$X(t,y) - Y(t,y) = \int_0^t [g(s,y,X(s,y)) - g(s,y,Y(s,y))]ds.$$

By Lemma 2.2 one has

$$\|X(t,y) - Y(t,y)\|^2 = \int_0^t < g(s,y,X(s,y)) - g(s,y,Y(s,y)), X(s,y) - Y(s,y) > ds.$$

Since $-g$ is monotone, the right hand side of the above equation is negative, so

$$X(t,y) = Y(t,y).$$

Q.E.D

2.2 The Measurability of the Solution in Finite-dimensional Space

Consider the integral equation

$$X(t,y) = \int_0^t h(s,y,X(s,y))ds, \tag{2.9}$$

where $h(\cdot,\cdot)$ satisfies the following hypothesis.

Hypothesis 2.2 **(a)** *h satisfies Hypothesis 2.1 (a), (b).*

(b) *For each $t \in S$ and $y \in G$, $-h(t,y,\cdot)$ is continuous and monotone.*

Since h is measurable and uniformly bounded in t, then $h(\cdot,y,x)$ is integrable. As $h(t,y,\cdot)$ is continuous, the integral equation (2.9) is a classical deterministic integral equation in $\mathbf{R}^n$ and the existence of its solution is well known. In section 2.1 we proved that (2.9) has a unique bounded solution, so we only need to prove the measurability of the solution.

The existence, uniqueness and measurability of the solution of (2.9) is known (see Krylov and Rozovskii (1979) for a proof in a more general situation). Since the measurability result is easy to prove in our setting, we will include a proof in the following theorem for the sake of completeness.

Theorem 2.3 *The solution of the integral equation (2.9) is measurable.*

Proof: For the proof of measurability we are going to construct a sequence of solutions of other integral equations which converge uniformly to a solution of (2.9).

First: Let $\psi(.)$ be a positive C^∞-function on $H_n \simeq \mathbf{R}^n$ with support $\{\|x\| \leq T\varphi(0) + 2\}$, which is identically equal to one on $\{\|x\| \leq T\varphi(y,0)+1\}$. Now define $\tilde{h}(t,x) = h(t,x)\psi(x)$.

$-\tilde{h}$ is semimonotone. This can be seen because. If $\|X\| > T\varphi(0)+2$ and $\|Z\| > T\varphi(0)+2$,then $\tilde{h}(t,X) = \tilde{h}(t,Z) = 0$ and so

$$< \tilde{h}(t,X) - \tilde{h}(t,Z),\, X - Z >= 0.$$

Let $\|Z\| \leq T\varphi(0)+2$. Then

$$\begin{aligned} < \tilde{h}(t,X) - \tilde{h}(t,Z)\,,\, X - Z > &= < h(t,X)\psi(X) - h(t,Z)\psi(X), X - Z > \\ &\quad + < h(t,Z)\psi(X) - h(t,Z)\psi(Z), X - Z > . \end{aligned}$$

By the Schwarz inequality this is

$$\leq \psi(X) < h(t,X) - h(t,Z), X - Z > +\|h(t,Z)\|\; |\psi(X) - \psi(Z)|\; \|X - Z\|.$$

Since $-h$ is monotone and ψ is positive, the first term of the right hand side of the inequality is negative. Now as Z is bounded and ψ is C^∞ with compact support, the second term is $\leq M(y)\|X-Z\|^2$ for some $M(y)$.

Since by Lemma 2.3 the solution of (2.9) is bounded by $T\varphi(0)$, it never leaves the set $\{\|x\| \leq T\varphi(0) + 1\}$, so the unique solution of (2.9) is also the unique solution of the equation $X(t) = \int_0^t \tilde{h}(s, X(s))ds$. Thus without loss of generality we can assume $h(t,.)$ has compact support.

Second: Define $k(x)$ to be equal to $C\exp\{\frac{1}{\|x\|^2-1}\}$ on $\{\|x\| < 1\}$ and equal to zero on $\{\|x\| \geq 1\}$. Then $k(x)$ is C^∞ with support in the unit ball $\{\|x\| \leq 1\}$. Choose C such that $\int_{\mathbf{R}^n} k(x)dx = 1$. Introduce, for $\varepsilon > 0$

$$I_\varepsilon u(x) = \varepsilon^n \int_{R^n} k(\frac{x-z}{\varepsilon})u(z)dz.$$

This is a C^∞-function called the mollifier of u.

Now define $h_\varepsilon(t,x) = I_\varepsilon h(t,.)(x)$. Since for any ε the first derivatives with respect to x of $J_\varepsilon u(x)$ and also $J_\varepsilon u(x)$ itself are bounded in terms of the maximum of $\|u(x)\|$, then h_ε and $D_x h_\varepsilon$ are bounded in terms of the maximum of $\|h(t,x)\|$. Thus there exist $K_1(y)$ and $K_2(y)$ independent of ε such that

$$K_1(y) \geq \sup_{\|x\| \leq T\varphi(y,0)+2} \|D_x h_\varepsilon(x)\| \quad \text{and} \quad K_2(y) \geq \sup_{\|x\| \leq T\varphi(y,0)+2} \|h_\varepsilon(x)\|.$$

By the mean value theorem we have

$$\|h_\varepsilon(t,y,x_2) - h_\varepsilon(t,y,x_1)\| \leq K_1(y)\|x_2 - x_1\|. \tag{2.10}$$

Now consider the following integral equation :

$$X_\varepsilon(t) = \int_0^t h_\varepsilon(s, X_\varepsilon(s))ds. \tag{2.11}$$

Equation (2.11) can be solved by the Picard method. Since $y \to h(t,y,x)$ is measurable in (t,y), $y \to h_\varepsilon(t,y,x)$ is measurable in (t,y). Then the solution X_ε of equation (2.11) is measurable and so is $\lim_{\varepsilon\to 0} X_\varepsilon$. To complete the proof of Theorem 2.3 we need to prove the following lemma.

Lemma 2.4 *The solution X_ε of (2.11) converges uniformly to a solution X of (2.9).*

Proof: From (2.9) and (2.11) we have

$$X_\varepsilon(t) - X(t) = \int_0^t (h_\varepsilon(s, X_\varepsilon(s)) - h(s, X(s)))ds.$$

Then

$$\begin{aligned}\|X_\varepsilon(t) - X(t)\| \leq & \int_0^t \|h_\varepsilon(s, X_\varepsilon(s)) - h_\varepsilon(s, X(s))\|ds \\ & + \int_0^t \|h_\varepsilon(s, X(s) - h(s, X(s))\|ds.\end{aligned}$$

By (2.10) we see this is

$$\begin{aligned}\leq & \; K_1(y)\int_0^t \|X_\varepsilon(s) - X(s)\|ds \\ & + \int_0^t \|h_\varepsilon(s, X(s)) - h(s, X(s))\|ds.\end{aligned}$$

By Gronwall's inequality we have

$$\sup_{0\leq t\leq T}\|X_\varepsilon(t)-X(t)\| \leq \exp(TK_1)\int_0^T \|h_\varepsilon(s,X(s))-h(s,X(s))\|ds.$$

But $h_\varepsilon(s,X(s)) \to h(s,X(s))$ pointwise and $\|h_\varepsilon(t,X(t))\| \leq K_2$ so by the dominated convergence theorem,

$$\sup_{0\leq t\leq T}\|X_\varepsilon(t)-X(t)\| \to 0.$$

Q.E.D

2.3 The Proof of the Measurability in Theorem 2.2

Now we shall briefly outline the proof of the existence from Vainberg(1973), Th(26.1), page 322 and give a proof of the measurability of the solution of equation (2.8).

Vainberg constructs a solution of this equation by first solving the finite-dimensional projections of the equation, and then taking the limit. Since the solution of the infinite-dimensional case is constructed as a limit of finite-dimensional solutions, one merely needs to trace the proof and check that the measurability holds at each stage. There is one extra hypothesis in [Vainberg, Th(26.1)], namely that $t \to g(t,x)$ is demicontinuous, whereas in our case, we merely assume g is measurable and uniformly bounded in t [Hypothesis 2.1 ((a) (b))]. However, the demicontinuity of g is not used in showing the existence of the solution of the integral equation (2.8). It is only used to show the inequality (2.4) for the finite-dimensional case. We have reproved (2.4) in Lemma 2.3.

Now let (H_n) be an increasing sequence of subspaces of H such that $\cup_n H_n$ is dense in H, and let J_n be the orthogonal projection of H onto H_n, so that $J_n \to I$ strongly. Consider the integral equation

$$X_n(t) = \int_0^t J_n\, g(s,X_n(s))\,ds. \tag{2.12}$$

First let us show that $J_n\, g$ satisfies Hypothesis 2.2.

• $J_n\, g(t, y, \cdot)$ is continuous .

Since $g(t, y, \cdot)$ is demicontinuous, $g(t, x_k) \to g(t, x)$ weakly when $\|x_k - x\| \to 0$. But $J_n\, g$ takes its values in the finite-dimensional space H_n, where weak and strong convergence coincide, therefore

$$\|J_n\, g(t, x_k) - J_n\, g(t, x)\| \to 0,$$

and $J_n\, g(t, y, \cdot)$ is continuous.

• $J_n\, g(t, y, \cdot)$ is monotone from H_n to H_n.

Let $X, Z \in H_n$. Then

$$< J_n\, g(t, X) - J_n\, g(t, Z), X - Z > = < g(t, X) - g(t, Z), J_n\, X - J_n\, Z > \tag{2.13}$$

since $J_n = J_n^*$. For $X, Z \in H_n$, $J_n\,(X - Z) = X - Z$ so the left hand side of (2.13) is negative, hence $J_n\, g(t, y, \cdot)$ is monotone.

• $J_n g(t, y)$ satisfies Hypothesis 2.1(a).

• $J_n g(t, y)$ is uniformly bounded by φ.

Now by Theorem 2.3, equation (2.12) has a unique continuous measurable solution which satisfies

$$\|X_n(t)\| \le 2\int_0^t \|J_n g(s, 0)\| ds \le 2\int_0^t \|g(s, 0)\| ds \le 2T\varphi(y, 0). \tag{2.14}$$

Now we are going to prove

Lemma 2.5 *For each y, $X_n(\cdot, y)$ converges weakly in $L^2(S, H)$ to a solution $X(\cdot, y)$ of (2.8). Furthermore $X(\cdot, y)$ is continuous for each y.*

Proof: Let (X_{n_k}) be an arbitrary subsequence of (X_n). By (2.14) and Hypothesis 2.1 (b) we have

$$\|g(t, y, X_{n_k}(y, t))\| \le \varphi(y, \|X_{n_k}(y, t)\|) \le \varphi(y, 2T\varphi(y, 0))$$

so $g(.,X_{n_k}(.))$ is a bounded sequence in $L^2(S,H)$. Then there is a further subsequence (n_{k_l}) such that $g(.,X_{n_{k_l}}(.)) \rightarrow Z(.)$ weakly in $L^2(S,H)$ as $l \rightarrow \infty$. Each X_n satisfies (2.12) and it can be proved that $X_{n_{k_l}}(.) \rightarrow \int_0^{\cdot} Z(s)ds$ weakly [see Vainberg]. We define X to be the weak limit of $X_{n_{k_l}}$ in $L^2(S,H)$. Vainberg proved that $X(y,.)$ is continuous and is a solution of (2.8) [see Vainberg, p 325-326].

Since the solution $X(\cdot,y)$ is unique, every subsequence of (X_n) has in turn a subsequence which converges to $X(y,\cdot)$ weakly, it follows that the whole sequence X_n converges weakly to X.

Q.E.D

To complete the proof of Theorem 2.2 we need to show the measurability of $X(\cdot,\cdot)$.

Fix $t \in S$, $h \in H$, since by Theorem 2.3 X_n is measurable in (t,y), then $\int_0^t < X_n(s,y),h > ds$ is measurable in (t,y). But $\int_0^t < X_n(s,y),h > ds$ converges to $\int_0^t < X(s,y),h > ds$ pointwise, so $\int_0^t < X(s,y),h > ds$ is measurable in (t,y) .

As the integrand $< X(s,y),h >$ is continuous in s, then

$$\frac{d}{dt}\int_0^t < X(s,y),h > ds =< X(t,y),h >$$

and since the integral is measurable in (t,y), the function $< X(t,y),h >$ is measurable. By the separablity of H, $X(t,y)$ is measurable in (t,y) .

Q.E.D

Chapter 3

STOCHASTIC CONVOLUTION INTEGRALS

3.1 Introduction and Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete stochastic basis with a right continuous filtration. This means that $\mathcal{F}$ is complete with respect to P and each $\mathcal{F}_t$ contains all P-null sets of $\mathcal{F}$ and $\mathcal{F}_s = \cap_{t \geq s}$ for all s.

Let $L(H)$ be the space of linear bounded operators on H with norm $\| \, \|_L$. Let $(X_t)_{t \in \mathbf{R}+}$ be an H-valued stochastic process. We say that X is a *locally square-integrable* process (l.s.i) if there is a sequence of stopping times (T_n) with $T_n \leq T_{n+1}$, $T_n \to \infty$ a.s. such that for all n, $E\{\|X_{t \wedge T_n}\|^2 1_{\{T_n > 0\}}\}$ is bounded in t. Note that a continuous adapted process is l.s.i, as is one with bounded jumps.

A process Z_t with values in H is a *semimartingale* if there exists an H-valued local martingale M_t and a process of finite variation V_t such that $Z_t = M_t + V_t$.

In this thesis we always assume $M_0 = Z_0 = V_0 = 0$. We shall say that the semimartingale Z is a *l.s.i. semimartingale* if M is a l.s.i. local martingale and V is a process of finite variation such that $|V|$ is l.s.i.

Let M be an H-valued, cadlag local martingale. Let V denote an H-valued, $\mathcal{F}_t$-adapted, cadlag process with total variation $|V|$. Let Z be an H-valued cadlag semimartingale. Let X_0 be an H-valued $\mathcal{F}_0$ -measurable random variable. Consider on H the linear stochastic evolution equation formally written as

$$\begin{cases} dX_t &= A(t)X_t dt + dZ_t \\ X(0) &= X_0, \end{cases} \tag{3.1}$$

where $\{A(t), \quad t \in S\}$ is a family of closed linear operators on H whose domain D is independent of $t \in S$ and is dense in H. (Useful background and motivation for stochastic evolution equations in Hilbert spaces can be found in Curtain and Pritchard (1978)).

We will define two types of solutions of Eq (3.1).

Definition 3.1 *An H-valued process X is a* strong solution *of (3.1) if and only if*

(i) $X_t \in D$ *for almost all* $t \in S$, $X_\cdot \in L^1(S,H)$ *a.s., and* $A(.)X \in L^1(S,H)$ *a.s..*

(ii) $X_t = X_0 + \int_0^t A(s)X_s ds + Z_t$ *a.s. for each* $t \in S$.

Suppose that $\{A(t) : t \in S\}$ generates a unique *evolution operator* $\{U(t,s) : 0 \leq s \leq t \leq T\}$, i.e, the $U(t,s)$ are bounded linear operators on H such that

$$U(t,t) = I, \quad U(t,s)\, U(s,r) = U(t,r) \quad \text{for} \quad 0 \leq r \leq s \leq t \leq T,$$

and $(t,s) \to U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$, and certain relationships between A and U hold, which we will introduce later on.

Definition 3.2 *An H-valued process X is a* mild solution *of (3.1) if and only if*

(i) $X_\cdot \in L^1(S,H)$ *a.s.*

(ii) $X_t = U(t,0)X_0 + \int_0^t U(t,s)dZ_s$ *a.s. for each* $t \in S$.

Definition 3.3 *We say the evolution operator $U(t,s)$ is an* almost strong evolution operator *with* generator $A(t)$ *if it satisfies the following:*

(a) *For almost all $s \leq t$ and for each $x \in D$*

$$U(t,s)x - x = \int_s^t U(t,r)A(r)x dr. \tag{3.2}$$

(b) *Let $x \in D$ and $s \in S$. For almost all $t > s$*

$$U(t,s)D \subseteq D; \tag{3.3}$$

$$\int_s^t A(r)U(r,s)x dr = (U(t,s) - I)x. \tag{3.4}$$

If U and A satisfy (3.2), (3.3), and (3.4) for every $s \in S$, u is called a *strong evolution operator.*

Remark 3.1 (i) *If $\{A(t) : t \in S\}$ is the generator of an almost strong evolution operator $U(t,s)$, then (3.1) (with $Z = 0$ and $X_0 \in D$) has a unique solution $X(t) = U(t,0)X_0$ which is differentiable almost everywhere.*

(ii) *For a.e. $0 \le s \le t \le T$ and each $x \in D$ we have*

$$\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x, \tag{3.5}$$

$$\frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x. \tag{3.6}$$

We say $U(t,s)$ is an *exponentially bounded with parameter λ on S* if there is $\lambda \in \mathbf{R}$ such that

$$\|U(t,s)\|_L \le e^{\lambda(t-s)} \quad \text{for a.e. } 0 \le s \le t \le T. \tag{3.7}$$

Note that if an almost strong evolution operator $U(t,s)$ is exponentially bounded on S with parameter λ, we have

$$< A(t)x, x > \le \lambda\|x\|^2, \quad \forall x \in D. \tag{3.8}$$

This can be seen because if $x \in D$ and $t > s$,

$$\frac{\|U(t,s)x\|^2 - \|x\|^2}{t-s} \le \frac{(e^{2\lambda(t-s)} - 1)\|x\|^2}{t-s}$$

or

$$\lim_{t\to s^+} \frac{\|U(t,s)x\|^2 - \|x\|^2}{t-s} \leq \lim_{t\to s^+} \frac{(e^{2\lambda(t-s)} - 1)\|x\|^2}{t-s} = 2\lambda\|x\|^2;$$

but

$$\lim_{t\to s} \frac{\|U(t,s)x\|^2 - \|x\|^2}{t-s} = \frac{d}{dt^+}\|U(t,s)x\|^2|_{t=s} = 2 < A(s)x, x > \text{ a.e..}$$

Let $B(t,s) := A(t)[\mu I - A(s)]^{-1}$. By (3.1c1), if $\mu > \lambda$ then $B(t,s)$ is a bounded operator [see Kato (1953) Lemma(2)].

The following are the relevant hypotheses concerning A and U:

Hypothesis 3.1 (a) *The domain $\mathcal{D}(A(t)) := D$ is independent of t for $t \in S$ and is dense in H;*

(b) *$\{A(t) : t \in S\}$ generates a unique almost strong evolution operator $U(t,s)$;*

(c) *$U(t,s)$ is exponentially bounded on S with parameter λ;*

(d) *$B(t,s)$ is uniformly bounded in (t,s), that is, for $\mu \geq \lambda$ there is a $K(\mu) > 0$ such that $\|B(t,s)\|_L \leq K(\mu)$ for every s,t (this is the case if $B(t,s)$ is continuous in t in the sense of the norm $\|\|_L$ at least for some s).*

We refer to Pazy (1983) and Tanabe (1979) for sufficient conditions for the existence of an evolution operator with the properties 3.1(a)–(d).

These conditions apply to a large class of delay equations, and to parabolic and hyperbolic equations [see for example Curtain and Pritchard (1978)].

Consider the *stochastic convolution integral* $X_t = \int_0^t U(t,s)dM_s$. Notice that because the integrand depends on t as well as on s, X_t is not necessarily a local martingale. However, it is possible to prove some results analogous to the well-known martingale inequalities. Kotelenez (1982, 1984) proved submartingale type and stopped-Doob inequalities for this object.

In this chapter we are going to prove an *Ito-type inequality* and a *Burkholder-type inequality.* For the definition and properties of stochastic integrals with respect to a semimartingale see Métivier (1982), and for the properties of stochastic convolution integrals see Kotelenez (1982, 1984).

Proposition 3.1 (Kotelenez (1982)) *Suppose U and A satisfy Hypothesis 3.1 (b)–(c). If the l.s.i. semimartingale Z is continuous (respectively cadlag), then $X_t := \int_0^t U(t,s)dZ_s := \int_{(0,t]} U(t,s)dZ_s$ has a version $\bar{X}$ with continuous (respectively cadlag) sample paths.*

Note that Kotelenez (1982) has proved the above theorem for martingales, but the extension to semimartingales is immediate.

Thus, in dealing with $\int_0^t U(t,s)dZ_s$ we may always assume that $\int_0^t U(t,s)dZ_s$ is cadlag (or continuous if Z is continuous).

Let A be an unbounded operator on H with dense domain D. Let $\| \ \|_D$ be the norm defined on D by

$$\|x\|_D^2 = \|Ax\|^2 + \|x\|^2, \quad x \in D.$$

This norm is called *the graph norm* on D. Note that it generated by the inner product

$$< x, y >_D = < Ax, Ay > + < x, y >$$

Remark 3.2 (i) *An operator A is closed if and only if its domain D is complete under the graph norm [see Reed and Simon(1972),problem 15(a), p314.]*

(ii) *Suppose A is a closed linear operator with dense domain D. Then D is a Hilbert space with graph norm $\| \ \|_D$. Let Z be an H-valued semimartingale. Suppose $F(.)$ is an $L(H,D)$-valued measurable function on S with*

$$\sup_{t\in S}\|F(t)\| < \infty. \tag{3.9}$$

Then $\int_0^t F(s)dZ_s$ is a D-valued semimartingale [see Métivier(1982) page 156 ,157]. Moreover

$$A\int_0^t F(s)dZ_s = \int_0^t AF(s)dZ_s \quad w.p.1 \tag{3.10}$$

(3.10) can be seen by approximating $F(s)$ by step functions in $L(H, D)$, and then taking limits.

Now we are going to prove our version of Ito's inequality.

3.2 Ito-Type Inequality

Theorem 3.1 (Ito's inequality) *Let $\{Z_t, t \in S\}$ be an H-valued cadlag l.s.i. semimartingale. Suppose U and A satisfy Hypothesis 3.1(a)–(c). If*

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)dZ_s, \tag{3.11}$$

then

$$\begin{aligned} \|X_t\|^2 \leq\ & e^{2\lambda t}\|X_0\|^2 + 2\int_0^t e^{2\lambda(t-s)} < X(s^-), dZ_s > \\ & + e^{2\lambda t}\left[\int_0 e^{-\lambda s}dZ_s\right]_t, \quad t \in S, \end{aligned} \tag{3.12}$$

where $[Z]_t$ is the real quadratic variation of Z.

Before proving Theorem (3.1) we are going to prove two lemmas. Suppose $U(t,s)$ satisfies (3.1c) for some $\lambda \in \mathbf{R}$. Define

$$U_1(t,s) = e^{-\lambda(t-s)}U(t,s), \quad A_1(t) = A(t) - \lambda I, \quad Z_t^1 = \int_0^t e^{-\lambda s}dZ_s$$

and $X_t^1 = e^{-\lambda t}X_t$.

Lemma 3.1 *If U and A satisfy Hypothesis 3.1, then U_1 and A_1 satisfy Hypothesis 3.1 with $\lambda \equiv 0$. Moreover, X_t satisfies (3.11) if and only if X_t^1 satisfies*

$$X_t^1 = U_1(t,0)X_0 + \int_0^t U_1(t,s)dZ_s^1. \tag{3.13}$$

Proof: $\|U_1(t,s)\|_L = e^{-\lambda(t-s)}\|U(t,s)\|_L \leq 1$ a.e. By the definition of U_1 we can rewrite (3.11) as $X_t = e^{\lambda t}U_1(t,0)X_0 + e^{\lambda t}\int_0^t e^{-\lambda s}U_1(t,s)dZ_s$.

