THE DIVISION TRANSFORMATION FOR MATRIC POLYNOMIALS WITH SPECIAL REFERENCE TO THE QUARTIC CASE

by

Ivan Morton Niven

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This thesis is based on results which were obtained by Dr. M. M. Flood of Princeton University and reported upon by him in a paper presented to the American Mathematical Society at its New York meeting, February 25th, 1933. (Cf. Abstract No. 39-3-92, The Bulletin of the American Mathematical Society). Professor Flood treated the problem for the linear, quadratic and cubic cases. The present paper extends the theory to the quartic case. For the sake of clarity it has been found desirable to outline Professor Flood's treatment[1] for the earlier cases, although in so doing, certain modifications and elaborations are made. For example, the proofs of Theorems (4.13), (5.21) and (6.27), as here given, are original. The entire treatment for the quartic case is original.

[1] Permission to do this has been very kindly granted by Professor Flood, who gave the author access to his manuscript.

I. Introduction.

A matrix $a(\lambda)$ whose elements are polynomials in a scalar variable λ, of which at least one element has the highest degree n, will be called a matric polynomial of degree n, and will be expressed in the form

$$(1.1) \qquad a(\lambda) = \sum_{i=0}^{n} a_i \lambda^i , \qquad a_n \neq 0,$$

where the a_i are constant matrices. We shall consider these matrices square and of order r. If now

$$(1.2) \qquad b(\lambda) = \sum_{j=0}^{m} b_j \lambda^j , \qquad \left| b_m \right| \neq 0,$$

is a second such matric polynomial, it is well known that there exist unique matric polynomials $q(\lambda)$, $r(\lambda)$, $q_1(\lambda)$ and $r_1(\lambda)$ of which $q(\lambda)$ and $q_1(\lambda)$ are of degree $n-m$ and $r(\lambda)$ and $r_1(\lambda)$ are of degree less than m, such that

$$\begin{aligned} (1.3) \qquad a(\lambda) &\equiv q(\lambda)\, b(\lambda) + r(\lambda) \\ &\equiv b(\lambda)\, q_1(\lambda) + r_1(\lambda). \end{aligned}$$

The scope of this result has been extended by J. H. M. Wedderburn[1] to include the case in which the leading coefficient, b_m, of the divisor is singular but not not zero. It will be shown later, moreover, that in this case the quotient and remainder are not of necessity each unique and

[1] Cf. J. H. M. Wedderburn, "Lectures on Matrices," American Mathematical Society, 1934.

that the degree of the quotient may be greater than $n-m$ (Theorem 3.1). This paper will consider this latter case more fully and will give results concerning the nature and degree of the quotient and remainder obtained under the stated conditions.

II. Definitions and Notation.

Small Roman letters with a subscript denote constant matrices, i.e., matrices with elements in a given field. The matrix of order r, all of whose elements are zero except the one in the i^{th} row and the i^{th} column which is 1, is denoted by e_i. It will be called a unit matrix. Similarly v_i represents the unit vector whose r components are all zero except the i^{th}, which is 1. The identity matrix of order r will be denoted by i_r.

If p_k are positive integers such that

$$(2.1) \qquad r = \sum_{k=1}^{s} p_k \qquad ,$$

we define E_i by means of the relation

$$(2.2) \qquad E_i = \sum_{k=1}^{p_i} e_{p_1 + p_2 + \cdots + p_{i-1} + k} \quad (i = 1, \cdots, s).$$

It follows that $i_r = \sum_{k=1}^{s} E_k$.

Analogously, we define N_i ;

$$(2.3) \qquad N_i = \sum_{k=1}^{p_i} v_{p_1 + p_2 + \cdots + p_{i-1} + k} \quad (i = 1, \cdots, s).$$

Having chosen the p_k so that (2.1) is satisfied, we may consider the matrix i_r as a matrix I_s of order s, whose elements Δ_{ij} are rectangular matrices with p_i

rows and p_j columns such that

$$\Delta_{ij} = 0, \quad i \neq j,$$

and Δ_{ii} is the identity matrix of order p_i. In a similar manner, the constant matrix a_k with elements a_{kij} in r - space may be considered as a constant matrix A_k with elements A_{kij} in s - space. Then $A_{k\alpha\beta}$ is a rectangular matrix with p_α rows and p_β columns appearing as a block of terms in a_k. Thus e_k and v_k, the unit matrices and unit vectors in r - space, correspond to E_k and N_k, the unit matrices and unit vectors in s - space. The following relations are easily proved:

(2.4) $E_i \, E_i = E_i$ and $E_i \, E_j = 0$, $i \neq j$.

(2.5) $E_i \, N_i = N_i$ and $E_i \, N_j = 0$, $i \neq j$.

(2.6) $A_k \, N_i = \sum_{\alpha=1}^{s} N_\alpha \, A_{k\alpha i}$.

III. Existence of Quotient and Remainder.

(3.1) Theorem:- If $a(\lambda)$ and $b(\lambda)$ are the polynomials defined by (1.1) and (1.2) except that $|b_m|=0$, but $b_m \neq 0$, then there exist polynomials $q(\lambda)$, $r(\lambda)$, $q_1(\lambda)$ and $r_1(\lambda)$ of which $r(\lambda)$ and $r_1(\lambda)$ are of degree less than m, such that

$$a(\lambda) \equiv q(\lambda)\, b(\lambda) + r(\lambda)$$
$$\equiv b(\lambda)\, q_1(\lambda) + r_1(\lambda).$$

Proof:-

Let the rank of b_m be $p_1 < r$. Using the development of paragraph 2 with $s = 2$ we have $p_2 = r - p_1$. By elementary matrix theory b_m can be expressed in the form

(3.2) $$B_m = P_1 E_1 Q_1 ;$$

where P_1 and Q_1 are non-singular matrices. Writing

(3.3) $$h_1(\lambda) = (E_2 \lambda + E_1) P_1^{-1}$$

we see that $b_1(\lambda)$, defined by

(3.4) $$b_1(\lambda) = h_1(\lambda)\, b(\lambda) ,$$

is a matric polynomial whose degree is not higher than the degree of $b(\lambda)$ since the coefficient of λ^{m+1}, $E_2 P_1^{-1} B_m$, vanishes. Now

