ON THE WHITEHEAD GROUPS OF SEMI-DIRECT PRODUCTS OF FREE GROUPS

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ABSTRACT

Let G be a group. We denote the Whitehead group of G by $\mathrm{Wh}\, G$ and the projective class group of the integral group ring $Z(G)$ of G by $\tilde{K}_0 Z(G)$. Then $\mathrm{Wh}\, G = 0$ if G is free abelian (Bass-Heller-Swan), free (Gersten-Stallings) or a semi-direct product of a free group and an infinite cyclic group (Farrell-Hsiang) and $\tilde{K}_0 Z(G) = 0$ if G is free abelian (Bass-Heller-Swan), free (Bass) or a direct product of a free abelian group and a free group (Gersten). In this thesis, we extend these results to a wider class of groups.

Let α be an automorphism of G and F a free group. We denote the semi-direct product of G and F with respect to α by $G \times_\alpha F$. Now, let D be a direct product of n free groups and α an automorphism of D which leaves all but one of the noncyclic factors in D pointwise fixed.

First, by using techniques of Bass-Heller-Swan on Whitehead groups of certain direct products, together with techniques of Stallings on Whitehead groups of free products, we prove $\mathrm{Wh}\, D = 0$ and $\tilde{K}_0 Z(D) = 0$.

Next, we establish a fundamental theorem for coherent rings : If R is a (right) Noetherian ring and if G_1 and G_2 are groups such that the group rings $R(G_1)$ and $R(G_2)$ are (right) coherent, then $R(G_1 * G_2)$ is (right) coherent, where $G_1 * G_2$ is the free product of G_1 and G_2. A similar theorem has been anounced by F. Waldhausen. From this fundamental theorem, we deduce that if A is a free abelian group and F is a free group, the integral group ring $Z(A \times F)$ of $A \times F$ is (right)

coherent. If A is of finite rank, then $Z(A \times F)$ has finite right global dimension. Combining these facts with techniques of Farrell-Hsiang on Whitehead groups of certain semi-direct products of groups and using the triviality of $Wh\ D$ and $\tilde{K}_o Z(D)$, we show that $Wh(D \times_\alpha T) = 0$ and $\tilde{K}_o Z(D \times_\alpha T) = 0$. The first result generalizes that of Farrell-Hsiang on semi-direct product $F \times_\alpha T$, and the second result implies, in particular, that for the fundamental group $\pi_1(M)$ of a closed surface M (other than the real projective plane), the projective class group of $Z(\pi_1(M))$ is trivial.

If M is a closed surface (other than the real projective plane) and $(S^1)^k$ is the k-dimensional torus, the fundamental group of $M \times (S^1)^k$ is of the form $D \times_\alpha T$. Then the triviality of $Wh(D \times_\alpha T)$ implies the following result in topology : If N is a differentiable or PL manifold of dim ≥ 5 which is h-cobordant to $M \times (S^1)^k$, then N is actually diffeomorphic or PL-homeomorphic to $M \times (S^1)^k$ respectively.

Finally, by adapting Gersten's discussion on Whitehead group of free associative algebra to the case of a twisted free associative algebra, and by using the facts that $Wh(D \times_\alpha T) = 0$ and $\tilde{K}_o Z(D \times_\alpha T) = 0$, we prove $Wh((D \times_\alpha T) \times_{\alpha \times id_T} F) = 0$. The factor T can presumably be dropped, although this is not entirely obvious.

There is also a separate chapter on combinatorial group theory in which we give certain necessary and sufficient conditions for a given one relator group to be of the form $F \times_\alpha T$.

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INTRODUCTION

This thesis deals with the Whitehead groups of certain semi-direct products of free groups.

Let G be a group. We denote the Whitehead group of G by $\mathrm{Wh}\, G$ and the projective class group of the integral group ring $Z(G)$ of G by $\tilde{K}_0 Z(G)$. Wh and $\tilde{K}_0$ are covariant functors from groups to abelian groups. The problem of computing $\mathrm{Wh}\, G$ is difficult but important in Algebraic K-theory. It is not even easy to decide when $\mathrm{Wh}\, G$ is trivial. It is known that $\mathrm{Wh}\, G = 0$ if G is one of the following forms :

(a) infinite cyclic or finite cyclic of order 2, 3 or 4 ([12]) ;

(b) free abelian ([4]) ;

(c) free ([18]) ;

(d) a semi-direct product $F \times_\alpha T$ of a free group F and an infinite cyclic group T with respect to an automorphism α of F ([8]) ;

(e) group of type n, that is semi-direct product of n infinite cyclic groups ([8]) ;

(f) free products of groups as given in (a) - (e) ([18]).

