**DYNAMICS AND CONTROL OF A FLEXIBLE TETHERED SYSTEM
WITH OFFSET**

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Abstract

A mathematical model of a platform based flexible tethered satellite system in an arbitrary orbit, undergoing planar motion, is obtained using the Lagrangian procedure. The governing equations of motion account for the platform and tether pitch, longitudinal tether oscillations, offset of the tether attachment point as well as deployment and retrieval of the tether.

A numerical parametric study of the highly nonlinear, nonautonomous and coupled equations of motion gives considerable insight into the system dynamics useful in its design. Of particular interest are the interactions involving orbital eccentricity, system librations, tether flexibility and offset, retrieval maneuvers and initial disturbances. Results show that the offset strongly couples tether and platform dynamics, and the resulting responses show high frequency modulations corresponding to the longitudinal tether oscillations. The system was found to be unstable during retrieval. The Linear Quadratic Regulator based offset control strategy, in conjunction with the platform mounted momentum gyros, is proposed to alleviate the situation. Results show that a strategy involving independent parallel control of low and high frequency responses can damp rather severe disturbances in a fraction of an orbit.

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List of Symbols

$[A]$	coefficient matrix of $\mathbf{x}$ and $\bar{\mathbf{x}}$
a_p	altitude at perigee
α_p	platform pitch angle
α_t	tether pitch angle
$[B], [\bar{B}]$	coefficient matrix of $\mathbf{u}$ and $\bar{\mathbf{u}}$
C	system center of mass
$[C]$	coefficient matrix of $\dot{\mathbf{q}}$
d_x, d_z	horizontal and vertical offsets, respectively
D_x, D_z	nondimensionalized offsets; $d_j/l_b, j = x, z$
$\mathbf{d}$	vector of offsets, $\mathbf{d} = d_x \mathbf{i}_p + d_z \mathbf{k}_p$
e	orbit eccentricity
ϵ	tether strain variable
G	universal gravitational constant
h_K	orbit constant
$\mathbf{i}_j, \mathbf{j}_j, \mathbf{k}_j$	unit vectors in frame $F_j, j = i, c, p, t$
I_{xx}, I_{yy}, I_{zz}	platform inertias
$[I], [0]$	identity and zero matrices, respectively
$[K]$	coefficient matrix of $\mathbf{q}$, stiffness matrix
$\mathbf{l}, l$	instantaneous tether line vector and magnitude, respectively
$\bar{l}$	nominal unstretched tether length
$L, \bar{L}$	nondimensional forms of l and $\bar{l}$; $L = l/l_b, \bar{L} = \bar{l}/l_b$
l_b	initial nominal tether length
M_e	mass of earth

$[M]$	coefficient matrix of $\ddot{\mathbf{q}}$, mass matrix
M_s	subsattellite mass
M_r	reel mass
M_t	deployed tether mass
M_p	platform mass
M_{srt}	$M_s + M_r + M_t$
M_{prt}	$M_p + M_r + M_t$
M	total mass of the system, $M = M_p + M_r + M_t + M_s$
$\bar{\mathbf{P}}, \mathbf{P}$	retrieval and eccentricity influence vectors, respectively
$\mathbf{q}$	vector of generalized coordinates
$[Q]$	matrix of weights for control variables
$[R]$	matrix of weights for state variables
ρ	tether line density
τ	nondimensional platform wheel torque
T	system kinetic energy
U_g, U_s	gravitational and strain energies, respectively
U	system potential energy, $U = U_g + U_s$
$\mathbf{u}$	vector of control variables
ω_t	angular velocity of the tether frame

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Chapter 1

Introduction

Mankind's venture into space began with the Soviet launch of Sputnik in 1957. Since then, the frequency and variety of missions has increased steadily. Apollo 11 sent men to the moon and back in 1969. Cosmonauts regularly work for months at a time aboard the space station Mir. The recently launched Galileo spacecraft is on its way to orbit Jupiter. Pioneer 11 is still transmitting signals to earth as it leaves the confines of our solar system. It seems clear that our fascination with space will continue to increase in the future.

The unusual low gravity environment of space allows for the use of unusual structures. Though most objects sent into space are compact and rigid, there is a trend towards larger more flexible satellites. The proposed U.S. space station, for example, would not be able to support its own weight on the earth's surface. Tethered satellite systems represent perhaps the most extreme case of both size and flexibility in a structure.

A tethered satellite is basically composed of two or more masses joined together by one or more tethers. A tether can be any rope-like object which offers only longitudinal tension. That is, a tether has little resistance to bending. It is fairly clear that such a structure has value only if the tether remains taught at all times. The principle which makes this state relatively easy to maintain is that of the gravity gradient (Figure 1.1). Essentially the combination of gravity and centrifugal forces combine to produce a force which tends to keep the system aligned along the local vertical while maintaining tension in the tether. A useful account of the basic physical principles governing tethered systems

is given by Arnold [1].

The proposed uses for tethered systems are surprisingly varied. A detailed description of many of them can be found in a N.A.S.A. report compiled by Cron [2]. A few of these are:

- transferring cargo;
- power generation (conducting tether);
- micro-gravity laboratory;
- atmospheric studies of planets;
- testing aerodynamic designs (flying wind tunnel).

In their utmost generality, tethered satellite systems possess very complex dynamics. Consider even the simple two body system of Figure 1.1. The end masses have their own rigid body degrees of freedom and may also be flexible. The tether, which may be deployed or retrieved also has rigid body degrees of freedom and may undergo longitudinal and transverse oscillations. Any offset of the tether attachment point from the center of mass of either end body introduces strong coupling between the above degrees of freedom. The entire system moves around an oblate earth being subjected to aerodynamic drag, solar radiation and other environmental forces. Modeling such a complex system is challenging. A compromise must be struck between retaining desired characteristics and making the problem amenable to useful study. A review of past investigations is given by Misra and Modi [3]. Much of the effort has been directed towards understanding the dynamics and developing control strategies to eliminate oscillations in the system. During retrieval, small disturbances can grow to the point where the tether wraps itself around the platform. The methods developed fall into three categories. Tension control

Figure 1.1: Tethered Satellite showing the working principle

was the first to be utilized. Here oscillations are controlled simply by changing the tension in the tether. Rupp [4] and Fan et al. [5] are among the many investigators who have demonstrated successful control through this approach. Unfortunately it is not effective when the tether tension becomes small, which can occur during longitudinal oscillations and with short tether lengths. The second method uses thrusters located on the subsatellite to guarantee tension in all conditions. However it is not recommended for short tether lengths because of possible damage to the platform by thruster plumes. The final method uses controlled motion of the tether attachment point. Lakshmanan and Modi [6] have shown this method effective for a platform based system. The study however, neglected flexibility and assumed the platform and system centers of mass to be coincident.

In the present study, a mathematical model of a two body tethered satellite system is considered which includes longitudinal flexibility of the tether and a movable offset of the tether attachment point. The kinetic and potential energy of the system are derived and the equations of motion obtained using the Lagrangian procedure [7]. A numerical parametric analysis is performed to study the uncontrolled dynamics (Chapter 3). Control of the system is considered using the tether offset and a platform mounted momentum wheel. Control gains are obtained using the Linear Quadratic Regulator approach (Chapter 4). Conclusions are drawn and recommendations for future work made (Chapter 5).

Chapter 2

Mathematical Model

2.1 System Description

2.1.1 Introduction

Figure 2.1 shows schematically the satellite system being considered. There are two main bodies joined by a tether. The platform may have an arbitrary three dimensional inertia distribution. The subsatellite is considered a point mass since in most proposed applications it is significantly smaller and less massive than the platform. The tether is treated as a continuum with longitudinal flexibility and may be deployed or retrieved at any specified rate. The tether reel mass is also included and is treated as a point mass. There were two reasons for including the reel mass in the formulation. The first was to conserve the total mass of the system. For example, during retrieval the tether mass decreases while the reel mass increases at the same rate. Secondly, since motion of the attachment point is to be considered, the presence of this mass may have an effect on the dynamics.

Degrees of freedom of the system include platform pitch, tether pitch, longitudinal tether vibration and controlled motion of the tether attachment point. In addition, the center of mass of the entire system follows an orbit of arbitrary eccentricity and altitude. Motion out of the plane of the orbit is not considered. It has been shown that for small oscillations, inplane and out of plane motions decouple and so may be studied separately [6].

Figure 2.1: Platform based Tethered Satellite System (TSS).

2.1.2 Reference Frames

Four reference frames are introduced in order to establish the orientation of the system with respect to an inertial reference (Figure 2.2). The inertial frame F_i is fixed to the earth's center with z_i axis passing through the perigee. The orbital frame F_c is fixed to the center of mass of the orbiting system with z_c axis along the local vertical. The platform frame F_p is fixed to the center of mass of the platform with axes along the principal axes of the platform. Finally the tether frame F_t is fixed at the attachment point with z_t axis along the tether.

2.1.3 Position Vectors

Using the reference frames described above, the location of any mass element can be represented as a sum of position vectors. From the inertial frame, the vector $\mathbf{R}_c$ locates the system center of mass. From the origin of the orbital frame, $\mathbf{R}_p$ locates the platform center of mass and $\mathbf{R}_s$ positions the subsatellite. With reference to the platform frame, $\mathbf{r}_p$ locates a platform mass element dm_p and $\mathbf{d}$ establishes the location of the tether attachment point. From the tether frame, $\mathbf{r}_t$ locates a tether mass element dm_t . Thus for example, the position of the reel mass at any time is given by $\mathbf{R}_c + \mathbf{R}_p + \mathbf{d}$ (Figure 2.3).

Note, the vector $\mathbf{r}_t$ must be a function of the mass element being considered as well as a function of time. That is, given a particular mass element of the tether, $\mathbf{r}_t$ changes to follow its longitudinal oscillations. To account for this the vector $\mathbf{r}_t$ is expressed as,

$$\mathbf{r}_t(z_t, t) = [z_t + w(z_t, t)]\mathbf{k}_t,$$

where

$$w(z_t, t) = \epsilon(t) \sum_{n=1}^N \phi_n(z_t).$$

Here $\phi_n(z_t)$ are independent functions satisfying the geometric boundary condition,

$$\mathbf{r}_t(0, t) = \mathbf{0},$$

For the relatively short tether lengths considered in this study, the strain variation can be approximated as linear [8]. This gives

$$w(z_t, t) = \epsilon(t)z_t.$$

Thus the position of a mass element before deformation, say $z_t \mathbf{k}_t$, is located after deformation by

$$\mathbf{r}_t(z_t, t) = z_t[1 + \epsilon(t)]\mathbf{k}_t,$$

and for $z_t = \bar{l}$,

$$\mathbf{l} = \bar{l}[1 + \epsilon]\mathbf{k}_t.$$

2.1.4 Generalized Coordinates

Before obtaining the equations of motion it is necessary to choose a set of generalized coordinates for the problem. Generalized coordinates are independent angles or displacements used to specify the orientation of a system.

There are five generalized coordinates chosen for this problem. The true anomaly θ is measured in radians from the line joining the earth's center to the perigee of the orbit (Figure 2.2). The radial distance r is measured in meters from the earth's center to the center of mass of the system. These two coordinates keep track of the orbital position of the system in an arbitrary orbit. The platform pitch angle α_p is measured in radians from the local vertical to the z_p axis. The tether pitch angle α_t is measured in radians from the local vertical to the z_t axis. Note, the rotations are considered positive in the clockwise sense. The variable ϵ described above is used to monitor the difference

Figure 2.2: Reference frames and generalized coordinates.

Figure 2.3: Position vectors.

between the actual tether length l and its nominal unstretched value $\bar{l}$. ϵ is defined by the expression:

$$l = \bar{l}(1 + \epsilon).$$

Notice,

$$\epsilon = \frac{l - \bar{l}}{\bar{l}} = \frac{\text{change in length}}{\text{original length}}$$

which is the expression for strain in a stretched wire.

The offset of the tether attachment point is considered a specified quantity and so is not included in the list of generalized coordinates

2.1.5 Constraints

The number of variables in the formulation can be reduced by utilizing various equality constraints. From mass considerations, the following relationships are clear.

$$\begin{aligned} M &= M_s + M_p + M_r + M_t. \\ M_r + M_t &= \text{constant}. \end{aligned} \tag{2.1}$$

From the geometry in Figure 2.2, the following relationship holds,

$$\mathbf{R}_s = \mathbf{R}_p + \mathbf{d} + \mathbf{l}. \tag{2.2}$$

Finally the definition of the center of mass of a system gives another constraint. Setting the first moment of mass about C to zero leads to the following vector equation,

$$M_s \mathbf{R}_s + \int (\mathbf{R}_p + \mathbf{r}_p) dm_p + M_r (\mathbf{R}_p + \mathbf{d}) + \int (\mathbf{R}_p + \mathbf{d} + \mathbf{r}_t) dm_t = 0.$$

This simplifies to

$$\mathbf{R}_p = -\frac{1}{M} [M_{srt} \mathbf{d} + M_s \mathbf{l} + \rho \int_0^{\bar{l}} \mathbf{r}_t dz_t], \tag{2.3}$$

with the use of equations (2.2), and (2.1) and the following relations:

$$\begin{aligned}\int \mathbf{R}_p dm_p &= \mathbf{R}_p \int dm_p = \mathbf{R}_p M_p; \\ \int \mathbf{r}_t dm_t &= \rho \int_0^l \mathbf{r}_t dz_t; \\ \int \mathbf{r}_p dm_p &= 0.\end{aligned}$$

The last equation is true since the F_p frame has its origin at the center of mass of the platform.