(3.5) $$|b_1(\lambda)| = |h_1(\lambda)| \cdot |b(\lambda)| = (\lambda^{p_2})\, |P_1^{-1}| \cdot |b(\lambda)|$$

so that the degree of $|b_1(\lambda)|$ exceeds the degree of $|b(\lambda)|$ by p_2. If the coefficient of λ^m in $b_1(\lambda)$ is singular, this process may be repeated giving $b_2(\lambda)$, $b_3(\lambda)$,---, where the degree of each $|b_i(\lambda)|$ exceeds that of $|b_{i-1}(\lambda)|$. But the degree of each $b_i(\lambda)$ is less than or equal to m and the degree of the determinant of a polynomial of the m^{th} degree cannot exceed r m. Hence the process must terminate, yielding a polynomial $b_j(\lambda)$, whose highest term has a non-singular coefficient, and from the law of formation (3.4) we have

$$b_j(\lambda) = h(\lambda)b(\lambda) \quad \text{where} \quad h(\lambda) = h_1(\lambda)h_2(\lambda)\text{---}\ h_j(\lambda).$$

Applying the division transformation (1.3) to $a(\lambda)$ and $b_j(\lambda)$ we have

$$\begin{aligned}(3.6) \quad a(\lambda) &= s(\lambda)\ b_j(\lambda) + r(\lambda) \\ &= s(\lambda)\ h(\lambda)\ b(\lambda) + r(\lambda) \\ &= q(\lambda)\ b(\lambda) + r(\lambda).\end{aligned}$$

Since P_1 and Q_1 in (3.2) are not unique, and since $q(\lambda)$ and $r(\lambda)$ depend upon the particular polynomial $h(\lambda)$ which is chosen, these latter polynomials are not unique. The above method of proof is obviously applicable to the dextro-lateral case.

IV. Association by a Linear Polynomial.

We shall say that the polynomial $h(\lambda)$ is associated with the polynomial $b(\lambda)$ if the degree of the polynomial $h(\lambda)b(\lambda)$ equals the degree of $b(\lambda)$ and if the coefficient of the leading term of $h(\lambda)\, b(\lambda)$ is non-singular.

As in section 3, let

$$(4.1) \quad b(\lambda) = \sum_{\alpha=0}^{m} b_\alpha \lambda^\alpha$$

be a matric polynomial for which b_m is singular but not zero, and suppose $h(\lambda)$ in (3.6) is linear and associated with $b(\lambda)$, so that

$$(4.2) \quad h(\lambda) = h_1 \lambda + h_0$$

and

$$(4.3) \quad h_1 b_m = 0.$$

Since we require the coefficient of λ^m in the product $h(\lambda)\, b(\lambda)$ to be non-singular, we may set this coefficient equal to i_r, without loss of generality, that is,

$$(4.4) \quad h_0 b_m + h_1 b_{m-1} = i_r .$$

Using (3.2) we get from (4.3)

$H_1\, P_1\, E_1\, Q_1 = 0$ or defining G_0 and G_1 by

$$(4.5) \quad G_0 = Q_1\, H_0\, P_1\,, \qquad G_1 = Q_1\, H_1\, P_1$$

we have

$$(4.6) \quad G_1\, E_1 = 0\,, \qquad G_1\, N_1 = 0.$$

Similarly, if we define

$$(4.7) \quad B_k^{(1)} = P_1^{-1}\, B_k\, Q_1^{-1} \qquad (k = 0, 1, ---, m-1),$$

then (3.2) and (4.4) give

$$H_0 P_1 E_1 Q_1 + H_1 P_1 B^{(1)}_{m-1} Q_1 = I_s ,$$

whence

$$(4.8) \quad G_0 E_1 + G_1 B^{(1)}_{m-1} = I_s .$$

Multiplying on the right by N_2, we obtain

$$(4.9) \quad G_1 B^{(1)}_{m-1} N_2 = G_1 N_2 B^{(1)}_{m-122} = N_2 ,$$

from which it follows that

$$(4.10) \quad \left| B^{(1)}_{m-122} \right| \neq 0.$$

If we set

$$(4.11) \quad \begin{cases} G_1 N_1 = 0; \quad G_1 N_2 = N_2 (B^{(1)}_{m-122})^{-1} ; \\ G_0 N_1 = N_1 - N_2 (B^{(1)}_{m-122})^{-1} B^{(1)}_{m-121} ; \ G_0 N_2 = 0. \end{cases}$$

we see that (4.3) and (4.4) are satisfied identically. Although P_1 and Q_1 in (3.2) are not of necessity unique, the rank of $B^{(1)}_{m-122}$ is independent of the manner in which they are chosen.

(4.12) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (4.1), the necessary and sufficient condition that there exist a linear polynomial associated with it is that the equation (4.10) be satisfied. One such polynomial is defined by equations (4.11).

(4.13) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (4.1), the necessary and sufficient condition that all polynomials associated with $b(\lambda)$ be linear is that

(4.10) be satisfied. That the condition is necessary follows from (4.12).

Assume now that $h(\lambda)$, as in (4.2), and

$$\ell(\lambda) = \sum_{k=0}^{t} \ell_k \lambda^k \, ; \, \ell_t \neq 0, \ (t > 1)$$

are each associated with $b(\lambda)$ and that equation (4.10) is satisfied. Then we have

$$0 = L_t B_m = L_t B_{m-1} + L_{t-1} B_m .$$

Setting

$$R_k = Q_1 L_k P_1 \qquad (k = 0, 1, -- t)$$

we have

$$0 = R_t E_1 = R_t B_{m-1}^{(1)} + R_{t-1} E_1$$

whence

$$R_t N_1 = 0 \text{ and } R_t B_{m-1}^{(1)} E_2 = R_t B_{m-122}^{(1)} = 0.$$

Since $B_{m-122}^{(1)}$ is non-singular,

$$Q_1 L_t P_1 = R_t = 0$$

giving $L_t = 0$, which contradicts the hypothesis.

