THE DETERMINATION OF SETS OF INTEGRAL ELEMENTS FOR CERTAIN RATIONAL DIVISION ALGEBRAS

by

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1. Introduction.

The purpose of this paper is to determine integral elements of a certain associative division algebra[1], D, of order nine over the field of rational numbers. The nine basal units of D are $y^i x^j \ (i, j = 0, 1, 2)$ where

$$y^3 = \delta,$$

with δ a rational integer having a rational prime factor of the form $9k \pm 2$ or $9k \pm 4$ that does not occur to a power which is a multiple of 3. Also, x satisfies the cubic

(1) $$x^3 - 3x + 1 = 0$$

which is cyclic, i.e., it is irreducible and has roots ξ, ξ', ξ'', of the form[2]

$$\xi, \quad \xi' = \theta(\xi) = \xi^2 - 2, \quad \xi'' = \theta(\xi') = \xi'^2 - 2.$$

Further,

$$xy = yx'$$

where $x' = \theta(x)$.

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[1] This is a special case of the algebra of order n^2 over a general field F, discovered by Dickson, and called by Wedderburn a Dickson Algebra. See Dickson's Algebras and Their Arithmetics, p. 66.

[2] F. S. Nowlan - Bulletin of the American Mathematical Society, Vol. 32, p. 375 (1926).

It follows that $x' = \theta(x)$ and $x'' = \theta(x') = \theta^2(x)$ satisfy (1), and that $x = \theta^3(x)$.

Integers of the cubic number field, $K(x)$, defined by a root of (1) are of the form

(2) $$Z = \alpha_0 + \alpha_1 x + \alpha_2 x^2,$$

with α_0, α_1, and α_2 rational integers. Professor F. S. Nowlan[1] has shown that the norm of a prime of $K(x)$, not associated with a rational prime is either 3, or a rational prime of the form $9k \pm 1$. In case the prime is associated with a rational prime, the norm is the cube of the rational prime and is thus of the form $9k \pm 1$. Rational primes other than 3, or those of the form $9k \pm 1$, are primes of $K(x)$. Further, every rational prime $9k \pm 1$ is factorable into three conjugate primes of $K(x)$, and so is the norm of a prime of $K(x)$.

Thus the restriction on δ insures that D is a division algebra[2].

In future developments, we shall need a corollary to the above, viz., the form F, given by

(3) $$F = \alpha_0^3 + 6\alpha_0^2\alpha_2 - 3\alpha_0\alpha_1^2 + 3\alpha_0\alpha_1\alpha_2 + 9\alpha_0\alpha_2^2 - \alpha_1^3 + 3\alpha_1\alpha_2^2 + \alpha_2^3,$$

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1 F. S. Nowlan - Bulletin of the American Mathematical Society, Vol. 32, p. 379 (1926).

2 Algebras and Their Arithmetics, p. 68.

of the norm $N(z)$ of the general integer, (z), of $K(x)$ represents 1, 3, all primes of the form $9k \pm 1$ and all products whose prime factors are 3 and primes $9k \pm 1$. The form represents no prime other than these. Other primes are divisors of the form only when they divide each of α_0, α_1, and α_2, and then they appear in powers which are multiples of 3.

The general element z of D satisfies a rank equation[1] viz., a certain cubic equation having unity for the coefficient of the term of third degree and having its other coefficients rational integral functions of the coordinates of z. Moreover, this element does not satisfy a like equation of lower degree.

The integral elements of an algebra are defined[2] as those elements belonging to some one set possessing the four following properties:

R (rank): For every element of the set, the coefficients in the rank equation are rational integers.

C (closure): The set is closed under addition, subtraction, and multiplication.

U (unity): The set contains the modulus 1.

M (maximal): The set is maximal, i.e., it is not con-

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[1] Algebras and Their Arithmetics, p. 111.

[2] Algebras and Their Arithmetics, p. 141.

tained in a larger set having properties R, C , U.

We restrict the investigation to those sets which contain the nine basal units $y^i x^j$. We write $\delta = \eta\varepsilon$ where η contains only prime factors 3 or those of the form $9k \pm 1$, while ε has only prime factors of the form $9k \pm 2$ or $9k \pm 4$, at least one of which occurs to a power which is not a multiple of 3 ; and further, ε is of the form $9k \pm 1$. This paper considers the problem when $\delta = \varepsilon$, ε having prime factors only of the forms $9k \pm 2$ or $9k \pm 4$, at least one of which occurs to a power which is not a multiple of 3 , but ε itself is of the forms $9k \pm 1$. Hull[1] considers, and treats completely, the problem when $\delta = \varepsilon$ is of the forms $9k \pm 2$ and $9k \pm 4$, and also when $\delta = \eta\varepsilon$, with ε restricted as above.

Hull gives in his thesis a full outline of the problem as it pertains to the general algebra G. Certain results and formulae which he obtained are needed for this paper and will be listed in the next section.

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[1] Hull - The Determination of Sets of Integral Elements for Certain Rational Division Algebras. M. A. Thesis in Library of the University of British Columbia.

2. Synopsis of Results and Formulae Obtained by Hull, which are Required in this Paper.

The rank equation of z for the algebra D is as follows (Hull, p. 7):

$$
\begin{aligned}
&\omega^3 - (3\alpha_0 + 6\alpha_2)\,\omega^2 \\
&+ \left[3\alpha_0^2 - 3\alpha_1^2 + 9\alpha_2^2 + 3\alpha_1\alpha_2 + 12\alpha_0\alpha_2 - \delta(3\beta_0\gamma_0 - 3\beta_1\gamma_1 \right.\\
&\quad \left. + 9\beta_2\gamma_2 + 6\beta_0\gamma_2 + 6\beta_2\gamma_0 + 6\beta_2\gamma_1 - 3\beta_1\gamma_2)\right]\omega \\
&- \left[\alpha_0^3 + 6\alpha_0^2\alpha_2 - 3\alpha_0\alpha_1^2 + 3\alpha_0\alpha_1\alpha_2 + 9\alpha_0\alpha_2^2 - \alpha_1^3 + 3\alpha_1\alpha_2^2 + \alpha_2^3 \right.\\
&\quad + \delta(\beta_0^3 + 6\beta_0^2\beta_2 - 3\beta_0\beta_1^2 + 3\beta_0\beta_1\beta_2 + 9\beta_0\beta_2^2 - \beta_1^3 + 3\beta_1\beta_2^2 + \beta_2^3) \\
&\quad + \delta^2(\gamma_0^3 + 6\gamma_0^2\gamma_2 - 3\gamma_0\gamma_1^2 + 3\gamma_0\gamma_1\gamma_2 + 9\gamma_0\gamma_2^2 - \gamma_1^3 + 3\gamma_1\gamma_2^2 + \gamma_2^3) \\
&\quad - \delta(3\alpha_0\beta_0\gamma_0 + 6\alpha_2\beta_0\gamma_0 + 6\alpha_0\beta_2\gamma_0 + 6\alpha_0\beta_0\gamma_2 - 3\alpha_1\beta_1\gamma_0 \\
&\qquad - 3\alpha_0\beta_1\gamma_1 - 3\alpha_1\beta_0\gamma_1 + 6\alpha_2\beta_1\gamma_0 + 6\alpha_1\beta_0\gamma_2 + 6\alpha_0\beta_2\gamma_1 \\
&\qquad - 3\alpha_1\beta_2\gamma_0 - 3\alpha_0\beta_1\gamma_2 - 3\alpha_2\beta_0\gamma_1 + 9\alpha_2\beta_2\gamma_0 + 9\alpha_0\beta_2\gamma_2 \\
&\qquad + 9\alpha_2\beta_0\gamma_2 - 3\alpha_1\beta_1\gamma_1 + 3\alpha_1\beta_2\gamma_2 + 3\alpha_2\beta_1\gamma_2 + 3\alpha_2\beta_2\gamma_1 \\
&\qquad \left. + 3\alpha_2\beta_2\gamma_2)\right] = 0 \qquad (4)
\end{aligned}
$$

A congruence of the form

(5) $$x + \varepsilon y + z \equiv 0 \pmod 3$$

yields, on manipulation (Hull, pp. 12 - 13):

(6) $$\begin{cases} x - \varepsilon y \equiv \varepsilon y - z \equiv z - x \pmod 3, \\ x^2 - \varepsilon yz \equiv y^2 - zx \equiv z^2 - \varepsilon yx \equiv (z-x)^2 \pmod 3. \end{cases}$$

The coefficients of ω^2 in the rank equations of $y^i z x^j$ $(i, j = 0, 1, 2)$ yield (Hull, pp. 9 & 14):

(7) $$\begin{cases} 3\alpha_0 + 6\alpha_2 = U_0 = u_0, \\ 6\alpha_1 - 3\alpha_2 = U_1 = u_1, \\ 6\alpha_0 - 3\alpha_1 + 18\alpha_2 = U_2 = u_2, \\ \delta(3\beta_0 + 6\beta_2) = V_0 = \varepsilon v_0, \\ \delta(6\beta_1 - 3\beta_2) = V_1 = \varepsilon v_1, \\ \delta(6\beta_0 - 3\beta_1 + 18\beta_2) = V_2 = \varepsilon v_2, \\ \delta(3\gamma_0 + 6\gamma_2) = W_0 = \varepsilon w_0, \\ \delta(6\gamma_1 - 3\gamma_2) = W_1 = \varepsilon w_1, \\ \delta(6\gamma_0 - 3\gamma_1 + 18\gamma_2) = W_2 = \varepsilon w_2. \end{cases}$$

On solving equations (7) for the coordinates of z , we obtain (Hull, p. 9):

(8) $$\begin{cases} 9\alpha_0 = 11U_0 - 2U_1 - 4U_2, \\ 9\alpha_1 = -2U_0 + 2U_1 + U_2, \\ 9\alpha_2 = -4U_0 + U_1 + 2U_2, \end{cases}$$

(9) $$\begin{cases} 9\delta\beta_0 = 11V_0 - 2V_1 - 4V_2, \\ 9\delta\beta_1 = -2V_0 + 2V_1 + V_2, \\ 9\delta\beta_2 = -4V_0 + V_1 + 2V_2, \end{cases}$$

$$(10)\quad \begin{cases} 9\delta\gamma_0 = 11W_0 - 2W_1 - 4W_2, \\ 9\delta\gamma_1 = -2W_0 + 2W_1 + W_2, \\ 9\delta\gamma_2 = -4W_0 + W_1 + 2W_2. \end{cases}$$

The following combinations simplify the reduction of the necessary conditions that the rank property holds (Hull, p.15):

$$(11)\quad \begin{cases} r = u_1 + u_2, & s = v_1 + v_2, & t = w_1 + w_2, \\ r' = -u_0 + u_2, & s' = -v_0 + v_2, & t' = -w_0 + w_2, \\ \rho = r + r', & \sigma = s + s', & \tau = t + t'. \end{cases}$$