2.2 Nonlinear Equations of Motion

2.2.1 Kinetic Energy

The kinetic energy of a general mechanical system is given by

$$T = \frac{1}{2} \int \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} dm,$$

where $\mathbf{r}$ locates the mass element dm and integration is over all such elements.

Recall that the system studied here consists of a platform, tether, reel mass and sub-satellite. Integrating over each of these separately, the kinetic energy can be written as

$$\begin{aligned}T &= \frac{1}{2} \int (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{r}}_p) \cdot (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{r}}_p) dm_p \\ &+ \frac{1}{2} \int (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{d}} + \dot{\mathbf{r}}_t) \cdot (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{d}} + \dot{\mathbf{r}}_t) dm_t \\ &+ \frac{1}{2} M_r (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{d}}) \cdot (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{d}}) \\ &+ \frac{1}{2} M_s (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{d}} + \dot{\mathbf{l}}) \cdot (\dot{\mathbf{R}}_c + \dot{\mathbf{R}}_p + \dot{\mathbf{d}} + \dot{\mathbf{l}}).\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}
T = & \frac{1}{2}M(\dot{\mathbf{R}}_c \cdot \dot{\mathbf{R}}_c) + \frac{1}{2}M(\dot{\mathbf{R}}_p \cdot \dot{\mathbf{R}}_p) + M_{srt}(\dot{\mathbf{R}}_p \cdot \dot{\mathbf{d}}) \\
& + \frac{1}{2}M_{srt}(\dot{\mathbf{d}} \cdot \dot{\mathbf{d}}) + \frac{1}{2} \int \dot{\mathbf{r}}_p \cdot \dot{\mathbf{r}}_p dm_p + M_s(\dot{\mathbf{R}}_p \cdot \dot{\mathbf{i}}) \\
& + M_s(\dot{\mathbf{d}} \cdot \dot{\mathbf{i}}) + \frac{1}{2}M_s(\dot{\mathbf{i}} \cdot \dot{\mathbf{i}}) + \int \dot{\mathbf{R}}_p \cdot \dot{\mathbf{r}}_p dm_p \\
& + \int \dot{\mathbf{R}}_p \cdot \dot{\mathbf{r}}_t + \dot{\mathbf{d}} \cdot \dot{\mathbf{r}}_t + \frac{1}{2}\dot{\mathbf{r}}_t \cdot \dot{\mathbf{r}}_t dm_t \\
& + \dot{\mathbf{R}}_c \cdot [M\dot{\mathbf{R}}_p + M_{srt}\dot{\mathbf{d}} + M_s\dot{\mathbf{i}} + \int \dot{\mathbf{r}}_t dm_t + \int \dot{\mathbf{r}}_p dm_p].
\end{aligned}$$

The last term in the above expression is equal to $-\dot{l}\bar{l}\rho$. This can be verified by taking the time derivative of equation (2.3) and applying Leibnitz's rule for differentiating the integral with $\bar{l}$ as an upper limit. Each of the remaining terms is written in terms of the generalized coordinates of the problem. To illustrate, the term $\frac{1}{2}M_s(\dot{\mathbf{i}} \cdot \dot{\mathbf{i}})$ is rewritten here.

Since

$$\mathbf{l} = l\mathbf{k}_t,$$

differentiating with respect to time in the inertial frame gives

$$\dot{\mathbf{l}} = \dot{l}\mathbf{k}_t + l(\boldsymbol{\omega}_t \times \mathbf{k}_t). \quad (2.4)$$

Now from the geometry of the problem,

$$\mathbf{k}_t = [-\sin(\theta - \alpha_t), 0, -\cos(\theta - \alpha_t)],$$

and

$$\boldsymbol{\omega}_t = [0, \dot{\theta} - \dot{\alpha}_t, 0].$$

Using these in equation (2.4) gives

$$\dot{\mathbf{l}} = [-\dot{l}\sin(\theta - \alpha_t) - l(\dot{\theta} - \dot{\alpha}_t)\cos(\theta - \alpha_t), 0, -\dot{l}\cos(\theta - \alpha_t) + l(\dot{\theta} - \dot{\alpha}_t)\sin(\theta - \alpha_t)].$$

Thus,

$$\frac{1}{2}M_s(\dot{\mathbf{i}} \cdot \dot{\mathbf{i}}) = \dot{l}^2 + l^2(\dot{\theta} - \dot{\alpha}_t)^2.$$

Continuing in this way, the kinetic energy for the system can be written as

$$\begin{aligned} T = & \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{M_{srt}M_p}{2M} \left[\dot{d}_x^2 + \dot{d}_z^2 + (d_x^2 + d_z^2)(\dot{\theta} - \dot{\alpha})^2 \right. \\ & - 2d_x\dot{d}_z(\dot{\theta} - \dot{\alpha}) + 2d_z\dot{d}_x(\dot{\theta} - \dot{\alpha}) \left. \right] + \frac{M_{prt}M_s}{2M} [\dot{l}^2 + l^2(\dot{\theta} - \dot{\beta})^2] \\ & + \frac{M_sM_p}{M} \left[d_x\dot{l}(\dot{\theta} - \dot{\alpha})\cos(\alpha_t - \alpha_p) + \dot{l}d_z(\dot{\theta} - \dot{\alpha})\sin(\alpha_t - \alpha_p) \right. \\ & - \dot{d}_x\dot{l}\cos(\alpha_t - \alpha_p) + d_xl(\dot{\theta} - \dot{\alpha})(\dot{\theta} - \dot{\beta})\sin(\alpha_t - \alpha_p) \\ & - d_zl(\dot{\theta} - \dot{\alpha})(\dot{\theta} - \dot{\beta})\cos(\alpha_t - \alpha_p) - \dot{l}d_x(\dot{\theta} - \dot{\beta})\cos(\alpha_t - \alpha_p) \\ & - \dot{d}_zl(\dot{\theta} - \dot{\beta})\sin(\alpha_t - \alpha_p) + d_x\dot{l}\sin(\alpha_t - \alpha_p) \left. \right] \\ & + \frac{M_p}{M}\rho \left[\left(\frac{1}{2}\dot{\epsilon}\bar{l}^2 + \dot{l}\bar{l} \right) (d_x(\dot{\theta} - \dot{\alpha})\cos(\alpha_t - \alpha_p) + d_z(\dot{\theta} - \dot{\alpha})\sin(\alpha_t - \alpha_p)) \right. \\ & - \dot{d}_x\cos(\alpha_t - \alpha_p) + \dot{d}_z\sin(\alpha_t - \alpha_p) \left. \right] + \frac{1}{2}\bar{l}\dot{l}(\dot{\theta} - \dot{\beta})(d_x(\dot{\theta} - \dot{\alpha})\sin(\alpha_t - \alpha_p) \\ & - d_z(\dot{\theta} - \dot{\alpha})\cos(\alpha_t - \alpha_p) - \dot{d}_x\cos(\alpha_t - \alpha_p) - \dot{d}_z\sin(\alpha_t - \alpha_p)) \left. \right] \\ & - \frac{M_s}{M}\rho \left[\left(\frac{1}{2}\dot{\epsilon}\bar{l}^2 + \dot{l}\bar{l} \right) \dot{l} + \frac{1}{2}l^2\bar{l}(\dot{\theta} - \dot{\beta})^2 \right] + \frac{1}{2}\rho \left[\frac{1}{3}\dot{\epsilon}^2\bar{l}^3 + \bar{l}\dot{l}^2(1 + \epsilon)^2 \right. \\ & + \frac{1}{3}l^2\bar{l}(\dot{\theta} - \dot{\beta})^2 + \dot{\epsilon}\dot{l}\bar{l}^2(1 + \epsilon) \left. \right] + \frac{1}{2}I_{yy}(\dot{\theta} - \dot{\alpha})^2 - \frac{\rho^2}{2M} \left[\frac{1}{4}\dot{\epsilon}^2\bar{l}^4 \right. \\ & + l^2\dot{l}^2 + \frac{1}{4}\bar{l}^2l^2(\dot{\theta} - \dot{\beta})^2 + \dot{\epsilon}\bar{l}^2\dot{l}\bar{l} \left. \right] - \rho\dot{l}\bar{l}(-\dot{r}\cos\alpha_t - r\dot{\theta}\sin\alpha_t) \\ & + \frac{1}{2M}l^2\dot{l}^2\rho^2 + \rho\dot{l} \left[-\frac{M_{srt}}{2M}l(d_x(\dot{\theta} - \dot{\alpha})\cos(\alpha_t - \alpha_p) \right. \\ & + d_z(\dot{\theta} - \dot{\alpha})\sin(\alpha_t - \alpha_p) + \dot{d}_x\sin(\alpha_t - \alpha_p) - \dot{d}_z\cos(\alpha_t - \alpha_p)) \\ & - \frac{M_s}{2M}l\dot{l} - \frac{1}{2M}\rho l \left(\frac{1}{2}\dot{\epsilon}\bar{l}^2 + \dot{l}\bar{l} \right) \left. \right]. \end{aligned}$$

2.2.2 Potential Energy

The potential energy of the system can be divided into the gravitational contribution associated with the masses (U_g), as well as the stored energy due to the elongation of the tether (U_s).

Gravitational Potential Energy

The gravitational potential energy of a general mechanical system is given by

$$U_g = - \int \frac{GM_e}{|r|} dm,$$

where G is the universal gravitational constant; M_e is the mass of the earth; and r locates the mass element dm of the system. Integration is over all such elements.

As before, integrating over the four regions of the system studied here gives

$$\begin{aligned} -\frac{1}{GM_e}U_g &= \int \frac{dm_p}{|\mathbf{R}_c + \mathbf{R}_p + \mathbf{r}_p|} + \frac{M_s}{|\mathbf{R}_c + \mathbf{R}_s|} \\ &+ \frac{M_r}{|\mathbf{R}_c + \mathbf{R}_p + \mathbf{d}|} + \int \frac{dm_t}{|\mathbf{R}_c + \mathbf{R}_p + \mathbf{d} + \mathbf{r}_t|}. \end{aligned} \quad (2.5)$$

Now,

$$\begin{aligned} \frac{1}{|\mathbf{R}_c + \mathbf{R}_s|} &= |r\mathbf{k}_c + \mathbf{R}_s|^{-1} \\ &= [(r\mathbf{k}_c + \mathbf{R}_s) \cdot (r\mathbf{k}_c + \mathbf{R}_s)]^{-\frac{1}{2}} \\ &= [r^2 + 2r(\mathbf{R}_s \cdot \mathbf{k}_c) + \mathbf{R}_s \cdot \mathbf{R}_s]^{-\frac{1}{2}} \\ &= \frac{1}{r} \left[1 + \frac{2(\mathbf{R}_s \cdot \mathbf{k}_c)}{r} + \frac{\mathbf{R}_s \cdot \mathbf{R}_s}{r^2} \right]^{-\frac{1}{2}} \\ &\approx \frac{1}{r} - \frac{\mathbf{R}_s \cdot \mathbf{k}_c}{r^2} + \frac{3(\mathbf{R}_s \cdot \mathbf{k}_c)^2 - \mathbf{R}_s \cdot \mathbf{R}_s}{2r^3}, \end{aligned}$$

where the binomial expansion is used, keeping only terms to order $1/r^3$.