Using the definition of X_t^1 and Z_t^1 we can rewrite the above as (3.13). Q.E.D

Lemma 3.2 *If inequality (3.12) is satisfied when*

(i) *M is globally square integrable,*

(ii) *the total variation $|V|_t$ of V satisfies*

$$E\{|V|_t^2\} < \infty,$$

(iii) *and $\|X_0\|$ is bounded,*

then (3.12) is also satisfied without these restrictions.

Proof: Since Z is l.s.i. then by definition there is a sequence of stopping times (T_n) with $T_n \leq T_{n+1}$, $T_n \to \infty$ such that

$$M_t^n := M_{t\wedge T_n}, \quad V_t^n := V_{t\wedge T_n} \quad \text{and} \quad Z_t^n := Z_{t\wedge T_n}$$

satisfy the above conditions. Let

$$X_0^n = X_0 1_{\{\|X_0\| \leq n\}}.$$

Now consider the integral equation

$$X_t^n = U(t,0)X_0^n + \int_0^t U(t,s)dZ_s^n.$$

Since X_0^n is bounded in norm, then X_0^n, M^n, and V^n satisfy the above conditions, so we have

$$\|X_t^n\|^2 \leq \|X_0^n\|^2 + 2\int_0^t < X_n(s^-), dZ_s^n > +[Z^n]_s. \tag{3.14}$$

Define

$$S_n = T_n \, 1_{\{\|X_0\| \leq n\}}.$$

Note that $X_0^n = X_0^{n+1} = X_0$ and $Z_t^n = Z_t^{n+1} = Z_t$ on $[0, S_n]$. Then by uniqueness

$$X_t^n = X_t^{n+1} = X_t \quad \text{on} \quad [0, S_n],$$

so $X_t = \lim_n X_t^n$. Then we can rewrite (3.14) as

$$\|X_t\|^2 1_{t\leq S_n} \leq \|X_0\|^2 1_{t\leq S_n} + 2\int_0^{t\wedge T_n} < X(s^-), dZ_s > +[Z]_s.$$

Since $P\{S_n = T\} \to 1$ this implies (3.12).

Q.E.D

Proof of Theorem 3.1 : By Lemma (3.1) we can assume $\lambda = 0$ in Hypothesis 3.1(c). Then for all $x \in D$, $< A(t)x, x > \leq 0$ for a.e. t.

Define a map $R_n(t) : H \to D$ by $R_n(t) = n(nI - A(t))^{-1}$. Then $R_n(t)$ is defined on all of H. Since $< A(t)x, x > \leq 0$ for a.e. t, then $< \frac{1}{n}(nI - A(t))x, x > \geq \|x\|^2$ for a.e. $t \in S$, $\forall x \in D$. By the Schwarz inequality we have for all $x \in D$ that

$$\|\frac{1}{n}(nI - A(t))x\| \geq \|x\|,$$

so $\|R_n(t)\|_L \leq 1$ for a.e. t .

We proceed as in Kotelenez (1984) and approximate X_t by Yosida's method. Define a semimartingale $Z_n(t) := \int_0^t R_n(s)dZ(s)$, a martingale $M_n(t) := \int_0^t R_n(s)dM(s)$ and a process $V_n(t) := \int_0^t R_n(s)dV(s)$. Note that since $R_n(t) : H \to D$,then $Z_n(t) \in D$, $M_n(t) \in D$ and $V_n(t) \in D$. Let $\{X_0^n\}$ be a sequence in D which converges almost surely to a limit X_0 such that $\|X_0^n\| \leq \|X_0\|$ for all n.

Define

$$X_n(t) := U(t,0)X_0^n + \int_0^t U(t,s)dZ_n(s). \tag{3.15}$$

We are going to prove that $\|X_n - X\|_\infty \to 0$ *in* L^2.

Note that by Doob's inequality for convolution integrals [see Kotelenez (1984)], we have

$$E\left\{\left\|\int_0^{\cdot} U(.,s)d(M_n(s) - M(s))\right\|_\infty^2\right\} \leq E\{[M_n - M]_T\}$$

while $[M_n - M]_T \to 0$ in L^1 [see Kotelenez (1984)]. Then

$$\left\| \int_0^{\cdot} U(.,s)d(M_n(s) - M(s)) \right\|_\infty \to 0 \quad \text{in} \quad L^2$$

and

$$\|U(.,0)(X_0^{(n)} - X_0)\|_\infty \leq \|X_0^n - X_0\| \to 0 \quad \text{boundedly.}$$

so that it is enough to show that

$$\sup_{0 \leq t \leq T} \| \int_0^t U(t,s)d(V_n(s) - V(s))\| \to 0 \quad \text{in} \quad L^2. \tag{3.16}$$

But since H is a separable Hilbert space then by the Radon-Nikodym property [see: Chatterji (1976)], we can write $dV(t) = Q^V(t)d|V|(t)$ for a.e. t, where $|V|(t)$ is the total variation of V on $[0,t]$ and $\{Q^V(t) \ , 0 \leq t \leq T\}$ is an integrable H-valued process with $\|Q^V\| \leq 1$ a.e.

Now

$$V_n(t) - V(t) = \int_0^t [R_n(s) - I]dV(s).$$

Then we have

$$d(V_n(s) - V(s)) = (R_n(s) - I)Q^V(s)d|V|(s).$$

Now $\|U(t,s)\|_L \leq 1$ so we have

$$\| \int_0^{\cdot} U(.,s)d(V_n(s) - V(s))\|_\infty \leq \int_0^T \|(R_n(s) - I)Q^V(s)\|d|V|(s). \tag{3.17}$$

Since $R_n(s) \to I$ strongly then $\|(R_n(s) - I)Q^V(s)\| \to 0$ for a.e. $s \in S$, and since $\|R_n(s) - I\|_L \leq 2$ then the integrand is ≤ 2 a.e. Then by the dominated convergence theorem, the right hand side of (3.17) approaches zero almost surely, and since it is bounded by square integrable process $|V|(T)$, we get (3.16). This can be seen by using dominated convergence theorem. Hence $\|X_n - X\|_\infty \to 0$ in L^2.

Let us first prove Ito's inequality (3.12) for

$$\bar{X}_t = U(t,0)\bar{X}_0 + \int_0^t U(t,s)d\bar{Z}_s, \tag{3.18}$$

where $\bar{Z}$ satisfies the following.

Hypothesis 3.2 **(a)** *N is a D-valued square integrable martingale;*

(b) *$\bar{V}$ is a D-valued process of bounded variation with*

$$E\{|\bar{V}|^2\} < \infty;$$

(c) *$\bar{X}_0$ is a D-valued bounded random variable;*

(d) $\bar{Z} = N + \bar{V}$.

Lemma 3.3 *If $\bar{X}_0$ and $\bar{Z}$ satisfy Hypothesis 3.2 and if $\bar{X}$ is a solution of (3.18), then*

$$\|\bar{X}_t\|^2 \leq \|\bar{X}_0\|^2 + \int_0^t < \bar{X}(s^-), d\bar{Z}_s > +[\bar{Z}]_t. \tag{3.19}$$

Proof: Define

$$\bar{Y}_t := U(t,0)\bar{X}_0 + \int_0^t U(t,s)d\bar{V}_s$$

and

$$Y_t := \int_0^t U(t,s)dN_s.$$

Since $d\bar{V}_s = Q^{\bar{V}}(s)d|\bar{V}|_s$ a.e. and because $\bar{V}$ and $\bar{X}_0$ satisfy Hypothesis 3.2, so by [Theorem 2.38 page 45 Curtain and Pritchard (1978)], $\bar{Y}_t$ satisfies

$$\bar{Y}_t = \bar{X}_0 + \int_0^t A(s)\bar{Y}_s + \bar{V}_t. \tag{3.20}$$

Let $\{e_i : i = 1, 2, ...\}$ be a basis for the Hilbert space D. Let J_k be the projection operator on the manifold generated by $\{e_1, e_2, ..., e_k\}$. Let $N_t^i =< N_t, e_i >_D$, $i = 1, ..., k$ be real-valued square integrable martingales such that

$$N_t = \sum_{i=1}^{\infty} N_t^i e_i.$$

Define $Y_k(t)$ by

$$Y_k(t) := \sum_{i=1}^{k} \int_0^t U(t,s)e_i dN_s^i = \int_0^t U(t,s)d(J_k N_s).$$

Define $\bar{X}_k := Y_k + \bar{Y}$ and $\bar{Z}_k := J_k N + \bar{V}$. Since $[(I - J_k)N]$ converges to zero in L^1, then by Doob inequality

$$\|\bar{X}_k - \bar{X}\|_\infty \to 0 \quad \text{in} \quad L^2.$$

$Y_k(t)$ satisfies

$$Y_k(t) = \int_0^t A(s)Y_k(s)ds + J_k N_t. \tag{3.21}$$

This can be seen by (3.4) and Fubini's theorem :

$$\begin{aligned}
\int_0^t A(r)Y_k(r)dr &= \sum_{i=1}^{k} \int_0^t A(r)(\int_0^r U(r,s)e_i dN_s^i)dr \\
&= \sum_{i=1}^{k} \int_0^t (\int_s^t A(r)U(r,s)e_i dr)dN_s^i \\
&= \sum_{i=1}^{k} \int_0^t [U(t,s) - I]e_i dN_s^i \\
&= Y_t^k - J_k N_t.
\end{aligned}$$

Now $\bar{Y}_k$ and Y_k satisfy (3.20) and (3.21) and A is linear so

$$\bar{X}_k(t) = X_0^k + \int_0^t A(s)\bar{X}_k(s)ds + \bar{Z}_k(t). \tag{3.22}$$

Since $A(.)\bar{X}_k(.) \in L^1(S,H)$, we can apply the usual Hilbert space form of Ito's formula [see Métivier (1982), page 184, Theorem (26.5)] to see that

$$\begin{aligned}
\|\bar{X}_k\|^2 &= \|\bar{X}_0^k\|^2 + 2\int_0^t < A(s)\bar{X}_k(s), \bar{X}_k(s) > ds \\
&\quad + 2\int_0^t < \bar{X}_k(s^-), d\bar{Z}_k(s) > + [\bar{Z}_k]_t.
\end{aligned} \tag{3.23}$$

But $[\bar{Z}_k]_t \leq [\bar{Z}]_t$. Moreover, $< A(s)\bar{X}_k(s), \bar{X}_k(s) > \leq 0$. a.e., so (3.23) implies that

$$\|\bar{X}_k(t)\|^2 \leq \|\bar{X}_0\|^2 + 2\int_0^t < \bar{X}_k(s^-), d\bar{Z}_k(s) > + [\bar{Z}]_t. \tag{3.24}$$

To complete the proof of the Lemma, we only need to show that

$$\int_0^t < \bar{X}_k(s^-), d\bar{Z}_k(s) > \rightarrow \int_0^t < \bar{X}(s^-), d\bar{Z}(s) > \quad \text{in probability.} \tag{3.25}$$

Now

$$\begin{aligned}
& |\int_0^t < \bar{X}_k(s^-), d\bar{Z}_k(s) > - \int_0^t < \bar{X}(s^-), d\bar{Z}(s)| \\
\leq \; & |\int_0^t < \bar{X}_k(s^-) - \bar{X}(s^-), dJ_kN(s) > | + |\int_0^t < \bar{X}(s^-), d(\{I - J_k\}N_k(s)) > | \\
& + |\int_0^t < \bar{X}_k(s^-) - \bar{X}(s^-), d\bar{V}(s) > | \\
:= \; & |\bar{I}_k^1(t)| + |\bar{I}_k^2(t)| + |\bar{I}_k^3(t)|.
\end{aligned}$$

- $\bar{I}_k^1$ is a local martingale and its quadratic variation process satisfies

$$[\bar{I}_k^1(t)] \leq \int_0^t \|\bar{X}_k(s^-) - \bar{X}(s^-)\|^2 d[J_kN]_s \leq \|\bar{X}_k - \bar{X}\|_\infty^2 [J_kN]_T. \tag{3.26}$$

Since $[J_kN]_T \leq [N]_T$ and $\|\bar{X}_k - \bar{X}\|_\infty$ converges to zero in L^2. It follows from (3.26) and boundedness of $E\{[N]_T\}$ that $[\bar{I}_k^1]^{1/2}$ converges to zero in L^1. and so by inequality of Burkholder-Gundy-Davis $\sup_{0 \leq t \leq T} |\bar{I}_k^1(t)| \rightarrow 0$ in L^1.

- $\bar{I}_k^2$ is also a local martingale, and its quadratic variation process satisfies

$$\begin{aligned}
[\bar{I}_k^2]_t \; & \leq \; \int_0^T \|\bar{X}(s^-)\|^2 d[(I - J_k)N]_s \\
& \leq \; \|\bar{X}\|_\infty^2 [(I - J_k)N]_T.
\end{aligned} \tag{3.27}$$

but $[(I - J_k)N]_T \rightarrow 0$ in L^1 and $E\{\|\bar{X}\|_\infty^2\}$ is finite, then (3.27) implies that $[\bar{I}_k^2]^{1/2}$ converges to zero in L^1, and so by inequality of Burkholder-Gundy-Davis $\sup_{0 \leq t \leq T} |\bar{I}_k^2(t)| \rightarrow 0$ in L^1.

- Since $d\bar{V}(s) = Q^{\bar{V}}(s) d|\bar{V}|(s)$ a.e. $s \in S$, then

$$\sup_{0 \leq t \leq T} |\bar{I}_k^3(t)| \leq \|\bar{X}_k - \bar{X}\|_\infty |\bar{V}|(T).$$

Since

$$\|\bar{X}_k - \bar{X}\|_\infty \to 0 \quad \text{in probability}$$

then

$$\sup_{0\le t\le T} |\bar{I}_k^3(t)| \to 0 \quad \text{in probability.}$$

Q.E.D

Now since X_0^n, $M_n(s)$, $V_n(s)$, and $Z_n(s)$ satisfy Hypothesis 3.2 then by Lemma 3.3 we have

$$\begin{aligned} \|X_t^n\|^2 &= \|X_0^n\|^2 + 2\int_0^t < A(s)X_n(s), X_n(s) > ds \\ &\quad + 2\int_0^t < X_n(s^-), dZ_n(s) > +[Z_n]_t. \end{aligned} \tag{3.28}$$

But $[Z_n]_t \le \int_0^t \|R_n(s)\|_L^2 d[Z]_s$. (See Métivier, Pellaumail (1980), 4.2, page (52)). Since $\|R_n(s)\|_L \le 1$ a.e, so $[Z_n]_t \le [Z]_t$. Moreover, $< A(s)X_n(s), X_n(s) > \le 0$ a.e. and $\|X_0^n\| \le \|X_0\|$, so that (3.28) implies that

$$\|X_n(t)\|^2 \le \|X_0\|^2 + 2\int_0^t < X_n(s^-), dZ_n(s) > + [Z]_t. \tag{3.29}$$

To complete the proof of the theorem, we only need to show that

$$\int_0^t < X_n(s^-), dZ_n(s) > \to \int_0^t < X(s^-), dZ(s) > \text{ in probability.} \tag{3.30}$$

Now

$$\begin{aligned} & |\int_0^t < X_n(s^-), dZ_n(s) > - \int_0^t < X(s^-), dZ(s) > | \\ \le\ & |\int_0^t < X_n(s^-) - X(s^-), dM_n(s) > | + |\int_0^t < X(s^-), d(M(s) - M_n(s)) > | \\ & + |\int_0^t < X_n(s^-) - X(s^-), dV_n(s) > | + |\int_0^t < X(s^-), d(V(s) - V_n(s)) > | \\ :=\ & |I_n^1(t)| + |I_n^2(t)| + |I_n^3(t)| + |I_n^4(t)|. \end{aligned}$$

- I_n^1 is a local martingale and its quadratic variation process satisfies

$$[I_n^1(t)] \leq \int_0^t \|X_n(s^-) - X(s^-)\|^2 d[M_n]_s \leq \|X_n - X\|_\infty^2 [M_n]_T. \tag{3.31}$$

But $[M_n]_T \leq [M]_T$. We have seen that

$$\|X_n - X\|_\infty \to 0 \quad \text{in} \quad L^2$$

So (3.31) implies that $[I_n^1]$ converges to zero in L^1 and by inequality of Burkholder-Gundy-Davis

$$\sup_{0\leq t\leq T}|I_n^1(t)| \to 0 \quad \text{in} \quad L^2$$

- I_n^2 is also a local martingale, and its quadratic variation process satisfies

$$\begin{aligned} [I_n^2]_t &\leq \int_0^T \|X(s^-)\|^2 d[M - M_n]_s \\ &\leq \|X\|_\infty^2 [M - M_n]_T. \end{aligned} \tag{3.32}$$

Since $[M - M_n]_T \to 0$ in L^1. It follows from (3.32) that $[I_n^2]_t^{\frac{1}{2}} \to 0$ in L^1 and so by inequality of Burkholder-Gundy-Davis

$$\sup_{0\leq t\leq T}|I_n^2(t)| \to 0 \quad \text{in probability.}$$

- Since $dV_n(s) = R_n(s)\,Q^V(s)d|V|(s)$ for a.e. s and $\|R_n(s)\,Q^V(s)\| \leq 1$ for a.e. s, then

$$\sup_{0\leq t\leq T}|I_n^3(t)| \leq \|X_n - X\|_\infty |V|(T).$$

Since $\|X_n - X\|_\infty \to 0$, in probability, $\sup_{0\leq t\leq T}|I_n^3(t)| \to 0$ in probability.

- Since $d(V(s) - V_n(s)) = (I - R_n(s))Q^V(s)d|V|(s)$ a.e. s, then

$$\sup_{0\leq t\leq T}|I_n^4(t)| \leq \|X\|_\infty \int_0^T \|(A_n(s) - I)Q^V(s)\|d|V|(s).$$

But $(A_n(s) - I)Q^V(s)$ converges to zero a.e. and its norm is bounded by 2, so by the bounded convergence theorem $\sup_{0\leq t\leq T}|I_n^4(t)|$ tends to zero. Q.E.D

Remark 3.3 *In the proof of Theorem 3.1 we have used the local square integrability of* $|V|_t$ *only to prove that* $\sup_{0\leq t\leq T}|I_n^1(t)|$ *and* $\sup_{0\leq t\leq T}|I_n^2(t)|$ *tend to zero in probability, so if* $M = I_n^1 = I_n^2 = 0$*, we don't need this restriction.*

3.3 Burkholder-Type Inequality

Before proving Burkholder's inequality for convolutions, we are going to prove the following lemma which we will need in Chapters 7 and 8.

Lemma 3.4 *Let* $p \geq 1$ *and let* C_p *be the constant in inequality of Burkholder-Gundy-Davis for real-valued martingales. Then for* $K > 0$ *we have*

$$\begin{aligned} E\left\{\sup_{0\leq\theta\leq t}\left|\int_0^\theta < X_{s-}, dM_s > \right|^p\right\} &\leq C_p E\left((X_t^\star)^p[M]_t^{\frac{p}{2}}\right) \\ &\leq \frac{C_p}{2K}E\left((X_t^\star)^{2p}\right) + \frac{C_p K}{2}E\left([M]_t^p\right), \quad (3.33) \end{aligned}$$

where $X_t^\star = \sup_{0\leq s\leq t}\|X_s\|$.

Proof: By Burkholder's inequality, we have

$$E\{\sup_{0\leq\theta\leq t}|\int_0^\theta < X_{s-}, dM_s > |^p\} \leq C_p E\{[\int_0^\cdot < X_{s-}, dM_s >]_t^{\frac{p}{2}}\}.$$

But $[\int_0^\cdot < X_{s-}, dM_s >]_t \leq (X_t^\star)^2[M]_t$ so this is $\leq C_p E\{(X_s^\star)^p[M]_t^{\frac{p}{2}}\}$. But since $ab \leq \frac{1}{2}(\frac{1}{K}a^2 + Kb^2)$, the proof of the lemma is complete. Q.E.D

Theorem 3.2 *Let* A *and* U *satisfy Hypothesis 3.1 with* $\lambda \equiv 0$*. If* $p \geq 1$*, then*

$$E[\|\int_0^\cdot U(.,s)dM_s\|_\infty^{2p}] \leq K_p E[[M]_T^p],$$

where

$$K_p = 2^{2p}C_p^2 + 2^p.$$

Proof: Without loss of generality we can assume that M_t is a cadlag globally square-integrable martingale. Let

$$X_t = \int_0^t U(t,s)dM_s, \quad X_n(t) = \int_0^t U(t,s)dM_n(s),$$

where

$$M_n(t) = \int_0^t R_n(s)dM(s).$$

We will prove that for all n

$$E\{\|X_n\|_\infty^{2p}\} \leq K_p E\{[M]_T^p\}. \tag{3.34}$$

Note that this implies the theorem, since by the proof of Theorem (3.1)

$$\|X_n - X\|_\infty \to 0 \quad \text{in probability,}$$

and we can apply Fatou's Lemma (using, if necessary, a subsequence) to obtain

$$E\{\|X\|_\infty^{2p}\} \leq K_p E\{[M]_T^p\}.$$

It remains to prove (3.34). Now by (3.12) we have

$$\|X_n(t)\|^2 \leq \int_0^t < X_n(s^-), dM_n(s) > + [M_n]_t.$$

Then

$$\|X_n(t)\|^{2p} \leq 2^p\{|\int_0^t < X_n(s^-), dM_n(s) > |^p + [M_n]_t^p\},$$

so

$$E\{\|X_n\|_\infty^{2p}\} \leq 2^p E\{\|\int_0^\cdot < X_n(s^-), dM_n(s) > \|_\infty^p\} + 2^p E\{[M_n]_T^p\}.$$

Now $[M_n]_t \leq [M]_t$, so by Lemma 3.4 this is

$$\leq \frac{2^p C_p}{2K} E\{\|X_n\|_\infty^{2p}\} + 2^p(\frac{C_p K}{2} + 1)E\{[M]_T\}.$$

Choose $K = 2^p C_p$; then we have

$$E\{\|X_n\|_\infty^{2p}\} \le \frac{1}{2}E\{\|X_n\|_\infty^{2p}\} + (\frac{(2^p C_p)^2}{2} + 2^p)E\{[M]_T^p\}.$$

To complete the proof of the Theorem we need to show that if $E\{[M]_T^p\} < \infty$, then $E\{\|X_n\|_\infty^{2p}\} < \infty$. By the stochastic version of integration by parts we have

$$X_n(t) = M_n(t) - \int_0^t \frac{\partial}{\partial s}(U(t,s))M_n(s)ds.$$

Since $M_n(s) \in D$, by (3.6) we have

$$\frac{\partial}{\partial s}(U(t,s))M_n(s) = -U(t,s)A(s)M_n(s), \quad \text{a.e. } s \in S.$$

Then

$$X_n(t) = M_n(t) + \int_0^t U(t,s)A(s)M_n(s)ds. \tag{3.35}$$

Define a martingale

$$N_n(t) := (I - A(0))M_n(t) = \int_0^t (I - A(0))R_n(s)dM(s).$$

Now we can rewrite (3.35) as

$$X_n(t) = M_n(t) + \int_0^t U(t,s)A(s)(I - A(0))^{-1}N_n(s)ds$$

and so

$$\|X_n(t)\|^{2p} \le 2^{2p}\left\{\|M_n(t)\|^{2p} + \int_0^t \|U(t,s)A(s)(I - A(0))^{-1}\|_L^{2p}\|N_n(s)\|^{2p}ds\right\}. \tag{3.36}$$

But by (3.1d) there exists K such that $\|A(s)(I - A(0))^{-1}\|_L \le K$ a.e, and since $\|U(t,s)\|_L \le 1$ a.e. we can write (3.36) as

$$E\left\{\|X_n\|_\infty^{2p}\right\} \le 2^{2p}\left\{E\left(\|M_n\|_\infty^{2p}\right) + K^{2p}TE\left(\|N_n\|_\infty^{2p}\right)\right\}.$$

But $[M_n]_t \le [M]_t$ and by (3.1d) there exists $K_n > 0$ such that $\|(I-A(0))R_n(s)\|_L \le K_n$. Then $[N_n]_T \le K_n^2[M]_T$. Since $E([M]_T^p) < \infty$ we are done.