Substitution of (11) in the coefficient of ω and in $N(z)$ as obtained from the rank equation for z, yields the following as necessary conditions that z has the rank property (Hull, pp. 15 - 16):

$$(12)\quad \begin{aligned} &-\rho^2 + 3ru_2 - \varepsilon\{-\sigma\tau + 3s't - 3st' \\ &\quad + 3w_2(s - s') + 3v_2 t\} \equiv 0 \pmod 9, \end{aligned}$$

and

$$(13)\quad \begin{aligned} &\rho^3 - 3rr'\rho - 3rr'^2 + 9(rr'\rho - \rho^2 u_2 \\ &\quad - rr'u_2 + 2\rho u_2^2 - u_2^3) \\ &+\varepsilon\{\sigma^3 - 3ss'\sigma - 3ss'^2 + 9(ss'\sigma - \sigma^2 v_2 \\ &\quad - ss'v_2 + 2\sigma v_2^2 - v_2^3)\} \\ &+\varepsilon^2\{\tau^3 - 3tt'\tau - 3tt'^2 + 9(tt'\tau - \tau^2 w_2 \\ &\quad - tt'w_2 + 2\tau w_2^2 - w_2^3)\} \\ &-3\varepsilon\{\rho\sigma\tau + 2(r'st + rs't + rst') + (rs't' + r'st' + r's't) \\ &\quad -3(\sigma\tau u_2 + \rho\tau v_2 + \rho\sigma w_2 + st'u_2 + tr'v_2 \\ &\quad + rs'w_2 - 2\tau u_2 v_2 - 2\rho v_2 w_2 - 2\sigma u_2 w_2) \\ &\quad - 9u_2 v_2 w_2\} \equiv 0 \pmod{81}. \end{aligned}$$

Congruences (12) and (13) yield the following relations (Hull, pp. 16 - 17):

(14) $$\rho \equiv \varepsilon\sigma \equiv \tau \pmod 3,$$

(15) $$r + \varepsilon s + t \equiv 0 \pmod 3,$$

(16) $$r' + \varepsilon s' + t' \equiv 0 \pmod 3,$$

(17) $$\begin{cases} r + r' = 3r_1 + m = \rho, \\ s + s' = 3s_1 + \varepsilon m = \sigma, \\ t + t' = 3t_1 + m = \tau. \end{cases}$$

3. The Solution of Congruences (12)(mod 9) and (13)(mod 27).

Using the results of the last section, we determine the necessary and sufficient conditions that z may have the rank property.

Substituting (17) in (12), and grouping, we require

(18) $$m(r_1+\varepsilon s_1+t_1)+ru_2-mt+\varepsilon ms - w_2(2\varepsilon s-m)-\varepsilon v_2(m-t)\equiv 0 \pmod 3).$$

We obtain the coefficient of ω in the rank equation for $z\chi$ from the corresponding coefficient in the rank equation for z [1]. The condition that it be integral is as follows:

(19) $$-\rho^2+3r^2-3u_2(r-r')-\varepsilon(-\sigma\tau+3st + 3s'w_2-3v_2t)\equiv 0 \pmod 9).$$

Substituting (17) in (19), and grouping, we require

(20) $$m(r_1+\varepsilon s_1+t_1)+m(u_2-w_2)+r^2-2ru_2 -\varepsilon st+\varepsilon s w_2+\varepsilon t v_2\equiv 0 \pmod 3).$$

On subtracting, (18) — (20), the following is required:

$$m(\varepsilon s-t)-m(u_2+\varepsilon v_2+w_2)-(r^2-\varepsilon st)\equiv 0 \pmod 3).$$

Introducing the notation

$$X = u_2+\varepsilon v_2+w_2 \quad , \qquad Y = t-r,$$

this last congruence becomes

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[1] For the necessary substitutions, see Hull, p. 8.

(21) $$mY - mX - Y^2 \equiv 0 \pmod 3.$$

Congruence (21) has the following solutions:

(22) $$\begin{cases} m=0,\ Y\equiv 0,\ X \text{ arbitrary} \pmod 3, \\ m=1 \begin{cases} Y\equiv 0,\ X\equiv 0 \pmod 3, \\ Y\equiv 1,\ X\equiv 0 \pmod 3, \\ Y\equiv -1,\ X\equiv 1 \pmod 3, \end{cases} \\ m=-1 \begin{cases} Y\equiv 0,\ X\equiv 0 \pmod 3, \\ Y\equiv -1,\ X\equiv 0 \pmod 3, \\ Y\equiv 1,\ X\equiv -1 \pmod 3, \end{cases} \end{cases}$$

Substitution of (17) in (13)(mod 27) yields a set of solutions, each of which is included in (22).

The following divisions of the problem are suggested by (22):

Case I. $m=0,\ Y\equiv 0,\ X$ arbitrary (mod 3),

Case II. $m=1,\ Y\equiv 0,\ X\equiv 0$ (mod 3),

Case III. $m=1,\ Y\equiv 1,\ X\equiv 0$ (mod 3),

Case IV. $m=1,\ Y\equiv -1,\ X\equiv 1$ (mod 3),

Case V. $m=-1,\ Y\equiv 0,\ X\equiv 0$ (mod 3),

Case VI. $m=-1,\ Y\equiv -1,\ X\equiv 0$ (mod 3),

Case VII. $m=-1,\ Y\equiv 1,\ X\equiv -1$ (mod 3).

In subsequent work, sub-cases will be denoted by subscripts, e.g., I_2 meaning case I, sub-case 2; and sets of elements will be denoted by I_i^j meaning the set contained in sub-case I_i for which $\mathcal{E}\equiv j$ (mod 9).

Since

$$\mathcal{E} \equiv \pm 1 \pmod 9,$$

then

$$\varepsilon^2 \equiv \gamma \pmod 9.$$

4. Case I

The conditions characterizing this case are as follows:

(23) $$\rho \equiv \varepsilon\sigma \equiv \tau \equiv 0 \pmod 3,$$

(24) $$r \equiv \varepsilon s \equiv t \pmod 3,$$

$u_2+\varepsilon v_2+w_2$ being arbitrary, (mod 3).

We make the following transformations, satisfying (23) and (24), on r , s , t , r' , s' , and t' :

(25) $$\begin{cases} r = 3r_1+n, & r' = 3r_1'-n, \\ s = 3s_1+\varepsilon n, & s' = 3s_1'-\varepsilon n, \\ t = 3t_1+n, & t' = 3t_1'-n. \end{cases}$$

The n used in (25) corresponds to the m used in Hull's transformations (59), p. 20.

Substitute (25) in (13)(mod 81), obtaining

(26) $$\begin{aligned} &3[(r_1'+\varepsilon s_1'+t_1')^3 + n(r_1'+\varepsilon s_1'+t_1')^2 \\ &+(r_1+\varepsilon s_1+t_1)^3 + n(r_1+\varepsilon s_1+t_1)^2 \\ &- m(r_1 r_1'+s_1 s_1'+t_1 t_1') - n(r_1' u_2+s_1' v_2+t_1' w_2) \\ &+ m(r_1 u_2+s_1 v_2+t_1 w_2) - (r_1+r_1')(u_2^2-\varepsilon v_2 w_2) \\ &- \varepsilon(s_1+s_1')(v_2^2-u_2 w_2) - (t_1+t_1')(w_2^2-\varepsilon u_2 v_2) \\ &- m(\varepsilon r_1' s_1+\varepsilon r_1 s_1'+r_1' t_1+r_1 t_1'+\varepsilon s_1' t+\varepsilon s_1 t_1') \\ &+ m u_2(t_1'-\varepsilon s_1)+\varepsilon m v_2(r_1'-t_1)+m w_2(\varepsilon s_1'-r_1)] \\ &- (u_2^3+\varepsilon v_2^3+w_2^3-3\varepsilon u_2 v_2 w_2) \equiv 0 \pmod 9. \end{aligned}$$

In order that congruence (26) may hold, we require

$$u_2^3+\varepsilon v_2^3+w_2^3 \equiv 0 \pmod 3,$$

or

(27) $$u_2+\varepsilon v_2+w_2 \equiv 0 \pmod 3,$$

since $u_2^3 \equiv u_2 \pmod 3$.

From congruence (27), we obtain, using (5) and (6),

(28) $$u_2^2 - \varepsilon v_2 w_2 \equiv v_2^2 - u_2 w_2 \equiv w_2^2 - \varepsilon u_2 v_2 \pmod 3 .$$

We use (28), after factoring and regrouping, to give the following:

(29) $$\begin{aligned} &u_2^3 + \varepsilon v_2^3 + w_2^3 - 3\varepsilon u_2 v_2 w_2 \\ &= (u_2 + \varepsilon v_2 + w_2)\{(u_2^2 - \varepsilon v_2 w_2) + (v_2^2 - u_2 w_2) \\ &+ (w_2^2 - \varepsilon u_2 v_2)\} \equiv 0 \pmod 9 . \end{aligned}$$

Writing

(30) $$\begin{cases} X_1 = r_1 + \varepsilon s_1 + t_1 , & X_1' = r_1' + \varepsilon s_1' + t_1' , \\ X_2 = X_1 + X_1' , & Y_1 = w_2 - u_2 , \end{cases}$$

and making use of (27) and (29), the congruence (26) reduces to

$$X_2 (1 + m X_2 - Y_1^2 - m Y_1) \equiv 0 \pmod 3 .$$

This yields solutions which suggest the following seven subdivisions of Case I:

(31) $$\begin{cases} \text{Sub-case } I_1 . & X_2 \equiv 0 , & n \text{ and } Y_1 \text{ arbitrary, (mod 3)}, \\ \text{Sub-case } I_2 . & X_2 \equiv 1 , & Y_1 \equiv 1 ,\ n \text{ arbitrary, (mod 3)}, \\ \text{Sub-case } I_3 . & X_2 \equiv 1 , & Y_1 \equiv -1 , \text{ (mod 3)}, \ n = 0 , \\ \text{Sub-case } I_4 . & X_2 \equiv 1 , & Y_1 \equiv 0 , \text{ (mod 3)}, \ n = -1 , \\ \text{Sub-case } I_5 . & X_2 \equiv -1 , & Y_1 \equiv -1 ,\ n \text{ arbitrary, (mod 3)}, \\ \text{Sub-case } I_6 . & X_2 \equiv -1 , & Y_1 \equiv 1 , \text{ (mod 3)}, \ n = 0 , \\ \text{Sub-case } I_7 . & X_2 \equiv -1 , & Y_1 \equiv 0 , \text{ (mod 3)}, \ n = 1 . \end{cases}$$

We now express the coordinates of z in terms of the parameters r , s , t , r', s', t' , u_2 , v_2 , w_2 , u_0 ,

v_0 , and w_0 , using (7), (8), (9), (10), and (11), and then grouping so that the substitution