V. Association by a Quadratic Polynomial.

As in section 4, let

$$(5.1)\qquad b(\lambda) = \sum_{\alpha=0}^{m} b_\alpha \lambda^\alpha$$

be a matric polynomial for which b_m is singular but not zero, and suppose $h(\lambda)$ is quadratic and associated with $b(\lambda)$; so that

$$(5.2)\qquad h(\lambda) = h_2 \lambda^2 + h_1 \lambda + h_0 ,$$

$$(5.3)\qquad h_2 b_m = 0 ,$$

$$(5.4)\qquad h_2 b_{m-1} + h_1 b_m = 0, \quad \text{and}$$

$$(5.5)\qquad h_2 b_{m-2} + h_1 b_{m-1} + h_0 b_m = i_r .$$

Since $h(\lambda)$ is not linear, it follows from Theorem (4.13) that

$$(5.6)\qquad \left| B^{(1)}_{m-122} \right| = 0.$$

In this section, we shall use $p_1^{(2)}$ in the same sense in which earlier we used p_1 and suppose that the rank of $B^{(1)}_{m-122}$ is $p_2^{(2)}$. Hence we have

$$r = p_1^{(2)} + p_2^{(2)} + p_3^{(2)} \text{ where } p_1 = p_1^{(2)} \text{ and } p_2 = p_2^{(2)} + p_3^{(2)}$$

In this new notation we may choose P_2 and Q_2 so that

$$(5.7)\qquad B_m = P_2 E_1 Q_2$$

and such that if

$$(5.8)\qquad B_k = P_2 B_k^{(2)} Q_2 \qquad (k = 0, 1, ---, m-1)$$

then the following relations are satisfied:-

$$(5.9)\begin{cases} B^{(2)}_{m-112} + B^{(2)}_{m-121} = 0 \\ B^{(2)}_{m-122} = \Delta_{22} \\ B^{(2)}_{m-132} = B^{(2)}_{m-133} = B^{(2)}_{m-123} = 0 . \end{cases}$$

This follows, since, as in (3.2), we can find P_1 and Q_1 such that $B_m = P_1 E_1 Q_1$.

If, also,

$$P_1^{-1} B_{m-1} Q_1^{-1} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

then, since $\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}$ is of rank $p_2^{(2)}$, we can find non-singular matrices X_1 and Y_1 each of order $p_2^{(2)} + p_3^{(2)}$ such that

$$X_1 \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} Y_1 = \begin{pmatrix} \Delta_{22} & 0 \\ 0 & 0 \end{pmatrix} .$$

Now, let

$$X = \begin{pmatrix} \Delta_{11} & 0 \\ 0 & X_1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \Delta_{11} & 0 \\ 0 & Y_1 \end{pmatrix},$$

then $X P_1^{-1} B_{m-1} Q_1^{-1} Y$ has the form $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & \Delta_{22} & 0 \\ C_{31} & 0 & 0 \end{pmatrix}$.

If

$$C_1 = \begin{pmatrix} \Delta_{11} & -C_{12} & 0 \\ 0 & \Delta_{22} & 0 \\ 0 & 0 & \Delta_{33} \end{pmatrix} \text{and} \quad C_2 = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ -C_{21} & \Delta_{22} & 0 \\ 0 & 0 & \Delta_{33} \end{pmatrix}$$

then

$$C_1 X P_1^{-1} B_{m-1} Q_1^{-1} Y C_2 = \begin{pmatrix} C_{11} - C_{12}C_{21} & 0 & C_{13} \\ 0 & \Delta_{22} & 0 \\ C_{31} & 0 & 0 \end{pmatrix}$$

Note that X, Y, C_1 and C_2 are each non-singular, and that since $X E_1 = E_1 Y = C_1 E_1 = E_1 C_2 = E_1$, then $X^{-1}E_1 = E_1 Y^{-1} = C_1^{-1}E_1 = E_1 C_2^{-1} = E_1$.

Place

$$P_2^{-1} = C_1 X P_1^{-1} \quad \text{and} \quad Q_2^{-1} = Q_1^{-1} Y C_2 \ ,$$

then

$$P_2 E_1 Q_2 = P_1 X^{-1} C_1^{-1} E_1 C_2^{-1} Y^{-1} Q_1 = P_1 E_1 Q_1 = B_m$$

and

$$P_2 B_{m-1}^{(2)} Q_2 = B_{m-1}$$

as desired in (5.7) and (5.8) where $B_{m-1}^{(2)}$ has a form such that conditions (5.9) are satisfied.

Using (5.7), we get from (5.3)

$H_2 P_2 E_1 Q_2 = 0$ or defining G_0, G_1, G_2 by

(5.10) $G_k = Q_2 H_k P_2 \quad (k = 0, 1, 2)$

we have

(5.11) $G_2 E_1 = 0 \ , \quad G_2 N_1 = 0$.

Using (5.8) and (5.10), we get from (5.4)

(5.12) $G_2 B_{m-1}^{(2)} + G_1 E_1 = 0$

Multiplication on the right by N_2 gives

$$G_2 B_{m-1}^{(2)} N_2 = 0$$

which yields on application of (2.6) and (5.9)

(5.13) $G_2 N_2 = 0$

Multiplication of (5.12) on the right by N_1 gives

$$G_2\, B^{(2)}_{m-1}\, N_1 \;+\; G_1\, N_1 \;=\; 0$$

which gives, as a consequence of (2.6), (5.9) and (5.11),

$$(5.14)\quad G_1\, N_1 \;=\; -\,G_2\, N_3\, B^{(2)}_{m-131}$$

Using (5.7), (5.8) and (5.10), we get from (5.5)

$$G_2\, B^{(2)}_{m-2} \;+\; G_1\, B^{(2)}_{m-1} + G_0\, E_1 \;=\; I_s$$

Multiplication on the right by N_3 gives

$$G_2\, B^{(2)}_{m-2}\, N_3 + G_1\, B^{(2)}_{m-1}\, N_3 = N_3$$

Applying (2.6), (5.9), (5.11), (5.13) and (5.14), we get

$$(5.15)\quad G_2\, N_3 \left[B^{(2)}_{m-233} - B^{(2)}_{m-131}\, B^{(2)}_{m-113} \right] = N_3$$