Q.E.D

We have proved the theorem when $\lambda \equiv 0$ in (3.1c). We can easily generalize to the case when $\lambda > 0$ by the following corollary.

Corollary 3.1 *If $\lambda > 0$ and $p \geq 1$ we have*

$$E\left[\left\|\int_0^{\cdot} U(.,s)dM_s\right\|_\infty^{2p}\right\} \leq K_p e^{2p\lambda T} E\left([M^1]_T^p\right), \tag{3.37}$$

where $M_t^1 := \int_0^t e^{-\lambda s} dM_s$.

Proof: Define $X_t = \int_0^t U(t,s)dM_s$. Then by Lemma 3.1 $X_t^1 = \int_0^t U_1(t,s)dM_s$, and by Theorem 3.2 one has $E\{\|X^1\|_\infty^{2p}\} \leq K_p E\{[M^1]_T^p\}$. By substituting $X_t^1 = e^{-\lambda t} X_t$ we get (3.37). Q.E.D

Theorem 3.2 also gives us Burkholder's inequality for H-valued local martingales by setting $A(t) \equiv 0$ and $U(t,s) = I$. To complete the proof of the inequality of Burkholder-Gundy-Davis for H-valued martingales we need to prove:

Theorem 3.3

$$E([M]_t^p) \leq 2^{2p}(1 + C_p^2)E((M_t^\star)^{2p}). \tag{3.38}$$

Proof: By Ito's formula we have $[M]_t = \|M_t\|^2 - \int_0^t < M_{s-}, dM_s >$. Then

$$[M]_t^p \leq 2^p \|M_t\|^{2p} + 2^p \left| \int_0^t < M_{s-}, dM_s > \right|^p,$$

so

$$E([M]_t^p) \leq 2^p E((M_t^\star)^{2p} + 2^p E(\sup_{0\leq\theta\leq t} \left| \int_0^\theta < M_s^-, dM_s > \right|^p).$$

By Lemma 3.4 this is

$$\leq 2^p E((M_t^\star)^{2p}) + \frac{2^p C_p}{2K} E((M_t^\star)^{2p}) + \frac{2^p C_p K}{2} E([M]_t^p).$$

Choose $K = \frac{1}{2^p C_p}$. To complete the proof of the theorem we need to show that if $E((M_t^\star)^{2p})$ is finite , then $E([M]_t^p)$ is finite. Define the stopping time

$$T_n = \inf\{t : [M]_t \geq n\}$$

then

$$[M]_{T_n \wedge t} \leq n + \sup_{s \leq t} \|\Delta M_s\|^2 \leq n + 4(M_t^\star)^2.$$

So $[M]_{T_n \wedge t} \in L^p$ and we have

$$E([M]_{T_n \wedge t}^p) \leq 2^{2p}(1 + C_p^2)E((M_{T_n \wedge t}^\star)^{2p}).$$

Now let $n \to \infty$.

Q.E.D

Remark 3.4 *Combining Theorem 3.2 and Theorem 3.3 we have*

$$E\left[\left\|\int_0^\cdot U(.,s)dM_s\right\|_\infty^{2p}\right] \leq K_p E\left([M]_T^p\right) \leq K_p' E\left(\|M\|_\infty^{2p}\right). \tag{3.39}$$

Chapter 4

A SEMILINEAR EQUATION

4.1 Introduction

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete stochastic basis with a right continuous filtration. Let Z be an H-valued cadlag semimartingale. Consider the initial value problem of the semilinear stochastic evolution equation of the form:

$$\begin{cases} dX_t &= A(t)X_t dt + f_t(X_t)dt + dZ_t \\ X(0) &= X_0, \end{cases} \tag{4.1}$$

where $A = \{A(t), t \in S\}$ is a family of operators satisfying the following hypothesis.

Hypothesis 4.1 **(a)** *There exists $\lambda \in \mathbf{R}$ such that for all $s > 0$, $(A(s) - \lambda I)$ is a generator of a contraction semigroup;*

(b) *the operator-valued function $(-A(t)+\mu I)^{-1}$ is strongly continuously differentiable with respect to t for $t \geq 0$ and $\mu > \lambda$;*

(c) *there exists a fundamental solution $U(t,s)$ of the linear equation $\dot{u}(t) = A(t)u(t)$. Moreover, if $u_0 \in H$ and $f \in C(S,H)$, then the equation*

$$\begin{cases} \dot{u}(t) &= A(t)u(t) + f(t) \\ u(0) &= u_0 \end{cases} \tag{4.2}$$

has a strong solution u given by

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds. \tag{4.3}$$

If $u_0 \in D(A(0))$ and $f \in C^1(S,H)$, then (4.3) is also a strong solution of (4.2).

Note that an evolution operator which satisfies the above condition is a strong evolution operator [see Curtain (1977)] which satisfies Hypothesis 3.1(b) and (c).

Remark 4.1 *Note that Hypothesis 4.1 holds, for example, if* $\{A(t), t \in \mathbf{R}^+\}$ *is a family of closed operators in* H *with domain* D *independent of* t*, satisfying the following conditions:*

(i) *considered as a mapping of* D *(with graph norm) into* H*,* $A(t)$ *is* C^1 *in* t *on* $\mathbf{R}^+$ *in the strong operator topology;*

(ii) *if* $A(t)^\star$ *is the adjoint of* $A(t)$*, then* $D(A(t)^\star) \subset D$ *for all* t*;*

(iii) $\exists \lambda \in R$ *such that*

$$< A(t)x, x > \leq \lambda \|x\|^2, \quad \forall x \in D(A(t)), \quad \forall t \in S.$$

Proof: See Browder (1964)

We say X_t is a *mild solution* of (4.1) if it is a strong solution of the integral equation

$$X_t = U(t,0)X_0 + \int_0^t U(t,s) f_s(X_s) ds + \int_0^t U(t,s) dZ_s. \tag{4.4}$$

Since Z is a cadlag semimartingale the stochastic convolution integral $\int_0^t U(t,s)dZ_s$ is known to be a cadlag adapted process (see Chapter 3). More generally, instead of (4.4) we are going to study

$$X_t = U(t,0)X_0 + \int_0^t U(t,s) f_s(X_s) ds + V_t, \tag{4.5}$$

where V_t is a cadlag adapted process.

In Theorem 4.1 we will study the integral equation (4.5) in a more abstract setting, where $V \equiv V(t,y)$ and $f \equiv f(t,y,x)$ satisfy the hypotheses of Theorem 2.1.

4.2 The Measurability of the Solution of the Semilinear Equation

Theorem 4.1 *Let* $X_0(.)$ *be* $\mathcal{G}$*-measurable. Suppose that* f *and* V *satisfy Hypothesis 2.1 and suppose that* $A(t)$ *and* $U(t,s)$ *satisfy Hypothesis 4.1. Then for each* $y \in G$*, (4.5) has* *a unique cadlag solution $X(.,y)$, and $X(.,.)$ is $\beta \times \mathcal{G}$-measurable. Furthermore*

$$\|X(t)\| \leq \|X_0\| + \|V(t)\| + \int_0^t e^{(\lambda+M)(t-s)} \|f(s, U(s,0)X_0 + V(s))\| ds, \tag{4.6}$$

$$\|X\|_\infty \leq \|X_0\| + \|V\|_\infty + C_T \varphi(\|X_0\| + \|V\|_\infty), \tag{4.7}$$

where

$$C_T = \begin{cases} \frac{1}{M+\lambda} e^{(M+\lambda)T} & \text{if } M + \lambda \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

If X_1 and X_2 are solutions corresponding to different initial values X_{01} and X_{02}, then

$$\|X_2(t) - X_1(t)\| \leq e^{(\lambda+M)t} \|X_{01} - X_{02}\|, \quad t \in S. \tag{4.8}$$

Proof: By using the transformations (2.6), and (2.7) we can assume by Lemma 2.1 that $X_0 = 0$, $M = 0$ and $V = 0$ in (4.5). By Lemma 3.1 we can also suppose $\lambda \equiv 0$ in Hypothesis 4.1(a). Thus we consider

$$X(t,y) = \int_0^t U(t,s) f(s,y,X(s,y)) ds, \quad t \in S, \quad y \in G. \tag{4.9}$$

y serves only as nuisance parameter. It only enters in the measurability part of conclusion. The proof of Theorem 4.1 in the case in which f is independent of y is a well-known theorem of Browder (1964) and Kato (1964).

The existence and uniqueness are therefore known. To establish the measurability and inequalities (4.6)–(4.8) we follow the proof of Vainberg (1973), Th (26.2) page 331. Let $A_n(t) := A(t)(I - n^{-1}A(t))^{-1}$, and consider the equation

$$X_n(t,y) = \int_0^t (A_n(s) X_n(s,y) + f(s,y,X_n(s,y))) ds. \tag{4.10}$$

A_n is a bounded operator with $\|A_n(t)\|_L \leq 2n$ which converges strongly to $A(t)$. Vainberg shows that (4.10) has a unique solution X_n, and moreover that there is a subsequence (X_{n_k}) of X_n which converges weakly in $L^2(S,H)$ to a limit X, which is a solution of (4.9); and for each y $X(.,y)$ is continuous.

But now by Lemma 2.5 X_n converges weakly to X in $L^2(S,H)$. Moreover $f_n(x) := A_n x + f(x)$ satisfies the hypotheses of Theorem 2.2 so that $X_n(\cdot,\cdot)$ is $\beta \times \mathcal{G}$-measurable. It follows by the proof of Theorem 2.2 that $X(.,.)$ is $\beta \times \mathcal{G}$-measurable.

The proof of the inequalities (4.6)–(4.8) in case $M = 0$, $\lambda = 0$ and $V \equiv 0$ are in Vainberg (1973), and the extension to the general case of Theorem 4.1 follows immediately from transformation (2.6) and (2.7).

Note that discontinuity of the solution in general comes from discontinuity of V.

Q.E.D

As an application of Theorem 4.1 we can show the existence and uniqueness of the solution of (4.5) when X_0 , f and V satisfy the following conditions.

Hypothesis 4.2 **(a)** $X_0 \in \mathcal{F}_0$.

(b) $f = f(t,\omega,x)$ *and* $V = V(t,\omega)$ *are optional;*

(c) *There exists a set* $G \subset \Omega$ *such that* $P(G) = 1$, *and if* $\omega \in G$, *then* f *and* V *satisfy Hypothesis 2.1.*

Corollary 4.1 *Suppose that* X_0, f *and* V *satisfy Hypothesis 4.2. Suppose* A *and* U *satisfy Hypothesis 4.1. Then (4.5) has a unique adapted cadlag (continuous, if* V_t *is continuous) solution.*

Proof: The existence and uniqueness of a cadlag solution is immediate from Theorem 4.1. We need only prove that it is adapted. To see this, fix $s < t$, take $S = [0,s]$, and take $\mathcal{G} = \mathcal{F}_t|_G$ in Theorem 4.1, where G is the set of Hypothesis 4.2. Now $\Omega - G$ has measure 0 so it is in $\mathcal{F}_0 \subset \mathcal{F}_t$.

Theorem 4.1 implies $X(s,.)|_G$ is $\mathcal{G}$-measurable; as all subsets of $\Omega - G$ are in $\mathcal{F}_t$ by completeness, $X(s,.)$ itself is $\mathcal{F}_t$-measurable. By right continuity of the filtration,

$$X(s,.) \in \mathcal{F}_s = \cap_{t>s}\mathcal{F}_t.$$

Thus $\{X(t,.), t \in S\}$ is adapted.

Q.E.D

4.3 Some Examples

Example (4.1)

Let A be a closed, self-adjoint, negative definite unbounded operator such that A^{-1} is nuclear. Let $U(t) \equiv e^{tA}$ be a semigroup generated by A. Since A is self-adjoint then U satisfies Hypotheses 3.1 and 4.1, so it satisfies all the conditions we impose on U.

Let $W(t)$ be a cylindrical Brownian motion on H. Consider the initial-value problem:

$$\begin{cases} dX_t &= AX_t dt + f_t(X_t)dt + dW(t), \\ X(0) &= X_0, \end{cases} \tag{4.11}$$

where X_0,and f satisfy Hypothesis 4.2 .

Let X be a mild solution of (4.11), i.e. a solution of the integral equation:

$$X_t = U(t)X(0) + \int_0^t U(t-s)f_s(X_s)ds + \int_0^t U(t-s)dW(s). \tag{4.12}$$

The existence and uniqueness of the solution of (4.12) have been studied in Marcus (1978). He assumed that f is independent of $\omega \in \Omega$ and $t \in S$ and that there are $M > 0$, and $p \geq 1$ for which

$$< f(u) - f(v), u - v > \leq -M\|u - v\|^p$$

and

$$\|f(u)\| \leq C(1 + \|u\|^{p-1}).$$

He proved that this integral equation has a unique solution in $L^p(\Omega, L^p(S, H))$.

As a consequence of Corollary 4.1 we can extend Marcus' result to more general f and we can show the existence of a strong solution of (4.12) which is continuous instead of merely being in $L^p(\Omega, L^p(S, H))$.

The Ornstein–Uhlenbeck process $V_t = \int_0^t U(t-s)dW(s)$ has been well-studied e.g. in [Iscoe et. al (1989)] where they show that V_t has a continuous version. We can rewrite (4.12) as

$$X_t = U(t)X(0) + \int_0^t U(t-s)f_s(X_s)ds + V_t,$$

where V_t is an adapted continuous process. Then by Corollary 4.1 the equation (4.12) has a unique continuous adapted solution.

Example (4.2) Let D be a bounded domain with a smooth boundary in $\mathbf{R}^d$. Let $-A$ be a uniformly strongly elliptic second order differential operator with smooth coefficients on D. Let B be the operator $B = d(x)D_N + e(x)$, where D_N is the normal derivative on ∂D, and d and e are in $C^\infty(\partial D)$. Let A (with the boundary condition $Bf \equiv 0$) be self-adjoint.

Consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au &= f_t(u) + \dot{W} \quad \text{on} \quad D \times [0, \infty) \\ Bu &= 0 \quad \text{on} \quad \partial D \times [0, \infty) \\ u(0, x) &= 0 \quad \text{on} \quad D, \end{cases} \tag{4.13}$$

where $\dot{W} = \dot{W}(t, x)$ is a white noise in space–time [for the definition and properties of white noise see J.B Walsh (1986)], and f_t is a non-linear function that will be defined below. Let $p > \frac{d}{2}$. W can be considered as a *Brownian motion* $\tilde{W}_t$ *on the Sobolov space* H_{-p} [see Walsh (1986), Chapter 4. Page 4.11]. There is a complete orthonormal basis $\{e_k\}$ for H_p.

The operator A (plus boundary conditions) has eigenvalues $\{\lambda_k\}$ with respect to $\{e_k\}$

i.e. $Ae_k = \lambda_k e_k$, $\forall k$. The eigenvalues satisfy $\Sigma_j(1+\lambda_j^{-p}) < \infty$ if $p > \frac{d}{2}$ [see Walsh (1986), Chapter 4, page 4.9]. Then $[A^{-1}]^p$ is nuclear and $-A$ generates a contraction semigroup $U(t) \equiv e^{-tA}$. This semigroup satisfies Hypotheses 3.1 and 4.1.

Now consider the initial-boundary-value problem (4.13) as a semilinear stochastic evolution equation

$$du_t + Au_t dt = f_t(u_t)dt + d\tilde{W}_t \tag{4.14}$$

with initial condition $u(0) = 0$, where $f : S \times \Omega \times H_{-p} \to H_{-p}$ satisfies Hypotheses (4.2b) and (4.2c) relative to the separable Hilbert space $H = H_{-p}$. Now we can define the mild solution of (4.14) (which is also a mild solution of (4.13)), as the solution of

$$u_t = \int_0^t U(t-s)f_s(u_s)ds + \int_0^t U(t-s)d\tilde{W}_s. \tag{4.15}$$

Since $\tilde{W}_t$ is a continuous local martingale on the separable Hilbert space H_{-p}, then $\int_0^t U(t-s)d\tilde{W}_s$ has an adapted continuous version [see Chapter 3]. If we define

$$V_t := \int_0^t U(t-s)d\tilde{W}_s,$$

then by Corollary 4.1, equation (4.15) has a unique continuous solution with values in H_{-p}.

4.4 A Second Order Equation

Let Z_t be a cadlag semimartingale on H. Let A satisfy the following:

Hypothesis 4.3 *A is a closed strictly positive definite self-adjoint operator on H with dense domain $D(A)$, so that there is a $K > 0$ such that $< Ax, x > \geq K\|x\|^2$, $\forall x \in D(A)$.*

Consider the Cauchy problem, written formally as

$$\begin{cases} \frac{\partial^2 x}{\partial t^2} + Ax &= \dot{Z} \\ x(0) &= x_0, \\ \frac{\partial x}{\partial t}(0) &= y_0. \end{cases} \tag{4.16}$$

Following Curtain and Pritchard (1978), we may write (4.16) formally as a first-order system

$$\begin{cases} dX(t) &= \mathcal{A}X(t)dt + d\tilde{Z}_t \\ X(0) &= X_0, \end{cases} \tag{4.17}$$

where $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\tilde{Z}_t = \begin{pmatrix} 0 \\ Z_t \end{pmatrix}$, $X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, and $\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$.

Introduce a Hilbert space $\mathcal{K} = D(A^{1/2}) \times H$ with inner product

$$< X, \bar{X} >_{\mathcal{K}} = < A^{1/2}x, A^{1/2}\bar{x} > + < y, \bar{y} >,$$

and norm

$$\|X\|_{\mathcal{K}}^2 = \|A^{1/2}x\|^2 + \|y\|^2,$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $\bar{X} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ [see Chapter 4, page, 93, Vilenkin (1972)].

Now for $X \in D(\mathcal{A}) = D(A) \times D(A^{1/2})$, we have

$$< X, \mathcal{A}X >_{\mathcal{K}} = < Ax, y > + < y, -Ax > = 0$$

Thus

$$< (\mathcal{A} - \lambda I)X, X >_{\mathcal{K}} = < \mathcal{A}X, X >_{\mathcal{K}} - \lambda\|X\|_{\mathcal{K}}^2 = -\lambda\|X\|_{\mathcal{K}}^2.$$

Since

$$| < (\mathcal{A} - \lambda I)X, X >_{\mathcal{K}} | \leq \|(\mathcal{A} - \lambda I)X\|_{\mathcal{K}}\|X\|_{\mathcal{K}},$$

we have

$$\|(\mathcal{A}-\lambda I)X\|_{\mathcal{K}} \geq \lambda\|X\|_{\mathcal{K}}.$$

The adjoint of $\mathcal{A}^*$ of $\mathcal{A}$ is easily shown to be $-\mathcal{A}$. With the same logic

$$\|(\mathcal{A}^*-\lambda I)X\|_{\mathcal{K}} \geq \lambda\|X\|_{\mathcal{K}}.$$

Then $\mathcal{A}$ generates a contraction semigroup $U(t) \equiv e^{t\mathcal{A}}$ on $\mathcal{K}$. [see Curtain and Pritichard (1978), Th (2.14), Page 22]. Moreover $\mathcal{A}$ and $U(t)$ satisfy Hypothesis 3.1 with $\lambda = 0$, and they also satisfy Hypothesis 4.1.

Now consider the mild solution of (4.17):

$$V_t = U(t)X_0 + \int_0^t U(t-s)d\tilde{Z}_s. \tag{4.18}$$

Since $\tilde{Z}_t$ is a cadlag semimartingale on $\mathcal{K}$, the stochastic convolution integral $\int_0^t U(t-s)d\tilde{Z}_s$ has a cadlag version (see Chapter 3), so V_t is a cadlag adapted process on $\mathcal{K}$.

Now let us consider the semilinear Cauchy problem, written formally as

$$\begin{cases} \frac{\partial^2 x(t)}{\partial t^2} + Ax(t) &= f(x(t), \frac{\partial x(t)}{\partial t}) + \dot{Z}_t \\ x(0) &= x_0, \\ \frac{\partial x}{\partial t}|_{t=0} &= y_0, \end{cases} \tag{4.19}$$

where $f : D(A^{1/2}) \times H \rightarrow H$ satisfies the following conditions:

Hypothesis 4.4 **(a)** *$-f(x,.) : H \rightarrow H$ is semimonotone i.e. $\exists M > 0$ such that for all $x \in D(A^{1/2})$ and all $y_1, y_2 \in H$*

$$< f(x,y_2) - f(x,y_1), y_2 - y_1 > \leq M\|y_2 - y_1\|^2;$$

(b) *for all $x \in D(A^{1/2})$, $f(x,.)$ is demicontinuous and there is a continuous increasing function $\varphi : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $\|f(0,y)\| \leq \varphi(\|y\|)$;*

(c) *$f(.,y) : D(A^{1/2}) \rightarrow H$ is uniformly Lipschitz i.e $\exists M > 0$ such that $\forall y \in H$*

$$\|f(x_2,y) - f(x_1,y)\| \leq M\|A^{1/2}(x_2 - x_1)\|.$$

[The completeness of $D(A^{1/2})$ under the norm $\|A^{1/2}x\|$ follows from the strict positivity of $A^{1/2}$.]

Note that any uniformly Lipschitz function $f : D(A^{1/2}) \times H \to H$ satisfies Hypothesis 4.4 .

Proposition 4.1 *If f satisfies Hypothesis 4.4, then the Cauchy problem (4.19) has a unique mild adapted cadlag solution $x(t)$ with values in $D(A^{1/2})$. Moreover $\frac{dx(t)}{dt}$ is an H-valued cadlag process. If Z_t is continuous, $(x, \frac{dx}{dt})$ is continuous in $\mathcal{K}$.*

Proof: Define a mapping F from $\mathcal{K}$ to $\mathcal{K}$ by $F(x,y) = \begin{pmatrix} 0 \\ f(x,y) \end{pmatrix}$. We are going to show that F satisfies the hypotheses of Corollary 4.1.

- F is semimonotone.

  Let $X_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Then

$$\begin{aligned} < F(X_2) - F(X_1), X_2 - X_1 >_{\mathcal{K}} &= < f(x_2,y_2) - f(x_1,y_1), y_2 - y_1 > \\ &= < f(x_2,y_2) - f(x_2,y_1), y_2 - y_1 > \\ &\quad + < f(x_2,y_1) - f(x_1,y_1), y_2 - y_1 > . \end{aligned}$$

By Hypothesis 4.4(a) and the Schwartz inequality this is

$$\leq M\|y_2 - y_1\|^2 + \|f(x_2,y_1) - f(x_1,y_1)\|\|y_2 - y_1\|.$$

By Hypothesis 4.4(c) this is

$$\begin{aligned} &\leq M\|y_2 - y_1\|^2 + M\|A^{1/2}(x_2 - x_1)\|\|y_2 - y_1\| \\ &\leq M\|y_2 - y_1\|^2 + M/2\|A^{1/2}(x_2 - x_1)\|^2 + M/2\|y_2 - y_1\|^2 \\ &\leq 3M/2(\|A^{1/2}(x_2 - x_1)\|^2 + \|y_2 - y_1\|^2) \\ &= 3M/2\|X_2 - X_1\|_{\mathcal{K}}^2. \end{aligned}$$

Thus $-F : \mathcal{K} \to \mathcal{K}$ is semimonotone.