(32) $$R = r_1 + r_1', \quad S = s_1 + s_1', \quad T = t_1 + t_1',$$

is advantageous, obtaining

$$\begin{aligned} \alpha_0 &= u_0 - \tfrac{2}{3}R, \\ \alpha_1 &= \frac{2R - u_2}{3}, \\ \alpha_2 &= r_1' + \frac{R - n - u_2}{3}, \\ \beta_0 &= v_0 - \tfrac{2}{3}S, \\ \beta_1 &= \frac{2S - v_2}{3}, \\ \beta_2 &= s_1' + \frac{S - \varepsilon n - v_2}{3}, \\ \gamma_0 &= w_0 - \tfrac{2}{3}T \\ \gamma_1 &= \frac{2T - w_2}{3}, \\ \gamma_2 &= t_1' + \frac{T - n - w_2}{3}. \end{aligned}$$

This substitution for the α's, β's, and γ's, yields for z :

$$\begin{aligned} z &= \left(u_0 - \tfrac{2}{3}R\right) + \left(\frac{2R - u_2}{3}\right)x + \left(r_1' + \frac{R - n - u_2}{3}\right)x^2 \\ &\quad + y\left\{\left(v_0 - \tfrac{2}{3}S\right) + \left(\frac{2S - v_2}{3}\right)x + \left(s_1' + \frac{S - \varepsilon n - v_2}{3}\right)x^2\right\} \\ &\quad + y^2\left\{\left(w_0 - \tfrac{2}{3}T\right) + \left(\frac{2T - w_2}{3}\right)x + \left(t_1' + \frac{T - n - w_2}{3}\right)x^2\right\} \\ &= z_1 + z_1' \end{aligned}$$

where

$$\begin{aligned} z_1 = &\left(u_0 - R + r_1'x^2\right) + y\left(v_0 - S + s_1'x^2\right) \\ &+ y^2\left(w_0 - T + t_1'x^2\right) \end{aligned}$$

has integral coordinates, and

(33) $$\begin{aligned} z_1' = &\tfrac{1}{3}\left(R + Sy + Ty^2\right)\left(1 + 2x + x^2\right) \\ &- \tfrac{1}{3}\left(u_2 + v_2 y + w_2 y^2\right)\left(x + x^2\right) \\ &- \tfrac{n}{3}\left(1 + \varepsilon y + y^2\right)x^2. \end{aligned}$$

Thus the elements of case I which have the rank property are obtained by annexing to the set $\mathcal{J}$ of elements having integral coordinates, the elements of the form z_1', given by (33), whose parameters satisfy (31).

Theorem I.

For case I, the necessary and sufficient conditions that the general element z of the algebras D with $\delta = \varepsilon$, and ε of the forms $9k \pm 1$, shall have the rank property, are that the congruences in (31) shall hold simultaneously in each sub-case.

Sub-case I .

Transformations (32) and relations (31) require

(34) $$R + \varepsilon S + T \equiv 0 \pmod 3,$$

and (27) requires

$$u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3.$$

Choose

$$T = w_2 = n = 0, \quad R = u_2 = 1, \quad S = v_2 = -\varepsilon,$$

which satisfy (34), obtaining a particular value of z_1',

$$A = \tfrac{1}{3}(1 - \varepsilon y)(1 + x).$$

Multiply A on the right by the two conjugates of $1 + x$, obtaining

$$\varepsilon y = 1 + A(-2 - x + x^2).$$

Consider $\varepsilon \equiv 1 \pmod 9$.

Using the above value of ε, we have

(35) $$y = 1 + A_1(-2 - x + x^2)$$

where

(36) $$A_1 = \tfrac{1}{3}(1-y)(1+x).$$

Multiplying (35) on the left by x and on the right by x', and then subtracting, we obtain

$$3(xA_1 - A_1x') = (1+x)(x-x'),$$

(37) $$xA_1 = 1 + A_1x'.$$

Multiply (37) on the left by x and substitute for xA_1 from (37), obtaining

(38) $$x^2A_1 = (-2+x+x^2) + A_1x'^2.$$

The remainder of the multiplication table may be computed by similar methods, but the results are not needed here.

Squaring (35), and using relations (37) and (38), we obtain

(39) $$y^2 = 1 + A_1(2-2x-x^2) + A_1^2(-9+3x+3x^2).$$

By means of (35) and (39), we now express z_1', given by (33), in terms of A_1 and x. We obtain

$$\begin{aligned} z_1' = {} & \frac{R+S+T}{3}(1+2x+x^2) - SA_1(1+x) + TA_1(2-3x-2x^2) \\ & + TA_1^2(-6+3x+3x^2) \\ & - \Big[\frac{u_2+v_2+w_2}{3}(x+x^2) - v_2A_1x + w_2A_1(1-2x-x^2) \\ & \quad + w_2A_1^2(-2+2x+x^2)\Big] \\ & - n\big[1 + A_1(1-3x) + A_1^2(-4+2x+3x^2)\big]. \end{aligned}$$

With $\varepsilon \equiv 1 \pmod 9$, reference to (34) shows that z_1' has been expressed, with integral coordinates, in terms of A_1 and x.

The rank equation for A_1, obtained by replacing the α's, β's, and γ's of (4) by their respective values in

(35), is

$$\omega^3 - \omega^2 \mp k = 0,$$

k being a rational integer.

These results show that in sub-case I_1, for $\varepsilon = 9h+1$, we have a set of elements possessing the rank property and having the basis $A_1^i x^j$ $(i, j = 0, 1, 2)$, containing the original basal units. The existence of the basis shows that the set is closed under addition, subtraction, and multiplication. The set is maximal (to be proved later, see section 10). The set contains the modulus 1. Hence, according to the definition, we have a set of integral elements.

Consider $\varepsilon \equiv -1$ (mod 9).

The conditions that the elements of set I shall have the rank property are, for this algebra,

(40) $$\begin{cases} R - S + T \equiv 0 \pmod 3, \\ u_2 - v_2 + w_2 \equiv 0 \pmod 3). \end{cases}$$

The general element of this set $I_1^{(-1)}$, with the term containing k removed to the set $\mathcal{J}$ since it has an integral coefficient, is

(41) $$\begin{aligned} z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 + 2x + x^2) \\ & - \tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) \\ & - \tfrac{n}{3}(1 - y + y^2)\, x^2. \end{aligned}$$

If, in (40) and (41), we replace $-S$ by S, $-v_2$ by v_2, and $-y$ by y, we obtain

(42) $$z_1' = \tfrac{1}{3}(R + Sy + Ty^2)(1 + 2x + x^2)$$

$$
\begin{aligned}
&-\tfrac{1}{3}(u_2+v_2y+w_2y^2)(x+x^2) \\
&-\tfrac{n}{3}(1+y+y^2)x^2,
\end{aligned}
\tag{42}
$$

where

$$R+S+T \equiv 0 \pmod 3,$$
$$u_2+v_2+w_2 \equiv 0 \pmod 3.$$

We now have the general element and the required conditions that we had for $\varepsilon = 9k+1$, and (42) may be expressed, with integral coefficients, in terms of the basis elements obtained there. By making the inverse transformation to the one above, we have a basis for the set $I_1^{(-1)}$.

Thus, for $\varepsilon = 9k-1$, we have

$$
\begin{aligned}
y &= -1+A_2(2+x-x^2), \\
y^2 &= 1+A_2(2-2x-x^2)+A_2^2(-9+3x+3x^2),
\end{aligned}
$$

where

$$A_2 = \tfrac{1}{3}(1+y)(1+x),$$

and

$$
\begin{aligned}
z_1' = {} & \frac{R-S+T}{3}(1+2x+x^2)+SA_2(1+x)+TA_2(2-3x-2x^2) \\
& +TA_2^2(-6+3x+3x^2) \\
& -\left[\frac{u_2-v_2+w_2}{3}(x+x^2)+v_2A_2x+w_2A_2(1-2x-x^2)\right. \\
& \left. +w_2A_2^2(-2+2x+x^2)\right] \\
& -n\left[1+A_2(1-3x)+A_2^2(-4+2x+3x^2)\right].
\end{aligned}
$$

The rank equation for A_2 is

$$\omega^3-\omega^2-k=0.$$

By reasoning similar to that used for $\varepsilon = 9k+1$, we have, for $\varepsilon = 9k-1$, a set of elements possessing the properties of rank, closure, unity, and maximality, and with the basis $A_2^{i} x^{j} (i, j = 0, 1, 2)$. This set is, thus, a set of integral elements according to the definition given in the introduction.

Sub-case I_2.

Transformations (32) and relations (31) require, for this sub-case,

$$(43) \quad \begin{cases} R + \varepsilon S + T \equiv 1 \pmod 3, \\ w_2 - u_2 \equiv 1 \pmod 3, \end{cases}$$

with r arbitrary (mod 3).

The set $I_2^{(1)}$ is not maximal, being contained in set $II_1^{(1)}$ (see section 6 for proof). From this, the set $I_2^{(-1)}$ is contained within the set $II_1^{(-1)}$.

Sub-case I_3.

The conditions required so that the elements of this set may have the rank property are as follows:

$$(44) \quad \begin{cases} R + \varepsilon S + T \equiv 1 \pmod 3, \\ u_2 - \varepsilon v_2 \equiv \varepsilon v_2 - w_2 \equiv w_2 - u_2 \equiv -1 \pmod 3, \\ u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \\ r = 0 . \end{cases}$$

Relation (33), under conditions (44), becomes

$$(45) \quad \begin{aligned} z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 + 2x + x^2) \\ & - \tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) . \end{aligned}$$

Choose

$$S = T = w_2 = 0 , \quad R = u_2 = 1 , \quad v_2 = -\varepsilon ,$$

satisfying (44), and substitute them in (45), giving, as a particular value of z_1' ,

$$C = \frac{1}{3}\left[1 + x + \varepsilon y(x + x^2)\right].$$

<u>Consider $\varepsilon \equiv 1$ (mod 9).</u>

Relations (44) become

$$(46)\quad \begin{cases} R + S + T - 1 \equiv 0 \pmod 3, \\ u_2 - v_2 \equiv v_2 - w_2 \equiv w_2 - u_2 \equiv -1 \pmod 3, \\ u_2 + v_2 + w_2 \equiv 0 \pmod 3. \end{cases}$$

Also, the above relations become

$$3C_1 = 1 + x + y(x + x^2)$$

from which

$$(47)\quad y = -3 + x^2 + C_1(7 - x - 2x^2),$$

$$(48)\quad y^2 = -2 + x + x^2 - C_1(1 + 2x + x^2) + C_1^2(12 - 3x^2),$$

$$xC_1 = 1 + C_1 x',$$

$$x^2 C_1 = (-2 + x + x^2) + C_1 x'^2.$$

The rank equation for C_1 is as follows:

$$\omega^3 - \omega^2 - k = 0$$

Substituting these values of and in (45), we obtain

$$\begin{aligned} z_1' = {} & \frac{R + S + T - 1}{3}(1 + x)^2 - S(2 + x) + SC_1(4 - x^2) \\ & + T(-2 + x + x^2) - TC_1(-1 + 5x + 3x^2) + TC_1^2(6 + 3x) \\ & - \Big[\frac{u_2 + v_2 + w_2}{3}(x + x^2) + v_2 C_1 - w_2 C_1(-1 + 3x + 2x^2) \\ & \quad + w_2 C_1^2(1 + 2x + x^2)\Big] \\ & + \frac{1 + v_2 - w_2}{3}(1 + x)^2 + w_2. \end{aligned}$$

Referring to (46), we see that the coefficients of

$C_1^{\,i} x^j\ (i,j=0,1,2)$ are rational integers. Thus $C_1^{\,i} x^j\ (i,j=0,1,2)$ form a basis of the set of integral elements of the division algebra D, with $\delta = \varepsilon = 9h+1$.

