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BING'S DOGBONE SPACE AND CURTIS' CONJECTURE

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ABSTRACT

Bing's dogbone space $\mathcal{D}$ is an upper semi continuous decomposition space of E^3 which fails to be E^3 although the associated decomposition consists only of points and tame arcs. It has proved difficult to find topological properties of $\mathcal{D}$ which distinguish it from E^3. In this paper, we prove a conjecture of Morton Curtis in 1961 that certain points of $\mathcal{D}$ fail to possess small simply connected neighbourhoods.

I wish to acknowledge my gratitude to my supervisor Dr. Whittaker for his unselfish and often indispensible aid during my graduate studies at UBC, and to Dr. Luft for his support and enthusiasm. I am grateful also for some conversations and a blizzard of letters from R. H. Bing.

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INTRODUCTION

Bing's dogbone space (which is denoted by $\mathcal{D}$ in this paper) is a decomposition space of E^3 which fails to be homeomorphic to E^3 even though the associated decomposition space is upper semicontinuous and point-like, and each element of the decomposition is either a point or a tame arc. The appearance of $\mathcal{D}$ in [12] caused some surprise since it was thought at the time that all usc point-like decomposition spaces of E^3 would turn out to be E^3. Although $\mathcal{D}$ dates from 1955 and has become rather well-known, it has been found hard to determine those topological properties of the space which distinguish it from E^3. Bing's original paper [12] showed that $\mathcal{D}$ is a non-manifold; but $\mathcal{D}$ is a simply connected homology manifold and locally simply connected. This paper contains a proof of a conjecture of Morton Curtis that $\mathcal{D}$ fails to possess small simply connected open neighbourhoods about certain points. This property is stronger than local simple connectivity (see our comments in II §1). A proof of Curtis' Conjecture was anounced in 1964 [14]; however the detailed proof has not appeared. Only one other topological property distinguishing $\mathcal{D}$ and E^3 is known: some points of $\mathcal{D}$ cannot be enclosed in 2-spheres [11], [13]. The general state of affairs seems to be that some points of $\mathcal{D}$ have no closed or open 3-cell neighbourhood systems, but do have systems of neighbourhoods bounded by double tori.

Our arguments use elementary methods exclusively (except for an easily circumvented reference to the Hopf property of knot groups) and may well appear old-fashioned. We are less than proud of much of the exposition, which was intended to combine the detail appropriate to a thesis with the directness of a journal paper and somehow didn't. The reader will probably share our pain at the length of the argument (the whole paper is essentially one theorem). The reader who is unfamiliar with pathological decomposition spaces is advised to read [3], which is brief and exceptionally entertaining, and then skim Ch. II. We will mention some notational peculiarities: we follow common practise in describing geometric constructions, even complicated ones, by the use of diagrams. "Theorem' in this paper means 'working theorem'; thus 'theorems' appear in the introductory chapter only.

CHAPTER ONE

0. Introduction.

This first chapter gives preliminary material for the arguments in Ch III and especially Ch IV. The reader who wishes to skim the paper will find that Ch II, which contains the discussion of Curtis' Conjecture, is largely independent of this first chapter. In this paper, our approach to elementary topology is along the lines of the easier chapters of [10[, in particular, we always assume a separable metric space. In this chapter, sections 1 and 2 are elementary, §3 contains working theorems for Ch IV, and §4 is essentially a comment on Bing's Theorems 6 and 7 of [12]. Section 5 is part of the argument of Ch IV which is self-contained and has been smuggled into the preliminary material, although it could have been left until it appeared naturally in the main argument.

1. Notation.

The arguments in this paper use elementary methods exclusively, so that notation should present no problems. We use $\emptyset$ for the null set and the symbol $\square$ for the end of the proof of a numbered result. The expression 'Bd A' may mean either the manifold boundary of the mainifold-with-boundary A, or the point-set boundary of the set A . A similar comment applies to the expression 'Int A' . This reflects common practise; we will comment whenever the meaning is unclear. As mentioned in the preface, our attitude to the construction of tame sets will be cavalier; we will construct many important tame sets simply by describing the set and perhaps giving a picture of it. We advise against the intuitive approach of imagining our constructions as straight-sided polyhedra whose structural detail is so fine that the polyhedra approximate the figures closely. Several of our arguments will require extensive repair if our geometric constructions are interpreted in this way. If neccessary, methods in [4] could be used to show that each of our constructions is in fact a curvilinear polyhedron.

2. Elementary Results.

In this section we give some 'obvious' results which we have found hard to justify by simple references. This may be a matter of ignorance, especially in the case of (2.1) and (2.3). We define an annulus to be a topological sphere with two holes. The proofs are omitted.

(2.1). Let a be an arc which intersects two disjoint closed sets S_1, S_2. Then there is a sub-arc a^* of a which connects S_1 and S_2 and meets $S_1 \cup S_2$ only at the end points of a^*.

(2.2). Any two annuli A_1, A_2 are homeomorphic. Any homeomorphism of one boundary component of A_1 onto a boundary component of A_2 may be extended to a homeomorphism of A_1 onto A_2.

(2.3). The union of two locally connected (lc) continua which intersect is a lc continuum.

(2.4). Let O be a bounded connected open set in the plane whose boundary is lc. Then any two points x and y in $\overline{O}$ may be connected by an arc which lies in O except possibly for its end points.

(2.5). Let A be a 2-manifold with boundary, and K a continuum in A. Then any two points of K may be connected by an arc in Int A (except possibly for end points) which lies within a distance ε of K.

(2.6). Let C_1, C_2 be disjoint simple closed curves in E^2. Then one of the following exclusive alternatives is true:

a) $C_1 \subset \text{Int } C_2$ or equivalently $\overline{\text{Int } C_1} \subset \text{Int } C_2$.

b) $C_2 \subset \text{Int } c_1$ or equivalently $\overline{\text{Int } C_1} \subset \text{Int } C_2$.

c) Each of C_1, C_2 lies in the others exterior, or equivalently $\overline{\text{Int } C_1} \cap \overline{\text{Int } C_2} = \emptyset$

(2.7). Let A be an annulus, and C a simple closed curve in $\text{Int } A$ which bounds no disk in A. Then C separates A into components B_1, B_2 such that $B_1 \cup C$ and $B_2 \cup C$ are annuli.

3. Sliding Curves on Spheres.

(3.1). We will often need to 'move' or 'deform' curves in E^3 . This will be done by sliding the curves on convenient spheres, disks and annuli in E^3 . The sort of thing that may be encountered is shown in fig. 1. A double ended lasso has loops p , q and 'middle' z . We may want to push z over to the position of z´ in the figure or expand p so that it looks like p´ . This can be done with a homeomorphism $H: E^3 \to E^3$ which carries, say, z onto z´ and can thus be said to move 'z to z´'.

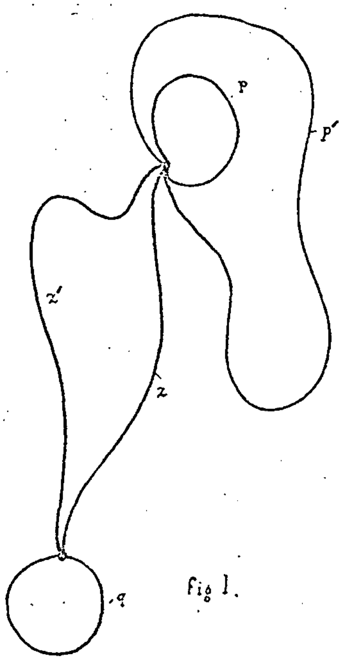


Fig 1.

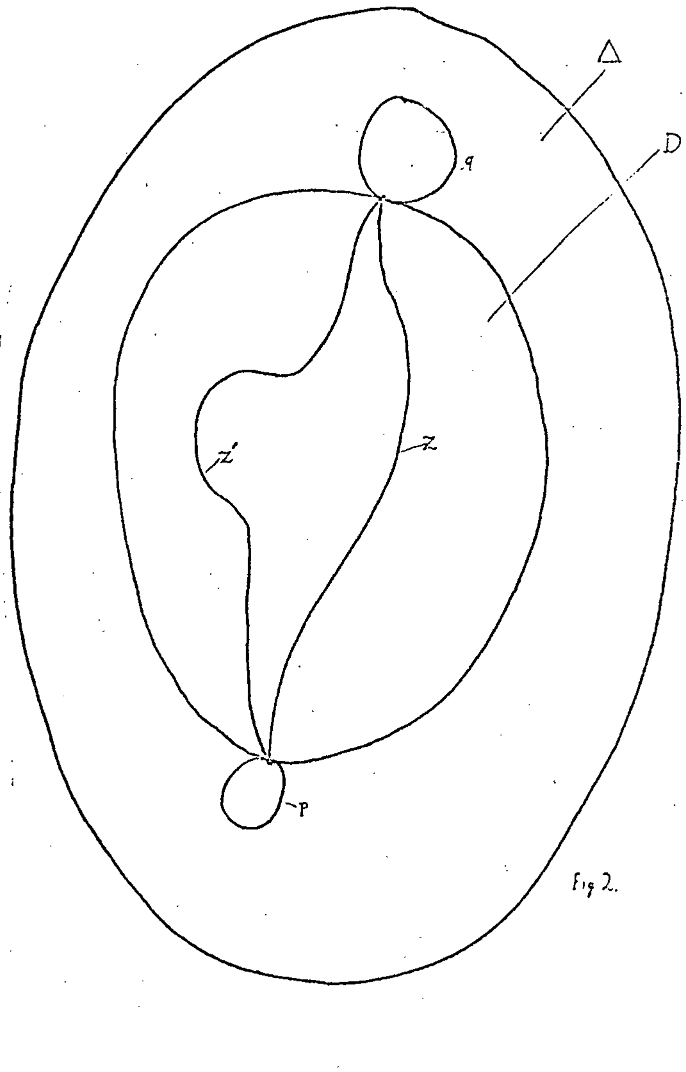


Fig 2.

Suppose $p \cup q \cup z$ lies on a disk $\Delta \subset E^3$. We ask what properties the homeomorphism H should have in order to reflect the intuitive idea of sliding z to z' on Δ while keeping $p \cup q$ fixed. One way to do this would be to construct a new disk $D \subset \Delta$ (see fig. 2) so that D contains $z \cup z'$ and misses p and q except where they hit $z \cup z'$. Then we could require that H carry z onto z' , $H[D] = D$, and that H be the identity on $\Delta - D$ and on Bd D , (thus H will fix $p \cup q$). It seems a good idea to specify a number of standard moves, prove that they can always be made and stick to these in the sequel. When, as commonly happens, an arc or loop moves only a short distance and has explicit initial and final locations, then our idea of 'standard moves' is probably too formal. However our standard moves are intended for the case that the initial position of the set is unknown. In this case the existence of the required move is less obvious, especially when, as in §5, Th 6 , a base point must be held fixed during the move. If S is a sphere with n holes in E^3 , then <u>a collar of S</u> is the image of an embedding h of $S \times [-1,1]$ into E^3 so that $h(x,0) = x$. Evidently a collar of S may not exist (S could be wild). A set upon which a collar has been constructed is called a <u>collared set</u>. Note that a collar of S is not a neighbourhood of S .

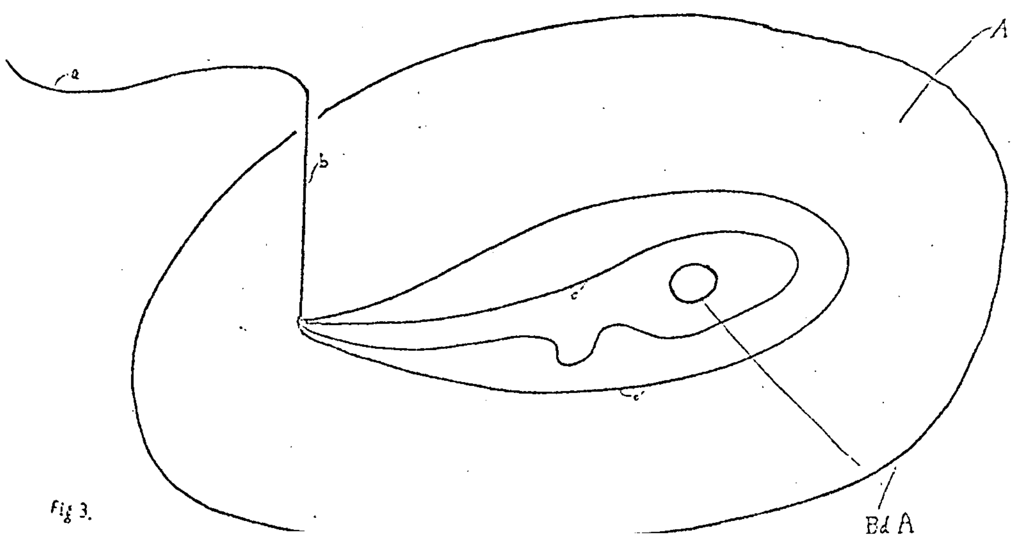


Fig 3.

(3.2). A- B- and B´ -moves.

We give three standard moves in Theorems 1 and 2 .

Theorem 1. Let D be a disk in E^3 , J a collar of D , and a´, a two arcs which have common end points and lie in Int D except for these end points, which lie in Bd D . Then there is a homeomorphism A(a,a´,D,J) of E^3 onto itself which carries a onto a´, D onto itself, and which fixes $\overline{E^3 - J}$.

We call A(a,a´,D,J) 'the A-move' and say that A(a,a´,D,J) moves a to a´ . (Of course the fact that a moves to a´ is only one of a number of things that have to be kept in mind. We write the move as a function of D and J to emphasize that the trick of using the move depends on the right definition of D and J).

Theorem 2. Let A be an annulus in E^3 . Let c, c´, be simple closed curves which lie in the interior of A and bound no disks in A . Let Q be a collar of A . Then there is a homeomorphism B(c,c´,A,Q) , also called a B-move, of E^3 onto itself which carries c onto c´ , A onto itself, and which fixes Bd Q and $\overline{E^3 - Q}$. If, in addition, c and c´ have a common base point y , then there is a homeomorphism B´(c,c´,A,Q) and the following additional property: if h is the embedding associated with Q , so that $Q = h[A \times [-1,1]]$, then the B´-move fixes y and in fact all of $h[y \times [-1,1]]$.

The B´move is a move 'keeping the base point fixed'. One could probably fix the base point by providing that $c \cup c'$ could hit $Bd\ A \subset Bd\ Q$

so that $y \in \text{Bd } Q$ (the B-move does not permit this), however the B´-move as given above fits the intended applications better and is easier to prove. We will give an example which shows why we want the B´-move to fix $h[y \times [-1,1]]$. Fig. 3 shows c, c', A, and an arc $a \cup b$ such that a misses A and b is a straight arc perpendicular to A . We want to move c to c' while leaving $a \cup b$ fixed. We do this with a B´-move $B'(c,c',y,A,Q)$ in which Q is defined so that all points of Q lie near A (i.e. for $x \in A$, $h[x \times [-1,1]]$ is short) and so that each arc $h[x \times [-1,1]]$ with $x \in A$ is perpendicular to A . For a sufficiently 'thin' Q , the B´-move will fix a because $a \subset \overline{E^3 - Q}$, and b will be fixed because b lies in $h[y \times [-1,1]]$ wherever it hits Q . Evidently the utility of the B´-move is limited. However subsequent use of the B´-move will be very much along the lines of this example.

4. The Phragmen-Brouwer Properties. The Zoretti Theorem.

The Phragmen-Brouwer Properties are usually given for the n-sphere, but hold also on a disk. We quote from Wilder, [I, II 4.1]. Let S be a locally connected metric space. Then the following properties of S are equivalent.

(4.11). If A, B are disjoint, closed subsets of S , and $x,y \in S$ such that neither A nor B separates x and y in S , then $A \cup B$ does not separate x and y in S . (By 'X separates x and y in S' is meant 'x and y are in different components of $S - X$').

(4.12). If $S = A \cup B$, where A, B are closed and connected, then

$A \cap B$ is connected.

(4.13). If A, B are disjoint closed subsets of S and $a \in A$, $b \in B$, then there exists a closed connected subset C of $S - (A \cup B)$ which separates a and b.

Theorem II 4.12 of [1] states that these properties are equivalent in a locally connected metric space. From VII, 9.3 of [1] (note also 9.2), a disk D will have properties (4.11), (4.12), (4.13), if its first Betti number is zero; thus (4.11) ... (4.13) hold on D.

(4.2). We get the following important working theorems from (4.11). These theorems resemble Theorems 6 and 7 of [12].

<u>Theorem 3</u>. Let D be a 2-cell in E^3 and F_1, F_2 closed disjoint subsets of E^3. Let pxq, pyq be arcs in D which share the end points p and q, and such that arc pxq misses F_1, arc pyq misses F_2. Then there exists an arc pzq with end points p, q such that arc $pzq \subset D - F_1 - F_2$.

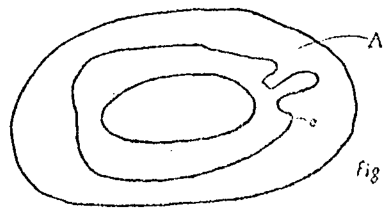
<u>Theorem 4</u>. Let D, F_1, F_2, p, q, arcs pxq, pyq be defined as in Th 3 except that arc $px \cup$ arc yq misses F_1, arc $py \cup$ arc xq misses F_2. Then there exists an arc pzq D with end points p, q, such that arc misses either F_1, or F_2.

Proofs of Th 3 and Th 4. Since D is simply connected, pxq and pyq are homotopic in D by a homotopy which fixes p and q. Using this fact, the proofs of Th 6 and Th 7 of [12] may be used word for word to prove Th 3 and Th 4 respectively, reading D for M in [12] $\square$.

(4.4). The Plane Separation Theorem and the Zoretti Theorem. We quote these results, slightly simplified, from [10,VI §3].

(4.41). The Plane Separation Theorem. Let A , B be compact sets in E^2 which intersect in at most one point. Let $a \in A - B$, $b \in B - A$, and let $\varepsilon > 0$. Then there is a simple closed curve J which separates a and b in E^2 , lies within an ε-neighbourhood of A , and misses $A \cup B$ except possibly at the point $A \cap B$.

(4.42). The Zoretti Theorem. If K is a component of a compact set M in the plane, then there is a simple closed curve J whose interior contains K , which misses M , and which lies in an ε-neighbourhood of K .



fig

5. Annulus Dodging Theorems.

Suppose A is an annulus and F is a closed set in A . When can we say that a simple closed curve which looks like c in fig. 4 exists so as to miss F ? The answer is about what would be expected. We say that <u>F bridges A</u> iff the two boundary components of A are in the same component of $\text{Bd}\, A \cup F$, or equivalently, iff some component of F meets both boundary components of A .

We will prove the equivalence. Let the boundary components of A be ℓ and m . '$\leftarrow$' is obvious. If no component of F meets both ℓ and m, then no component of F meets both $\ell \cap F$ and $m \cap F$, and by I(9.3) of [10] (taking A, B, K to be $\ell \cap F$, $m \cap F$, F), there is a separation of F into compacta F_ℓ, F_m such that F_ℓ meets only ℓ , F_m meets only m in $\text{Bd}\, A$. Evidently this denies the existence of a connected subset of

$F \cup \ell \cup m$ which meets both ℓ and m .

(5.1). If F fails to bridge A, then there is a simple closed curve c in Int A such that c bounds no disk in A and c misses F .

Proof: We can assume that A is the set $1 \leq x^2 + y^2 \leq 2$ in E^2 . Let D be the set $x^2 + y^2 \leq 1$. Let ℓ, m be the boundary components $x^2 + y^2 = 1$, $x^2 + y^2 = 2$ respectively.

Consider the component K of $\ell \cup m \cup F$ which contains (the connected set) ℓ . The set $\ell \cup m \cup F$ is clearly compact, and by the Zoretti theorem (4.4.2) there is a simple closed curve c which lies in $E^2 - F - \ell - m$, contains K in its interior and lies in an ε-neighbourhood of K . We will show that c has the properties required by (5.1). To see that c Int A : K contains ℓ and misses m, since otherwise F bridges A . Thus $K \subset (A \quad D) - m = \mathrm{Int}(A \cup D)$. Since K is compact, K has an ε-neighbourhood in $\mathrm{Int}(A \cup D)$, and we can assume that c lies in this neighbourhood. Thus $c \subset \mathrm{Int}(A \cup D)$. But c encloses $K \supset \ell$ and hence D (by (2.6)); therefore $c \subset \mathrm{Int}(A \cup D) - D$ $= \mathrm{Int}\, A$. We know that c bounds no disk in A because, from the Schoenflies theorem, c bounds just one disk in E^2 . This disk is $\overline{\mathrm{Int}\, c}$ which is not a subset of A since it contains D . Since c misses F (by construction), lies in Int A, and bounds no disk in A, the proof of (5.1) is complete. □.

Remark: the converse of 5.1 is true and easily proved.

We will look at some generalizations, the choice being influenced by later applications.

Theorem 5. Let F_1, F_2 be disjoint closed sets in the annulus A . If each of F_1, F_2 fails to bridge A , then there is a simple closed curve c in Int $A - F_1 - F_2$ such that c bounds no disk in A .

Proof: This result is trivial once we show that if neither of F_1 , F_2 bridges A , then $F_1 \cup F_2$ fails to bridge A . Once this is done, the proof of Th. 5 is completed by applying (5.1) taking F to be $F_1 \cup F_2$. To see that $F_1 \cup F_2$ fails to bridge A : since F_1 does not bridge A , no component of $\ell \cup m \cup F_1$ intersects both ℓ and m , (for otherwise some component of $\ell \cup m \cup F_1$ would contain ℓ and m). By I (9.3) of [10], taking A , B , K in that theorem to be ℓ , m , $\ell \cup m \cup F_1$, there is a separation of $\ell \cup m \cup F_1$ into disjoint compact sets U_1 ,

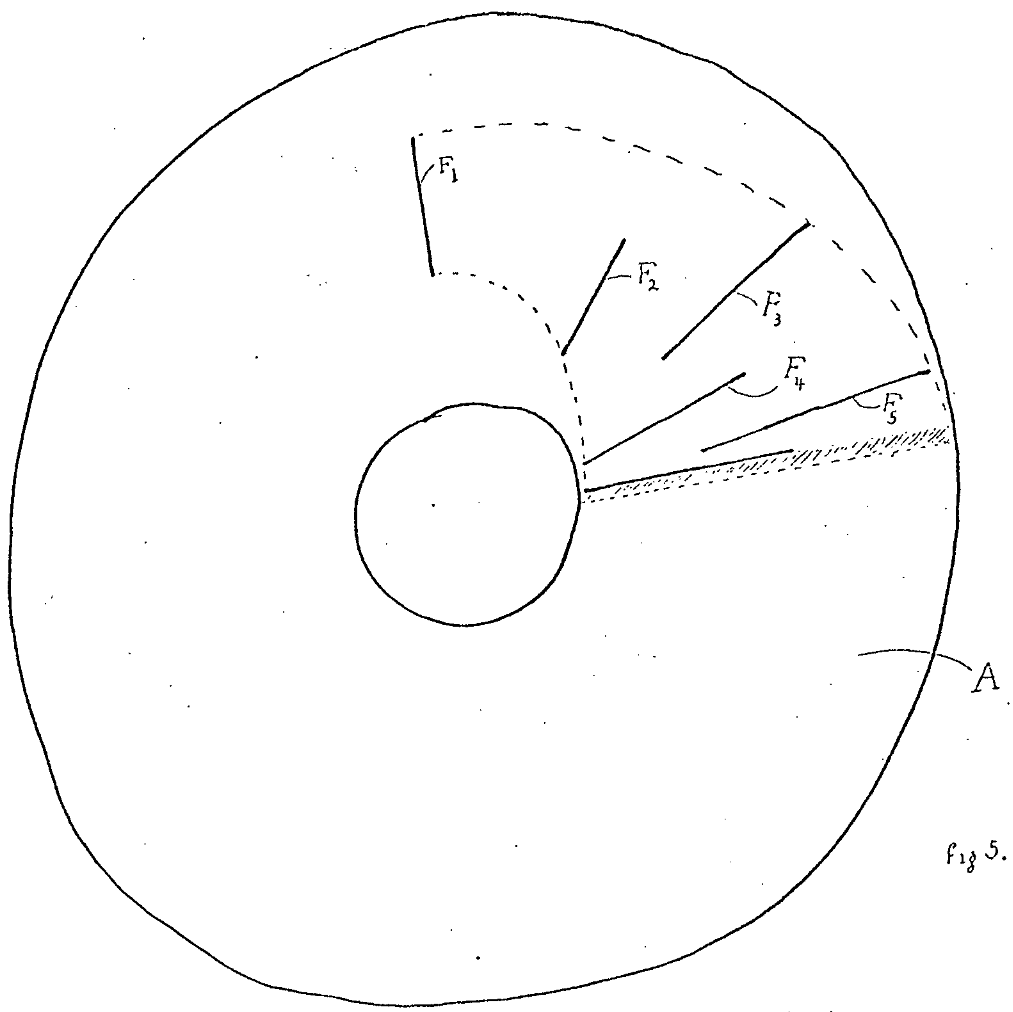


Fig 5.

U_2 so that $\ell \subset U_1$, $m \subset U_2$. Similarly there is a separation of $\ell \cup m \cup F_2$ into disjoint compact sets V_1 , V_2 , with $\ell \subset V_1$, $m \subset V_2$. It is easily checked that $U_1 \cup V_1$ misses $U_2 \cup V_2$. Evidently $\ell \cup m \cup F_1 \cup F_2$ may be separated into the disjoint closed sets $U_1 \cup V_1$ and $U_2 \cup V_2$ with $\ell \subset U_1 \cup V_1$, $m \subset U_2 \cup V_2$. Therefore ℓ and m are not in the same component of $\ell \cup m \cup F_1 \cup F_2$ and $F_1 \cup F_2$ fails to bridge A $\square$.

We remark that '$F_1 \cup F_2$' may be replaced by a finite union of disjoint closed sets with a few trivial changes in the proof. Theorem 5 is false for a non-compact union of sets F_1, F_2, ... Fig. 5 shows A and a collection F_1, F_2, ... such that A is the set $1 \leq r \leq 2$ in polar coordinates and for $1 = 1, 2, 3, \ldots$, F_i is a subset of the ray $\theta = 1/i$. Although each F_i does not bridge A (nor does the union $\bigcup_{i=1}^{\infty} F_i$) , the curve c in Th. 5 cannot be constructed.

We next look at the case where the curve c is constructed as in Th. 5 but with the further property that c contains a given base point x . In this case c cannot in general miss either of F_1, F_2 , as Fig. 6 shows.

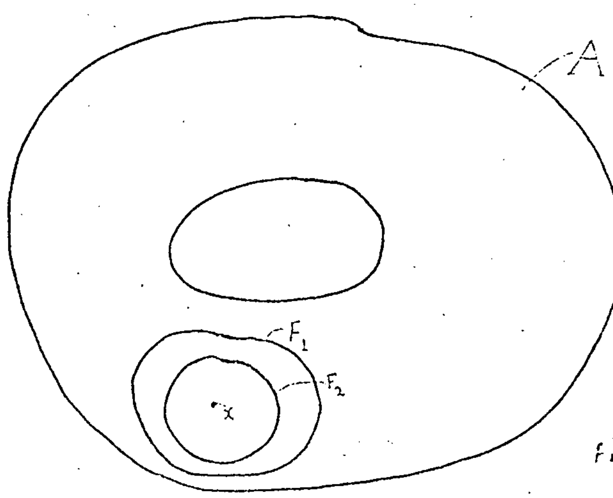


Fig 6.

We will give a characterization of those placements of x, F_1, F_2, so that c can be made to miss one of F_1, F_2. We say that a simple closed curve c with base point x has Property $\sim P$ (read 'property not - P') with respect to closed sets F_1, F_2 iff one of the following is true:

$\sim P(a)$: c misses on of F_1, F_2.

$\sim P(b)$: There exists a point $y \in c - x$ and a decomposition of c into arcs c_1, c_2, with $c_1 \cup c_2 = c$ and $c_1 \cap c_2 = \{x,y\}$, such that F_1 misses c_1, F_2 misses c_2 (see fig. 7).

This is an ugly and awkward definition. An equivalent and prettier statement is 'c has Property $\sim P$ iff any point in $c - x$ may be joined to x by an arc which misses one of F_1, F_2' ; however we will not prove this, and we will use the earlier statement exclusively. The odd name of this

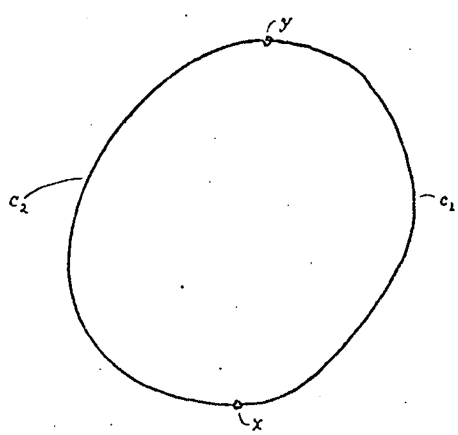


fig 7.

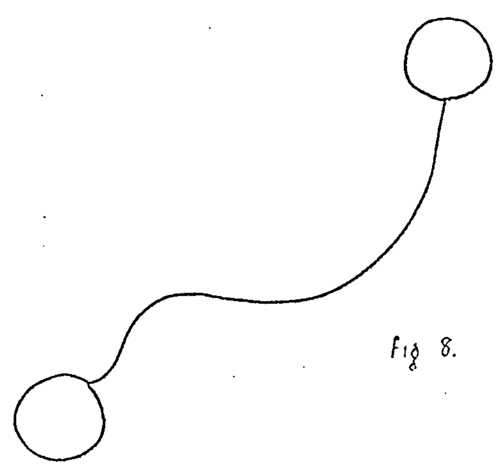


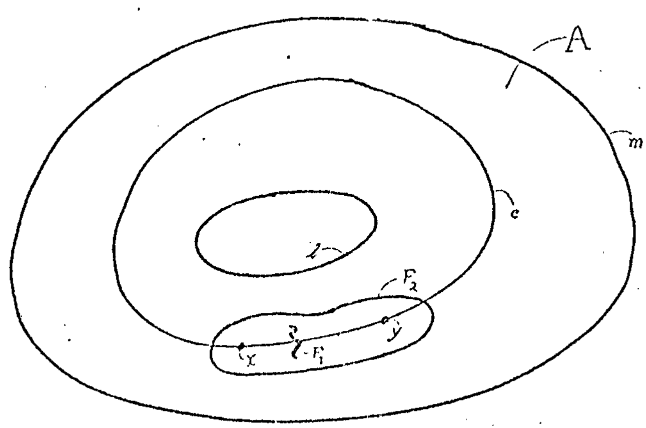
fig 8.

property is intended to recall Bing's Property P in [12]. This property is defined on double ended lassos (see fig. 8). Later we will define Property $\sim P$ on double ended lassos and it will turn out that the loops of such lassos, with the obvious base points, have Property $\sim P$ in the present sense. The next theorem says that if c with base point x has Property $\sim P$, then there is a loop c' which behaves like c and misses one of F_1, F_2.

<u>Theorem 6</u>. Let A, F_1, F_2 be defined as in Th. 5, including the condition that neither F_1 nor F_2 bridges A. Let $x \in \operatorname{Int} A$. Let c be a simple closed curve which lies in $\operatorname{Int} A$ and bounds no disk in A and contains x. If c has Property $\sim P$ with respect to x, F_1, F_2, then there exists a simple closed curve c' which lies in $\operatorname{Int} A$, bounds no disk in A, has base point x, and misses one of F_1, F_2.

This result cannot be improved so as to allow us to specify which of F_1, F_2 is to be missed by c'. Fig. 9 shows a case where c' in Th. 6 cannot be made to miss F_2 although $F_1 \cup F_2$ fails to bridge A, and c exists with Property $\sim P$. (There are simpler counter examples in which only F_2 hits c. One of these may be derived by removing F_1

Fig 9.



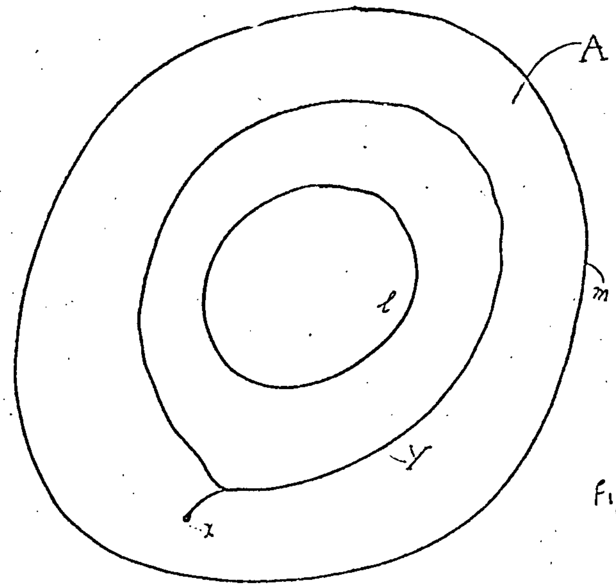


fig 10.

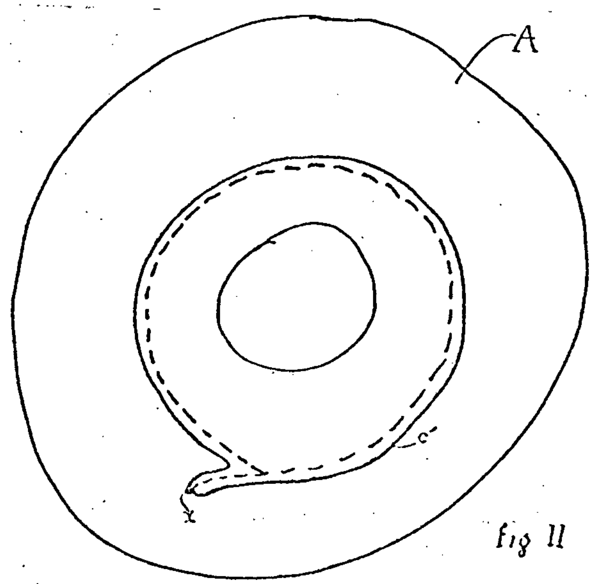


fig 11

from fig. 9. However fig 9 shows that matters do not improve if we insist that both F_1 and F_2 hit c.)

Proof of Th 6. We can assume that A is the set $1 \leq x^2 + y^2 \leq 2$ in E^2. The inner and outer boundary components of A will be called ℓ and m respectively. Since neither of F_1, F_2, bridges A, it follows from Th 5 that there is a simple closed curve $e \subset \text{Int } A$ which bounds no disk in A and misses $F_1 \cup F_2$. If $x \in e$, then the proof is completed by letting e be c'; thus we assume that $x \notin e$. We make the further assumption that $x \in \text{Int } e$; it turns out that this restriction is easy to remove. Assuming that $x \in \text{Int } e$, we construct c' by first defining a lasso Y as shown in fig 10. The loop of Y is either e or a curve which behaves like e and is constructed similarly, while the 'handle' of Y joins the loop to x. The whole of Y misses one of F_1, F_2. The curve c' lies near Y and meets x as shown in fig 11.

Construction of Y. The lasso Y consists of the union of a simple closed curve r and an arc s, and is constructed so as to have the following properties:

$Y \subset \text{Int } A$,

Y misses one of F_1, F_2,

the circle r bounds no disk in A,

the end points of s are x and a point $z \in r$,

and $s - z$ misses r.

The construction of Y is divided into two cases.

Case one: e meets $c - x$. We assume that c satisfies Property $\sim P(b)$, since if c satisfies Property $\sim P(a)$, we immediately let

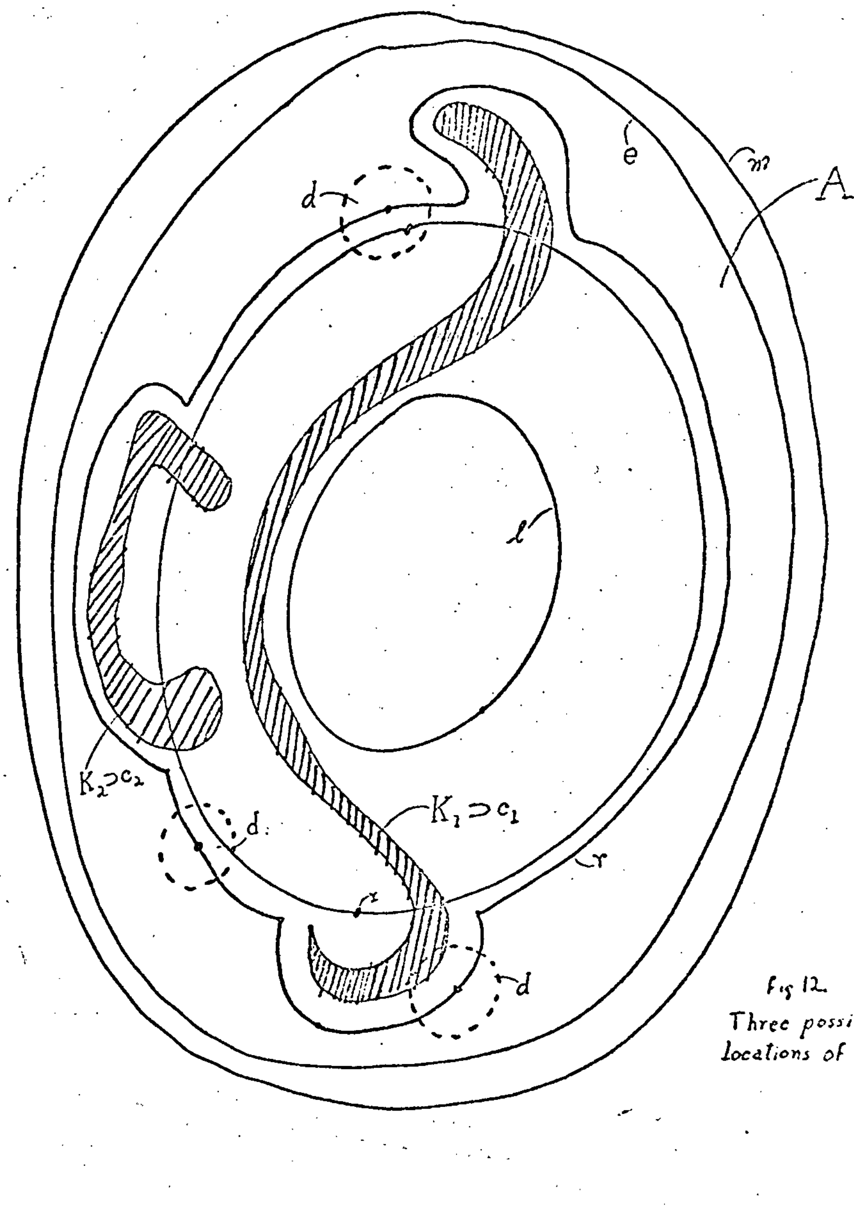


Fig 12.
Three possible locations of d.