Setting

$$(5.16)\quad V_{33} = B^{(2)}_{m-233} - B^{(2)}_{m-131} B^{(2)}_{m-113}$$

we have

$$(5.17)\quad G_2\, N_3\, V_{33} \;=\; N_3\ .$$

It necessarily follows that

$$(5.18)\quad \left| V_{33} \right| \;=\; \left| B^{(2)}_{m-233} - B^{(2)}_{m-131}\, B^{(2)}_{m-113} \right| \neq 0\ .$$

If we set arbitrarily

$$(5.19)\begin{cases} G_2\, N_1 = G_2\, N_2 = 0\ ; \quad G_2\, N_3 = N_3\, V_{33}^{-1} \\ G_1\, N_1 = -\,N_3\, V_{33}^{-1}\, B^{(2)}_{m-131}\ ; \quad G_1\, N_2 = N_2 - N_3\, V_{33}^{-1}\, B^{(2)}_{m-232} \\ G_1\, N_3 = G_0\, N_2 = G_0\, N_3 = 0 \\ G_0\, N_1 = N_1 - N_3\, V_{33}^{-1} \left[B^{(2)}_{m-231}\ \ B^{(2)}_{m-131}\ \ B^{(2)}_{m-111} \right] \end{cases}$$

we see that (5.3), (5.4) and (5.5) are satisfied identically. Again, although P_2 and Q_2 are not of necessity unique, the rank of V_{33} is independent of the manner in which they are chosen.

(5.20) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (5.1), the necessary and sufficient condition that there exist a quadratic polynomial associated with it is that (5.18) be satisfied. One such polynomial is defined by equations (5.19).

(5.21) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (5.1), the necessary and sufficient conditions that all polynomials associated with it be quadratic is that (5.18) be satisfied.

That this condition is necessary follows from (5.20). Assume now that $h(\lambda)$, as in (5.2), and

$$\ell(\lambda) = \sum_{k=0}^{t} \ell_k \lambda^k, \quad \ell_t \neq 0, \quad (t > 2)$$

are each associated with $b(\lambda)$. Then we have

$$0 = L_t B_m = L_t B_{m-1} + L_{t-1} B_m = L_t B_{m-2} + L_{t-1} B_{m-1} + L_{t-2} B_m$$

Setting

$$R_k = Q_2 L_k P_2 \qquad (k = 0, 1, \text{--} \; t)$$

we have

$$0 = R_t E_1 = R_t B^{(2)}_{m-1} + R_{t-1} E_1 = R_t B^{(2)}_{m-2} + R_{t-1} B^{(2)}_{m-1} + R_{t-2} E_1$$

$$R_t N_1 = 0$$

$$R_t N_2 = 0 \qquad R_{t-1} N_1 = - R_t N_3 B^{(2)}_{m-131}$$

$$R_t N_3 \left[B^{(2)}_{m-233} - B^{(2)}_{m-131} B^{(2)}_{m-113} \right] = 0 \quad \text{whence} \quad R_t N_3 = 0.$$

It follows that

$$Q_2 L_t P_2 \equiv R_t = 0 .$$

Hence

$$L_t = 0,$$

which contradicts the hypothesis.

VI. Association by a Cubic Polynomial.

As previously, let

$$(6.1)\quad b(\lambda) = \sum_{\alpha=0}^{m} b_\alpha \lambda^\alpha$$

be a matric polynomial for which b_m is singular but not zero, and suppose $h(\lambda)$ is cubic and associated with $b(\lambda)$, so that

$$(6.2)\quad h(\lambda) = h_3\lambda^3 + h_2\lambda^2 + h_1\lambda + h_0$$

$$(6.3)\quad h_3 b_m = 0$$

$$(6.4)\quad h_3 b_{m-1} + h_2 b_m = 0$$

$$(6.5)\quad h_3 b_{m-2} + h_2 b_{m-1} + h_1 b_m = 0$$

$$(6.6)\quad h_3 b_{m-3} + h_2 b_{m-2} + h_1 b_{m-1} + h_0 b_m = i_r .$$

Since $h(\lambda)$ is neither linear nor quadratic, we see by Theorems (4.13) and (5.21) that neither (4.10) nor (5.18) can be satisfied.

In this section, we shall use $p_1^{(3)}$ and $p_2^{(3)}$ in the same sense as, in section 5, we used $p_1^{(2)}$ and $p_2^{(2)}$ respectively, and suppose that the rank of V_{33} is $p_3^{(3)}$. Hence we have $r = p_1^{(3)} + p_2^{(3)} + p_3^{(3)} + p_4^{(3)}$ where $p_1^{(3)} = p_1^{(2)}$ and $p_2^{(3)} = p_2^{(2)}$ and $p_3^{(3)} + p_4^{(3)} = p_3^{(2)}$.

In this new notation we may choose P_3 and Q_3 so that

$$(6.7)\quad B_m = P_3 E_1 Q_3$$

and such that if

$$(6.8)\quad B_k = P_3 B_k^{(3)} Q_3 \qquad (k = 0, 1, \cdots, m-1).$$

then the following relations are satisfied:-

$$(6.9)\quad \begin{cases} B^{(3)}_{m-112} = B^{(3)}_{m-121} = B^{(3)}_{m-123} = B^{(3)}_{m-124} = 0 \\ B^{(3)}_{m-13k} = B^{(3)}_{m-14k} = 0 \qquad (k = 2, 3, 4) \\ B^{(3)}_{m-122} = \Delta_{22} \end{cases}$$

$$(6.10)\quad \begin{cases} B^{(3)}_{m-233} - B^{(3)}_{m-131} B^{(3)}_{m-113} = \Delta_{33} \\ B^{(3)}_{m-234} - B^{(3)}_{m-131} B^{(3)}_{m-114} = 0 \\ B^{(3)}_{m-243} - B^{(3)}_{m-141} B^{(3)}_{m-113} = 0 \\ B^{(3)}_{m-244} - B^{(3)}_{m-141} B^{(3)}_{m-114} = 0 \end{cases}$$

Conditions (6.9) correspond to conditions (5.9) and are derived by methods which are identical with those following (5.9). In order to derive (6.10), consider

$$V_{33} = B^{(2)}_{m-233} - B^{(2)}_{m-131} B^{(2)}_{m-113} ,$$

which is of rank $p_3^{(3)}$ and of order $p_3^{(2)} = p_3^{(3)} + p_4^{(3)}$.