- F is demicontinuous in the pair (x, y) because it is demicontinuous in y and uniformly continuous in x.
- F is bounded since

$$\|F(x)\|_{\mathcal{K}} = \|f(x,y)\| \leq \|f(x,y) - f(0,y)\| + \|f(0,y)\|;$$

by Hypotheses(4.4b) and (4.4c) this is

$$\leq M\|A^{1/2}x\| + \varphi(\|y\|),$$

and since $\|A^{1/2}x\| \leq \|X\|_{\mathcal{K}}$ and $\|y\| \leq \|X\|_{\mathcal{K}}$ then

$$\|F(X)\|_{\mathcal{K}} \leq M\|X\|_{\mathcal{K}} + \varphi(\|X\|_{\mathcal{K}}).$$

Thus F is bounded by the function $\psi(r) = Mr + \varphi(r)$. Then F satisfies the hypotheses of Corollary 4.1 on $\mathcal{K}$.

Now as in the linear case we may write (4.19) as a first order initial value problem:

$$\begin{cases} dX_t &= \mathcal{A}(t)X_t dt + F(X(t))dt + d\tilde{Z}_t, \\ X(0) &= X_0. \end{cases}$$

Since A generates a contraction semigroup $U(t)$ we can write the above initial value problem as

$$X(t) = U(t)X(0) + \int_0^t U(t-s)F(X(s))ds + \int_0^t U(t-s)d\tilde{Z}_t.$$

By (4.18) we can rewrite this as

$$X(t) = \int_0^t U(t-s)F(X_s)ds + V_t.$$

Since V_t is cadlag and adapted then F, U and V satisfy all the conditions of Corollary 4.1. then there is an adapted cadlag solution on $\mathcal{K}$. If Z_t is continuous, V_t is continuous too and X_t is a continuous solution of (4.19) on $\mathcal{K}$. Q.E.D

Remark 4.2 *We assume $f : D(A^{1/2}) \times H \to H$. We could let f depend on $\omega \in \Omega$ and $t \in S$ as well. This would not involve any essential modification of the proof.*

Example (4.3): Let D, A, B, and W be as in Example (4.2). Let $p > d/2$ and consider a mixed problem of the form:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + Au &= f(u, \frac{\partial u}{\partial t}) + \dot{W} \quad \text{on}\, D \times [0, \infty) \\ Bu &= 0 \quad \text{on}\, \partial D \times [0, \infty) \\ u(x, 0) &= 0 \quad \text{on}\, D \\ \frac{\partial u}{\partial t}(x, 0) &= 0 \quad \text{on}\, D, \end{cases} \tag{4.20}$$

where $f : H_{-p+1} \times H_{-p} \to H_{-p}$.

As in Example (4.2) we consider W as a Brownian motion $\tilde{W}_t$ on the Sobolev space H_{-p}. Now A is a strictly positive definite self-adjoint operator on H_{-p}, and $[A^{-1}]^p$ is nuclear. Since all of the eigenvalues of A are strictly positive, then

$$< Ax, x >_{H_{-p}} \geq \lambda_0 \|x\|^2_{H_{-p}}, \tag{4.21}$$

for all $X \in D(A) = H_{-p+2}$.

Then we can write (4.20) as the following Cauchy problem on the Sobolev space H_{-p}:

$$\begin{cases} du_t &= \dot{u}_t dt \\ d\dot{u}_t &= -Au_t dt + f(u_t, \dot{u}_t)dt + d\tilde{W}_t \\ u(0) &= 0 \\ \dot{u}(0) &= 0. \end{cases} \tag{4.22}$$

Now A satisfies (4.21) and it is a positive definite self-adjoint operator on H_{-p}. Note that if $f \in H_n$, then $A^{1/2} f \in H_{n-1}$ [see, Walsh (1986), Example 3, Page 4.10]. Then $D(A^{1/2}) = H_{-p+1}$. Since $\tilde{W}_t$ is continuous then by Proposition 4.1, (4.22) has a continuous mild solution $u_t \in C(S, H_{-p+1})$ and, moreover, $u_t \in C^1(S, H_{-p})$ i.e., the mild solution of (4.20) is continuous process in H_{-p} for any $p > d/2 - 1$, and it is a differentiable process in H_{-p} for any $p > d/2$.

4.5 A Semilinear Integral Equation on the Whole Real Line

Recall the integral equation (4.12) of Example (4.1). Marcus (1974) has studied the existence of the solution of (4.12) where the parameter set of the processes extended to the whole real line, i.e. the integral equation

$$X_t = \int_{-\infty}^{t} U(t-s)f(X_s)ds + \int_{-\infty}^{t} U(t-s)dW(s). \tag{4.23}$$

This motivated us to study the existence of the solution of (4.23) when $-f$ is only monotone rather than being Lipschitz. We are going to use this in Chapter 6 to prove that the solution X_t of (4.23) is stationary.

Instead of (4.23) we are going to study the slightly more general equation

$$X_t = \int_{-\infty}^{t} U(t-s)f(X_s)ds + V_t. \tag{4.24}$$

We will impose the following conditions on f, V and the generator A of the semigroup U.

Hypothesis 4.5 (a) *$U(t)$ is a semigroup generated by a strictly negative definite, self-adjoint unbounded operator A such that A^{-1} is compact. Then there is $\lambda > 0$ such that $\|U(t)\| \le e^{-\lambda t}$.*

(b) *Let $\varphi(t) = K(1+t^p)$ for some $p > 0$, $K > 0$. $-f$ is a monotone demicontinuous mapping from H to H such that $\|f(x)\| \le \varphi(\|x\|)$ for all $x \in H$.*

(c) *Let $r = 2p^2$. V_t is cadlag adapted process such that $\sup_{t\in\mathbf{R}} E\{\|V_t\|^r\} < \infty$.*

Let us first study the integral equation:

$$X_t = \int_{-\infty}^{t} U(t-s)f(X_s+V_s)ds. \tag{4.25}$$

The following theorem translates Corollary 4.1 to the case when parameter set of the process is the whole real line.

Theorem 4.2 *If A, f and V satisfy Hypotheses 4.5, then the integral equation (4.25) has a unique continuous solution X such that*

$$\|X_t\| \le \int_{-\infty}^{t} e^{-\lambda(t-s)}\varphi(\|V_s\|)ds; \tag{4.26}$$

$$E\{\|X_t\|\} \le \frac{1}{\lambda}\sup_{s\in R} E\{\varphi(\|V_s\|)\} := K_1. \tag{4.27}$$

Proof: Consider a sequence of solutions (X_n) of the integral equation

$$X_n(t) = \int_{-n}^{t} U(t-s)f(X_n(s)+V_s)ds. \tag{4.28}$$

The solution of (4.28) exists by Corollary 4.1. It satisfies

$$\|X_n(t)\| \le \int_{-n}^{t} e^{-\lambda(t-s)}\varphi(\|V_s\|)ds \le \int_{-\infty}^{t} e^{-\lambda(t-s)}\varphi(\|V_s\|)ds \quad \text{for} \quad t > -n.$$

Since by Hypothesis 4.5 $\sup_{s\in R} E\{\varphi(\|V_s\|)\} < \infty$, then by Fubini's theorem

$$E\{\int_{-\infty}^{t} e^{-\lambda(t-s)}\varphi(\|V_s\|)ds\} = \int_{-\infty}^{t} e^{-\lambda(t-s)}E\{\varphi(\|V_s\|)\}ds.$$

Then

$$E[\|X_n(t)\|] \le K_1 \quad \text{for} \quad t > -n. \tag{4.29}$$

But

$$X_{n+1}(t) = \int_{-n-1}^{t} U(t-s)f(X_{n+1}(s)+V_s)ds,$$

so by using the semigroup property of $U(t)$ we can rewrite this as

$$X_{n+1}(t) = U(t+n)X_{n+1}(-n) + \int_{-n}^{t} U(t-s)f(X_{n+1}(s)+V_s)ds. \tag{4.30}$$

But (4.30) is the same equation as (4.28) with different initial conditions. Then by Corollary 4.1 we have

$$\|X_{n+1}(t) - X_n(t)\| \le e^{-\lambda(t+n)}\|X_{n+1}(-n) - 0\|,$$

or

$$\|X_{n+1} - X_n\|_T \le e^{\lambda T} e^{-\lambda n} \|X_{n+1}(-n)\|,$$

where $\|X\|_T = \sup_{-T \le t \le T} \|X_t\|$. Then

$$E\{\|X_{n+1} - X_n\|_T\} \le e^{\lambda T} e^{-\lambda n} E\{\|X_{n+1}(-n)\|\}.$$

But by (4.29) $E\{\|X_{n+1}(-n)\|\} \le K_1$, so

$$\sum_{k=0}^{\infty} E\{\|X_{n+1} - X_n\|_T\} \le K_1 e^{\lambda T} \sum_{k=0}^{\infty} e^{-\lambda k} < \infty.$$

Thus (X_n) is a convergent sequence in $L^1(\Omega, C([-T,T], H)$ for each $T > 0$. Define $X = \lim_{n\to\infty} X_n$. Since $X_n(t)$ satisfies (4.26) and(4.27) and since $E[\|X_n(t) - X(t)\|] \to 0$ then X also satisfies (4.26) and (4.27) .

To complete the proof of the theorem one needs to show that X satisfies (4.25). Now we can rewrite equation (4.28) as

$$X_n(t) = U(t+T)X_n(-T) + \int_{-T}^{t} U(t-s) f(X_n(s) + V_s) ds. \tag{4.31}$$

Consider the integral equation

$$Y(t) = U(t+T)X(-T) + \int_{-T}^{t} U(t-s) f(Y(s) + V_s) ds. \tag{4.32}$$

By Corollary 4.1 this equation has a unique solution. Comparing (4.32) with (4.31), we have by Corollary 4.1 that

$$\|X_n(t) - Y(t)\| \le e^{-\lambda(t+T)} \|X_n(-T) - X(-T)\|.$$

Now

$$\begin{aligned} E[\|Y(t) - X(t)\|] &\le E[\|X(t) - X_n(t)\|] + E[\|X_n(t) - Y(t)\|] \\ &\le E[\|X(t) - X_n(t)\|] + e^{-\lambda(t+T)} E[\|X_n(-T) - X(-T)\|], \end{aligned}$$

and since $E[\|X(t) - X_n(t)\|] \to 0$ as $n \to \infty$ then we have $X(t) = Y(t)$ a.s., i.e., $X(t)$ is a solution of (4.32). We can rewrite (4.32) as

$$X(t) = U(t+n)X(-n) + \int_{-n}^{t} U(t-s)f(X_s + V_s)ds. \tag{4.33}$$

By (4.26), (4.27) and Hypothesis 4.5(b) we have

$$\begin{aligned}\|U(t-s)f(X_s+V_s)\| &\leq e^{-\lambda(t-s)}\varphi(\|X_s+V_s\|) \\ &\leq 2^p K_1 e^{-\lambda(t-s)}\left\{1 + \left(\int_{-\infty}^{s} e^{-\lambda p(s-u)}\|V_u\|^p du\right)^p + \|V_s\|^p\right\}\end{aligned}$$

Then by hypothesis 4.5(c) and Fubini's theorem it is easy to see that

$$\int_{-\infty}^{t} \|U(t-s)f(X_s+V_s)\|ds < \infty.$$

Then by the dominated convergence theorem we have

$$\lim_{n\to\infty} \int_{-n}^{t} U(t-s)f(X_s+V_s)ds = \int_{-\infty}^{t} U(t-s)f(X_s+V_s)ds.$$

Since X satisfies (4.27) then $E[\|X(-n)\|] \leq K$, so

$$E[\|X(t+n)X(-n)\|] \leq e^{-\lambda(t+n)}K_1,$$

which implies that $U(t+n)X(-n) \to 0$ and so (4.33) implies

$$X_t = \int_{-\infty}^{t} U(t-s)f(X_s+V_s)ds.$$

Q.E.D.

Chapter 5

THE CONTINUITY OF THE SOLUTION

5.1 Introduction

Consider the integral equation

$$X(t,y) = U(t,0)X_0(y) + \int_0^t U(t,s)f(s,y,X(s,y))ds + V(t,y), \quad t \in S, \quad y \in S. \quad (5.1)$$

Faris and Jona-Lasinio (1982) have proved that the solution X of (5.1) is a continuous function of V in the special case when the generator of U is $\frac{d^2}{dx^2}$ and $f(x) = -\lambda x^3 - \mu x$. Da Prato and Zabczyk (1988) generalized this result to the case where U is a general analytic semigroup and f is a locally Lipschitz function on a Banach space.

We are going to generalize the previous result by proving that the solution of (5.1) changes continuously when any or all of V, f, A and X_0 are varied. As a corollary we will prove a generalization of Faris and Jona-Lasino's theorem for semimonotone f and more general U; this was open after Faris and Jona-Lasinio (1982) [see for example Smolenski et al (1986), page 230]. We will also prove the strong convergence of the finite dimensional Galerkin approximation to the solution of (5.1).

5.2 The Main Theorem and its Corollary

Theorem 5.1 *Let f^1 and f^2 be two mappings satisfying Hypothesis 2.1 with parameters $M_1(y)$ and $M_2(y)$ respectively and bounded by functions φ_1 and φ_2 respectively.*

Suppose V^1 and V^2 satisfy Hypothesis 2.1. Suppose A and U satisfy Hypotheses 4.1 *and 3.1. Let $X^i(t)$, $i = 1, 2$ be solutions of the integral equations:*

$$X^i(t,y) = U(t,0)X_0^i + \int_0^t U(t,s)f^i(s,y,X^i(s,y))ds + V^i(t,y). \tag{5.2}$$

Then we have

$$\begin{aligned} \|X^2(t) - X^1(t)\|^2 \ \leq\ & 2\|V^2(t) - V^1(t)\|^2 \\ & + 2e^{(2\lambda+4M_2+1)t}\Big\{\|X_0^2 - X_0^1\|^2 \\ & + \left\{\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds\right\}^{1/2} I \\ & + \int_0^t e^{-2\lambda s}\left\|f^2(X^1(s)) - f^1(X^1(s))\right\|^2 ds\Big\}, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} I \ =\ & 2\left\{\int_0^T e^{-2\lambda s}\left\|f^2(X^2(s)) - f^1(X^1(s))\right\|^2 ds\right\}^{1/2} \\ & + 4M_2\left\{\int_0^T e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds\right\}^{1/2} \end{aligned}$$

Note that, since by Theorem 4.1 X_1 and X_2 are bounded by V_1 and V_2, then I is bounded by function of V_1 and V_2 .

Proof: Since U satisfies Hypothesis 4.1, then by Theorem 4.1 the solution of (5.2) exists. Define $Y^i(t) = X^i(t) - V^i(t)$, $i = 1, 2$. Then we can write (5.2) in the form

$$Y^i(t) = U(t,0)X_0^i + \int_0^t U(t,s)f^i(X^i(s)ds, \quad i = 1, 2,$$

so that

$$Y^2(t) - Y^1(t) = U(t,0)(X_0^2 - X_0^1) + \int_0^t U(t,s)[f^2(X^2(s)) - f^1(X^1(s)]ds.$$

Since U satisfies Hypothesis 3.1(a)-(d), then by Remark 3.3 we have

$$\begin{aligned} \|Y^2(t) - Y^1(t)\|^2 \ \leq\ & e^{2\lambda t}\|X_0^2 - X_0^1\|^2 \\ & + 2\int_0^t e^{2\lambda(t-s)} < Y^2(s) - Y^1(s), f^2(X^2(s)) - f^1(X^1(s)) > ds. \end{aligned} \tag{5.4}$$

To complete the proof of this theorem we need the following lemma.

Lemma 5.1 *Let $K > 0$. Then*

$$\begin{aligned}
&2\int_0^t e^{-2\lambda s} < Y^2(s) - Y^1(s), f^2(X^2(s)) - f^1(X^1(s)) > ds \\
\le\ & (K + 4M_2)\int_0^t e^{-2\lambda s}\|Y^2(s) - Y^1(s)\|^2 ds \\
&+ I\left(\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds\right)^{\frac{1}{2}} \\
&+ \frac{1}{K}\int_0^t e^{-2\lambda s}\left\|f^2(X^1(s)) - f^1(X^1(s))\right\|^2 ds.
\end{aligned} \tag{5.5}$$

Note that because Y^i and X^i are cadlag and the f^i are bounded by φ_i, then the integrands are dominated by cadlag functions and hence are integrable.

Proof: The left hand side of (5.4) is

$$\begin{aligned}
\le\ & 2\int_0^t e^{-2\lambda s} < Y^2(s) - Y^1(s), f^2(X^2(s)) - f^1(X^1(s)) > ds \\
&+ 2\int_0^t e^{-2\lambda s} < Y^2(s) - Y^1(s), f^2(X^1(s)) - f^1(X^1(s)) > ds
\end{aligned}$$

since $Y^i = X^i - V^i$ and $-f^2$ is semimonotone. By the Schwartz inequality this is

$$\begin{aligned}
\le\ & 2\int_0^t e^{2\lambda s}\|V^2(s) - V^1(s)\|\ \|f^2(X^2(s)) - f^2(X^1(s))\| ds \\
&+ 2M_2\int_0^t e^{-2\lambda s}\|X^2(s) - X^1(s)\|^2 ds \\
&+ 2\int_0^t e^{-2\lambda s}\|Y^2(s) - Y^1(s)\|\ \|f^2(X^1(s)) - f^1(X^1(s))\| ds.
\end{aligned}$$

Apply the Schwartz inequality to the first integral, use the inequality $2ab < Ka^2 + \frac{1}{K}b^2$ in the third, and write $X^i = Y^i + V^i$ and use the inequality again in the second to see that this is

$$\begin{aligned}
\le\ & 2\left\{\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2) ds\right\}^{1/2}\left\{\int_0^t e^{-2\lambda s}\|f^2(X^2(s)) - f^2(X^1(s)\|^2 ds\right\}^{1/2} \\
&+ 4M_2\int_0^t e^{-2\lambda s}\|Y^2(s) - Y^1(s)\|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+ 4M_2 \int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds \\
&+ K \int_0^t e^{-2\lambda s}\|Y^2(s) - Y^1(s)\|^2 ds \\
&+ \frac{1}{K} \int_0^t e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s))\|^2 ds.
\end{aligned}
$$

This proves the lemma.

Q.E.D

To finish the proof of Theorem 5.1, let us define $g(t) := e^{-2\lambda s}\|Y^2(t) - Y^1(t)\|^2$. By Lemma 5.1 one has

$$
\begin{aligned}
g(t) \quad \leq \quad & \|X_0^2 - X_0^1\| + (1 + 4M_2) \int_0^t g(s) ds \\
& + \left(\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds \right)^{1/2} I \\
& + \int_0^t e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s)\|^2 ds.
\end{aligned}
$$

By Gronwall's inequality

$$
\begin{aligned}
g(t) \quad \leq \quad & e^{(1+4M_2)t} \Big[\|X_0^2 - X_0^1\|^2 \\
& + I \left(\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds \right)^{\frac{1}{2}} \\
& + \int_0^t e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s))\|^2 ds \Big].
\end{aligned}
$$

Substituting for $g(t)$ and using the following inequality

$$
\|X^2(t) - X^1(t)\|^2 \leq 2\|Y^2(t) - Y^1(t)\|^2 + 2\|V^2(t) - V^1(t)\|^2,
$$

to get inequality (5.3).

Q.E.D

Remark 5.1 *We can extend Theorem 5.1 to the case where the evolution operator* $U(t,s)$ *varies too, but unfortunately the inequality becomes more complicated.*

Let $U^i(t,s)$, $i = 1,2$ *be an evolution operator satisfying the hypotheses of Theorem 5.1. Let* $X^i(t)$, $i = 1,2$ *be the solutions of the integral equations*

$$X^i(t) = U^i(t,0)X_0^i + \int_0^t U^i(t,s)f_s^i(X^i(s))ds + V^i(t). \tag{5.6}$$

Define

$$V^3(t) := (U^2(t,0) - U^1(t,0))X_0^1 + \int_0^t (U^2(t,s) - U^1(t,s))f^2(X^1(s))ds,$$

and define

$$\begin{aligned}\bar{I}: &= 2\left(\int_0^T e^{-2\lambda s}\|f^2(X^2(s)) - f^1(X^1(s))\|^2 ds\right)^{1/2} I \\ &+ 4M_2\left(\int_0^T e^{-2\lambda s}\|V^3(s) + V^1(s) - V^2(s))\|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

Then we have

$$\begin{aligned}\|X^2(t) - X^1(t)\|^2 \le\ & 4\|V^2(t) - V^1(t)\|^2 + 4\|V^3(t)\|^2 \\ & + 2e^{(2\lambda+K+4M_2)t}\Big[\|X_0^2 - X_0^1\|^2 \\ & + \bar{I}\left(\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds\right)^{\frac{1}{2}} \\ & + \bar{I}\left(\int_0^t e^{-2\lambda s}\|V^3(s)\|^2 ds\right)^{\frac{1}{2}} \\ & + \int_0^t e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s))\|^2 ds\Big].\end{aligned} \tag{5.7}$$

Proof: From (5.6) and the definition of $V^3(t)$ we have

$$X^1(t) = U^2(t,0)X_0^1 + \int_0^t U^2(t,s)f^1(X^1(s))ds + V^1(t) + V^3(t).$$

Compare this equation with (5.6) for $i = 1$ to get (5.7). Q.E.D

Corollary 5.1 *Consider equations (5.6). There is a constant d_N such that on the set where* $\max_{i=1,2}(\|X_0^i\|, M_2(y), \|V^i\|_\infty) < N$ *one has*

$$\begin{aligned}\|X^2 - X^1\|_\infty^2 &\leq d_N \Big[\|X_0^2 - X_0^1\|^2 + \|V^2 - V^1\|_\infty + \|V^3\|_\infty \\ &\quad + \int_0^T e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s))\|^2 ds\Big]. \end{aligned} \tag{5.8}$$

Proof: From the definition of $V^3(t)$ and inequality (4.7) there is a constant $d_N^1 > 0$ such that $\|V^3\|_\infty \leq d_N^1$. From inequality (4.7) and the definition of $\bar{I}$ there is $d_N^2 > 0$ such that $\bar{I} \leq d_N^2$ and $\|f^2(X^2(s))| \leq d_N^2$. Define $d_N^3 = \max(d_N^1, d_N^2)$. Using (5.7) we have

$$\begin{aligned}\|X^2 - X^1\|_\infty^2 &\leq 4\|V^2 - V^1\|_\infty^2 + 4\|V^3\|_\infty^2 2e^{(2\lambda+1+4N)T}\Big\{\|X_0^2 - X_0^1\|^2 \\ &\quad + [\|V^2 - V^1\|_\infty + \|V^3\|_\infty] d_N^3 [\int_0^T e^{-2\lambda s} ds] \\ &\quad + \int_0^T e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s))\|^2 ds\Big\}.\end{aligned}$$

Using the facts that $\|V^1\|_\infty \leq N$, $\|V^2\|_\infty \leq N$, $\|V^2\|_\infty \leq d_N^3$ and the above inequality, we get (5.8). Q.E.D

Remark 5.2 *Let $D(S,H)$ be the set of H-valued cadlag functions on S with norm*

$$\|f\|_\infty = \sup_{t \in S}\|f(t)\|.$$

By Corollary 5.1 there is a continuous mapping $\psi : S \times D(S,H) \to D(S,H)$ such that if $X(t)$ is a solution of

$$X(t) = \int_0^t U(t,s) f(X(s)) ds + V(t),$$

then $X(t) = \psi(t,V)(t)$. Moreover there is a constant d_N such that on the set where $\max_{i=1,2}(M_2(y), \|V^i\|_\infty) < N$, *we have*

$$\|\psi(.,V^2) - \psi(.,V^1)\|_\infty \leq d_N \|V^2 - V^1\|_\infty^{\frac{1}{2}},$$

so ψ is Hölder continuous with exponent $1/2$.