$c' = c$. Thus we take c to be the union of arcs c_1, c_2 which meet only at their end points, and for $i = 1, 2$, c_i misses F_i. If e meets, say, c_1 (it will do no harm if e meets both c_i), then use (2.5) to construct an arc s which joins x and $e \cap c_1$, and lies so near c_1 that s misses F_1 (or take the obvious sub-arc of c_1). Let $r = e$, $Y = r \cup s$. To check that Y has the required properties; Y misses one of F_1, F_2 because s misses one of F_1, F_2 and r misses both; $r = e$ bounds no disk in A by construction; and $Y \subset \text{Int } A$ because $e \cup c \subset \text{Int } A$. Finally, from (2.1), we can assume that s meets r only at a single point z.

Case two: e misses c. As before, we assume that c has Property $\sim P(b)$. Outline of proof: a) As usual we take c to be the union of arcs c_1, c_2; let K_1 be c_1 plus those components of F_2 which hit c_1 and let K_2 be c_2 plus those components of F_1 which hit c_2. $K_1 \cup K_2$ is a component of $c \cup F_1 \cup F_2$. b) A 'Zoretti curve' r is constructed so that r misses $c \cup F_1 \cup F_2$, lies in Int A, encloses $K_1 \cup K_2$, and bounds no disk in A. c) Some care needs to be taken to attach the tail s to r so that s misses one of the F_i. Construct a disk $d \subset \text{Int } A$ with centre on r, (see fig 12) so that d is big enough to hit $K_1 \cup K_2$ but small enough to miss one F_i. This is managed by a careful choice of the ε associated with the Zoretti curve. d) There is an arc s near $K_1 \cup K_2 \cup d$ which has the required properties.

Details of proof.

a) Let $K_1 = c_1$ plus those components of F_2 which hit c_1. Let $K_2 = c_2$ plus those components of F_1 which hit c_2. We will show that $K_1 \cup K_2$ is a component of $c \cup F_1 \cup F_2$. Let K be the component of $c \cup F_1 \cup F_2$ which contains the connected set $K_1 \cup K_2$, and suppose that some point p exists in $K - (K_1 \cup K_2)$. Then p lies in one of F_1, F_2, say F_1 . Since $p \notin K_1 \cup K_2$, no component of F_1 meets both p and $c \cap F_1$. By I(9.3) of [10], there is a separation of F_1 into compacta U_1, U_2 containing $c \cap F_1$ and p respectively. Evidently U_2 misses not only U_1 but F_2 and the whole of c ; thus $U_1 \cup F_2 \cup c$ is a compactum disjoint from U_2, and there is therefore a separation of $c \cup F_1 \cup F_2 = U_1 \cup U_2 \cup c \cup F_2$ into compacta containing c and p severally. This denies the assumption that p lies with c in a connected subset of $c \cup F_1 \cup F_2$.

b) Since $x \in \text{Int } e$ and e misses $c \cup F_1 \cup F_2$, all of $K_1 \cup K_2$ lies in Int e by the usual argument. Since $K_1 \cup K_2$ is a component of $c \cup F_1 \cup F_2$ we can construct a Zoretti curve r which misses $c \cup F_1 \cup F_2$, encloses $K_1 \cup K_2$, and lies withing a distance ε of $K_1 \cup K_2$. The following argument shows that r bounds no disk in A : since c bounds no disk in A, $\overline{\text{Int } c}$, which is a disk, must meet points of $E^2 - A$. Since r encloses $K_1 \cup K_2 \supset c$, by (2.6) r encloses $\overline{\text{Int } c}$. Hence the unique disk bounded by r meets points not in A . To see that $r \subset \text{Int } A$: we saw that c encloses points of $E^2 - A$. These points cannot be in Ext m by (2.6), since m encloses $\text{Int } A \supset c$. Hence Int c meets Int ℓ, and by a connectedness argument, since c misses $\overline{\text{Int } \ell}$, $\text{Int } c \supset \overline{\text{Int } \ell}$. Since r encloses Int c, r misses $\overline{\text{Int } \ell}$. Since r lies close to the compactum $K_1 \cup K_2 \subset \text{Int } m$, r can be assumed to lie in Int m . Therefore $r \subset \text{Int } m - \overline{\text{Int} \ell} = \text{Int } A$.

c) We construct a (closed) disk of radius 2ε with centre anywhere on r . Clearly d will hit $K_1 \cup K_2$. We show that $d \subset \text{Int } A$ by showing that $d \subset \text{Int } m$ and d misses $\overline{\text{Int } \ell}$. The distance ε could have been chosen so that 4ε (i.e. the diameter of d) is less than the distance separating c and $\text{Int } \ell$, and the distance separating $K_1 \cup K_2$ and m; and we assume that this was done (the last distance is positive because $K_1 \cup K_2$ is compact and lies in $\text{Int } m$). Since d hits $K_1 \cup K_2$, $d \subset \text{Int } m$ by the choice of ε . If d hits $\overline{\text{Int } \ell}$, then d must also hit c, since points of $d \cap r$ lie in the exterior of c which encloses $\overline{\text{Int } \ell}$ as we saw. By the choice of ε, d cannot meet both c and $\overline{\text{Int } \ell}$; thus d misses $\overline{\text{Int } \ell}$.

We also assume that ε was chosen so that for $i = 1, 2$, 4ε is less than the distance separating K_i from F_i . The disk d must hit one of K_1, K_2, say K_1 (it does no harm if d hits both K_i). Since d hits K_1, d misses F_1 since otherwise F_1 would be closer to K_1 than the diameter of d .

d) The continuum $K_1 \cup d$ meets x and r and misses F_1 . Using (2.5), let s be an arc in Int A which joins x and r and lies so near $K_1 \cup d$ that s misses F_1 . (note that although K_1 may not miss Bd A, (2.5) provides that s misses Bd A). To see that $Y = r \cup s$ has the required properties: $r \cup s \subset \text{Int } A$ by construction; Y misses one of F_1, F_2 because s misses one F_i and r misses both. The circle r bounds no disk in A as we saw in b); and finally we can assume that s has end points x and $z \in r$ with $r \cap s - z = \emptyset$ by (2.1).

This completes the construction of Y assuming that $x \in \text{Int } e$.

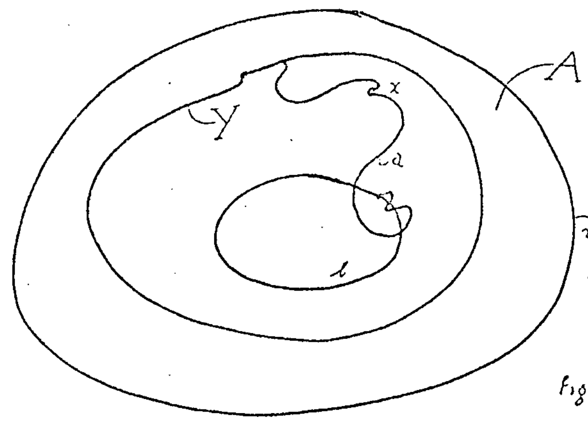


fig 13.

Construction of c'. We maintain the assumption that $x \in \text{Int } e$ during this construction. We will first construct a continuum $a \subset A$ which joins $\overline{\text{Int } \ell}$ to x so that a meets Y only at x. As suggested by fig 13, the plane separation theorem can then be used to separate $Y - x$ and $\overline{\text{Int } \ell} \cup a - x$. Construction of a: since r encloses c, $x \in \text{Int } r$. Let Q be the open set $\text{Int } r - s$. Q is connected because $s - r$ does not disconnect $\text{Int } r$ ([10, VI(3.4)]). Evidently $\text{Bd } Q = Y$, and since each of r, s is a lc continuum, so is Y. Using (2.4), connect x to a point in $\overline{\text{Int } \ell}$ by an arc a which lies in Q except for x. Since Y misses $\overline{\text{Int } \ell}$, Y and $\overline{\text{Int } \ell} \cup a$ are continua in $\text{Int } m$ which meet only at the point x. We now use (4.41) to separate $Y - x$ and $(\overline{\text{Int } \ell} \cup a) - x$ by a circle c' which lies so close to Y that it misses one of F_1, F_2. Evidently c' must pass through x (since otherwise $(\overline{\text{Int } \ell} \cup a) - x$ and $Y - x$ are subsets of a connected set in $E^2 - c'$). We know that $c' \subset \text{Int } A$ because c' misses $\overline{\text{Int } \ell}$ by construction and c' lies so near $Y \subset \text{Int } m$ that $c' \subset \text{Int } m$. It remains to show that c' bounds no disk in A. To see this: we know that r encloses ℓ. This means that c' cannot enclose r, since this would imply that $\text{Int } c' \subset r \cup \ell$ (from (2.6)), whereas we know that c' separates r and ℓ.

Thus Ext $c' \supset r$ and Int $c' \supset \ell$. The fact that Int $c' \supset \ell$ implies that c' bounds no disk in A by the usual argument.

The construction of c' is now complete except that the restriction $x \in$ Int e must be removed. Since the proof is easy if $x \in e$, we only look at the case that $x \in$ Ext e. Since we know that A is homeomorphic to a nice annulus, it is easy to construct a homeomorphism ϕ of A onto itself which exchanges ℓ and m, i.e. $\phi[\ell] = m$, $\phi[m] = \ell$. Then $\phi[A \cap \text{Ext } e] = \text{Int } \phi[e] \cap A$ and $\phi[A \cap \text{Int } e] = \text{Ext } \phi[e] \cap A$, using the fact that $A \cap$ Ext e, $A \cap$ Int e are connected to m, ℓ respectively in $A - e$. Then if $x \in$ Ext e, apply earlier arguments to $\phi[A]$, using the fact that $\phi(x) \in \text{Int } \phi[e]$, etc. □.

Theorem 7. Let A, F_1, F_2 be defined as in Th 5 and Th 6 except that F_1 bridges A while F_2 does not. Let c be a simple closed curve in Int A which bounds no disk in A and contains the base point x. Then if c has Property $\sim$P with respect to x, F_1 F_2, there exists a simple closed curve c' which meets x, lies in Int A, bounds no disk in A, and misses F_2.

Th 7 is proved in the same way as Th 6. At first glance one might think that one of Th 6, Th 7 is stronger than the other; but in fact this is not true. If F_1, F_2 fail to bridge A, one might wish to add pieces to, say, F_1 so that the enlarged F_1 would bridge A; this would obtain the conclusion of Th 7 which is stronger than that of Th 6 (since it predicts which F_i is hit by c'). However it may not be possible to do this ($F_1 \cup F_2$ might be a number of circles concentric with m in A).

Proof of Th 7. Use (5.1) with F taken to be F_2 to construct a simple closed curve e which lies in Int A, misses F_2, and bounds no disk in A. If e meets x, then e is the required c'. If $x \notin e$, assume that $x \in$ Int e as before. Since e bounds no disk in A, e separates ℓ and m by (2.7); in particular e meets some component k of F_1 which hits both ℓ and m (there must be at least one since F_1 bridges A). For similar reasons, c meets the same component k. Let $y' \in k \cap e$. Because c has Property $\sim$ P, there is an arc $b \subset c$ such that b misses one of F_1, F_2 and connects a point of k to x. Since $k \subset F_1$, evidently $b \cup k$ misses F_2. Since $b \cup k$ is a continuum which connects $y' \in e$ to x and misses F_2 (as does e), we can construct the lasso Y as in the proof of Th 6, reading $b \cup k$ for $c \cup d$ and e for r. In the proof of Th 6, the curve r misses $F_1 \cup F_2$, whereas here e misses just F_2; however following the procedure of the proof of Th 6 will yield a lasso Y which misses F_2. The lasso Y is used to construct c' precisely as in the proof of Th 6, keeping in mind the fact that Y misses F_2, so that the resulting c' also misses F_2. The assumption that $x \notin$ Int c is removed just as in the proof of Th 6 □.

CHAPTER TWO: BING'S DOGBONE SPACE AND CURTIS' CONJECTURE.

1. An upper semicontinuous decomposition G of E^3 into compact sets (or simply a decomposition of E^3) is a collection of disjoint compact sets Λ of E^3 such that the union of the elements of the decomposition is E^3, and each element $\Lambda \in G$ possesses a system of open neighbourhoods which are unions of elements of G. The decomposition space $\mathcal{G}$ associated with G is a topological space in which each point is as element $\Lambda \in G$, and the open sets are just those subsets of G the union of whose elements is open when considered as subset of E^3. Thus each point Λ of $\mathcal{G}$ has a system of neighbourhoods each of which is open 'both in $\mathcal{G}$ and in E^3'. One can use this intuitive idea to get a certain geometric grasp of the topology of $\mathcal{G}$ simply by remembering that 'some points are sets' and keeping an eye on the neighbourhoods; for example one often does geometry on a torus or Klein bottle by looking at the equivalent decomposition space of a rectangle 'with certain sides identified'. If an element $\Lambda \in G$ contains more than one point of E^3, then Λ is called a big element of G. If Λ is a singleton, then Λ is a small element of G. In $\mathcal{G}$, the corresponding points are called big and small points. The decompositions G in which we will be interested are all pointlike, which is to say that the complement of each $\Lambda \in G$ is topologically equivalent to that of a point; in particular, each Λ is connected. We definitely assume some acquaintance with these ideas and do not regard the present text as an adequate introduction. The classical approach to decompositions and decomposition spaces may be found in Ch VII of [10]. Our approach will be more along the lines of [3,§6]. We will use two main classical results: i) an upper semicontinuous decomposition space (i.e. the decomposition space associated with an upper semicontinuous decomposition) of E^3 is a separable metric space. ii) there is an

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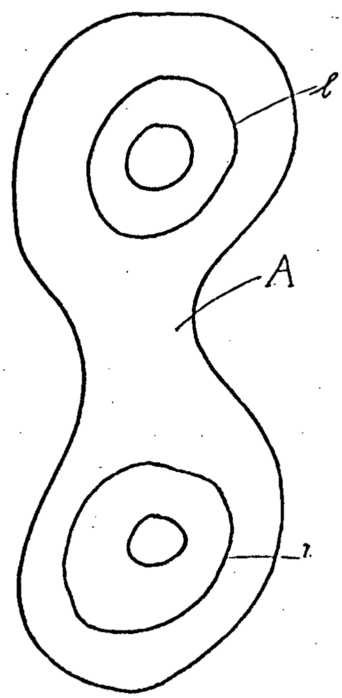


Fig 16a.

obvious way of expressing G as a quotient space. In this case the quotient topology turns out to be the decomposition space topology, and the canonical mapping ϕ of the quotient space carries each $\Lambda \in G$ onto the corresponding point $\phi[\Lambda]$ in G . We will often write A^* for $\phi[A]$ if A is a subset of E^3 . In the sequel, 'decomposition space' will mean 'pointlike upper semi continuous decomposition space of E^3.'

An important question is: if G is a decomposition space, is G homeomorphic to E^3? That G is homeomorphic to E^3 is <u>Wardwell's conjecture</u> (in [8]) and is known to be false. R. H. Bing showed this in 1957 with a celebrated example ([12]) which reinforced everyone's worst prejudices against the analytic topology of E^3 . In Bing's example, the dogbone space of our title, most of the elements of the decomposition are small. Each big element is a tame arc (so that the example refutes a very strong form of Wardwell's conjecture), and the big points in the decomposition space form a totally disconnected set.

Detailed construction of the dogbone decomposition.

We will describe an infinite sequence of compact sets whose elements intersect to form the set of big elements of the dogbone decomposition G . Our construction differs slightly from Bing's, but we assume an acquaintance with the original construction in [12] and will not prove, for example, that the various embeddings to be described can be assumed to be polyhedral.

Dogbone space takes its name from the distinctive shape of the double handlecube A depicted in fig 16a. We imagine A imbedded in E^3 . A path $\ell \subset \text{Int } A$, which makes one circuit of the circle marked ℓ

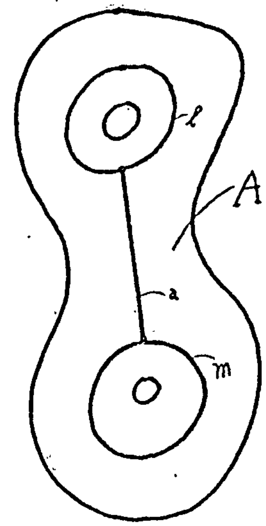


fig 16 b.

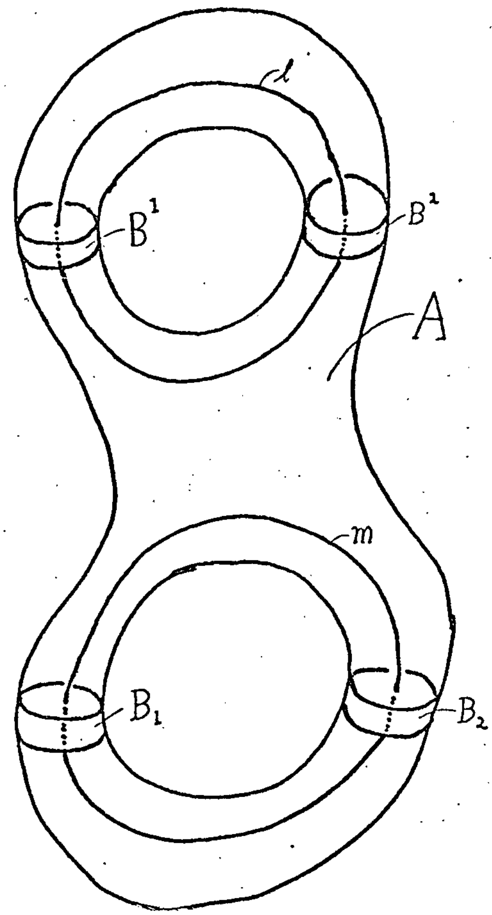
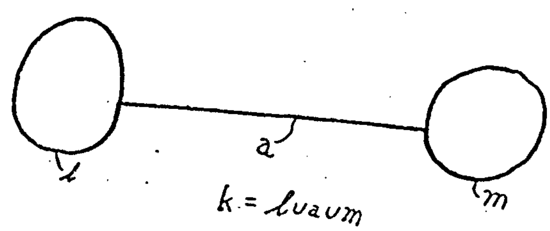
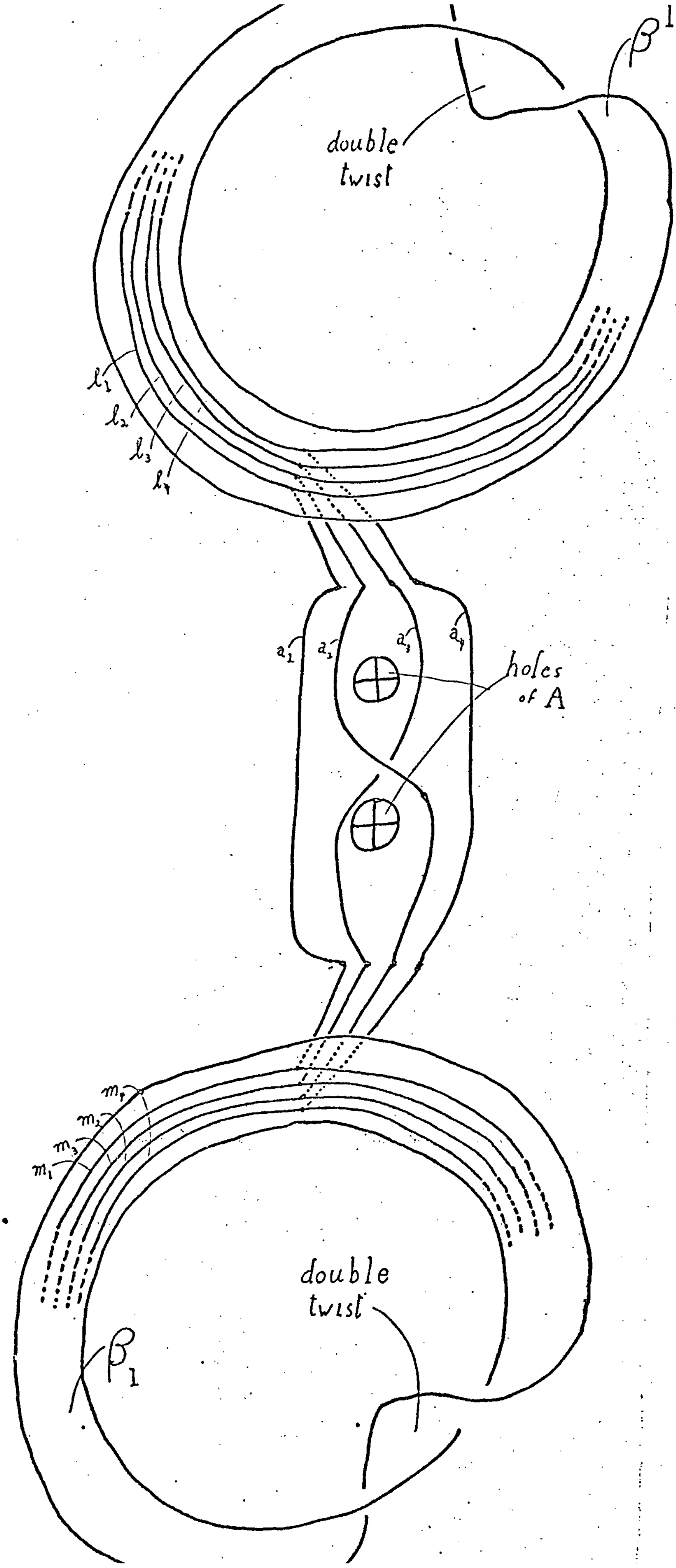


fig 17.

fig 18.





β^1
double twist
l_1
l_2
l_3
l_4
a_1
a_2
a_3
a_4
holes of A
m_4
m_2
m_3
m_1
double twist
β_1

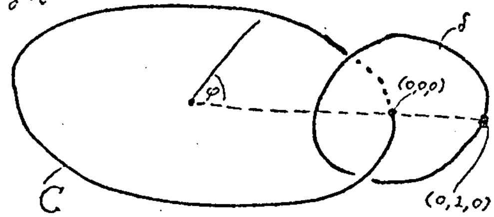
Fig 19

in fig 16a is called the upper eye of A. A path $m \subset \text{Int } A$ which makes one circuit of the curve marked m in the figure is called the lower eye of A (we imagine the dogbone placed vertically in E^3 so that it makes sense to talk about 'upper' and 'lower' here). One could imagine A to be a closed r-neighbourhood of a planar double ended lasso consisting of the eyes ℓ and m laid out as nice circles plus a straight connecting arc a (with r of course, taken sufficiently small, say less than one-third of the common diameter of the nice circles ℓ and m). We call $\ell \cup m \cup a$ the <u>centre</u> of A. The centre of a dogbone will not be important in this chapter (but will be needed in Chapters III, IV). The idea of A as an r-neighbourhood of its centre k is introduced mainly to pin down the embedding of k in A; we usually draw k and A as in fig 16b. Fig 17 shows four short solid cylinders B^1, B^2, B_1, B_2, which are subsets of A and cut into the eyes of A as the figure suggests. The removal of one of B^1, B^2 and one of B_1, B_2 from A leaves a set whose closure is a cube. A dogbone can be imagined in the topologically equivalent form of a thick double ended lasso as shown in fig 18. In a sense, we are pictorially confusing the dogbone with its centre. Let A_1, A_2, A_3, A_4 be four dogbones embedded as shown in fig 19 by embeddings $h_j: A \to A$, $j = 1, 2, 3, 4$ so that the $A_j = h_j[A]$ are mutually disjoint and lie in $\text{Int } A$. In fig 19, two double twisted bands β^1 and β_1 are placed so that $\beta^1(\beta_1)$ lies in the interior of the upper (lower) component of $A - B_1 - B_2 - B^1 - B^2$. In the obvious way, the centre of A_j is called k_j, $j = 1, 2, 3, 4$, with upper loop ℓ_j and lower loop m_j. The ℓ_j are placed so as to lie as parallels on β_1. The connecting arcs a_j are laid out in a peculiar

Toroidal Coordinates (φ, r, θ).

(r,θ) defines a point on the disk δ. As φ increases in $0 \le \varphi \le 2\pi$, δ sweeps a toroid which is a figure of revolution about the planar circle C.

Fig 20.



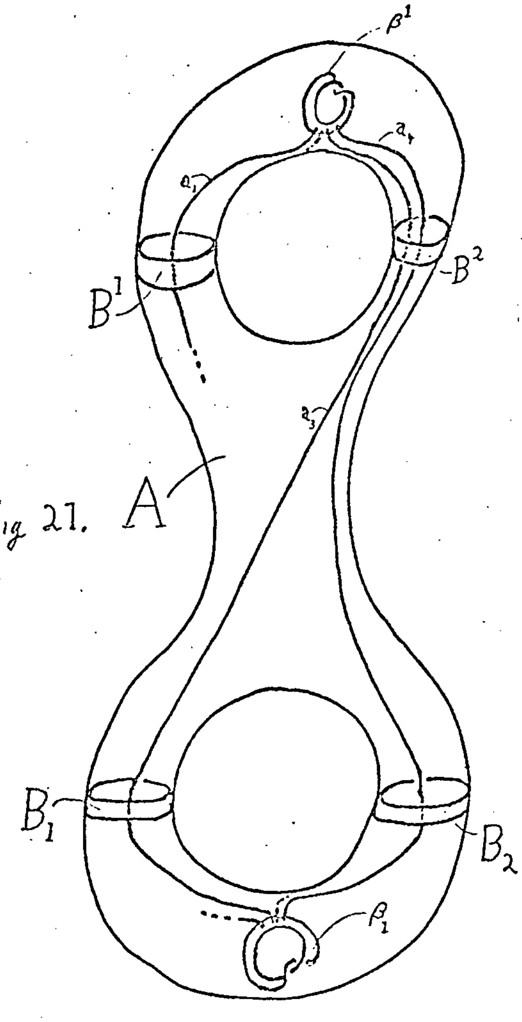


Fig 21.

way which is characteristic of the dogbone construction. Using toroidal coordinates (which we recall in fig 20), we could define β^1 and β_1 to be appropriate translations of the set $r \leq 1$, $\theta = \phi$ and thus construct a band with an even double twist. However the bands in the drawing are translations of the set

$$r \leq 1, \quad \theta = 0: \quad \pi/3 \leq \phi \leq 2\pi$$

$$r \leq 1, \quad \theta = 6\phi: \quad 0 \leq \phi \leq \pi/3$$

This gives a 'flatter' band and a better picture. Another concession to art appreciation is the placing of β^1 and β_1 so that their 'flat' parts lie on the plane of k. This necessitates a right angled bend in the a_j near β^1 and again near β_1. The additional conditions are imposed on a_j that a_j misses $\beta^1 \cup \beta_1$ except at $a_j \cap \ell_j$ and $a_j \cap m_j$, and that the part of a_j lying within a distance ε of $\beta^1(\beta_1)$ consists of a single straight arc perpendicular to $\beta^1(\beta_1)$. Note that the order of ℓ_j's on β^1 is $\ell_1, \ell_2, \ell_3, \ell_4$ while the order of m_j's on β_1, due to the unusual embedding of the A_j, is m_1, m_3, m_2, m_4. The B_i and B^i locate the A_j in the following way (see fig. 21):

$$A_1 \text{ lies in } \mathrm{Int}(A - B^2 - B_2)$$
$$A_2 \text{ lies in } \mathrm{Int}(A - B^2 - B_1)$$
$$A_3 \text{ lies in } \mathrm{Int}(A - B^1 - B_2)$$
$$A_4 \text{ lies in } \mathrm{Int}\,(A - B^1 - B_1)$$

The closure of the component of $A - B^1 - B^2 - B_1 - B_2$ which contains $\beta^1(\beta_1)$ is called $K_1(K_2)$. Finally we let $A_1 \cup A_2 \cup A_3 \cup A_4 = \mathcal{A}_1$.

Now since each dogbone A_j is homeomorphic to A, we can embed four dogbones $A_{j1}, A_{j2}, A_{j3}, A_{j4}$ in each A_i just as the A_j are embedded in A. We could write $A_{jk} = h_j h_k[A]$. The union of the 16 A_{jk}, j, k, chosen from 1, 2, 3, 4, is called $\mathcal{A}_2$. The construction proceeds as in [12] with the definition of 64 $A_{jk\ell} = h_j h_k h_\ell[A]$ where $h_j h_k h_\ell$ embeds A in A_{jk} just as A_ℓ is embedded in A. The union of the 64 $A_{jk\ell}$ is called $\mathcal{A}_3$. The construction proceeds in this way, defining at each m-th stage 4^m dogbones whose union is $\mathcal{A}_m$. Let the intersection $A \cap \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \ldots = A_0$. The components of A_0 are compact and are defined to be the big elements of G while the remaining points of E^3 are the small elements. The dogbone space $\mathcal{D}$ is the associated decomposition space of G.

Remark 1. In $k_1 \cup \ldots \cup k_4$, each upper (lower) eye fails to shrink to a point in the complement of any other upper (lower) eye. This is easily checked using, say, Ch XV of [6].

Remark 2. We are sure that the construction of $\mathcal{D}$ here is the same as that given by Bing in [12]. In the Appendix we show a deformation of the upper part of $k_1 \cup \ldots \cup k_4$ to look like the upper part of Bing's construction. We think that the reader will see the plausibility, but we give no strict proof that our embedding of $A_1 \cup \ldots \cup A_4$ is the same as the corresponding embedding in [12], and our attitude in this paper will be that Dogbone space has been redefined.

Remark 3. We know little about the h_j except that they embed A in certain ways. We cannot, for example be sure of the location of the 64 $h_j h_k h_\ell[k]$. However the various subsets of $A_{jk\ell}$ are images of sub-

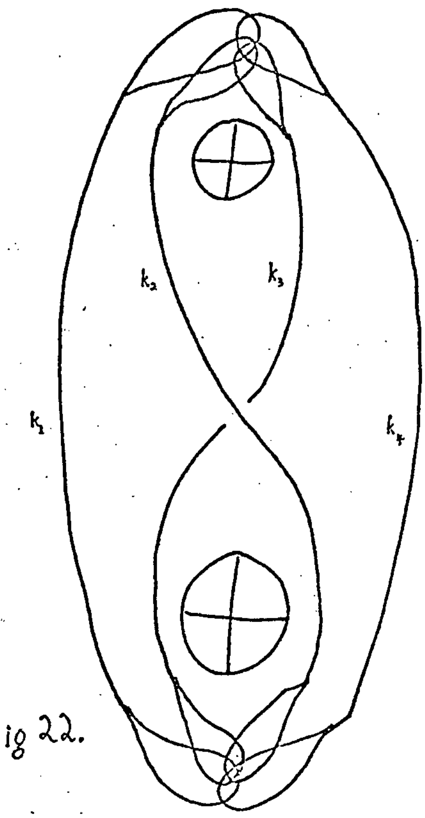


Fig 22.

sets of $A_{jk\ell}$ are images of subsets of A and continue to be related to each other in all the ways which are preserved by a homeomorphism of A; and we will usually apply results obtained for A to any $A_{jk...r}$ without further justification. Note that k_j has a property which is not preserved by homeomorphism: a_j is perpendicular to β^1 or β_1 wherever it lies near these sets. This property is lost after the first stage of the dogbone construction. This does not prevent the construction of $\mathcal{D}$, but further comment will be required when we use the property in Chapters III and IV.

Remark 4. Partly out of adherence to the traditional representation in [3] and partly because the use of β^1 and β_1 will not become apparent until Ch III (apart from the fact that they cause the eyes to link together) we will often use the picture in fig 22 to describe the embedding of $k_1 \cup \ldots \cup k_4$ in A. We will use pictures like fig 22 in which the crossovers of the links are ignored, whenever the exact manner of linking is unimportant. In this chapter, the only thing which needs to be kept in mind concerning the linking of the ℓ_j and m_j is that no $\ell_j(m_j)$ will shrink to a point in the complement of any other $\ell_{j'}(m_{j'})$. Another pictorial abbreviation shown in fig 22 is the omission of much of the boundary of A, even though the figure purports to describe the embedding of the four centres in A. As in fig 22, we will often show only the holes of A which will be represented by the symbol $\oplus$.

Intuitively it often helps to see a decomposition space as E^3 with certain sets identified. One typically finds the small elements distributed so that it is easy to define a neighbourhood system for the big elements. Thus a lot can be learned about the topology of the decomposition space by looking at elements of the associated decomposition. However if we try to approach $\mathcal{D}$ in this way, we find that the components of A_0, which constitute the big elements of G, are hard to see. To find a big element, note that each big element of G is the limit of a sequence of dogbones A, A_j, A_{jk}, ... Evidently each big element may be specified by an infinite sequence j, k, ... of integers chosen from 1, 2, 3, 4; and the A, A_j, A_{jk}, ... constitute a neighbourhood system of this big element. Because A is compact, we know that if Λ is a big element of G, then if Λ lies in an open set V, some member $A_{jk...r}$ of the neighbourhood system lies in V (see I, 7.2 of [10]). It is known that each big element of G is a tame arc (see [12, §2]). The canonical mapping ϕ is a local homeomorphism near small elements of G (because A_0 is compact) but not of course in general. The fact that ϕ is monotone means that ϕ^{-1} preserves connectedness (VIII (2.2) of [10]). Simple connectivity properties are more complicated. As will appear later, any open set $V^* \subset \mathcal{D}$ which lies in A^* and contains a big point of $\mathcal{D}$ cannot be simply connected. We must expect a proof of this property to be delicate since it is known that $\mathcal{D}$ is locally simply connected. ([5]). (Roughly, what happens is that any mapping of S^1 into small neighbourhood V^* of a big point of $\mathcal{D}$ will shrink to a point in the <u>second</u> smallest dogbone which contains V^* . Thus one can satisfy the definition of 'locally simply connected' by taking a smaller neighbourhood V^* although V^* itself will never be simply connected.)

For the rest of this section we will prove a result which relates simple connectivity in $\mathcal{D}$ to the same property in E^3. A mapping f of S^1 into a space X shrinks to a point in X iff f is homotopic in X to a constant mapping or represents the identity in $\pi_1(X)$ for an appropriate base point. A third equivalent statement is: consider S^1 to be the boundary of a disk Δ : then $f:S^1 \to X$ shrinks to a point iff f can be extended to a mapping $\overline{f}$ of Δ into X .

(1.1). Let V* be an open set in $\mathcal{D}$. If $f:S^1 \to V^*$ so that rng f consists of small points, then $\phi^{-1}f$ will shrink to a point in V, where ϕ is the canonical mapping of E^3 onto $\mathcal{D}$.

Corollary: if V* is simply connected then so is V .

We can use this result to examine sets V* which we suspect not to be simply connected, by looking at the associated $V \subset E^3$. The result (1.1) and its corollary are not new and are particular cases of Lemma 1 of [2]. The proof the (1.1) introduces methods which will recur frequently in the sequel, and we will complicate the (pretty easy) proof slightly by introducing more generality in the method than is needed for the present argument.

Outline of proof. a) Assume that f maps the boundary of a disk Δ into V* . Since f shrinks to a point, there is a mapping $\overline{f}:\Delta \to V^*$ such that $\overline{f}_{|Bd\Delta} = f$. Recall that A_0^* is the union of the big points of $\mathcal{D}$. The set $f^{-1}[A_0^*]$ is compact. Let Q be a disk with holes such that $Q \subset \Delta$, the outer boundary of Q is $Bd\Delta$, and the (open) holes of Q contain $f^{-1}[A_0^*]$. b) The mapping $\overline{f}$ maps Q into small points of $\mathcal{D}$; thus '$\phi^{-1}\overline{f} = \overline{f}$' on Q . Let the (open) holes of Q be $u_1, \dots u_n$. For each u_r extend $\phi^{-1}\overline{f}_{|Bd\, u_r}$ to a mapping γ_r into V by shrinking $\phi^{-1}\overline{f}_{|Bd\, u_r}$ to a point in a certain cube in V .

c) Glue the γ_r, $r = 1, \dots n$, to $\phi^{-1}\overline{f}_{|Q}$ to form a mapping of Δ into V .

Details of proof. a) We know that $\overline{f}:\Delta \to V^*$ so that $f = \overline{f}_{|Bd\Delta}$. Since A_o is compact, so is A_o^* and $\overline{f}^{-1}[A_o^*]$. Note that $\overline{f}^{-1}[A_o^*]$ misses $Bd\Delta$ because $\overline{f}[Bd\Delta]$ consists of small points, from the hypothesis. To obtain the disk with holes Q, we use the following result which will be needed several times in the sequel.

(1.2) Let Δ be a disk in E^2 and S a compact set in Δ . Then there exist n disks $W_1, \dots W_n$ such that W_r Δ and

i) $W_r \cap W_s = \emptyset$, $r \neq s$.