When we pass to the notation of this section, V_{33} becomes

$$\begin{pmatrix} B^{(3)}_{m-233} & B^{(3)}_{m-234} \\ B^{(3)}_{m-243} & B^{(3)}_{m-244} \end{pmatrix} - \begin{pmatrix} B^{(3)}_{m-131} & 0 \\ B^{(3)}_{m-141} & 0 \end{pmatrix} \begin{pmatrix} B^{(3)}_{m-113} & B^{(3)}_{m-114} \\ 0 & 0 \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} B^{(3)}_{m-233} - B^{(3)}_{m-131} B^{(3)}_{m-113} & B^{(3)}_{m-234} - B^{(3)}_{m-131} B^{(3)}_{m-114} \\ B^{(3)}_{m-243} \quad B^{(3)}_{m-141} B^{(3)}_{m-113} & B^{(3)}_{m-244} \quad B^{(3)}_{m-141} B^{(3)}_{m-114} \end{pmatrix}$$

We can find non-singular matrices H_1 and K_1 of order $p_3^{(3)} + p_4^{(3)}$ such that

$$H_1 V_{33} K_1 = \begin{pmatrix} \Delta_{33} & 0 \\ 0 & 0 \end{pmatrix}$$

whence conditions (6.10) arise. If now, we write

$$H = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & \Delta_{22} & 0 \\ 0 & 0 & H_1 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & \Delta_{22} & 0 \\ 0 & 0 & K_1 \end{pmatrix}$$

it will be noted that the H and K can be absorbed by the P_3 and Q_3, as were the X, Y, C_1 and C_2 in the last section, thus leaving the forms of $B^{(3)}_{m-1}$ and $B^{(3)}_{m} = E_1$ unaltered.

Using (6.7), we get from (6.3)

$H_3 P_3 E_1 Q_3 = 0$ or defining G_0, G_1, G_2 ; G_3 by

$$(6.11) \quad G_k = Q_3 H_k P_3 \qquad (k = 0, 1, 2, 3)$$

we have

$$(6.12) \quad G_3 E_1 = 0 , \qquad G_3 N_1 = 0 .$$

Using (6.7), (6.8) and (6.11), the relation (6.4) gives

$$(6.13) \quad G_3 B^{(3)}_{m-1} + G_2 E_1 = 0$$

Multiplication on the right by N_2 gives

$$G_3 \; B_{m-1}^{(3)} \quad N_2 = 0$$

which gives, on application of (2.6) and (6.9),

$$(6.14) \quad G_3 \, N_2 \quad = \quad 0 \; .$$

Multiplication of (6.13) on the right by N_1 yields

$$G_3 \; B_{m-1}^{(3)} \, N_1 \quad + \quad G_2 \, N_1 \quad = \quad 0$$

whence

$$(6.15) \quad G_2 \; N_1 \;=\; - \left[G_3 \; N_3 \; B_{m-131}^{(3)} + G_3 \; N_4 \; B_{m-141}^{(3)} \right] .$$

Using (6.7), (6.8) and (6.11), we get from (6.5)

$$(6.16) \quad G_3 \; B_{m-2}^{(3)} \; + \; G_2 \; B_{m-1}^{(3)} \; + \; G_1 \; E_1 \quad = \quad 0 \quad .$$

Multiplication on the right by N_3 yields

$$G_3 \; B_{m-2}^{(3)} \quad N_3 \; + \; G_2 \; B_{m-1}^{(3)} \quad N_3 \quad = \quad 0$$

which, on application of (2.6), (6.9), (6.15) and (6.10), gives

$$(6.17) \quad G_3 \, N_3 \quad = \quad 0 \; .$$

Multiplication of (6.16) on the right by N_2 yields

$$G_3 \; B_{m-2} \quad N_2 \quad + \quad G_2 \; B_{m-1} \quad N_2 \quad = \quad 0$$

which, on application of (2.6), (6.9), (6.15), (6.17), (6.12), (6.14) and (6.10), gives

$$(6.18) \qquad G_2 \, N_2 \quad = \quad - \; G_3 \; N_4 \; B_{m-242}^{(3)} \; .$$

Multiplication of (6.16) on the right by N_1 gives, in a similar manner

$$(6.19)\quad G_1\, N_1 = G_3\, N_4 \left[B^{(3)}_{m-141}\, B^{(3)}_{m-111} - B^{(3)}_{m-241} \right] - G_2\, N_3\, B^{(3)}_{m-131} - G_2\, N_4\, B^{(3)}_{m-141}$$

Using (6.7), (6.8) and (6.11), we get from (6.6)

$$(6.20)\quad G_3\, B^{(3)}_{m-3} + G_2\, B^{(3)}_{m-2} + G_1\, B^{(3)}_{m-1} + G_0\, E_1 = I_s$$

Multiplication on the right by N_4 gives, upon simplification

$$(6.21)\quad G_3\, N_4 \Big[B^{(3)}_{m-344} - B^{(3)}_{m-141}\, B^{(3)}_{m-214} - B_{m-242}\, B_{m-224} + B_{m-141}\, B_{m-111}\, B_{m-114} - B_{m-241}\, B_{m-114} \Big] = N_4$$