In this thesis, we extend (b), (c), (d) to a wider class of groups.

Also, it is known that $\tilde{K}_0 Z(G) = 0$ if G is one of the following forms :

(a) free abelian ([4]) ;

(b) free ([3]) ;

(c) direct product of a free abelian group and a free group ([11]).

We also extend these results in the thesis.

Our work is divided into four chapters. In section 1 of Chapter 1, we state those definitions and terminology which are used throughout the thesis. We will make use of two techniques in computing $\mathrm{Wh}\, G$. One of them is due to Stallings ([18]) on free products and the other is due to Farrell and Hsiang ([8]) on semi-direct products $G \times_\alpha T$ of groups, or Bass-Heller-Swan ([4]) when α is the identity. In fact, Farrell and Hsiang obtained a decomposition formula for $\mathrm{Wh}(G \times_\alpha T)$, which is due to Bass-Heller-Swan when α is the identity. We recall, in Section 2 and Section 3 of Chapter 1, these techniques and those formulae which will be subsequently used.

Chapter 2 is devoted to the study of a special class of rings, called coherent rings, which is of some importance in Algebraic K-theory, especially in computing $\mathrm{Wh}\, G$. A ring R is called right coherent if any finitely generated submodule of a free right R-module is finitely presented. The importance of the coherent property can be explained as follows : If the integral group ring $Z(G)$ of G is right coherent and has finite right global dimension, the exotic summand $\tilde{C}(Z(G),\alpha)$ in the Farrell-Hsiang (or Bass-Heller-Swan when α = identity) decomposition formula for $\mathrm{Wh}(G \times_\alpha T)$ becomes zero and this greatly simplifies the determination of $\mathrm{Wh}(G \times_\alpha T)$. We establish in this chapter the following fundamental theorem for coherent rings : Let R be a right Noetherian ring and let G_1 and G_2 be groups such that the group rings $R(G_1)$ and $R(G_2)$ are right coherent. Then $R(G_1 * G_2)$ is right coherent where $G_1 * G_2$ is the free product of G_1

and G_2 . A similar theorem has been anounced in ([19]). From this fundamental theorem, we deduce that the ring $R(F)$ of a free group F over a right Noetherian ring R is right coherent and that the integral group ring $Z(A \times F)$ is right coherent for any abelian group A.

In Chapter 3, we show the triviality of Wh G and $\tilde{K}_0 Z(G)$ for certain semi-direct products G of free groups. Let D be a direct product of n free groups. By using the Bass-Heller-Swan decomposition formula together with Stallings' technique, we show in Section 1 of Chapter 3 that Wh D = 0. This extends the results of Bass-Heller-Swan ([4]) for free abelian groups and of Gersten-Stallings ([18]) for free groups. In addition to the triviality of Wh D, we have $\tilde{K}_0 Z(D) = 0$. This generalizes those results for $\tilde{K}_0$ previously mentioned.

Next, let α be an automorphism of D which leaves all but one of the noncyclic factors in D pointwise fixed and $D \times_\alpha T$ the semi-direct product of D and T with respect to α. In Section 2 of Chapter 3, we show that $Wh(D \times_\alpha T) = 0$. In the proof, we need to use the coherence property of $Z(A \times F)$, which we have established in Chapter 2. This generalizes that of Farrell-Hsiang ([8]) on $F \times_\alpha T$. As a consequence, we have $\tilde{K}_0 Z(D \times_\alpha T) = 0$. This implies, in particular, that for the fundamental group $\pi_1(M)$ of a closed surface M (other than the real projective plane), the projective class group of $Z(\pi_1(M))$ is trivial.

If M is a closed surface (other than the real projective plane) and $(S^1)^k$ is the k-dimensional torus, the fundamental group of $M \times (S^1)^k$

is of the form $D \times_\alpha T$. Then the triviality of $Wh(D \times_\alpha T)$ implies the following result in topology : If N is a differentiable or PL manifold of $\dim \geq 5$ which is h-cobordant to $M \times (S^1)^k$, then N is actually diffeomorphic or PL-homeomorphic to $M \times (S^1)^k$ respectively.