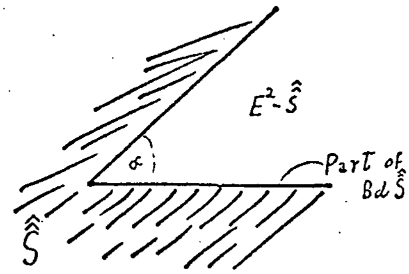
ii) $S \subset W_1 \cup \dots \cup W_n$.

iii) Each point of $Bd\, W_r$ lies withing a positive distance ε of S .

iv) If S misses $Bd\Delta$, then $S \subset Int\, W_1 \cup Int\, W_2 \cup \dots \cup Int\, W_n$, and $\Delta - Int\, W_1 - \dots - Int\, W_n$ is a disk with holes. If S hits $Bd\Delta$, then S misses $Bd\, W_r - Bd\Delta$ for each $r = 1, \dots n$.

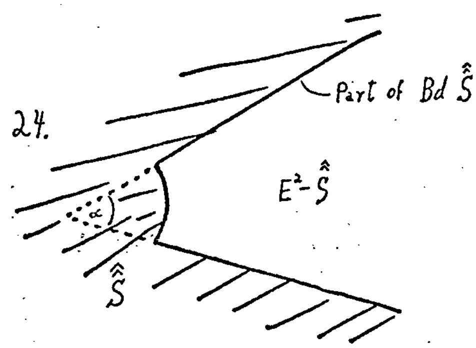
Proof of (1.2). We can assume that $S \neq \emptyset$ and that Δ has the form of an equilateral triangle. Triangulate Δ into a finite number of 2 - simplexes (i.e. closed triangular disks) whose diameter is less than $\varepsilon/2$, and whose edges are parallel to the three sides of the big triangle Δ . Note that the three vertices of Δ each belong to one 2 - simplex only so that the three vertices of Δ cannot be cut points of any union of 2 - simplexes. The only properties of the 2 - simplexes which will be used are that each 2 - simplex has an edge of length less than $\varepsilon/2$, and if two 2 - simplexes meet, they meet either along the whole of one edge or only at a vertex. Let $\hat{S}$ be the finite union of those 2 - simplexes which meet S . Evidently $\hat{S}$ is lc and each component of $\hat{S}$ is a lc continuum. For later reference, note that S cannot meet $\mathrm{Bd}\,\hat{S}$ at a point interior to Δ; for assume that S meets $\mathrm{Bd}\,\hat{S}$ at a vertex $v \in \mathrm{Int}\,\Delta$. Then by construction of $\hat{S}$, the entire star of v lies in $\hat{S}$ and $v \notin \mathrm{Bd}\,\hat{S}$. S cannot meet $\mathrm{Bd}\,\hat{S}$ at the interior of an edge in $\mathrm{Int}\,\Delta$ by a similar argument. We alter $\hat{S}$ to a set $\hat{\hat{S}}$ which has no cut points in this way: a cut point of $\hat{S}$ cannot lie in the interior of a 2-simplex in $\hat{S}$, nor in the interior of an edge belonging to one 2-simplex, nor in the interior of an edge belonging to two 2-simplexes. Thus the cut points of $\hat{S}$ are a (finite) subset of the vertexes. Let the cut points be $t_1,\ldots t_k$, and cover each t_s with a set b_s, $s = 1,\ldots k$, which is a disk of radius $\varepsilon/6$ and centre t_s if $t_s \subset \mathrm{Int}\,\Delta$; and is a semi-disk of the same centre and radius if t_s lies on $\mathrm{Bd}\Delta$ and is not a vertex of the big triangle Δ (thus b_s is a 'disk relative to Δ'). We do not define b_s for the three remaining points of Δ since these points are never cut points of $\hat{S}$. Note that the b_s are disjoint. Define $\hat{\hat{S}}$ to be $\hat{S} \cup b_1 \cup \ldots \cup b_k$. It will turn out that the $\mathrm{Bd}\,W_r$ are some of the components of $\mathrm{Bd}\,\hat{\hat{S}}$. We know the following facts about $\hat{\hat{S}}$: the components of $\hat{\hat{S}}$ are lc continua and are consequently bounded apart. Every point of $\hat{\hat{S}}$ (in particular every point of $\mathrm{Bd}\,\hat{\hat{S}}$) lies within a distance ε of S; the boundary of $\hat{\hat{S}}$ consists of the union of a finite number of straight arcs (which are either edges of 2 - simplexes or edges minus the interior of one or two b_s) and a finite number of segments of circles (i.e. proper subsets of various $\mathrm{Bd}\,b_s$). Such a subset is precisely $\mathrm{Bd}\,b_s - \mathrm{Bd}\Delta$ intersected with a connected subset of $\mathrm{St}\,t_s$; a suitable upper bound for the number of segments is the number of b_s

Fig 23.



The angle α may be $r\pi/3$; $r = 1, 2, 3, 4, 5$.

Fig 24.



times the number of subsets of 2 - simplexes). Two straight arcs in Bd S meet as in fig 23; a straight arc meets a segment as in fig 24. Segments never meet because the b_s are disjoint. Evidently $\hat{\hat{S}}$ has no cut point on its boundary and hence no cut point at all. Let the components of $\hat{\hat{S}}$ be $W_1', \ldots W_m'$. These will be reordered so that the first n components will lie in the disks required in (1.2). Since each W_a', $a = 1, \ldots m$, is a lc continuum with no cut point, by IV (9.3) and VI (2.5) of [10], the unbounded complementary domain of W_a' is bounded by a simple closed curve which will be called c_a . Evidently $c_a \subset W_a'$. Reorder the W_a' and corresponding c_a so that $c_1, \ldots c_n$ are contained in the interior of no other circle c_a, while each of $c_{n+1}, \ldots c_m$, if they exist, is contained in the interior of a c_a . Let $W_a = \overline{\text{Int } c_a}$, $a = 1, \ldots m$. We will show that $W_1, \ldots W_n$ satisfy i), ii), iii), iv) of the statement of (1.2).

Proof of i). If $r \neq s$ then neither of c_r, c_s is contained in the interior of the other. Then $\overline{\text{Int } c_r} \cap \overline{\text{Int } c_s} = W_r \cap W_s = \emptyset$ by I(1.6).

Proof of ii). $S \subset \hat{\hat{S}} = W_1' \cup \ldots \cup W_m'$. Each $W_a' \subset \overline{\text{Int } c_a}$, $a = 1, \ldots m$, since Ext c_a is the unbounded complementary domain of W_a' . Hence $S \subset W_1 \cup \ldots \cup W_m$. And in fact $S \subset W_1 \cup \ldots \cup W_n$ because for $n + 1 \leq a \leq m$, c_a lies in some Int c_r, $r = 1, \ldots n$, and by I(1.6), $\text{Int } c_a = W_a \subset W_r$.

Proof of iii). Each Bd W_r is a $c_r \subset \text{Bd } \hat{\hat{S}}$, and we saw earlier that all of (the closed set) $\hat{\hat{S}}$ lies near S .

Proof of iv). Take ε less than the distance from (compact) S to $Bd\Delta$ if S misses $Bd\Delta$. The rest follows from the definition of a disk with holes, from i), and from the fact that a point of $Bd\,\hat{S}$ and hence a point of $Bd\,\hat{\hat{S}}$ misses S wherever it lies in $Int\,\Delta$ $\square$.

We now return to a) in the proof of (1.1). Since $f^{-1}[A_0^*]$ is compact, from (1.2) there are disks $W_1, \ldots W_n$ which lie in Δ and such that $f^{-1}[A_0^*] \subset W_1 \cup \ldots \cup W_n$ and each $Bd\,W_r$ lies within a distance ε of $f^{-1}[A_0^*]$. Since $f^{-1}[A_0^*]$ misses $Bd\Delta$, by (1.2)iv, the set $\Delta - Int\,W_1 - \ldots - Int\,W_n = Q$ is a disk with holes. Let u_r be the 'holes' of Q, $r = 1, \ldots n$, i.e. $u_r = Int\,W_r$.

b) Since $\overline{f}^{-1}[A_0^*]$ lies in the holes u_r of Q, $\overline{f}[Q]$ consists only of small points. Thus ϕ^{-1} is a well defined mapping when restricted to $\overline{f}[Q]$ and $\phi^{-1}\overline{f}_{|Q}$ maps Q into $V \subset E^3$. We now find cubes in which to shrink $\phi^{-1}\overline{f}_{|Bd\,u_r}$, $r = 1, \ldots n$. Since $\overline{f}[\Delta] \cap A_0^*$ is compact and the dogbones (considered as sets in $\mathcal{D}$) evidently form a neighbourhood system of A_0^*, there is a covering of $f[\Delta] \cap A_0^*$ by a finite number of dogbones $J_1^*, \ldots J_q^*$ each of which lies in V^*. Look at the corresponding $J_1, \ldots J_q$ in E^3. If J_s, $s = 1, \ldots q$, belongs to the mth stage of the dogbone construction, define J_{s1}, J_{s2}, J_{s3}, J_{s4} to be the four dogbones of the $m + 1$st stage lying in J_s. Note that since $J_s \subset V$, each J_{sj} lies in a cube M_{sj} which is a subset of J_s and hence of V (if J_s were the dogbone $A \subset V$ then $J_{s1} \subset M_{s1} = A - B^2 - B_2$, J_{s2} lies in $M_{s2} = A - B^2 - B_1$ etc.) Now in a) above, we could have chosen ε so small that each $\overline{f}[Bd\,W_r] = \overline{f}[Bd\,u_r]$ lies in some J_{sj}^* (for $\mathcal{D}$ is a separable metric space, and there is a minimum distance in the dogbone metric separating the compact set $\overline{f}[\Delta] \cap A_0^*$ from the complement of the union of the J_{sj}^*). Clearly $\phi^{-1}\overline{f}[\text{Bd } u_r]$ is defined and lies in the union of the J_{sj}. We can assume the J_{sj} are disjoint because we could have removed from the covering $J_1, \ldots J_q$ any J_s which was contained in any other member of the covering. Since $\phi^{-1}\overline{f}[\text{Bd } u_r]$ is connected and the closed sets J_{sj} are separated, $\phi^{-1}\overline{f}[\text{Bd } u_r]$ lies entirely in some one J_{sj} and $\phi^{-1}\overline{f}|_{\text{Bd } u_r}$ shrinks to a point in $M_{sj} \subset V$. Thus there is an extension γ_r of $\phi^{-1}\overline{f}|_{\text{Bd } u_r}$ to all of $\overline{u}_r$, i.e. $\gamma_r : \overline{u}_r \to M_{sj} \subset V$ and $\gamma_r|_{\text{Bd } u_r} = \phi^{-1}\overline{f}|_{\text{Bd } u_r}$.

c) In view of the set-theoretic definition of function we can express the idea of 'mappings glued together' by unions of mappings. Consider the union $\phi^{-1}f|_Q \cup \gamma_1 \cup \ldots \cup \gamma_n$. This is a well-defined mapping of $Q \cup \text{dom } \gamma_1 \cup \ldots \cup \text{dom } \gamma_n = Q \cup u_1 \cup \ldots \cup u_n$ into V because each mapping in the union has its image in V and because where the domains intersect the intersection is closed and the mappings agree on the intersection; in fact every point of domain intersection occurs on a $\text{Bd } u_r$ where we know that γ_r agrees with $\phi^{-1}\overline{f}|_{\text{Bd } u_r}$ by construction of γ_r. Finally we note that the new mapping $\phi^{-1}\overline{f}|_Q \cup \gamma_1 \cup \ldots \cup \gamma_n$ agrees with $\phi^{-1}\overline{f}$ on $\text{Bd}\Delta \subset Q$ and is thus a homotopy which shrinks $\phi^{-1}f$ to a constant mapping into V. This completes the proof of (1.1) □. We will record the argument in this paragraph as a separate result.

(1.3). Let Δ, $W_1, \ldots W_n$ be defined as in (1.2) including the fact that $\Delta - \text{Int } W_1 - \ldots - \text{Int } W_n$ is not neccessarily a disk with holes.

Let $g:\Delta - \text{Int } W_1 - \ldots - \text{Int } W_n \to E^3$. Then each $g_{|\text{Bd } W_r}$ is defined; and if $g_{|\text{Bd } W_r}$ shrinks to a constant mapping in a space P_r, $r = 1, \ldots n$, then there is a mapping of Δ into rng $g \cup P_1 \cup \ldots \cup P_n$. In particular, $g_{|\text{Bd}\Delta}$ will shrink to a point in rng $g \cup P_1 \cup \ldots \cup P_n$.

Proof: The argument of c) in the proof of (1.1) is used and is valid even if $\Delta - \text{Int } W_1 - \ldots - \text{Int } W_n$ is not a disk with holes. It is easy to see that $\text{Bd}\Delta \subset \Delta - \text{Int } W_1 - \ldots - \text{Int } W_n$ since $W_r \subset \Delta$; then g is defined on $\text{Bd}\Delta$. Since $\text{Bd } W_r \subset \Delta$, $g_{|\text{Bd } W_r}$ is always defined $\square$.

Proof of the Corollary to (1.1). Let $\psi:S^1 \to V$. If V^* is simply connected, we can use (1.1) to show that ψ shrinks to a point in V only if rng ψ misses A_0. Evidently in order to apply (1.1), it is sufficient to show that ψ is homotopic in V to a mapping $\psi':S^1 \to V - A_0$. We use '$\simeq$' to mean 'is homotopic in V to'. To construct ψ': using an argument like that of b) in the proof of (1.1), cover $\psi[S^1] \cap A_0$ with dogbones $J_1, \ldots J_q$ which are disjoint and lie in V. Dogbones J'_{sj}, $j = 1, 2, 3, 4,$ are defined just as in b) of (1.1) so that $\bigcup_{rj} J_{rj}$ covers $\psi[S^1] \cap A_0$ and each J_{sj} lies in a cube $M_{rj} \subset V$ (the construction of the J_{sj} here is not identical to that of the J_{sj} in b) of (1.1), but the construction here is easier since we need not consider sets in $\mathcal{D}$). We assume that some point z exists in $\psi[S'] \cap (E^3 - \bigcup_{sj} \text{Int } J_{sj})$, for otherwise, since S^1 is connected and the Int J_{sj} are separated, rng ψ lies in one J_{sj} M_{sj}, and the proof is concluded by shrinking ψ to a point in M_{sj} V. Now choose $\delta > 0$ so that if x and y are closer together on S^1 than the distance δ, then $\psi(x)$ and $\psi(y)$ are closer together than the distance from $\Psi[S'] \cap A_0$ to $E^3 - \bigcup_{sj} J_{sj}$ (remember that $\bigcup_{sj} J_{sj}$ is a neighbourhood of $\psi[S'] \cap A_0$ so that this distance is positive). If every point of S^1 lies closer than δ to $\psi^{-1}[E^3 - \bigcup_{sj} J_{sj}]$, then by the definition of δ no point of S^1 maps under ψ into A_0, and we can let $\psi' = \psi$. If some point of S^1 fails to lie withing δ of $\psi^{-1}[E^3 - \bigcup_{sj} J_{sj}]$, then there is an open interval; e_1' in $\psi^{-1}[\bigcup_{sj} \text{Int } J_{sj}]$ such that the length of e_1' is greater than δ. Let e_1 be the largest open interval such that $e_1' \subset e_1 \subset \psi^{-1}[\bigcup_{sj} \text{Int } J_{sj}]$. Then $\overline{e}_1$ is a closed interval of length greater than δ whose end points p_1 and q_1 map into Bd $\bigcup_{sj} J_{sj}$ by the usual continuity argument. (Since S^1 is a circle, we must make an easy allowance for the possibility that $\psi(p_1) = \psi(q_1) = z$.) Because the J_{sj} are separated and $\psi[\overline{e}_1]$ is connected, $\psi[e_1]$ lies in the interior of just one J_{sj} which we will call R_1, while $\psi(p_1)$ and $\psi(q_1)$ lie in Bd R_1. Define the mapping $\psi_1 : S^1 \to E^3$ so that $\psi = \psi_1$ on $S^1 - e_1$ while $\psi_1|\overline{e}_1$ is a path in (connected) Bd R_1 with end points $\psi(p_1)$ and $\psi(q_1)$ (this is well defined because p_1 and q_1 map into Bd R_1 under ψ). Both ψ and ψ_1 are paths in V and $\psi|\overline{e}_1 \simeq \psi_1|\overline{e}_1$ because they share end points and both map into the same cube $M_{sj} \supset R_1$ with $M_{sj} \subseteq V$. Evidently $\psi \simeq \psi_1$. Since rng $\psi_1|\overline{e}_1 \subset$ Bd $R_1 \subset E^3 - A_0$, the homotopy has moved images of points in e_1 away from A_0. We now look for an open interval e_2' in $S^1 - \overline{e}_1$ where e_2' is of length greater than δ and such that e_2' maps into $\bigcup_{sj} \text{Int } J_{sj}$ under ψ_1 (and in fact under ψ, since $\psi = \psi_1$ on $S^1 - e_1$). If e_2' does not exist, let $\psi_1 = \psi'$. If e_2' exists, then there is an open interval e_2 of maximal length such that $e_2' \subset e_2 \subset S^1 - \overline{e}_1$ and $\psi_1[e_2] \subset \bigcup_{sj} \text{Int } J_{sj}$. The end points p_2, q_2 of e_2 map under ψ_1 into $E^3 - \bigcup_{sj} \text{Int } J_{sj}$, either because of the maximality of e_2 if the end point is in $S^1 - \overline{e}_1$, or, if the end point is in $\mathrm{Bd}\,(S^1 - \overline{e}_1) = \mathrm{Bd}\ e_1$, because $\psi[\mathrm{Bd}\ e_1] \subset \mathrm{Bd}\ R_1$. By a continuity argument, $\psi_1[e_2]$ lies in the interior of some one J_{sj} called R_2 and $\psi_1(p_2)$ and $\psi_1(q_2)$ lie in $\mathrm{Bd}\ R_2$. Define $\psi_2 : S^1 \to E^3$ so that ψ_2 agrees with ψ_1 on $S^1 - e_2$ (note that this means that ψ_2 agrees with ψ on $S^1 - e_1 - e_2$), and so that $\psi_{2|e_2}$ is a path in $\mathrm{Bd}\ R_2$ with end points $\psi_1(p_2)$ and $\psi_1(q_2)$. By a previous argument, $\psi_2 \simeq \psi_1 \simeq \psi$. Note that the fact that $\psi_2 = \psi$ on $S^1 - e_1 - e_2$ means that $\psi_2 = \psi$ on the end points of both e_1 and e_2.

In general, suppose that mappings $\psi_1 \simeq \ldots \simeq \psi_{r-1}$ of S^1 into V, intervals $e_1, \ldots e_{r-1}$ and components $R_1, \ldots R_{r-1}$ of $\bigcup_{sj} J_{sj}$ have been defined so that each $e_s \subset S^1 - \overline{e}_1 - \ldots - \overline{e}_{s-1}$, $s = 1, \ldots r - 1$, and for each ψ_s, $\psi_s = \psi_{s-1}$ on $S^1 - e_s$, $\psi_s[\overline{e}_s] \subset \mathrm{Bd}\ R_s \subset E^3 - A_o$. Now look for an open interval $e_r' \subset S^1 - \overline{e}_1 - \overline{e}_2 - \ldots - \overline{e}_{r-1}$ such that the length of e_r' is greater than δ and $\psi_{r-1}[e_r'] \subset \bigcup_{sj} \mathrm{Int}\ J_{sj}$, or equivalently $\psi_{r-1}[e_r] \subset \mathrm{Int}\ R_r$, where R_r is a single J_{sj} (and hence lies in a cube $M_{sj} \subset V$). If there is no such interval, let ψ_{r-1} be ψ'. If e_r' exists, then let e_r be the largest open interval in $S^1 - \overline{e}_1 - \ldots - \overline{e}_{r-1}$ such that $\psi_{r-1}[e_r] \subset \mathrm{Int}\ R_r$. We know that ψ_{r-1} carries the end points p_r, q_r of e_r into $E^3 - \mathrm{Int}\ R_r$ by the maximality of e_r if the end point is in $S^1 - \overline{e}_1 - \ldots - \overline{e}_{r-1}$, or, if the end point is in $\mathrm{Bd}(S^1 - \overline{e}_1 - \ldots - \overline{e}_{r-1}) \subset \overline{e}_1 \cup \ldots \cup \overline{e}_{r-1}$, because ψ_{r-1} carries $\overline{e}_1 \cup \ldots \cup \overline{e}_{r-1}$ into $\bigcup_{sj} \mathrm{Bd}\ J_{sj}$. (To see that $\psi_{r-1}[\overline{e}_s]$ lies in some $\mathrm{Bd}\ J_{sj}$: for $s = 1, \ldots r - 1$, $\psi_s[\overline{e}_s] \subset \mathrm{Bd}\ R_s$ by construction. ψ_{s+1} agrees with ψ_s on $S^1 - e_{s+1} \supset \overline{e}_s$ since

e_{s+1} lies in $S^1 - \overline{e}_1 - \ldots - \overline{e}_s$; ψ_{s+2} agrees with ψ_{s+1} on $S^1 - e_{s+2} \supset \overline{e}_s$ since $e_{s+2} \subset S^1 - \overline{e}_1 - \ldots - \overline{e}_s - \overline{e}_{s+1}$; ψ_{s+3} agrees with ψ_{s+2} on $\overline{e}_s$, etc. until ψ_{r-1} agrees with ψ_{r-2} on $\overline{e}_s$. Since $\psi_s = \psi_{s+1} = \ldots = \psi_{r-1}$ on $\overline{e}_s$, $\psi_{r-1}[\overline{e}_s] = \psi_s[\overline{e}_s] \subset \operatorname{Bd} R_s$). Since we know that the end points of e_r are mapped by ψ_{r-1} outside of $\operatorname{Int} R_r$, and $\psi_{r-1}[e_r] \not\subset \operatorname{Int} R_r$, by continuity, $\psi_{r-1}(p_r)$ and $\psi_{r-1}(q_r)$ lie in $\operatorname{Bd} R_r$. Let $\psi_r = \psi_{r-1}$ on $S^1 - e_r$ and $\psi_r|\overline{e}_r$ be a path in $\operatorname{Bd} R_r$ with end points $\psi_{r-1}(p_r)$ and $\psi_{r-1}(q_r)$. Evidently $\psi_r|e_r \simeq \psi_{r-1}|e_r$ and $\psi_r \simeq \psi_{r-1}$.

Since each e_r is of length greater than δ and $r\delta$ must be less than the circumference of S^1, (it is easily checked that the e_r are disjoint) the sequence $\psi_1, \ldots \psi_r, \ldots$ ends at ψ_k. Let ψ' be ψ_k. We know that $\psi_k : S^1 \to V$ because each ψ_r does this; and $\psi \simeq \psi_1 \simeq \ldots \simeq \psi_k = \psi'$ in V. Before we can show that $\operatorname{rng} \psi'$ misses A_0, we must show that $\psi = \psi' = \psi_k$ on $S^1 - e_1 - \ldots - e_k$. To see this:

$$\psi_1 = \psi \text{ on } S^1 - e_1 ,$$

$$\psi_2 = \psi_1 \text{ on } S^1 - e_2 \text{ and } \psi_2 = \psi \text{ on } S^1 - e_1 - e_2 ,$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\psi_k = \psi_{k-1} \text{ on } S^1 - e_k \text{ and } \psi_k = \psi \text{ on } S^1 - e_1 - \ldots - e_k .$$

It is now easy to show that $\operatorname{rng} \psi' = \operatorname{rng} \psi_k$ misses A_0, for ψ_k carries every $\overline{e}_s$ into $\operatorname{Bd} R_s \subset E^3 - A_0$ by a previous result; and if $x \in S^1 - \overline{e}_1 - \ldots - \overline{e}_k$, then x lies within a distance δ of a point y such that $\psi_k(y) \in E^3 - \bigcup_{sj} \operatorname{Int} J_{sj}$. We can assume that $y \in S^1 - e_1 - e_2 - \ldots - e_k$ because otherwise $y \in e_1 \cup \ldots \cup e_k$ and

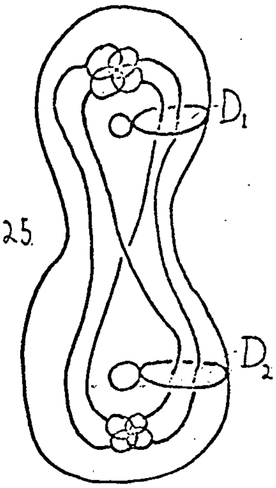


Fig 25.

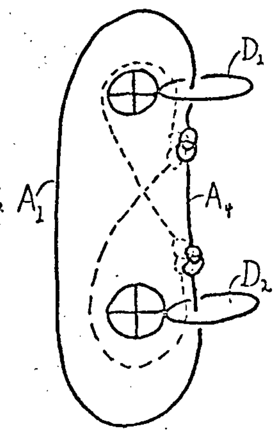


Fig 26.

A_4 may be shrunk; A_2, A_3, A_4 now miss at least one D_i. But A_1 must hit both.

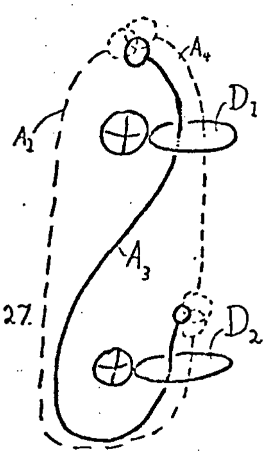


Fig 27.

A_1 & A_4 may be moved so that each of A_1, A_4 misses one D_i. But now A_3 meets both.

some point y' of $Bd(e_1 \cup \ldots \cup e_k)$ $S^1 - e_1 - \ldots - e_k$ lies closer to x than y does. Since ψ_k carries $y' \in Bd(e_1 \cup \ldots \cup e_k)$ $\subset \overline{e}_1 \cup \ldots \cup \overline{e}_k$ into $\bigcup_{sj} Bd\ J_{sj} \subset E^3 - \bigcup_{sj} Int\ J_{sj}$, we could have originally chosen y' instead of y. But if both x and y' lie in $S^1 - e_1 - \ldots - e_k$, then since $\psi = \psi_k$ on $S^1 - e_1 - \ldots - e_k$ and $\psi_k(y') \subset E^3 - \bigcup_{sj} Int\ J_{sj}$, $\psi(x) = \psi_k(x)$ lies so near to $\psi(y') = \psi_k(y')$ that $\psi(x)$ misses A_0 by our definition of δ $\square$.

2. In his paper [12], Bing was concerned with an interesting property of G which we will make use of here and in Ch. IV. The formidable aspect of G lies in what might be called its 'topological idiom', as shown in fig 25: four double ended lassos strung in a special way inside a 2-holed torus. Bing's intent in using this idiom to construct G was to utilize this property: let D_1, D_2, be the planar disks shown in fig 25. Then, no matter how the four lassos are deformed (provided that they remain linked and stay in the interior of the 2-holed torus), some one lasso will hit both D_1 and D_2. Figs 26, 27 show unsuccesful attempts by the lassos to avoid this necessity, and there is a proof of a very similar idea in §7 of [12]. Bing hoped to show that this property was induced through the construction of G in the following sense: assume that fig 25 shows D_1, D_2 in relation to the first stage of the construction of G, then, no matter how A_1, A_2, A_3, A_4 were deformed, one of these, say A_1, would hit both of D_1, D_2. Additionally, however, it might turn out that for any deformation of A_2 inside A, one of the 16 A_{jk} would hit both D_1 and D_2, and so on for the 64 $A_{jk\ell}$ etc. Bing found that there was no easy

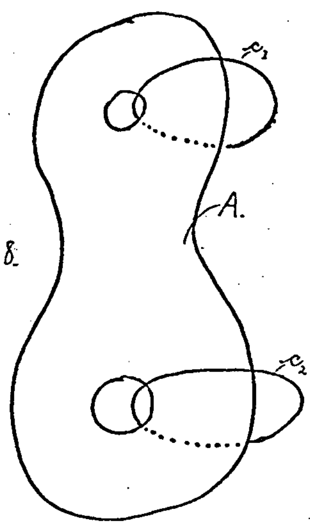


Fig 28.

proof of this (see §7 of [12]); however he was able to define a property which he called Q on the dogbones of the decomposition and show that A had this property. If a dogbone had property Q, this implied trivially that it intersected both of D_1, D_2; at the same time it could be shown that if a dogbone B had property Q, then one of the four dogbones of the next stage of the dogbone construction lying in B would have property Q . Evidently there would be a descending intersection chain of dogbones each with property Q and the limit of the chain would be a big element of G which touched both D_1 and D_2 . We can express this idea in a slightly different way:

(2.1) (Bing). Let D_1, D_2 be topological disks whose boundaries c_1, c_2 lie in A and link the upper and lower eyes respectively of A as shown in fig 28. Then either D_1 metts D_2 in A, or some big element of G meets both D_1 and D_2 .

We will refer frequently to fig 28, which shows the relationship of c_1, c_2 to A . Strictly speaking, we take c_i, $i = 1, 2,$ to be an embedding of S^1 in E^3; however we frequently will confuse the embedding with the circle which is its range (at the same time reserving the right to write rng c_i when we wish to make the distinction clear).

Bing showed that (2.1) was inconsistant with the existence of a homeomorphism between $\mathcal{D}$ and E^3 (Th 12 of [12]). In this paper we will be interested in this conjecture:

(2.2). Let Δ be a 2 - simplex. For $i = 1, 2,$ let f_i: $\Delta \to E^3$ so that $f_{i|Bd\Delta} = c_1$ and $f_{2|Bd\Delta} = c_2$ are paths whose ranges lie in $E^3 - A$ and which will not shrink to a point in the complements of the upper and lower eyes respectively of A . Then either $f_1[\Delta]$ and $f_2[\Delta]$ intersect in A, or some big element of G meets both $f_1[\Delta]$ and $f_2[\Delta]$.

In (2.2), we replace the disks D_i of (2.1) with singular disks $f_i[\Delta]$. The conjecture is plausible and lacks earthshaking surprise. It is interesting because it leads directly to the following topological property of $\mathcal{D}$:

(2.3). If (2.2) is true, then $\mathcal{D}$ fails to possess arbitrarily small simply connected open neighbourhoods about any big point.

The conclusion of (2.3) is called <u>Curtis' conjecture</u> (see 3, §6), and (2.3) reduces it to the somewhat more plausible conjecture (2.2). The remainder of this chapter consists of a proof of (2.3). The pleasures of (2.2) will be deferred to Chapters III and IV.

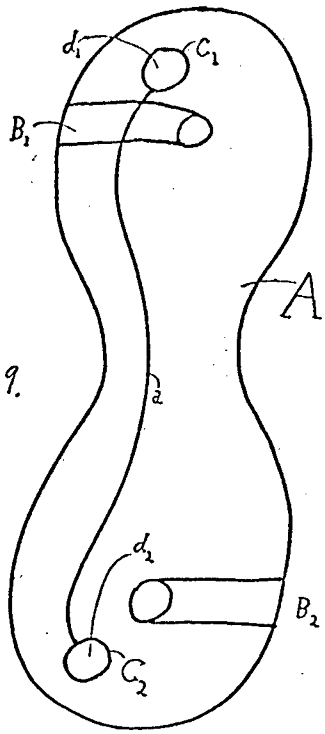
Proof of (2.3). Supppose that Λ is a big element of G and that in $\mathcal{D}$, $\Lambda^* = \phi[g]$ lies in a simply connected open neighbourhood V^* such that $\Lambda^* \subset V^* \subset A^*$. Clearly $\Lambda \subset V \subset A$, and V is open in E^3 . We could write $\Lambda = A \cap A_j \cap A_{jk} \cap \ldots$ for some sequence of dogbones A, A_j, A_{jk}, ... By the Corollary to (1.1) (of lemma 1 of [2]), V is simply connected if V^* is. Thus our assumption implies that V is simply connected. We will demonstrate that this is false by showing that $\Lambda \subset V \subset A$ with V simply connected, implies that the upper eye ℓ and the lower eye m of A shrink to a point in A . We define an upper (lower) principal path of A_j to be a mapping of S^1 into Int A_j which is homotopic in A_j to the upper eye ℓ_j (the lower eye m_j) of A_j . Upper and lower principal paths of other dogbones, including

A, are defined analogously; this a mapping of S^1 into Int A_{jk} which is homotopic in A_{jk} to $h_{jk}[\ell]$ is an upper principal path of A_{jk}. As usual, we will often confuse the mapping with its range. We know that A, A_j, A_{jk}, ... is a neighbourhood system of Λ and, by a previous remark, that some member $A_{jk \ldots rs}$ of the system lies in V. However this fact plus the following lemma leads to a contradiction.

Lemma for (2.3). If one of A_j, j = 1, 2, 3, 4 contains an upper principal path e_1 and a lower principal path e_2 which intersect and lie in V, then A contains upper and lower principal paths which intersect and lie in V. In general, if $A_{j \ldots rs}$ contains intersecting upper and lower principal paths which lie in V, then so does $A_{j \ldots r}$.

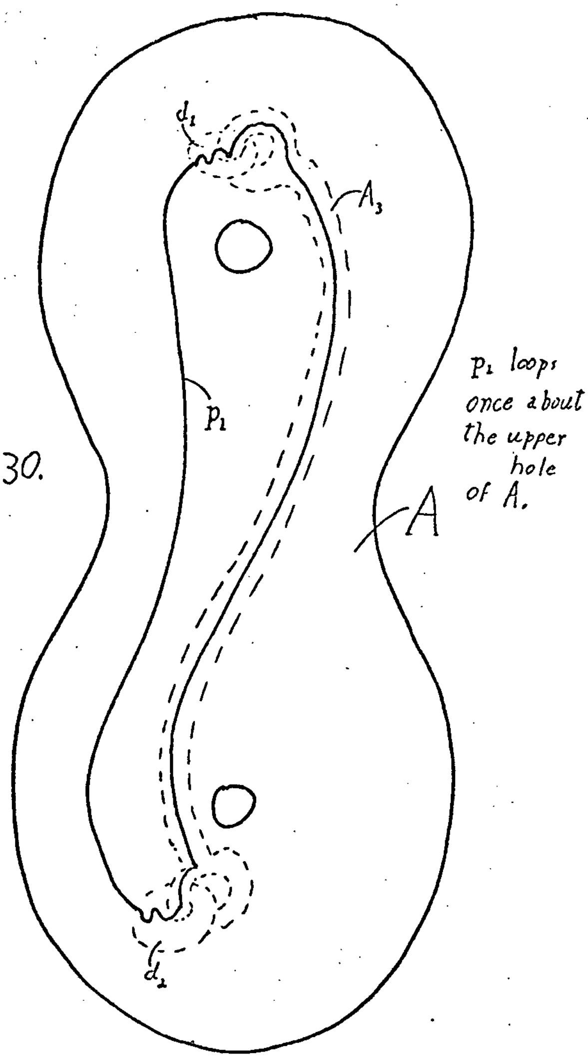
To apply the lemma we look at the neighbourhood $A_{j \ldots rs}$ which we know to be a neighbourhood of Λ in V. Obviously $V \cap A_{j \ldots rs}$ contains intersecting upper and lower principal paths of $A_{j \ldots rs}$ since any intersecting principal paths will qualify. The lemma implies that the dogbone $A_{ij \ldots r}$ contains intersecting principal paths in V. Repeated application of the lemma leads to the conclusion that A contains an upper principal path which lies in V. Since $V \subset A$, V is simply connected, and the upper principal paths of A are all homotopic to ℓ in A, therefore ℓ must shrink to a point in A. This is clearly false from fig 16a. Thus the proof of (2.3) will be complete when we have proved the lemma.

Proof of the lemma for (2.3): Simplified version. Suppose that e_1 and e_2 lie in A_1. The following outline reflects our original intuition of the proof. Although the 'proof' we give now is



d_1
C_1
B_1
a
A
d_2
B_2
C_2

Fig 29.



d_1
A_3
p_1
p_1 loops once about the upper hole of A.
A
d_2

Fig 30.

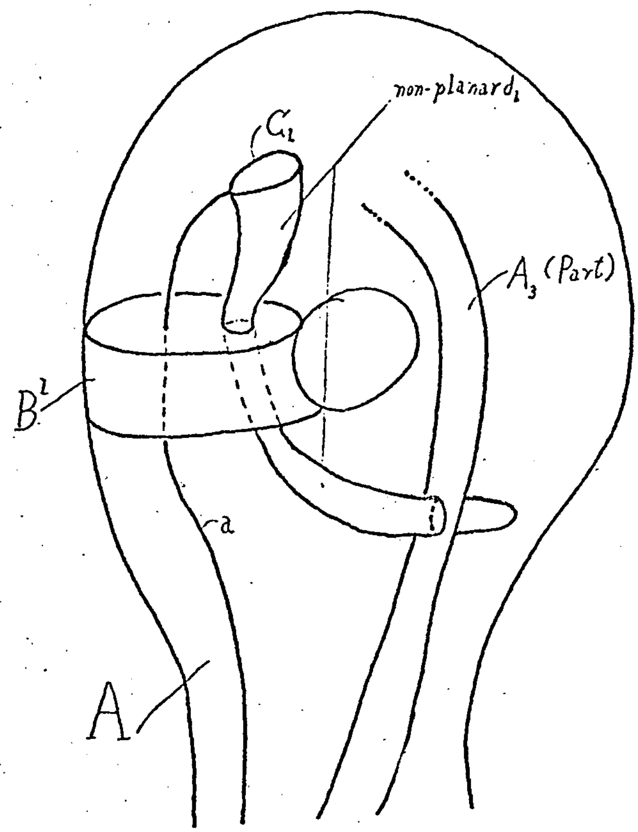


Fig 31a.

A non-planar d_1 may meet A_3 as shown. It is now possible to construct p_1 so that p_1 does not loop about the upper hole of A as required.