After rewriting each of the quotients in (2.5) this way, the first terms from each quotient add to give the orbital potential energy (that is, the energy due to the position of the center of mass of the system). The second term in each quotient, which is of order $1/r^2$, vanishes due to the center of mass constraint. This leaves

$$\begin{aligned}
 -\frac{U_g}{GM_e} &\approx \frac{M}{r} + \frac{1}{r^3} \left[\frac{3}{2}(\mathbf{k}_c \cdot \mathbf{R}_s)^2 - \frac{1}{2}(\mathbf{R}_s \cdot \mathbf{R}_s)M_s \right. \\
 &+ \int \left(\frac{3}{2}(\mathbf{k}_c \cdot (\mathbf{R}_p + \mathbf{r}_p))^2 - \frac{1}{2}(\mathbf{R}_p + \mathbf{r}_p) \cdot (\mathbf{R}_p + \mathbf{r}_p) \right) dm_p \\
 &+ \int \left(\frac{3}{2}(\mathbf{k}_c \cdot (\mathbf{R}_p + \mathbf{d} + \mathbf{r}_t))^2 - \frac{1}{2}(\mathbf{R}_p + \mathbf{d} + \mathbf{r}_t) \cdot (\mathbf{R}_p + \mathbf{d} + \mathbf{r}_t) \right) dm_t \\
 &\left. + \left(\frac{3}{2}(\mathbf{k}_c \cdot (\mathbf{R}_p + \mathbf{d}))^2 - \frac{1}{2}(\mathbf{R}_p + \mathbf{d}) \cdot (\mathbf{R}_p + \mathbf{d}) \right) M_r \right].
 \end{aligned}$$

Collecting terms in common dot product gives

$$\begin{aligned}
 -\frac{U_g}{GM_e} &\approx \frac{M}{r} + \frac{1}{r^3} \left[-\frac{3}{2}M(\mathbf{k}_c \cdot \mathbf{R}_p)^2 + \frac{1}{2}M(\mathbf{R}_p \cdot \mathbf{R}_p) \right. \\
 &- \frac{3}{2}M_{srt}(\mathbf{k}_c \cdot \mathbf{d})^2 - 3M(\mathbf{k}_c \cdot \mathbf{d})(\mathbf{k}_c \cdot \mathbf{R}_p) + \frac{1}{2}M_{srt}(\mathbf{d} \cdot \mathbf{d}) \\
 &+ M(\mathbf{d} \cdot \mathbf{R}_p) + \frac{3}{2}M_s(\mathbf{k}_c \cdot \mathbf{l})^2 - \frac{1}{2}(\mathbf{l} \cdot \mathbf{l}) \\
 &+ \int \frac{3}{2}(\mathbf{k}_c \cdot \mathbf{r}_p)^2 - \frac{1}{2}(\mathbf{r}_p \cdot \mathbf{r}_p) dm_p \\
 &\left. + \int \frac{3}{2}(\mathbf{k}_c \cdot \mathbf{r}_t)^2 - \frac{1}{2}(\mathbf{r}_t \cdot \mathbf{r}_t) dm_t \right].
 \end{aligned}$$

Introducing the generalized coordinates results in the third-order approximation gives

$$\begin{aligned}
 U_g &\approx -\frac{GM_e M}{r} - \frac{GM_e}{r^3} \left[-\frac{3}{2M} \left(M_{srt}^2 (d_x \sin \alpha_p + d_z \cos \alpha_p)^2 \right. \right. \\
 &+ M_s^2 l^2 \cos^2 \alpha_t + \frac{1}{4} \rho^2 l^2 \bar{l}^2 \cos^2 \alpha_t + M_s \rho l^2 \bar{l} \cos^2 \alpha_t \\
 &\left. \left. - 2M_{srt} l \cos \alpha_t (d_x \sin \alpha_p + d_z \cos \alpha_p) (M_s + \rho \frac{\bar{l}}{2}) \right) \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2M}(M_{srt}^2(d_x^2 + d_z^2) + M_s^2 l^2 + \frac{1}{4}\rho^2 l^2 \bar{l}^2) \\
& + M_{srt} l (d_x \sin(\alpha_t - \alpha_p) - d_z \cos(\alpha_t - \alpha_p))(2M_s + \rho \bar{l}) + M_s l^2 \bar{l} \rho) \\
& - \frac{3}{2} M_{srt} (d_x \sin \alpha_p + d_z \cos \alpha_p)^2 + 3(d_x \sin \alpha_p \\
& + d_z \cos \alpha_p)(M_{srt}(d_x \sin \alpha_p + d_z \cos \alpha_p) - M_s l \cos \alpha_t - \frac{1}{2} \bar{l} l \cos \alpha_t) \\
& - \left(\frac{1}{2} M_{srt} (d_x^2 + d_z^2) + (d_x \sin(\alpha_t - \alpha_p) - d_z \cos(\alpha_t - \alpha_p))(M_s l \right. \\
& + \left. \frac{1}{2} \rho l \bar{l}) \right) + \frac{3}{2} M_s l^2 \cos^2 \alpha_t - \frac{1}{2} M_s l^2 + \frac{I_{yy}}{2} + \frac{I_{zz}}{2} \\
& - I_{xx} + \frac{3}{2} (I_{xx} - I_{zz}) \cos^2 \alpha_p + \frac{1}{2} \rho \bar{l} l^2 \cos^2 \alpha_t - \frac{1}{6} \rho \bar{l} l^2 \Big].
\end{aligned}$$

Strain Energy

The strain energy in a deformed, elastic body is given by

$$U_s = \frac{1}{2} \int \sigma \delta dV,$$

where σ is the stress in a differential element of volume dV , and δ represents the strain in a differential element of volume dV .

It is assumed that the properties of the tether are the same along its length. Now, by definition,

$$E = \frac{\sigma}{\delta},$$

where E is Young's Modulus. Hence

$$U_s = \frac{1}{2} \int E \delta^2 dV.$$

As shown by Nayfeh and Mook [9], given an element of the tether of initial length dz_t

with one end located at $z_t \mathbf{k}_t$ and the other at $[z_t + w(z_t, t)] \mathbf{k}_t$,

$$\begin{aligned}
 \delta|_{z_t} &= \left. \frac{\Delta \text{length}}{\text{length}} \right|_{z_t} \\
 \Rightarrow \delta|_{z_t} &= \lim_{dz_t \rightarrow 0} \frac{w(z_t + dz_t, t) - w(z_t, t) + dz_t - dz_t}{dz_t} \\
 &= \frac{\partial w(z_t, t)}{\partial z_t} \\
 &= \frac{\partial [\epsilon z_t]}{\partial z_t} \\
 &= \epsilon.
 \end{aligned}$$

So,

$$\begin{aligned}
 U_s &= \frac{1}{2} \int E \epsilon^2 dV \\
 &= \frac{1}{2} EA \int_0^l \epsilon^2 dz_t \\
 &= \frac{1}{2} EA \epsilon^2 l.
 \end{aligned}$$

The total potential energy of the system is now given to third order by,

$$U = U_g + U_s.$$

2.2.3 Lagrange's Method

In 1788 Lagrange published his book, Mécanique Analytique [7], in which he describes an energy approach to obtain equations of motion. The method requires writing the kinetic energy (T), and potential energy (U), in terms of a set of generalized coordinates (q_j). The equations of motion are then given by the following ordinary differential

equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} = Q_j, \quad (2.6)$$

where Q_j represents the effect of any external forces on the coordinate q_j . The result is one second order, ordinary differential equation related to each of the generalized coordinates. Substitution of the above energy expressions into equation (2.6) leads to the desired equations of motion. The equations for r and θ turn out to be the classical Keplerian equations with small perturbation terms due to the finite dimensions of the system. It is assumed that the effect of these terms on the system dynamics is negligible. Modi and Misra[10] have shown that even after a full year of longitudinal oscillations of a 1 km tethered system, the coupling effect on r and θ is small. Since this paper is concerned with disturbances which are quickly controlled, it is assumed that the effect of these terms is negligible. This assumption allows us to use Kepler's relations to change the independent variable of the problem from time to the true anomaly θ , which is more convenient for satellite problems.

Using:

$$\begin{aligned} r^2 \dot{\theta} &= h_K; \\ r &= \frac{h_K^2}{GM_e(1 + e \cos \theta)}; \end{aligned}$$

where h_K is constant for a given orbit (angular momentum per unit mass), one obtains the following substitution for time derivatives:

$$\begin{aligned} \frac{d}{dt} &= \frac{d\theta}{dt} \frac{d}{d\theta}; \\ \frac{d^2}{dt^2} &= \frac{d\theta^2}{dt} \left(\frac{d^2}{d\theta^2} - F \frac{d}{d\theta} \right); \end{aligned}$$

where

$$F = \frac{2e \sin \theta}{(1 + e \cos \theta)}.$$

Nondimensionalizing with respect to $M_s l_b^2 \frac{d\theta^2}{dt}$, the resulting nonlinear, nonautonomous and coupled equations are:

Platform Pitch Equation:

$$\begin{aligned} \frac{M_{srt} M_p}{2MM_s} [& -4(D_x \dot{D}_x + D_z \dot{D}_z)(1 - \dot{\alpha}_p) - 2(D_x^2 + D_z^2)(-F - \ddot{\alpha}_p + F \dot{\alpha}_p) \quad (2.7) \\ & - 2(\ddot{D}_x - F \dot{D}_x) D_z + 2(\ddot{D}_z - F \dot{D}_z) D_x] \\ & + \frac{M_p}{M} [-D_x(\ddot{L} - F \dot{L}) \cos(\alpha_t - \alpha_p) - 2D_x \dot{L} \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_t) \\ & - D_z(\ddot{L} - F \dot{L}) \sin(\alpha_t - \alpha_p) + 2D_z \dot{L} \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_t) \\ & - D_x L(-F - \ddot{\alpha}_t + F \dot{\alpha}_t) \sin(\alpha_t - \alpha_p) + D_x L(1 - \dot{\alpha}_t)^2 \cos(\alpha_t - \alpha_p) \\ & + D_z L(-F - \ddot{\alpha}_t + F \dot{\alpha}_t) \cos(\alpha_t - \alpha_p) + D_z L \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_t)^2] \\ & + \rho \frac{M_p}{MM_s} [(\frac{1}{2}(\ddot{\epsilon} - F \dot{\epsilon}) \bar{L}^2 + 2\dot{\epsilon} \bar{L} \dot{\bar{L}} \\ & + (1 + \epsilon)((\ddot{\bar{L}} - F \dot{\bar{L}}) \bar{L} + \dot{\bar{L}}^2))(-D_x \cos(\alpha_t - \alpha_p) - D_z \sin(\alpha_t - \alpha_p)) \\ & - (\frac{1}{2}\dot{\epsilon} \bar{L}^2 + L \bar{L})(1 - \dot{\alpha}_t)(D_x \sin(\alpha_t - \alpha_p) - D_z \cos(\alpha_t - \alpha_p)) \\ & + (\frac{1}{2}(-F - \ddot{\alpha}_t + F \dot{\alpha}_t) L \bar{L} + (1 - \dot{\alpha}_t) L \dot{\bar{L}} + \frac{1}{2} \bar{L}^2 \dot{\epsilon})(D_z \cos(\alpha_t - \alpha_p) \\ & - D_x \sin(\alpha_t - \alpha_p)) + \frac{1}{2}(1 - \dot{\alpha}_t)^2 L \bar{L} (D_z \sin(\alpha_t - \alpha_p) \\ & + D_x \cos(\alpha_t - \alpha_p))] - \frac{I_{yy}}{M_s l_b^2} (-F - \ddot{\alpha}_p + F \dot{\alpha}_p) \\ & - \frac{1}{M_s(1 + e \cos \theta)} [-\frac{3}{2} M (2M_{srt}^2 (D_x \sin \alpha_p - D_z \cos \alpha_p)(D_x \cos \alpha_p \\ & + D_z \sin \alpha_p - 2M_{srt} \cos \alpha_t (D_x \cos \alpha_p + D_z \sin \alpha_p)(M_s L + \frac{1}{2} \rho L \bar{L})) \\ & - \frac{M_{srt}}{2M} (2M_s L + \rho L \bar{L})(D_x \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p)) \end{aligned}$$

$$\begin{aligned}
& + 3M_{srt}(D_x \sin \alpha_p - D_z \cos \alpha_p)(D_x \cos \alpha_p + D_z \sin \alpha_p) \\
& + (M_s + \frac{1}{2}\rho\bar{L})L(D_x \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p)) \\
& - \frac{3}{M_s I_b^2}(I_{xx} - I_{zz}) \cos \alpha_p \sin \alpha_p \\
& - 3(D_x \cos \alpha_p + D_z \sin \alpha_p) \cos \alpha_t (M_s L + \frac{1}{2}\rho L \bar{L}) \\
& + \rho \dot{\bar{L}} [-\frac{M_{srt}}{2MM_s} L(-D_x \cos(\alpha_t - \alpha_p) - D_z \sin(\alpha_t - \alpha_p))] \\
& + \rho \dot{\bar{L}} [\frac{M_{srt}}{2MM_s} L(D_x \cos(\alpha_t - \alpha_p) - D_z \sin(\alpha_t - \alpha_p))] \\
& + \frac{M_{srt}}{2MM_s} L(\dot{D}_x \cos(\alpha_t - \alpha_p) - D_x \sin(\alpha_t - \alpha_p)(\dot{\beta} - \dot{\alpha}) \\
& + D_z \cos(\alpha_t - \alpha_p)(\dot{\beta} - \dot{\alpha}) + \dot{D}_z \sin(\alpha_t - \alpha_p)) = \tau;
\end{aligned}$$