In the last section of Chapter 3, we establish the result $Wh((D \times_\alpha T) \times_{\alpha \times id_T} F) = 0$, where F is a free group. This generalizes the result in §3.2. In proving this assertion, we come across the so-called "twisted free associative algebras". We adapt Gersten's discussion on free associative algebras ([10]) to this case of twisted free associative algebras in Section 3 of Chapter 3.

Chapter 4 is a separate chapter dealing with groups with one defining relator. We obtain certain necessary and sufficient conditions for such a one relator group to be of the form $F \times_\alpha T$, with F free.

CHAPTER I

PRELIMINARIES

§1.1. Definitions and Terminology

Throughout the thesis, a ring R always mean an associative ring with identity, and ring homomorphisms are assumed to map the identity into the identity. The ring of integers is denoted by Z. If G is a group, the group ring of G over R is denoted by R(G).

The purpose of this section is to recall those definitions and terminology from Algebraic K-theory that will be used in the thesis. A general reference for these will be [2], [8] and [14].

The Whitehead group K_1R of a ring

Let R be a ring. Denote the group of all nonsingular $n \times n$ matrices over R by $GL(n,R)$. We have a natural inclusion $GL(n,R) \subset GL(n+1,R)$. The commutator subgroup of $GL(n,R)$ is denoted by $[GL(n,R), GL(n,R)]$ and the Whitehead group K_1R of R is defined by

$$K_1R = \underset{n \to \infty}{\text{direct limit}}\ GL(n,R)/[GL(n,R), GL(n,R)].$$

If $a \in GL(n,R)$, denote the corresponding element in K_1R by $[a]$. Clearly, K_1 is a covariant functor from rings to abelian groups. That is, a ring homomorphism $f : R_1 \longrightarrow R_2$ induces a homomorphism $f_* : K_1R_1 \longrightarrow K_1R_2$.

The Whitehead group Wh G of a group

Let G be a group. Let $J(G)$ be the subgroup of $K_1Z(G)$ generated by the elements $[(\pm g)]$ for $g \in G$ where $(\pm g)$ is the 1×1 matrix with single entry g or $-g$. The quotient group $K_1Z(G)/J(G)$ is called <u>the Whitehead group of G</u>, denoted by <u>Wh G</u>. Clearly, Wh is a covariant functor from groups to abelian groups. In other words, any group homomorphism $f : G_1 \longrightarrow G_2$ induces a homomorphism $f_* : \mathrm{Wh}\, G_1 \longrightarrow \mathrm{Wh}\, G_2$.

In general, it is a difficult problem to compute Wh G ; it is even not easy to determine when is Wh G trivial. In [12], Higman proved that $\mathrm{Wh}\, G = 0$ if G is infinite cyclic, or is finite of order 2, 3 or 4. Bass, Heller and Swan ([4]) have shown that the Whitehead group of any free abelian group is zero, while Stallings ([18]) and Gersten ([10]) have proved that the Whitehead group of any free group is zero. In Chapter 3, we will determine some classes of groups G in which $\mathrm{Wh}\, G = 0$, and this will generalize the results mentioned above.

The Grothendieck group K_0R of a ring

Closely related to K_1 is the functor K_0 which is defined as follows. Let $\mathbb{P}(R)$ be the category whose objects are finitely generated projective right R-modules and whose morphisms are R linear homomorphisms. Then K_0R is the Grothendieck group of $\mathbb{P}(R)$; i.e. K_0R is the abelian group generated by the isomorphism classes of objects in $\mathbb{P}(R)$ modulo the relations $(P_2 - P_1 - P_3)$ for short exact sequences $0 \to P_1 \to P_2 \to P_3 \to 0$

in $\mathbb{P}(R)$. The class of $P \in \mathbb{P}(R)$ in K_0R is denoted by $[P]$.

The class of the free right R-module of rank 1 generates a cyclic subgroup of K_0R. The quotient

$$K_0R/(\text{subgroup generated by free right R-modules})$$

is called the <u>projective class group</u> $\tilde{K}_0R$ of R. Clearly, $\tilde{K}_0$ is also a covariant functor from rings to abelian groups.