<u>Consider $\varepsilon \equiv -1 \pmod 9$.</u>

Relations (44) become

$$(49)\qquad \begin{cases} R-S+T-1 \equiv 0 \pmod 3, \\ u_2+v_2 \equiv -v_2-w_2 \equiv w_2-u_2 \equiv -1 \pmod 3, \\ u_2-v_2+w_2 \equiv 0 \pmod 3. \end{cases}$$

On a change of notation identical with that used in sub-case I_1, the following relations are obtained from the corresponding ones for $\varepsilon \equiv 1 \pmod 9$ of this sub-case:

$$y = 3-x^2-C_2(7-x-2x^2),$$
$$y^2 = (-2+x+x^2)-C_2(1+2x+x^2)+C_2^{\,2}(12-3x^2),$$

where

$$3C_2 = 1+x-y(x+x^2),$$

and

$$\begin{aligned} z_1' = {} & \frac{R-S+T-1}{3}(1+x)^2+S(2+x)-SC_2(4-x^2) \\ & +T(-2+x+x^2)-TC_2(-1+5x+3x^2)+TC_2^{\,2}(6+3x) \\ & -\Big[\frac{u_2-v_2+w_2}{3}(x+x^2)-v_2C_2-w_2C_2(-1+3x+2x^2) \\ & \quad +w_2C_2^{\,2}(1+2x+x^2)\Big] \\ & +\frac{1-v_2-w_2}{3}(1+x)^2+w_2. \end{aligned}$$

The rank equation for C_2 is

$$\omega^3-\omega+h=0.$$

Referring to (49), we see that z_1' is expressed in terms of $C_2^{\,i} x^j\ (i,j=0,1,2)$ with rational integral coefficients. Thus $C_2^{\,i} x^j$ $(i, j = 0, 1, 2)$ form a basis of the set of integral elements of the division algebras D, with $\delta = \varepsilon = 9k - 1$ which are obtained from (33) by putting $n = 0$ and values for parameters satisfying (49).

Sub-case I_4.

Consider $\varepsilon \equiv 1$ (mod 9).

The following are necessary and sufficient conditions that the set $I_4^{(1)}$ of elements of the algebra D, with $\delta = \varepsilon = 9k + 1$ shall have the rank property:

$$(50)\qquad \begin{cases} R + S + T \equiv 1 \pmod 3, \\ u_2 + v_2 + w_2 \equiv 0 \pmod 3, \\ u_2 \equiv v_2 \equiv w_2 \pmod 3, \\ n = -1 \end{cases}$$

We shall show that this set, $I_4^{(1)}$, of elements is contained within the set $I_3^{(1)}$ of integral elements. Form two distinct elements of $I_3^{(1)}$ by choosing two sets of values for R, S, and T, say R_1, S_1, T_1 and R_2, S_2, T_2, and one set of values for u_2, v_2, and w_2, say u_2', v_2', and w_2', where

$$(51)\qquad \begin{cases} R_1 + S_1 + T_1 \equiv 1 \pmod 3, \\ R_2 + S_2 + T_2 \equiv 1 \pmod 3, \end{cases}$$

and

$$u_2' - v_2' \equiv v_2' - w_2' \equiv w_2' - u_2' \equiv -1 \pmod 3.$$

These two elements of $I_3^{(1)}$ are

$$D_1 = \frac{1}{3}(R_1 + S_1 y + T_1 y^2)(1 + 2x + x^2)$$
$$-\frac{1}{3}(u_2' + v_2' y + w_2' y^2)(x + x^2),$$

and

$$D_2 = \frac{1}{3}(R_2 + S_2 y + T_2 y^2)(1 + 2x + x^2)$$
$$-\frac{1}{3}(u_2' + v_2' y + w_2' y^2)(x + x^2).$$

Subtracting, and replacing $R_1 - R_2$ by R', $S_1 - S_2$ by S', $T_1 - T_2$ by T', we obtain an element $D_1 - D_2$ which belongs in $I_3^{(1)}$ by closure:

$$D_1 - D_2 = \frac{1}{3}(R' + S' y + T' y^2)(1 + 2x + x^2).$$

Subtracting the relations (51) we obtain

$$(R_1 - R_2) + (S_1 - S_2) + (T_1 - T_2) \equiv 0 \pmod 3,$$

or

$$R' + S' + T' \equiv 0 \pmod 3.$$

Referring to relations (46), we have

$$u_2 \equiv w_2 + 1 \pmod 3,$$
$$v_2 \equiv w_2 - 1 \pmod 3,$$

and therefore

$$u_2 = w_2 + 1 + 3l_1,$$
$$v_2 = w_2 - 1 + 3l_2.$$

Substituting these values of u_2 and v_2 in (33), and transferring the terms containing l_1 and l_2 to the set $\mathcal{J}$ we obtain

$$Z_1' = \frac{1}{3}(R + S y + T y^2)(1 + 2x + x^2)$$
$$-\frac{1}{3}w_2(1 + y + y^2)(x + x^2) - \frac{1}{3}(1 - y)(x + x^2).$$

Choosing $w_2 = 0$, we see that

$$\frac{1}{3}(R + S y + T y^2)(1 + 2x + x^2) - \frac{1}{3}(1 - y)(x + x^2)$$

is an element of $I_3^{(1)}$, and then by closure, subtracting this from z_1' , we see that $\frac{1}{3} w_2 (1+y+y^2)(x+x^2)$ is an element of $I_3^{(1)}$.

From (50) we obtain the following:

$$(R-1) + S + T \equiv 0 \pmod 3,$$
$$u_2 = w_2 + 3 l_3,$$
$$v_2 = w_2 + 3 l_4 .$$

Substituting these values in (33), and replacing $R-1$ by R_3 , the general element for $I_4^{(1)}$ becomes

$$z_1' = \frac{1}{3}(R_3 + Sy + Ty^2)(1+2x+x^2) + \frac{1}{3}(1+2x+x^2),$$
$$-\frac{1}{3} w_2 (1+y+y^2)(x+x^2) + \frac{1}{3}(1+y+y^2)x^2,$$

where

$$R_3 + S + T \equiv 0 \pmod 3,$$

and in which the terms containing h , l_3, and l_4 have been transferred to the set $\mathcal{J}$.

Since

$$\frac{1}{3}(R_3 + Sy + Ty^2)(1+2x+x^2),$$
$$\frac{1}{3} w_2 (1+y+y^2)(x+x^2),$$

where

$$R_3 + S + T \equiv 0 \pmod 3$$

w_2 is arbitrary $\pmod 3$,

belong in $I_3^{(1)}$, the necessary and sufficient condition that the general element of $I_4^{(1)}$ be contained within $I_3^{(1)}$ is that

$$\frac{1}{3}(1+2x+x^2) + \frac{1}{3}(1+y+y^2)x^2$$

be contained within $I_3^{(1)}$ (from the property of closure), or that this element be expressible, with rational integral coefficients, in terms of C_1 and x. Substituting (47) and (48) in

$$\tfrac{1}{3}(1+2x+x^2)+\tfrac{1}{3}(1+y+y^2)x^2,$$

we obtain

$$(x+x^2)+C_1(2-x-x^2)x^2+C_1^{\,2}(4-x^2)x^2.$$

Thus the set $I_4^{(0)}$ is contained within the set $I_3^{(0)}$ and so is not maximal.

Consider $\varepsilon \equiv -1$ (mod 9).

The necessary and sufficient conditions that the set $I_4^{(-1)}$ of elements of the algebra D, with $\delta = \varepsilon = 9k-1$, shall have the rank property are the following:

$$R - S + T \equiv 1 \pmod 3,$$
$$u_2 - v_2 + w_2 \equiv 0 \pmod 3,$$
$$u_2 \equiv -v_2 \equiv w_2 \pmod 3,$$
$$n = -1.$$

The general element of $I_4^{(-1)}$, obtained from (33), is of the form

$$\begin{aligned} z_1' = {} & \tfrac{1}{3}(R+Sy+Ty^2)(1+2x+x^2) \\ & -\tfrac{1}{3}(u_2+v_2y+w_2y^2)(x+x^2) \\ & +\tfrac{1}{3}(1-y+y^2)x^2, \end{aligned}$$

after the term containing k has been transferred to the set $\mathcal{J}$.

For the set $I_4^{(-1)}$, the general element and the necessary and sufficient conditions determining the rank property bear the same relation to those of $I_3^{(-1)}$ as the general element and necessary and sufficient conditions of $I_4^{(1)}$ bear to those of $I_3^{(1)}$. From this we may show that

$$\frac{1}{3}(R_4 + S_4 y + T_4 y^2)(1+2x+x^2),$$
$$\frac{w_2}{3}(1-y+y^2)(x+x^2),$$

where

$$R_4 - S_4 + T_4 \equiv 0 \pmod 3,$$
$$w_2 \text{ is arbitrary} \pmod 3,$$

are elements of the set $I_3^{(-1)}$. Thus the necessary and sufficient condition that the set $I_4^{(-1)}$ be contained within the set $I_3^{(-1)}$ is that

$$\frac{1}{3}(1+2x+x^2) + \frac{1}{3}(1-y+y^2)x^2$$

be expressible, with rational integral coefficients, in terms of C_2 and x, the basis elements of $I_3^{(-1)}$. In terms of C_2 and x, this latter element becomes

$$x + x^2 + C_2(2-x-x^2)x^2 + C_2^2(4-x^2)x^2.$$

Thus the set $I_4^{(-1)}$ is contained within the set $I_3^{(-1)}$ and so is not maximal.

Sub-cases I_5, I_6, and I_7.

The necessary and sufficient conditions that the sets of elements in sub-cases I_5, I_6, and I_7 of the algebras D, with $\delta = \varepsilon = 9h \pm 1$, shall have the rank property are as follows:

(52)
$$\begin{cases} R + \varepsilon S + T \equiv -1 \pmod 3, \\ w_2 - u_2 \equiv -a \pmod 3, \\ u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \end{cases}$$

$$n = -b,$$

where a and b are equal to 0, 1, or 2.

The necessary and sufficient conditions that the sets of elements in sub-cases I_2, I_3, and I_4 of the algebras D, with $\delta = \varepsilon = 9k \pm 1$, shall have the rank property are as follows:

$$(53)\quad \begin{cases} R + \varepsilon S + T \equiv 1 \pmod 3, \\ w_2 - u_2 \equiv a \pmod 3, \\ u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3; \\ n = b, \end{cases}$$

where a and b have identically the same values as in (52).

The general element z_1' is given by (33) for all sub-cases.