5.3 Application to the Large Deviation Principles

Da Prato and Zabczyk (1988) have studied large-deviation principles for the Ornstein - Uhlenbeck process $V_t = \int_0^t U(t-s)dW(s)$. In the case when f is locally Lipschitz, they also studied large-deviation principles for the solution of

$$\begin{cases} dX_t &= AX_t + f(t, X(t))dt + \epsilon dW(t) \\ X(0) &= x, \end{cases} \tag{5.9}$$

where $\epsilon > 0$. As a consequence of Remark 5.2 we can generalize their result to the case when f is semimonotone.

Suppose A, U, f, and W are as in Example (4.1). Then we can write the mild solution of (5.9) as

$$X_t = U(t)x + \int_0^t U(t-s)f(s, X(s))ds + \epsilon V_t. \tag{5.10}$$

Let $\eta \in L^2(S, H)$. Consider a system of the form

$$\begin{cases} \frac{d\xi}{dt} &= A\xi(t) + f(t, \xi(t)) + A^{-1}\eta(t) \\ \xi(0) &= x. \end{cases} \tag{5.11}$$

Note that A^{-1} is a nuclear operator. We can write the mild solution of (5.11) as

$$\xi(t) = U(t)x + \int_0^t U(t-s)f(s, \xi(s))ds + \xi^\eta(t), \tag{5.12}$$

where $\xi^\eta(t) = \int_0^t U(t-s)A^{-1}\eta(s)ds$.

By Remark 5.2 we can write the solutions of (5.10) and (5.12) as

$$X_t^{x,\epsilon} = \psi(t, U(.)x + \epsilon V.)(t),$$

and

$$\xi^{\eta,x} = \psi(t, U(.)x + \xi^\eta)(t).$$

Then we have the following proposition.

Proposition 5.1 (i) *For arbitrary $\delta > 0$, $\alpha > 0$, $\beta > 0$ and $C > 0$ there exists $\epsilon_0 > 0$ such that for η, x satisfying $\int_0^t \|\eta(s)\|^2 ds \le C$, $\|x\| \le \beta$ and $\epsilon \in (0, \epsilon_0)$,*

$$P(\|X^{x,\epsilon} - \xi^{\eta,x}\|_\infty < \delta) \ge \exp\left[-\frac{1}{2}\epsilon^{-2}(\int_0^T \|\eta(s)\|^2 ds + \alpha)\right].$$

(ii) *For arbitrary $\delta > 0$, $\alpha > 0$, $\beta > 0$ and*

$r_0 > 0$ there exists $\epsilon_0 > 0$ such that for arbitrary $r \in (0, r_0)$ and $\epsilon \in (0, \epsilon_0)$ and $\|x\| \le \beta$

$$P(distance_H(X^{x,\epsilon}, K(x,r)) > \delta) \le \exp[-\frac{1}{2}\epsilon^{-2}(r^2 - \alpha)]$$

where $K(x,r)$ stands for the set for all $\xi^{\eta,x}$ for which $\int_0^T \|\eta(s)\|^2 ds \le r^2$.

Proof: The continuity of ψ in Remark 5.2 allows us to reduce the problem to the linear case ($f \equiv 0$) and zero initial condition [see Freidlin and Wentzell (1984), Theorem 3.1, Page (81)]. But when $f \equiv 0$, the theorem has been proved by Da Prato and Zabczyk [Da Prato and Zabczyk (1988), Theorem 5].

Q.E.D

5.4 Galerkin Approximations

Let $U(t)$ be a semigroup generated by a strictly negative definite closed unbounded self-adjoint operator A such that A^{-1} is compact.

Then there is a complete orthonormal basis (ϕ_n) and eigenvalues $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$ with $\lambda_n \to \infty$, such that $A\phi_n = -\lambda_n \phi_n$.

Let H_n be the subspace of H generated by $\{\phi_0, \phi_1, \dots, \phi_{n-1}\}$ and let J_n be the projection operator on H_n.

Define $f_n = J_n f$, $V_n(t) = J_n V(t)$ and $U_n(t) = J_n U(t) J_n$, where f and V satisfy Hypothesis 4.1.

Let $X_n(t)$ be the solution of

$$X_n(t,y) = \int_0^t U_n(t-s) f_n(s,y,X_n(s,y))ds + V_n(t,y), \ t \geq 0 \tag{5.13}$$

and let $X(t)$ be the solution of

$$X(t,y) = \int_0^t U(t-s) f(y,s,X(s,y))ds + V(t,y). \tag{5.14}$$

We will prove

Proposition 5.2 *For all $y \in G$, we have*

$$\|X_n(\cdot,y) - X(\cdot,y)\|_\infty \to 0.$$

Proof

By Corollary 5.1 we have

$$\begin{aligned} \|X_n - X\|_\infty^2 \ \leq \ & d_N \Big[\|V_n - V\|_\infty + \|\bar{V}_n\|_\infty \\ & + \int_0^T e^{-2\lambda s} \|f_n(X(s)) - f(X(s))\|^2 ds \Big], \end{aligned} \tag{5.15}$$

where

$$\bar{V}_n = \int_0^t [U_n(t-s) - U(t-s)] f(s,X(s))ds.$$

Since $f_n = J_n f$ and $V_n = J_n V$, then the first and 3th term in the right hand side of (5.15) approach zero, so to complete the proof we need to show that

$$\|\bar{V}_n\|_\infty \ \to 0.$$

But

$$\|\bar{V}_n(t)\| \leq \varphi(\|X\|_\infty) \int_0^t \|U_n(s) - U(s)\|_L ds,$$

so

$$\|\bar{V}_n\|_\infty \leq \varphi(\|X\|_\infty) \int_0^T \|U_n(s) - U(s)\|_L ds. \tag{5.16}$$

Since $\|U_n(t) - U(t)\|_L$ equals $e^{-\lambda_n t}$ for $t > 0$ and equals zero for $t = 0$, then by the bounded convergence theorem the left hand side of (5.16) approaches zero.

Q.E.D

5.5 Galerkin Approximations for the Integral Equation on the Whole Real Line

We can prove the convergence of similar Galerkin approximations to the solution of equation (4.24) of Chapter 4. Define

$$f_n = J_n f, \quad V_n(t) = J_n V(t), \quad U_n(t) = J_n V(t) J_n$$

and define $X_n(t)$ and $X(t)$ as solutions of

$$X_n(t) = \int_{-\infty}^t U_n(t-s) f_n(X_n(s)) ds + V_n(t). \tag{5.17}$$

and

$$X(t) = \int_{-\infty}^t U(t-s) f(X(s)) ds + V(t) \tag{5.18}$$

Now we can prove

Theorem 5.2 *If A, U, f and V satisfy Hypotheses 4.5, then one has*

$$E(\|X_n(t) - X(t)\|) \to 0.$$

Proof:

Define

$$X_n^k(t) = \int_{-k}^t U_n(t-s) f_n(X_n^k(s)) ds + V_n(t),$$

$$X^k(t) = \int_{-k}^{t} U(t-s)f(X^k(s))ds + V(t),$$

and

$$\bar{V}_{n,k}(t) = \int_{-k}^{t} (U_n(t-s) - U(t-s)f(X^k(s))ds.$$

By Remark 5.1 we have

$$\begin{aligned} \|X_n^k(t) - X^k(t)\|^2 &\leq 4\|V_n(t) - V(t)\|^2 + 4\|\bar{V}_{n,k}(t)\|^2 \\ &+ \bar{I}\left(\int_{-k}^{t} e^{2\lambda_0 s}\|V_n(s) - V(s)\|^2 ds\right)^{\frac{1}{2}} \\ &+ \bar{I}\left(\int_{-k}^{t} e^{2\lambda_0 s}\|\bar{V}_{n,k}(s)\|^2 ds\right)^{\frac{1}{2}} \\ &+ \int_{-k}^{t} e^{2\lambda_0 s}\|f_n(X(s)) - f(X(s))\|^2 ds. \end{aligned} \tag{5.19}$$

Taking expectations and using the Schwartz inequality and Fubini's theorem, (5.19) implies that

$$\begin{aligned} E\{\|X_n^k(t) - X^k(t)\|^2\} &\leq 4E\{\|V_n(t) - V(t)\|^2\} + 4E\{\|\bar{V}_{n,k}(t)\|^2\} \\ &+ (E\{\bar{I}^2\})^{\frac{1}{2}}\left(\int_{-\infty}^{t} e^{2\lambda_0 s}E(\|V_n(s) - V(s)\|^2)ds\right)^{\frac{1}{2}} \\ &+ (E\{\bar{I}^2\})^{\frac{1}{2}}\left(\int_{-\infty}^{t} e^{2\lambda_0 s}E(\|\bar{V}_{n,k}(s)\|^2)ds\right)^{\frac{1}{2}} \\ &+ \int_{-\infty}^{t} e^{2\lambda_0 s}E(\|f_n(X(s)) - f(X(s))\|^2)ds. \end{aligned} \tag{5.20}$$

We first show

$$E\{\|X_n^k(t) - X^k(t)\|^2\} \to 0 \quad \text{uniformly in } k. \tag{5.21}$$

Since $V_n = J_n V$ and $f_n = J_n f$, the first,third, and 5th term of the right hand side of inequality (5.20) converge to zero. Then to prove (5.21) it is enough to show $E(\|\bar{V}_{n,k}(t)\|^2)$ converges to zero uniformly in k and $t \in (-\infty, T]$.

By using $\|f(x)\| \leq C(1 + \|x\|^p)$ and inequality (4.6), we see that

$$\sup_{t\in R} E(\|V(t)\|^{2p}) < \infty,$$

and, using Fubini's theorem, one has

$$\sup_{t\in R} E(\|\bar{V}_{n,k}(t)\|^2) \leq \sup_{t\in R} E\{1 + \|V(t)\|^p\} \int_{-\infty}^{0} \|U(-s) - U_n(-s)\|_L^2 ds.$$

Since $\|U(-s) - U_n(-s)\|_L \to 0$ for $s < 0$ and

$$\|U(-s) - U_n(-s)\|_L \leq e^{2\lambda_0 s},$$

then by the dominated convergence theorem

$$\sup_{t\in R} E(\|\bar{V}_{n,k}(t)\|^2) \to 0 \quad \text{uniformly in } k.$$

Then $E(\|X_{n,k}(t) - X^k(t)\|^2) \to 0$ uniformly in k. By Theorem 4.2, then $E(\|X_{n,k}(t) - X_n(t)\|) \to 0$ as $k \to \infty$, hence $E(\|X^k(t) - X(t)\|) \to 0$ and we have $E(\|X_n(t) - X(t)\|) \to 0$ Q.E.D

Chapter 6

STATIONARY PROCESSES

6.1 Introduction

Consider an integral equation of the form:

$$X(t) = \int_{-\infty}^{t} U(t-s)f(X(s))ds + V(t), \tag{6.1}$$

where U and f satisfy Hypothesis 4.5 and V satisfies the following condition:

Hypothesis 6.1 *V is a cadlag adapted stationary processes on H, such that*

$$E(\|V(0)\|^r) < \infty \text{ for } r = 2p^2, \tag{6.2}$$

where $p \geq 1$.

In the special case when $f(x) = -\frac{1}{2}\nabla F(x)$ is the Fréchet derivative of a potential $F(x)$ on H and V_t is the stationary Ornstein-Uhlenbeck processes $\int_{-\infty}^{t} U(t-s)dW(s)$, we may consider the integral equation (6.1) as a mild solution of the infinite dimensional Einstein-Smoluchowski equation:

$$dX(t) = -AX(t)dt - \frac{1}{2}\nabla F(x_t)dt + dW(t). \tag{6.3}$$

In finite-dimensions, the solutions are diffusion processes and the stationary measures of these diffusion processes were studied by Kolmogorov (1937).

Infinite-dimensional Einstein-Smoluchowski equations have been studied by many authors, e.g. Marcus (1974, 1978, 1979), Funaki (1983) and Iwata (1987). The stationary measure associated to this equation has important applications in stochastic quantization [see Marcus (1979), Jona-Lasinio and Mitter (1985), Albeverio and Röckner (1989) and Iwata (1987)].

Marcus (1974) studied (6.1) when f is Lipschitz, V is an Ornstein-Uhlenbeck process, and A^{-1} is nuclear. He proved that the solution of (6.1) is a stationary process; when $f(x) = \frac{1}{2}\nabla F(x)$, he characterized its stationary measures explicitly. This result was generalized somewhat in Marcus (1978) to the case where $f : B \to B^{\star}$, where $B \subseteq H \subseteq B^{\star}$ is a Gelfand triple and f satisfies

$$< f(x) - f(y), x - y >_{B^\star \times B} \leq -C\|x-y\|_B^p \text{ and}$$
$$\|f(x)\|_{B^\star} \leq C(1 + \|x\|_B^{p-1}) \quad \text{for some } C \geq 0, \quad \text{and} \quad p \geq 1$$

Unfortunately we were unable to follow his proof of the stationarity of the solution of (6.1).

In this chapter we extend his setting to a slightly more general case in which f, U and V satisfy Hypothesis 4.5 (a), (b) and Hypothesis 6.1 on a Hilbert space H. Our method of proof is different from that of Marcus (1978). We are going to use the results of Chapter 4 and Chapters 5. We will give the stationary distribution of (6.3) when $\nabla F(x)$ is monotone.

Since V satisfies Hypothesis 6.1, it also satisfies Hypothesis 4.5(c), so by Theorem 4.2 a solution of (6.1) exists. By Theorem 5.3 this solution is the L^1-limit of the solutions of the finite-dimensional equations:

$$X_n(t) = \int_{-\infty}^{t} U_n(t-s) f_n(X_n(s))ds + V_n(t), \tag{6.4}$$

where $U_n(t) = J_n U(t) J_n$, $f_n = J_n f$, and $V_n = J_n V$.

Thus to prove that the solution of (6.1) is stationary, it is enough to prove that the solution of (6.4) is stationary.

Let $f : \mathbf{R} \to Y$, where Y is a topological space. Define $(\theta_s f)(t) = f(t+s)$.

Definition 6.1 *A process $X = \{X(t) : t \in \mathbf{R}\}$, taking values in a topological space Y, is called* strongly stationary *if for each h and real numbers $t_1, t_2, \ldots, t_n$ the families $(X(t_1), X(t_2), \ldots, X(t_n))$ and $((\theta_h X)(t_1), \ldots, (\theta_h X)(t_n))$ have the same joint distribution.*

Let $D(\mathbf{R}, H)$ be the space of H-valued cadlag functions on $\mathbf{R}$ with the metric of uniform convergence on compacts

$$d(f,g) = \sum_{k=1}^{\infty} \frac{\|f-g\|_k}{2^k(1+\|f-g\|_k)},$$

where

$$\|f\|_k = \sup_{-k \le t \le k} \|f(t)\|.$$

Let H_n be a finite dimensional subspace of H. If $f \in D(\mathbf{R}, H_n)$, $\theta.f$ is a function from $\mathbf{R}$ to $D(\mathbf{R}, H_n)$.

Now we are going to prove the following lemma:

Lemma 6.1 *If $V = \{V(t),\ t \in \mathbf{R}\}$ is an H_n-valued cadlag stationary process on $\mathbf{R}$ then $\theta.V, = \{\theta_s V\ s \in \mathbf{R}\}$ is a $D(\mathbf{R}, H_n)$-valued stationary process on $\mathbf{R}$.*

Proof: To prove this, it is enough to prove that for all real $t_1 < t_2 < \ldots < t_n$, all real $s_1 < s_2 < \ldots < s_m$, and all real h,

$$\{(\theta_{t_1}V)(s_1), (\theta_{t_2}V)(s_1), \ldots, (\theta_{t_n}V)(s_1), \ldots, (\theta_{t_1}V)(s_m), \ldots, (\theta_{t_n}V)(s_m)\}$$

and

$$\{(\theta_{t_1} + hV)(s_1), (\theta_{t_2} + hV)(s_1), \ldots, (\theta_{t_n} + hV)(s_1), \ldots, (\theta_{t_1} + hV)(s_m), \ldots, (\theta_{t_n} + hV)(s_m)\}$$

have the same joint distribution. But by definition $(\theta_{t_i+h}V)(s_j) = V(t_i+h+s_j)$ and since V is an H_n-valued stationary process, then we have equality of the joint distributions, and the proof is complete. Q.E.D

6.2 The Continuity of the Solution with Respect to V_n

Let K be $D(\mathbf{R}, H_n)$, with metric

$$d_K(f,g) = \sum_{k=1}^{\infty} \frac{\|f-g\|_k}{2^k(1+\|f-g\|_k)} + \left(\int_{-\infty}^{\infty} e^{-|s|\lambda_0}\|f(s)-g(s)\|^r ds\right)^{\frac{1}{r}}, \quad \lambda_0 > 0.$$

To prove that the solution $X_n(t)$ of (6.4) is a stationary process, we need to prove a result similar to Remark 5.2 for equation (6.4), i.e., that there is a continuous mapping

$$\psi : \mathbf{R} \times K \to D(\mathbf{R}, H_n) \text{ such that } X_n(t) = \psi(t, V_n(\cdot))(t).$$

To prove this we first need to prove the existence of a solution of (6.4) when $V_n \in K$. Then instead of equation (6.4) we consider the following integral equation:

$$Y(t) = \int_{\infty}^{t} \mathcal{U}(t-s) f(Y(s)+g(s))ds, \tag{6.5}$$

under the following hypothesis.

Hypothesis 6.2 **(a)** *$\mathcal{U}(t) =: U_n(t) = J_n U(t) J_n$, and U satisfies Hypothesis 4.5;*

(b) *$-f : H_n \to H_n$ is a continuous monotone function such that $\|f(x)\| \le C(1 + \|x\|^r)$, for $r = 2p^2$;*

(c) *$g \in K$.*

Note that because H_n is a finite-dimensional space, the $\mathcal{U}(t)$ form a group and $\mathcal{U}(t)$ is well-defined for all $t \in \mathbf{R}$ and $\mathcal{U}(-t)\mathcal{U}(t) = I$.

Now we are going to prove two purely deterministic lemmas:

Lemma 6.2 *If f, $\mathcal{U}$, and g satisfy Hypothesis 6.2, then (6.5) has a unique continuous solution.*

Proof: As in Theorem 4.2, define

$$Y_k(t) = \int_{-k}^{t} \mathcal{U}(t-s)f(Y_k(s)+g(s))ds. \tag{6.6}$$

Then we have

$$\|Y_k(t)\| \leq C\int_{-\infty}^{t} e^{-\lambda_0(t-s)}(1+\|g(s)\|^r)ds,$$

and by Hypothesis 6.2(c) there are $C(T) > 0$ and $C_1(T) > 0$ such that for all $t \in (-\infty, T]$,

$$\|Y_k(t)\| \leq C(T)e^{-\lambda_0 t} \leq C_1(T). \tag{6.7}$$

Let $a \leq t_1 \leq t_2 \leq T$. By (6.6) one has

$$\mathcal{U}(-t_2)\,Y_k(t_2) - \mathcal{U}(-t_1)\,Y_k(t_1) = \int_{t_1}^{t_2} \mathcal{U}(-s)f(Y_k(s)+g(s))ds.$$

Now it is easy to see from (6.7) and Hypothesis 6.2(c) that there is $C(T,a) > 0$ such that

$$\|\mathcal{U}(-t_2)\,Y_k(t_2) - \mathcal{U}(-t_1)\,Y_k(t_1)\| \leq C(T,a)|t_2 - t_1|.$$

Then $\mathcal{U}(-t)\,Y_k(t)$ is uniformly equicontinuous on $[a,T]$ so $Y_k(t)$ is uniformly equicontinuous on $[a,T]$. Since $Y_k(t)$ is uniformly bounded by (6.7), then by the Arzela-Ascoli theorem there is a subsequence (k_l) such that Y_{k_l} converges uniformly to a continuous function Y on $[a,T]$.

To complete the proof of the Lemma we need to prove that $Y(t)$ is a solution of (6.5). As in the proof of Theorem 4.2 we can show that $Y(t)$ is a solution of the equation

$$Y(t) = \mathcal{U}(t+T)Y(-T) + \int_{-T}^{t} \mathcal{U}(t-s)f(Y(s)+g(s))ds, \; t \geq -T.$$

Then it is enough to prove that

$$Y(-T) = \int_{-\infty}^{-T} \mathcal{U}(-T-s)f(Y(s)+g(s))ds. \tag{6.8}$$

But

$$\begin{aligned} Y_k(-T) &= \int_{-k}^{-T} \mathcal{U}(-T-s)f(Y_k(s)+g(s))ds \\ &= \int_{-\infty}^{-T} \mathcal{U}(-T-s)f(Y_k(s)+g(s))1_{[-k,-T]}(s)ds. \end{aligned}$$

By Hypothesis 6.1(c)

$$\|\mathcal{U}(-T-s)f(Y_k(s)+g(s))1_{[-k,-T]}(s)\|$$

is dominated by an integrable function. Since $Y_{k_l}(s) \to Y(s)$ and since f is continuous, then by the dominated convergence theorem we get (6.8). Q.E.D

Lemma 6.3 *Suppose $\mathcal{U}, f,$ and g_i satisfy the conditions of Lemma 6.2. If X_i, $i = 1,2$ are solutions of*

$$Y_i(t) = \int_{-\infty}^{t} p(t-s)f(Y_i(s)+g(s))ds,$$

then there is a constant $C(T) > 0$ such that

$$\|Y_2 - Y_1\|_T^2 \leq C(T, g_1, g_2)\left(\int_{-\infty}^{t} e^{2\lambda s}\|g_2(s)-g_1(s)\|^2 ds\right)^2. \tag{6.9}$$

Proof: Define

$$Y_i^k(t) = \int_{-k}^{t} \mathcal{U}(t-s)f(Y_i^k(s)+g(s))ds, \quad i = 1,2.$$

By Theorem 5.1 we have

$$\|Y_2^k(t) - Y_1^k(t)\| \leq 2e^{(-\lambda_0+1)t}I\left(\int_{-k}^{t} e^{2\lambda_0 s}\|g_2(s)-g_1(s)\|^2 ds\right)^{\frac{1}{2}}, \tag{6.10}$$

where

$$I = 2\left(\int_{-k}^{t} e^{2\lambda_0 s}\|f(Y_2^k(s)+g_2(s)) - f(Y_1^k(s)+g_1(s))\|^2 ds\right)^{\frac{1}{2}}.$$

First we show that I is uniformly bounded in k. Because $\|f(x)\| \leq C(1+\|x\|^p)$ and $\int_{-\infty}^{T} e^{2\lambda_0 s}\|g_i(s)\|^{2p^2}ds < \infty$, it is enough to show that $\int_{-k}^{T} e^{2\lambda_0 s}\|Y_k^k(s)\|^{2p}ds$ is uniformly bounded in k. But by (4.7)

$$\|Y_i^k(t)\| \leq e^{-\lambda_0 t}\int_{-\infty}^{t} e^{\lambda_0 s}(1+\|g_i(s)\|^p)ds, \quad i=1,2.$$

By using Fubini's Theorem we can show that

$$\int_{-\infty}^{T} e^{2\lambda_0 s}\|Y_i^k(s)\|^{2p}ds \leq \int_{-\infty}^{T} e^{2p\lambda_0 u}(1+\|g_i(u)\|^{2p^2})\left(\int_{u}^{T} e^{2\lambda_0(1-p)s}ds\right)du.$$

Then $\int_{-k}^{T} e^{2\lambda_0 s}\|Y_i^k(s)\|^{2p}ds$ is uniformly bounded in k, so there is $C_1(T)$ such that $I \leq C_1(T)$, and we can rewrite (6.10) as

$$\|Y_2^k(t)-Y_1^k(t)\|^2 \leq 2C_1(T)e^{(-2\lambda_0+1)t}\left[\int_{-\infty}^{t} e^{2\lambda_0 s}\|g_2(s)-g_1(s)\|^2 ds\right]^{\frac{1}{2}}.$$

Since by the proof of Lemma 6.2 $Y_i^{k_l}(t) \to Y_i(t)$, then by taking the limit over the subsequence (k_l) and taking the supremum on $[-T,T]$ we get (6.9). Q.E.D

Remark 6.1 *Let $\phi(g) := \int_{-\infty}^{T} e^{\lambda_0 s}\|g(s)\|^r ds$ for $g \in K$. Then:*

(i) *if $\phi(g_i) \leq N$, $i=1,2$, there is a constant $C_N > 0$ such that*

$$\|Y_2-Y_1\|_T^2 \leq C_N\left[\int_{-\infty}^{T} e^{2\lambda_0 s}\|g_2(s)-g_1(s)\|^2)ds\right]^{\frac{1}{2}}. \tag{6.11}$$

(ii) *By Theorem 5.2 equation (6.4) has a unique cadlag adapted solution, and by (i) there is a constant $C_N > 0$ such that on the set where $\phi(V_i) \leq N$, $i=1,2$,*

$$\|X_2-X_1\|_T^2 \leq C_N\left(d_K(V_2,V_1)\right)^{\frac{1}{2}}, \tag{6.12}$$

where $d_K(\cdot,\cdot)$ is a metric on K.