Bass, Heller and Swan ([4]) proved that $\tilde{K}_0Z(A) = 0$ for any free abelian group A, while Bass ([3]) has shown that $\tilde{K}_0Z(F) = 0$ for any free group F. Moreover, Gersten ([11]) proved that $\tilde{K}_0Z(A \times F) = 0$ where A is free abelian and F is free. We will also obtain some generalization of these results in Chapter 3.

<u>The group $\tilde{C}(R,\alpha)$</u>

There is another class of group (introduced by Farrell in [9]) associated to a given ring with an automorphism, which is also closely related to K_1 and K_0, and is defined as follows.

Let R be a ring and α an automorphism of R. First, recall that an additive map ϕ from a right R-module M_1 to a right R-module M_2 is <u>α-linear</u> if $\phi(mr) = \phi(m)\alpha(r)$ for $m \in M_1$ and $r \in R$. Let $\mathcal{C}(R,\alpha)$ be the category whose objects are pairs (P,ϕ) where $P \in \mathbb{P}(R)$ and ϕ is an α-linear nilpotent endomorphism of P, and whose morphisms

$g : (P_1,\phi_1) \longrightarrow (P_2,\phi_2)$ are R linear homomorphisms $g : P_1 \longrightarrow P_2$ such that the following diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{g} & P_2 \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ P_1 & \xrightarrow[g]{} & P_2 \end{array}$$

is commutative. We have the forgetting functor $\mathcal{F} : \mathcal{C}(R,\alpha) \longrightarrow \mathbb{P}(R)$ defined by $\mathcal{F}(P,\phi) = P$ for $(P,\phi) \in \mathcal{C}(R,\alpha)$, and the zero functor $\mathcal{J} : \mathbb{P}(R) \longrightarrow \mathcal{C}(R,\alpha)$ defined by $\mathcal{J}(P) = (P,0)$ for $P \in \mathbb{P}(R)$. Both $\mathcal{F}$ and $\mathcal{J}$ are covariant functors and $\mathcal{F}\circ\mathcal{J}$ is the identity functor of $\mathbb{P}(R)$.

Let $C'(R,\alpha)$ be the Grothendieck group of the category $\mathcal{C}(R,\alpha)$. The class of an element $(P,\phi) \in \mathcal{C}(R,\alpha)$ in $C'(R,\alpha)$ is denoted by $[P,\phi]$. The class $[R,0]$ generates a cyclic subgroup $\mathbb{F}(R)$ of $C'(R,\alpha)$. Let $C(R,\alpha) = C'(R,\alpha)/\mathbb{F}(R)$ and let $\underline{\tilde{C}(R,\alpha)}$ be the subgroup of $C(R,\alpha)$ generated by $[R^n,\phi]$ for $(R^n,\phi) \in \mathcal{C}(R,\alpha)$.

The following result gives us more precisely the relations between K_o, $C'(R,\alpha)$, $C(R,\alpha)$ and $\tilde{C}(R,\alpha)$:

<u>Theorem 1.1.1.</u> ([9]) The following sequences are split exact :

$$0 \longrightarrow \tilde{C}(R,\alpha) \xrightarrow{I} C'(R,\alpha) \underset{\mathcal{J}_*}{\overset{\mathcal{F}_*}{\rightleftarrows}} K_oR \longrightarrow 0,$$

$$0 \longrightarrow \tilde{C}(R,\alpha) \longrightarrow C(R,\alpha) \underset{\mathcal{J}_*}{\overset{\mathcal{F}_*}{\rightleftarrows}} \tilde{K}_oR \longrightarrow 0,$$

where $\mathcal{F}_*$ and $\mathcal{J}_*$ are homomorphisms induced by $\mathcal{F}$ and $\mathcal{J}$ respectively, and $I[R^n,\phi] = [R^n,\phi] - [R^n,0]$.

Semi-direct product of groups

Let G be a group and α an automorphism of G. Let F be a free group generated by $\{x_\lambda\}$. If w is a word in x_λ defining an element in F, we denote by $|w|$ the total exponent sum of the x_λ appearing in w. The semi-direct product $G \times_\alpha F$ of G and F with respect to α is defined as follows : $G \times_\alpha F = G \times F$ as sets and multiplication in $G \times_\alpha F$ is given by

$$(g,w)(g',w') = (g\alpha^{-|w|}(g'), ww')$$

for $(g,w), (g',w') \in G \times_\alpha F$. In particular, if F is an infinite cyclic group $T = \langle t\rangle$ generated by t, we have the semi-direct product $G \times_\alpha T$ of G and T with respect to α.