If we replace

$$(54)\quad \begin{cases} R \text{ by } -R, & u_2 \text{ by } -u_2, \\ S \text{ by } -S, & v_2 \text{ by } -v_2, \quad n \text{ by } -n, \\ T \text{ by } -T, & w_2 \text{ by } -w_2, \end{cases}$$

in (52), we obtain the conditions (53).

Since this substitution replaces any basis element, such as C_1 of sub-case $I_3^{(1)}$, by its negative, we will have $y = f(x, -C_1)$ and $y^2 = \varphi(x, -C_1)$ for these sub-cases, where $y = f(x, C_1)$ and $y^2 = \varphi(x, C_1)$ are the substitutions used in $I_3^{(1)}$. Any general element z_1' which may be expressed in terms of x and C_1, with integral coefficients, may be so expressed in terms of x and $-C_1$. Thus the sets of elements obtained in these sub-cases are coincident with those obtained in sub-cases I_2, I_3, and I_4.

Theorem II.

In case I there exist two sets of integral elements for each of the algebras D, defined by $\delta = \varepsilon = 9k+1$ and $\delta = \varepsilon = 9k-1$. For $\varepsilon = 9k+1$, the elements of the two sets are formed by annexing elements of the form (33), and linear combinations of these elements, where the parameters satisfy (34) and (46) respectively, to elements of the set $\mathcal{J}$. For $\varepsilon = 9k-1$, the elements of the two sets are obtained by annexing elements of the form (33) and linear combinations of these elements, where the parameters satisfy (40) and (49) respectively, to elements of the set $\mathcal{J}$.

5. Case II

The conditions characterizing this case are the following:

$$(55)\quad \begin{cases} u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \\ r \equiv \varepsilon s \equiv t \pmod 3, \\ \rho \equiv \varepsilon \sigma \equiv \tau \equiv 1 \pmod 3; \end{cases}$$

and as a consequence of the last two

$$r' \equiv \varepsilon s' \equiv t' \pmod 3,$$

Choose the following transformations on the parameters r, s, t, r', s', and t', so that conditions (55) are satisfied:

$$(56)\quad \begin{cases} r = 3r_1 + n, & r' = 3r_1' + (1-n), \\ s = 3s_1 + \varepsilon n, & s' = 3s_1' + \varepsilon(1-n), \\ t = 3t_1 + n, & t' = 3t_1' + (1-n). \end{cases}$$

These transformations give for ρ, σ, and τ,

$$(57)\quad \begin{cases} \rho = 3(r_1 + r_1') + 1 = 3R + 1, \\ \sigma = 3(s_1 + s_1') + \varepsilon = 3S + \varepsilon, \\ \tau = 3(t_1 + t_1') + 1 = 3T + 1, \end{cases}$$

Substitute (56) and (57) in (13)(mod 81), using (30) and (32), thus obtaining

$$(58)\quad \begin{aligned} 3\Big[&(R + \varepsilon S + T) + n(R^2 + S^2 + T^2) + (r_1'^2 + s_1'^2 + t_1'^2) \\ &+ (r_1 r_1' + s_1 s_1' + t_1 t_1') + 2n(\varepsilon s_1 t_1 + \varepsilon r_1 s_1 \\ &+ r_1 t_1 + \varepsilon s_1 t_1' + \varepsilon r_1' s_1 + \varepsilon s_1' t_1 + \varepsilon r_1 s_1' \\ &+ r_1' t_1 + r_1 t_1' + \varepsilon s_1' t_1' + \varepsilon r_1' s_1' + r_1' t_1') \\ &- 2(\varepsilon s_1 t_1' + \varepsilon r_1' s_1 + \varepsilon s_1' t_1 + \varepsilon r_1 s_1' + r_1' t_1 + r_1 t_1') \end{aligned}$$

$$- (\varepsilon s_1't_1' + \varepsilon r_1's_1' + r_1't_1')$$
$$+(n-1)(u_2 r_1 + v_2 s_1 + w_2 t_1) - n(u_2 r_1' + v_2 s_1' + w_2 t_1')$$
$$+n(u_2 t_1' + \varepsilon v_2 r_1' + \varepsilon w_2 s_1') + (1-n)(\varepsilon u_2 s_1 + \varepsilon v_2 t_1 + w_2 r_1)$$
$$+2(R+\varepsilon S+T)(w_2 - u_2)^2]$$
$$+2[(u_2^2 - \varepsilon v_2 w_2) + (v_2^2 - u_2 w_2) + (w_2^2 - \varepsilon u_2 v_2)]$$
$$- (u_2^3 + \varepsilon v_2^3 + w_2^3 - 3\varepsilon u_2 v_2 w_2) \equiv 0 \pmod 9.$$

From $u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3$ we obtain

$$(u_2^2 - \varepsilon v_2 w_2) + (v_2^2 - u_2 w_2) + (w_2^2 - \varepsilon u_2 v_2)$$
$$= (u_2 + \varepsilon v_2 + w_2)^2 - 3(\varepsilon v_2 w_2 + u_2 w_2 + \varepsilon u_2 v_2)$$
$$\equiv -3(\varepsilon v_2 w_2 + u_2 w_2 + \varepsilon u_2 v_2)$$

(59) $$\equiv 3(w_2 - u_2)^2 \pmod 9.$$

Using (29), (30), and (59), congruence (58) becomes

(60) $$X_2 + n X_2^2 + X_1'^2 + X_1 X_1' - n Y_1 X_2 + 2X_2 Y_1^2 + 2 Y_1^2 \equiv 0 \pmod 3.$$

Substituting (56) and (57) in (12), and using (30), we obtain as a necessary and sufficient condition that the coefficient of ω in the rank equation of z be integral,

(61) $$X_2 \equiv Y_1 \pmod 3.$$

Substituting (61) in (60), congruence (60) becomes

$$X_1' + X_1 X_1' + 2X_2^2 \equiv 0 \pmod 3,$$

or

(62) $$X_1 X_2 \equiv 0 \pmod 3.$$

The solutions of (62) are as follows:

$$X_2 \equiv 0, \quad X_1 \text{ is arbitrary}, \pmod 3,$$
$$X_2 \equiv \pm 1, \quad X_1 \equiv 0 \pmod 3.$$

~~is arbitrary~~,

This suggests the following division into sub-cases:

Sub-case II_1 . $X_2 \equiv 0 \equiv Y_1 \pmod 3$,

Sub-case II_2 . $X_2 \equiv 1 \equiv Y_1 \pmod 3$,

Sub-case II_3 . $X_2 \equiv -1 \equiv Y_1 \pmod 3$.

<u>Theorem III.</u>

For case II, the necessary and sufficient conditions that elements of the algebras D, with $\delta = \varepsilon = 9k \pm 1$, shall have the rank property are given by the congruences characterizing the above sub-cases.

Using (8), (9), and (10), with (11) and (32), we express the α's, β's, and γ's in terms of the parameters R, S, T, u_2, v_2, w_2, and n, as follows:

$$\alpha_0 = u_0 - \tfrac{2}{3}R - \tfrac{2}{9},$$

$$\alpha_1 = \frac{2R - u_2}{3} + \frac{2}{9},$$

$$\alpha_2 = r_1' + \frac{R - u_2 - n}{3} + \frac{4}{9},$$

$$\beta_0 = v_0 - \tfrac{2}{3}S - \frac{2\varepsilon}{9},$$

$$\beta_1 = \frac{2S - v_2}{3} + \frac{2\varepsilon}{9},$$

$$\beta_2 = s_1' + \frac{S - v_2 - \varepsilon n}{3} + \frac{4\varepsilon}{9},$$

$$\gamma_0 = w_0 - \tfrac{2}{3}T - \tfrac{2}{9},$$

$$\gamma_1 = \frac{2T - w_2}{3} + \frac{2}{9},$$

$$\gamma_2 = t_1' + \frac{T - w_2 - n}{3} + \frac{4}{9},$$

The substitution of the above in the expression for z yields

$$
\begin{aligned}
z = {} & u_0 - R + Rx + s_1' x^2 \\
& + y\{v_0 - S + Sx + s_1' x^2\} \\
& + y^2\{w_0 - T + Tx + t_1' x^2\} \\
& + \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\
& - \tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) \\
& + \tfrac{1}{9}(1 + \varepsilon y + y^2)(-2 + 2x + 4x^2) - \tfrac{n}{3}(1 + \varepsilon y + y^2)x^2 \\
= {} & z_1 + z_1'
\end{aligned}
$$

where

$$
\begin{aligned}
z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\
& - \tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) \\
& + \tfrac{1}{9}(1 + \varepsilon y + y^2)(-2 + 2x + 4x^2) - \tfrac{n}{3}(1 + \varepsilon y + y^2)x^2 .
\end{aligned}
$$

Thus the sets of integral elements of case II are formed by annexing to the set $\mathcal{J}$ of elements with integral coefficients, the elements given by (63), where the parameters obey the conditions required in each sub-case, and linear combinations of such elements.

Sub-case II$_1$.

The necessary and sufficient conditions that a set of elements, given by (63), of the algebras D, with $\delta = \varepsilon = 9k \pm 1$, shall have the rank property are as follows:

$$
(64) \qquad
\begin{cases}
R + \varepsilon S + T \equiv 0 \pmod 3, \\
u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \\
u_2 \equiv \varepsilon v_2 \equiv w_2 \pmod 3, \\
n \text{ being arbitrary } \pmod 3.
\end{cases}
$$

~~being arbitrary~~

Consider $\varepsilon \equiv 1 \pmod 9$.

With $\varepsilon = 9k + 1$, the conditions (64) become

$$(65) \quad \begin{cases} R + S + T \equiv 0 \pmod 3 \\ u_2 + v_2 + w_2 \equiv 0 \pmod 3 \\ u_2 \equiv v_2 \equiv w_2 \pmod 3 \\ \qquad n \text{ being arbitrary} \pmod 3; \end{cases}$$

and the general element z_1', given by (63), after the terms containing k have been removed to $\mathfrak{J}$, becomes

$$(66) \quad \begin{aligned} z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\ & - \tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) \\ & + \tfrac{1}{9}(1 + y + y^2)(-2 + 2x + 4x^2) - \tfrac{n}{3}(1 + y + y^2)x^2. \end{aligned}$$

From (65) we obtain

$$(67) \quad \begin{cases} v_2 = u_2 + 3l_1, & S = R - k_1 + 3k_2, \\ w_2 = u_2 + 3l_2, & T = R + k_1 + 3k_3, \end{cases}$$

the expressions for S and T following from

$$R - S \equiv S - T \equiv T - R \equiv k_1,$$

which, in turn, follows from the first congruence of (65).

Substitute (67) in (66); we obtain

$$(68) \quad \begin{aligned} z_1' = {} & \tfrac{1}{3}[R(1 + y + y^2) - k_1(y - y^2)](1 - x + x^2) \\ & - \tfrac{1}{3}u_2(1 + y + y^2)(x + x^2) \\ & + \tfrac{1}{9}(1 + y + y^2)(-2 + 2x + 4x^2) - \tfrac{n}{3}(1 + y + y^2)x^2, \end{aligned}$$

after the terms with integral coefficients have been removed to $\mathfrak{J}$.