Tether Pitch Equation:

$$\begin{aligned}
\frac{M_{srt}}{2M} \quad [& -2L^2(-F - \ddot{\alpha}_t + F\dot{\alpha}_t) - 4L\dot{L}(1 - \dot{\alpha}_t)] \\
& + \frac{M_p}{M} [(\ddot{D}_x - F\dot{D}_x)L \cos(\alpha_t - \alpha_p) + (\ddot{D}_z - F\dot{D}_z)L \sin(\alpha_t - \alpha_p) \\
& - 2L\dot{D}_x \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) + 2\dot{D}_z L \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) \\
& - D_x L(-F - \ddot{\alpha}_p + F\dot{\alpha}_p) \sin(\alpha_t - \alpha_p) + D_z L(-F - \ddot{\alpha}_p + \dot{\alpha}_p F) \cos(\alpha_t - \alpha_p) \\
& - D_x L \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2 - D_z L \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2] \\
& - \frac{M_p}{2MM_s} \rho \bar{L} L [-(\ddot{D}_x - F\dot{D}_x) \cos(\alpha_t - \alpha_p) - (\ddot{D}_z - F\dot{D}_z) \sin(\alpha_t - \alpha_p) \\
& + 2\dot{D}_x \sin(\alpha_t - \alpha_p)(\dot{\theta} - \dot{\alpha}) - 2\dot{D}_z \cos(\alpha_t - \alpha_p)(\dot{\theta} - \dot{\alpha}) + D_x(-F - \ddot{\alpha}_p \\
& + F\dot{\alpha}_p) \sin(\alpha_t - \alpha_p) - D_z(-F - \ddot{\alpha}_p + F\dot{\alpha}_p) \cos(\alpha_t - \alpha_p) \\
& + D_x \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2 + D_z \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2] \\
& + \frac{1}{M} \rho [(-F - \ddot{\alpha}_t F \dot{\alpha}_t) L^2 \bar{L} + 2(1 - \dot{\alpha}_t) L \dot{L} \bar{L} + (1 - \dot{\alpha}_t) L^2 \dot{\bar{L}}]
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
& - \frac{\rho}{2M} [2L^2 \dot{\bar{L}}(1 - \dot{\alpha}_t) + \frac{2}{3} L^2 \bar{L}(-F - \ddot{\alpha}_t + F\dot{\alpha}_t) + \frac{4}{3}(1 - \dot{\alpha}_t)\dot{\epsilon}] \\
& + \frac{\rho^2}{2MM_s} [2L^2 \bar{L}\dot{\bar{L}}(1 - \dot{\alpha}_t) + L\bar{L}^3 \dot{\epsilon}(1 - \dot{\alpha}_t) + \frac{1}{2} L^2 \bar{L}^2(-F - \ddot{\alpha}_t + F\dot{\alpha}_t)] \\
& - \frac{1}{M_s(1 + e \cos \theta)} \left[-\frac{3}{2MM_s} (-2 \cos \alpha_t \sin \alpha_t (M_s^2 L^2 + \frac{1}{4} \rho^2 L^2 \bar{L}^2 \right. \\
& + M_s \rho L^2 \bar{L}) + 2(D_x \sin \alpha_p - D_z \cos \alpha_p) M_{srt} \sin \alpha_t (M_s L + \frac{1}{2} \rho L \bar{L})] \\
& + \frac{M_{srt}}{2MM_s} [(2M_s L + \rho L \bar{L})(D_x \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p))] \\
& + 3(D_x \sin \alpha_p - D_z \cos \alpha_p) \sin \alpha_t (M_s L + \frac{1}{2} \rho L \bar{L}) - (M_s L \\
& + \frac{1}{2} \rho L \bar{L})(D_x \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p)) - 3M_s L^2 \cos \alpha_t \sin \alpha_t \\
& - \rho L^2 \bar{L} \cos \alpha_t \sin \alpha_t] - \rho \frac{\dot{\bar{L}}}{l_b} L(-\dot{r} \sin \alpha_t + r \dot{\theta} \cos \alpha_t) \\
& - \rho \dot{l} \left[-\frac{M_{srt}}{2M} L(-D_x(\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) \right. \\
& + D_z(\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) + \dot{D}_x \cos(\alpha_t - \alpha_p) + \dot{D}_z \sin(\alpha_t - \alpha_p))] = 0;
\end{aligned}$$

Tether Length Equation:

$$\begin{aligned}
\frac{M_{prt}}{M} \quad [& (\ddot{\bar{L}} - F\dot{\bar{L}})\bar{L} - L\bar{L}(1 - \dot{\alpha}_t)^2] + \frac{M_p}{M} [(\ddot{D}_x - F\dot{D}_x)\bar{L} \sin(\alpha_t - \alpha_p) \\
& - (\ddot{D}_z - F\dot{D}_z)\bar{L} \cos(\alpha_t - \alpha_p) + 2\dot{D}_x \bar{L} \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) \\
& + D_x \bar{L} \cos(\alpha_t - \alpha_p)(-F - \ddot{\alpha}_t + f\dot{\alpha}_t) + D_z \bar{L} \sin(\alpha_t - \alpha_p)(-F - \ddot{\alpha}_t + F\dot{\alpha}_t) \\
& - D_x \bar{L} \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2 + 2\dot{D}_z \bar{L} \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) \\
& + D_z \bar{L} \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2] + \frac{M_p}{2MM_s} \rho \bar{L}^2 [(\ddot{D}_x - F\dot{D}_x) \sin(\alpha_t - \alpha_p) \\
& - (\ddot{D}_z - F\dot{D}_z) \cos(\alpha_t - \alpha_p) + 2\dot{D}_x \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) \\
& + D_x \cos(\alpha_t - \alpha_p)(-F - \ddot{\alpha}_t + F\dot{\alpha}_t) - D_x \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2 \\
& + 2\dot{D}_z \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) + D_z(-F - \ddot{\alpha}_t + F\dot{\alpha}_t) \sin(\alpha_t - \alpha_p)
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
& + D_z \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2] - \frac{1}{M} \rho [\bar{L}^3(\ddot{\epsilon} - F\dot{\epsilon}) + 3\dot{\bar{L}}\bar{L}^2\dot{\epsilon} \\
& + \frac{3}{2}L(\ddot{\bar{L}} - F\dot{\bar{L}})\bar{L} + L\dot{\bar{L}}^2 - L\bar{L}^2(1 - \dot{\alpha}_t)^2] + \frac{\rho}{2}[\frac{2}{3}(\ddot{\epsilon} - F\dot{\epsilon})\bar{L}^3 \\
& + 2\dot{\bar{L}}^2\dot{\bar{L}} + (\ddot{\bar{L}} - F\dot{\bar{L}})\bar{L}L - \frac{2}{3}L\bar{L}^2(1 - \dot{\alpha}_t)^2] - \frac{\rho^2 M_s}{2M}[\frac{1}{2}(\ddot{\epsilon} - F\dot{\epsilon})\bar{L}^4 \\
& + (\ddot{\bar{L}} - F\dot{\bar{L}})\bar{L}^2L + 2\dot{\bar{L}}^3\dot{\bar{L}} + L\bar{L}\dot{\bar{L}}^2 - \frac{1}{2}\bar{L}^3L(1 - \dot{\alpha}_t)^2] \\
& - \frac{1}{M_s(1 + e \cos \theta)}[-\frac{3}{2}M(\cos^2 \alpha_t(2M_s^2 L\bar{L} + \frac{1}{2}\rho L\bar{L}^3 + 2M_s \rho L\bar{L}^2) \\
& - 2M_{srt}(D_x \sin \alpha_p - D_z \cos \alpha_p)(M_s \bar{L} + \frac{1}{2}\rho \bar{L}^2) \cos \alpha_t] + \frac{1}{2M}[2M_s^2 L\bar{L} \\
& + \frac{1}{2}L\bar{L}^2 \rho^2 + (2M_s M_{srt} \bar{L} + M_{srt} \rho \bar{L}^2)(D_x \sin(\alpha_t - \alpha_p) - D_z \cos(\alpha_t - \alpha_p)) \\
& + 2M_s \rho L\bar{L}^2] - 3(M_s \bar{L} + \frac{1}{2}\rho \bar{L}^2) \cos \alpha_t (D_x \sin \alpha_p - D_z \cos \alpha_p) \\
& - (M_s \bar{L} + \frac{1}{2}\rho \bar{L}^2)(D_x \sin(\alpha_t - \alpha_p) - D_z \cos(\alpha_t - \alpha_p)) - 3M_s \bar{L}L \cos^2 \alpha_t \\
& - M_s L\bar{L} + \rho L\bar{L}^2 \cos^2 \alpha_t - \frac{1}{3}\rho L\bar{L}^2] + EA\epsilon \frac{\bar{L}}{l_b M_s} \\
& + \rho \ddot{\bar{L}}(-\frac{1}{2M}L\bar{L} - \frac{1}{4MM_s}\rho L\bar{L}) + \rho \dot{\bar{L}}(-\frac{1}{2M}L\bar{L} - \frac{1}{2M}L\dot{\bar{L}} \\
& - \frac{1}{4MM_s}\rho \dot{\bar{L}}\bar{L}^2 - \frac{1}{2MM_s}\rho L\bar{L}\dot{\bar{L}}) + \rho \frac{\ddot{\bar{L}}\bar{L}}{l_b}(-\dot{r} \cos \alpha_t - r\dot{\theta} \sin \alpha_t) \\
& - \frac{1}{MM_s}L\bar{L}\dot{\bar{L}}^2 \rho^2 + \rho \dot{\bar{L}}[-\frac{M_{srt}}{2MM_s}\bar{L}(D_x(\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) \\
& + D_z(\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) + \dot{D}_x \sin(\alpha_t - \alpha_p) - \dot{D}_z \cos(\alpha_t - \alpha_p)) \\
& - \frac{1}{2M}(\bar{L}\dot{\bar{L}} + L\dot{\bar{L}}) - \frac{1}{2MM_s}\rho \bar{L}(\frac{1}{2}\dot{\bar{L}}\bar{L}^2 + L\dot{\bar{L}}) - \frac{1}{2MM_s}\rho L(\bar{L}\dot{\bar{L}})] = 0.
\end{aligned}$$

2.3 Linearized System

Since the goal is to use the Linear Quadratic Regulator approach to control the system, it is desired to see how closely a linearized version of equations 2.7-2.9 can approximate the system dynamics. Linearizing about the state $\alpha_p = \alpha_t = \epsilon = 0$ gives

$$[M]\ddot{\mathbf{q}} = [C]\dot{\mathbf{q}} + [K]\mathbf{q} + [\bar{\mathbf{B}}]\bar{\mathbf{u}} + \bar{\mathbf{P}},$$

i.e.

$$\ddot{\mathbf{q}} = [\mathbf{M}]^{-1}[\mathbf{C}]\dot{\mathbf{q}} + [\mathbf{M}]^{-1}[\mathbf{K}]\mathbf{q} + [\mathbf{M}]^{-1}[\tilde{\mathbf{B}}]\bar{\mathbf{u}} + [\mathbf{M}]^{-1}\bar{\mathbf{P}}, \quad (2.10)$$

where

$$\mathbf{q} = \begin{Bmatrix} \alpha_p \\ \alpha_t \\ \epsilon \end{Bmatrix}, \quad \bar{\mathbf{u}} = \begin{Bmatrix} \ddot{D}_x \\ \tau \\ \ddot{D}_z \end{Bmatrix}.$$

Here, τ is the nondimensional torque produced by the platform momentum wheel. Details of the matrices involved are given in Appendix A. Note that if $\ddot{D}_x$ and $\ddot{D}_z$ are known then the offset positions and velocities can be determined by integration. Thus they need not be included in u and are time dependent functions in the above matrices.

Now letting

$$\bar{\mathbf{x}} = \begin{Bmatrix} \dot{\mathbf{q}} \\ \dots \\ \mathbf{q} \end{Bmatrix},$$

equation (2.10) becomes

$$\dot{\bar{\mathbf{x}}} = [\mathbf{A}]\bar{\mathbf{x}} + [\mathbf{B}]\bar{\mathbf{u}} + \mathbf{P}, \quad (2.11)$$

where:

$$[\mathbf{A}] = \begin{bmatrix} [\mathbf{I}] & \vdots & [\mathbf{0}] \\ \dots & \dots & \dots \\ [\mathbf{M}]^{-1}[\mathbf{C}] & \vdots & [\mathbf{M}]^{-1}[\mathbf{K}] \end{bmatrix};$$

$$[\mathbf{B}] = \begin{bmatrix} [\mathbf{0}] \\ \dots \\ [\tilde{\mathbf{B}}] \end{bmatrix}; \quad \mathbf{P} = \begin{bmatrix} [\mathbf{0}] \\ \dots \\ \bar{\mathbf{P}} \end{bmatrix}.$$

To assess the accuracy of the linearization, both the linear and nonlinear sets of equations were integrated numerically. Figure 2.4 compares the response after a severe disturbance in each of the degrees of freedom. Note that the behaviour is closely approximated by the linear equations.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
ORBIT PARAMETERS	<div> <div>LEGEND</div> <div> <div>NONLINEAR</div> <div>LINEAR</div> </div> </div>	
$e = .01$		
$h = 500 \text{ KM}$

Figure 2.4: Comparison between nonlinear and linear responses to a fixed initial disturbance

Chapter 3

Parametric Study

3.1 Introduction

Studying the uncontrolled dynamics of the system is important for several reasons. First, it provides a better understanding of the system. For example the existence and strength of coupling and the presence of resonance due to similar frequencies in the system can be discovered. Secondly, the dynamics will reveal if there is a need for control of the system and may suggest what type would be the best. Finally, if control is necessary, the uncontrolled dynamics provide a comparison as to how well the control strategies are working. The study is initiated with the system in stationkeeping mode. That is the tether is neither being deployed nor retrieved. Some important initial system parameters are listed in the following table.