Twisted group rings

Let R be a ring and α an automorphism of R. Let F be a free group (or free semigroup) generated by $\{x_\lambda\}$. The α-twisted R group ring of F, denoted by $R_\alpha[F]$, is defined as follows : additively $R_\alpha[F] = R[F]$ so that its elements are finite linear combinations of elements in F with coefficients in R. Multiplication in $R_\alpha[F]$ is given by

$$(rw)(r'w') = r\alpha^{-|w|}(r')ww'$$

for any $rw, r'w' \in R_\alpha[F]$. In particular, if F is a free group (resp. free semigroup) generated by t, we have $R_\alpha[T]$ (resp. $R_\alpha[t]$) and we call it the <u>α-twisted finite Laurent series ring</u> (resp. <u>α-twisted polynomial ring</u>).

Let $R = Z(G)$ and α an automorphism of G. Then α is also used to denote the induced automorphism on $Z(G)$ defined by

$$\alpha\left(\sum_{g\in G} \lambda_g g\right) = \sum_{g\in G} \lambda_g \alpha(g)$$

where $g \in G$ and $\lambda_g \in Z$. Note that there is a standard isomorphism between $Z(G)_\alpha[F]$ (resp. $Z(G)_\alpha[T]$) and $Z(G \times_\alpha F)$ (resp. $Z(G \times_\alpha T)$) which is the identity map on $Z(G)$ and maps $x_\lambda \in Z(G)_\alpha[F]$ (resp. $t \in Z(G)_\alpha[T]$) onto $x_\lambda \in Z(G \times_\alpha F)$ (resp. $t \in Z(G \times_\alpha T)$).

In [8], Farrell and Hsiang obtained a formula for $K_1R_\alpha[T]$, and, as an application, they deduced a decomposition formula for $Wh(G \times_\alpha T)$, which we will review in the next section.

§1.2. The Farrell-Hsiang Decomposition Formula for $Wh(G \times_\alpha T)$

In this section, we recall some of the results in Farrell and Hsiang ([8]) which will be subsequently used. For more detail, we refer to [8].

First, we recall the following terminology. If M is a right R-module, we denote by $r_s : M \longrightarrow M$ a right multiplication by $s \in R$, i.e. $r_s(m) = ms$ for all $m \in M$. If M is a right module over $R_\alpha[T]$, then $r_t : M \longrightarrow M$ is an α-linear endomorphism.

The following inclusion maps are ring homomorphisms :

$$j : R \longrightarrow R_\alpha[T],$$

$$k : R \longrightarrow R_\alpha[t] \quad \text{and} \quad k^- : R \longrightarrow R_\alpha[t^{-1}],$$

$$i : R_\alpha[t] \longrightarrow R_\alpha[T] \quad \text{and} \quad i^- : R_\alpha[t^{-1}] \longrightarrow R_\alpha[T] \ ;$$

and the following two ring homomorphisms are augmentations :

$$\varepsilon : R_\alpha[t] \longrightarrow R \quad \text{defined by} \quad \varepsilon(t) = 0,$$

$$\varepsilon^- : R_\alpha[t^{-1}] \longrightarrow R \quad \text{defined by} \quad \varepsilon^-(t^{-1}) = 0.$$

For any abelian group G with an automorphism α, we introduce the following two subgroups :

$$G^\alpha = \{ g \mid \alpha(g) = g, \ g \in G \}$$

$$I(\alpha) = \{ g - \alpha(g) \mid g \in G \}.$$

Consider now the homomorphism $i : R_\alpha[t] \longrightarrow R_\alpha[T]$. We identify the element $P \in \mathbb{P}(R_\alpha[t])$ with

$$P \otimes_{R_\alpha[t]} R_\alpha[t] \subset P \otimes_{R_\alpha[t]} R_\alpha[T],$$

by sending x to $x \otimes 1$ for $x \in P$. We will then give a description of a homomorphism from $K_1R_\alpha[T]$ into $C'(R,\alpha)$.