Setting

$$(6.22)\quad V_{44} = B^{(3)}_{m-344} - B^{(3)}_{m-141}\, B^{(3)}_{m-214} - B^{(3)}_{m-242}\, B^{(3)}_{m-224} + B^{(3)}_{m-141}\, B^{(3)}_{m-111}\, B^{(3)}_{m-114} - B^{(3)}_{m-241}\, B^{(3)}_{m-114}$$

we have

$$(6.23)\quad G_3\, N_4\, V_{44} = N_4 \,.$$

It necessarily follows that

$$(6.24)\quad \left| V_{44} \right| \neq 0 \,.$$

If we set arbitrarily

$$(6.25)\quad G_3 N_1 = G_2 N_2 = G_3 N_3 = 0 \qquad G_3 N_4 = N_4 V_{44}^{-1}$$

$$G_2 N_1 = - N_4 V_{44}^{-1} B^{(3)}_{m-141} \qquad G_2 N_2 = - N_4 V_{44}^{-1} B^{(3)}_{m-242}$$

$$G_2 N_3 = N_3 + N_4 V_{44}^{-1} V_{43} \qquad G_2 N_4 = 0$$

$$G_1 N_1 = N_4 V_{44}^{-1} \left[B^{(3)}_{m-141} B^{(3)}_{m-111} - B^{(3)}_{m-241} \right] - \left[N_3 + N_4 V_{44}^{-1} V_{43} \right] B^{(3)}_{m-131}$$

$$G_1 N_2 = N_2 - N_3 B^{(3)}_{m-232} - N_4 V_{44}^{-1} \left[B^{(3)}_{m-342} - B^{(3)}_{m-141} B^{(3)}_{m-212} - B^{(3)}_{m-242} B^{(3)}_{m-222} + V_{43} B^{(3)}_{m-232} \right]$$

$$G_1 N_3 = G_1 N_4 = 0$$

$$G_0 N_1 = N_1 + N_3 \left[B^{(3)}_{m-131} B^{(3)}_{m-111} - B^{(3)}_{m-231} \right] + N_4 V_{44}^{-1} \left[B^{(3)}_{m-141} B^{(3)}_{m-211} + B^{(3)}_{m-242} B^{(3)}_{m-221} - B^{(3)}_{m-341} + V_{43} B^{(3)}_{m-131} B^{(3)}_{m-111} - V_{43} B^{(3)}_{m-231} - B^{(3)}_{m-141} B^{(3)}_{m-111} B^{(3)}_{m-111} + B^{(3)}_{m-241} B^{(3)}_{m-111} \right]$$

$$G_0 N_2 = G_0 N_3 = G_0 N_4 = 0 \text{ , where}$$

$$V_{43} = B^{(3)}_{m-141} B^{(3)}_{m-213} + B^{(3)}_{m-242} B^{(3)}_{m-223} - B^{(3)}_{m-343} + B^{(3)}_{m-241} B^{(3)}_{m-113} - B^{(3)}_{m-141} B^{(3)}_{m-111} B^{(3)}_{m-113}$$

we see that (6.3), (6.4), (6.5) and (6.6) are satisfied identically. Again, although P_3 and Q_3 are not of necessity unique, the rank of V_{44} is independent of the manner in which they are chosen.

(6.26) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (6.1), the necessary and sufficient condition that there exist a cubic polynomial associated with it is that equation (6.24) be satisfied. One such polynomial is defined by equations (6.25).

(6.27) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (6.1), the necessary and sufficient condition that all polynomials associated with it be cubic is that the equation (6.24) be satisfied. The proof of this is analogous to the proofs of Theorems (5.21) and (4.13), and so will not be given.

VII. Association by a Quartic Polynomial.

As previously, let

$$(7.1) \quad b(\lambda) = \sum_{\alpha=0}^{m} b_\alpha \lambda^\alpha$$

be a matric polynomial for which b_m is singular but not zero, and suppose $h(\lambda)$ is quartic and associated with $b(\lambda)$, so that

$$(7.2) \quad h(\lambda) = h_4 \lambda^4 + h_3 \lambda^3 + h_2 \lambda^2 + h_1 \lambda + h_0 ,$$

$$(7.3) \quad h_4 b_m = 0 ;$$

$$(7.4) \quad h_4 b_{m-1} + h_3 b_m = 0 ,$$

$$(7.5) \quad h_4 b_{m-2} + h_3 b_{m-1} + h_2 b_m = 0 ,$$

$$(7.6) \quad h_4 b_{m-3} + h_3 b_{m-2} + h_2 b_{m-1} + h_1 b_m = 0 ,$$

$$(7.7) \quad h_4 b_{m-4} + h_3 b_{m-3} + h_2 b_{m-2} + h_1 b_{m-1} + h_0 b_m = i_r .$$

Since $h(\lambda)$ is not linear, quadratic or cubic we see by Theorems (4.13), (5.21) and (6.27) that none of (4.10), (5.18) and (6.24) can be satisfied.

In this section, we shall use $p_1^{(4)}$, $p_2^{(4)}$ and $p_3^{(4)}$ in the same sense as, in section 6, we used $p_1^{(3)}$, $p_2^{(3)}$ and $p_3^{(3)}$, respectively, and suppose that the rank of V_{44} is $p_4^{(4)}$. Hence we have

$r = p_1^{(4)} + p_2^{(4)} + p_3^{(4)} + p_4^{(4)} + p_5^{(4)}$ where $p_1^{(4)}, p_2^{(4)}, p_3^{(4)}$ $= p_1^{(3)}, p_2^{(3)}, p_3^{(3)}$; respectively, and $p_4^{(4)} + p_5^{(4)} = p_4^{(3)}$.