Substitute in (66) the following values of the parameters;

$$R = S = T = n = 0, \quad u_2 = v_2 = w_2 = 1.$$

These values satisfy (65). We then obtain as a special value of z_1',

(69) $$H_1 = \frac{1}{9}(1 + y + y^2)(-2 - x + x^2).$$

Multiplying H_1 on the right by the conjugates of $-2 - x + x^2$ we obtain

(70) $$1 + y + y^2 = 3H_1(-1 - x).$$

Squaring $9H_1$ from (69), and using $y^3 = \varepsilon$ we obtain

$$27H_1^2 = (2 - x)(1 - \varepsilon) + y(-1 + x^2)(1 - \varepsilon).$$

We may take $\varepsilon = 10$, since 10 is of the form $9h + 1$ and has prime factors of the form $9h + 2$ and $9h + 5$, neither of which occurs to a power which is a multiple of 3. With $\varepsilon = 10$, $27H_1^2$ becomes

$$3H_1^2 = -2 + x + y(1 - x^2),$$

which, on right-handed multiplication by the conjugates of $1 - x^2$, yields

(71) $$y = 2 - x^2 + H_1^2(4 - x - 2x^2).$$

Substitute (71) in (70), obtaining

(72) $$y^2 = -3 + x^2 + 3H_1(-1 - x) + H_1^2(-4 + x + 2x^2).$$

Substitution of (70), (71), and (72) in (68) yields, for the general element,

$$\begin{aligned} z_1' = &-3RH_1x - h_1\{(1 - x + x^2) + 3H_1x + H_1^2(2 - 2x^2)\} \\ &- [u_2H_1(1 - 4x - 2x^2) + (l_1y + l_2y^2)(x + x^2)] \\ &+ H_1(2 - 4x - 2x^2) + nH_1(1 + x)x^2. \end{aligned}$$

Thus z_1' is expressible, with rational integral coordinates, in terms of H_1 and x.

The rank equation of H_1 is

$$\omega^3 + k\omega - k^2 = 0.$$

We have, now, a set of elements, given by (66) where the parameters satisfy (65), of the algebra D with $\delta = \varepsilon = 9k+1$, which have the properties of rank, closure, unity, and maximality (to be proved later, see section 10). By definition, this set is a set of integral elements with the basis $H_1^{\,i} x^j \; (i, j = 0, 1, 2)$.

Consider $\varepsilon \equiv -1 \pmod 9$.

With $\varepsilon = 9k - 1$, the conditions (64) become

(73) $$\begin{cases} R - S + T \equiv 0 \pmod 3, \\ u_2 - v_2 + w_2 \equiv 0 \pmod 3, \\ u_2 \equiv -v_2 \equiv w_2 \pmod 3, \\ \qquad n \text{ being arbitrary} \pmod 3. \end{cases}$$

The general element z_1', given by (63), after the terms containing k have been removed to ϑ, becomes

(74) $$\begin{aligned} z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\ & - \tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) \\ & + \tfrac{1}{9}(1 - y + y^2)(-2 + 2x + 4x^2) - \tfrac{n}{3}(1 - y + y^2)x^2. \end{aligned}$$

Replacing

$$S \text{ by } -S, \quad v_2 \text{ by } -v_2, \quad y \text{ by } -y,$$

(73) and (74) become (65) and (66) respectively. Making these replacements in (69),

$$H_2 = \tfrac{1}{9}(1 - y + y^2)(-2 - x + x^2)$$

will serve as a basis element for the set of elements in this sub-case.

We have, then, a set of elements, given by (74) where the parameters satisfy (73), of the algebra D with $\delta = \varepsilon = 9k-1$, which have the properties of rank, closure, unity, and maximality (to be proved later, see section 10). By definition this is a set of integral elements with the basis $\mathcal{H}_2^i x^j (i,j=0,1,2)$.

Sub-case II$_2$.

The necessary and sufficient conditions that a set of elements of the algebras D, with $\delta = \varepsilon = 9k \pm 1$, shall have the rank property are as follows:

$$(75) \quad \begin{cases} R + \varepsilon S + T \equiv 1 \pmod 3, \\ u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \\ u_2 - \varepsilon v_2 \equiv \varepsilon v_2 - w_2 \equiv w_2 - u_2 \equiv 1 \pmod 3, \\ n \text{ being arbitrary} \pmod 3. \end{cases}$$

Consider $\varepsilon \equiv 1 \pmod 9$.

With $\varepsilon = 9k+1$, congruences (75) become

$$(76) \quad \begin{cases} R + S + T \equiv 1 \pmod 3, \\ u_2 + v_2 + w_2 \equiv 0 \pmod 3, \\ u_2 - v_2 \equiv v_2 - w_2 \equiv w_2 - u_2 \equiv 1 \pmod 3; \end{cases}$$

and the general element is given by (66).

From (76), with $R - 1 = R'$ we obtain the following:

$$R' - S \equiv S - T \equiv T - R' \equiv k_1 \pmod 3,$$

and therefore

$$S = R' - k_1 + 3k_2,$$
$$T = R' + k_1 + 3k_3,$$

and also

$$v_2 = w_2 + 1 + 3l_1,$$
$$u_2 = w_2 - 1 + 3l_2.$$

Substituting these in the general element and transferring the terms containing h_1, h_3, l_1, and l_2 to the set $\mathcal{J}$, relation (66) becomes

$$\begin{aligned} z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\ & -\tfrac{1}{3}\left[w_2(1+y+y^2) + (-1+y)\right](x + x^2) \\ & +\tfrac{1}{9}(1+y+y^2)(-2+2x+4x^2) - \tfrac{n}{3}(1+y+y^2)x^2. \end{aligned} \tag{77}$$

In the same manner, the general element for the set $II_1^{(1)}$ may be written in the form:

$$\begin{aligned} z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\ & -\tfrac{1}{3}w_2(1+y+y^2)(x+x^2) \\ & +\tfrac{1}{9}(1+y+y^2)(-2+2x+4x^2) - \tfrac{n}{3}(1+y+y^2)x^2, \end{aligned}$$

the parameters satisfying (65).

Taking $w_2 = 0$ in (78), and any set of values satisfying (65) for the remaining parameters, we obtain an element of $II_1^{(1)}$. Taking w_2 variable and the same set of values for the remaining parameters, we obtain another element of $II_1^{(1)}$. On subtracting these two elements, the element

$$\frac{w_2}{3}(1+y+y^2)(x+x^2)$$

is determined as being in the set $II_1^{(1)}$. With $R = S = T = w_2 = n = 0$ in (78), the element

$$\tfrac{1}{9}(1+y+y^2)(-2+2x+4x^2)$$

is in the set $II_1^{(1)}$. Using an argument similar to that used for $\frac{w_2}{3}(1+y+y^2)(x+x^2)$, we see that

$$\frac{n}{3}(1+y+y^2)x^2$$

is also in the set $II_1^{(1)}$. With $R = S = T = 0$ and then $R = S = T = 1$, determine two elements of $II_1^{(1)}$ which, on subtraction, yield

$$\frac{1}{3}(R + Sy + Ty^2)(1 - x + x^2)$$

where $R + S + T \equiv 0 \pmod 3$, as an element of $II_1^{(1)}$.

Since the term

$$\frac{1}{3}(R + Sy + Ty^2)(1 - x + x^2)$$

with $R + S + T \equiv 1 \pmod 3$, of (77), may be written as

$$\frac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) + \frac{1}{3}(1 - x + x^2),$$

where $R + S + T \equiv 0 \pmod 3$, it is obvious, using the results of the last paragraph, that the necessary and sufficient condition that the set $II_2^{(1)}$ be contained within the set $II_1^{(1)}$, is that this latter element be contained within $II_1^{(1)}$. Using (71), this element becomes

$$x^2 + H_1^2(1 - x + x^2)$$

and so is an element of $II_1^{(1)}$.

Thus the set $II_2^{(1)}$ is contained within the set $II_1^{(1)}$ and so is not maximal.

<u>Consider $\varepsilon \equiv -1 \pmod 9$.</u>

With $\varepsilon = 9k - 1$ relations (75) become

$$(79)\quad \begin{cases} R - S + T \equiv 1 \pmod 3, \\ u_2 - v_2 + w_2 \equiv 0 \pmod 3, \\ u_2 + v_2 \equiv -v_2 - w_2 \equiv w_2 - u_2 \equiv 1 \pmod 3, \\ \qquad n \text{ being arbitrary} \pmod 3. \end{cases}$$

The general element is given by (74).

The replacement of

$$S \text{ by } -S, \quad v_2 \text{ by } -v_2, \quad y \text{ by } -y,$$

transforms (79) into (76), (73) into (65), and (74) into (66). Thus the discussion of the sets $II_1^{(-1)}$ and $II_2^{(-1)}$ may be reduced to the discussion of sets $II_1^{(1)}$ and $II_2^{(1)}$ respectively. As a result the set $II_2^{(-1)}$ is contained within the set $II_1^{(-1)}$ and so is not maximal.

<u>Sub-case II_3.</u>

The necessary and sufficient conditions that a set of elements of the algebras D, with $\delta = \varepsilon = 9k \pm 1$, shall have the rank property are that the following congruences shall hold simultaneously:

$$(80) \quad \begin{cases} R + \varepsilon S + T \equiv -1 \pmod 3, \\ u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \\ u_2 - \varepsilon v_2 \equiv \varepsilon v_2 - w_2 \equiv w_2 - u_2 \equiv -1 \pmod 3, \\ \qquad n \text{ being arbitrary} \pmod 3. \end{cases}$$

The general element is given by (63).

Employing reasoning similar to that used in sub-case II_1, the necessary and sufficient condition that the set $II_3^{(1)}$ be contained within the set $II_1^{(1)}$ is that

$$-\tfrac{1}{3}\left[(1 - x + x^2) + (1 - y)(x + x^2)\right]$$

be expressible, with rational integral coordinates, in terms of H_1 and x. As seen in sub-case II_2, this is possible. Therefore the set $II_3^{(1)}$ is not maximal.

Similarly $II_3^{(-1)}$ is not maximal.

Theorem IV.

For case II, there exist two sets, II and II , of integral elements, one in each of the algebras D, defined by $\delta = \varepsilon = 9h+1$ and $\delta = \varepsilon = 9h-1$ respectively.

6. The Non-Maximality of Sets $I_2^{(1)}$ and $I_2^{(-1)}$.

We deduced in the last section that the following elements were contained in the set $II_1^{(1)}$:

$$\tfrac{1}{3}(R' + Sy + Ty^2)(1 - x + x^2)$$

and so, adding a multiple of 3 ,

$$\tfrac{1}{3}(R' + Sy + Ty^2)(1 + 2x + x^2),$$

$$\tfrac{w_2}{3}(1 + y + y^2)(x + x^2),$$

$$\tfrac{n}{3}(1 + y + y^2)x^2,$$

where

$$R' + S + T \equiv 0 \pmod 3,$$

w_2 and n being arbitrary (mod 3).