Table 3.1 System characteristics.

<i>Platform and Subsatellite Characteristics</i>	<i>Tether Characteristics</i>
Platform Mass: 100,000 kg	Status: Station Keeping
Platform Inertias:	Diameter: 0.002 m
$I_{xx} = 1.2 \times 10^8 \text{kgm}^2$	Young's Modulus: $1.25 \times 10^8 \text{Nm}$
$I_{yy} = 2.0 \times 10^8 \text{kgm}^2$	Linear Density: 2 kg/km
$I_{zz} = 8.3 \times 10^7 \text{kgm}^2$	Equilibrium Length: 1 km
Satellite Mass: 100 kg	

It should be noted here that the tether oscillations are at a higher frequency than the pitch oscillations. For the 1 km tether modeled here, the frequency is about 100 cycles per orbit, while the pitch motions have a frequency of about 1 cycle per orbit. It is thus difficult to resolve both of the motions on the same chart. If the high frequency response is of interest in a particular case, a second graph with a larger scale is included.

3.2 Basic Response

Figure 3.1 shows the response of the system to an initial disturbance in each of the degrees of freedom. Figure 3.1(a) shows periodic oscillations in the pitch motions as the gravity gradient torque attempts to bring the system back to the equilibrium configuration. The oscillations have constant amplitude since the inherent damping of the system was purposely ignored to accentuate the response. If necessary of course, the energy dissipation can easily be modelled through the corresponding generalized force. Figure 3.1(b) demonstrates the decoupling of platform and tether dynamics for the case of zero offsets. As expected the tether oscillations have no effect on the platform pitch.

3.3 Offsets

3.3.1 Horizontal Offset

Figure 3.2 shows the system response with the tether attachment point displaced 20 meters along the local horizontal. Notice that the platform no longer oscillates about zero(Figure 3.2(a)). The offset causes the system to rotate to a new equilibrium configuration. Coupling between the platform and longitudinal tether dynamics results in small modulations of the platform response at the tether frequency(Figure 3.2(b)). Even with a small subsatellite mass which is a tiny fraction of the platform mass, the pitch response of the platform is modulated.

<u>OFFSETS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$D_x = 0 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 1^\circ$
$D_z = 0 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>ORBIT PARAMETERS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$e = 0$		
$h = 500 \text{ KM}$

Figure 3.1: Response of the system during the reference stationkeeping configuration to a prescribed disturbance: (a) low frequency platform and tether pitch oscillations.

<u>OFFSETS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$D_x = 0 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 1^\circ$
$D_z = 0 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
<u>ORBIT PARAMETERS</u>		
$e = 0$		
$h = 500 \text{ KM}$

Figure 3.1: Response of the system during the reference stationkeeping configuration to a prescribed disturbance: (b) relatively high frequency longitudinal oscillations of the tether.

<u>OFFSETS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 1^\circ$
$D_z = 0 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>ORBIT PARAMETERS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$e = 0$		
$h = 500 \text{ KM}$

Figure 3.2: Effect of the tether attachment point's offset along the local horizontal on the system response: (a) time history of the pitch motion.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 1^\circ$
$D_z = 0 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
<u>ORBIT PARAMETERS</u>	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$e = 0$		
$h = 500 \text{ KM}$

Figure 3.2: Effect of the tether attachment point's offset along the local horizontal on the system response: (b) coupling between the tether longitudinal dynamics and the pitch motions.

3.3.2 Vertical Offset

Figure 3.3 shows the response for a vertical offset (i.e. offset along the local vertical) of 20 meters. The equilibrium position of the platform remains unaffected in this case. Note the effect of the tether stretch on the platform is now less pronounced than that for the horizontal offset case. This can be expected as the torque applied to the platform is primarily governed by the offset along the local horizontal.

3.4 Eccentricity

Eccentricity has the effect of introducing a periodic (at the orbital frequency) forcing term into the pitch equations. To study the effect of eccentric orbits the offsets and initial disturbances are set to zero. Figure 3.4 compares the response for orbits with $e = 0.01$ and $e = 0.05$. As anticipated, the higher eccentricity increases the amplitude of the response, particularly in the platform pitch. For $e = 0.05$, α_p reaches 30° which may not be acceptable. However, the tether pitch response is confined to 4° even for $e = 0.05$. As expected, the tether's longitudinal mode remained virtually unexcited due to the eccentricity.

3.5 Subsatellite Mass

Figure 3.5 shows the effect of doubling the subsatellite mass. Note, the platform pitch angle reaches a much lower value for $M_s = 200$ kg. This is to be expected since the extra mass increases the restoring gravity gradient moment. With the horizontal offset and the subsatellite mass, the platform equilibrium position is also affected. The period of the longitudinal tether oscillations increases by about 30 percent for $M_s = 200$ kg (Figure 3.5(b)). The increased mass also causes the high frequency platform pitch modulation to be a little more pronounced.

<u>OFFSETS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$D_x = 0 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 1^\circ$
$D_z = 20 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>ORBIT PARAMETERS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$e = 0$		
$h = 500 \text{ KM}$

Figure 3.3: Effect of the tether attachment point's offset along the local vertical on the system response: (a) time history of the pitch motion.

<u>OFFSETS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$D_x = 0 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 1^\circ$
$D_z = 20 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
<u>ORBIT PARAMETERS</u>	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$e = 0$		
$h = 500 \text{ KM}$

Figure 3.3: Effect of the tether attachment point's offset along the local vertical on the system response: (b) small influence of the tether's longitudinal dynamics on its pitch motion.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 0 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 0^\circ$
$D_z = 0 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 0^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = 0$
ORBIT PARAMETERS	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$h = 500 \text{ KM}$	<div> <div>LEGEND</div> <div> $e = .01$ $e = .05$ </div> </div>

Figure 3.4: System pitch response as influenced by the orbit eccentricity.

<u>OFFSETS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_r = 50 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$\rho = .002 \text{ KG/M}$	$\varepsilon(0) = .01$
		$l_b = 1000 \text{ M}$
<u>ORBIT PARAMETERS</u>	<u>LEGEND</u>	
$e = .01$	$M_s = 100 \text{ KG}$	$M_s = 200 \text{ KG}$
$h = 500 \text{ KM}$

Figure 3.5: System dynamics as affected by the subsatellite mass: (a) pitch response over a long duration.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_r = 50 \text{ KG}$	$\alpha_t(0) = 1^\circ$
<u>ORBIT PARAMETERS</u>	$\rho = .002 \text{ KG/M}$	$\varepsilon(0) = .01$
$e = .01$		$l_b = 1000 \text{ M}$
$h = 500 \text{ KM}$		
	<u>LEGEND</u>	
	$M_s = 100 \text{ KG}$	$M_s = 200 \text{ KG}$

Figure 3.5: System dynamics as affected by the subsatellite mass: (b) enlarged view over a short duration showing the coupling effects.

3.6 Tether Mass

The effect of a more massive tether on the dynamics was studied by increasing its line density from 0.002 kg/m to 0.2 kg/m (Figure 3.6) . This increases the tether mass from 2 kg to 200 kg. Notice that the influence is very similar to that of increasing the subsatellite mass (Figure 3.5). In fact, many investigators approximate the effect of the tether mass simply by increasing the mass of the subsatellite. This is called the lumped mass approach. Note, however, that the high frequency platform pitch modulations are not as much effected by the change as they were in the subsatellite mass variation case. In both of the cases the tether pitch oscillations remain relatively unaffected.

3.7 Platform Inertias

The inertias used thus far are the same as in reference [6] and are intended to model a space station. Since the mass is spread out (the platform is modelled as a rectangular plate), the inertias are relatively large causing resistance to high frequency disturbances. Here the inertias are changed to approximate those of the U.S. Space Shuttle:

$$I_{xx} = 8.5 \times 10^6 \text{kgm}^2$$

$$I_{yy} = 8.5 \times 10^6 \text{kgm}^2$$

$$I_{zz} = 1.1 \times 10^6 \text{kgm}^2$$

The inertias correspond to an orientation which has the Shuttle's nose pointed directly away from the earth and the wings in the plane of the orbit (Lagrange configuration; Minimum moment of inertia along the local vertical, maximum moment of inertia along the orbit normal). This has been shown to be the most stable configuration [11]. The platform mass is also changed to match that of the shuttle. The smaller inertias mean

that the restoring moment due to the gravity gradient is smaller. For a given offset, the shuttle deviates from the reference equilibrium by a significant amount, as shown in Figure 3.7. To partially compensate for this the offsets were reduced to 10 meters in each direction. The smaller inertias also cause the frequency of the platform pitch to increase from 0.9 cycles/orbit to 1.5 cycles/orbit. Figure 3.7(b) shows how decreasing the inertias can increase the coupling between the shuttle pitch and the tether stretch. Even with smaller offsets the high frequency shuttle pitch oscillations have a much larger amplitude than those of the platform (Figure 3.5(b)).

3.8 Reel Mass

The reel mass was increased from 50 kg to 500 kg and the results plotted in Figure 3.8. Notice that the platform pitch angle is affected because of the reel location offset from the platform center of mass. The increased reel mass changes the inertias of the platform thus affecting its equilibrium orientation. Of course the tether pitch equilibrium is not affected.

3.9 Tether Length

The effect of tether length on the dynamics is studied by comparing the dynamics for $l_b = 1000$ m and $l_b = 100$ m (Figure 3.9). The shorter tether length results in a reduced gravity gradient torque. Also the platform equilibrium position is less affected by the tether offset (Figure 3.9(a)). Another effect of the decreased gravity gradient force is an increase in the period of the tether pitch. Figure 3.9(b) shows that the frequency of the longitudinal tether oscillations for a 100 m tether is three times that of a 1000 m tether. Note also that the amplitude of the high frequency superimposed modulations decreases as the length decreases for a given initial strain ($\epsilon = 0.01$).

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
		$l_b = 1000 \text{ M}$
ORBIT PARAMETERS	LEGEND	
$e = .01$	$\rho = .002 \text{ KG/M}$	$\rho = .2 \text{ KG/M}$
$h = 500 \text{ KM}$

Figure 3.6: Effect of tether mass on the system response: (a) time history of the platform and tether pitch dynamics.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
		$l_b = 1000 \text{ M}$
ORBIT PARAMETERS		
$e = .01$	<div>LEGEND</div> <div> $\rho = .002 \text{ KG/M}$ $\rho = .2 \text{ KG/M}$ </div>	
$h = 500 \text{ KM}$

Figure 3.6: Effect of tether mass on the system response: (b) longitudinal dynamics of the tether and its coupling effects.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 10 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
$D_z = 10 \text{ M}$	$M_r = 50 \text{ KG}$	$\epsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
ORBIT PARAMETERS		
$e = .01$	<div> <div>LEGEND</div> <div> <div>PLATFORM</div> <div>SHUTTLE</div> </div> </div>	
$h = 500 \text{ KM}$

Figure 3.7: System response showing the effect of platform inertias: (a) pitch response.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 10 \text{ M}$	$M_p = 79,000 \text{ KG}$	$\alpha_p(0) = -6.9^\circ$
$D_z = 10 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>ORBIT PARAMETERS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$e = .01$		
$h = 500 \text{ KM}$		
SHUTTLE INERTIAS

Figure 3.7: System response showing the effect of platform inertias: (b) high frequency coupling effects of the tether longitudinal dynamics.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -6.9^\circ$
$D_z = 20 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$\rho = .002 \text{ KG/M}$	$\varepsilon(0) = .01$
		$l_b = 1000 \text{ M}$
ORBIT PARAMETERS	LEGEND	
$e = .01$	$M_r = 50 \text{ KG}$	$M_r = 500 \text{ KG}$
$h = 500 \text{ KM}$

Figure 3.8: Effect of the reel mass on the system dynamics.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_r = 50 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_s = 100 \text{ KG}$	$\varepsilon(0) = .01$
ORBIT PARAMETERS	$\rho = .002 \text{ KG/M}$	
$e = .01$		
$h = 500 \text{ KM}$		

LEGEND

$l_b = 1000 \text{ M}$ $l_b = 100 \text{ M}$

Figure 3.9: Effect of the tether length on the response of the system: (a) pitch motion.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_r = 50 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_s = 100 \text{ KG}$	$\varepsilon(0) = .01$
ORBIT PARAMETERS	$\rho = .002 \text{ KG/M}$	
$e = .01$		
$h = 500 \text{ KM}$		

LEGEND

$l_b = 1000 \text{ M}$ $l_b = 100 \text{ M}$

Figure 3.9: Effect of the tether length on the response of the system: (b) coupling effects due to change in the tether longitudinal oscillation frequency.