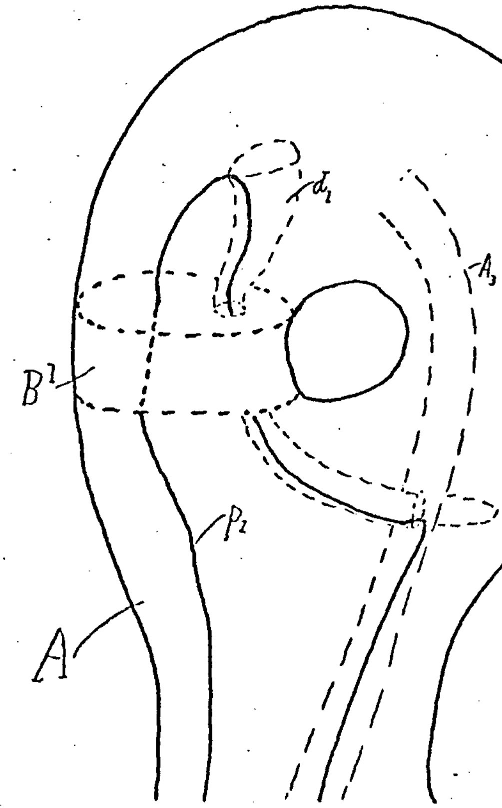


Fig 31 b.

The curve p_1 may lie in $a \cup d_1 \cup A_3 \cup g$ as constructed in fig 31a, and fail to loop once about the hole.

simple minded and needs much patching, we give the crude version because we think that it clarifies basic ideas which tend to be submerged in the final version of the proof. Suppose that by good fortune the paths e_1, e_2 take the form of the double ended lasso J in fig 29. J consists of circles C_1, C_2 and connecting arc a as shown. The circle C_1 will not shrink to a point in the complement of the upper eye of A_3. Similarly C_2 will not shrink to a point in the complement of the lower eye of A_3. We also pretend that J lies in V and that disjoint planar disks d_1, d_2 bounded by C_1, C_2 also lie in V. By (2.1) some big element g in A_3 meets both d_1 and d_2; and g lies in $V \supset d_i$ since V contains every element of G that it intersects (remember that V is the pre-image of an open set in $\mathcal{D}$). We can now construct the upper principal path p_1 shown in fig 30 from parts of V lying in a, d_1, d_2, g. A similar procedure using A_2 instead of A_3 will yield the lower principal path p_2. The paths p_1, p_2 intersect in A so that $p_1 \cup p_2$ is the set required by the conclusion of the lemma.

The above 'proof' is far to easy and will fail if we allow d_1, d_2 to be non-planar, for then p_1 may not be a principal path as figs 31a, b show. We ensure that p_1 makes one circuit about the upper hole of A by trapping $p_1 \cap a$ in the cube $\overline{A - B^2 - B^2}$ (which is easy) and $p_1 \cap (d_1 \cup g \cup d_2) = q$ in the cube $\overline{A - B^1 - B_2}$ (see fig 32). This last step is hard since one would fear that the connectivity would be spoilt by parts of $d_1 \cup d_2$ projecting from the cube. The trick of controlling the homotopy class of p_1 by constructing certain arcs in

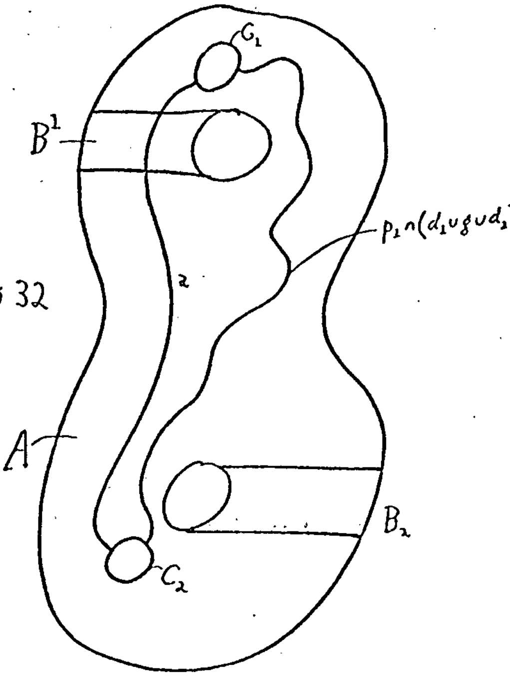


fig 32

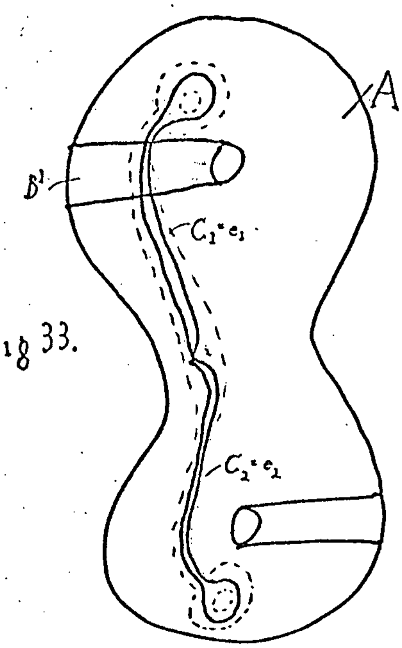


fig 33.

cubes only works if the C_i lie in $\overline{A - B^1 - B_2}$. But if we use the obvious candidate for J, viz. $C_i = e_i$ with arc a degenerate, then fig 33 shows that this may not happen, and in fact J cannot usually be $e_1 \cup e_2$. However we show that, provided that intersecting principal paths exist in $A_1 \cap V$, there is a double ended lasso (perhaps with singularities) in $A \cap V$ which has just the properties which we assigned to J . We will now give the final version of the proof of the lemma for (2.3). This proof uses the ideas of the earlier crude version, but incorporates the various improvements suggested in this paragraph.

Outline of final version of proof. We first give the proof assuming that $e_1 \subset e_2 \subset A_1$, then indicate alterations in the case that $e_1 \cup e_2$ lies in A_2, A_3 or A_4 . a) Let e_1, e_2 be upper and lower principal paths of A_1 which lie in V and intersect at least at p . We follow the sketch of the 'proof' already given, but as previously explained, we cannot use $e_1 \cup e_2$ for J in fig 32. We construct $J = C_1 \cup C_2 \cup a$ so as to satisfy five properties i), ... v) . Sometimes we will regard C_i as a mapping (not necessarily an embedding) of S^1 and sometimes as the range of this mapping. The set J must satisfy the following properties: for i = 1, 2,

i) $C_i[\mathrm{Bd}\Delta] \subset V \cap \mathrm{Int}(\overline{A - B^1 - B_2}$

ii) rng C_i misses A_3 .

iii) C_1 (C_2) fails to shrink to a point in $E^3 - \ell_3(E^3 - m_3)$.

iv) There is a point $y_1 \in \mathrm{rng}\, C_1 \cap \mathrm{rng}\, e_1 \cap K_1$ and a point y_2 in $\mathrm{rng}\, C_2 \cap \mathrm{rng}\, e_2 \cap \overline{A - B - B_2 - K_1}$ (recall that K_1 is the topological cube which is the closure of the upper component of $A - B^1 - B^2$, see fig 34) .

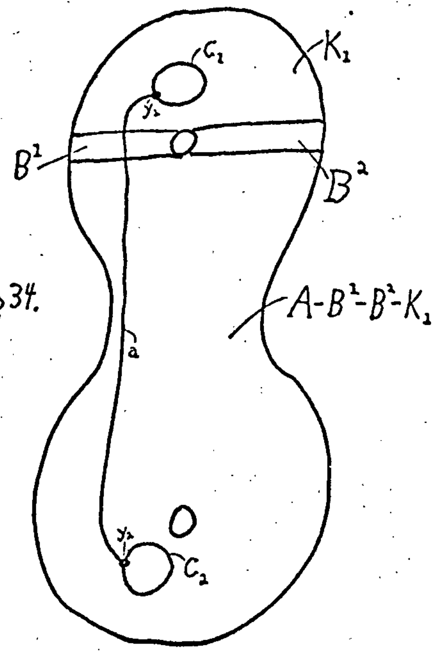


Fig 34.

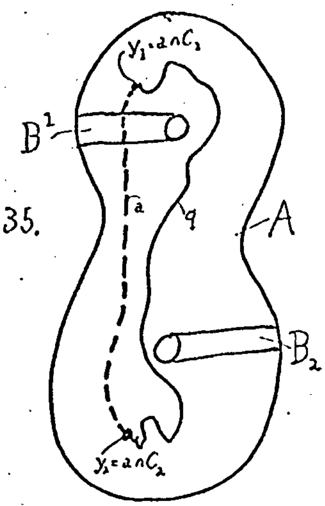


Fig 35.

v) the arc $a \subset V \cap A_1$ and the points y_1, y_2 of iv) are the end points of a .

The idea of ii) and iii) is that we want the C_i to act like the circles c_1, c_2 in fig 28 with respect to A_3 . Property iv) provides the end-points of a 'above and below B^1 ' . This plus v) and the fact that the C_i are trapped in $\overline{A - B^1 - B_2}$ alows us eventually to construct an upper principal path of A which winds one around the upper hole of A . This happens because we will join y_1 and y_2 by a path like $q \subset \overline{A - B^1 - B_2}$ in fig 35. b) For $i = 1, 2$, let $f_i : \Delta \to V$ so that $f_i|_{Bd\Delta} = C_i$. Using (1.2) and (1.3), obtain a new mapping $\overline{f}_i$ which agrees with f_i on $\Delta - \text{Int } W_1 - \ldots - \text{Int } W_n$, where $\text{Int } W_r$, $r = 1, \ldots n$, are holes in Δ; in particular $\overline{f}_i = f_i$ in $Bd\Delta$. The $\overline{f}_i[W_r]$ may leave V (!) but this does not harm the proof. c) By (2.2), $\overline{f}_1[\Delta]$ and $\overline{f}_2[\Delta]$ either intersect in A_3 or hit the same big element g in A_3 . There is a path q from $a \cap C_1$ to $a \cap C_2$ which resembles q in fig 35 and lies in V and in $\text{Int } (\overline{A - B^1 - B_2})$. The path q travels to A_3 in $\overline{f}_1[\Delta]$, passes from $\overline{f}_1[\Delta]$ to $\overline{f}_2[\Delta]$ in A_3 either at the intersection of $\overline{f}_1[\Delta]$ and $\overline{f}_2[\Delta]$ or using the element g, and then proceeds to $a \cap C_2$ by means of $\overline{f}_2[\Delta]$. d) The path which begins at $a \cap C_1$, travels to $a \cap C_2$ on q and returns to $a \cap C_1$ on a, is an upper principal path of A which lies in V . e) The lower principal path of A in V may be constructed as in a), b), c), d) above, using A_2 and $\overline{A - B_1 - B^2}$ instead of A_3 and $\overline{A - B^1 - B_2}$. f) If $k = 2, 3, 4$ the lemma remains true.

Details of Proof. Suppose that e_1, e_2 are upper and lower principal paths respectively of A_1 which lie in V and intersect at p. For $i = 1, 2,$ since e_i shrinks to a point in V, there are mappings $\bar{e}_i : \Delta \to V$ such that $\bar{e}_{i|Bd\Delta} = e_i$. We do not claim that rng $\bar{e}_i$ lies in A_1 or even A. We use (1.2), taking f, S, to be e_1, $\overline{e_1^{-1}[A - B^1 - B_2]}$. Thus there are disjoint disks $W_1, \ldots W_m$ in Δ such that $\overline{e_1^{-1}[A - B^1 - B_2]} \subset W_1 \cup W_2 \cup \ldots \cup W_m$, each point x of Bd W_r lies near $\overline{e_1^{-1}[A - B^1 - B_2]}$, and $x \in$ Bd W_r misses $\overline{e_1^{-1}[A - B^1 - B_2]}$ if $x \notin Bd\Delta$. Those W_r which hit $Bd\Delta$ are called $W_1, \ldots W_n$; those W_r which miss $Bd\Delta$ are $W_{n+1}, \ldots W_m$ (with obvious adjustments of one or the other class does not exist). We now apply (1.3) with g taken to be the restriction of $\bar{e}_1$ to Δ - Int W_1 - ... - Int W_m. For $r = n+1, \ldots m$, $\bar{e}_{1|Bd\,W_r} = g_{|Bd\,W_r}$ maps into $\text{Ext}(\overline{A - B^1 - B_2})$ because Bd W_r misses $Bd\Delta$ for $r > n$ and hence misses $\overline{\bar{e}_1^{-1}[A - B^1 - B_2]}$. Thus for $r = n+1, \ldots m$, $\bar{e}_{1|Bd\,W_r}$ shrinks to a point in $\text{Ext}(\overline{A - B^1 - B_2})$ which is the exterior of a cube in E^3; and we can let $\text{Ext}(\overline{A - B^1 - B_2})$ be $P_{n+1} = \ldots = P_m$ in the hypothesis of (1.3). There is no chance that $\bar{e}_{1|Bd\,W_r}$ is C_1 for $r > n$, since ℓ_3 misses $\text{Ext}(\overline{A - B^1 - B_2}) = P_r$. We suspect that C_1 is an $\bar{e}_{1|Bd\,W_r}$ for $r \leq n$. Assume that every $\bar{e}_{1|Bd\,W_r} = g_{|Bd\,W_r}$ will shrink to a point in $E^3 - \ell_3$. Use (1.3) again, letting $P_1 = P_2 = \ldots = P_n$ be $E^3 - \ell_3$. Then with $\text{Ext}(\overline{A - B^1 - B_2})$ taken to be $P_{n+1} = \ldots = P_m$, $g_{|Bd\Delta} = e_1$ will shrink to a point in rng $g \cup P_1 \cup \ldots \cup P_n \cup P_{n+1} \cup \ldots \cup P_m$. Each P_r misses ℓ_3 either by definition of because P_r misses $\overline{A - B^1 - B_2}$. And rng g misses

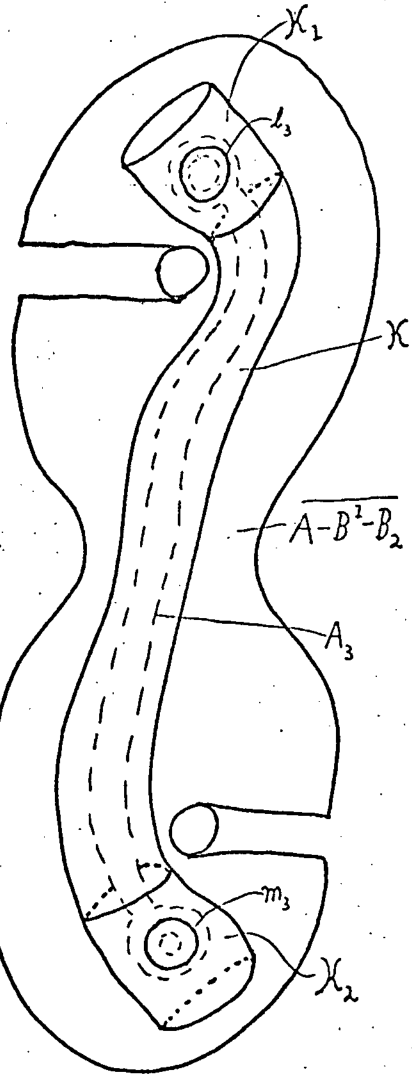


fig 36.

ℓ_3 as well; for $g = \bar{e}_1|_{\tilde{\Delta}} - \text{Int } W_1 - \ldots - \text{Int } W_m$ and from (1.2) ii, iv, the only points of $\Delta - \text{Int } W_1 - \ldots - \text{Int } W_m$ which can map into $\overline{A - B^1 - B_2}$ are those in $\text{Bd}\Delta$. Such points are in dom e_1 and map into A_1. Hence $\text{rng } g \subset A_1 \cup \text{Ext}(\overline{A - B^1 - B_2}) \subset E^3 - \ell_3$. Therefore $g_{|\text{Bd}\Delta} = e_1$ shrinks to a point in $\text{rng } g \cup P_1 \cup \ldots \cup P_m \subset E^3 - \ell_3$, which contradicts the fact the e_1 is an upper principal path. Thus it is false that every $\bar{e}_{1|\text{Bd } W_r}$, $r \leq n$, shrinks to a point in $E^3 - \ell_3$. Let C_1 be one of the $\bar{e}_{1|\text{Bd } W_r}$ which fails to shrink to a point in $E^3 - \ell_3$. As regards C_1: the above argument plus the fact that $\text{rng } C_i \subset \text{rng } \bar{e}_i \subset V$ shows that iii) is true; ii) is true because from (1.2) iv, every point x in $\text{Bd } W_r$ is either in $\text{Int } \Delta$, in which case $C_1(x) \in E^3 - \overline{(A - B^1 - B_2)} \subset E^3 - A_3$, or $x \in \text{Bd}\Delta$, when $C_1(x) = e_1(x) \in A_1 \subseteq E^3 - A_3$. In general, i) is not true because some candidates for $C_1(x)$ lie outside of $\overline{A - B^1 - B_2}$ as we have just seen. However we can assume that $C_1[\text{Bd } W_r]$ lies in $\text{Int } \overline{(A - B^1 - B_2)}$ by the following argument: By (1.2), we assume that dom C_1 (which is one of the $\text{Bd } W_r$) lies so near $\bar{e}_1^{-1}[\overline{A - B^1 - B_2}]$ that rng C_1 lies within ε of $\overline{A - B^1 - B_2}$ (remember that $C_1 = g = \bar{e}_1$ on dom C_1). In this paragraph a) so far, we could have replaced $\overline{A - B^1 - B_2}$ by a cube $K \supset A_3$ such that an ε-neighbourhood of K lies in $\text{Int } \overline{(A - B^1 - B_2)}$. Such a cube is shown in fig 36. If this had been done, we would have rng C_1 in the ε-neighbourhood of K, i.e. $\text{rng } C_1 \subset \text{Int } \overline{(A - B^1 - B_2)}$. We assume that this was done and that $\text{rng } C_1 \subset \text{Int } \overline{(A - B^1 - B_2)}$. Proof of iv): In (1.2) iv, each $\text{Bd } W_r$ misses S (in (1.2)) except where $\text{Bd } W_r$ hits $\text{Bd}\Delta$. In the present context, with $e_1^{-1}[K]$ for S (i.e. continuing to use K for $\overline{A - B^1 - B_2}$), the domain of C_1 is a $\text{Bd}\, W_r$ and $C_1[\text{Bd}\, W_r - \text{Bd}\Delta]$ misses K. To show that there is a $y_1 \in \text{rng}\, C_1 \cap \text{rng}\, e_1 \cap K_1$ $= C_1[\text{Bd}\, W_r] \cap e_1[\text{Bd}\Delta] \cap K_1 = C_1[\text{Bd}\, W_r \cap \text{Bd}\Delta] \cap K_1$, assume that $C_1[\text{Bd}\Delta \cap \text{Bd}\, W_r] \cap K_1 = \emptyset$. In fig 36, note the two cubes $\mathcal{K}_1$, $\mathcal{K}_2$ which are placed so that $\ell_3 \subset \mathcal{K}_1 \subset K \cap K_1$ and $m_3 \subset \mathcal{K}_2 \subset K \cap K_2$. Then $C_1[\text{Bd}\, W_r \cap \text{Bd}\Delta] \cap \mathcal{K}_1 = \emptyset$ because $K_1 \supset \mathcal{K}_1$; and $C_1[\text{Bd}\, W_r - \text{Bd}\Delta] \cap \mathcal{K}_1 = \emptyset$ because $C_1[\text{Bd}\, W_r - \text{Bd}\Delta]$ misses K as we just saw. This means that all of $\text{rng}\, C_1 \subset E^3 - \mathcal{K}_1$ and C_1 shrinks to a point in $E^3 - \mathcal{K}_1 \subset E^3 - \ell_3$, which contradicts the choice of C_1.

We repeat the entire procedure of this paragraph a) taking e_2, $\overline{e}_2$, m_1, m_3, $\mathcal{K}_2$, $\overline{A - B^1 - B_2 - K_1}$, for e_1, $\overline{e}_1$, ℓ_1, ℓ_3, $\mathcal{K}_1$, K_1. This is just the preceding argument 'upside down' and constructs the path $C_2 \ni y_2$ as required. The only unexpected thing is the use of the cube $\overline{A - B^1 - B_2 - K_1}$ for the original cube K_1; this reflects the fact that y_1 should be found in K_1 and y_2 not necessarily in K_2 but merely 'in A_1 and below B^1'. We now have y_1 and y_2 as required by iv). To construct a, join y_1 to p by a path a_1 lying in $\text{rng}\, e_1 \subset A_1 \cap V$; and y_2 to p by a path a_2 in $\text{rng}\, e_2 \subset A_1 \cap V$. Let $a = a_1 \cup a_2$.

b) We can assume that $\text{dom}\, C_i = \text{Bd}\Delta$, $i = 1, 2$. Since $\text{rng}\, C_i \subset V$, C_i shrinks to a point in V and there is a mapping $f_i : \Delta \to V$ such that $f_i|_{\text{Bd}\Delta} = C_i$. It happens to be true that (2.2) gives us a big element g' in $V \cap A_3$ which hits both $f_1[\Delta]$ and $f_2[\Delta]$

(unless they intersect), but we are not sure that there is a connected set in $f_i[\Delta]$ that will join g' and y_i and stay in $\overline{A - B^1 - B_2}$, so that it is not yet possible to build $q \subset V \cap (\overline{A - B^1 - B_2})$ as in fig 35. By (1.2), taking S to be $\mathrm{Ext}(\overline{A - B^1 - B_2})$, there are disks $W_1^i, \dots W_n^i$ in Δ such that $f_i^{-1}[\mathrm{Ext}(\overline{A - B^1 - B_2})] \subset W_1^i \cup \dots \cup W_n^i$. Since $\mathrm{Bd}\Delta$ misses $f_i^{-1}[\mathrm{Ext}(\overline{A - B^1 - B_2})]$ (because $f_i[\mathrm{Bd}\Delta] \subset \mathrm{Int}(A - B^1 - B_2)$), $\Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i$ is a disk with holes. We assume that each $\mathrm{Bd}\, W_r^i$ lies so near $f_i^{-1}[\mathrm{Ext}(\overline{A - B^1 - B_2})]$ that $f_i[\mathrm{Bd}\, W_r^i]$ lies close to $\mathrm{Ext}(\overline{A - B^1 - B_2})$. Since we know exactly what $\overline{A - B^1 - B_2}$ looks like, we can construct an ε-neighbourhood N of $\mathrm{Ext}(\overline{A - B^1 - B_2})$ so that N is simply connected. We can assume that each $f_i[\mathrm{Bd}\, W_r^i] \subset N$; then $f_i[\mathrm{Bd}\, W_r^i]$ shrinks to a point in N; and by (1.3), taking $N = P_1 = P_2 = \dots = P_n$, and $g = f_i|\Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i$, there is a mapping $\overline{f}_i : \Delta \to \mathrm{rng}\, g \cup P_1 \cup \dots \cup P_n = f_i[\Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i] \cup N$ such that $\overline{f}_i = f_i$ on $\Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i$. In particular $\overline{f}_i = f_i$ on $\mathrm{Bd}\Delta$. It is important that $\overline{f}_i[\Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i]$ $= f_i[\Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i] \subset V$.

c) Since $\mathrm{rng}\, C_i = f_i[\mathrm{Bd}\Delta] = \overline{f}_i[\mathrm{Bd}\Delta]$ misses A_3 and fails to shrink to a point in the absence of the appropriate eyes of A_3, by (2.2), $\overline{f}_1[\Delta]$ and $\overline{f}_2[\Delta]$ either intersect in A_3 or hit the same big element λ in A_3. We can combine these ideas by saying '$f_1[\Delta]$ and $f_2[\Delta]$ meet the same element λ in A_3' and allowing λ to be either a big element or a small element. Since for a small ε, N misses A_3, $\lambda \cap \mathrm{rng}\, \overline{f}_i$ must lie in $\mathrm{rng}\, \overline{f}_i - N \subset \overline{f}_i[\Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i] \subset V$, and $f_i^{-1}[\lambda] \subset \Delta - \mathrm{Int}\, W_1^i - \dots - \mathrm{Int}\, W_n^i$, which we saw was a disk with

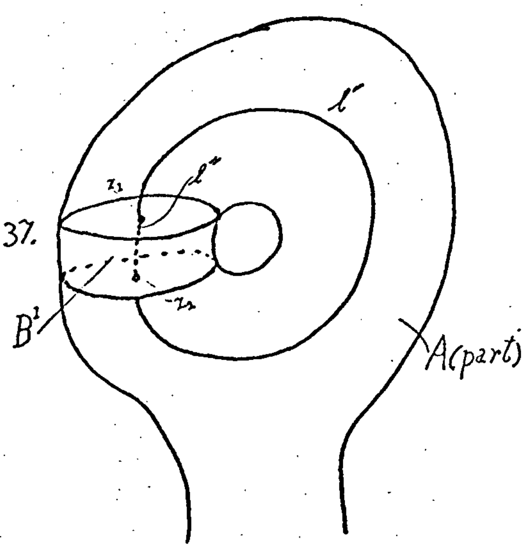


Fig 37.

holes and which contains $C_i^{-1}(y_i) \subset \text{dom } C_i = \text{Bd}\Delta$. Since $\lambda \cap f_i[\Delta]$ and y_i lie in the image under $\overline{f}_i$ of a disk with holes which maps into V, there is a path ν_i which joins y_i and λ in V. Futhermore $\nu_i \subset A - B^1 - B_2$, because ν_i may be constructed in $\overline{f}_i[\Delta - \text{Int } W_1 - \ldots - \text{Int } W_n] = f_i[\Delta - \text{Int } W_1 - \ldots - \text{Int } W_n]$ which misses $\text{Ext}(\overline{A - B^1 - B_2})$ by the construction of the W_r. Hence $\nu_i \subset V \cap (\overline{A - B^1 - B_2})$. Let q be a path joing y_1 and y_2 in $\nu_1 \cup \lambda \cup \nu_2$. Clearly $q \subset V \cap (\overline{A - B^1 - B_2})$.

d) We will show that the path ξ_1 which travels from y_1 to y_2 in q and returns to y_1 in a is an upper principal path of A in V by showing that $\xi_1 \subset A \cap V$ and that ξ_1 is homotopic to ℓ in A. Let ℓ be decomposed into two paths ℓ' and ℓ'' such that $\ell'' \subset B^1$ and $\ell' \subset \overline{E^3 - B^1}$. We assume that ℓ pierces Bd B^1 in just two points z_1, z_2 as shown in fig 37. We can do this because Bd B^1 is horizontal near ℓ and because ℓ can be a nice circle. Construct arcs $z_1 y_1$ and $z_2 y_2$ in the cubes K_1 and $\overline{A - B^1 - B_2 - B^2 - K_1}$ respectively. (The idea here is that both $z_i y_i$ will lie in $\overline{A - B^1 - B_2}$, the cube which locates $A_3 \supset q$, and in $\overline{A - B^2 - B_2}$, the cube which locates $A_1 \supset a$). The path which begins at z_1 and travels to z_2 through $z_1 y_1$, a, and $z_2 y_2$ is homotopic in the cube $\overline{A - B^2 - B_2}$ to ℓ''. The path which begins at z_2 and travels to z_1 via $z_2 y_2$, q, and $z_1 y_2$ is homotopic in the cube $\overline{A - B^1 - B_2}$ to ℓ'. Combining homotopies, the path ξ_1' which begins at z_1, travels to z_2 in $z_1 y_1 \cup q \cup z_2 y_2$ and returns to z_1 in $z_2 y_2 \cup a \cup z_1 y_1$, is

Fig 38

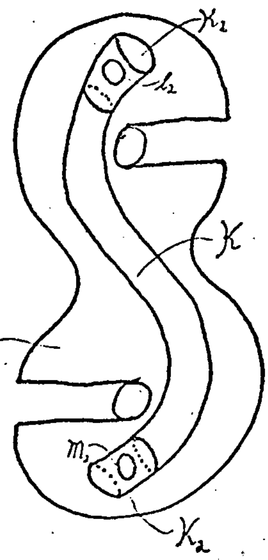
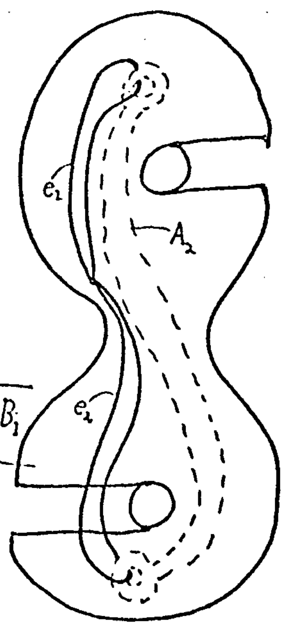


Fig 39



homotopic in A to ℓ . The path ξ_1' is evidently homotopic to ξ_1. Note that ξ_1 passes through the point $p \in a$. Eventually p will be the 'official' intersection point of the principal paths ξ_1 and ξ_2 of A .

e) There is no difficulty in altering the argument to construct a lower principal path ξ_2 if one keeps in mind the fact that 'the pictures are different' and that everything in the construction of ξ_1 must be repeated. We cannot, for example, use the C_i from a) because they were defined with respect to $\overline{A - B^1 - B_2}$ and we must replace $\overline{A - B^1 - B_2}$ (the cube which located A_3 and 'shaped' the right side of ξ_1) with $\overline{A - B^2 - B_1}$ which locates A_2 . The idea is to start with e_1 and e_2 as before, but to use A_2 rather than A_3 as suggested in fig 38 which, in a sense, is a replacement for fig 35. The new ξ_2 turns out to contain $p \in e_1 \cap e_2$ just as ξ_1 does; this establishes that $\xi_1 \cap \xi_2 \neq \emptyset$. We begin by finding a new lasso $J' = C_1' \cup C_2' \cup a'$ so that $C_1' \cup C_2' \subset V \cap \operatorname{Int}(\overline{A - B^2 - B_1})$, $C_i' \subset A_3$, and C_i contains a point y_i' such that $y_1' \in \operatorname{rng} C_1' \cap \operatorname{rng} e_1 \cap (\overline{A - B^2 - B_1 - K_2})$ and $y_2' \in \operatorname{rng} C_2' \cap \operatorname{rng} e_2 \cap K_2$. The arc a' lies in $V \cap A_1$ and has end points y_1', y_2' . One finds $C_1' \cup C_2' \cup a'$ by adapting the procedure in a); there is very little more involved than reading m, m_j, B_1, B^2, K_2, $\overline{A - B^2 - B_1 - K_2}$ for ℓ, ℓ_j, B^1, B_2, $\overline{A - B^1 - B_2 - K_1}$, K_2, and priming every new construction. It will be found that the arc a' contains p just as the original a does. For the construction of K', K_1', K_2', replace fig 36 by fig 39. It is quite easy to adapt b) and

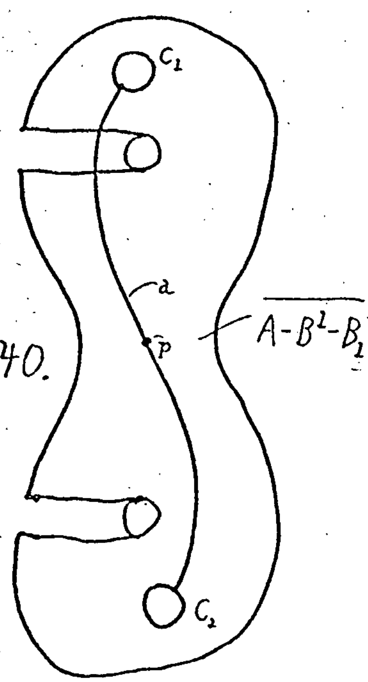


Fig 40.

c) by keeping in mind that the important cube is $\overline{A - B^2 - B_1}$ which replaces $\overline{A - B^1 - B_2}$ in the construction of ξ_1 . (The point is that in b), c), one must use a cube whose boundary encloses the 'important' dogbone A_2, see fig 38). Finally we construct a path q' which joins y_1', y_2' in $\overline{A - B^2 - B_1}$. This plus $a' \subset \overline{A - B^2 - B_2}$ can be combined into the path ξ_2 which can be shown to be homotopic to m by adapting d) above, decomposing m into paths $m'' \subset B_1$ and $m' \subset \overline{E^3 - \overline{B_1}}$ etc. The path ξ_2 lies in V by an argument which should appear naturally from the adaptation of a), b), c) to construct ξ_2; and ξ_2 is clearly in A . Since the point p lies in both a and a' and hence in both ξ_1 and ξ_2 ; therefore $\xi_1 \cap \xi_2 \neq \emptyset$.

f) The proof is now complete if $e_1 \cup e_2$ lies in A_1, i.e. if j = 1 . The is no difficulty in constructing a proof for the lemma when j = 4 in view of the symmetry of the construction of the A_j . We will give only an outline of the proof for j = 2 (and by symmetry for j = 3) for these reasons: 1) the details can be filled in along the lines of a) ... e) above, and, 2) the argument in a), ... e) is sufficient to prove the 'meat' of (2.2), viz. that there are uncountably many big points of $\mathcal{D}$ which fail to possess arbitrarily small simply connected open neighbourhoods, these being images of elements of the form $A \cap A_i \quad A_{ij} \quad A_{ijk}$ ... where i, j, k, ... are chosen from 1 or 4. To construct the principal path ξ_1 when j = 2, use the paths e_1, e_2, which we now assume to lie in $A_2 \cap V$, and the cube $\overline{A - B^1 - B_1}$ (see fig 40) which acts toward A_2 just as $\overline{A - B^1 - B_2}$ acts toward A_1 in a) (i.e. $\overline{A - B^1 - B_1}$ separates A_2 just under the

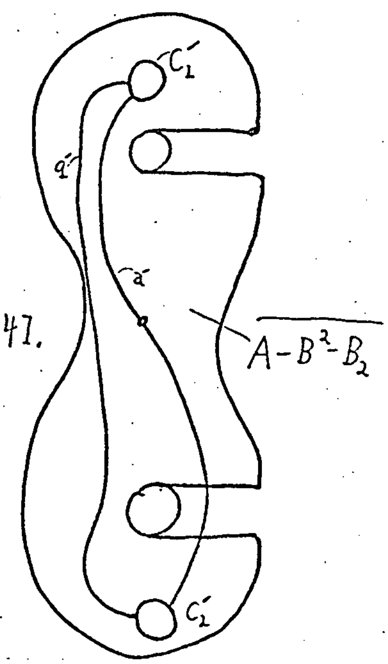


Fig 47.

upper eye while $\overline{A - B^1 - B_2}$ does the same for A_1). Using the argument of a), construct a lasso $J = C_1 \cup a \cup C_2$ which is related to V, $e_1 \cup e_2$, A_4, and $\overline{A - B^1 - B_1}$ just as J in a) was related to V, $e_1 \cup e_2$, A_3, $\overline{A - B_1^1 - B_2}$. Fig 40 shows the new J. When C_1 and C_2 shrink to a point in V they hit the same element λ of A_4 (λ may be a big element or a point). The path q joining the end points of a in $V \cap (\overline{A - B^1 - B_1})$ may be constructed by adapting the argument of b), c), and $\xi_1 = a \cup q$ may be shown to be an upper principal path of A by an argument like that of d). Just as in the case $j = 1$, the arc a contains a point $p \in e_1 \cap e_2$. Thus $p \in \xi_1$. To construct the lower principal path ξ_2 when $j = 2$, we start as before with $e_1 \cup e_2 \subset A_2$, but we use the cube $\overline{A - B^2 - B_2} \supset A_1$ and construct $J' = C_1' \cup a' \cup C_2''$ so that J' is related to V, $e_1 \cup e_2$, A_1, $\overline{A - B^2 - B_2}$, just as J' in e) is related to V, $e_1 \cup e_2$, A_2, $\overline{A - B^2 - B_1}$, see fig. 41. Fig. 41 also shows q'' which is used with a' to form ξ_2. The arc a' and hence the path ξ_2 turns out to contain p; hence $\xi_1 \cap \xi_2 \neq \emptyset$ as before. □.

CHAPTER THREE: GENERALIZATION OF A THEOREM OF BING: LEMMAS.

1. In this chapter, we give two lemmas for the proof of II (2.2), the generalization of Bing's theorem II (2.1). In proving II (2.1), Bing defined a property Q such that A had Property Q, and if a dogbone $A_{jk \dots r}$ had Property Q, then one of $A_{jk \dots r1}$, $A_{jk \dots r2}$, $A_{jk \dots r3}$, $A_{jk \dots r4}$ had Property Q. This meant that there was a chain $A \supset A_j \supset A_{jk} \supset \dots$ of dogbones with Property Q. Since the possession of Property Q implied intersection with both disks D_i in II (2.1), the limit of the chain was a big element Λ which hit both D_1 and D_2 (see the discussion in II§2). We follow Bing's proof closely (in spite of the fact that we alter Property Q to a property which has to be applied to a whole $\mathcal{A}_m$ to be of any use) and in fact depend on the reader's familiarity with [12] for the motivation in this chapter and the next. In the remainder of this paper, $i = 1, 2$, and $j = 1, 2, 3, 4$.