In this new notation, we may choose P_4 and Q_4 so that

$$(7.8)\quad B_m = P_4 E_1 Q_4$$

and such that if

$$(7.9)\quad B_k = P_4 B_k Q_4 \qquad (k = 0, 1, ---, m-1)$$

then the following relations are satisfied:-

$$(7.10)\quad \begin{cases} B^{(4)}_{m-112} = B^{(4)}_{m-121} = B^{(4)}_{m-123} = B^{(4)}_{m-124} = B^{(4)}_{m-125} = 0 \\ B^{(4)}_{m-13k} = B^{(4)}_{m-14k} = B^{(4)}_{m-15k} = 0 \qquad (k = 2, 3, 4, 5) \\ B^{(4)}_{m-122} = \Delta_{22} \end{cases}$$

$$(7.11)\quad \begin{cases} B^{(4)}_{m-2jk} - B^{(4)}_{m-1j1} \quad B^{(4)}_{m-11k} = 0 \qquad (j, k = 3, 4, 5) \\ \text{except when } j = k = 3 \text{, in which case} \\ B^{(4)}_{m-233} - B^{(4)}_{m-131} \quad B^{(4)}_{m-113} = \Delta_{33} \end{cases}$$

$$(7.12)\quad \begin{cases} B^{(4)}_{m-3jk} - B^{(4)}_{m-1j1} \quad B^{(4)}_{m-21k} - B^{(4)}_{m-2j2} \quad B^{(4)}_{m-22k} \\ + B^{(4)}_{m-1j1} \quad B^{(4)}_{m-111} \quad B^{(4)}_{m-11k} - B^{(4)}_{m-2j1} \quad B^{(4)}_{m-11k} = 0 \\ \qquad\qquad (j, k = 4, 5) \\ \text{except when } j = k = 4 \text{, in which case the} \\ \text{above expression equals } \Delta_{44}\,. \end{cases}$$

Conditions (7.10) and (7.11) correspond to conditions (6.9) and (6.10) and are derived by methods which are identical with those used in the last section. In order to obtain (7.12), consider V_{44}, which is of rank $p_4^{(4)}$ and of order

$p_4^{(3)} = p_4^{(4)} + p_5^{(4)}$; in the same manner as V_{33} was considered in the previous section, remembering, however, that V_{44} reduces to

$$\begin{pmatrix} \Delta_{44} & 0 \\ 0 & 0 \end{pmatrix}$$

upon multiplication right and left by suitable non-singular matrices.

Due to the complexity of the above and following relations, it is expedient to introduce further notation. Henceforth, the matrix $B^{(4)}_{m-ijk}$ will be designated by (ijk). Thus the relations (7.10), (7.11) and (7.12) are written

$$(7.13)\begin{cases} (112) = (121) = (123) = (124) = (125) = 0 . \\ (13k) = (14k) = (15k) = 0 \quad (k = 2, 3, 4, 5) . \\ (122) = \Delta_{22} . \\ (2jk) - (1j1)(11k) = 0 \quad (j, k = 3, 4, 5) \\ \text{except when } j = k = 3, \text{ in which case } (233)-(131)(113) \\ \quad = \Delta_{33} . \\ (3jk)-(1j1)(21k)-(2j2)(22k)+(1j1)(111)(11k)-(2j1)(11k) \\ \quad = 0 \\ \text{except when } j = k = 4, \text{ in which case the above expres-} \\ \text{sion equals } \Delta_{44} . \end{cases}$$

Using (7.8) we get, from (7.3),

$H_4\ P_4\ E_1\ Q_4 = 0$, or defining G_0, G_1, G_2, G_3, G_4, by

$$(7.14) \quad G_k = Q_4\ H_k\ P_4 \qquad (k = 0, 1, 2, 3, 4) ,$$

we have

$$(7.15) \qquad G_4 E_1 = 0 \ , \qquad G_4 N_1 = 0 \ .$$

Similarly, (7.4) gives

$$(7.16) \qquad G_4 B^{(4)}_{m-1} + G_3 E_1 = 0 \ .$$

Multiplication on the right by N_2 yields

$$G_4 B^{(4)}_{m-1} N_2 = 0$$

which gives, on application of (2.6) and (7.13),

$$(7.17) \qquad G_4 N_2 = 0 \ .$$

Multiplication of (7.16) on the right by N_1 yields

$$(7.18) \qquad G_3 N_1 = - \left[G_4 N_3 (131) + G_4 N_4 (141) + G_4 N_5 (151) \right] .$$

Again, (7.5) gives

$$(7.19) \qquad G_4 B^{(4)}_{m-2} + G_3 B^{(4)}_{m-1} + G_2 E_1 = 0 \ .$$

Multiplication on the right by N_3, together with an application of (7.13), (7.15), (7.17) and (7.18) gives

$$(7.20) \qquad G_4 N_3 = 0$$

Multiplication of (7.19) on the right by N_2 yields

$$(7.21) \qquad G_3 N_2 = - \left[G_4 N_4 (242) + G_4 N_5 (252) \right].$$

Multiplication of (7.19) on the right by N_1 produces

$$(7.22) \qquad G_2 N_1 = - G_3 N_3 (131) - G_3 N_4 (141) - G_3 N_5 (151) - G_4 N_4 \left[(241) - (141)(111) \right]$$

over

$$- G_4\, N_5 \Big[(251) - (151)(111) \Big] \;.$$

Again (7.6) transforms into

$$(7.23) \quad G_4\, B_{m-3}^{(4)} + G_3\, B_{m-2}^{(4)} + G_2\, B_{m-1}^{(4)} + G_1\, E_1 = 0 \;.$$

Multiplication on the right by N_4 simplifies this into

$$(7.24) \quad G_4\, N_4 = 0 \;.$$

Multiplication of (7.23) on the right by N_2 yields

$$(7.25) \quad G_2\, N_2 = G_4\, N_5 \Big[(151)(212) + (252)(222) - (352) \Big]$$
$$- G_3\, N_3 (232) - G_3\, N_4 (242) - G_3\, N_5 (252) \;.$$

Multiplication of (7.23) on the right by N_1 gives

$$(7.26) \quad G_1\, N_1 = G_4 N_5 \Big[(251)(111) + (252)(221) + (151)(211)$$
$$- (151)(111)(111) - (351) \Big]$$
$$+ G_3\, N_3 \Big[(131)(111) - (231) \Big] - G_2\, N_3 (131)$$
$$+ G_3\, N_4 \Big[(141)(111) - (241) \Big] - G_2\, N_4 (141)$$
$$+ G_3\, N_5 \Big[(151)(111) - (251) \Big] - G_2\, N_5 (151) \;.$$