3.10 Retrieval

Retrieval of a deployed tether is a difficult task. As the length decreases any disturbance in the tether pitch must increase in order to conserve the angular momentum of the system. In the equations of motion, the coefficient of $\dot{\alpha}_t$ becomes negative during retrieval thus imparting, effectively, negative damping to this degree of freedom. Retrieval is achieved by supplying the desired nominal length function ($\bar{l}$) in the equations of motion. Decaying exponential schemes are desirable in applications since they avoid quick decelerations at the end of the maneuver. In this study, the nominal length is given by

$$\bar{l} = l_b \exp[ct],$$

where t is time in seconds and c is a constant (negative for retrieval, positive for deployment).

Since it is more convenient to specify the retrieval time in orbits ($t \approx \theta a_p^{1.5} (GM_e)^{-0.5}$), the above equation can be rewritten as

$$\bar{l} = l_b \exp \left[\frac{c\theta a_p^{1.5}}{\sqrt{GM_e}} \right],$$

where θ represents the true anomaly in orbital units.

For example, if it is desired to reduce a tether's nominal length from 100 m to 10 m in 0.4 orbit, one can solve for c from

$$10 = 100 \exp \left[\frac{.4ca_p^{1.5}}{\sqrt{GM_e}} \right]$$

to give $c = -0.00638$.

In order to study the effect of the retrieval rate on the system dynamics, a small initial disturbance of 1° is given to the tether pitch without affecting the platform. Note, in Figure 3.10, the tether pitch angle quickly reaches 13° . The platform also librates due to

coupling through the offset. The offset here is in the z direction, a relatively less critical situation. As mentioned earlier, retrieval maneuvers represent a critical phase leading to instability if uncontrolled. Any practical application of tethers will have to address this problem effectively.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 0 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 0^\circ$
$D_z = 5 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = 0$
ORBIT PARAMETERS	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
$e = .01$		
$h = 500 \text{ KM}$

Figure 3.10: Retrieval from 100 m to 10 m in .4 orbits

Chapter 4

Control

The dynamical study clearly showed situations leading to unacceptable response, under critical combinations of system parameters, suggesting a need for control. This chapter develops a control procedure based on offset of the tether attachment point.

As discussed in Chapter 2 the equations of motion can be written as

$$\dot{\bar{\mathbf{x}}} = [\mathbf{A}]\bar{\mathbf{x}} + [\mathbf{B}]\mathbf{u} + \mathbf{P}, \quad (4.1)$$

where $\bar{\mathbf{x}}^T = (\dot{\alpha}_p, \dot{\alpha}_t, \dot{\epsilon}, \alpha_p, \alpha_t, \epsilon)$, and $\mathbf{u}^T = (\ddot{D}_x, \tau, \ddot{D}_z)$.

Setting the generalized coordinates, velocities and control quantities to zero gives

$$\mathbf{x}_{eq} = -[\mathbf{A}]^{-1}\mathbf{P}, \quad (4.2)$$

where $\mathbf{x}_{eq}$ is the quasistatic equilibrium orientation of the system. $\mathbf{x}_{eq}$ is a slowly varying function of time since the only time dependent elements left in $[\mathbf{A}]$ and $\mathbf{P}$ are due to eccentricity and retrieval effects.

Now, $\bar{\mathbf{x}}$ can be partitioned as

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{x}_{eq}, \quad (4.3)$$

where $\mathbf{x}$ represents deviation from the equilibrium. Using equation (4.3) in equation (4.1) gives,

$$\dot{\mathbf{x}} + \dot{\mathbf{x}}_{eq} = [\mathbf{A}](\mathbf{x} + \mathbf{x}_{eq}) + [\mathbf{B}]\mathbf{u} + \mathbf{P}.$$

Now substituting from equation (4.2) and using the quasistatic assumption gives finally,

$$\dot{\mathbf{x}} = [\mathbf{A}]\mathbf{x} + [\mathbf{B}]\mathbf{u}. \quad (4.4)$$

4.1 Linear Quadratic Regulator (LQR)

The LQR approach to control is useful here since it works for systems with multiple inputs and complex outputs or dynamics. It is basically a mathematical method and can be stated as follows.

Minimize the functional

$$J = \int_0^{\infty} (\mathbf{x}^T [\mathbf{Q}] \mathbf{x} + \mathbf{u}^T [\mathbf{R}] \mathbf{u}) dt,$$

subject to the constraints

$$\dot{\mathbf{x}} = [\mathbf{A}] \mathbf{x} + [\mathbf{B}] \mathbf{u},$$

with initial conditions

$$\mathbf{x}(0) = \mathbf{x}_o$$

$$\mathbf{u}(0) = \mathbf{0}.$$

The reason for the term “Quadratic” is clear from the form of the functional J . The term $\mathbf{x}^T [\mathbf{Q}] \mathbf{x}$ represents the deviation of the system from equilibrium while $\mathbf{u}^T [\mathbf{R}] \mathbf{u}$ represents the control effort being applied. Minimizing J then controls the states of the system while simultaneously keeping cumulative energy expenditures at a minimum.

The diagonal matrices $[\mathbf{Q}]$ and $[\mathbf{R}]$ provide weights to the state and control variables respectively. The design of the controller basically involves selection of the weighting matrices so as to receive the desired system response. For example if $Q(1,1)$ is large relative to the other elements of $[\mathbf{Q}]$, then the state $x(1)$ will have relatively higher restrictions put on it so that J is kept small. Similarly if $R(1,1)$ is relatively large, the control variable $u(1)$ will be used more sparingly.

4.2 Parallel Control

The design process described above becomes difficult if there are many state and control variables. Assume for example that the system is controlled well except that $x(1)$ becomes slightly too large. It is true that increasing $Q(1,1)$ will decrease $x(1)$ however all other states may increase since their weights have become relatively smaller.

The situation can be improved if there is a large separation of natural frequencies in the problem. As seen in Chapter 2, the longitudinal tether oscillations are at a much higher frequency than the pitch motions. The coupling is such that the tether oscillations superimpose high frequency motions on the pitch angles. Figure 4.1 shows the effect of eliminating the coupling terms between the high and low frequency degrees of freedom. Notice the tether pitch is closely approximated and the platform pitch and tether stretch remain virtually unaffected. It seems reasonable then, to control the high and low frequency motions separately. Thus one solves two smaller control problems at each time step instead of a single large one. Equation (4.4) is separated into low and high frequency groups as follows:

$$\dot{\mathbf{x}}_s = [\mathbf{A}_s]\mathbf{x}_s + [\mathbf{B}_s]\mathbf{u}_s;$$

$$\dot{\mathbf{x}}_f = [\mathbf{A}_f]\mathbf{x}_f + [\mathbf{B}_f]\mathbf{u}_f;$$

where:

$$\mathbf{x}_s = \begin{Bmatrix} \dot{\alpha}_p \\ \dot{\alpha}_t \\ \alpha_p \\ \alpha_t \end{Bmatrix}, \quad \mathbf{u}_s = \begin{Bmatrix} \ddot{D}_x \\ \tau \end{Bmatrix};$$

$$\mathbf{x}_f = \begin{Bmatrix} \dot{\epsilon} \\ \epsilon \end{Bmatrix}, \quad \mathbf{u}_f = \ddot{D}_z.$$

Here $[\mathbf{A}_f]$, $[\mathbf{A}_s]$, $[\mathbf{B}_s]$, $[\mathbf{B}_f]$ are obtained from $[\mathbf{A}]$ and $[\mathbf{B}]$ by omitting the coupling terms.

Notice that the control variables are different for each group. By studying the equations of motion it was determined that the vertical offset has the the greatest effect on the tether stretch. Similarly, for the pitch motions, the horizontal offset and platform torque have the most effect. These facts are taken advantage of in improving the speed of the control program. Notice that now one can choose the weights of $[\mathbf{R}_s]$, $[\mathbf{R}_f]$, $[\mathbf{Q}_s]$, $[\mathbf{Q}_f]$, more easily since they are specialized.

4.3 Numerical Solution

The method used to solve the minimization problem in Section 4.1 is described in a text by Kuo [13]. An addition was required to ensure that the tether attachment point offset would return to its original central starting position at the completion of control. At each time step the offset accelerations are adjusted as

$$\ddot{\mathbf{D}} = \ddot{\mathbf{D}} - [\mathbf{V}]\dot{\mathbf{D}} - [\mathbf{W}]\mathbf{D},$$

where $\mathbf{D} = (D_x, D_z)$ and $[\mathbf{V}]$ and $[\mathbf{W}]$ are constant diagonal matrices. So the offset accelerations are not allowed to settle to zero until both the offset positions and velocities are also zero. A block diagram showing the closed loop system is shown in Figure 4.2.

OFFSETS	MASS PARAMETERS	INITIAL CONDITIONS
$D_x = 20 \text{ M}$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = -2.12^\circ$
$D_z = 20 \text{ M}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 1^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>ORBIT PARAMETERS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 1000 \text{ M}$
$e = .01$		
$h = 500 \text{ KM}$		

LEGEND

COUPLED UNCOUPLD

Figure 4.1: Effect of decoupling high and low frequency motions

4.2: Block diagram showing closed loop system with parallel control and offset feedforward

4.4 Varying Weights

To begin the control analysis, the system is in stationkeeping mode with a nominal tether length of 100 m. The pitch angles are given an initial disturbance of 10° . In practice this would be considered a very large disturbance. The tether is also stretched by 1 m.

It was found that using fixed weights in $[\mathbf{R}_s]$ placed limitations on how quickly the tether pitch oscillations could be damped. This is due to the physical constraint on the magnitude of the offsets used. Available telerobotics technology limits the offsets to 20 m from the central position. Therefore $R_s(1,1)$, which is the penalty weight corresponding to D_x , could only be decreased to the point where D_x reached a maximum of 20 m. The result of this design is shown in Figure 4.3 with legend label “FIXED WEIGHTS”. Note that the tether pitch requires more than 5 orbits to damp sufficiently. In Figure 4.3(b) the amplitude of oscillations of D_x decreases quite quickly, approaching the extreme of 20 m only in the first half orbit. It was concluded that the penalty weight on D_x could be safely decreased after this point, thus further exploiting the potential of the offset in controlling the tether pitch. To test this idea, $R_s(1,1)$ was held constant until 0.5 orbits and then decreased in a linear fashion for the remainder of the control effort. The result is shown in Figure 4.3 with legend label “VARIABLE WEIGHTS”. Even with this simple design the control of the tether pitch is now achieved in less than 3 orbits and the physical constraint on the offset is not violated. Examples of the weighting and feedforward matrices used are given in Appendix B.

Figure 4.3 also shows that the longitudinal oscillations are damped quickly by the vertical offset. Note, D_z reaches only 2 m and damps the disturbance in 0.05 orbit. The platform pitch is controlled at about the same rate as the tether pitch with a maximum platform wheel torque of 7 Nm being required.

ORBIT PARAMETERS	MASS PARAMETERS	INITIAL CONDITIONS
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^\circ$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^\circ$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>TETHER STATUS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
STATION KEEPING	<div> <div>LEGEND</div> <div> <div>VARIABLE WEIGHTS</div> <div>FIXED WEIGHTS</div> </div> </div>

Figure 4.3: Control of the system in the stationkeeping mode: (a) time history of the pitch and tether longitudinal motions.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
<u>TETHER STATUS</u>	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
<u>STATION KEEPING</u>	<div> <u>LEGEND</u> <u>VARIABLE WEIGHTS</u> <u>FIXED WEIGHTS</u> </div>

Figure 4.3: Control of the system in the stationkeeping mode: (b) associated offset motions and momentum gyro output.

4.5 Subsatellite Mass

Using the same control procedure as above, the effect of doubling the subsatellite mass is studied (Figure 4.4). The higher mass results in more energy being stored in the stretched tether. The longitudinal oscillations thus take longer (about twice as long) to control. It is encouraging that the tether pitch is controlled as well as before and the platform pitch control is improved. However, the demands on the platform based gyro-momentum wheel have increased to 10 Nm. This can be partially attributed to the increased effect which the offset has on the platform pitch due to the larger gravity gradient effect.

4.6 Eccentricity

In chapter 3 it was shown that an eccentric orbit introduces a forcing term in the platform and tether pitch equations. Note, in Figure 4.5, a constant amplitude cyclic effort is required from the horizontal offset and platform wheel to keep the pitch oscillations under 2° . However, the longitudinal tether oscillations remain unaffected and are controlled quickly.