In the proof of II (2.1) in [12], it is evident that the crucial part of the argument is the proof of [12, Th 10], where it is shown that if the four centres of A_1, A_2, A_3, A_4 fail to have Property P, then some set homotopic to the centre of A also fails to have Property P, (The precise definition of Property P is unimportant until Ch IV). In Ch IV we will prove just this result with the disks D_i in [12§7] replaced by the singular disks $f_i[\Delta]$ in II (2.2). Our proof will differ from the proof of [12, Th 10] in that whereas in [12 Th 10] the disks D_i remain unchanged during the proof, in our proof of the analogous result the $f_i[\Delta]$ are replaced by new singular disks $f_i'[\Delta]$ which retain the desirable properties of the $f_i[\Delta]$. Although

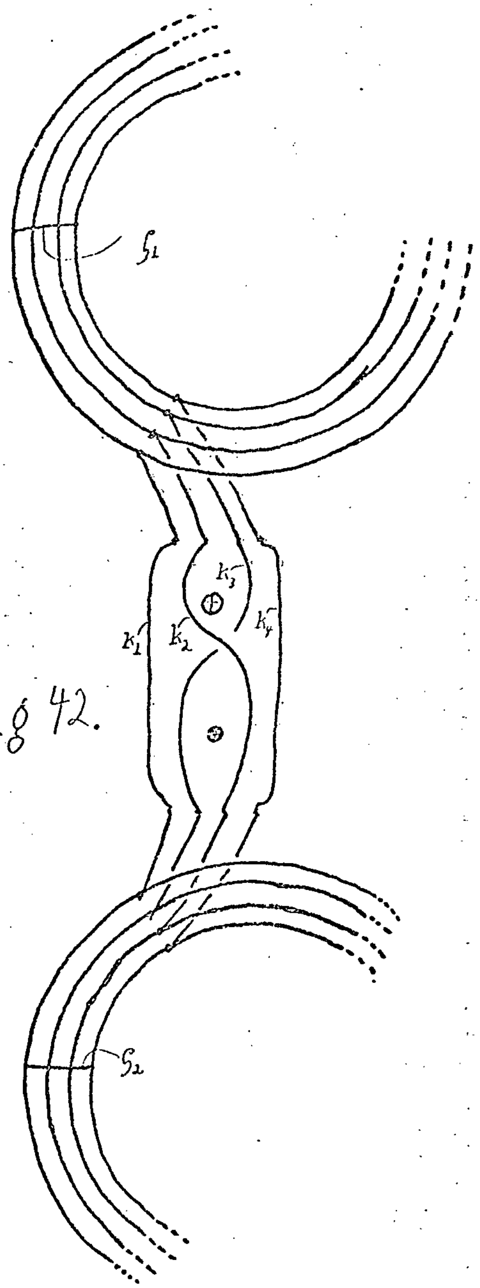


Fig 42.

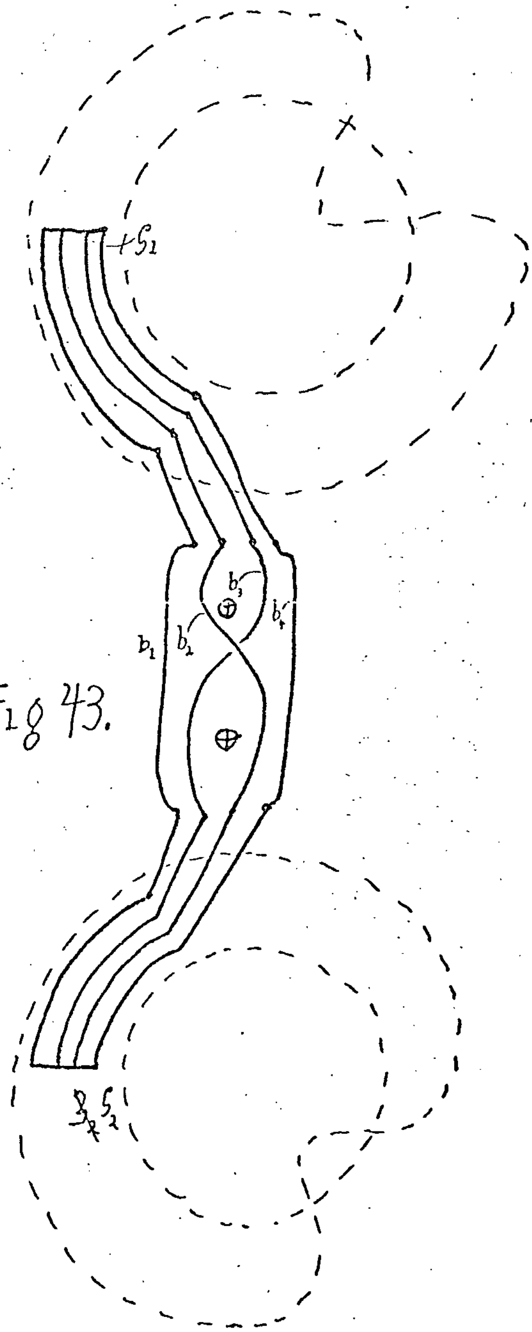


Fig 43.

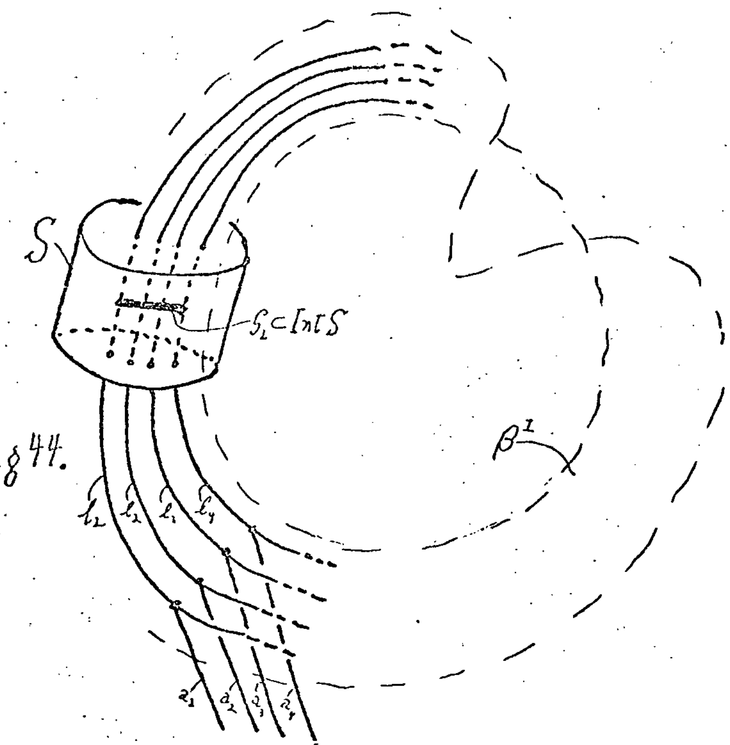


Fig 44.

this is a considerable change, it turns out that our Ch IV resembles the argument of [12§7] very closely. In the present chapter, we prove an important lemma which shows that if each k_j (in fig 19) misses one of $f_1[\Delta]$, $f_2[\Delta]$, then the new $f_i[\Delta]$ may be constructed so that not only does each k_j miss one $f_i'[\Delta]$, but both $f_i'[\Delta]$ miss each of the arcs ζ_1 and ζ_2 shown in fig 42. The ζ_i lie in β^1 and β_1 and tie the upper and lower loops of the k_j together as shown in the figure. If we can obtain such singular disks $f_i'[\Delta]$, the reward is considerable, for then parts of the k_j can be erased as shown in fig 43, leaving the set $b_1 \cup b_2 \cup b_3 \cup b_4 \cup \zeta_1 \cup \zeta_2$ shown in this figure. Since each $b_j \subset k_j$, each b_j misses one $f_i'[\Delta]$ while $\zeta_1 \cup \zeta_2$ misses both. One can now apply Part II of the proof of Th 10 of [12] to $b_1 \cup \ldots \cup b_4 \cup \zeta_1 \cup \zeta_2$ instead of to $\bigcup pq_j r_j s$ in [12, fig 2]. This can be done with very little change in the argument of [12] and results in the construction of a centre of A which fails to have Property P. We say that mappings $g_i:\Delta \to E^3$ are Z-disjoint iff $Z \subset E^3$ and $\text{rng } g_1 \cap \text{rng } g_2 \cap Z = \emptyset$, i.e. iff the ranges are disjoint at least in Z.

Lemma One. Consider A, A_j, β^1, k_j as defined in Ch II (see fig 19). Let $Z \supset A$ and let $c_1:\text{Bd}\Delta \to E^3 - Z$. Let $g_i:\Delta \to E^3$ be Z-disjoint mappings such that $g_{i|\text{Bd}\Delta} = c_i$. Let S be the sphere shown in fig 44 consisting of the cylindrical annulus Ω with disks d_1, d_2 for end caps. Each ℓ_j pierces each d_i exactly once and β^1 misses Ω. Let $S \subset \text{Int } A$ and let N be an η - neighbourhood of S such that $N \subset \text{Int } A$. The arc ζ_1 shown in fig 42 lies in Int A - N. Then there exist Z - disjoint mappings $\bar{g}_i:\Delta \to E^3$ such that

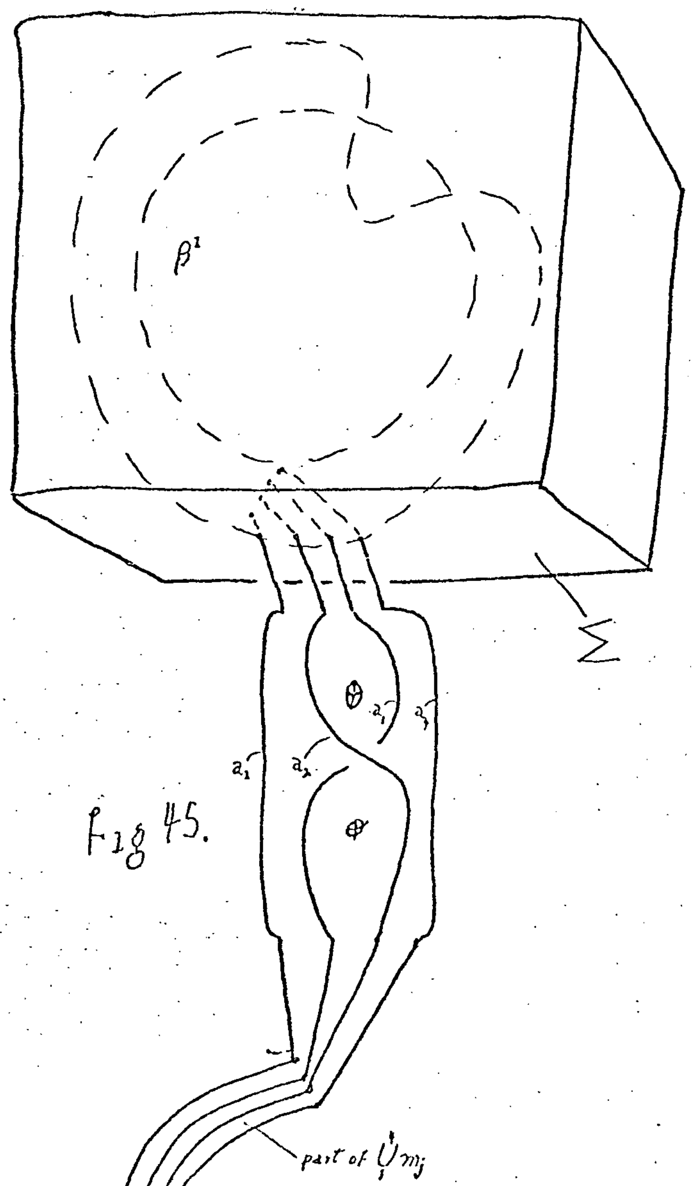


Fig 45.

i) $\overline{g}_i = c_i$ on $\mathrm{Bd}\Delta$,

ii) $\overline{g}_i : \Delta \to (\mathrm{rng}\ g_i - \mathrm{Int}\ S) \cup N$,

iii) If $\ell_j \cup \Omega$ misses rng g_i, then ℓ_j misses rng $\overline{g}_i$.

Corollary. Let K_1 be the cube defined in Ch II (see fig 21). Then ii) and iii) in Lemma One may be replaced by

ii)´ $\overline{g}_i : \Delta \to \mathrm{rng}\ g_i \cup K_1$,

iii)´ If $k_j \cup \Omega$ misses rng g_i, then k_j misses rng $\overline{g}_i$.

The proof of Lemma One is delayed to §2, which may be read after Ch IV if desired.

We give a second lemma which is intended to **repair** a gap which would otherwise appear in the proof in Ch IV. This lemma is quite specialized, but appears here because its proof is just a variation of the proof of Lemma One. As before, the proof is delayed to §2 and may be omitted on a first reading.

Lemma Two. Consider A, Z, ζ_i, k_j, a_j, β^1, as defined in fig 42. Let Σ be the sphere shown in fig 45. The sphere Σ together with an η - neighbourhood N of Σ lies in Int A; $\beta^1 \subset \mathrm{Int}\ \Sigma - n$, and each of a_1, a_2, a_3 pierces Σ as shown. Let mappings $g_i : \Delta \to E^3$ be Z-disjoint with $g_{i|\mathrm{Bd}\Delta} = c_i$, where c_i is defined as in Lemma One. Both rng g_i miss the set $\zeta_1 \cup k_1 \cup k_2 \cup k_3$. Let u_{12}, u_{13} be arcs in β^1 which join $a_1 \cap \beta^1$ and $a_2 \cap \beta^1$, $a_1 \cap \beta^1$ and $a_3 \cap \beta^1$ respectively and miss rng g_1 . Let v_{12}, v_{13} be arcs in β^1 which join $a_1 \cap \beta^1$ and $a_2 \cap \beta^1$, $a_2 \cap \beta^1$ and $a_3 \cap \beta^1$ respectively and miss rng g_2 . The arcs u_{12}, u_{13}, v_{12}, v_{13} are not necessarily

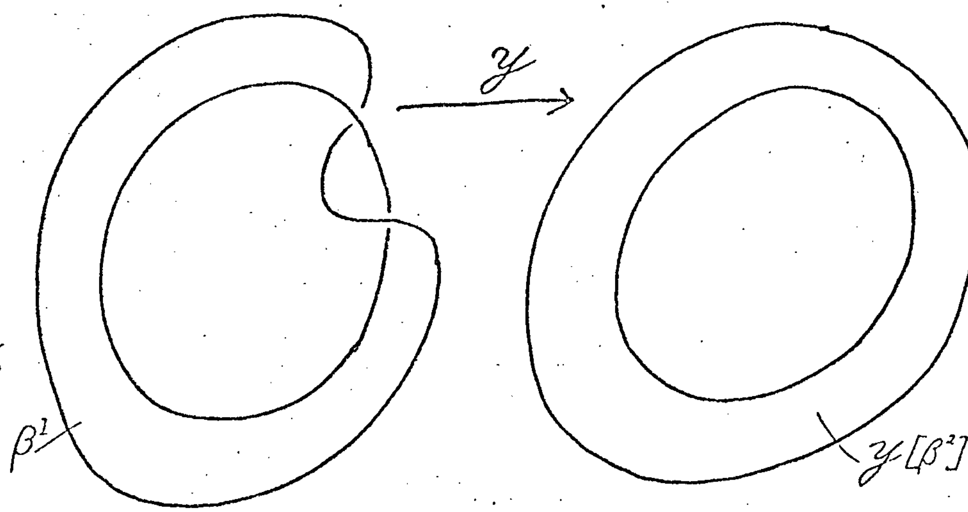


Fig 46

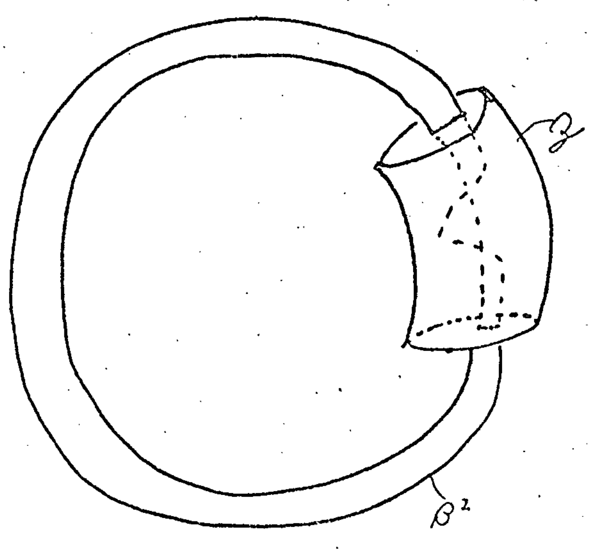


Fig 47.

disjoint. Then there exist Z-disjoint mappings $g_i':\Delta \to (\text{rng } g_i - \text{Int}\Sigma) \cup N$ such that $g_i' = c_i$ on $\text{Bd}\Delta$, and one of rng g_1', rng g_2' misses $k_1 \cup k_2 \cup k_3$.

Corollary. One of rng g_1', rng g_2' misses $\zeta_1 \cup b_1 \cup b_2 \cup b_3 \cup \zeta_2$, and $g_i':\Delta \to \text{rng } g_i \cup K_1$.

Although u_{12} lies in an annulus and is joined to Σ by the orderly arcs $a_1 \cap \overline{\text{Int } \Sigma}$, $a_2 \cap \overline{\text{Int } \Sigma}$, the arc $(a_1 \cup u_{12} \cup a_2) \cap \overline{\text{Int } \Sigma}$ may be knotted in $\overline{\text{Int } \Sigma}$, as a few moments experiment will show (an arc ab in a cube K with $ab \subset \text{Bd } D = a \cup b$ is knotted if there is no disk $D \subset K$ with $ab \subset \text{Bd} D \subset \text{Bd} K \cup ab$). To be knotted the arc must make more than one circuit on the twisted annulus. A similar comment applies to u_{13}, v_{12}, v_{13} .

2. Proof of Lemma One.

(2.1). As a preliminary, we describe an untwisting function $y:E^3 \to E^3$ which is onto and one-to-one and which unwinds the twist in β^1, i.e. $y(\beta^1)$ is the planar annulus shown in fig 46. For well known reasons y cannot be a mapping, but we ensure that y will be discontinuous only on the curved cylindrical surface z shown in fig 47. In fig 47, the end caps of z are called M_1, M_2, and the cube $\overline{\text{Int}(z \cup M_1 \cup M_2)}$ is called K . Eventually y will be composed with a mapping whose range misses z . Thus the result of the composition will be a mapping. The function y is defined to be the identity on $E^3 - K$ and on both M_i . To define y in Int K: Imagine K to be cut free of the space by means of a cut on z and on M_2, remaining attached only on M_1 .

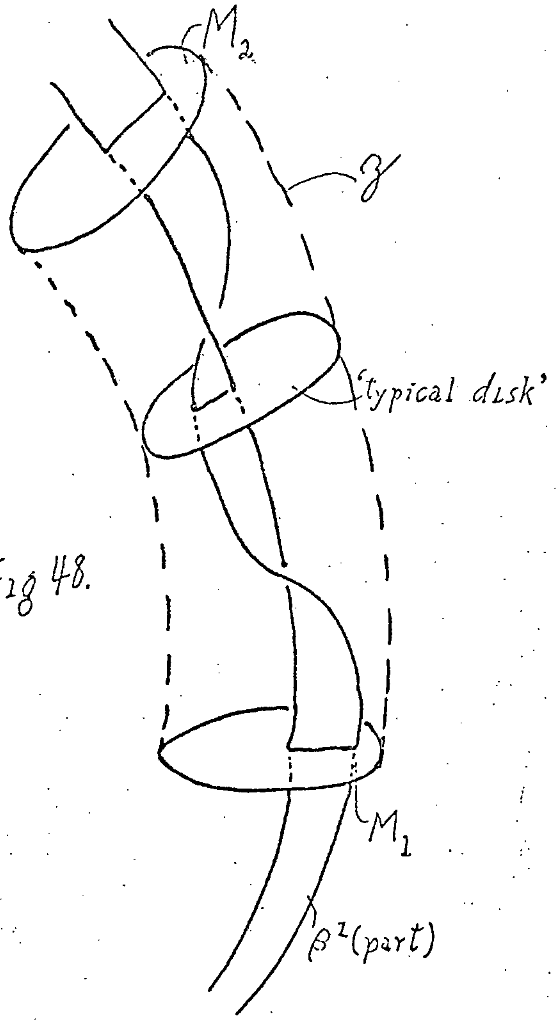


Fig 48.

K may be thought of as a stack of circular disks of infinitesmal thickness. These disks span the cylinder z and each meets β^1 in a straight arc. Fig 48 shows M_1, which is called the initial disk; M_2, which is called the final disk; and a 'typical disk' in the stack between M_1 and M_2. Now apply a twist (which may be thought of as an isotopy of K) to M_2 so that M_2 rotates once (i.e. through an angle of 2π) in place. When this happens, the disk M_1, which is attached to the space, necessarily remains fixed and does not rotate. Each disk intermediate between M_1 and M_2 rotates through an angle which is close to zero for disks close to M_1 and approaches 2π for disks whose location approaches that of M_2. The rotations of the various disks in the stack can be contrived so that $\beta^1 \cap K$ is carried onto the plane which contains $\beta^1 - K$, and so that the final result is homeomorphic to K. In fig 48, the 'typical disk', which is located half-way between M_1 and M_2 will rotate through an angle of π. This carries its intersection with β^1 on to the desired plane. Since M_2 has returned to its original position, we can restore the cut at M_2. Evidently y is one-to-one and continuous in Int K, Ext K, and on $M_1 \cup M_2$. The fact that we cannot sew up the cut on z appears in the definition of y as a discontinuity on z. Clearly y carries β^1 into the plane containing $\beta^1 - K$.

(2.2). We will prove a simpler version of Lemma One to show the general approach.

(2.21). Let S be a sphere in E^3 having a simply connected neighbourhood N. Let $g:\Delta \to E^3$ so that $g[\mathrm{Bd}\Delta] \subset \mathrm{Ext}\, S$. Then there exists a mapping $\bar{g}:\Delta \to (\text{rng } g - \text{Int } S) \cup N$ which agrees with g on BdΔ .

When simplified in this way, (2.21) is insignificant, for there are easier proofs of stronger results, as the reader doubtless sees. However our proof is intended to show how II(1.2) is used in the proof of Lemma One.

Proof. Apply II(1.2) to obtain disks $W_1, \dots W_n$ in Δ such that $g^{-1}[\overline{\text{Int } S}] \subset W_1 \cup \dots \cup W_n$. Since $g^{-1}[\overline{\text{Int } S}]$ misses BdΔ, $\Delta - \text{Int } W_1 - \dots - \text{Int } W_n$ is a disk with holes and $g^{-1}[\overline{\text{Int } S}] \subset$ Int W_1 ... Int W_n . If ε in II(1.2) is sufficiently small, then g carries each Bd W_r into N, for g[Bd W_r] lies close to, but not in $\overline{\text{Int } S}$ and hence close to S . In II(1.3), take (simply connected) N to be $P_1 = P_2 = \dots = P_n$ to obtain the mapping $\bar{g} = g|_{\Delta - \text{Int } W_1 - \dots - \text{Int } W_n} \cup \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_n : \Delta \to \text{rng } g \cup N$. Since each point x in Δ lies either in $\Delta - \text{Int } W_1 - \dots - \text{Int } W_n$ in which case $\bar{g}(x) \in E^3 - \overline{\text{Int } S}$, or in some W_r, in which case $\bar{g}(x) \in N$, rng $\bar{g}$ misses Int S - N . Thus rng $\bar{g}$ lies in $(\text{rng } g - \text{Int } S) \cup N$. Finally $\bar{g} = g$ on BdΔ because the two mappings differ only in $W_1 \cup W_2 \cup \dots \cup W_n$, which misses Bd$\Delta$.

(2.3). We will now give a formal proof of Lemma One.

Case one: neither $g_i[\Delta]$ meets S . Let $\bar{g}_i = g_i$. Since $\bar{g}_i[\Delta]$ meets Ext S, a connectivity argument shows that $\bar{g}_i[\Delta]$ misses Int S . The rest of the requirements of Lemma One are clear.

In the next two cases we insist that one of the rng g_i touch Ω while the other does not.

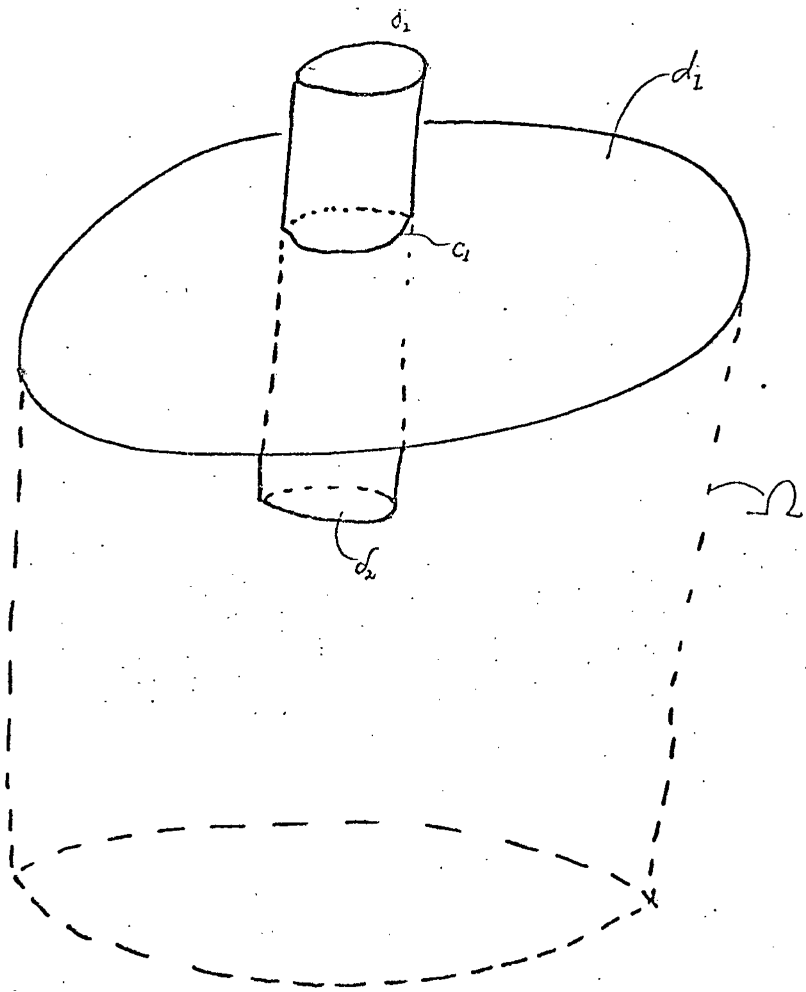


Fig 49.

Case two: exactly one rng g_i meets S . The rng g_i which meets S also meets Ω . Assume that rng g_1 meets $\Omega \subset S$. Let $\overline{g}_2 = g_2$. Evidently i), ii), iii) of Lemma One are true of $\overline{g}_2$. Apply the argument of 2.2, taking g in 2.2 to be g_1, and construct a mapping $\overline{g}_1 : \Delta \to (\text{rng } g_1 - \text{Int } S) \cup N$ which agrees with c_1 on BdΔ . With regard to $\overline{g}_1$, i) and ii) are satisfied, and iii) is vacuously satisfied since $g_1[\Delta]$ hits Ω. The $\overline{g}_i$ are Z-disjoint because we could have taken N small enough to miss rng g_2. Thus
$\emptyset = Z \cap \text{rng } g_1 \cap \text{rng } g_2 = Z \cap (\text{rng } g_1 \cup N) \cap \text{rng } g_2 \supset Z \cap \text{rng } \overline{g}_1 \cap \text{rng } \overline{g}_2$.

Case three: both rng g_i meet S . One rng g_i, say rng g_1, meets Ω; the other (rng g_2) does not. The aim of the proof will be to construct an intermediate pair of mappings g_i^k such that rng g_2^k misses S although rng g_1^k may not. The argument then reduces to an easy variation of either case one or case two.

Outline of proof. a) Choose a component z_1 of rng $g_2 \cap S$. It is important that, since rng g_2 misses Ω, rng $g_2 \cap S \subset \text{Int } d_1 \cup \text{Int } d_2$. Using this fact, we construct a circle $c_1 \subset \text{Int } d_1 \cup \text{Int } d_2 - \text{rng } g_1 - \text{rng } g_2$ which encloses (on one of the d_i) points of exactly one of rng $g_1 \cap S$, rng $g_2 \cap S$. Although we choose $z_1 \subset \text{rng } g_2 \cap S$, c_1 may turn out to enclose points of rng $g_1 \cap S$.

b) Assume that $\overline{\text{Int } c_1} \subset \text{Int } d_1$. Construct a sphere $\omega \cup \delta_1 \cup \delta_2$ in the shape of a pill-box as shown in fig 49 so that c_1 is the equator of $\omega \cup \delta_1 \cup \delta_2$.

c) An argument like that of case two but using $\omega \cup \delta_1 \cup \delta_2$ instead of S yields a pair of Z-disjoint mappings

$g_i^1: \Delta \to \text{rng } g_i \cup N$ such that $g_i^1 = c_i$ on $\text{Bd}\Delta$, $\text{Int } c_1$ misses $\text{rng } g_1^1 \cup \text{rng } g_2^1$; and if $\text{rng } g_i$ misses $\Omega \cup \ell_j$, then so does $\text{rng } g_i^1$. The argument of case two is used virtually as is if c_1 encloses points of $\text{rng } g_1 \cap S$. If c_1 encloses points of $\text{rng } g_2 \cap S$ the argument must be modified somewhat, since the method of case two would not ordinarily ensure that $\text{rng } g_2^1$ would miss all the ℓ_j that $\text{rng } g_2$ misses.

d) It turns out that $\text{rng } g_2^1$ misses Ω. If some component $z_2 \subset \text{rng } g_2^1 \cap S$ exists in, say, $\text{Int } d_1$, then we repeat steps a), b) c) to define mappings $g_i^2: \Delta \to \text{rng } g_i^1 \cup N$ with properties analogous to those of the g_i^1. We continue to construct mappings g_i^3, g_i^4, ... , first finding a component $z_{r+1} \subset \text{rng } g_2^r \cap (\text{Int } d_1 \cup \text{Int } d_2)$ and then constructing the pair $g_i^{r+1}: \Delta \to \text{rng } g_i^r \cup N$ such that $g_i^{r+1} = c_i$ on $\text{Bd}\Delta$ and if $\text{rng } g_i$ misses $\Omega \cup \ell_j$, then so do $\text{rng } g_i^1$, $\text{rng } g_i^2$, ... $\text{rng } g_i^r$, $\text{rng } g_i^{r+1}$. We show that $\text{rng } g_i^r \cap S \supset \text{rng } g_i^{r+1} \cap S$ and that the sequence of mappings ends at a pair g_i^k for which $\text{rng } g_2^k \cap S = \emptyset$ although $\text{rng } g_1^k$ may hit S.

e) The situation now reduces to case one or case two. An additional argument shows that iii) of Lemma One is satisfied.

Details of proof. a) If we assume that $\text{rng } g_2$ misses Ω but hits S, then there is a component z_1 of $\text{rng } g_2 \cap S$ lying in $\text{Int } d_1$ or $\text{Int } d_2$, say $\text{Int } d_1$. By the Zoretti Theorem, there is a circle χ_1 in $\text{Int } d_1$ which misses $\text{rng } g_2$, encloses z_1, and lies so near to z_1 that it misses $\text{rng } g_1$ (by 'encloses' we mean 'encloses relative to d_1'). If χ_1 encloses points of just one of $\text{rng } g_1 \cap S$, $\text{rng } g_2 \cap S$, then let χ_1 be c_1. We must expect that χ_1 will enclose points of both rng $g_1 \cap S$ and rng $g_2 \cap S$ (z_1 could be itself a circle enclosing points of rng $g_1 \cap S$). In this case use the Plane Separation Theorem in Int $\chi_1 \subset$ Int d_1 to construct a circle $\chi_2 \subset$ Int χ_1 which misses $(g_1[\Delta] \cup g_2[\Delta]) \cap S$ and separates the component z_1 of $g_2[\Delta] \cap S$ from a component z' of $g_1[\Delta] \cap S$ in Int χ_1. Since z_1 may be in Int χ_1 - $\overline{\text{Int } \chi_2}$, we cannot predict that Int χ_2 contains points of rng $g_2 \cap S$; but by the Plane Separation Theorem we know that χ_2 encloses points of (rng $g_1 \cup$ rng $g_2) \cap S$. If χ_2 encloses points of both rng $g_1 \cap S$ and rng $g_2 \cap S$, then separate Int $\chi_2 \cap$ (rng $g_1 \cup$ rng g_2) still further by means of another application of the Plane Separation Theorem. We repeat this procedure, defining circles χ_3, χ_4, ... ; χ_r being constructed in Int χ_{r-1} whenever Int χ_{r-1} contains points of both rng $g_i^{r-1} \cap S$. The following argument shows that the sequence χ_1, χ_2, ... must be finite: each annulus $\overline{\text{Int } \chi_r}$ - Int χ_{r+1} contains points of (rng $g_1 \cup$ rng $g_2) \cap S$. Without loss of generality in the construction of the χ_r, we could have replaced the sets rng $g_i \cap S$ with 'thickened' sets obtained by covering the rng $g_i \cap S$ by small disks of area σ (from compactness, the thickened rng $g_1 \cap S$ can be assumed to remain disjoint from the thickened rrng $g_2 \cap S$). But since each of the disjoint open annuli must therefore have area at least σ, the number of χ_r must be finite and the sequence ends at some χ_t. Since χ_{t+1} could be defined if χ_t encloses points of both rng $g_1 \cap S$ and rng $g_2 \cap S$, χ_t must enclose points of only one of rng $g_1 \cap S$, rng $g_2 \cap S$. Let χ_t be c_1. We repeat that we do not know which of rng $g_1 \cap S$, rng $g_2 \cap S$ is intersected by Int χ_t = Int c_1.

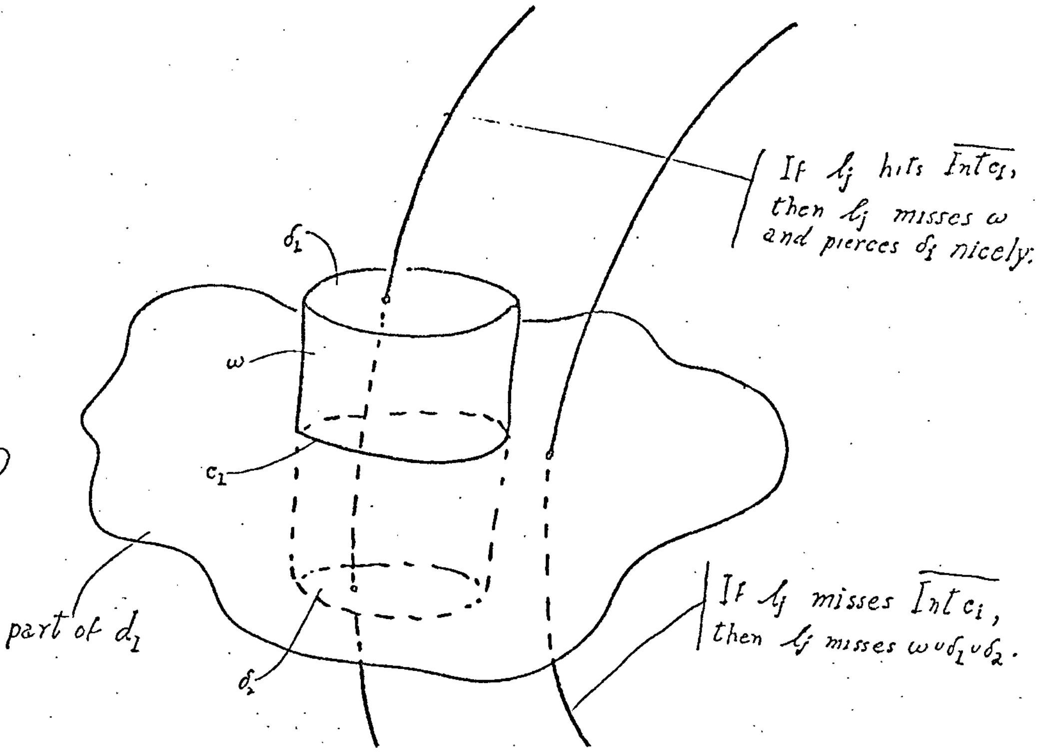

If l_j hits $\overline{Int c_1}$, then l_j misses ω and pierces δ_1 nicely.
δ_1
ω
c_1
part of d_1
δ_2
If l_j misses $\overline{Int c_1}$, then l_j misses $\omega \cup \delta_1 \cup \delta_2$.