(iii) *There is a continuous mapping $\psi_N : \mathbf{R}\times K \to D(\mathbf{R},H_n)$ such that if $X_n(t)$ is the solution of (6.4), then $X_n(t) = \psi(t,V_n(\cdot))(t)$ on the set $\{\phi(V_n) < N\}$.*

6.3 The Main Theorem

Theorem 6.1 *If f and V satisfy Hypothesis 4.5 and if V satisfies Hypothesis 6.1, then the solution of (6.1) is a stationary processes.*

Proof: Since V_t is an H-valued stationary process then $V_n(t) := J_n V(t)$ is also an H_n-valued stationary process. From (6.4) we have

$$X_n(t+h) = \int_{-\infty}^{t+h} U_n(t+h-s)f(X_n(s))ds + V_n(t+h);$$

by changing variables we see this is

$$\int_{-\infty}^{t} U_n(t-s)f(X_n(s+h))ds + \theta_h V_n(t).$$

Then by Remark 6.1 we have $X_n(t+h) = \psi_N(t,(0,(\theta_h V_n))(t)$ on the set $\{\phi(v) < N\}$, and in particular $X_n(h) = \psi_N(0,(\theta_h V_n))(0)$ on the set $\{\phi(v) < N\}$. But by Lemma 6.1 $\theta_h V_n$ is a $D(\mathbf{R}, H_n)$-valued stationary process; since $\varphi(f) = \psi_N(0,f)(0)$ is a continuous function from K to H_n then $X_n(t) = \psi(\theta_t V_n) = \psi_N(0,(\theta_t V_n)(0)$ is an H_n-valued stationary process. Since $X(t)$ is the limit of $X_n(t)$ by Lemma 6.3, then $\{X(t) : t \in \mathbf{R}\}$ is also a stationary process.

Q.E.D

6.4 The Einstein-Smoluchowski Equation

Now consider (6.3) where $-\nabla F(x)$ satisfies Hypothesis 4.5. The stationary solution of (6.3) satisfies the following integral equation:

$$X(t) = \frac{-1}{2}\int_{-\infty}^{t} U(t-s)\nabla F(X(s))ds + \int_{-\infty}^{t} U(t-s)dW(s). \qquad (6.13)$$

By Theorem 5.3 the solution of (6.13) is a limit of solutions of the finite dimensional equations

$$X_n(t) = \frac{-1}{2}\int_{-\infty}^{t} U_n(t-s)\nabla F(X_n(s))ds + \int_{-\infty}^{t} U_n(t-s)dW(s). \tag{6.14}$$

The stationary distribution of (6.14) is well-known from Kolmogorov (1937) and can be given explicitly [see Marcus (1974), (1978)]. But instead of (6.14) we are interested in a slightly different equation. Consider

$$Y_n(t) = \frac{-1}{2}\int_{-\infty}^{t} U_n(t-s)\nabla F(J_nY_n(s))ds + \int_{-\infty}^{t} U_n(t-s)dW(s) \tag{6.15}$$

It is clear that $J_nY_n(t) = X_n(t)$. Since $Y_n(t) = J_nY_n(t) + (Y_n(t) - J_nY_n(t))$ and

$$Y_n(t) - J_nY_n(t) = \int_{-\infty}^{t}(I-J_n)U(t-s)(I-J_n)dW(s)$$

and $X_n(t) \to X(t)$, then we have $Y_n(t) \to X(t)$. By Theorem 6.1 $Y_n(t)$ is a stationary process. Let M be the stationary Gaussian measure of $\int_{-\infty}^{t} U(t-s)dW(s)$ on H. Then

Lemma 6.4 *If U and -$\nabla F(x)$ satisfy Hypothesis 4.5, the stationary distribution of $Y_n(t)$ has a Radon-Nikodym derivative* $\exp(-F(J_n.))\int_H \exp(-F(.))dM(.)$ *with respect to M on H.*

Proof: See Marcus (1978), Lemma (10).

Now we can prove

Theorem 6.2 *If U and -$\nabla F(x)$ satisfy Hypothesis 4.5, then the distribution of the solution $X(t)$ of (6.13) has the Radon-Nikodym derivative* $\exp(-F((.))\int_H \exp(-F(.))dM(.)$ *with respect to M on H.*

Proof: Since $E(\|Y_n(t) - X(t)\|) \to 0$ it is sufficient to show that

$$\lim_{n\to\infty}\int_H |\exp(-F(x)) - \exp(-F(J_nx))|dM(x) = 0$$

since this implies weak convergence. Note that $\lim_{n\to\infty} F(J_n.) = F(.)$ on the set with M-measure equal to 1.

Without loss of generality let $V(0) = 0$. Then the monotonicity of $\nabla F(x)$ ensures that F is nonnegative and $\exp(-F(.)) \leq 1$. The Lebesgue bounded convergence theorem can now be applied to show that the limit of the integral is equal to 0.

Q.E.D

Chapter 7

THE GENERAL SEMILINEAR EQUATION

7.1 Introduction

Let H and K be two real separable Hilbert spaces. Let $L_2(K, H)$ be the space of Hilbert-Schmidt operators from K to H with Hilbert-Schmidt norm $\| \|_2$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete stochastic basis with a right continuous filtration. Let W_t be cylindrical Brownian motion on K with respect to $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let $g : \mathbf{R}^+ \times \Omega \times C(\mathbf{R}^+, H) \to L_2(K, H)$ be a predictable functional on the H-valued continuous adapted processes. We say g is a *predictable functional* if, whenever X and Y are H-valued continuous adapted processes and τ is a stopping time such that $X1_{[0,\tau)} = Y1_{[0,\tau)}$, then $1_{[0,\tau]}g(.,.,X) = 1_{[0,\tau]}g(.,.,Y)$. See Métivier and Pellaumail (1980b).

Consider a semilinear stochastic evolution equation of the form

$$dX_t = A(t)X_t dt + f_t(X_t)dt + g_t(X_.)dW_t \tag{7.1}$$

with initial condition $X(0) = X_0$.

In the case when f and g are Lipschitz, the existence and uniqueness of the solution of (7.1) has been studied using semigroup theory [see for example Kotelenz (1982, 1984)]. In this chapter we will use semigroup theory to prove the existence and uniqueness of the solution of (7.1), when $-f$ is semimonotone.

Let us write the *mild* form of (7.1) as the integral equation:

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)f_s(X_s)ds + \int_0^t U(t,s)g_s(X)dW_s.$$

We are going to study a slightly more general equation:

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)f_s(X_s)ds + \int_0^t U(t,s)g_s(X)dW_s + V_t, \qquad (7.2)$$

where V_t is a continuous adapted process.

The following are the relevant hypotheses concerning X_0, f, g, A, U and V:

Hypothesis 7.1 *There exists a set $G \subseteq \Omega$ of probability one and constants $q \geq 1$ and $C > 0$ with the following properties:*

(a) *f satisfies Hypothesis 4.2, with $\varphi(x) = C(1 + x^q)$, $x \in \mathbf{R}^+$ and the constant M is independent of ω in Ω;*

(b) *$g : \mathbf{R}^+ \times \Omega \times D(\mathbf{R}^+, H) \to L_2(K,H)$ is a predictable functional on the H-valued continuous adapted processes.*

(c)

$$\|g(s,\omega,X) - g(s,\omega,Y)\|_2 \leq C\sup_{0\leq s\leq t}\|X_t - Y_t\| = C(X-Y)^{\star}_t,$$

for all $t \in S$, $\omega \in G$, $X, Y \in C(S,H)$;

(d) *A, and U satisfy Hypotheses 4.1 and Hypotheses 3.1;*

(e) *$V = \{V_t : t \in S\}$ is an H-valued continuous adapted process.*

(f) *X_0 is an H-valued $\mathcal{F}_0$-measurable random variable ;*

(g) *for all $p \geq 1$ and all $t \in S$ $E\{\|X_0\|^p\}$, $E\{(V_t^{\star})^p\}$ and $E\{\sup_{0\leq s\leq t}\|g_s(0)\|_2^p\}$ are finite.*

From this chapter and the following chapter C will denote a positive constant whose exact value is unimportant and may change from line to line.

7.2 The Main Theorem

Theorem 7.1 *If Hypothesis 7.1 is satisfied, then the integral equation (7.2) has a unique* *continuous strong adapted solution X with*

$$E\{(X_t^*)^p\} < \infty \quad \textit{for all} \quad p \geq 1 \quad \textit{and} \quad t \in S.$$

Before proving Theorem 7.1 we are going to prove

Lemma 7.1 *Suppose that there is a unique solution to (7.2) in the case where $X_0 \equiv 0$, $g(s,\omega,0) \equiv 0$ and $\lambda \equiv 0$ in Hypothesis 3.1(c). Then there is a unique solution in the general case.*

Proof: We will prove this in two stages. First define $\bar{g}_s(x) = g_s(x) - g_s(0)$ and set

$$\bar{V}_t = U(t,0)X_0 + \int_0^t U(t,s)g_s(0)dW_s + V_t.$$

Then we can rewrite equation (7.2) as

$$X_t = \int_0^t U(t,s)f_s(X_s)ds + \int_0^t U(t,s)\bar{g}_s(X)dW_s + \bar{V}_t. \tag{7.3}$$

Note that $\bar{g}(s,0) = 0$ and that there is no X_0 term on the right hand side of equation (7.3). We claim that f, $\bar{g}$ and $\bar{V}$ satisfy Hypothesis 7.1. Indeed, f, A, U have not changed so (a) and (d) still hold; $g_s(0)$ is predictable and

$$\bar{g}(s,x) - \bar{g}(s,y) = g(s,x) - g(s,y)$$

so (b) and (c) still hold. (f) is trivial since there is no X_0 term. We need only check $\bar{V}$ to verify (e) and (g).

Since $g_s(0)$ is an $L_2(K,H)$-valued predictable process which satisfies Hypothesis 7.1(b), and W_t is a K-valued cylindrical Brownian motion, then $\int_0^t g_s(0)dW_s$ is an H-valued continuous local martingale with quadratic variation $\int_0^t \|g_s(0)\|_2^2 ds$ [see Yor (1974)]. By Proposition 3.1 the stochastic convolution integral

$$\int_0^t U(t,s)g_s(0)dW_s$$

is adapted and continuous in t. By Burkholder's inequality (Theorem 3.2) we have

$$E\left\{\sup_{0\le t\le T}\left\|\int_0^t U(t,s)g_s(0)dWs\right\|^p\right\} \le K_p.$$

Then

$$E\left\{\left(\int_0^t \|g_s(0)\|_2^2\, ds\right)^p\right\} \le TK_pE\left\{\sup_{0\le s\le T}\|g_s(0)\|_2^{2p}\right\} < \infty$$

for all $p \ge 1$ by Hypothesis 7.1(g).

Thus the stochastic convolution integral is L^p bounded for all $p \ge 1$. Next, $U(t,0)X_0$ is adapted, continuous and L^p-bounded by (f), (g) and Hypothesis 3.1(c). Finally, V_t is continuous and L^p-bounded by (e) and (g), hence $\bar{V}$ also satisfies (e) and (g). Since (7.3) and (7.2) are the same equation — only the notation has been changed— then (7.2) has a unique solution iff (7.3) does. Finally, by Lemma 3.1, the map $X \to X_1$ reduces (7.3) to an equivalent equation which $\lambda = 0$ in Hypothesis 3.1(c).

Q.E.D

Proof of Theorem

•Uniqueness:

By Lemma 7.1 we may assume X_0, $g(s,\omega,0)$ and λ are zero. Let X and Y be two adapted continuous solutions of (7.2). Then we have

$$\begin{aligned} X_t - Y_t &= \int_0^t U(t,s)(f_s(X_s) - f_s(y_s))ds \\ &\quad + \int_0^t U(t-s)(g_s(X) - g_s(Y))dW_s. \end{aligned}$$

By Theorem 3.1 (Ito's inequality)

$$\begin{aligned} \|X_t - Y_t\|^2 &\le 2\int_0^t < X_s - Y_s, f_s(X_s) - f_s(Y_s) > ds \\ &\quad + 2\int_0^t < X_s - Y_s, (g_s(X) - g_s(Y))dW_s > \\ &\quad + \int_0^t \|(g_s(X) - g_s(Y))\|_2^2 ds. \end{aligned} \tag{7.4}$$

Since $-f$ is semimonotone,the first term of the right hand side is bounded by

$$2M\int_0^t \|X_s - Y_s\|^2 ds \le 2M\int_0^t ((X-Y)_s^\star)^2\, ds,$$

and the second term is bounded by

$$2\sup_{0\le r\le t}\left|\int_0^r < X_s - Y_s\,, (g_s(X) - g_s(Y))dW_s > \right|,$$

and by Hypothesis 7.1(c) the third term is bounded by $C\int_0^t((X-Y)_s^\star)^2 ds$. Define the stopping time

$$T_n := inf\{t : \|X_t\| + \|Y_t\| > n\} \wedge T.$$

Then from (7.4) and the above

$$\begin{aligned} & E\left\{\left((X-Y)_{t\wedge T_n}^\star\right)^2\right\} \\ \le\ & (2M + C)\, E\left[\int_0^t \left((X-Y)_{s\wedge T_n}^\star\right)^2 ds\right] \\ & + 2E\left[\sup_{0\le r\le t\wedge T_n}\left|\int_0^r < X_s - Y_s, (g_s(X) - g_s(Y))dW_s > \right|\right]. \end{aligned} \tag{7.5}$$

The expectations are all finite since $\|X_t\|$ and $\|Y_t\|$ are bounded on $[0, T_n]$. By using Fubini's theorem on the first term, and Lemma 3.2 with $p = 1$ and $K = 2C_1$ on the second term we have

$$\begin{aligned} E\left\{\left((X-Y)_{t\wedge T_n}^\star\right)^2\right\} \ \le\ & C\int_0^t E\left\{\left((X-Y)_{s\wedge T_n}^\star\right)^2 ds\right\} \\ & + \frac{1}{2}E\{((X-Y)_{t\wedge T_n}^\star)^2\} \\ & + C\int_0^t E\left\{\left((X-Y)_{s\wedge T_n}^\star\right)^2 ds\right\}, \end{aligned} \tag{7.6}$$

so

$$\frac{1}{2}E\{((X-Y)_{t\wedge T_n}^\star)^2\} \le C\int_0^t E\left\{\left((X-Y)_{s\wedge T_n}^\star\right)^2\right\} ds.$$

By Gronwall's inequality,

$$E\{(X-Y)_{t\wedge T_n}^\star\}^2 = 0, \quad \forall n.$$

But $P\{T_n = T\} \to 1$ so $X_t = Y_t$ a.s.

Q.E.D

• **Existence:**

By Lemma 7.1 we can assume that X_0, $g(s,\omega,0)$ and λ are zero. We proceed as in Pardoux (1975). Define $X_t^0 \equiv 0$ and define X_t^n by induction.

Suppose for $k = 0,\ldots,n$ that X_t^k is an adapted continuous process such that $(X^k)_t^* \in L^p$ for all $p \geq 0$. Define

$$V_t^k = V_t + \int_0^t U(t,s) g_s(X^k) dW_s + M \int_0^t U(t,s) X_s^k ds. \tag{7.7}$$

Lemma 7.2 *For $k \leq n$, V^k is an adapted continuous process, and $(V^k)_t^* \in L^p$ for all $p \geq 0$.*

Proof: The stochastic integral exists since $g_s(X^k)$ is predictable and

$$\|g_s(X^k)\|_2 \leq C(X^k)_s^*$$

by Hypothesis 7.1(c) and the fact that $g_s(0) = 0$. But $(X^k)_s^* \in L^p$ so $\|g_s(X^k)\|_2 \in L^p$. Set

$$M_t^k = \int_0^t g_s(X^k) dW_s.$$

This is a continuous H-valued martingale with quadratic variation $[M^k]_t$. By Proposition 3.1, the stochastic convolution integral in (7.7) is adapted and continuous in t, and since $(X^k)_t^* \in L^p$ for all $p \geq 0$ then by Theorem 3.2

$$E\{\sup_{r \leq t} \| \int_0^r U(r,s) dM_s^k \|^{2p}\} < \infty,$$

for all $p \geq 0$, and $E\{\sup_{r \leq t} \| \int_0^r U(r,s) X_s^k ds \|^{2p}\} < \infty$. Since V satisfies Hypothesis 7.1(e) and (g) the lemma is proved.

Q.E.D

Now consider

$$X_t^{n+1} = \int_0^t U(t,s)\bar{f}_s(X_s^{n+1})ds + V_t^n, \tag{7.8}$$

where for all $x \in H$ $\bar{f}_s(x) := f_s(x) - Mx$. Note that $\bar{f}$ and V^n satisfy the hypotheses of Corollary 4.1 and $-\bar{f}$ is monotone, so (7.8) has a unique cotinuous adapted solution which satisfies

$$\|X_t^{n+1}\| \le \|V_t^n\| + \int_0^t \|\bar{f}_s(V_s^n)\|ds. \tag{7.9}$$

Since f is dominated by $\varphi(x) = C(1+x^q)$, then $\bar{f}$ is dominated by $2C(1+x^q)$, so

$$(X^{n+1})_t^\star \le (V^n)_t^\star + 2tC(1+(V^n)_t^{\star q}).$$

Since $(V^n)_t^\star \in L^p$ for all n, p and $t \in S$ then

$$(X^n)_t^\star \in L^p, \quad \text{for all} \quad n, \quad p \quad \text{and} \quad t. \tag{7.10}$$

Let

$$N_t^n = \int_0^t (g_s(X^n) - g_s(X^{n-1})dW_s$$

and note that

$$\begin{aligned} X_t^{n+1} - X_t^n &= \int_0^t U(t,s)[\bar{f}_s(X_s^{n+1}) - \bar{f}_s(X_s^n)]ds \\ &\quad + M\int_0^t U(t,s)(X_s^n - X_s^{n-1})ds + \int_0^t U(t,s)dN_s^n. \end{aligned} \tag{7.11}$$

Moreover

$$\begin{aligned} d[N^n]_t &\le \left\|g_s(X^n) - g_s(X^{n-1})\right\|_2^2 dt \\ &\le C\left(\left(X^n - X^{n-1}\right)_t^\star\right)^2 dt, \end{aligned}$$

so by the Ito inequality of Chapter 3,

$$\begin{aligned}
\left\|X_t^{n+1}-X_t^n\right\|^2 &\leq 2\int_0^t < X_s^{n+1}-X_s^n, \bar{f}_s(X_s^{n+1})-\bar{f}_s(X_s^n) > ds \\
&\quad + 2M\int_0^t < X_s^{n+1}-X_s^n, X_s^n - X_s^{n-1} > ds \\
&\quad + 2\int_0^t < X_s^{n+1}-X_s^n, dN_s^n > \\
&\quad + C\int_0^t \left\{\left(X^n - X^{n-1}\right)_s^\star\right\}^2 ds \\
&:= I_1(t)+I_2(t)+I_3(t) \\
&\quad + C\int_0^t \left\{\left(X^n - X^{n-1}\right)_s^\star\right\}^2 ds. \qquad (7.12)
\end{aligned}$$

Now $-\bar{f}$ is monotone, so $I_1(t) \leq 0$. We can bound I_2:

$$\begin{aligned}
I_2 &\leq 2M\int_0^t \|X_s^{n+1}-X_s^n\|\,\|X_s^n - X_s^{n-1}\| ds \\
&\leq \frac{1}{2}\left\{\left(X^{n+1}-X^n\right)_t^\star\right\}^2 + 2M\int_0^t \left\{\left(X^n - X^{n-1}\right)_s^\star\right\}^2 ds.
\end{aligned}$$

By Lemma 3.2, for any $K > 0$,

$$\begin{aligned}
E\left\{(I_3^\star(t))^p\right\} &\leq \frac{C}{K}E\left\{\left(\left(X^{n+1}-X^n\right)_t^\star\right)^{2p}\right\} \\
&\quad + KC\, E\left\{\int_0^t \left(\left(X^n - X^{n-1}\right)_s^\star\right)^{2p} ds\right\}.
\end{aligned}$$

Using the bounds of $I_1(t)$ and $I_2(t)$ we can rewrite (7.12) as

$$\left\{\left(X_t^{n+1}-X^n\right)_t^\star\right\}^2 \leq C\int_0^t \left\{\left(X^n - X^{n-1}\right)_s^\star\right\}^{2p} ds + 2I_3^\star(t).$$

Using the above and the bound of $I_3^\star(t)$, there is $C > 0$ such that

$$\begin{aligned}
E\left\{\left((X^{n+1}-X^n)_t^\star\right)^{2p}\right\} &\leq C\Big[(K+1)\int_0^t E\left\{\left((X^n - X^{n-})_s^\star\right)^{2p}\right\} ds \\
&\quad + \frac{1}{2K}E\left\{\left(X^{n+1}-X^n\right)_t^\star\right\}^{2p}\Big].
\end{aligned}$$

Set

$$h_n(t) = E\left\{\left((X_t^{n+1}-X^n)_t^\star\right)^{2p}\right\}$$

and choose $K = C$. The $h_n(t)$ are finite by (7.10), so we can subtract:

$$\frac{1}{2}h_n(t) \leq C(C+1)\int_0^t h_{n-1}(s)ds.$$

Then from (7.10) there exists $D_0 \geq 0$ such that

$$h_0(t) \leq D_0 \quad \text{if} \quad t \leq T,$$

and if $D = 2C(C+1)$, then

$$h_n(t) \leq D\int_0^t h_{n-t}(s)ds.$$

By induction

$$h_n(t) \leq D_0\frac{(Dt)^n}{n!}.$$

Thus

$$\sum(h_n(t))^{\frac{1}{2p}} < \infty,$$

and we conclude that (X^n) is a Cauchy sequence in $L^{2p}(\Omega, C(S,H))$ for all $p \geq 1$. Take $p = 1$: there exists a process $\{X_t, \quad 0 \leq t \leq T\}$ such that

$$\lim_{n\to\infty} E\{\sup_{t\in S}\|X_t - X_t^n\|^2\} = 0.$$

Then $t \to X_t$ is continuous (it is the uniform limit of continuous functions) and adapted. Moreover

$$E\{(X_t^*)^p\} < \infty, \ \forall p \geq 1.$$

We must show it satisfies the equation (7.3). Consider

$$R(t) := X_t - \int_0^t U(t,s)f_s(X_s)ds - \int_0^t U(t,s)g_s(X)dW_s - V_t.$$

R is well-defined, for both integrals make sense. It is continuous in t. Let us also consider

$$\begin{aligned} R_n(t) \quad := \quad & X_t^{n+1} - \int_0^t U(t,s) f_s(X_s^{n+1}) ds \\ & + M \int_0^t U(t,s)(X_s^{n+1} - X_s^n) ds \\ & - \int_0^t U(t,s) g_s(X^n) dW_s - V_t. \end{aligned}$$

($R_n(t) = 0$, of course). Let $x \in H$. We claim that $< x, R(t) >= 0$ a.s. This will do it since then for all $x \in H$, $< x, R(t) >= 0$ a.s., which implies $R(t) = 0$ a.s. Let t range over all rationals and use the continuity of R to see that $R(t) = 0$, for all t w.p.1.