4.7 Platform Inertias

The effect of using smaller platform inertias on control is shown in Figure 4.6. As can be expected the platform pitch can be controlled using a smaller extreme value of the gyro torque (3 Nm). Note, however that the time taken to completely damp the platform pitch oscillations is still around 4 orbits. This is so because the smaller platform is more sensitive to the effects of the horizontal offset.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 200 \text{ KG}$	$\alpha_t(0) = 10^0$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>TETHER STATUS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
STATION KEEPING

Figure 4.4: Plots showing effectiveness of the LQR control strategy in the presence of an increased subsatellite mass: (a) time variation of the pitch and tether length.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 200 \text{ KG}$	$\alpha_t(0) = 10^0$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>TETHER STATUS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
STATION KEEPING

Figure 4.4: Plots showing effectiveness of the LQR control strategy in the presence of an increased subsatellite mass: (b) offset dynamics and momentum gyro output.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0.01$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
<u>TETHER STATUS</u>	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
STATION KEEPING

Figure 4.5: Controlled response during stationkeeping in the presence of an orbital eccentricity of $e = 0.01$: (a) platform and tether motions.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0.01$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
<u>TETHER STATUS</u>	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
STATION KEEPING

Figure 4.5: Controlled response during stationkeeping in the presence of an orbital eccentricity of $e = 0.01$: (b) offset and momentum-gyro output time histories.

ORBIT PARAMETERS	MASS PARAMETERS	INITIAL CONDITIONS
$e = 0$	$M_p = 79,000 \text{ KG}$	$\alpha_p(0) = 10^\circ$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^\circ$
<u>TETHER STATUS</u>	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
STATION KEEPING		
SHUTTLE INERTIAS

Figure 4.6: Effect of the platform inertia on the controlled motion of the system in stationkeeping: (a) platform and tether responses.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0$	$M_p = 70,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>TETHER STATUS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
STATION KEEPING		
SHUTTLE INERTIAS

Figure 4.6: Effect of the platform inertia on the controlled motion of the system in stationkeeping: (b)time histories of the tether attachment point and momentum gyro output.

4.8 Tether Length

It is intuitively clear that since the motion of the offset is constrained to about 20 m, its performance during control would deteriorate for longer tether lengths. In Figure 4.7, the controller effectiveness is studied when the tether length is increased to 500 m. The tether pitch now requires 10 orbits to be controlled. The longer length leads to higher weight of the tether and larger elongation. The increased tether stretch puts a much larger demand on the vertical offset. D_z reaches a maximum of about 12 m and the control takes five times longer than for the 100 m tether.

4.9 Control During Retrieval

In Chapter 3 it was shown that the tether pitch oscillations increase in amplitude during retrieval. In fact for large enough initial disturbance and retrieval rates it is possible for the tether to completely wrap itself around the platform. Effectiveness of the offset control method is tested by retrieving the tether from 100 m to 10 m with the same severe initial disturbance of 10° in both the platform and tether pitch. Two different retrieval rates are used. First the retrieval is completed in 0.4 orbit and then the process is repeated with the retrieval time being 1 orbit. Figure 4.8 shows that in both cases the tether pitch angle approximately doubles but is then controlled by the offset. The tether pitch disturbance becomes larger for the faster retrieval rate, reaching 19.9° for retrieval in 0.4 orbit and 18° for retrieval in 1.0 orbit. There seems to be no special difficulty in damping the longitudinal oscillations during retrieval. The vertical offset reaches only 2 m and control is complete in 0.04 orbit.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>TETHER STATUS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 500 \text{ M}$
STATION KEEPING

Figure 4.7: Effectiveness of the offset-control strategy as affected by a tether length of 500 m: (a) pitch and longitudinal oscillations response.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
<u>TETHER STATUS</u>	$\rho = .002 \text{ KG/M}$	$l_b = 500 \text{ M}$
STATION KEEPING

Figure 4.7: Effectiveness of the offset-control strategy as affected by a tether length of 500 m: (b) offset and momentum-gyro output time histories.

<u>ORBIT PARAMETERS</u>	<u>MASS PARAMETERS</u>	<u>INITIAL CONDITIONS</u>
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
<u>TETHER STATUS</u>	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
RETRIEVAL	<div> <div><u>LEGEND</u></div> <div> <div>0.4 ORBITS</div> <div>1.0 ORBITS</div> </div> </div>

Figure 4.8: System response as affected by the retrieval rates: (a) pitch dynamics and the exponential retrieval profiles.

ORBIT PARAMETERS	MASS PARAMETERS	INITIAL CONDITIONS
$e = 0$	$M_p = 100,000 \text{ KG}$	$\alpha_p(0) = 10^0$
$h = 500 \text{ KM}$	$M_s = 100 \text{ KG}$	$\alpha_t(0) = 10^0$
	$M_r = 50 \text{ KG}$	$\varepsilon(0) = .01$
TETHER STATUS	$\rho = .002 \text{ KG/M}$	$l_b = 100 \text{ M}$
RETRIEVAL	<div> <div>LEGEND</div> <div> <div>0.4 ORBITS</div> <div>1.0 ORBITS</div> </div> </div>

Figure 4.8: System response as affected by the retrieval rates: (b) offset and longitudinal oscillation time histories.

Chapter 5

Concluding Comments

The platform based tethered satellite model, although relatively simple, is useful in understanding complex interactions between the librational dynamics, tether flexibility, offset of the attachment point and initial disturbance. The parametric analysis of the system dynamics should prove useful at least in the preliminary design phase. The model is also helpful in assessing merits and limitations of the offset control in the presence of tether flexibility. It should be noted that because of the inclusion of the longitudinal tether oscillations, the equations of motion derived here demand considerable time and effort to solve numerically. This is especially true during retrieval since the frequency of the tether oscillations increases at shorter lengths.

The equations of motion and the parametric analysis reveal that platform and tether dynamics are coupled through the offset of the attachment point. This coupling increases with longer tether lengths, smaller platform inertias and more massive subsatellites. Longitudinal tether oscillations superimpose high frequency oscillations on the platform pitch response which could disrupt sensitive experiments or even damage equipment. Retrieval of the tether results in large tether pitch oscillations even for small initial disturbances.

The offset control method developed is effective in damping both rigid body pitch oscillations of the platform and the tether, as well as the tether's longitudinal vibrations due to its flexibility. Its performance improves with shorter tether lengths. For a 100 meter tether, relatively large pitch disturbances are damped in about 3 orbits.

Longitudinal oscillations are damped quickly by the vertical offset. This is encouraging since applications such as NASA's proposed microgravity laboratory [2] would require precise vertical positioning.

The feasibility of controlling high and low frequency motions separately is established. The approach improves the speed of the control program and allows the control weights to be determined more easily. It is shown that improvement in the control performance can be obtained by varying the weights in the Linear Quadratic Regulator method. This is especially useful when the physical limit on the offset motion is reached.

Recommendations for Future Work

The model used here could be generalized to include the out of plane motion and transverse oscillations of the tether. The presence of transverse oscillations in the model will help evaluate effectiveness of the offset control method in regulating this degree of freedom. It will be of interest to assess the effect of offset motions in actually inducing tether transverse oscillations. Flexibility of the platform could also have a large effect on an offset control strategy and should therefore be investigated. An optimal method of choosing the LQR weights could dramatically improve the control performance. Satoh and Yuhara [12] did some work on this but for a fixed tether length using tension control.

Ultimately, ground or space based experiments of this and other models will be necessary to verify that they capture the system dynamics.

Appendix A: Details of the Linearized Equations of Motion

Equations 2.7-2.9 are linearized in the form,

$$[\mathbf{M}]\ddot{\mathbf{q}} = [\mathbf{C}]\dot{\mathbf{q}} + [\mathbf{K}]\mathbf{q} + [\bar{\mathbf{B}}]\bar{\mathbf{u}} + \bar{\mathbf{P}}$$

The details of the coefficient matrices are as follows:

Mass Matrix $[\mathbf{M}]$

$$\begin{aligned} [\mathbf{M}](1,1) &= -\frac{M_{prt}M_p}{MM_s}(D_x^2 + D_z^2) - \frac{I_{yy}}{M_sl_b} \\ [\mathbf{M}](1,2) &= D_z\bar{L}\left(\frac{M_p}{M} + \frac{1}{2}\frac{M_p}{MM_s}\rho\bar{L}\right) \\ [\mathbf{M}](1,3) &= D_x\left(\frac{M_p}{M}\bar{L} + \frac{M_p}{2MM_s}\rho\bar{L}^2\right) \\ [\mathbf{M}](2,1) &= M(1,2) \\ [\mathbf{M}](2,2) &= \bar{L}^2\left(-\frac{M_{prt}}{MM_s} + \frac{1}{M}\rho\bar{L} - \frac{1}{3}\bar{L}\rho + (\rho\bar{L})^2\frac{1}{4MM_s}\right) \\ [\mathbf{M}](2,3) &= 0 \\ [\mathbf{M}](3,1) &= [\mathbf{M}](1,3) \\ [\mathbf{M}](3,2) &= 0 \\ [\mathbf{M}](3,3) &= [\mathbf{M}](2,2) \end{aligned}$$

Stiffness Matrix $[\mathbf{K}]$

$$\begin{aligned} [\mathbf{K}](1,1) &= 3\frac{H}{M_sl_b^2}(I_{xx} - I_{zz}) - \frac{M_p}{M}\bar{L}D_z + \frac{M_p}{M_sM}\rho\left[D_z(\bar{L}(\ddot{\bar{L}} - F\dot{\bar{L}}) + \dot{\bar{L}}^2)\right. \\ &\quad \left.+ 2D_x\bar{L}\dot{\bar{L}} - \frac{1}{2}D_xF\bar{L}^2 + \frac{1}{2}\bar{L}^2D_z\right] + \frac{M_p}{M_sM}\left[D_z(\ddot{\bar{L}} - F\dot{\bar{L}}) + 2D_x\dot{\bar{L}} - F\bar{L}D_x\right] \end{aligned}$$

$$\begin{aligned}
& - H \left[-\frac{3}{M_{srt}^2 M M_s} (D_x^2 - D_z^2) - \frac{1}{2} \frac{M_{srt}}{M M_s} (D_z \bar{L} (\rho \bar{L} + 2M_s)) \right. \\
& + \left. 3 \frac{M_{srt}}{M_s} (D_x^2 - D_z^2) + \frac{2}{M_s} (M_s + \frac{1}{2} \rho \bar{L}) D_z \bar{L} \right] \\
[\mathbf{K}](1, 2) &= \frac{M_p}{M} [D_z \bar{L} - 2\bar{L} D_x] + \frac{M_p}{M_s M} \rho [-D_z (\bar{L} (\ddot{\bar{L}} - F \dot{\bar{L}}) \\
& + \dot{\bar{L}}^2) - 2D_x \bar{L} \dot{\bar{L}} + \frac{1}{2} D_x F \bar{L}^2 + \frac{1}{2} \bar{L}^2 D_x] - \frac{M_p}{M} [D_z (\ddot{\bar{L}} - F \dot{\bar{L}}) \\
& + 2D_x \dot{\bar{L}} - F \bar{L} D_x] - H \left[-\frac{1}{2} \frac{M_{srt}}{M M_s} D_z \bar{L} (\rho \bar{L} + 2M_s) + (M_s + \frac{1}{2} \rho \bar{L}) D_z \bar{L} \right] \\
[\mathbf{K}](1, 3) &= \frac{M_p}{M} [-D_x (\ddot{\bar{L}} - F \dot{\bar{L}}) + 2\dot{\bar{L}} D_z - F D_z \bar{L} + D_z \ddot{\bar{L}}] \\
& + \frac{M_p}{M M_s} \rho [-D_x \bar{L} (\ddot{\bar{L}} - F \dot{\bar{L}}) - D_x \dot{\bar{L}}^2 - \frac{1}{2} D_z F \bar{L}^2 + \frac{1}{2} D_x \bar{L}^2 + D_z \bar{L} \dot{\bar{L}}] \\
& - H \left[\frac{3}{M M_{srt} M_s} (D_x M_s \bar{L} + \frac{1}{2} D_x \rho \bar{L}^2) - \frac{M_{srt}}{2 M_s M} (D_x (2M_s \bar{L} + \rho \bar{L}^2)) \right. \\
& + \left. D_x \bar{L} (M_s + \frac{1}{2} \rho \bar{L}) - 3D_x (M_s \bar{L} + \frac{1}{2} \rho \bar{L}^2) \right] \\
[\mathbf{K}](2, 1) &= \left[\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{M M_s} \rho \bar{L} \right] [\bar{L} F \dot{D}_z + 2\bar{L} \dot{D}_x - F \bar{L} D_x \\
& + \bar{L} D_z] - \frac{H}{M_s} D_z (M_s \bar{L} + \frac{1}{2} \rho \bar{L}^2) \\
[\mathbf{K}](2, 2) &= -[\mathbf{K}](2, 1) - H \left[-\frac{3}{2 M M_s} (-2(M_s^2 \bar{L}^2 + \frac{1}{4} \rho^2 \bar{L}^4 + M_s \rho \bar{L}^3) \right. \\
& + \left. 2D_z M_{srt} (M_s \bar{L} + \frac{1}{2} \rho \bar{L}^2) + 2(M_s \bar{L} + \frac{1}{2} \rho \bar{L}^2) D_z - (3M_s \bar{L}^2 + \rho \bar{L}^3) \right] \\
[\mathbf{K}](2, 3) &= \frac{M_{prt}}{2M} [4\bar{L}^2 F - 8\bar{L} \dot{\bar{L}}] + (\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{M M_s} \rho \bar{L}) (\bar{L} F \dot{D}_x + 2\bar{L} \dot{D}_z \\
& - F \bar{L} D_z - \bar{L} D_x) + \frac{1}{M} \rho [-2F \bar{L}^3 + 6\bar{L}^2 \dot{\bar{L}}] - \frac{1}{2} \rho [4\dot{\bar{L}} \bar{L}^2 - \frac{4}{3} F \bar{L}^3] \\
& + \frac{1}{2 M M_s} \rho^2 [4\bar{L}^3 \dot{\bar{L}} - F \bar{L}^4] - H \left[-D_x (\bar{L} + \frac{1}{2 M_s} \rho \bar{L}) \right] \\
[\mathbf{K}](3, 1) &= \left[\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{M M_s} \rho \bar{L} \right] [\bar{L} F \dot{D}_x - 2\bar{L} \dot{D}_z + \bar{L} F D_z + \bar{L} D_x]
\end{aligned}$$