Fig 50

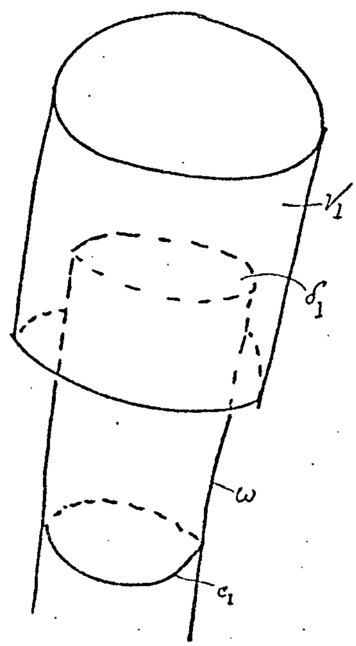

V_1
δ_1
ω
c_1

Fig 51.

b) Now assuming d_1 to be horizontal, we build a small sphere in the shape of a pill box consisting of vertical cylinder ω and end caps δ_1, δ_2, which are parallel to d_1 (see fig 49). The cylinder ω intersects d_1 only at c_1 and extends equal distances above and below d_1. Thus Int c_1 (considered as a subset of d_1) lies in $\text{Int}(\omega \cup \delta_1 \cup \delta_2)$. Since c_1 misses rng $g_1 \cup$ rng g_2 we build ω so near c_1 that ω misses rng $g_1 \cup$ rng g_2. Fig 50 shows $\omega \cup \delta_1 \cup \delta_2$ and part of d_1. We assume that c_1 has been moved slightly if necessary so as to miss $(\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4) \cap d_1$. We also assume that ω misses every ℓ_j although this may necessitate making ω smaller or even curving ω slightly to follow the curve of ℓ_j. The sphere $\omega \cup \delta_1 \cup \delta_2$ is constructed as in fig 50 so that if some ℓ_j meets d_1 then ℓ_j (misses ω and) pierces each Int δ_i just once. Let ν_i be the simply connected neighbourhood of δ_i shown in fig 51. We construct ν_i so that $\delta_i - \ell_1 - \ell_2 - \ell_3 - \ell_4$ is a deformation retract of $\nu_i - \ell_1 - \ell_2 - \ell_3 - \ell_4$, and so that ν_i misses $S \supset$ Int c_1 and any ℓ_j which misses Int c_1, i.e. any ℓ_j which misses $\omega \cup \delta_1 \cup \delta_2$. We assume that $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ has been constructed so that $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ lies in N, and (since $c_1 \subset \text{Int } d_2$) so that $\overline{\text{Int}(\omega \cup \delta_1 \cup \delta_2)} \cup \nu_1 \cup \nu_2$ misses $\Omega \cup d_2$.

c) Assume for the moment that Int c_1 contains points of rng $g_1 \cap S$. Because rng g_1 hits Ω, we can ignore iii) in Lemma One as far as $\bar{g}_1$ is concerned, i.e. the g_1^k which we are about to construct need not miss any ℓ_j. We assume that $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ lies so near Int c_1 that $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ misses rng g_2. Since

ω also misses rng g_1, rng g_1 meets $\omega \cup \delta_1 \cup \delta_2$ only in δ_1 or δ_2. Apply an argument like that of 2.2 to $\omega \cup \delta_1 \cup \delta_2$, i.e. use II(1.2), taking S in II(1.2) to be $\omega \cup \delta_1 \cup \delta_2$ and N to be $\nu_1 \cup \nu_2$, to obtain disks $W_1, \ldots W_n$ in Δ such that $g_1^{-1}[\overline{\text{Int } (\omega \cup \delta_1 \cup \delta_2)}]$ lies in Int $W_1 \cup \ldots \cup$ Int W_n. Since $g_1[\text{Bd } W_r]$ lies near rng $g_1 \cap (\omega \cup \delta_1 \cup \delta_2)$ as we saw in 2.2, and since ω misses rng g_i, therefore $g_1[\text{Bd } W_r]$ lies near $\delta_1 \cup \delta_2$, i.e. in $\nu_1 \cup \nu_2$. Evidently each $g_1[\text{Bd } W_r]$ lies in one ν_i. Since each $g_1[\text{Bd } W_r]$ lies in a simply connected subset of $\nu_1 \cup \nu_2$, we can construct the mappings γ_r in II(1.3) and, following the argument of 2.2, define a mapping $g_1^1 : \Delta \to [\text{rng } g_1 - \text{Int}(\omega \cup \delta_1 \cup \delta_2)] \cup \nu_1 \cup \nu_2$ such that $g_1^1 = c_1$ on BdΔ. Let $g_2 = g_2^1$. Then it is true of both g_i^1 that $g_i^1 : \Delta \to \text{rng } g_i \cup N$, $g_i^1 = c_1$ on BdΔ, and if rng g_i misses $\Omega \cup \ell_j$ then so does rng g_i^1 (this last property is only true because rng g_1 hits Ω and because g_2, which can miss some $\Omega \cup \ell_j$, is identical to g_2^1). Additionally Int c_1 on d_1 misses rng g_2 because $g_2 = g_2^1$ and misses rng g_1 because Int c_1 misses $\nu_1 \cup \nu_2$. The g_i^1 are Z-disjoint because rng g_2^1 = rng g_2, and rng g_1^1 exceeds rng g_1 only in $\nu_1 \cup \nu_2$ which misses rng g_2^1. (By 'rng g_1^1 exceeds rng g_1 only in N', we mean that rng $g_1 \cup N \supset$ rng g_1^1.)

Suppose that instead of rng $g_1 \cap S$, Int c_1 encloses points of rng $g_2 \cap S$. Let $g_1 = g_1^1$, and construct $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ as in b), this time so that $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ misses rng g_1 and ω misses rng g_2. Constructing g_2^1 is harder than constructing g_1^1 as we did in the last paragraph for we must ensure that rng g_2^1 misses any set $\Omega \cup \ell_j$ that rng g_2 misses. We must take careful account of the various ℓ_j. Some ℓ_j are not missed by rng g_2 and can be ignored.

Some ℓ_j are missed by rng g_2 but do not meet $\overline{\text{Int } c_1}$; we note that $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ has been constructed so that any ℓ_j which misses $\overline{\text{Int } c_1}$ also misses $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$. With this precaution, it is safe to ignore those ℓ_j which miss rng g_2 and also miss $\overline{\text{Int } c_1}$. In the remainder of this paragraph we will assume that ℓ_1 and ℓ_2 are those ℓ_j which miss rng g_2 and hit $\overline{\text{Int } c_1}$. We think that the procedure in the general case that some subset ℓ_{j_1}, ℓ_{j_2}, ... ℓ_{j_s} of ℓ_1, ℓ_2, ℓ_3, ℓ_4 misses rng g_2 and hits $\overline{\text{Int } c_1}$ will be evident. We proceed to define g_2^1 using II(1.2) and II(1.3) as before. The only difficulty occurs when we wish to shrink $g_{2|\text{Bd } W_r}$ to a point so as to define γ_r. It was easy to shrink $g_{1|\text{Bd } W_r}$ to a point in one component, say ν_1, of $\nu_1 \cup \nu_2$ in the course of defining g_1^1. But in this case we must shrink $g_{2|\text{Bd } W_r}$ to a point in $\nu_1 - \ell_1 - \ell_2$; otherwise rng γ_r and hence rng g_2^1 will hit $\ell_1 \cup \ell_2$. The reason that $g_{2|\text{Bd } W_r}$ will shrink to a point on $\nu_1 - \ell_1 - \ell_2$ is that $g_2[W_r]$ misses $\ell_1 \cup \ell_2$ (because rng g_2 does) and can be assumed to miss z in fig 47 without loss of generality. There is a retract R (though not a deformation retract) of $E^3 - z - \ell_1 - \ell_2$ onto $\delta_1 - \ell_1 - \ell_2$. Additionally it turns out that R restricted to $\nu_1 - \ell_1 - \ell_2$ is a deformation retract of $\nu_1 - \ell_1 - \ell_2$ onto $\delta_1 - \ell_1 - \ell_2$. This means that $Rg_{2|\text{Bd } W_r}$ is homotopic to $g_{2|\text{Bd } W_r}$ in $\nu_1 - \ell_1 - \ell_2$; and since $Rg_{2|\text{Bd } W_r}$ shrinks to a point in $Rg_2[W_r] \subset \nu_1 - \ell_1 - \ell_2$, therefore $g_{2|\text{Bd } W_r}$ shrinks to a point in $\nu_1 - \ell_1 - \ell_2$ as required. We delay the description of the retract R and the proof that $g_2[\text{Bd } W_r]$ can miss z until the end of this proof. Except for the use of R to make $g_{2|\text{Bd } W_r}$ shrink to a point, the construction of g_2^1 is like that of g_1^1, and we have $g_2^1:\Delta \to [\text{rng } g_2 - \text{Int}(\omega \cup \delta_1 \cup \delta_2)] \cup (\nu_1 \cup \nu_2 - \ell_1 - \ell_2)$. The g_i^1 are Z-disjoint by an argument like that in the previous paragraph, and $g_i^1:\Delta \to \text{rng } g_i \cup N$, $g_i^1 = c_i$ on $\text{Bd}\Delta$ as before. We know that rng g_2^1 misses Ω because rng g_2^1 exceeds rng g_2 only in $\nu_1 \cup \nu_2$ which is remote from Ω . If ℓ_j misses rng g_2, then either ℓ_j meets Int c_1, in which case ℓ_j misses rng g_2^1 because ℓ_j is one of ℓ_1, ℓ_2 above; or else ℓ_j misses Int c_1, in which case ℓ_j misses rng g_2^1 because ℓ_j is remote from $\nu_1 \cup \nu_2$.

Note that the g_i^1 satisfy the hypothesis of Lemma One. The important difference between g_i and g_i^1 is that Int $c_1 \cap \text{rng } g_i^1 = \emptyset$ whereas Int c_1 hits one of rng g_1, rng g_2 . Since rng $g_i^1 \subset$ rng $g_i \cup \nu_1 \cup \nu_2$, rng $g_i^1 \cap S \subset \text{rng } g_i \cap S$ (because $\nu_1 \cup \nu_2$ misses S). Evidently we can write $(\text{rng } g_1^1 \cup \text{rng } g_2^1) \cap S \subset (\text{rng } g_1 \cup \text{rng } g_2) \cap S$ where the inclusion is proper.

d) Since the g_i^1 satisfy the hypothesis of Lemma One, we look for a component z_2 of rng $g_2^1 \cap S$ in Int $d_1 \cup$ Int d_2 and repeat a), b), c) to obtain a circle $c_2 \subset \text{Int } d_1 \cup \text{Int } d_2$ and Z-disjoint mappings $g_i^2:\Delta \to \text{rng } g_i^1 \cup N$ with $g_i^2 = c_i$ on $\text{Bd}\Delta$, and if rng g_i misses $\Omega \cup \ell_j$, then rng g_i^1 and rng g_i^2 also miss $\Omega \cup \ell_j$. Furthermore $(\text{rng } g_1^2 \cup \text{rng } g_2^2) \cap S \subset (\text{rng } g_1^1 \cup \text{rng } g_2^1) \cap S \subset (\text{rng } g_1 \cup \text{rng } g_2) \cap S$, both inclusions being proper. We can continue in this way, defining mappings g_i^3, g_i^4, ... and components $z_3 \subset \text{rng } g_2^2 \cap (\text{Int } d_1 \cup \text{Int } d_2)$, $z_4 \subset \text{rng } g_2^3 \cap (\text{Int } d_1 \cup \text{Int } d_2)$, ... so that $g_i^r:\Delta \to \text{rng } g_i^{r-1} \cup N$, $g_i^r = c_i$ on $\text{Bd}\Delta$, and if rng g_i misses $\Omega \cup \ell_j$, then so does rng g_i^r .

Furthermore $(\text{rng } g_1^r \cup \text{rng } g_2^r) \cap S \subset (\text{rng } g_1^{r-1} \cup \text{rng } g_2^{r-1}) \cap S$ where the inclusion is proper. An argument from compactness like that used in a) to show that the number of χ_r was finite can be used to show that there must be a final pair of Z-disjoint mappings g_i^k. Since g_i^{k+1} could be defined if z_{k+1} existed in $\text{rng } g_2^k \cap S$, therefore $\text{rng } g_2^k$ must miss S.

Since $\text{rng } g_i^r \subset \text{rng } g_i^{r-1} \cup N$, evidently $g_i^k : \Delta \to \text{rng } g_i^{k-1} \cup N \subset \text{rng } g_i^{k-2} \cup N \subset \ldots \subset \text{rng } g_i \cup N$; and $g_i^k = c_i$ on $\text{Bd}\Delta$, while if $\Omega \cup \ell_j$ misses some $\text{rng } g_i$, then $\Omega \cup \ell_j$ misses $\text{rng } g_i^k$. Since $\text{rng } g_2^k$ misses S, the argument reduces to either Case one or Case two. In the course of the argument of Case one or two, g_2^k, whose range already misses S, will be set equal to $\bar{g}_2$. This means that $\bar{g}_2$ has the properties of g_2^k; thus i), ii), iii) of Lemma One are true for $\bar{g}_2$. The argument of either Case one or of Case two will now construct a new $\bar{g}_1 : \Delta \to \text{rng } g_1^k \cup N \subset \text{rng } g_1 \cup N$ with $\bar{g}_1 = c_1$ on $\text{Bd}\Delta$, so that $\bar{g}_1$, $\bar{g}_2$ are Z-disjoint. This proves i), ii) of Lemma One for $\bar{g}_1$, while iii) is vacuously true by the Case three assumption.

Case four: Exactly one $\text{rng } g_i$ meets S but misses Ω. The reader will find that the method of case three works here almost word or word if it is assumed that the $\text{rng } g_i$ which hits S is $\text{rng } g_2$. When we arrive at the point in Case three where g_i^k is defined, we can let g_i^k be $\bar{g}_i$ immediately (or go to Case one). Actually the retract R works on S in Case two just as well as on $\omega \cup \delta_1 \cup \delta_2$ in Case three. Thus a quick proof is possible by adapting Case two.

Case five: Both $\text{rng } g_i$ hit S; both $\text{rng } g_i$ miss Ω.

Proceed as in Case three to the point where the g_i^k are defined, allowing for the fact that iii) in Lemma One applies to both $\bar{g}_i$ rather than only to $\bar{g}_2$ as in Case three (thus one may have to use the retract R to construct both g_i^r, whereas in Case three, R was used only to construct g_2^r). When the g_i^k are defined, the argument reduces to Case four or Case one.

Case six: Both rng g_i hit S; both rng g_i hit Ω. This case is not used in the applications of Lemma One, which always require every set $\Omega \cup \ell_j$ to miss one rng g_i. It is not hard to prove Case six using the ideas of the other cases.

(2.4). The retract R.

This retract was used in 2.3 Case three c). We will show how to define $R: E^3 - z - \ell_1 - \ell_2 \to \delta_1 - \ell_1 - \ell_2$; the definition of R when δ_1 is replaced by δ_2 is similar. Strictly speaking, the proof of Lemma One requires a retraction onto $d_1 - \ell_{j_1} - \dots - \ell_{j_s}$, where $\ell_{j_1}, \dots, \ell_{j_s}$ are some subset of ℓ_1, ℓ_2, ℓ_3, ℓ_4; however we continue the assumption in 2.3 Case three c) that rng g_2 misses $\ell_1 \cup \ell_2$. Assume that the unique boundary component of β^1 which is a planar circle lies on the Y - Z plane and that the centre of this circle is the origin. The idea is that if we untwist β^1 by means of y, all the circles $y[\ell_j]$ will be nice circles on the Y - Z plane with centre the origin. We assume further that δ_1 lies on the left-hand X - Y half-plane. We describe R in terms of several mappings which are applied in sequence to $E^3 - z - \ell_1 - \ell_2$. Each mapping will leave

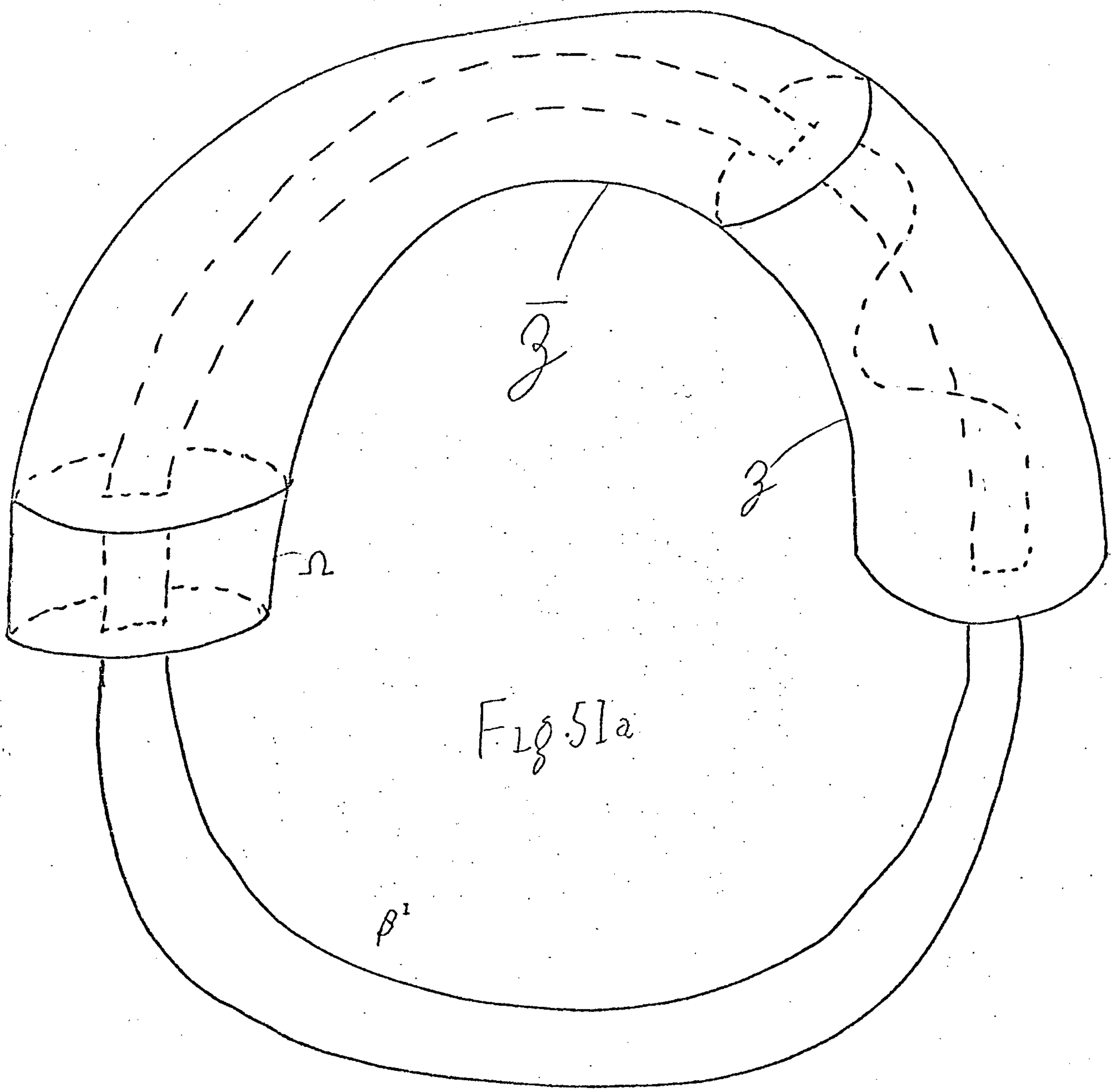


Fig. 51a

$\delta_1 - \ell_1 - \ell_2$ fixed, and the last will be onto $\delta_1 - \ell_1 - \ell_2$.

First: untwist β^1 by applying the mapping $y|_{E^3 - z - \ell_1 - \ell_2}$ (y becomes a mapping by restricting it so that the domain misses the 'bad' set z). Each $y[\ell_j]$ is a plane circle with centre the origin. Second: using the symmetry of $E^3 - y(\ell_1 \cup \ell_2)$ across the X - Z plane, reflect the right-hand half-space minus $y(\ell_1 \cup \ell_2)$ onto the left-hand half-space minus $y(\ell_1 \cup \ell_2)$. This reflection carries $E^3 - z - y(\ell_1 \cup \ell_2)$ into those points in E^3 with non-positive coordinates which do not lie on $\ell_1 \cup \ell_2$. Third: retract the left-hand half-space minus $y(\ell_1 \cup \ell_2)$ (which is the same as the left-hand half-space minus $\ell_1 \cup \ell_2$) onto the left-hand X - Y half-plane minus $\ell_1 \cup \ell_2$. This is easy because the remaining parts of ℓ_1, ℓ_2 are nice semicircles with centre the origin; one could imagine the X - Z plane hinged along the X axis. Using this hinge, topple the upper half of the X - Z plane onto the left-hand X - Y plane; simultaneously bring the lower half of the X - Z plane up to meet the left-hand X - Y plane. These movements define a (deformation) retract which crushes the left-hand half space minus $\ell_1 \cup \ell_2$ onto the left-hand X - Y half-plane minus $\ell_1 \cup \ell_2$. Finally retract the left-hand X - Y half-plane minus $\ell_1 \cup \ell_2$ onto $\delta_1 - \ell_1 - \ell_2$. The four successive mappings define R . Note that R acts on ν_1 as a deformation retract (this was used in §2.3 Case three c)) .

Finally we will show that $g_2[W_r]$ in 2.3 Case three c) can be assumed to miss the curved cylinder z . Since $g_2[\Delta]$ misses Ω, by the case three assumption, we can construct a mapping $\hat{g}_2:\Delta \to E^3$ whose range misses the curved cylinder $\overline{z}$ in fig 51a and which agrees with g_2 on every point of Δ which maps under g_2 outside of a small neighbourhood of $\bar{z}$. In fig 51a, $\bar{z}$ is constructed so as to contain $\Omega \cup \mathcal{Z}$ and to miss $\ell_1 \cup \ell_2$. Thus rng $\hat{g}_2$ can be assumed to miss $\ell_1 \cup \ell_2$. The sets ν_1, ν_2 (see 2.3 Case Three b)) miss Ω and could have been constructed so as to miss all of $\bar{z}$. Hence we assume that $\hat{g}_2 = g_2$ on Bd W_r since $g_2[\text{Bd } W_r] \subset \nu_1$ by the definition of W_r. We do not intend that $\hat{g}_2$ should replace g_2 since g_1, $\hat{g}_2$ may not be Z-disjoint; but if $g_2[W_r]$ hits z, we can apply the retract R not to $g_{2|W_r}$ but to $\hat{g}_{2|W_r}$ and use the fact that $g_{2|\text{Bd } W_r}$ shrinks to a point in $\nu_1 - \ell_1 - \ell_2$ iff $\hat{g}_{2|\text{Bd } W_r}$ does. Essentially the same argument applies to the construction of the other mappings in the sequence g_2, $g_2^{(1)}$, $g_2^{(2)}$, ... $\square$.

3. Proof of Lemma Two.

We will modify the argument of §2 so as to serve as a proof of Lemma Two. We assume familiarity with §2 in what follows. Modifying the argument of §2 to fit fig 45 presents a small and a large difficulty. The small problem is that we cannot build pillboxes according to the nice picture in fig 49, where c_1 is planar and the δ_1 can be considered to be horizontal while ω is vertical. It will be appreciated that the problem is more apparent than real; we have room to construct $\Sigma \subset \text{Int } A$ with some obvious smoothness conditions so that if c is any circle on Σ and Int c is defined, then a sphere $\omega \cup \delta_1 \cup \delta_2$ can be constructed together with neighbourhoods ν_1, ν_2 so that $\omega \cup \delta_1 \cup \delta_2 \cup \nu_1 \cup \nu_2$ behaves like the corresponding set in §2, i.e. $\omega \cap \Sigma = c$, δ_1, δ_2 are disks in $E^3 - \Sigma$ which meet ω only at its

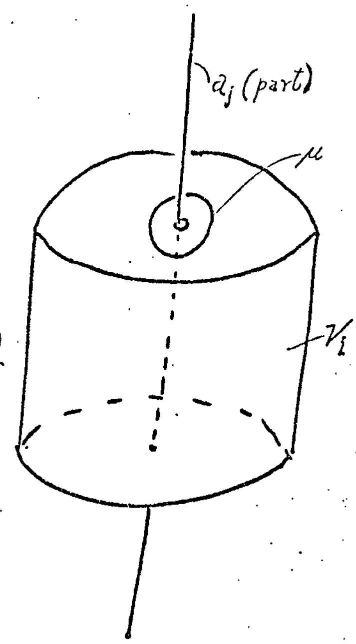


Fig 52

two boundary components, while the ν_i are simply connected neighbourhoods of the δ_i which miss Σ. Furthermore, if a_j hits $\overline{\text{Int } c}$, then a_j pierces each Int δ_i just once and misses ω; while if a_j misses $\overline{\text{Int } c}$, then a_j misses $\overline{\text{Int}(\omega \cup \delta_1 \cup \delta_2)} \cup \nu_1 \cup \nu_2$. We require that when Int c and hence δ_i hits just one a_j, then $\nu_i - a_j$ is homeomorphic to the structure shown in fig 52. This requirement is easy to manage; for Σ can be made to meet the a_j near β^1, where (according to the definition on Ch II) the a_j are straight and parallel and perpendicular to β^1 (in fact it is easy to make $a_j \cap N$ a straight arc perpendicular to Σ).

The hard problem is that in Lemma Two we cannot use the retract R, which was crucial to the proof of iii) of Lemma One. The reason is that we permit the arcs $(a_1 \cup u_{12} \cup a_2) \cap \overline{\text{Int } \Sigma}$, etc. to be knotted, and in general deviate from the specialized geometry in fig 19. Recall that R was used to show that certain mappings $g_i|\text{Bd } W_r$ will shrink to a point in $\nu_i - \ell_1 - \ell_2 - \ell_3 - \ell_4$. Instead of R, we use the following easy but very weak result.

(3.1). Let ν_i be the usual neighbourhood of δ_i. Let ν_i intersect only a_1 as shown in fig 52. Let $f: \text{Bd}\Delta \to \nu_i - a_1$. Let $F: \Delta \to E^3$ so that $f = F$ on BdΔ and rng F misses a simple closed curve L such that $L \supset a_1$. The curve L may be knotted. Then f shrinks to a point in $\nu_i - a_1$. A similar result is true if a_2 or a_3 replaces a_1.

Proof. Let μ be the small circle shown in fig 52. Then μ can be considered to represent the sole generator y of $\nu_i - a_1$, and also (by consulting, say, the definition in [6 Ch VI]) a generator of the Wirtinger presentation of $E^3 - L$ (we specify the particular presentation only to be sure that μ does not represent a trivial generator). If $i:\pi_1(\nu_1 - a_1) \to \pi_1(E^3 - L)$ is the inclusion homomorphism, then, with a change of basepoint, $f \in y^m$ for some integer m, and $i(y)$ is an element of $\pi_1(E^3 - L)$. Then $f \in i(y^m) = (i(y))^m$, which is the identity of $\pi_1(E^3 - L)$ because f shrinks to a point in $E^3 - L$. Since $i(y)$ is a non-trivial element (in fact a Wirtinger generator) of $\pi_1(E^3 - L)$, either $m = 0$ or $i(y)$ is an element of finite order. It is known ([7, (31.9)]) that the fundamental group of the complement of a knot has no element of finite order; therefore $m - 0$, and f represents the identity y^0 in $\pi_1(\nu_i - a_1)$ $\square$.

To prove Lemma Two, we will apply arguments like those of §2 to a disk D rather than the sphere S. We will first define some simple closed curves to play the part of L in (3.1): Let u_{12}, u_{13} be the unique simple closed curves which are subsets of $\zeta_2 \cup a_1 \cup u_{12} \cup a_2$, $\zeta_2 \cup a_1 \cup u_{13} \cup a_3$, respectively. Let ν_{12}, ν_{13} be identical to u_{12}, u_{13} respectively except that u_{12} is replaced by v_{12} and u_{13} by v_{13}. From the hypothesis of Lemma Two, it is clear that u_{12} and u_{13} miss rng g_1 while ν_{12}, ν_{13} miss rng g_2. Now if D is a disk which is a subset of Σ and Bd D misses rng $g_1 \cup$ rng $g_2 \cup a_1 \cup a_2 \cup a_3$, then if rng $g_1 \cup$ rng g_2 meets D, we can define a circle c_1 just as in §2 so that $c_1 \subset$ Int D and c_1 encloses points of just one of rng $g_1 \cap \Sigma$, rng $g_2 \cap \Sigma$. Then a pillbox $\omega \cup \delta_1 \cup \delta_2$ can be constructed as usual, and finally a pair of mappings $g_i^1:\Delta \to$ rng $g_i \cup \nu_1 \cup \nu_2$, where the ν_i are the usual neighbourhoods of the δ_i, and the g_i^1 have properties like the g_i^1 in §2, Case three. If Int c_1 and hence $\omega \cup \delta_1 \cup \delta_2$ meets just one of a_1, a_2,

a_3 say a_1 , then we use (3.3) instead of the retract R to shrink the various mappings $g_i|_{Bd\ W_r}$ to a point in $\nu_1 \cup \nu_2$ as was done in §2, Case three. Thus if rng g_i hits Int c_1, rng g_i^1 misses a_1 because Int c_1 meets only one of a_1, a_2, a_3; while if rng g_i misses Int c_1, rng g_i^1 misses a_1 because $g_i^1 = g_i$. And by the usual arguments, a_2 and a_3, which are remote from $\omega \cup \delta_1 \cup \delta_2$, continue to miss both rng g_i^1 . In applying (3.1), we let L be u_{12} or ν_{12} (assuming that Int c_1 hits a_1) depending as rng g_1 or rng g_2 hits Int c_1 . If Int c_1 hits a_2 only, let L be u_{12} or ν_{12} again; if Int c_1 hits a_3 only, let L be u_{13} or ν_{13} . Unfortunately, as the reader doubtless sees, if c_1 encloses more than one of $a_1 \cap \Sigma$, $a_2 \cap \Sigma$, $a_3 \cap \Sigma$, then the present argument fails (because the argument with (3.1) is weaker than the original argument in §2 which used the retract R), and g_i^1 cannot be constructed so that rng g_i^1 misses all of a_1, a_2, a_3 . The trick of proving Lemma Two is to apply the argument of §2 so that none of Int c_1, Int c_2, ... ever hits more than one of a_1, a_2, a_3 . By extending the above ideas to further pairs g_i^2, g_i^3, ... and using methods from §2, we can prove

(3.2). In the context of Lemma Two, let D be a disk such that $D \subset \Sigma$, and let Bd D miss rng $g_1 \cup$ rng $g_2 \cup a_1 \cup a_2 \cup a_3$. Then there exist circles c_1, c_2, ... c_m in Int D and Z-disjoint mappings g_i^1, g_i^2, ... g_i^m such that $g_i^r : \Delta \to$ rng $g_i^{r-1} \cup (n - \Sigma)$, $g_i^1 = g_i^2 = \ldots = g_i^m = c_i$ on BdΔ, c_r encloses (relative to D) points of just one of rng $g_1^{r-1} \cap \Sigma$, rng $g_2^{r-1} \cap \Sigma$, and rng g_i^m misses D . If, additionally, each c_r can be constructed so that c_r encloses just one of $a_1 \cap \Sigma$, $a_2 \cap \Sigma$,

$a_3 \cap \Sigma$, then rng g_i^m can be constructed so as to miss $a_1 \cup a_2 \cup a_3$.

Proof. (3.2) is proved in the same way as Lemma One. We can ignore Cases two and three in the proof of Lemma One because the fact that Bd D misses rng $g_1 \cup$ rng g_2 evidently takes the place of the condition in Lemma One that Ω misses rng $g_1 \cup$ rng g_2. Clearly we cannot have rng g_i^m miss Int Σ in this version of the argument because D is a proper subset of Σ. The only part of the proof which does not have an exact counterpart in §2 is the statement that rng $g_i^r \subset$ rng $g_i^{r-1} \cup (n - \Sigma)$. The reason that rng g^r exceeds rng g_i^{r-1} only in $n - \Sigma$ is, as usual, that $\nu_1 \cup \nu_2$ is remote from Σ.

(3.3). Corollary. If, additionally to the hypothesis of (3.2), D misses rng g_1, then rng g_1^m misses $a_1 \cup a_2 \cup a_3$ regardless of the number of a_1, a_2, a_3 which are hit by the Int c_r. Similarly, if rng g_2 misses D, then rng g_2^m misses $a_1 \cup a_2 \cup a_3$.

Proof. According to the argument of §2, if rng g_i misses D, then we let $g_i = g_i^m$ immediately.

We will now give the proof of Lemma Two. The following question does not look like a simplification at first glance: Do there exist Z-disjoint mappings $f_i:\Delta \to (\text{rng } g_i \cup n) - a_1 - a_2 - a_3$ with $f_i = c_i$ on BdΔ, and a decomposition of Σ into disks D_1, D_2 so that $D_1 \cup D_2 = \Sigma$ and $D_1 \cap D_2 = \text{Bd } D_1 = \text{Bd } D_2$, and so that Int D_2 misses one rng f_i, say rng f_1, and hits a_2 and a_3; while Int D_1 hits a_1?

Case one: the mappings f_i exist as described above. Look at the decomposition $\Sigma = D_1 \cup D_2$. Apply (3.2) to D_1 to convert the f_i to Z-disjoint mappings $\overline{f}_i : \Delta \to \text{rng } f_i \cup (n - \Sigma)$ with $\overline{f}_i = c_i$ on $\text{Bd}\Delta$ and such that $\text{rng } \overline{f}_1$ misses a_1 and D_1. In (3.2), the condition that $\text{Int } c_r$ hit at most one of a_1, a_2, a_3 is satisfied because a_2, a_3, miss $D_1 \supset \text{Int } c_r$. Furthermore $\text{rng } \overline{f}_i$ misses a_2 and a_3 by the usual argument. Now apply (3.3) to D_2, to replace the $\overline{f}_i$ with Z-disjoint mappings $g_i' : \Delta \to \text{rng } \overline{f}_i \cup (n - \Sigma)$ with $g_i'|_{\text{Bd}\Delta} = c_i$. We know that $\text{rng } \overline{f}_1$ misses D_2 since $\text{rng } f_1$ does, and $\overline{f}_i$ evidently satisfies the hypothesis of (3.2) and (3.3). By (3.3), $\text{rng } g_1'$ misses $a_1 \cup a_2 \cup a_3$, although $\text{rng } g_2'$ probably does not. Since $\text{rng } g_i'$ now misses all of Σ, $\text{rng } g_i'$ misses $\text{Int } \Sigma$ except perhaps in n.

Case two: no mappings f_i exist as described. Let $d \subset \Sigma$ be a disk pierced by a_1 which is small enough to miss both $\text{rng } g_i$ and $a_2 \cup a_3$. Let $D = \overline{\Sigma - d}$. Using (3.2), construct a sequence of circles and mappings c_1, g_i^1, c_2, g_i^2, c_3, g_i^3, ... as described in (3.2), ending in the construction of Z-disjoint mappings $g_i^m = g_i' : \Delta \to \text{rng } g_i \cup (n - \Sigma)$ with $g_i' = c_i$ on $\text{Bd}\Delta$ and such that $\text{rng } g_i'$ misses D and $a_2 \cup a_3$. We know that every c_r encloses at most one of $a_1 \cap \Sigma$, $a_2 \cap \Sigma$, $a_3 \cap \Sigma$, as required by (3.2): for otherwise g_i^{r-1}, $\overline{\text{Int } c_r}$, $\Sigma - \text{Int } c_r$ satisfy the definition of f_i, D_1, D_2 given above, which means that (since g^{r-1} exists) c_r contradicts the Case two assumption. Evidently $\text{rng } g_i'$ misses not only $D \cup a_2 \cup a_3$, but also $d \cup a_1$, so that $\text{rng } g_i'$ misses $a_1 \cup a_2 \cup a_3$ and all of Σ, etc $\square$.

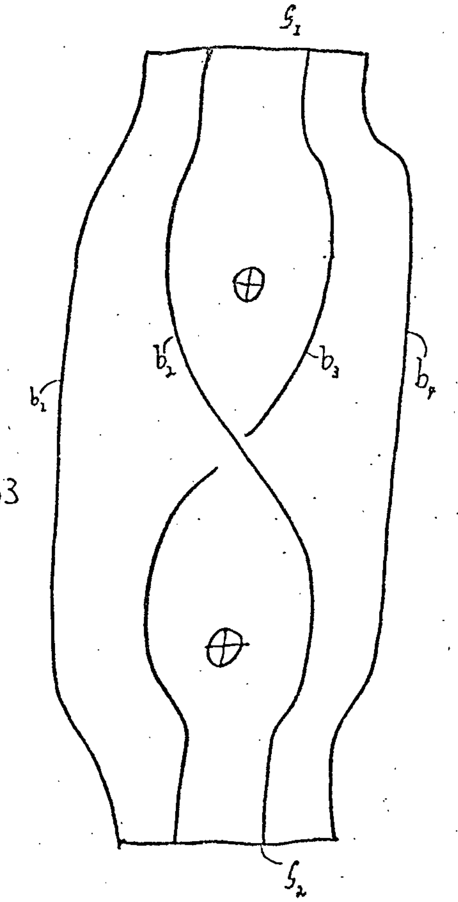


Fig 53

CHAPTER FOUR. GENERALIZATION OF A THEOREM OF BING: MAIN PROOF.

1. We will use Lemma One, Lemma Two and I§5 to prove II(2.2). The organization of the proof is much like that of [12 §7] and we depend on the reader's familiarity with [12] for orientation (although a detailed reading is required only of the section called 'Part II of Proof' in [12 §7]). As in [12 §7], we first give a (somewhat altered) definition of Property Q, then induce Property Q through the steps of the dogbone construction. This argument occupies most of the length of this chapter. As in [12 §7], it follows immediately (and for more or less the same reasons) that some big element of the decomposition hits both singular disks $f_i[\Delta]$ in II(2.2).

Still following [12], we will not present a formal induction, but will show that if A has Property Q, then so does $A_1 \cup A_2 \cup A_3 \cup A_4 = \mathcal{a}_1$ (Bing proves that one A_j has Property Q; our version of Property Q is only useful when applied to $\mathcal{a}_1$, $\mathcal{a}_2$, $\mathcal{a}_3$, ... , of Ex 2 in §2). The proof of this is divided into Part I and Part II as in [12 §7]. In Part I, we look at the set $\zeta_1 \cup b_1 \cup b_2 \cup b_3 \cup b_4 \cup \zeta_2$ (see fig 43) which serves a purpose like that of the set $\bigcup_j pq_jr_js$ in [12, fig 2]. We show that the f_i in II(2.2) can be replaced by mappings g_i' such that each rng g_i' misses one b_j and both ζ_i. We call the set $\zeta_1 \cup b_1 \cup b_2 \cup b_3 \cup b_4 \cup \zeta_2$ <u>the cradle of A</u>, and later represent it as in fig 53, which preserves the embedding of $\zeta_1 \cup b_1 \cup b_2 \cup b_3 \cup b_4 \cup \zeta_2$ in A . In [12], $\bigcup_j pq_jr_js$ behaves like the cradle of A in that each pq_jr_js misses one of the disks D_i in II(2.1) . In Part II of our proof we follow [12] very closely and require a detailed reading of the corresponding part of [12, Th 10]. There are a few alterations; these are required by the fact that some homotopies are replaced by isotopies.

2. Properties P and Q.

We will define a Property P on double ended lassos $\ell \cup a \cup m$ with respect to closed sets Y_1, Y_2. The lasso $\ell \cup a \cup m$ consists of circles ℓ and m connected by an arc a. In Ch II we often specified constructions only up to homotopy (e.g. the intersecting principal paths of Ch II). The consequence was that we ignored singularities in these constructions. In this chapter, this practice is emphatically not allowed; in particular, in the lasso $\ell \cup a \cup m$, the circles ℓ and m are disjoint simple closed curves and a meets $\ell \cup m$ only at its end points. One of the things that make the present chapter harder than Ch II is that geometric constructions have to be moved isotopically, whereas in Ch II homotopy was good enough.

Properties P and Q are defined in terms of their negatives, which we write Property $\sim P$ and Property $\sim Q$. A double ended lasso $\ell \cup a \cup m$ has Property $\sim P$ with respect to closed sets Y_1, Y_2 iff one of the following two conditions obtains.