Multiplication of (7.23) on the right by N_3 yields

$$(7.27) \quad G_3 N_3 = G_4 N_5 \Big[- (353) + (252)(223) + (151)(213) + (251)(113)$$
$$- (151)(111)(113) \Big] \;.$$

Designating the coefficient of $G_4 N_5$ above by V_{54} we have

$$G_3\, N_3 = G_4\, N_5\, V_{54} \;.$$

Finally, (7.7) gives

$$(7.28) \quad G_4 B_{m-4}^{(4)} + G_3 B_{m-3}^{(4)} + G_2 B_{m-2}^{(4)} + G_1 B_{m-1}^{(4)} + G_0 E_1 = I_s \;.$$

Multiplication on the right by N_5 yields

$$G_4 N_5 \Big[V_{52} \Big] + G_3 N_3 \Big[V_{53} \Big] = N_5$$

in which

$$V_{52} = (455) - (151)(315) - (252)(325) - (251)(215)$$
$$- (151)(111)(215) + (151)(212)(225) + (252)(222)(225)$$
$$- (352)(225) + (251)(111)(115) + (252)(221)(115)$$
$$+ (151)(211)(115) - (151)(111)(111)(115) - (351)(115)$$

and

$$V_{53} = (335) - (131)(215) - (232)(225) + (131)(111)(115) - (231)(115).$$

Substituting for G_3N_3 from (7.27), we obtain

$$(7.29) \qquad G_4 \; N_5 \; V_{55} \; = \; N_5$$

in which

$$V_{55} = V_{52} + V_{54}V_{53}$$
$$= (455) - (252)(325) - (151)(315) - (352)(225)$$
$$+ (252)(222)(225) + (151)(212)(225) + (151)(111)(215)$$
$$- (151)(111)(111)(115) + (251)(111)(115) + (252)(221)(115)$$
$$+ (151)(211)(115) - (351)(115) + (151)(213)(335)$$
$$- (151)(213)(232)(225) - (151)(213)(131)(215) -$$
$$(151)(213)(231)(115)$$
$$+ (151)(213)(131)(111)(115) - (151)(111)(113)(335)$$
$$+ (151)(111)(113)(232)(225) + (151)(111)(113)(131)(215)$$
$$+ (151)(111)(113)(231)(115) - (151)(111)(113)(131)(111)(115)$$
$$+ (252)(223)(335) - (252)(223)(232)(225) - (252)(223)(131)(215)$$
$$- (252)(223)(231)(115) + (252)(223)(131)(111)(115)$$
$$+ (251)(113)(335) - (251)(113)(232)(225) - (251)(113)(131)(215)$$
$$- (251)(113)(231)(115) + (251)(113)(131)(111)(115) - (353)(335)$$
$$+ (353)(232)(225) + (353)(131)(215) + (353)(231)(115)$$
$$- (353)(131)(111)(115) .$$

It necessarily follows from (7.29) that

(7.30) $$\left| V_{55} \right| \neq 0 .$$

If we set arbitrarily

(7.31) $$G_4N_1 = G_4N_2 = G_4N_3 = G_4N_4 = 0$$

$$G_4N_5 = N_5V_{55}^{-1} .$$

$$G_3N_1 = -N_5V_{55}^{-1}(151) \qquad G_3N_2 = -N_5V_{55}^{-1}(252)$$

$$G_3N_3 = N_5V_{55}^{-1}V_{54} \qquad G_3N_4 = G_3N_5 = 0 .$$

$$G_2N_1 = -N_5V_{55}^{-1}V_{54}(131) - N_5V_{55}^{-1}\left[(251)-(151)(111)\right] .$$

$$G_2N_2 = N_5V_{55}^{-1}\left[(151)(212)+(252)(222)-(352)\right]$$

$$G_2N_3 = G_2N_4 = G_2N_5 = 0 .$$

$$G_1N_1 = N_5V_{55}^{-1}V_{51} \qquad G_1N_2 = N_2 - N_5V_{55}^{-1}V_{50}$$

$$G_1N_3 = G_1N_4 = G_1N_5 = 0$$

$$\begin{aligned} G_0N_1 = {} & N_1 - N_5V_{55}^{-1}(451) + N_5V_{55}^{-1}(151)(311) \\ & + N_5V_{55}^{-1}(252)(321) - N_5V_{55}^{-1}V_{54}(331) \\ & - N_5V_{55}^{-1}\left[(151)(212)+(252)(222)-(352)\right] (221) \\ & + N_5V_{55}^{-1}\left[V_{54}(131)+(251)-(151)(111)\right] (211) \\ & - N_5V_{55}^{-1}V_{51}(111) - \left[N_2 - N_5V_{55}^{-1}V_{50}\right] (121) , \end{aligned}$$

$$G_0 N_2 = G_0 N_3 = G_0 N_4 = G_0 N_5 = 0,$$

in which

$$V_{51} = (251)(111)+(252)(221)+(151)(211)-(151)(111)(111)-(351) + V_{54}\left[(131)(111)-(231)\right] .$$

and

$$\begin{aligned} V_{50} = {} & (452)-(151)(312)-(252)(322) + V_{54}(332) \\ & + V_{54}(131)(212) + \left[(251)-(151)(111)\right](212) \\ & - \left[(151)(212) + (252)(222)-(352)\right](222) \end{aligned}$$

we see that (7.3), (7.4), (7.5), (7.6) and (7.7) are satisfied identically. Again, although P_4 and Q_4 are not of necessity unique, the rank of V_{55} is independent of the manner in which they are chosen.

(7.32) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (7.1), the necessary and sufficient condition that there exist a quartic polynomial associated with it is that equation (7.30) be satisfied. One such polynomial is defined by equations (7.31).

(7.33) Theorem:- If $b(\lambda)$ is the matric polynomial defined by (7.1), the necessary and sufficient condition that all polynomials associated with it be quartic is that equation (7.30) be satisfied. The proof of this is analogous to the proofs of Theorems (5.21) and (4.13) and so will not be given.