Since the general element for the set $I_2^{(1)}$ may be written in the form

$$Z_1' = \tfrac{1}{3}(R' + Sy + Ty^2)(1 + 2x + x^2) + \tfrac{1}{3}(1 + 2x + x^2)$$
$$- \tfrac{1}{3}w_2(1 + y + y^2)(x + x^2) + \tfrac{1}{3}(1 - y)(x + x^2)$$
$$- \tfrac{n}{3}(1 + y + y^2)x^2,$$

and since the above elements are contained in $II_1^{(1)}$, it is necessary and sufficient to prove that

$$\tfrac{1}{3}\left[(1 + 2x + x^2) + (1 - y)(x + x^2)\right]$$

is an element of the set $II_1^{(1)}$ in order to prove that the set $II_1^{(1)}$ contains the set $I_2^{(1)}$.

Using (71), this latter element becomes

$$x + x^2 - H_1^2(1 - x - x^2).$$

Thus the set $I_2^{(1)}$ is contained within the set $II_1^{(1)}$.

Since both $I_2^{(-1)}$ and $II_1^{(-1)}$ are obtained from $I_2^{(1)}$ and $II_1^{(1)}$ respectively by replacing S by $-S$, ν_2 by $-\nu_2$, y by $-y$, the set $I_2^{(-1)}$ is contained within the set $II_1^{(-1)}$ and so is not maximal.

7. Case III.

This case is characterized by

$$(81)\qquad \begin{cases} u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \\ r - \varepsilon s \equiv \varepsilon s - t \equiv t - r \equiv 1 \pmod 3, \\ \rho \equiv \varepsilon\sigma \equiv \tau \equiv 1 \pmod 3. \end{cases}$$

Satisfying (81), choose the following transformations on the parameters r, s, t, r', s', and t' :

$$(82)\qquad \begin{cases} r = 3r_1 + n, & r' = 3r_1' - (n-1), \\ s = 3s_1 + \varepsilon(n-1), & s' = 3s_1' - \varepsilon(n+1), \\ t = 3t_1 + (n+1), & t' = 3t_1' - n. \end{cases}$$

The substitution of (82) in (13)(mod 81) yields, after using (29) and

$$(u_2^2 - \varepsilon v_2 w_2) + (-2v_2^2 + u_2 w_2) + (w_2^2 - \varepsilon u_2 v_2) \equiv 0 \pmod 9,$$

and the substitutions (30):

$$(83)\qquad \begin{aligned} &m X_2 + n X_2^2 + n X_2 Y_1 + \varepsilon v_2 X_2 + 2 X_2 Y_1^2 \\ &+ [(r_1'^2 - t_1'^2) + (r_1 r_1' - s_1 s_1') + (t_1^2 - s_1^2) \\ &\quad + \varepsilon(s_1 t_1' + s_1' t_1 + r_1' s_1' - r_1 s_1 - s_1' t_1' \\ &\qquad + \varepsilon r_1 t_1 - \varepsilon r_1' t_1 - \varepsilon r_1 t_1')] \equiv 0 \pmod 3. \end{aligned}$$

Substitution of (82) in (12) yields

$$(84)\qquad X_2 \equiv 0 \pmod 3$$

as the only solution, and as a consequence

$$R - \varepsilon S \equiv \varepsilon S - T \equiv T - R \pmod 3.$$

Combining (84) with (83), we obtain

$$(85)\qquad X_2[m + n X_2 - n Y_1 + \varepsilon v_2 + 2Y_1^2 + T - R] \equiv 0 \pmod 3,$$

all of the solutions being included in (84).

Theorem V.

A comparison of (84) and (85) reveals that the necessary and sufficient conditions that the elements of the algebras D, with $\delta = \varepsilon = 9h \pm 1$, whose parameters, given by (82), satisfy (81), have the rank property are as follows:

$$(86)\qquad \begin{cases} R + \varepsilon S + T \equiv 0 \pmod 3, \\ u_2 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3, \\ w_2 - u_2 \text{ and } n \text{ being arbitrary} \pmod 3. \end{cases}$$

Employing (7) - (11) and (82), we obtain for the coordinates of the general element Z in this case:

$$\alpha_0 = u_0 - \tfrac{2}{3} R - \tfrac{2}{9},$$
$$\alpha_1 = \frac{2R - u_2}{3} + \frac{2}{9},$$
$$\alpha_2 = r_1' + \frac{R - u_2 - n}{3} + \frac{4}{9},$$
$$\beta_0 = v_0 - \tfrac{2}{3} S + \frac{4\varepsilon}{9},$$
$$\beta_1 = \frac{2S - v_2}{3} - \frac{4\varepsilon}{9},$$
$$\beta_2 = s_1' + \frac{S - v_2 - \varepsilon n}{3} - \frac{5\varepsilon}{9},$$
$$\gamma_0 = w_0 - \tfrac{2}{3} T - \tfrac{2}{9},$$
$$\gamma_1 = \frac{2T - w_2}{3} + \frac{2}{9},$$
$$\gamma_2 = t_1 + \frac{T - w_2 - n}{3} + \frac{1}{9}.$$

The general element is $Z = Z_1 + Z_1'$, where Z_1 has rational integral coordinates, and

$$(87)\qquad \begin{aligned} Z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\ & - \tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) \\ & + \tfrac{1}{9}(1 - 2\varepsilon y + y^2)(-2 + 2x + 4x^2) + \tfrac{1}{3}\varepsilon y x^2 \\ & - \tfrac{n}{3}(1 + \varepsilon y + y^2)x^2 - \tfrac{1}{3} y^2 x^2. \end{aligned}$$

Thus the elements belonging in this case and having the rank-property are obtained by annexing elements of the form (87) whose parameters satisfy conditions (86) to the set $\mathcal{J}$ of elements having integral coordinates.

Consider $\varepsilon \equiv 1 \pmod 9$.

With $\varepsilon = 9k+1$, the conditions (86) become

$$(88)\quad \begin{cases} R+S+T \equiv 0 \pmod 3, \\ u_2+v_2+w_2 \equiv 0 \pmod 3, \\ u_2-v_2 \equiv v_2-w_2 \equiv w_2-u_2 \equiv l \pmod 3, \\ \qquad n \text{ being arbitrary} \pmod 3, \end{cases}$$

where l is 0, 1, or 2.

The general element z_1', after transferring the terms with integral coefficients to $\mathcal{J}$, becomes

$$(89)\quad \begin{aligned} z_1' = {} & \tfrac{1}{3}(R+Sy+Ty^2)(1-x+x^2) \\ & -\tfrac{1}{3}(u_2+v_2y+w_2y^2)(x+x^2) \\ & +\tfrac{1}{9}(1+y+y^2)(-2+2x+4x^2) - \tfrac{n}{3}(1+y+y^2)x^2 \\ & -\tfrac{1}{3}y(-2+2x) - \tfrac{1}{3}y^2x^2. \end{aligned}$$

It may easily be shown by the methods used in former cases that the following elements, whose parameters satisfy the conditions noted, are contained in set $I_1^{(1)}$:

$$\tfrac{1}{3}(R+Sy+Ty^2)(1-x+x^2), \quad R+S+T \equiv 0 \pmod 3,$$
$$\tfrac{1}{3}(u_2+v_2y+w_2y^2)(x+x^2), \quad u_2+v_2+w_2 \equiv 0 \pmod 3,$$
$$\tfrac{n}{3}(1+y+y^2)x^2, \quad n \text{ arbitrary} \pmod 3.$$

From this it is necessary and sufficient to show that

$$\frac{1}{9}(1+y+y^2)(-2+2x+4x^2)-\frac{1}{3}y(-2+2x)-\frac{1}{3}y^2$$

is an element of set $I_1^{(1)}$ in order to show that the set $III^{(1)}$ is contained in $I_1^{(1)}$. Substitute (35) in this element, obtaining

$$A_1(-1+x) + A_1^2 .$$

Since the general element for the set of elements $III^{(1)}$ having the rank property is expressible, with rational integral coordinates, in terms of the basis elements of set $I_1^{(1)}$, the set $III^{(1)}$ is contained within the set $I_1^{(1)}$, and so is not maximal.

Consider $\varepsilon \equiv -1 \pmod 9$.

With $\varepsilon = 9k-1$, the conditions (86) become

$$R - S + T \equiv 0 \pmod 3,$$
$$u_2 - v_2 + w_2 \equiv 0 \pmod 3;$$

and the general element becomes

$$\begin{aligned} z_1' = {} & \tfrac{1}{3}(R+Sy+Ty^2)(1-x+x^2) \\ & -\tfrac{1}{3}(u_2+v_2y+w_2y^2)(x+x^2) \\ & +\tfrac{1}{9}(1-y+y^2)(-2+2x+4x^2)-\tfrac{1}{3}y^2x^2 \\ & +\tfrac{1}{3}y(-2+2x)-\tfrac{n}{3}(1-y+y^2)x^2. \end{aligned}$$

Replacing

$$S \text{ by } -S, \quad v_2 \text{ by } -v_2, \quad y \text{ by } -y,$$

the discussion of sets $I_1^{(-1)}$ and $III^{(-1)}$ may be reduced to the discussion of sets $I_1^{(1)}$ and $III^{(1)}$ respectively. Thus the set $III^{(-1)}$ is contained within the set $I_1^{(-1)}$ and so is not maximal.

<u>Theorem VI.</u>

There exist no sets of integral elements in either algebra D in case III. The elements, belonging in this case, which have the rank property occur in the integral sets $I_1^{(1)}$ and $I_1^{(-1)}$.

8. Case IV.

The conditions characterizing this case are as follows:

$$(90)\qquad \begin{cases} u_2 + \varepsilon v_2 + w_2 \equiv 1 \pmod 3, \\ r - \varepsilon s \equiv \varepsilon s - t \equiv t - r \equiv -1 \pmod 3, \\ \rho \equiv \varepsilon \sigma \equiv \tau \equiv 1 \pmod 3. \end{cases}$$

Choose the following transformations on the variables r, s, t, r', s', t', and u_2 such that conditions (90) are satisfied:

$$(91)\qquad \begin{cases} r = 3r_1 + n, & r' = 3r_1' - (n-1), \\ s = 3s_1 + \varepsilon(n+1), & s' = 3s_1' - \varepsilon n, \\ t = 3t_1 + (n-1), & t' = 3t_1' - (n+1), \end{cases}$$
$$u_2 = u_3 + 1.$$

Substituting $u_2 = u_3 + 1$ in $u_2 + \varepsilon v_2 + w_2 \equiv 1 \pmod 3$, we obtain

$$u_3 + \varepsilon v_2 + w_2 \equiv 0 \pmod 3.$$

These transformations (91) necessitate the following:

$$(92)\qquad \begin{cases} \rho = 3(r_1 + r_1') + 1 = 3R + 1, \\ \sigma = 3(s_1 + s_1') + \varepsilon = 3S + \varepsilon, \\ \tau = 3(t_1 + t_1') - 2 = 3T - 2. \end{cases}$$

Substitute (91) and (92) in (13)(mod 81), using (29), (30), and (32), and replacing

$$w_2 - u_3 = Y_2,$$

we obtain

$$(93)\qquad \begin{aligned} &X_2^2 + nX_2^2 + 2X_2Y_2^2 - Y_2^2 + X_1' + X_1'Y_2 - nY_2 + Y_2 \\ &+ Y_2(\varepsilon S - T) + (T - R) + n(-X_1 + \varepsilon S - T) - nX_2Y_2 - X_1X_2 \\ &+ X_2(\varepsilon S - T) \equiv 1 + n \pmod 3. \end{aligned}$$

The substitution of (91) and (92) in (12) yields as a necessary and sufficient condition that the coefficient of ω in the rank equation of z be integral:

$$(94) \qquad X_2 + Y_2 \equiv -1-n \pmod 3 .$$

Combining this expression with (93), we obtain

$$(95) \qquad X_2\left[n(X_2 - Y_2 - 1) - Y_2^2 + Y_2\right] \equiv 0 \pmod 3 .$$

The set of solutions of (94), which contains all the solutions of (95), is as follows:

$$(96) \qquad \begin{cases} X_2 \equiv -1 , & n + Y_2 \equiv 0 \pmod 3 , \\ X_2 \equiv 1 , & n + Y_2 \equiv 1 \pmod 3 , \\ X_2 \equiv 0 , & n + Y_2 \equiv -1 \pmod 3 . \end{cases}$$

<u>Theorem VII.</u>

For case IV, the necessary and sufficient conditions that elements of the algebras D, with $\delta = \varepsilon = 9k \pm 1$, shall have the rank property are given by (96).