$\sim P(a)$: $\ell \cup a \cup m$ misses Y_1 or Y_2 (or both),

$\sim P(b)$: $\ell \cup a \cup m$ meets both Y_1 and Y_2. The set $a \cup m$ misses $Y_1 \cup Y_2$; ℓ contains a point $y \neq \ell \cap a$ such that of the two distinct arcs in ℓ with end points y and $\ell \cap a$, one misses Y_1 while the other misses Y_2.

We intend that Property $\sim P(b)$ should be symmetric, i.e. $a \cup \ell$ may miss $Y_1 \cup Y_2$ and the point y may be in $m - a$. Regardless of whether $\ell \cup a \cup m$ has Property $\sim P(a)$ or Property $\sim P(b)$, each of ℓ and m has Property $\sim P$ as defined in I §5 for circles with base point (the base points here are taken to be $\ell \cap a$, $m \cap a$). This statement, which is important, is easily checked. Evidently $\ell \cup a \cup m$ may have both Property $\sim P(a)$ and Property $\sim P(b)$.

Property $\sim P$ is the negative of Bing's Property P in [12]. It is easy to see that our Property $\sim P$ implies the negative of Bing's property, i.e. our Property $\sim P$ implies that if $x_1 \in \ell$ and $x_2 \in m$ and $\ell \cup a \cup m$ has Property $\sim P$ (by our definition), then there is an arc in $\ell \cup a \cup m$ with end points x_1, x_2 which misses one of Y_1, Y_2. We will neither use nor prove the complete equivalence of the two definitions here, although a proof will be found to be straightforward.

Property Q_{Z, c_1, c_2} is defined on dogbones. If a dogbone X has Property $\sim Q_{Z, c_1, c_2}$, this means roughly that the centre of X has Property $\sim P$ with respect to the ranges of certain mappings f_1, f_2. To be precise, let $Z \supset X$ and for $i = 1, 2$, c_i: $Bd\Delta \to E^3 - Z$. Then X has Property $\sim Q_{Z, c_1, c_2}$ iff there exist Z-disjoint mappings g_1, g_2 such that $g_i: \Delta \to E^3$, $g_i = c_i$ on $Bd\Delta$, and the centre of X has Property $\sim P$ with respect to rng g_1, rng g_2. We also say 'X has Property $\sim Q_{Z, c_1, c_2}$ with respect to g_1, g_2' with the obvious meaning. We define X to have Property Q_{Z, c_1, c_2} iff X fails to have Property $\sim Q_{Z, c_1, c_2}$ (i.e. with respect to every qualified pair of mappings g_i). Note that a statement like 'X has Property Q_{Z, c_1, c_2}

with respect to g_1, g_2' means very little.

Example 1). Suppose $Z = X = A$ and c_1, c_2 are the two circles shown in fig 28. Then A has Property Q_{Z, c_1, c_2}. For if c_1 (c_2) shrinks to a point, it must hit the upper (lower) eye ℓ (m) of A. Thus if f_i is an extension of c_i to all of Δ, then rng f_1 hits ℓ and rng f_2 hits m. This 'kills' Property $\sim P$ for $k = \ell \cup a \cup m$ with respect to rng f_1, rng f_2, since Property $\sim P$ would require either that one $f_i[\Delta]$ miss both ℓ and m or that one of ℓ or m miss both $f_i[\Delta]$.

Example 2). Let $Z = A_1$; c_1, c_2 as in Ex. 1). Then A_1 has Property $\sim Q_{Z, c_1, c_2}$; for the c_i can shrink to a point so as to miss Z and A_1. We emphasize that 'X has Property Q_{Z, c_1, c_2}' does not imply that c_1, c_2 link the eyes of X.

Evidently if X has Property Q_{Z, c_1, c_2} and f_1, f_2 are any Z-disjoint mappings of Δ into E^3 with $f_i = c_i$ on BdΔ, then the centre of X fails to have Property $\sim P(a)$ with respect to $f_1[\Delta]$, $f_2[\Delta]$, and consequently both $f_1[\Delta]$ and $f_2[\Delta]$ meet (the centre of) X. This suggests that the obvious way to attack the proof of II(2.2) is to let $Z = A$ and let c_1, c_2 be the c_i in II(2.2), and we will eventually do this. But it turns out that in this case there is no sequence $A \supset A_j \supset A_{jk} \supset \ldots$ such that each of A, A_j, A_{jk}, ... has Property Q_{A, c_1, c_2}, with the c_i defined as in II(2.2); in fact every dogbone $X \neq A$ has Property $\sim Q_{A, c_1, c_2}$. We overcome this difficulty with the next definition.

A set $\{X_1, \dots X_m\}$ of dogbones has Property $\sim Q_{Z,c_1, c_2}$ iff each X_r, $r = 1, 2, \dots m$ has Property $\sim Q_{Z,c_1, c_2}$ with respect to the same pair of mappings f_1, f_2, and the same triple Z, c_1, c_2. If $\{X_1, \dots X_m\}$ fails to have Property $\sim Q_{Z, c_1, c_2}$, then we will say that $\{X_1, \dots X_m\}$ has Property Q_{Z,c_1, c_2}. If the set of components of some $\mathcal{a}_s$ has Property Q_{Z,c_1, c_2} and if $g_i : \Delta \to E^3$ is an extension of c_i, $i = 1, 2$, and the g_i are Z-disjoint; then some component X of $\mathcal{a}_s$ fails to have Property $\sim Q_{Z,c_1, c_2}$ with respect to the g_i. As we saw earlier, this means that both $g_i[\Delta]$ meet X. We will say that $\mathcal{a}_s$ has Property Q_{Z,c_1, c_2} iff the set of components of $\mathcal{a}_s$ has Property Q_{Z,c_1, c_2}. Eventually we will show that each of $\mathcal{a}_1, \mathcal{a}_2, \mathcal{a}_3, \dots$ has Property Q_{Z,c_1, c_2}.

3. We now give our version of [12, Th 10].

(3.1). Let $Z \supset A$ and c_1, c_2 by any circles whatever in $E^3 - Z$. In particular, the c_i do not necessarily link the eyes of A. Then if $\{A_1, A_2, A_3, A_4\}$ has Property $\sim Q_{Z,c_1, c_2}$, so has A.

We remark that in [12], the proof of Th 10 does not use the fact that Bd D_1, Bd D_2 (in fig 1 of [12]) link the eyes of A, even though a short proof of [12, Th 10] can be constructed along the lines of the second paragraph of [12 §7]. The reason is that in later applications of the argument of the proof of [12, Th 10] (which is a disguised induction step) to, say, A_1 and A_{11}, A_{12}, A_{13}, A_{14}, the Bd D_i do not in fact link the eyes of A_1. For a similar reason we state (3.1) for very general circles c_i rather than the c_i in fig 28. We assume that

Z, c_1, c_2 have been chosen once and for all before the proof of (3.1) begins, and will now write Property $\sim Q$ for Property $\sim Q_{Z,c_1,c_2}$. We will not refer to Bing's Property Q again in this paper. We will continue the convention in Ch III that $i = 1, 2$, and $j = 1, 2, 3, 4$.

Proof of (3.1): Part I

In this part of the proof we assume that $\{A_1, A_2, A_3, A_4\}$ has Property $\sim Q$ with respect to mappings g_1, g_2 and show that the g_i can be replaced by Z-disjoint mappings $g_i': \Delta \to E^3$ with $g_i' = c_i$ on $\mathrm{Bd}\Delta$ and with the property that in the cradle $\zeta_1 \cup b_1 \cup b_3 \cup b_4 \cup \zeta_2$ (see fig 43), each b_j misses one rng g_i' while $\zeta_1 \cup \zeta_2$ misses both. By the definition of Property $\sim Q$, each k_j has Property $\sim P$ with respect to the rng g_i. Look at $\beta^1 \cup \beta_1$, and recall the definition of bridging in I §5. The construction of the g_i' divides into three cases depending on the way that the sets rng $g_i \cap (\beta^1 \cup \beta_1)$ bridge β_1 and β^1. If rng $g_1 \cap \beta^1$ or rng $g_2 \cap \beta^1$ bridges β^1, but not both, then we say that β^1 is bridged once by rng g_1 or rng g_2 respectively. If both sets rng $g_1 \cap \beta^1$, rng $g_2 \cap \beta^1$, bridge β^1, then β^1 is said to be bridged twice. The bridging of β_1 is defined anologously. The three cases (not exclusive) are

Case one. Each k_j has Property $\sim P(a)$; neither of β^1, β_1 is bridged twice.

Case two. Some k_j have Property $\sim P(b)$; neither of β^1, β_1 is bridged twice.

Case three. One of β^1, β_1 is bridged twice.

Fig 54a.

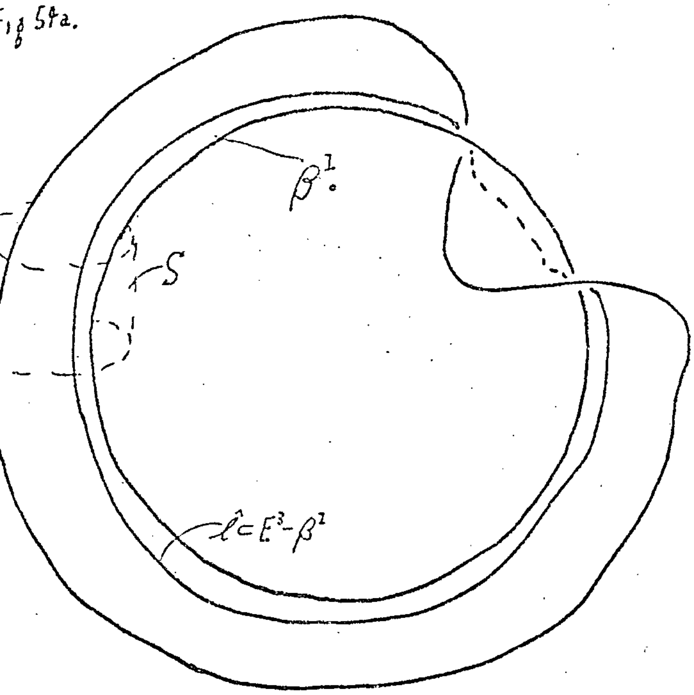
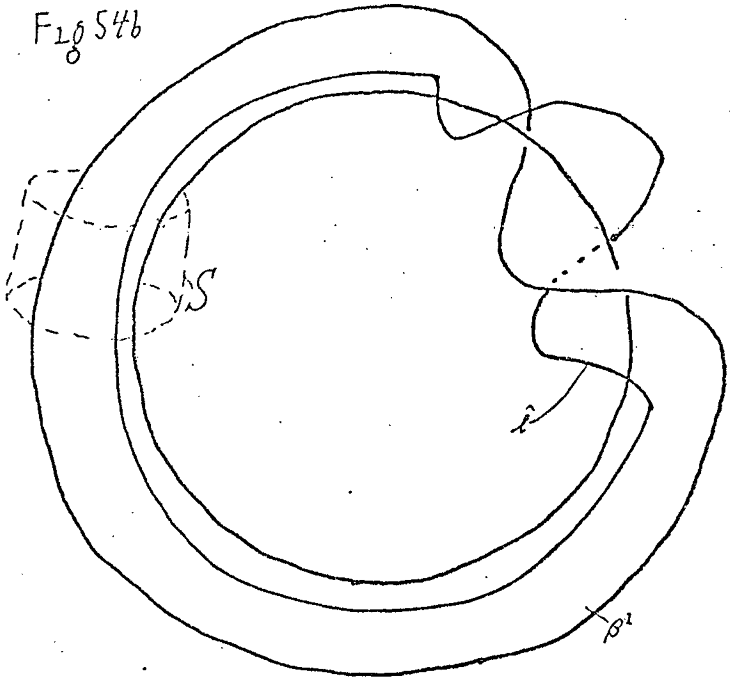


Fig 54b



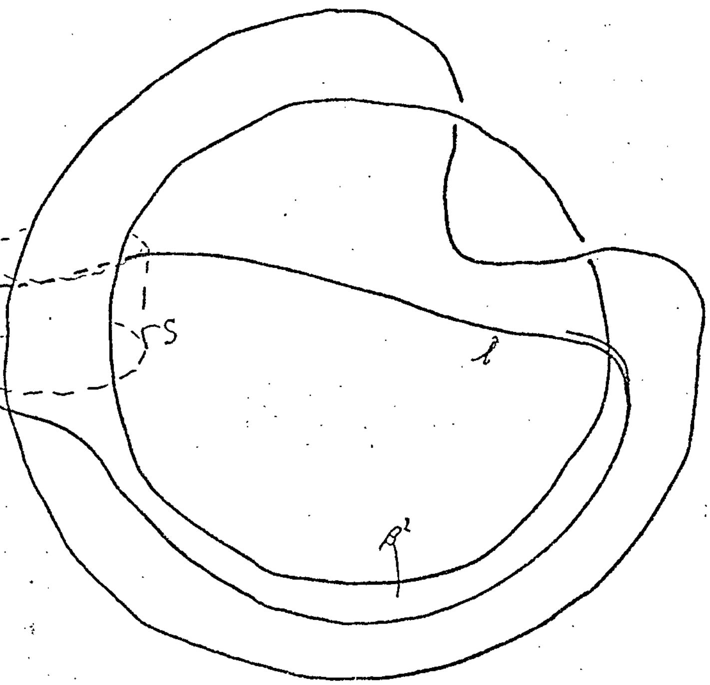


Fig 54c.

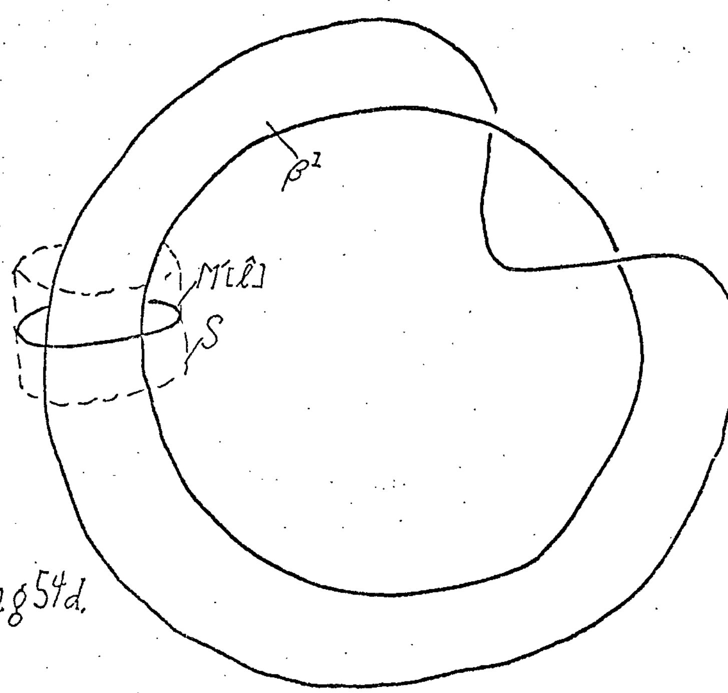


Fig 54d.

These cases are clearly exhaustive (taking 'one' in case three to mean 'at least one'; however the reader has probably noticed that if one of β^1, β_1 is bridged twice, the other cannot be bridged even once).

Case one. Since each k_j misses one rng g_i, this case suggests an immediate application of Lemma One. It is easily seen that the hypothesis of Lemma One is satisfied except for the fact that the rng g_i may hit Ω . If this happens, we alter the g_i by means of the following argument: assume that k_1 misses rng g_1 and k_4 misses rng g_2 (if another pair of k_j miss the rng g_i or if all four miss the same rng g_i, the method is similar or easier). Since ℓ_1 misses rng g_1, there is a circle $\ell \subset E^3 - \beta^1$ which lies near ℓ_1 and approximates it so that ℓ misses rng g_1 . We imagine ℓ sliding on the surface of the twisted band β^1 and eventually coming to rest directly over ℓ_4 . Although we use the term 'slide', we intend that ℓ stays close to but does not touch β^1 . By sliding ℓ on the side of β^1 which is free of the arcs a_j, we are assured that ℓ can move without touching the a_j . This shows that there is a homeomorphism M of E^3 onto itself such that M is fixed on $E^3 - K_1$ and on $\beta^1 \cup a_1 \cup a_2 \cup a_3 \cup a_4$; and carries ℓ to a position directly over $M[\ell_4] = \ell_4$. Clearly $M[\ell]$ misses rng Mg_1, ℓ_4 misses rng Mg_2 . Construct a small annulus α so that its boundary components are $M[\ell]$ and ℓ_4 . This can be done so that α misses k_1, k_2, k_3 and $k_4 - \ell_4$. By Th 5 (in Ch I), Int α contains a simple closed curve $\hat{\ell}$ which bounds no disk in α and which misses both rng Mg_i . Figs 54a, ..., d show how $\hat{\ell}$ may be moved to the location of an equator of S without hitting $\beta^1 \cup k_1 \cup k_2 \cup k_3 \cup k_4$.

This shows that there is a homeomorphism M' of E^3 onto itself which fixes β^1, every k_j, $E^3 - K_1$, and carries $\hat{\ell}$ onto the location $M'[\ell]$ shown in fig 54d. Evidently $M'[\hat{\ell}]$ misses both $M'Mg_i[\Delta]$; and in fact we can assume that all of Ω misses both $M'Mg_i[\Delta]$, since otherwise an obvious homeomorphism can be used to push the $M'Mg_i$ away from Ω. Note that the $M'Mg_i$ continue to be Z-disjoint and $M'Mg_i = c_i$ on $Bd\Delta$, while each k_j has Property $\sim P(a)$ with respect to the $M'Mg_i[\Delta]$ because both M' and M are fixed on each k_j. We can now apply Lemma One to construct Z-disjoint mappings $\bar{g}_i$ with $\bar{g}_i = c_i$ on $Bd\Delta$, such that $\text{rng}\ \bar{g}_i \subset \text{rng}\ M'Mg_i \cup K_1$ and (since Ω misses both $\text{rng}\ M'Mg_i$) $\text{rng}\ \bar{g}_i$ misses every k_j that $\text{rng}\ g_i$ misses. Since $\zeta_1 \subset \text{Int}\ S - N$, both $\text{rng}\ \bar{g}_i$ miss ζ_1. Since M and M' are fixed outside of K_1, $\text{rng}\ \bar{g}_i \subset \text{rng}\ g_i \cup K_1$. Now apply a result like Lemma One to 'the β_1 end' of $\bigcup_j k_j$ to construct Z-disjoint mappings g_i' such that $g_i = c_i$ on $Bd\Delta$, $\text{rng}\ g_i' \subset \text{rng}\ \bar{g}_i \cup K_2$, and $\text{rng}\ g_i'$ misses ζ_2 as well as any k_j that $\text{rng}\ \bar{g}_i$ misses. It may be necessary to alter the $\bar{g}_i$ with homeomorphisms which act like M, M' above, in order to make $\text{rng}\ g_i'$ miss those k_j which $\text{rng}\ \bar{g}_i$ misses. Evidently $\text{rng}\ g_i'$ misses both ζ_i and every k_j that $\text{rng}\ g_i$ misses. Since each k_j misses one $\text{rng}\ g_i$, the cradle of A has the required property.

Case two. In this case we allow some of the k_j to have Property $\sim P(b)$ with respect to the $\text{rng}\ g_i$. We reduce this case to Case one by converting the k_j with Property $\sim P(b)$ to Property $\sim P(a)$ or, more accurately, we will define mappings $G_{i0} = g_i$, G_{i1}, G_{i2}, G_{i3}, G_{i4} with the usual properties such that k_j has Property $\sim P(a)$ with respect to $\text{rng}\ G_{1j}$, $\text{rng}\ G_{2j}$, and in fact each k_j misses one $\text{rng}\ G_{i4}$.

The argument then reduces to Case one.

We will show how G_{i1} is constructed and indicate the construction of the other G_{ij}. If k_1 has Property $\sim P(a)$ with respect to the rng g_i, then let $g_i = G_{i0} = G_{i1}$. If k_1 has Property $\sim P(b)$ with respect to rng G_{1o}, rng G_{2o}, we assume that $a_1 \cup m_1$ misses rng $G_{1o} \cup$ rng G_{2o} and that ℓ_1 has Property $\sim P(b)$ (as defined in I §5 for a circles with basepoint $\ell_1 \cap a_1$), since otherwise we simply 'turn the picture upsidedown'. Now by Th 6 or Th 7 (in I §5), since at most one of rng $G_{1o} \cap \beta^1$, rng $G_{2o} \cap \beta^1$, bridges β^1 there is a circle $\ell' \subset \text{Int } \beta^1$ which bounds no disk in β^1, contains the base point $\ell_j \cap a_j$ and misses one of the rng G_{io}, say, rng G_{1o}. We now have a centre (or at least a double ended lasso) with Property $\sim P(a)$ since $\ell' \cup a_1 \cup m_1$ misses rng G_{1o}; but ℓ' is likely to be a ver disorderly circle and among other delinquencies, probably hits $k_2 \cup k_3 \cup k_4$ (which means that $\ell' \cup a_1 \cup m_1$ can't be used in Lemma One (the construction of R in Lemma One absolutely requires disjoint ℓ_j). We get disjoint loops and a picture like fig 44 by the following procedure which recalls the manipulation of ℓ in Case one. Let λ be a simple closed curve which lies near and approximates ℓ' but misses β^1. A short straight arc a connects $\ell_1 \cap a_1$ to a base point on λ so that a meets λ at only one point. We can assume that $\lambda \cup a$ misses rng G_{10}, and we now regard $\lambda \cup a \cup a_1 \cup m_1$ as a double ended lasso which misses rng G_{10} = rng g_1. Now slide λ over β^1, keeping the base point fixed, so that the final position of λ is directly over ℓ_1. As before, we choose the 'right' side of β^1 to slide λ on so that λ will miss $a_1 \cup a_2 \cup a_3 \cup a_4$.

We now have a double ended lasso which looks like k_1 except that the upper loop rides near but not on β^1, and it remains only to telescope $a \cup \lambda$ so that a collapses and λ moves to the location of ℓ_1. We conclude that there is a homeomorphism M'' of E^3 onto itself which fixes $E^3 - K_1$, k_2, k_3, k_4, and carries $\lambda \cup a \cup a_1 \cup m_1$ onto k_1. Evidently $M''G_{i0}$ has the required properties of G_{i1}. Clearly k_1 misses G_{11}, and since $\text{rng } G_{i1} \cap k_2 = \text{rng } G_{i0} \cap k_2, = k_2$ continues to have Property $\sim$P with respect to the $\text{rng } G_{i1}$, and a similar argument applies to k_3, k_4. Since M is not fixed on β^1, we must ask how the $\text{rng } G_{i1}$ bridge β^1. It is clear that β^1 is bridged at most once by the $\text{rng } G_{i1}$, since ℓ_1, which misses $\text{rng } G_{11}$, separates the boundary components of β^1; and of course β_1 is bridged by the $\text{rng } G_{i1}$ just as it was bridged by the $\text{rng } G_{i0}$. Thus neither of β^1, β_1 is bridged twice by the $\text{rng } G_{i1}$. If k_2 misses one of $\text{rng } G_{11}$, $\text{rng } G_{21}$, then let $G_{i1} = G_{i2}$. Otherwise k_3 has Property $\sim$P(b) with respect to the $\text{rng } G_{i1}$, and we construct G_{i2} so that $\text{rng } G_{i2} \cap (k_1 \cup k_3 \cup k_4) = \text{rng } G_{i1} \cap (k_1 \cup k_3 \cup k_4)$, and k_2 misses one of the $\text{rng } G_{i2}$ (note that we may have to work at the lower end of the figure; the fact that neither band β^1 or β_1 is bridged twice by the $\text{rng } G_{i1}$ is used in the second application of Th 6 or Th 7). Evidently k_1 misses one of the $\text{rng } G_{i2}$. Proceeding in the same way, we define G_{i3} so that k_3 misses one $\text{rng } G_{i3}$, and since G_{i3} can be constructed so that $\text{rng } G_{i3} \cap (k_1 \cup k_2) = \text{rng } G_{i2} \cap (k_1 \cup k_2)$ each of k_1, k_2 misses one $\text{rng } G_{i3}$. Finally define G_{i4} so that each k_j misses one $\text{rng } G_{i4}$. Evidently the G_{i4} can be used in the argument of Case one to construct the g_i'. When altering the g_i to G_{i1}, G_{i1} to G_{i2},

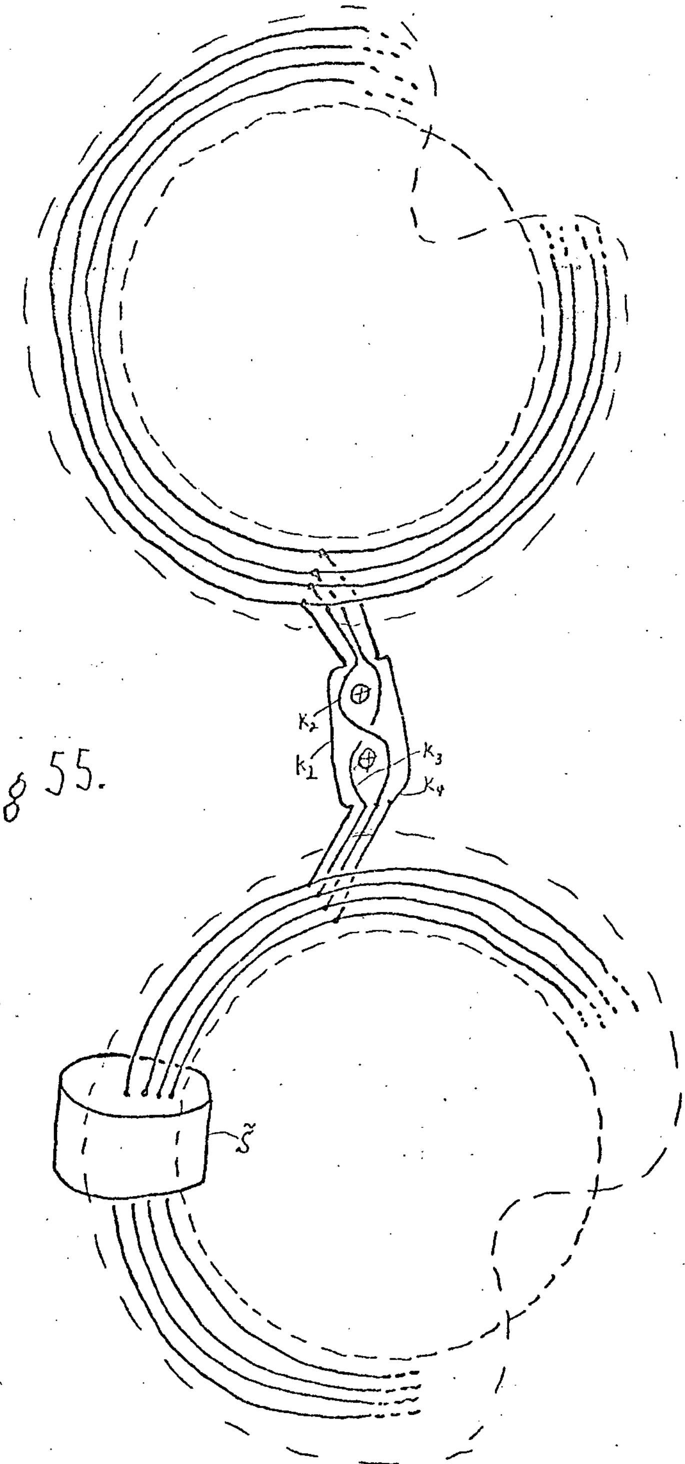


Fig 55.

etc., we preserve 'Z-disjointness' because we adjust only points in Z. Similarly each $G_{ij} = c_i$ on $\mathrm{Bd}\Delta$.

Case three. In this case we know only that one of β^1, β_1, say β^1 is bridged twice. It is easy to see that if β^1 is bridged twice, then no k_j can have Property $\sim P(a)$. For this would mean that some ℓ_{j_0} misses, say, rng g_1; then by I(1.7), rng $g_1 \cap \beta^1$ cannot bridge β^1, so that the number of bridges is at most one. But if each k_j has Property $\sim P(b)$, then in every case, m_j must miss rng $g_1 \cup$ rng g_2 and ℓ_j must have Property $\sim P(b)$. For evidently if any $\ell_j \subset E^3$ - rng g_1 - rng g_2, then there can be no bridges at all. We are thus led to the conclusion that when case three holds, there is just one possible configuration (assuming that β^1 is bridged twice): β^1 is bridged twice, β_1 is bridged not even once, and each k_j has Property $\sim P(b)$ with respect to rng g_1, rng g_2, with $m_j \cup a_j \subset E^3$ - rng g_1 - rng g_2. Except for the fact that ζ_2 may not miss rng $g_1 \cup$ rng g_2, the picture begins to resemble fig 45, (though we still must construct the arcs u_{13}, etc.). We first alter the g_i so that ζ_2 misses rng $g_1 \cup$ rng g_2. This is done just as in Case one. Fig 55 shows the k_j and a sphere $\tilde{S}$ placed in the usual way with respect to ζ_2 and the m_j. In fig 55, the k_j do not have Property $\sim P(a)$, so that we use Lemma One itself and not the corollary. Using the method of Case one, construct Z-disjoint mappings $\bar{g}_i : \Delta \to (\mathrm{rng}\ g_i \cup K_2) - \zeta_2$ such that rng $\bar{g}_i$ misses every m_j that rng g_i misses. This simply means that rng $\bar{g}_1 \cup$ rng $\bar{g}_2$ misses each m_j. An examination of the method of Case one shows that if rng g_i misses a_j, so does rng $\bar{g}_i$;

thus $\text{rng } \bar{g}_1 \quad \text{rng } \bar{g}_2$ misses all four $m_j \cup a_j$. We also know that each point of ℓ_j which misses $\text{rng } g_i$ also misses $\text{rng } \bar{g}_i$; this means that ℓ_j has Property $\sim P(b)$ with respect to $\text{rng } \bar{g}_1$, $\text{rng } \bar{g}_2$. Therefore the four k_j have Property $\sim P$ with respect to the rrng $\bar{g}_i$. On the other hand, the fact that the inclusion $\text{rng } \bar{g}_i \subset (\text{rng } g_i \cup K_2) - \zeta_2$ may be proper means that the number of bridges on β^1 with respect to $\text{rng } \bar{g}_1 \cap \beta^1$, $\text{rng } \bar{g}_2 \cap \beta^1$ may not be two, but may be one or zero. If this happens, then, since the number of bridges on β_1 with respect to $\text{rng } \bar{g}_1 \cap \beta_1$, $\text{rng } \bar{g}_2 \cap \beta_1$ is zero (because of the presence of, say, $m_1 \subset E^3 - \text{rng } \bar{g}_1 - \text{rng } \bar{g}_2$, using a previous argument) we have reduced the situation to either Case one or Case two, i.e. we have each k_j with Property $\sim P$ with respect to the $\text{rng } \bar{g}_i$ and neither β^1 nor β_1 is bridged twice. However in the 'worst case', β^1 continues to be bridged twice.

If β^1 is bridged twice by the $\text{rng } \bar{g}_i \cap \beta^1$, then we use Lemma Two. The hypothesis of Lemma Two is satisfied except that we must construct u_{12}, u_{13}, v_{12}, v_{13} . Since $\text{rng } \bar{g}_2 \cap \beta^1$ bridges β^1, there is a component Q of $\text{rng } \bar{g}_2 \cap \beta^1$ which connects the boundary components of β^1 . Q is compact and misses $\text{rng } \bar{g}_1$. By the definition of Property $\sim P(b)$, Q meets a continuum $e_1 \subset \ell_1$ and a continuum e_2 in ℓ_2 such that e_i contains $a_i \cap \ell_i$ and misses one of $\text{rng } \bar{g}_1$, $\text{rng } \bar{g}_2$. Since e_i hits $Q \subset \text{rng } \bar{g}_2$, e_i must miss $\text{rng } \bar{g}_1$. Since the whole continuum $e_1 \cup Q \cup e_2$ misses $\text{rng } \bar{g}_2$, we use I(2.5) to construct an arc u_{12} which joins $\ell_1 \cap a_1$ and $\ell_2 \cap a_2$ in β^1 and misses $\text{rng } \bar{g}_1$. The constructions of u_{13}, v_{12}, v_{13} are similar.

Now by Lemma Two there are Z-disjoint mappings $g_i':\Delta \to (\text{rng } \overline{g}_i - \text{Int } \Sigma) \cup n$ (where Σ, n are the sets described in Lemma Two) with $g_i' = \overline{g}_i = c_i$ on $\text{Bd}\Delta$, and such that one $\text{rng } g_i'$, say $\text{rng } g_1'$, misses $b_1 \cup b_2 \cup b_3$ while both $\text{rng } g_i'$ miss $\zeta_1 \cup \zeta_2$. Evidently $\text{rng } g_i' \subset \text{rng } g_i \cup K_1$.

In the argument of Case three we did not suceed in constructing the g_i' so that $\zeta_1 \cup \zeta_2$ misses $\text{rng } g_1' \cup \text{rng } g_2'$ and each b_j misses one $\text{rng } g_i'$; instead $\zeta_1 \cup \zeta_2$ misses both $\text{rng } g_i'$ and three b_j miss the same $\text{rng } g_i'$. In Part II of the proof of (3.1), it turns out that it is sufficient to define the g_i' so that three b_j miss the same $\text{rng } g_i'$ (the same thing happens in the proof of [12, Th 10]). With some additional complication, it is possible to improve the argument of Lemma Two so as to yield the usual result, i.e. to construct g_i' so that each b_j misses one $\text{rng } g_i'$; however we omit this argument.

We have now completed the three cases of the proof of Part I of (3.1). Note that in each Case, we constructed g_i' so that $\text{rng } g_i' \subset \text{rng } g_i \cup K_1 \cup K_2$. Thus we can write $\text{rng } g_i' \subset \text{rng } g_i \cup A$. This will be important when we apply the argument of (3.1) to the components of $\mathcal{a}_2$, $\mathcal{a}_3$, etc. To summarize the situation: if $\{A_1, A_2, A_3, A_4\}$ has Property $\sim$Q with respect to g_1, g_2, then there exist Z-disjoint mappings $g_i':\Delta \to E^3$ such that

> $g_i' = c_i$ on $\text{Bd}\Delta$,
>
> $\text{rng } g_i' \subset \text{rng } g_i \cup A$,
>
> if $\zeta_1 \cup \zeta_2 \cup b_1 \cup b_2 \cup b_3 \cup b_4$ is the cradle of A, then both $\text{rng } g_i'$ miss $\zeta_1 \cup \zeta_2$ and either each b_j misses one $\text{rng } g_i'$ or three b_j miss the same $\text{rng } g_i'$.

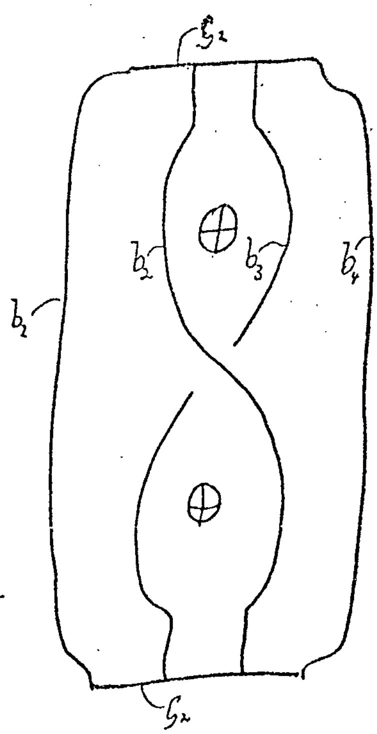
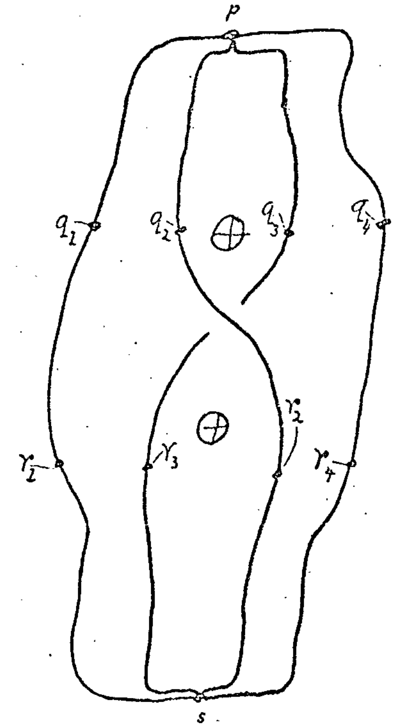


Fig 56.

Each arc pq_ir_js lies near $\zeta_1 \cup b_j \cup \zeta_2$



Part II of the proof of (3.1).

We remind the reader that we are proving a result much like Bing's Th 10 of [12], which is also divided into a Part I and Part II. Our Part II is very similar to Part II in Bing's proof and we absolutely require familiarity in detail with Bing's Part II (this is only a matter of half a page). We think it likely that the reader sees from the proof in [12], how to complete Part II here, and instead of a formal proof, we will give what amounts to a gloss on Bing's method, plus a few comments required by the fact that our Property Q is not quite identical to Bing's.

We begin by replacing $\zeta_1 \cup b_1 \cup \ldots \cup b_4 \cup \zeta_2$ by the figure $\bigcup_j pq_j r_j s$ shown in fig 56. This can be done so that either each arc $pq_j r_j s$ misses one rng g_i' or three $pq_j r_j s$ miss the same rng g_i'. Our terminology is now like that of [12] except that rng g_i' replaces D_i in [12]. We follow the division into cases found in [12]. We will not prove that the three cases given in [12] exhaust the possibilities, but remark for plausibility that the case division ... 1) Three $pq_j r_j s$ miss one rng g_i', 2) $pq_1 r_1 s$ plus $pq_2 r_2 s$ misses rng g_1', $pq_3 r_3 s$ plus $pq_4 r_4 s$ misses rng g_2', 3) $pq_1 r_1 s$ plus $pq_4 r_4 s$ misses rng g_1'; $pq_2 r_2 s$ plus $pq_3 r_3 s$ misses rng g_2' ... seems at first glance to ignore the possibility: $pq_1 r_1 s$ plus $pq_3 r_3 s$ misses rng g_1', $pq_2 r_2 s$ plus $pq_4 r_4 s$ misses rng g_2'. However this last variation is just Case Two with the diagram inverted. We will now describe how Bing's Part II can be altered to show that there exist Z-disjoint mappings $F_i : \Delta \to E^3$ such that $F_i = c_i$ on BdΔ and the centre of A has Property $\sim$P with respect to rng F_1, rng F_2.