Let $a \in GL(n, R_\alpha[T])$ and let

$$v : R_\alpha[T]^n \longrightarrow R_\alpha[T]^n$$

be the linear isomorphism associated with a. We have the natural inclusion

$$R_\alpha[t]^n \subset R_\alpha[T]^n ,$$

Thus, there is an integer $N \geq 0$ such that $r_{t^N} v(R_\alpha[t]^n) \subset R_\alpha[t]^n$. Let

$$M = R_\alpha[t]^n / r_{t^N} v(R_\alpha[t]^n).$$

Then $M \in \mathbb{P}(R)$ and r_t induces an α-linear nilpotent endomorphism on M, i.e. $(M, r_t) \in \mathcal{C}(R,\alpha)$ (cf. [8], Theorem 8 (b)). Let $p : K_1R_\alpha[T] \longrightarrow C'(R,\alpha)$ be defined by

(1) $$p[a] = [M, r_t] - [R_\alpha[t]^n / r_{t^N}(R_\alpha[t]^n), r_t].$$

Then one can verify that p is a homomorphism (cf. [8], Theorem 8 (c)). In particular, if $a = (t)$, the 1×1 matrix determined by the generator t of T, then $n = 1$ and $N = 0$ so that in (1), $M = R$ and the second term on the right hand side is zero. Thus

$$p[(t)] \;=\; [R, r_t].$$

By combining p with the map $F_* : C'(R,\alpha) \longrightarrow K_oR$, we get the homomorphism $F_*p : K_1R_\alpha[T] \longrightarrow K_oR$ and that

$$F_*p[(t)] \;=\; [R].$$

Next, let $p' = pi_*^- : K_1R_\alpha[t^{-1}] \longrightarrow C'(R,\alpha)$. Then

<u>Theorem 1.2.1</u>. ([8], Theorem 13) Image $p' = \tilde{C}(R,\alpha)$ and the following sequence is split exact :

$$0 \longrightarrow K_1R \underset{\varepsilon_*^-}{\overset{k_*^-}{\rightleftarrows}} K_1R_\alpha[t^{-1}] \longrightarrow \tilde{C}(R,\alpha) \longrightarrow 0$$

Likewise, the sequence

$$0 \longrightarrow K_1R \underset{\varepsilon_*}{\overset{k_*}{\rightleftarrows}} K_1R_\alpha[t] \longrightarrow \tilde{C}(R,\alpha^{-1}) \longrightarrow 0$$

is split exact. (Here, we identify $\tilde{C}(R,\alpha)$ with the image $I(\tilde{C}(R,\alpha))$ in $C'(R,\alpha)$ where I is given in Theorem 1.1.1.).

Note that the kernel of j_* is $I(\alpha_*)$ where $j_* : K_1R \to K_1R_\alpha[T]$ is the homomorphism induced by the inclusion $j : R \longrightarrow R_\alpha[T]$. (cf. [8], Theorem 14). Moreover, we have :

Theorem 1.2.2. ([8], Theorem 19 and [4]) (Farrell-Hsiang decomposition formula for $K_1R_\alpha[T]$; Bass-Heller-Swan decomposition formula when α = identity)

$$K_1R_\alpha[T] \cong X \oplus \tilde{C}(R,\alpha) \oplus \tilde{C}(R,\alpha^{-1})$$

and X is given by the following exact sequence

$$0 \longrightarrow K_1R/I(\alpha_*) \xrightarrow{\phi} X \xrightarrow{\psi} (K_0R)^{\alpha_*} \longrightarrow 0,$$

where ϕ is induced by j_* and ψ is induced by F_*p.

Remark 1.2.3. Note that, in Theorem 1.2.2,

$$\psi[(t)] = [R].$$

Finally, we recall :

Theorem 1.2.4. ([8], Theorem 21 and [4]) (Farrell-Hsiang decomposition formula for $Wh(G \times_\alpha T)$; Bass-Heller-Swan decomposition formula when α = identity)

$$Wh(G \times_\alpha T) = X \oplus \tilde{C}(Z(G),\alpha) \oplus \tilde{C}(Z(G),\alpha^{-1})$$

where X is given by the following exact sequence

$$0 \longrightarrow Wh\ G/I(\alpha_*) \xrightarrow{\phi} X \xrightarrow{\psi} (\tilde{K}_0Z(G))^{\alpha_*} \longrightarrow 0$$

in which ϕ and ψ are induced by the corresponding maps in Theorem 1.2.2.