$$\begin{aligned}
& - H \left[\frac{2}{M_{srt} M M_s} (M_s \bar{L} + \frac{1}{2} \rho \bar{L}^2) D_x - 2 D_x (\bar{L} + \frac{1}{2} \rho \bar{L}^2) \right] \\
[\mathbf{K}](3, 2) &= \left[\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{M M_s} \rho \bar{L} \right] [\bar{L} F \dot{D}_x - 2 \bar{L} \dot{D}_z + \bar{L} F D_z + \bar{L} D_x] \\
& - H \left[D_x \bar{L} \frac{M_{srt}}{2 M M_s} (2 M_s + \rho \bar{L}) - D_x (M_s \bar{L} + \frac{1}{2} \rho \bar{L}^2) \right] \\
[\mathbf{K}](3, 3) &= \frac{M_{prt}}{M} [(\ddot{\bar{L}} - F \dot{\bar{L}}) \bar{L} - \bar{L}^2] - \frac{1}{M} \rho [\bar{L} \dot{\bar{L}}^2 \\
& + \frac{3}{2} \bar{L}^2 (\ddot{\bar{L}} - F \dot{\bar{L}}) - \bar{L}^3] + \frac{\rho}{M_s} \left[\frac{1}{2} \bar{L}^2 (\ddot{\bar{L}} - F \dot{\bar{L}}) \right. \\
& - \left. \frac{1}{3} \bar{L}^3 \right] - \frac{1}{2 M_s M} \rho^2 [\bar{L}^2 \dot{\bar{L}}^2 + \bar{L}^3 (\ddot{\bar{L}} - F \dot{\bar{L}}) - \frac{1}{2} \bar{L}^4] \\
& - H \left[-\frac{1}{M M_s} (2 M_s^2 \bar{L}^2 + \frac{1}{2} \rho^2 \bar{L}^4 + 2 M_s \rho \bar{L}^3) \right] \\
& + E A H^4 \frac{h_K^6}{M_e^4 G^4} \frac{\bar{L}}{l_b M_s}
\end{aligned}$$

Gyroscopic Matrix [C]

$$\begin{aligned}
[\mathbf{C}](1, 1) &= \frac{M_{srt} M_p}{2 M M_s} [-4 (D_x \dot{D}_x + D_z \dot{D}_z) - 2 (D_x^2 + D_z^2) F] - I_{yy} F \frac{1}{M_s l_b^2} \\
[\mathbf{C}](1, 2) &= \frac{M_p}{M} [-2 D_z \dot{\bar{L}} + F D_z \bar{L} + 2 \bar{L} D_x] \\
& + \frac{M_p}{M M_s} \rho [-2 D_z \dot{\bar{L}} \bar{L} + \frac{1}{2} D_z F \bar{L}^2 + D_x \bar{L}^2] \\
[\mathbf{C}](1, 3) &= \frac{M_p}{M} [-D_x (2 \dot{\bar{L}} - F \bar{L}) + D_z \bar{L}] \\
& + \frac{M_p}{M_s M} \rho \left[D_x (-\frac{1}{2} F \bar{L} - \bar{L} \dot{\bar{L}}) + D_z \bar{L}^2 \right] \\
[\mathbf{C}](2, 1) &= \left[\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{M M_s} \rho \bar{L} \right] [-2 \bar{L} \dot{D}_z + F \bar{L} D_z - 2 \bar{L} D_x] \\
[\mathbf{C}](2, 2) &= -\frac{M_{prt}}{M} F \bar{L}^2 + \frac{1}{M} \rho (F \bar{L}^3 - 3 \bar{L}^2 \dot{\bar{L}}) + 2 \frac{M_{prt}}{M} \bar{L} \dot{\bar{L}}
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{2MM_s} \rho^2 (2\bar{L}^3 \dot{\bar{L}} - \frac{1}{2} F \bar{L}^4) - \rho (-\dot{\bar{L}} \bar{L}^2 + \frac{1}{3} \bar{L}^2 F) \\
[\mathbf{C}](2,3) &= -2 \frac{M_{prt}}{M} \bar{L}^2 - \frac{2}{3} \rho \bar{L}^3 + \frac{2}{M} \rho \bar{L}^3 \\
& + \frac{1}{2MM_s} \rho^2 \bar{L}^4 \\
[\mathbf{C}](3,1) &= \left[\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{MM_s} \rho \bar{L} \right] [-2\bar{L} \dot{D}_x + F \bar{L} D_x - 2\bar{L} D_z] \\
[\mathbf{C}](3,2) &= -[\mathbf{C}](2,3) \\
[\mathbf{C}](3,3) &= \frac{M_{prt}}{M} \bar{L} (2\dot{\bar{L}} - F \bar{L}) - \frac{1}{M} \rho (3\bar{L}^2 \dot{\bar{L}} - F \bar{L}^3) \\
& + \rho (\bar{L}^2 \dot{\bar{L}} - \frac{1}{3} \bar{L}^3 F) + \frac{1}{4} \rho^2 \frac{1}{MM_s} F \bar{L}^4 - \rho^2 \frac{1}{MM_s} \bar{L}^3 \dot{\bar{L}}
\end{aligned}$$

Control Influence Matrix $[\bar{\mathbf{B}}]$

$$\begin{aligned}
[\bar{\mathbf{B}}](1,1) &= -\frac{M_{srt} M_p}{M_s M} D_z \\
[\bar{\mathbf{B}}](1,2) &= -H^4 \\
[\bar{\mathbf{B}}](1,3) &= \frac{M_{srt} M_p}{M_s M} D_x \\
[\bar{\mathbf{B}}](2,1) &= \left[\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{MM_s} \rho \bar{L} \right] \bar{L} \\
[\bar{\mathbf{B}}](2,2) &= 0 \\
[\bar{\mathbf{B}}](2,3) &= 0 \\
[\bar{\mathbf{B}}](3,1) &= 0 \\
[\bar{\mathbf{B}}](3,2) &= 0 \\
[\bar{\mathbf{B}}](3,3) &= -\left[\frac{M_p}{M} + \frac{1}{2} \frac{M_p}{MM_s} \rho \bar{L} \right] \bar{L}
\end{aligned}$$

Retrieval Influence Vector $\bar{\mathbf{P}}$

$$\begin{aligned}
\bar{\mathbf{P}}(1) = & \frac{M_{srt}M_p}{2M_sM} \left[-4(D_x\dot{D}_x + D_z\dot{D}_z) + 2F(D_x^2 + D_z^2) \right] + \frac{M_p}{M} \left[(F\dot{\bar{L}} - \ddot{\bar{L}})D_x \right. \\
& + 2D_z\dot{\bar{L}} - FD_z\bar{L} + \bar{L}D_x \left. \right] + \frac{M_p}{MM_s} \rho \left[-D_x\bar{L}(\ddot{\bar{L}} - F\dot{\bar{L}}) \right. \\
& - D_x\dot{\bar{L}}^2 + D_z(2\bar{L}\dot{\bar{L}} - \frac{1}{2}F\bar{L}^2) + \frac{1}{2}D_x\bar{L}^2 \left. \right] + \frac{I_{yy}}{M_sl_b^2} F \\
& - H \left[-\frac{3}{MM_s} D_xD_zM_{srt}^2 + \frac{2M_{srt}}{MM_s} (D_xM_s\bar{L} + \frac{1}{2}D_x\rho\bar{L}^2) \right. \\
& + 3\frac{M_{srt}}{M_s} D_xD_z - 2D_x\bar{L}(M_s + \frac{1}{2}\rho\bar{L}) \left. \right]
\end{aligned}$$

$$\begin{aligned}
\bar{\mathbf{P}}(2) = & \frac{M_{prt}}{2MM_s} \left[2\bar{L}^2F - 4\bar{L}\dot{\bar{L}} \right] + \left[\frac{M_p}{M} + \frac{1}{2}\frac{M_p}{MM_s}\rho\bar{L} \right] \left[-\bar{L}F\dot{D}_x \right. \\
& + 2\bar{L}\dot{D}_z - FD_z\bar{L} - D_x\bar{L} \left. \right] + \frac{1}{M} \rho (-F\bar{L}^3 + 3\bar{L}^2\dot{\bar{L}}) \\
& - \frac{\rho}{M_s} (\dot{\bar{L}}\bar{L}^2 - \frac{1}{3}\bar{L}^3F) + \frac{1}{2MM_s} \rho^2 (2\bar{L}^3\dot{\bar{L}} - \frac{1}{2}F\bar{L}^4) \\
& - \frac{H}{M_s} \left[-D_x(M_s\bar{L} + \frac{1}{2}\rho\bar{L}^2) \right]
\end{aligned}$$

$$\begin{aligned}
\bar{\mathbf{P}}(3) = & [\mathbf{K}](3,3) + \left[\frac{M_p}{M} + \frac{1}{2}\frac{M_p}{MM_s}\rho\bar{L} \right] \left[\bar{L}F\dot{D}_z \right. \\
& + 2\bar{L}\dot{D}_x - FD_x\bar{L} + D_z\bar{L} \left. \right] - H \left[\frac{2M_{srt}}{MM_s} D_z(M_s\bar{L} + \frac{1}{2}\rho\bar{L}^2) \right. \\
& - \frac{M_{srt}}{2MM_s} \bar{L}(2M_s + \rho\bar{L})D_z - 2D_z(M_s\bar{L} + \frac{1}{2}\rho\bar{L}^2) \left. \right] \\
& - EAH^4 \frac{h_K^6}{M_e^4G^4} \frac{\bar{L}}{l_bM_s}
\end{aligned}$$

Appendix B: Typical Weighting Matrices

The decomposition described in Section 4.2 amounts to writing the functional J of Section 4.1 as $J = J_s + J_f$, where J_s corresponds to the lower frequency pitch motions and J_f pertains to the tether stretch. These two quantities are given by:

$$\begin{aligned} J_s &= \int_0^\infty \mathbf{x}_s^T [\mathbf{Q}_s] \mathbf{x}_s + \mathbf{u}_s^T \mathbf{R}_s \mathbf{u}_s dt; \\ J_f &= \int_0^\infty \mathbf{x}_f^T [\mathbf{Q}_f] \mathbf{x}_f + \mathbf{u}_f^T [\mathbf{R}_f] \mathbf{u}_f dt. \end{aligned}$$

The nonzero elements of the weighting matrices are given here for the stationkeeping case shown in Figure 4.3 (fixed weight case). Note that $[\mathbf{R}_f]$ and $\mathbf{u}_f$ are scalars in this case since D_z is the only control variable for the high frequency motion. The feedforward matrices used to return the offsets to the starting position are also shown.

Weights for State Variables:

$$\begin{aligned} Q_s(1,1) &= 100 \\ Q_s(2,2) &= 1000 \\ Q_s(3,3) &= 100 \\ Q_s(4,4) &= 1000 \\ \\ Q_f(1,1) &= 100 \\ Q_f(2,2) &= 10 \end{aligned}$$

Weights for Control Variables:

$$R_s(1,1) = 5$$

$$R_s(2,2) = 0.1$$

$$R_f = 0.0001$$

Feedforward Matrices:

$$V(1,1) = 14$$

$$V(2,2) = 250$$

$$W(1,1) = 6$$

$$W(2,2) = 240$$

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