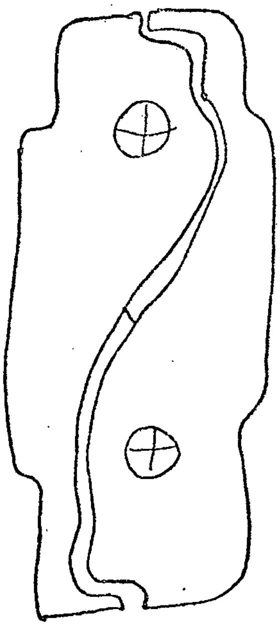


Fig 57

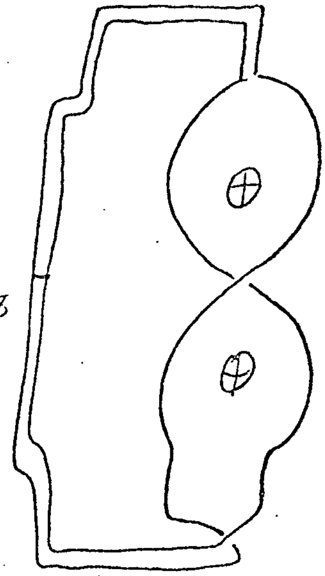


Fig 58

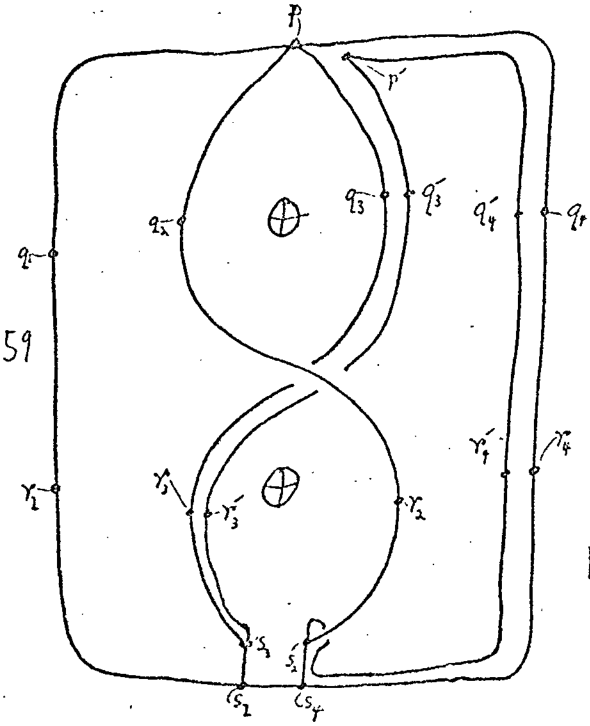


Fig 59

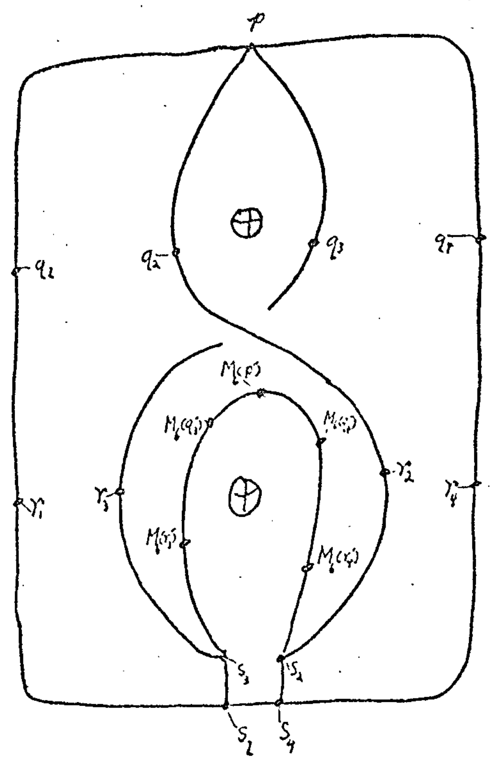
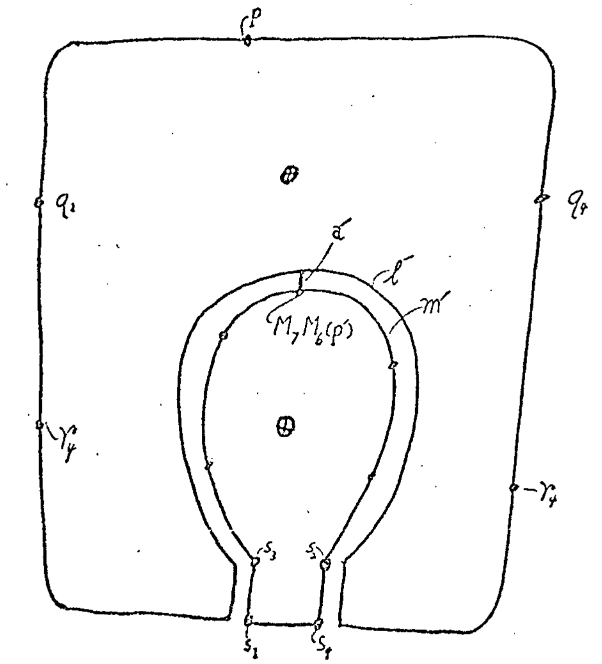


Fig 60.

Case One: any three of pq_jr_js $(j = 1, 2, 3, 4)$ miss the same $\mathrm{rng}\ g'_{i_o}$. If pq_2r_2s is an arc which fails to miss $\mathrm{rng}\ g'_{i_o}$, then the structure shown in fig 57 lies near $pq_1r_1s \cup pq_3r_3s \cup pq_4r_4s$ and misses $\mathrm{rng}\ g'_{i_o}$. The structure in fig 57 can be moved to the position of the centre k of A by a homeomorphism M_5 which fixes $E^3 - A$. Evidently $M_5g'_1$, $M_5g'_2$ are the required F_1, F_2. If pq_4r_4s is an arc which fails to miss $\mathrm{rng}\ g'_{i_o}$, then one uses the structure in fig 58 which lies near $pq_1r_1s \cup pq_2r_2s \cup pq_3r_3s$ and misses $\mathrm{rng}\ g'_{i_o}$. If pq_1r_1s or pq_3r_3s fail to miss $\mathrm{rng}\ g'_{i_o}$, the method is like one of those already given. If all four pq_jr_js miss $\mathrm{rng}\ g'_{i_o}$, then 'forget' one of them.

Case Two. pq_1r_1s plus pq_2r_2s misses $\mathrm{rng}\ g'_1$, pq_3r_3s plus pq_4r_4s misses $\mathrm{rng}\ g'_2$. We replace $\bigcup_j pq_jr_js$ with the more complicated construction in fig 59. In fig 59, s has been replaced by s_1, s_2, s_3, s_4 which lie near s so that the s_j and arcs s_1s_3, s_1s_4, s_2s_4 miss $\mathrm{rng}\ g'_1 \cup \mathrm{rng}\ g'_2$. Abusing the notation slightly, we have arcs $pq_jr_js_j$ with $pq_1r_1s_1 \cup pq_2r_2s_2 \subset E^3 - \mathrm{rng}\ g'_1$, $pq_3r_3s_3 \cup pq_4r_4s_4 \subset E^3 - \mathrm{rng}\ g'_2$. We build two new arcs: $p'q'_3r'_3s_3$, which lies near $pq_3r_3s_3$ and misses $\mathrm{rng}\ g'_2$, and $p'q'_4r'_4s_2$ which lies near $pq_4r_4s_4$ and also misses $\mathrm{rng}\ g'_2$. Apply a move M_6 which carries $s_3r'_3q'_3p'q'_4r'_4s_4$ to the location shown in fig 60 and fixes $pq_jr_js_j$, s_1s_3, s_2s_4, and s_1s_4. Look at a disk in A bounded by the circle $pq_1r_1s_1s_3M_6(r'_3)M_6(q'_3)M_6(p')M_6(q'_4)M_6(r'_4)s_2r_2q_2p$. We will call this disk T and assume that it is just the obvious disk suggested by the figure. Thus T misses all but the end points of $pq_4r_4s_4s_2$. Later we will need the fact that T can be constructed so as also to miss all but the end points of $pq_3r_3s_3$ (in Case 3). There is an arc

Fig 61.



$\lambda \subset T$ with end points s_2 and s_3 which misses both rng $M_6 g_i'$ because arc $s_3 M_6(r_3')M_6(q_3')M_6(p')M_6(q_4')M_6(r_4')s_2$ misses rng $M_6 g_2'$ and arc $s_3 s_1 r_1 q_1 p q_2 r_2 s_2$ misses rng $M_6 g_1'$. (We will now begin to abbreviate our arc nomenclature). Define a move M_7 which moves λ to the position of arc $s_3 M_6(p')s_2$ and fixes each $pq_j r_j s_j$ and $s_3 s_1 s_4 s_2$. Although we do not know the location of λ in T, this can be done by means of the A - move defined in I §3. Evidently rng $M_7 M_6 g_1'$ misses $pq_1 r_1 s_1$, rng $M_7 M_6 g_2'$ misses $pq_4 r_4 s_4$, and both rng $M_7 M_6 g_i'$ miss the circle $s_1 s_3 M_6(p')s_2 s_4 s_1$. Fig 61 shows $pq_1 r_1 s_1 \cup pq_4 r_4 s_4 \cup s_1 s_3 M_6(p')s_2 s_4 s_1$ replaced by a set $\ell' \cup a' \cup m'$ which lies very near the first set so that $m' \cup a'$ misses both rng $M_7 M_6 g_i'$, and ℓ' has Property $\sim P(b)$ with respect to the rng $M_7 M_6 g_i'$ (it is easy to give ℓ' this property since much of ℓ' can coincide with $s_1 r_1 q_1 p q_4 r_4 s_4$). Evidently $\ell' \cup a' \cup m'$ can be moved to the position of the centre $\ell \cup a \cup m$ of A. If this is accomplished by a move M_8, then the centre of A has Property $\sim P(b)$ with respect to the rng $M_8 M_7 M_6 g_i'$, which we define to be the required F_i.

Case three. $pq_1 r_1 s$ plus $pq_4 r_4 s$ misses rng g_1', $pq_2 r_2 s$ plus $pq_3 r_3 s$ misses rng g_2'. The mechanism of this case resembles that of Case two. We repeat the construction in fig 59 and define M_6 precisely as in Case two, so that we arrive once more at fig 60. However, since the rng g_i are related differently to the various parts of the figure, we have this time: $s_3 s_1 s_4 s_2 \subset E^3$ - rng $M_6 g_1$ - rng $M_6 g_2$ as usual, but $pq_1 r_1 s_1 \cup pq_4 r_4 s_4 \cup M_6(p')M_6(r_4')s_2 \subset E^3$ - rng $M_6 g_1$, $pq_2 r_2 s_2 \cup pq_3 r_3 s_3 \cup M_6(p')M_6(r_3')s_3 \subset E^3$ - rng $M_6 g_2$. In this case we

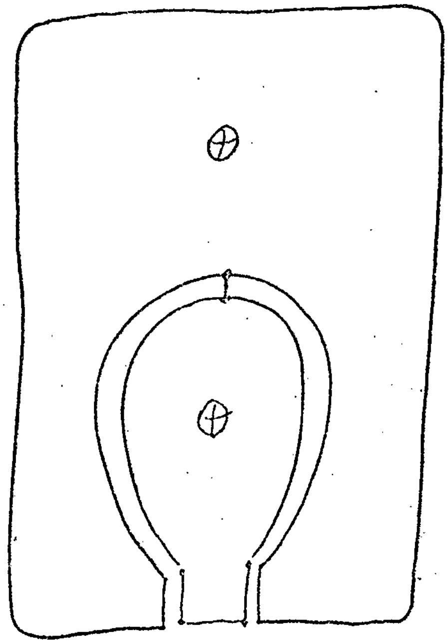


Fig 62.

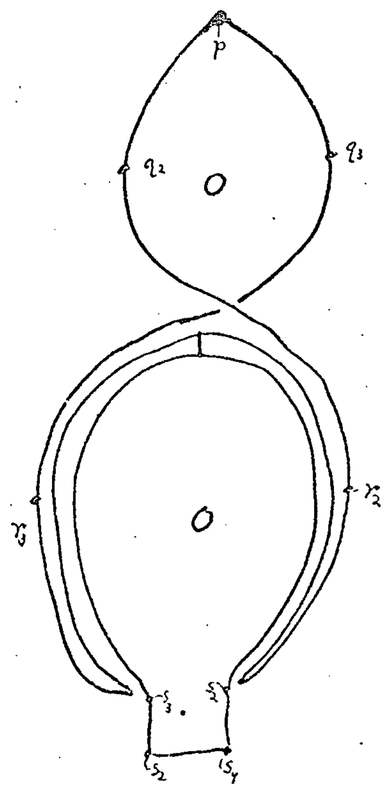


Fig 63.

must use a fact that we stated but did not completely use in Case 2, viz. that M_6 fixes all four pq_jr_js. We assume tht the disk T is placed so as to miss $pq_3r_3s_3$. We use Th 4 from I §4 at this point; at the analogous place in [12], Th 7 of [12] is used. By Th 4, since $s_3M_6(q_3')M_6(p')$ and $pq_2r_2s_2$ miss rng M_6g_2 and $M_6(p')M_6(q_4')s_2$ and $s_3s_1r_1q_1p$ miss rng M_6g_1, there is an arc $\overline{\lambda} \subset T$ with end points s_3, s_2, such that $\overline{\lambda}$ misses either rng M_6g_1 or rng M_6g_2. Apply a move M_7', similar to M_7, to move $\overline{\lambda}$ to $s_3M_6(p')s_2$. This can be done by an A - move as before; but some care should be taken so that M_7' fixes every $pq_jr_js_j$ (as well as, of course $s_3s_1s_4s_2$); the reader might first prefer to move $pq_3r_3s_3$ to a new location where it cannot interfere with the collar of T used in the A - move. The proof is now completed along the lines of the previous cases, using the fact that if $s_3M_6(p')s_2$ misses rng $M_7'M_6g_1$, then the set shown in fig 62 lying near $s_1s_3M(p')s_2s_4s_1 \cup s_1r_1q_1pq_4r_4s_4$ misses rng $M_7M_6g_1$; while if $s_3M_6(p')s_2$ misses rng $M_7M_6g_2$, then the set in fig 63 lying near $s_1s_3M_6(p')s_2s_4s_1 \cup s_3r_3q_3pq_2r_2s_2$ misses rng $M_7M_6g_2$. This completes part II of the proof of III(2.1) $\square$.

Corollary to III(2.1). If $\{A_1, \ldots A_4\}$ has Property $\sim Q$ with respect to mappings g_1, g_2, then A has Property $\sim Q$ with respect to mappings F_1, F_2 such that rng $F_i \subset$ rng $g_i \cup A$.

Proof. We know that rng $g_i' \subset$ rng $g_i \cup A$. And all the moves given in Part II of the proof of III(2.1) can be defined so as to fix $E^3 - A$ $\square$.

4. Proof of II(2.2).

We have now shown that if $\{A_1, \ldots A_4\}$ has Property $\sim Q$ then A has Property $\sim Q$. Our argument now diverges somewhat from Bing's in [12]. Suppose that $f_i : \Delta \to E^3$ such that $f_i|_{Bd\Delta} = c_i$ and the f_i are Z-disjoint (we continue to take Z, c_1, c_2 to be assigned arbitrarily according to the remark at the beginning of §3). We will show that if A has Property Q, then each of a_1, a_2, a_3, ... has Property Q; the proof of II(2.2) follows directly from this fact. If A has Property Q, then by (3.1), $\{A_1, \ldots A_4\}$ has Property Q. i.e. a_1 has Property Q. (3.1) does not imply that some A_j has Property Q for the luminous reason that each A_j has Property $\sim Q$, as the argument in §1 Ex 2 shows. However we can show that a_2 has Property Q by adapting the argument of the proof of (3.1) to show that for each A_j, if $\{A_{j1}, \ldots A_{j4}\}$ has Property $\sim Q$ then so does A_j. This is easy to do since the proof is simply restated in terms of images under the embedding h_j of various subsets of A. Occasionally in the proof of III(2.1) we constructed arcs which were perpendicular to certain surfaces. While h_j does not preserve this property, the reader will appreciate that we used such constructions for topological purposes, e.g. to make one arc lie along another, or to miss certain subsets, and these properties are preserved by h_j. We do not re-define Z, c_1, c_2 of course, since we intend to show that the same Property Q_{Z, c_1, c_2} is possessed by each of a_1, a_2, a_3, ... We originally defined Z to contain A so that we have $Z \supset A_j$ as required. We intend of course to let $Z = A$ eventually. To show that a_2 has Property Q, assume that the set of components of a_2 has Property $\sim$Q with respect to qualified mappings g_1, g_2. Apply a result like the corollary of (3.1) to $\{A_{11}, \ldots A_{14}\}$ to obtain Z-disjoint mapping $F_{i1}:\Delta \to \text{rng } g_i \cup A_1$ such that $F_{i1} = c_i$ on BdΔ, and A_1 has Property $\sim$Q with respect to the F_{i1}. We can see that since rng F_i does not exceed rng f_i in $E^3 - A_1$, the dogbones $A_{21}, \ldots A_{24}, A_{31}, \ldots A_{34}, A_{41}, \ldots A_{44}$ continue to have Property $\sim$Q with respect to the F_{i1}, for as we saw earlier, possession of Property $\sim$P depends on the fact that rng g_i misses certain continua in various dogbones, and this property is inherited by rng F_i at least for dogbones in $E^3 - A_1$. Construct Z-disjoint mappings $F_{i2}:\Delta \to$ rng $F_{i1} \cup A_2$ such that $F_{i2} = c_i$ on BdΔ and A_2 has Property $\sim$Q with respect to the F_{i2}. Once again, dogbones in $E^3 - A_2$ which have Property $\sim$Q with respect to the F_{i1} continue to have Property $\sim$Q with respect to the F_{i2}. This means that not only A_2, but $A_1, A_{31}, \ldots A_{34}, A_{41}, \ldots A_{44}$ have Property $\sim$Q with respect to the F_{i2}. Evidently we can continue in this way and finally derive Z-disjoint mappings $F_{i4}:\Delta \to E^3$ which agree with c_i on BdΔ and with respect to which, all of $A_1, \ldots A_4$ have Property $\sim$Q. Assume that the set of components of a_3 have Property $\sim$Q. Then an argument like that of (3.1) Corollary can be applied to each A_{jk} in (perhaps lexicographic) order to show eventually that a_2 has Property $\sim$Q. If A has Property Q, then by induction, a_1, a_2 have Property Q and a_3 must also have Property Q. We think that it is now evident how to proceed in the case that $m = 4, 5, \ldots$.

We will show how the induction argument above implies II(2.2). If the f_i in the hypothesis have ranges that intersect in A, then II(2.2) is true; thus we consider only the case that rng $f_1 \cap$ rng $f_2 \cap A = \emptyset$, i.e. the case that the f_i are A - disjoint. In the preceding argument we showed that for a fixed choice of Z, c_1, c_2, if A has Property Q_{Z,c_1,c_2}, then so does each $\mathcal{a}_m$. If $Z = A$ and c_1, c_2 are the c_i in II(2.2) then $A \subset Z \subset E^3 - \text{rng } c_1 - \text{rng } c_2$ as required, and A has Property Q by an argument like that of §1 Ex 1. By the induction argument, every $\mathcal{a}_m$ has Property Q_{Z,c_1,c_2}. As we saw earlier, this means that both rng f_i hit some component of $\mathcal{a}_m$ for m however large.

Finally we will show that both rng f_i must hit a big element Λ of the dogbone decomposition G . Let $\hat{G}$ be the set of all elements of the dogbone construction (i.e. all components of $\mathcal{a}_1, \mathcal{a}_2, \mathcal{a}_3, \ldots$) which meet both rng f_1 and rng f_2. Evidently $\hat{G}$ is infinite, for by the arguments of this chapter, each $\mathcal{a}_m$ must contain an element of $\hat{G}$. Clearly one of $A_1, \ldots A_4$ must contain an infinite subset of $\hat{G}$, for the four A_j contain all of $\hat{G}$. If A_j contains an infinite subset of $\hat{G}$, then one of $A_{j1}, \ldots A_{j4}$, say A_{jk}, contains an infinite subset of $\hat{G}$. There is a sequence $A \supset A_j \supset A_{jk} \supset A_{jk\ell} \ldots$ each of which contains infinitely many dogbones which meet both rng f_i. Obviously each member of the sequence meets both rng f_i, and the intersection $A \cap A_j \cap A_{jk} \cap A_{jk\ell} \cap \ldots$ meets both rng f_i. One can also use the dogbone metric to show that if the images of the rng f_i are disjoint in $\mathcal{D}$, then there is a neighbourhood system of the points of $\mathcal{D}$ consisting of small 3-cells around the small points and images of dogbones about the big points such that no neighbourhood of diameter smaller than ε (in the dogbone metric) meets both images of the rng f_i. This implies that some $\mathcal{a}_m$ has Property $\sim Q$, cf. proof of Th 12 of [12] $\square$.

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Appendix.

Figs 64a, ... j show how to deform the upper part of Fig 19 so that it looks like the upper part of Fig 1 of [12].

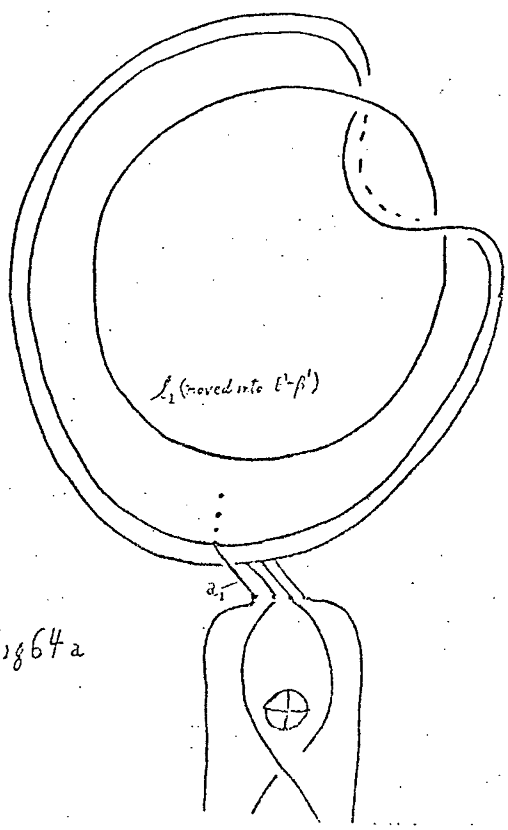

Fig 64a

Push ℓ_1 off β^1. Move a_1 so that a_1 passes through β^1 and now approaches β^1 from the upper side, while a_2, a_3, a_4 continue to approach from the <u>lower</u> side.

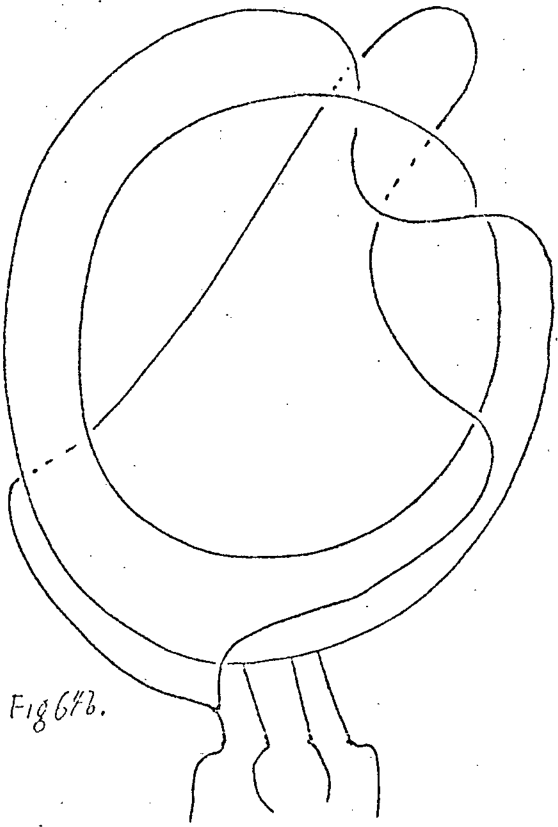

Fig 64b.

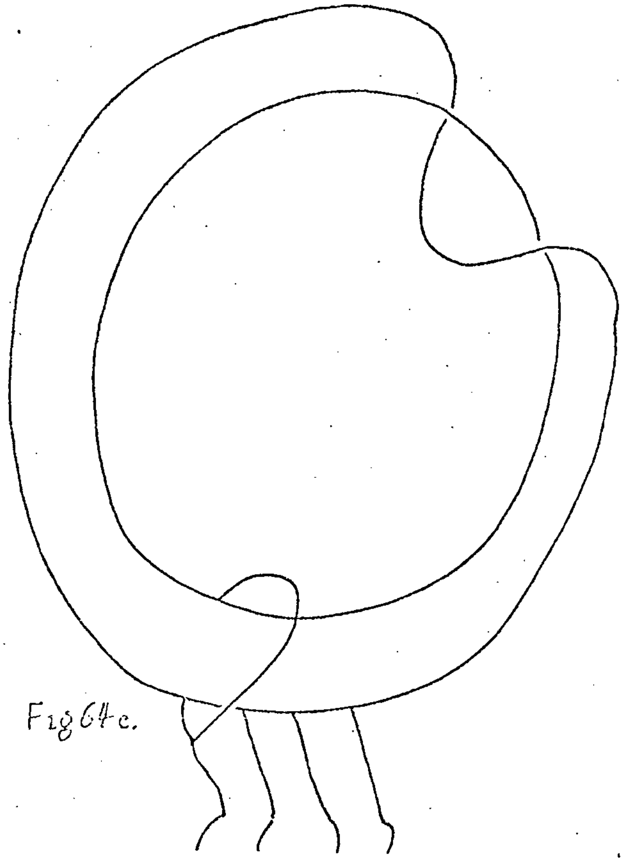

Fig 64c.

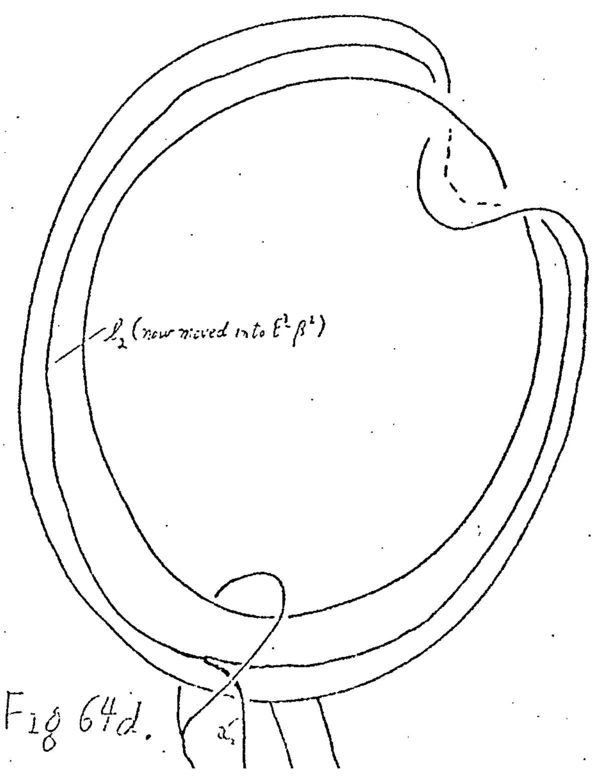

Fig 64d.

Push off ℓ_2 from β^1.
Bring a_2 through β^1
so that a_2 now approaches
β^1 from 'above.

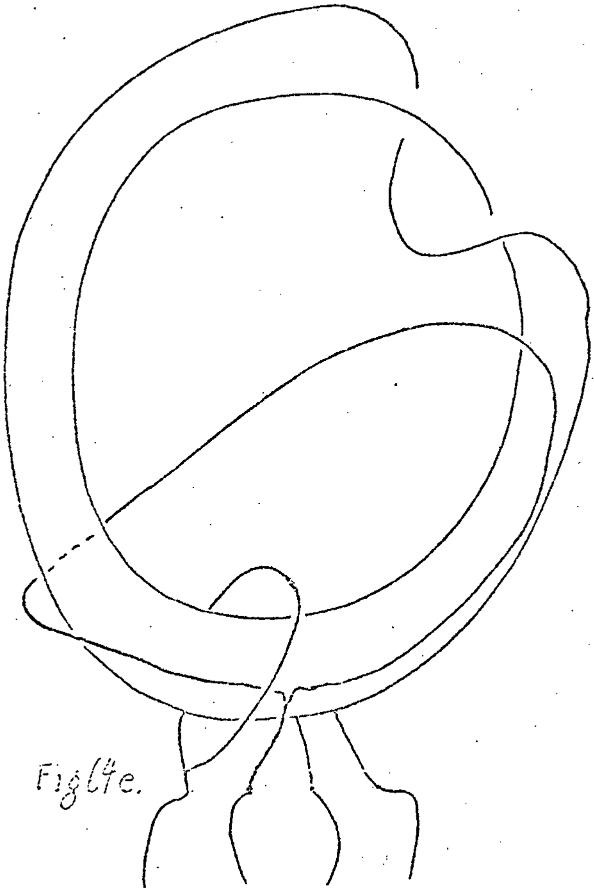

Fig 64e.

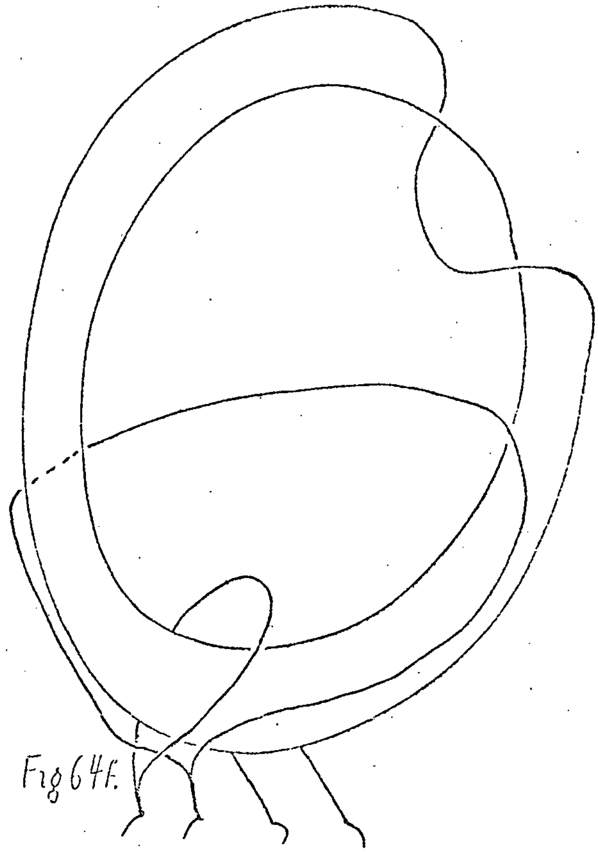

Fig 64f.

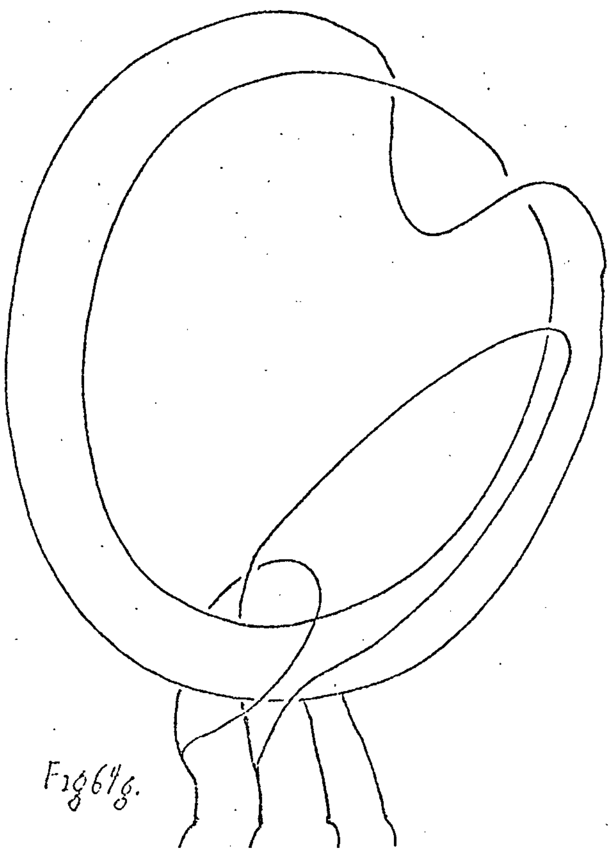

Fig 64g.

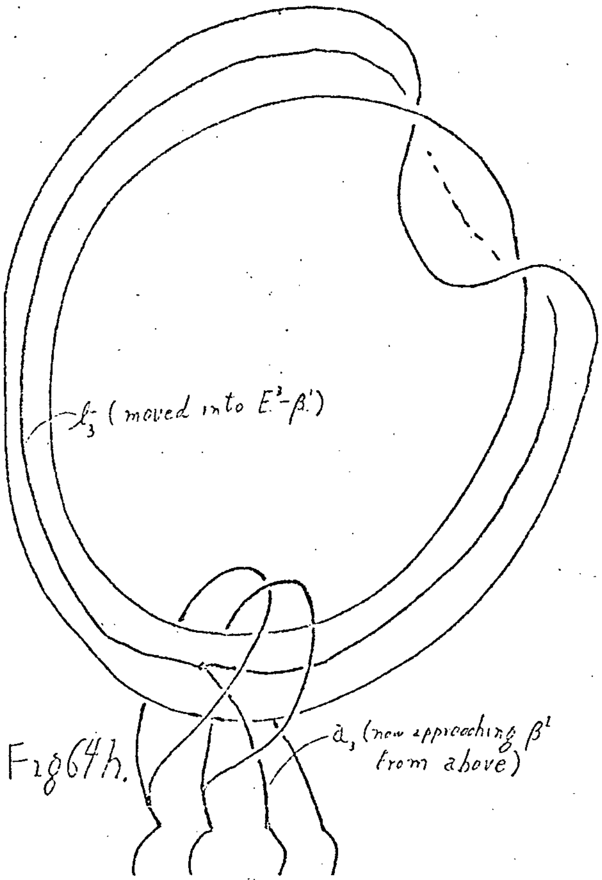

Fig 64h.

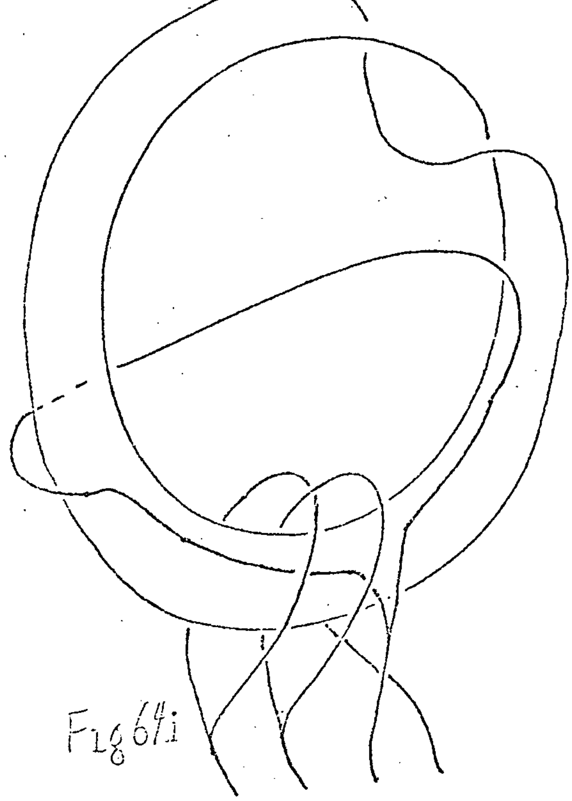

Fig 64i

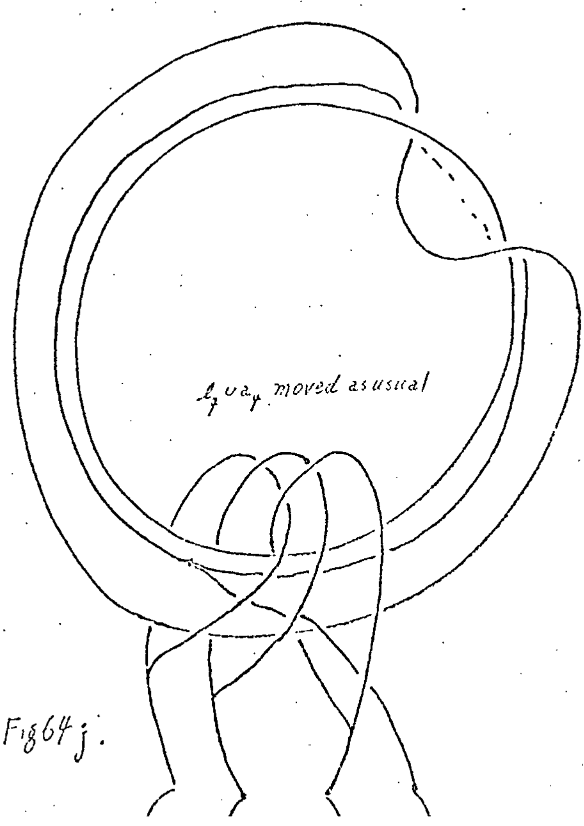

Fig 64j.

Etc.