The coordinates of the general element z for this case are the following:

$$\begin{aligned}
\alpha_0 &= u_0 - \tfrac{2}{3} R - \tfrac{2}{9} , \\
\alpha_1 &= \frac{2R - u_3 - 1}{3} + \frac{2}{9} , \\
\alpha_2 &= r_1' + \frac{R - u_3 - n}{3} + \frac{4}{9} , \\
\beta_0 &= v_0 - \tfrac{2}{3} S - \frac{2\varepsilon}{9} , \\
\beta_1 &= \frac{2S - v_2}{3} + \frac{2\varepsilon}{9} , \\
\beta_2 &= s_1' + \frac{S - v_2 - \varepsilon n}{3} + \frac{\varepsilon}{9} , \\
\gamma_0 &= w_0 - \tfrac{2}{3} T + \tfrac{4}{9} , \\
\gamma_1 &= \frac{2T - w_2}{3} - \frac{4}{9} , \\
\gamma_2 &= t_1' + \frac{T - w_2 - n}{3} - \frac{5}{9} ;
\end{aligned}$$

And in terms of these, the part of the general element which has fractional coefficients becomes

$$
\begin{aligned}
z_1' = {} & \tfrac{1}{3}(R + Sy + Ty^2)(1 - x + x^2) \\
& -\tfrac{1}{3}(u_2 + v_2 y + w_2 y^2)(x + x^2) \\
& +\tfrac{1}{9}(1 + \varepsilon y - 2y^2)(-2 + 2x + x^2) - \tfrac{1}{3} y^2 x^2 \\
& -\tfrac{n}{3}(1 + \varepsilon y + y^2)x^2 - \tfrac{1}{3}x .
\end{aligned}
\tag{97}
$$

By substituting for y and y^2 the relations expressing them in terms of $\mathcal{C}_1$, the basis element of set $I_3^{(1)}$, we may express z_1', given by (97) with $\varepsilon = 9k+1$, in terms of $\mathcal{C}_1$ and x with integral coefficients, for each of the sub-cases defined by (96).

Theorem VIII.

With $\varepsilon = 9k+1$, all the elements in case IV having the rank property, given by (97) where the parameters satisfy conditions (96), are to be found in the set of integral elements $I_3^{(1)}$. As a result, case IV contains no sets of integral elements for $\varepsilon = 9k+1$. Similarly it may be shown that it contains no such sets for $\varepsilon = 9k-1$.

9. Cases V. VI, and VII.

The conditions characterizing cases V, VI, and VII, and the logical substitutions to be used in each case may be generalized as follows:

$$(98)\quad \begin{cases} u_2 + \varepsilon v_2 + w_2 \equiv -b \pmod 3, \\ r - \varepsilon s \equiv \varepsilon s - t \equiv t - r \equiv -a \pmod 3, \\ \rho \equiv \varepsilon\sigma \equiv \tau \equiv -1 \pmod 3\ ; \end{cases}$$

$$(99)\quad \begin{cases} r = 3r_1 - n\ , & r' = 3r_1' + (n-1)\ , \\ s = 3s_1 - \varepsilon(n-a)\ , & s' = 3s_1' + \varepsilon(n-1-a)\ , \\ t = 3t_1 - (n+a)\ , & t' = 3t_1' + (n-1+a)\ , \end{cases}$$

a , b, and n being 0 , 1 , or 2 .

The conditions characterizing cases II, III, and IV, and the substitutions used in each case may be generalized as follows:

$$(100)\quad \begin{cases} u_2 + \varepsilon v_2 + w_2 \equiv b \pmod 3, \\ r - \varepsilon s \equiv \varepsilon s - t \equiv t - r \equiv a \pmod 3, \\ \rho \equiv \varepsilon\sigma \equiv \tau \equiv 1 \pmod 3\ ; \end{cases}$$

$$(101)\quad \begin{cases} r = 3r_1 + n\ , & r' = 3r_1' + (1-n)\ , \\ s = 3s_1 + \varepsilon(n-a)\ , & s' = 3s_1' + \varepsilon(1-n+a)\ , \\ t = 3t_1 + (n+a)\ , & t' = 3t_1' + (1-n-a)\ , \end{cases}$$

where a , b, and n have the same values as above.

The conditions (100) become conditions (98), and the substitutions (101) become (99) if we make the following replacements in (100) and (101) respectively:

$$r \text{ by } -r, \quad r' \text{ by } -r', \quad u_2 \text{ by } -u_2,$$
$$s \text{ by } -s, \quad s' \text{ by } -s', \quad v_2 \text{ by } -v_2,$$
$$t \text{ by } -t, \quad t' \text{ by } -t', \quad w_2 \text{ by } -w_2,$$
$$r_1 \text{ by } -r_1, \quad r_1' \text{ by } -r_1',$$
$$s_1 \text{ by } -s_1, \quad s_1' \text{ by } -s_1',$$
$$t_1 \text{ by } -t_1, \quad t_1' \text{ by } -t_1',$$

which obviously necessitate that the following replacements be made:

$$\rho \text{ by } -\rho, \quad R \text{ by } -R,$$
$$\sigma \text{ by } -\sigma, \quad S \text{ by } -S,$$
$$\tau \text{ by } -\tau, \quad T \text{ by } -T.$$

The same substitutions reduce the general element in each of the cases V, VI, and VII to the general elements of II, III, and IV respectively.

Theorem IX.

The sets of integral elements contained in cases V, VI, and VII are identical with the sets obtained in cases II, III, and IV.

10. The Maximality of Sets $I_1^{(1)}$, $I_1^{(-1)}$, $I_3^{(1)}$, $I_3^{(-1)}$, $II_1^{(1)}$, and $II_1^{(-1)}$.

The maximality of each of the sets mentioned in the heading is not obvious. This is due to the fact that the parameters, in terms of which the necessary and sufficient conditions that the elements have the rank property are expressed, are not independent, since r_1, s_1; t_1; r_1', s_1'; t_1', are functions of the u's, v's, and w's.

We determine the maximality of the above mentioned sets by expressing the basis element of each set in terms of the basis elements of each of the other sets. Substitute (47) in (36), obtaining

$$A_1 = \frac{1}{3}\left[5 + x - x^2 + C_1(9 - 3x^2)\right].$$

Thus A_1 is not in the set $I_3^{(1)}$. Similarly A_1 is not in the set $II_1^{(1)}$, and so the set $I_1^{(1)}$ is maximal.

In like manner each of the sets $I_3^{(1)}$ and $II_1^{(1)}$ may be shown to be maximal.

Similarly, for $\varepsilon = 9h - 1$, we may prove that the sets $I_1^{(-1)}$, $I_3^{(-1)}$, and $II_1^{(-1)}$, are maximal.

11. General Case, $\delta = \eta\varepsilon$.

Hull shows in his paper (Hull, pp. 28 - 31) that the necessary and sufficient conditions that the elements of the algebras D, with $\delta = \eta\varepsilon$, where η is the product of positive integral powers of 3 and like powers of rational primes of the form $9k \pm 1$, shall have the rank property are the same as for the algebras D with $\delta = \varepsilon$, but with respect to a new set of basal units given by $y_1^i x^j$ $(i, j = 0, 1, 2)$ where

$$y = y_1 E,$$
$$E = e_0 + e_1 x + e_2 x^2$$

being a number of $K(x)$, and $y_1^3 = \varepsilon$. In terms of these new basal units

$$Z = (\alpha_0 + \alpha_1 x + \alpha_2 x^2) + (\beta_0' y_1 + \beta_1' y_1 x + \beta_2' y_1 x^2)$$
$$+ (\gamma_0' y_1^2 + \gamma_1' y_1^2 x + \gamma_2' y_1^2 x^2)$$

the β's and γ's being rational integral functions of the original coordinates of Z .

Since Hull does this without taking into consideration the form of ε , his result holds when ε is of the forms $9k \pm 1$, as in this paper.

Theorem X.

For the algebras D, with $\delta = \eta\varepsilon$, η being the product of integral powers of 3 and like powers of rational primes of the forms $9k \pm 1$, and ε is the product of rational primes of the forms $9k \pm 2$ and $9k \pm 4$, at least one of which occurs to a power not a multiple of 3, but ε itself is of the forms $9k \pm 1$, there exist sets of integral elements each of which corresponds to a set obtained for $\delta = \varepsilon$, ε being restricted as above. In each case the set contains the original basal units

$y^i x^j \ (i, j = 0, 1, 2)$.

12. Conclusion.

The following theorems sum up the results obtained in this paper:

Theorem XI.

For the algebra D with $\delta = \varepsilon = 9k+1$, ε having rational prime factors of the forms $9k \pm 2$, and $9k \pm 4$, at least one of which occurs to a power not a multiple of 3, there exist three sets of integral elements, $I_1^{(1)}$, $I_3^{(1)}$, and $II_1^{(1)}$. The elements in each of these sets are given by (32) for $I_1^{(1)}$ and $I_3^{(1)}$, and by (66) for $II_1^{(1)}$, the parameters satisfying the conditions (34) with $\varepsilon \equiv 1 \pmod 9$, ~~(mod 9)~~, (46), and (65), respectively.

Theorem XII.

For the algebra D with $\delta = \varepsilon = 9k-1$, ε being restricted as in Theorem XI, there exist three sets of integral elements, $I_1^{(-1)}$, $I_3^{(-1)}$, and $II_1^{(-1)}$. The elements in each of these sets are given by (41) for the first two sets, and by (74) for the third set, the parameters satisfying the conditions (40), (49), and (73) respectively.

13. Bibliography.

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