First

$$E\{\|X_t^{n+1} - X_t\|^2\} \to 0 \Rightarrow E\{< X, X_t^{n+1} - X^t >^2\} \to 0$$

or $< X, X_t^{n+1} > \to < X, X_t >$ in L^2.

Next:

$$\begin{aligned} < x, \int_0^t U(t,s) f_s(X_s^n) ds > \quad &= \quad \int_0^t < x, U(t,s) f_s(X_s^n) > ds \\ &= \quad \int_0^t < U^*(t,s)x, f_s(X_s^n) > ds \end{aligned}$$

Let $y_s = U^*(t,s)x$ (U^* is the adjoint of U, not the sup, here).

Now $f_s(z)$ is demicontinuous in z, hence $z \to < y_s, f_s(z) >$ is continuous. Since $X_s^n \to X_s$ in L^2, it also converges in probability, so that

$$< y_s, f_s(X_s^n) > \to < y_s, f_s(X_s) > \qquad \text{in probability.}$$

Since $X_s^n \to X_s$ in $L^2(\Omega, C(S,H))$, then there is a subsequence (n_k) such that

$$(X^{n_k})_t^* \to X_t^* \quad \text{w.p.1}$$

Then for large enough k,

$$(X^{n_k})_t^* \leq X_t^* + 1 < \infty.$$

The convergence is bounded, since

$$\begin{aligned}
< y_s, f_s(X_s^{n_k}) > &\leq \|y_s\|\varphi(\|X_s^{n_k}\|) \\
&\leq \|y_s\|\varphi((X^{n_k})_t^\star) \\
&\leq \|y_s\|\varphi(X_t^\star + 1).
\end{aligned}$$

We can go to the limit under the integral to see

$$< x, \int_0^t U(t,s) f_s(X_s^{n_k}) ds > \to < x, \int_0^t U(t,s) f_s(X_s) ds > .$$

Since X_s^n converges to X_s in $L^2(\Omega, C(S,H))$,

$$\int_0^t U(t,s)(X_s^{n+1} - X_s^n) ds \to 0 \quad \text{in} \quad L^2.$$

The third integral also converges in L^2, since

$$\begin{aligned}
E\left\{\| \int_0^t U(t,s)\left(g_s(X^n) - g(X)\right) dW_s\|^2\right\} &\leq E\left\{\int_0^t \|g_s(X^n) - g(X)\|_2^2 ds\right\} \\
&\leq CtE\{((X^n - X)_t^\star)^2\} \to 0.
\end{aligned}$$

The last term doesn't depend on n. Thus

$$< x, R(t) > = \lim_n < x, R_n(t) > = 0.$$

Q.E.D

Note: the proof above gives L^2-bounds on X_t:

$$\begin{aligned}
\|X_t\|_{L^2} &\leq \sum_0^n \|X^{n+1} - X^n\|_{L^2} \\
&\leq \sum \sqrt{h(t)} \\
&\leq D_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(DT)^{n/2}}{\sqrt{n!}}.
\end{aligned}$$

Remark 7.1 *Theorem 7.1 remains valid if we replace the cylindrical Brownian motion W by a K-valued Brownian motion $\tilde{W}_t$, and if we let the predictable functional g be in $L(K,H)$ instead of $L_2(K,H)$.*

Proof: This comes from the fact that a K-valued Brownian motion $\tilde{W}_t$ has a covariance Q which is nuclear [see Métivier (1982)], so we can write $\tilde{W}_t = Q^{\frac{1}{2}}W_t$, where W_t is a cylindrical Brownian motion on K. Now $Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on K so if $g_s(x)$ is $L(K,H)$-valued then $g_s(\cdot)Q^{\frac{1}{2}}$ is a $L_2(K,H)$-valued predictable functional of Hypothesis 7.1, so we can apply Theorem 7.1. Q.E.D

7.3 Some Examples

Example (7.1): Let D, A, B, ∂D, and W be as in Example (4.2). Consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f_t(u) + g_t(u)\dot{W} & \text{on } D \times [0,\infty) \\ Bu = 0 & \text{on } \partial D \times [0,\infty) \\ U(0,X) = 0 & \text{on } D. \end{cases} \tag{7.13}$$

Since $\dot{W}$ can be considered as a Brownian motion $\tilde{W}_t$ on a Sobolev space H_p, $p > \frac{d}{2}$ [See Walsh (1986), Chapter 4, Page 4.11], we can let $K = H_p$, for some $p > \frac{d}{2}$ and let H be the Sobolev space H_n for a fixed $n \in Z$.

Let $g_s(.) : D(S, H_n) \to L(H_p, H_n)$ satisfy Hypothesis 7.1(b), (c), and (g). Let $f_s(.) : H_n \to H_n$ satisfy Hypothesis 7.1(a) and rewrite (7.13) as

$$du_t = -Au_t + f_t(u)dt + g_t(u)d\tilde{W}_t.$$

Since $-A$, and U satisfy Hypothesis 7.1(d) then by Remark 7.1 there is a unique continuous solution with values in H_n.

Definition 7.1 *An $\mathbf{R}^m$-valued function $f(x,u)$ of two variables $x \in D \subset \mathbf{R}^d$, $u \in \mathbf{R}^N$ is said to satisfy the Caratheodory condition if it is continuous with respect to u for almost all $x \in D$ and measurable with respect to x for all values of u.*

Example 7.2 (Zakai Equation) Let D, A, B, and ∂D be as in Example (7.1). Let W_i, $i = 1, \ldots, l$ be independent standard scalar Brownian motions.

Now consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au &= f(x, u(t,x)) + \sum_{i=l}^{l} g_i(x, u(t,x))\dot{W}_i(t) \quad \text{on } D \times [0, \infty), \\ Bu &= 0 \quad \text{on } \partial D \times [0, \infty), \\ u(0,x) &= 0 \quad \text{on } D, \end{cases} \tag{7.14}$$

where f and g_i satisfy the following:

Hypothesis 7.2 (a) *$f, g_i : D \times \mathbf{R} \to \mathbf{R}$, $i = 1, \ldots, l$ satisfy the Caratheodory condition;*

(b) *there exists a function $a \in L^2(D)$ and a constant $C > 0$ such that*

$$|f(x,u)| \leq a(x) + C|u|, \quad u \in \mathbf{R}, \quad x \in D \subset \mathbf{R}^d,$$

$$|g(x,u)| \leq a(x) + C|u|, \; i = 1, \ldots, l;$$

(c) *the $g_i(x, .)$, $i = 1, \ldots, l$ are uniformly Lipschitz i.e. there is a constant $C > 0$ such that*

$$|g_i(x, u_2) - g_i(x, u_1)| \leq C|u_2 - u_1|, \quad \forall x \in D, \quad u_2, u_1 \in \mathbf{R}, \quad i = 1, \ldots, l;$$

(d) *$-f(x, .)$ is semimonotone i.e. $\exists M > 0$ such that*

$$(f(x, u_2) - f(x, u_1))(u_2 - u_1) \leq M(u_2 - u_1)^2.$$

Define $H = L^2(D)$ and let $\| \quad \|$ be the L^2-norm. By Example 4.1, the operator A (with boundary conditions) generates a contraction semigroup $U(t)$ on H. Define $\bar{g}_i$ and $\bar{f} : L^2(D) \to L^2(D)$ by:

$$(\bar{f}(u))(x) = f(u(x))$$

$$(\bar{g}_i(u))(x) = g_i(u(x)),$$

$$u \in L^2(D), \quad x \in D \subset \mathbf{R}^d \quad \text{and} \quad i = 1, \ldots, l.$$

Since f and g_i satisfy Hypothesis 7.2(a) and (b), then by Theorem (2.1) of Krasnoel'skii (1964), $\bar{f}$ and $\bar{g}_i \quad i = 1, \ldots, l$ are continuous and there is $C > 0$ such that

$$\|\bar{f}(u)\| \leq C(1 + \|u\|) \quad \text{and} \quad \|\bar{g}_i(u)\| \leq C(1 + \|u\|).$$

and since g_i, $i = 1, \ldots, l$ satisfy Hypothesis 7.2(c), then

$$\begin{aligned}
\|\bar{g}_i(u_2) - \bar{g}_i(u_1)\|^2 &= \int_D [g_i(x, u_2(x)) - g_1(x, u_1(x))]^2 dx \\
&\leq C^2 \int_D (u_2(x) - u_1(x))^2 dx \\
&= C^2 \|u_2 - u_1\|^2.
\end{aligned}$$

Since f satisfies Hypothesis 7.2(d), then

$$\begin{aligned}
< f(u_2) - f(u_1), u_2 - u_1 > &= \int_D (f(x, u_2(x)) - f(x, u_1(x))\,(u_2(x) - u_1(x))\,dx \\
&\leq M \int_D (u_2(x) - u_1(x))^2\,dx \\
&= M\,\|u_2 - u_1\|^2.
\end{aligned}$$

Define a map $\bar{g} = (\bar{g}_1, ., \bar{g}_l)$ from $H = L^2(D)$ to $(L^2(D))^l \simeq L(\mathbf{R}^l, L^2(D))$. Then $K = \mathbf{R}^l$ and we can write (7.14) as

$$\begin{cases} du(t) &= -Au(t)dt + \bar{f}(u(t))dt + \sum_{i=1}^{l} \bar{g}_i(u(t))dW_i(t) \\ u(0) &= 0. \end{cases} \tag{7.15}$$

Since $-A$, U, $\bar{f}$,and $\bar{g}$ satisfy the conditions of Remark 7.1, there is a unique mild continuous adapted solution of (7.15) with values in $H = L^2(D)$ i.e. the SPDE (7.14) has a unique continuous mild solution with values in $L^2(D)$.

7.4 Initial-Value Problem of the Semilinear Hyperbolic System

Example (7.3) Consider the following initial-value problem of the system of semilinear stochastic partial differential equations

$$\begin{cases} \frac{\partial u}{\partial t} &= \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j} + b(x)u + f(x,u) + g(x,u)\dot{W}, \\ u(0,x) &= u_0(x), \quad u_0(x) \in L^2(\mathbf{R}^n)^N, \quad x \in \mathbf{R}^n, \end{cases} \tag{7.16}$$

where $\dot{W}$ is an m-dimensional Brownian motion, $u = (u_1, ., u_N)^t$ (the superscript t denotes a transpose) is the set of unknowns, and for each j and x, $a_j(x)$ and $b(x)$ are square matrices of order N. We will assume the following:

Hypothesis 7.3 **(a)** *The matrices $a_j(x)$, for $j = 1, ., n$, and $x \in \mathbf{R}^n$ are symmetric;*
(b) *each component of a_j, $j = 1, \ldots, n$ and its first order derivatives are continuous and bounded, and b is continuous and bounded;*
(c) *$f : \mathbf{R}^n \times \mathbf{R}^N \to \mathbf{R}^N$ satisfies the Caratheodory conditions;*
(d) *$-f$ is semimonotone in the second variable i.e. $\exists M > 0$ such that for all $x \in D = \mathbf{R}$ and for all $u_1, u_2 \in \mathbf{R}^N$ one has*

$$< f(x,u_2) - f(x,u_1), u_2 - u_1 >\leq M\|u_2 - u_1\|^2;$$

(e) *there exist a function $a \in L^2(\mathbf{R}^n)$ and a constant $C > 0$ such that*

$$\|f(x,u)\| \leq a(x) + C\|u\|, \quad x \in \mathbf{R}^n,\ u \in \mathbf{R}^N,$$

$$\|g(x,u)\|_{L(\mathbf{R}^m,\mathbf{R}^n)} \leq a(x) + c\|u\|;$$

(f) *$g : \mathbf{R}^n \times \mathbf{R}^N \to L(\mathbf{R}^m, \mathbf{R}^n)$ satisfies the Caratheodory condition and is uniformly Lipschitz in the second variable.*

Let $H = L^2(\mathbf{R}^n)^N$, $K = \mathbf{R}^m$ and define a closed unbounded operator A on H by

$$Au = \sum_{j=l}^{n} a_j(x)\frac{\partial u}{\partial x_j} + b(x)u, \quad u \in D(A) \subset H.$$

By Theorem (3.51), page 75 of Tanabe (1979), A generates a semigroup $U(t)$ on H which satisfies all of the conditions of Theorem 7.1.

Now define

$$\bar{f} : H \to H$$

by

$$\bar{f}(u)(x) = f(x, u(x)), \ u \in H, \ x \in \mathbf{R}^n,$$

and

$$\bar{g} : H \to L(K, H), \qquad \bar{g}(u)(x) = g(x, u(x)).$$

As in Example 7.2, $\bar{f}$ and $\bar{g}$ are continuous and there is $C > 0$ such that

$$\begin{aligned} \|\bar{f}(u)\| &\leq C(1 + \|u\|), \\ \|\bar{g}(u)\|_{L(K,H)} &\leq C(1 + \|u\|). \end{aligned}$$

Moreover, $-\bar{f}$ is semimonotone on H and $\bar{g}$ is uniformly Lipschitz. Then $\bar{f}$ and $\bar{g}$, satisfy the conditions of Remark 7.1 and we can write (7.16) as

$$\begin{cases} du(t) = Au(t)dt + \bar{f}(u(t))dt + \bar{g}(u(t))dW_t \\ u(0) = u_0. \end{cases} \tag{7.17}$$

Since A, $\bar{f}$, $\bar{g}$, u_0, and W satisfy the conditions of Remark 7.1, then equation (7.17) has a continuous adapted mild solution with values in $H = L^2(\mathbf{R}^n)^N$. Thus problem (7.16) has a unique mild continuous adapted solution with values in $L^2(\mathbf{R}^n)^N$.

Remark 7.2 *We assumed in Examples 7.2 and 7.3 that f, g and g_i did not depend on $\omega \in \Omega$ or $t \in S$. In fact we could have let them depend on ω and t; this would not have involved any essential modification of the proof.*

7.5 Second Order Equations

Let us consider the semilinear Cauchy problem on the Hilbert space H, written formally as

$$\begin{cases} \frac{\partial^2 x(t)}{\partial t^2} + Ax(t) &= f\left(x(t), \frac{\partial x(t)}{\partial t}\right) + g\left(x(t), \frac{\partial x(t)}{\partial t}\right)\dot{W}_t \\ U(0,x) &= u_0(x), \\ \frac{\partial x(t)}{\partial t}\big|_{t=0} &= y_0, \end{cases} \tag{7.18}$$

where W_t is a cylindrical Brownian motion on K. Let A, f, and g satisfy the following:

Hypothesis 7.4 **(a)** *A satisfies Hypothesis 4.3;*

(b) *there are $p > 0$ and $C > 0$ such that $f : D(A^{\frac{1}{2}}) \times H \to H$ satisfies Hypotheses 4.4 with $\varphi(x) = C(1 + x^p)$, $x \in \mathbf{R}^+$;*

(c) *$g : D(A^{\frac{1}{2}}) \times H \to L^2(K, H)$ is uniformly Lipschitz i.e $\exists C > 0$ such that*

$$\|g(x,y) - g(\bar{x},\bar{y})\| \leq C\left(\|A^{\frac{1}{2}}(x-\bar{x})\|^2 + \|y-\bar{y}\|^2\right)^{\frac{1}{2}}.$$

As in Chapter 4 we define $X_t = \begin{pmatrix} x_t \\ \frac{\partial x}{\partial t} \end{pmatrix}$ and

$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$ on the Hilbert space $\mathcal{K} = D(A^{\frac{1}{2}}) \times H$. We can rewrite (7.18) as

$$\begin{cases} dX_t &= \mathcal{A}X_t dt + F(X_t)dt + G(X_t)dW_t, \\ X(0) &= X_0, \end{cases} \tag{7.19}$$

where $X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $F(x,y) = \begin{pmatrix} 0 \\ f(x,y) \end{pmatrix}$ and $G(x,y) = \begin{pmatrix} 0 \\ g(x,y) \end{pmatrix}$. Note that $G : \mathcal{K} \to L_2(K, \mathcal{K})$. W_t is still a cylindrical Brownian motion on K. Now $\mathcal{A}$ satisfies Hypothesis 4.3 and by Chapter 4 it also satisfies Hypothesis 4.1 and Hypothesis 3.1, so it satisfies Hypothesis 7.3(d). Since F is bounded by a polynomial, then by Proposition

4.1 it satisfies Hypothesis 7.3(a). Since G is a uniformly Lipschitz operator, it satisfies Hypothesis 7.3(b), (c) and (g). Then all conditions of Theorem 7.1 are satisfied and we have

Proposition 7.1 *If A, f, and g satisfy Hypothesis 7.4, then the equation (7.19) has a unique mild solution so that $x_t \in C(S, D(A^{\frac{1}{2}})) \cap C^1(S, H)$, i.e., the mild solution of (7.19) is a continuous process in $D(A^{\frac{1}{2}})$ and it is a differentiable process in H.*

Example 7.4: Let D, A, B and $\dot{W}$ be as in Examples 4.2 and 4.3. Consider a mixed problem of the form

$$\begin{cases} \frac{\partial^2 u(t)}{\partial t^2} + Au &= f(u, \frac{\partial u}{\partial t}) + g(u, \frac{\partial u}{\partial t})\dot{W} \quad \text{on } D \times [0, \infty) \\ Bu &= 0 \quad \text{on } \partial D \times [0, \infty) \\ u(x, 0) &= 0 \quad \text{on } D \\ \frac{\partial u}{\partial t}(x, 0) &= 0 \quad \text{on } D, \end{cases} \tag{7.20}$$

where $n \in \mathbf{Z}$ and $g : H_{n+1} \times H_n \to L(K, H_{n+1} \times H_n)$ is uniformly Lipschitz and $f : H_{n+1} \times H_n \to H_n$.

As in Example 4.2, we consider W as a Brownian motion $\tilde{W}_t$ on the Sobolev space H_{-p}, for some $p > \frac{d}{2}$. Now A is a strictly positive definite self-adjoint operator on H_n. As in Example (4.3) we can write (7.20) as the following Cauchy problem on the Sobolev space H_n:

$$\begin{cases} du_t &= \dot{u}_t dt \\ \dot{u} &= -Au_t dt + f(u_t, \dot{u}_t)dt + g(u_t, \dot{u}_t)dW_t \\ u(0) &= 0 \\ \dot{u}(0) &= 0 \end{cases} \tag{7.21}$$

Now f, g and A satisfy the conditions of Proposition 7.1. Then (7.21) has a unique continuous mild solution $u_t \in C(S, H_{n+1})$ and, moreover, $u_t \in C^1(S, H_n)$. Q.E.D

Chapter 8

GENERALIZATION AND THE CONTINUITY

8.1 Introduction

In this chapter we first generalize Theorem 7.1 by relaxing the L^p-boundedness Hypothesis 7.1(g). Then we prove a theorem about the continuity of the solution of the integral equation (7.2) with respect to a parameter. We also give a bound for the pth moments of the solution of (7.2).

The continuity and smoothness of the solution of a stochastic equation depending on a parameter have been well-studied by several authors. [see e.g. Emery (1978)].

Métivier (1982) has proved continuity and smoothness of the solution of an H-valued stochastic differential equation of Lipschitz type with respect to a parameter. We will generalize his result for evolution equations, i.e., we will prove that the solution of (7.2) changes continuously when any or all of V, f, g and X_0 are varied. This is also a generalization of Theorem 5.1.

8.2 Boundedness of the Solutions

Lemma 8.1 *Let $p \geq 1$. If X_t is a solution of (7.2) and if Hypothesis 7.1 is satisfied, then*

$$\begin{aligned} E\left\{(X_t^\star)^{2p}\right\} \;&\leq\; C\left\{1 + E\left(\|X_0\|^{2p}\right)\right. \\ &\left. + E\left(\int_0^t \|g_s(0)\|_2^{2p}\, ds\right) + E\left((V_t^\star)^{2pq}\right)\right\}. \end{aligned} \tag{8.1}$$

In particular $X_t^\star \in L^p$ for all $p \geq 1$.

Proof: Without loss of generality we can assume that $\lambda = 0$ in Hypothesis 3.1(c) and that $g_s(0) = 0$ (by Lemma 7.1). Define $Y_t = X_t - V_t$. Then we can rewrite (7.2) as

$$Y_t = U(t,0)X_0 + \int_0^t U(t,s)f_s(X_s)ds + \int_0^t U(t,s)g_s(X)dW_s.$$

By the Ito's inequality of Chapter 3 one has

$$\begin{aligned} \|Y_t\|^2 &\leq \|X_0\|^2 + 2\int_0^t < Y_s, f_s(X_s) > ds \\ &\quad + 2\int_0^t < Y_s, dN_s > + [N]_t, \end{aligned} \tag{8.2}$$

where $N_t = \int_0^t g_s(X)dW_s$ is an H-valued martingale. Now

$$\begin{aligned} 2\int_0^t < Y_s, f_s(X_s) > ds &= 2\int_0^t < Y_s, f_s(Y_s + V_s) - f_s(V_s) > ds \\ &\quad + 2\int_0^t < Y_s, f_s(V_s) > ds. \end{aligned}$$

Since f_s is semimonotone with parameter M and since it is bounded by $\varphi(x) = C(1+x^q)$ for some $q \geq 1$, the right hand side of the above equation is

$$\begin{aligned} &\leq 2M\int_0^t \|Y_s\|^2 ds + 2CTY_t^\star(1 + \{V_t^\star\}^q) \\ &\leq 2M\int_0^t (Y_s^\star)^2 ds + \frac{1}{2}(Y_t^\star)^2 + 2(2CT)^2\left(1 + \{V_t^\star\}^{2q}\right), \end{aligned}$$

so we can rewrite (8.2) as

$$\begin{aligned} \frac{1}{2}(Y_t^\star)^2 &\leq \|X_0\|^2 + 2M\int_0^t (Y_s^\star)^2 ds \\ &\quad + C\left(1 + \{V_t^\star\}^{2q}\right) + 2\sup_{0\leq r\leq t}\left|\int_0^r < Y_s, dN_s > \right| + [N]_t. \end{aligned}$$

Using

$$(a_1 + a_2 + \ldots + a_5)^p \leq 5^p(a_1^p + \ldots + a_5^p)$$

for $p \geq 1$, taking expectations and using Fatou's lemma, we can see that there is $C > 0$ such that

$$\begin{aligned} E\left\{(Y_t^\star)^{2p}\right\} &\leq C\Big\{1 + E\left(\|X_0\|^{2p}\right) + \int_0^t E\left\{(Y_s^\star)^{2p}\right\} ds \\ &\quad + E\left(\{V_t^\star\}^{2pq}\right) + E\left([N]_t^p\right). \\ &\quad + E\left(\sup_{0\leq r\leq t}\left|\int_0^r < Y_s, dN_s >\right|^p\right)\Big\}. \end{aligned}$$

Using Lemma 3.2 on the last term of the above inequality to see that for for all $K > 0$ this is

$$\begin{aligned} &\leq C\Big\{1 + E\left(\|X_0\|^{2p}\right) + \int_0^t E\left\{(Y_s^\star)^{2p}\right\} ds \\ &\quad + E\left\{(V_t^\star)^{2pq}\right\} + \frac{C_p}{2K} E\left\{(Y_s^\star)^{2p}\right\} \\ &\quad + (1 + \frac{C_p K}{2}) E\left\{[N]\right\}_t^p\Big\}. \end{aligned}$$

Choose $K = CC_p$ and note that $E\{(Y_t^\star)^{2p}\} < \infty$. Then

$$\begin{aligned} \frac{1}{2} E\left\{(Y_t^\star)^{2p}\right\} &\leq C\Big\{1 + E\left\{\|X_0\|^{2p}\right\} + \int_0^t E\left\{(Y_s^\star)^{2p}\right\} ds \\ &\quad + E\left\{(V_t^\star)^{2pq}\right\} + \left(1 + \frac{CC_p^2}{4}\right) E\left([N]_t^p\right)\Big\}. \end{aligned} \tag{8.3}$$

But $[N]_t = E(\int_0^t \|g_s(X)\|_2^2 ds)$, so

$$\begin{aligned} E\left\{[N]_t^p\right\} &\leq E\left\{\int_0^t \|g_s(X)\|_2^{2p} ds\right\} \\ &\leq C\Big\{\int_0^t E\left((Y_s^\star)^{2p}\right) ds \\ &\quad + E\left((V_t^\star)^{2p}\right)\Big\} \end{aligned}$$

by Hypothesis 7.1(c) and the fact that $g_s(0) \equiv 0$. Since there is $C > 0$ such that

$$E\left\{(V_t^\star)^{2p}\right\} \leq C\left(1 + E\left\{(V_t^\star)^{2pq}\right\}\right),$$

we can rewrite (8.3) as

$$\begin{aligned} E\left\{(Y_t^\star)^{2p}\right\} \;\le\; & C\left\{1+E\left\{\|X_0\|^{2p}\right\}\right. \\ & \left. +\int_0^t E\left\{(Y_s^\star)^{2p}\right\}ds + E\left\{(V_t^\star)^{2pq}\right\}\right\}. \end{aligned}$$

By Gronwall's inequality we have

$$E\left\{(Y_t^\star)^{2p}\right\} \le e^{CT}\left[1+E\left\{\|X_0\|^{2p}\right\}+E\left\{(V_t^\star)^{2pq}\right\}\right].$$

But now

$$(X_t^\star)^{2p} \le (2)^{2p}\left\{(Y_t^\star)^{2p}+\left(V_t^\star\right)^{2p}\right\}$$

so there is $C>0$ such that

$$E\left\{(X_t^\star)^{2p}\right\} \le CT\left\{1+E\left\{\|X_0\|^{2p}\right\}+E\left\{(V_t^\star)^{2pq}\right\}\right\}.$$

Q.E.D

8.3 Generalization of Theorem 7.1

In this section we are going to relax Hypothesis 7.1(g) as follows.

Hypothesis 8.1 **(a)** *Let X_0, f, g, A, U and V satisfy Hypothesis 7.1(a)-(f);*

(b) $E\{\|X_0\|^2\}$, $E\{(V_t^\star)^{2q}\}$, $E\{\sup_{0\le s\le t}\|g_s(0)\|_2^{2p}\}$ *are bounded.*

Theorem 8.1 *If Hypothesis 8.1 is satisfied, then the integral equation (7.2) has a unique continuous adapted strong solution with $E\{(X_t^\star)^2\}<\infty$. Moreover it satisfies (8.1).*

Proof: Uniqueness is trivial from Theorem 7.1.

Existence

Just as in Theorem 7.1 we can assume without loss of generality that $g_s(0)=0$.

Define the stopping time

$$T_n := \inf\{t : \|V_t\| > n\} \wedge T$$

and define

$$V_t^n := V_{t\wedge T_n} \quad \text{and} \quad X_0^n := X_0 1_{\{\omega : \|X_0\| < n\}}$$

Now consider the integral equation

$$\begin{aligned} X_t^n &= U(t,0)X_0^n + \int_0^t U(t,s)f_s(X_s^n)ds \\ &\quad + \int_0^t U(t,s)g_s(X^n)dW_s + V_t^n. \end{aligned} \tag{8.4}$$

Since X_0^n and V_t^n are bounded in norm by n, then X_0^n and V_t^n satisfy Hypothesis 7.1(g), so that all of the conditions of Theorem 7.1 are satisfied, and there is a unique continuous solution on $S = [0, T]$.

Define

$$S_n = T_n 1_{\{\omega : \|X_0\| \le n\}}.$$

Note that $V_t^n = V_t^{n+1}$ and $X_0^n = X_0^{n+1}$ on $[0, S_n]$, so by uniqueness $X_t^{n+1} = X_t^n$ if $t < S_n$. Now by Lemma 8.1 we have

$$\begin{aligned} E\left\{((X_t^n)^\star)^2\right\} &\le C\left\{1 + E\left\{\|X_0^n\|^2\right\}\right. \\ &\quad \left. + E\left\{((V_t^n)^\star)^{2q}\right\}\right\}. \end{aligned}$$

But $(V_t^n)^\star \le V_t^\star$ and $\|X_0^n\| \le \|X_0\|$ so we have

$$\begin{aligned} E\left\{((X_t^n)^\star)^2\right\} &\le C\left\{1 + E\left\{\|X_0\|^2\right\}\right. \\ &\quad \left. + E\left\{((V_t)^\star)^{2q}\right\}\right\} < \infty, \end{aligned}$$

by Hypothesis 8.1(b). Define $X_t := \lim_{n\to\infty} X_t^n$. Since $X_t = X_t^n$ for $0 \le t \le S_n$, $X_t^\star = (X_t^n)^\star$ for $0 \le t \le S_n$, and

$$E\left\{1_{\{0\le t\le S_n\}}(X_t^\star)^2\right\} = E\left\{1_{\{0\le t\le S_n\}}\left[(X_t^n)^\star\right]^2\right\}$$

$$\begin{aligned} &\leq E\left\{[(X_t^n)^\star)]^2\right\} \\ &\leq C\left[1 + E\left\{\|X_0\|^2\right\} + E\left\{(V_t^\star)^{2q}\right\}\right] \\ &< \infty. \end{aligned}$$

Since $P\{S_n = T\} \to 1$ this implies that

$$E\left\{(X_t^\star)^2\right\} \leq C\left\{1 + E\left\{\|X_0\|^2\right\} + E\left\{(V_t^\star)^{2q}\right\}\right\}.$$

Now since X_t^n is the solution of (8.4) we have

$$\begin{aligned} X_t^n 1_{\{t \leq S_n\}} &= 1_{\{t \leq S_n\}}\left[U(t,0)X_0^n + \int_0^t U(t,s)f_s(X_s^n)ds \right. \\ &\quad \left. + \int_0^t U(t,s)g_s(X^n)dW_s + V_t^n\right]. \end{aligned} \tag{8.5}$$

Since $X_t = X_t^n$, $V_t = X_t^n$ and $X_0^n = X_0$ on $[0, S_n]$, then we can write (8.5) as

$$\begin{aligned} X_t 1_{\{t \leq S_n\}} &= 1_{\{t \leq S_n\}}\left[U(t,0)X_0 + \int_0^t U(t,s)f_s(X_s)ds \right. \\ &\quad \left. + \int_0^t U(t,s)g_s(X)dW_s + V_t\right]. \end{aligned}$$

Since $P\{S_n = T\} \to 1$ this proves that X_t is a solution of (7.2) on $[0,T]$. To complete the proof of the theorem we need to prove (8.1).

First we show without loss of generality we can assume $g_s(0) = 0$. If $g_s(0) \neq 0$ we can define $\bar{g}_s(x) = g_s(x) - g_s(0)$ and

$$\bar{V}_t = \int_0^t U(t,s)g_s(0)dWs + V_t.$$

Then

$$E\{((\bar{V})_t^\star)^{2pq}\} \leq C\{E(\sup_{0 \leq r \leq t}\left|\int_0^r U(r,s)g_s(0)dW_s\right|^{2pq}) + E((V_t^\star)^{2pq})\}. \tag{8.6}$$

By Theorem 3.2 (Burkholder's inequality) there is $C > 0$ such that the first term on the right hand side of (8.6) is bounded by $CE[\int_0^t \|g_s(0)\|_2^{2pq} ds]$. Then X_t satisfies (8.1) without $g_s(0) = 0$.

Then let $g_s(0) = 0$. By Lemma 8.1 we have

$$\begin{aligned} E\{((X_t^n)^\star)^{2p}\} &\leq C\{1 + E\{\|X_0^n\|^{2p}\} \\ &\qquad +E\{((V^n)_t^\star)^{2pq}\}. \end{aligned}$$

But $\|X_0^n\| \leq \|X_0\|$ and $(V_n)_t^\star \leq V_t^\star$ so

$$\begin{aligned} E\{1_{\{t\leq S_n\}}(X_t^\star)^{2p}\} &= E\{1_{\{t\leq S_n\}}(X^n)_t^\star)^{2p}\} \\ &\leq E\{((X^n)_t^\star)^{2p}\} \leq C\{1 + E\{\|X_0\|^{2p}\} + E\{(V_t^\star)^{2pq}\}\}. \end{aligned}$$

Since $P\{S_n = T\} \to 1$ we have

$$E\{(X_t^\star)^{2p}\} \leq C\{1 + E\{\|X_0\|^{2p}\} + E\{(V_t^\star)^{2pq}\}\}.$$

Q.E.D

Remark 8.1 *Let $p \geq 1$. It follows from (8.1) that if*

$$E\{\|X_0\|^{2p}\} < \infty, \quad E\{(V_t^\star)^{2pq}\} < \infty \quad \textit{and} \quad E\{\int_0^T \|g_s(0)\|_2^{2pq}\} < \infty,$$

then $X_t^\star \in L^{2p}$.

8.4 The Continuity of the Solution with Respect to the Parameter

Theorem 8.2 *If f^i, g^i, V^i, and X_0^i, $i = 1, 2$ satisfy the conditions of Theorem 7.1 and if X_t^i, $i = 1, 2$ are solutions of the integral equations*

$$\begin{aligned} X_t^i &= U(t,0)X_0^i + \int_0^t U(t,s)f_s^i(X_s^i)ds \\ &\qquad + \int_0^t U(t,s)g_s^i(X^i)dWs + V_t^i, \; i = 1, 2, \end{aligned} \tag{8.7}$$

then there is a constant $C > 0$ such that

$$\begin{aligned} E\{((X^2 - X^1)_t^*)^{2p}\} \;\le\; & C[E\{\|X_0^2 - X_0^1\|^{2p}\} \\ & + E\{\int_0^T \|g_s^2(X^1) - g_s^1(X^1)\|_2^{2p} ds\} \\ & + E\{\int_0^T \|f_s^2(X_s^1) - f^1(X_s^1)\|^{2p} ds\} \\ & + K(E\{\|V^2 - V^1\|_\infty^{2p}\})^{\frac{1}{2}}], \end{aligned} \tag{8.8}$$

where

$$\begin{aligned} K \;\le\; & C\left(1 + E\{\|X_0^2\|^{2pq}\} + E\{\|X_0^1\|^{2pq}\}\right. \\ & \left. + E\{\|V^2\|_\infty^{4pq^2}\} + E\{\|V^1\|_\infty^{4pq^2}\}\right). \end{aligned} \tag{8.9}$$

Proof: By Lemma 3.1 we can assume $\lambda \equiv 0$ without loss of generality. Define $Y_t^i = X_t^i - V_t^i$, $i = 1, 2$. Then

$$Y_t^i = U(t,0)X_0^i + \int_0^t U(t,s)f_s^i(X_s^i)ds + \int_0^t U(t,s)g_s^i(X^i)dW_s,$$

hence

$$\begin{aligned} Y_t^2 - Y_t^1 \;=\; & U(t,0)(X_0^2 - X_0^1) + \int_0^T U(t,S)[f_s^2(X_s^2) - f_s^1(X_s^1)]ds \\ & + \int_0^T g_s^2(X^2) - g_s^1(X^1)]dW_s. \end{aligned} \tag{8.10}$$

Define an H-valued local martingale N_t by

$$N_t := \int_0^T g_s^2(X^2) - g_s^1(X^1)]dW_s.$$

This has quadratic variation

$$[N]_t = \int_0^T \|g_s^2(X^2) - g_s^1(X^1)\|_2^2 ds.$$

Define

$$Y_t := Y_t^2 - Y_t^1, \quad V_t := V_t^2 - Y_t^1, \quad X_0 := X_0^2 - X_0^1, \quad \text{and} \quad X_t := X_t^2 - X_t^1.$$

Using Ito's inequality of Chapter 3, (8.10) implies

$$\begin{aligned}\|Y_t\|^2 &\leq \|X_0\|^2 + 2\int_0^t < Y_s, f_s^2(X_s^2) - f_s^1(X_s^1) > ds \\ &\quad + 2\int_0^t < Y_s, dN_s > +[N]_t \\ &:= \|X_0\|^2 + I_1(t) + I_2(t) + [N]_t. \end{aligned} \tag{8.11}$$

By Lemma 5.1

$$\begin{aligned} I_1(t) &\leq (1+4M)\int_0^t \|Y_s\|^2 ds + TI\|V\|_\infty \\ &\quad + \int_0^t \|f^2(X_1) - f^1(X_s^1)\|^2 ds, \end{aligned} \tag{8.12}$$

where

$$I = 2\{\int_0^T \|f^2(X_s^2) - f^1(X_s^1)\|^2 ds\}_{\frac{1}{2}} + 4MT\|V\|\infty.$$

But since the f^i are bounded by $\varphi(x) = C(1+x^q)$ then

$$\|f^2(X_s^2) - f^1(X_s^1)\| \leq C(1 + \|X^2\|_\infty^q) + C(1 + \|X^1\|_\infty^q),$$

and

$$I \leq TC\{2 + \|X^2\|_\infty^q + \|X^1\|_\infty^q\} + 4MT\|V\|\infty. \tag{8.13}$$

Now by taking the supremum over $t \in S$ in (8.11) and using the inequality

$$(a_1 + a_2 + a_3 + a_4)^p \leq 4^p(a_1^p + \ldots + a_4^p),$$

we get

$$\begin{aligned} E\{(Y_t^\star)^{2p}\} &\leq 4^p\{E\{\|X_0\|^{2p}\} + E\{(I_1^\star(t))^p\} \\ &\quad + E\{(I_2^\star(t))^p\} + E\{[N]_t^p\}\}. \end{aligned} \tag{8.14}$$

By Lemma 3.2

$$E\{I_2^\star(t)^p\} \leq \frac{C_p}{K}E\{(Y_t^\star)^{2p}\} + C_pKE\{[N]_t^p\}. \tag{8.15}$$

Choose $K = \frac{C_p}{2^{2p+1}}$ in (8.15). Since $E\{(Y_t^\star)^{2p} < \infty$, (8.14) implies that there is $C > 0$ such that

$$\begin{aligned} E\{(Y_t^\star)^{2p}\} &\leq C\{E\{\|X_0\|^{2p}\} \\ &\quad + E\{(I_2^\star(t))^p\} + E([N]_t^p)\}. \end{aligned} \tag{8.16}$$

But

$$\begin{aligned} [N]_t &\leq \int_0^t \left\|g_s^2(X^2) - g_s^1(X^1)\right\|_2^2 ds \\ &\leq 2\int_0^t \left\|g_s^2(X^2) - g_s^2(X^1)\right\|_2^2 ds \\ &\quad + 2\int_0^t \left\|g_s^2(X^1) - g_s^1(X^1)\right\|_2^2 ds, \end{aligned}$$

and by Hypothesis 7.1(c)

$$\left\|g_s^2(X^2) - g_s^2(X^1)\right\|_2 \leq LX_s^\star \leq LY_s^\star + L\|V\|_\infty.$$

Then there is $C > 0$ such that

$$\begin{aligned} E\{[N]_t^p\} &\leq C\left\{\int_0^t E\left((Y_s^\star)^{2p}\right) ds + E\left\{\|V\|_\infty^{2p}\right\}\right. \\ &\quad \left. + E\left\{\int_0^T \|g_s^2(X^1) - g_s^1(X^1)\|_2^{2p} ds\right\}\right\} \end{aligned} \tag{8.17}$$

Using the Schwartz inequality in (8.12) we can see there is $C > 0$ such that

$$\begin{aligned} E\{(I_2^\star(t))^p\} &\leq C\left[\int_0^t E\left\{(Y_s^\star)^{2p}\right\} ds\right. \\ &\quad + \left(E\left\{I^{2p}\right\}\right)^{\frac{1}{2}}\left(E\left\{\|V\|_\infty^{2p}\right\}\right)^{\frac{1}{2}} \\ &\quad \left. + E\left\{\int_0^T \|f_s^2(X_s^1) - f_s^1(X_s^1)\|^{2p} ds\right\}\right]. \end{aligned} \tag{8.18}$$

Combining (8.16), (8.17) and (8.18) we can see that there is $C > 0$ such that

$$E\left\{(Y_t^\star)^{2p}\right\} \leq C\left[E\left\{\|X_0\|^{2p}\right\} + \int_0^t E\left\{(Y_t^\star)^{2p}\right\} ds\right.$$

$$
\begin{aligned}
&+ \left(E\left\{I^{2p}\right\}\right)^{\frac{1}{2}} \left(E\left\{\|V\|_\infty^{2p}\right\}\right)^{\frac{1}{2}} \\
&+ E\left\{\int_0^T \|f_s^2(X_s^1) - f_s^1(X_s^1)\|^{2p} ds\right\} + E\left\{\|V\|_\infty^{2p}\right\} \\
&+ E\left\{\int_0^T \|g_s^2(X^1) - g_s^1(X^1)\|_2^{2p} ds\right\}\Bigg].
\end{aligned}
$$

By Gronwall's inequality

$$
\begin{aligned}
E\left\{(Y_t^\star)^{2p}\right\} \;\le\; & C\Big[E\left\{\|X_0\|^{2p}\right\} \\
&+ E\left\{\int_0^T \|f_s^2(X_s^1) - f_s^1(X_s^1)\|^{2p} ds\right\} \\
&+ E\left\{\int_0^T \|g_s^2(X^1) - g_s^1(X^1)\|_2^{2p} ds\right\} \\
&+ \left(E\left\{\|V\|_\infty^{2p}\right\}\right)^{\frac{1}{2}} \left(E\left\{I^{2p}\right\}\right)^{\frac{1}{2}} \\
&+ \left(E\left\{\|V\|_\infty^{2p}\right\}\right)^{\frac{1}{2}}\Big]. \qquad (8.19)
\end{aligned}
$$

Since $X_t^\star \le Y_t^\star + \|V\|_\infty$, (8.19) implies that there is $C > 0$ such that

$$
\begin{aligned}
E\left\{(X_t^\star)^{2p}\right\} \;\le\; & C\Big[E\left\{\|X_0\|^{2p}\right\} \\
& E\left\{\int_0^T \|f_s^2(X_s^1) - f_s^1(X_s^1)\|^{2p} ds\right\} \\
&+ E\left\{\int_0^T \|g_s^2(X^1) - g_s^1(X^1)\|_2^{2p} ds\right\} \\
&+ K\left(E\left\{\|V\|_\infty^{2p}\right\}\right)^{\frac{1}{2}}\Big],
\end{aligned}
$$

where

$$K = E(I^{2p})^{\frac{1}{2}} + E\left\{\|V\|_\infty^{2p}\right\}^{\frac{1}{2}}.$$

To complete the proof of Theorem 8.2 we need to show that K satisfies (8.9). Indeed (8.13) implies that

$$E\left(I^{2p}\right) \le C\left\{1 + E\left\{\|X^2\|_\infty^{2pq}\right\} + E\left\{\|X^1\|_\infty^{2pq}\right\} + E\left\{\|V\|_\infty^{2pq}\right\}\right\}.$$

By (8.1) we have

$$E\left\{\|X^i\|_\infty^{2pq}\right\} \le C\left[1 + E\left\{\|X_0^i\|^{2pq}\right\} + E\left\{\|V^i\|_\infty^{4pq^2}\right\}\right]$$

so

$$\begin{aligned} E\{I^{2p}\} \;\le\;& C\left[1 + E\left\{\|X_0^2\|^{2pq}\right\} + E\left\{\|X_0^1\|^{2pq}\right\}\right. \\ & \left. + E\left\{\|V^2\|_\infty^{4pq^2}\right\} + E\left\{\|V^1\|_\infty^{4pq^2}\right\}\right]. \end{aligned}$$

But

$$E\left\{\|V\|_\infty^{2p}\right\}^{\frac{1}{2}} \le C\left\{1 + E\left\{\|V^2\|_\infty^{2pq^2}\right\} + E\left\{\|V^1\|_\infty^{2pq^2}\right\}\right\},$$

so K satisfies (8.9).

Q.E.D

Remark 8.2 **(i)** *If $V^2 = V^1$ then (8.8) implies*

$$\begin{aligned} E\left\{(X^2 - X^1)_t^{*2p}\right\} \;\le\;& C\left[E\left\{\|X_0^2 - X_0^1\|^{2p}\right\}\right. \\ & + E\left\{\int_0^T \|g_s^2(X^1) - g_s^1(X^1)\|_2^{2p} ds\right\} \\ & \left. + E\left\{\int_0^T \|f_s^2(X_s^1) - f_s^1(X_s^1)\|^{2p} ds\right\}\right]. \end{aligned} \qquad (8.20)$$

By localization this inequality holds without Hypothesis 7.1(g).

(ii) *By the localization method we can generalize inequality (8.8) by replacingthe condition*

$$E\left\{\|X_0^i\|^{2pq}\right\} < \infty, \quad E\left\{\|V^i\|_\infty^{4pq^2}\right\} < \infty$$

instead Hypothesis 7.1(h).

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