Note that $\tilde{C}(Z(G),\alpha) \cong \tilde{C}(Z(G),\alpha^{-1})$ in the case of a group ring $Z(G)$.

As an application, Farrell and Hsiang have also shown that :

<u>Theorem 1.2.5</u>. ([8], Theroem 31) Let F be a free group. Then

$$Wh\ (F \times_\alpha T) = 0.$$

§1.3. <u>Stallings' Decomposition Formula for Free Products</u>

In this section, we recall those definitions and results in [18] which we need in our later work. For more detail, we refer to [18].

Let R be a ring. A ring Λ is called an <u>R-ring</u>, if Λ contains R, the inclusion $i : R \longrightarrow \Lambda$ is a ring homomorphism, and there exists a ring homomorphism $\varepsilon_\Lambda : \Lambda \longrightarrow R$ such that $\varepsilon_\Lambda(r) = r$ for all $r \in R$. ε_Λ is called an <u>augmentation</u> of Λ. Any group ring $R(G)$ of a group G is an R-ring with augmentation $\varepsilon_G : R(G) \longrightarrow R$ defined by $\varepsilon_G(g) = 1$ for all $g \in G$. If Λ is an R-ring, we denote by $\overline{K}_1\Lambda$ the cokernel of the homomorphism $i_* : K_1R \longrightarrow K_1\Lambda$, induced by the inclusion $i : R \longrightarrow \Lambda$.

Let Λ and Γ be R-rings with augmentations ε_Λ and ε_Γ respectively. If a ring homomorphism $f : \Lambda \longrightarrow \Gamma$ is such that $f(r) = r$ for all $r \in R$, we call f a <u>homomorphism of R-rings</u> or simply just <u>R-homomorphism</u>. We say that the R ring Ω is the <u>free product</u> of Λ and Γ if there are given R-homomorphisms $f : \Lambda \longrightarrow \Omega$ and $g : \Gamma \longrightarrow \Omega$

such that for any R-ring Σ and R-homomorphisms $f' : \Lambda \longrightarrow \Sigma$ and $g' : \Gamma \longrightarrow \Sigma$, there exists a unique consistent R-homomorphism $h : \Omega \longrightarrow \Sigma$. We abbreviate it as $\Omega = \Lambda * \Gamma$. It is clear that $R(G_1 * G_2) = R(G_1) * R(G_2)$ for any two groups G_1 and G_2, where $G_1 * G_2$ denotes the free product of G_1 and G_2. Note that the free product of R-rings is just the coproduct in the category whose objects are R-rings and whose morphisms are R-homomorphisms.

Now, let $\overline{\Lambda} = \text{Ker } \varepsilon_\Lambda$ and $\overline{\Gamma} = \text{Ker } \varepsilon_\Gamma$. Then $\overline{\Lambda}$ and $\overline{\Gamma}$ are R-bimodules and as bimodules

$$\Lambda = R \oplus \overline{\Lambda}, \qquad \Gamma = R \oplus \overline{\Gamma}.$$

Moreover, the multiplications on Λ and Γ define associative maps $\overline{\Lambda} \otimes_R \overline{\Lambda} \longrightarrow \overline{\Lambda}$ and $\overline{\Gamma} \otimes_R \overline{\Gamma} \longrightarrow \overline{\Gamma}$. We have the following structure theorem for $\Lambda * \Gamma$ ([18], §3.2) :

$$\Lambda * \Gamma = R \oplus \overline{\Lambda} \oplus \overline{\Gamma} \oplus (\overline{\Lambda} \otimes_R \overline{\Gamma}) \oplus (\overline{\Gamma} \otimes_R \overline{\Lambda}) \oplus (\overline{\Lambda} \otimes_R \overline{\Gamma} \otimes_R \overline{\Lambda}) \oplus (\overline{\Gamma} \otimes_R \overline{\Lambda} \otimes_R \overline{\Gamma}) \oplus \cdots .$$

The multiplicative structure is determined by multiplying components by the tensor product and then collapsing if possible using the multiplications $\overline{\Lambda} \otimes_R \overline{\Lambda} \longrightarrow \overline{\Lambda}$ and $\overline{\Gamma} \otimes_R \overline{\Gamma} \longrightarrow \overline{\Gamma}$ derived from Λ and Γ.

Recall that, if M is an R-bimodule, the <u>tensor algebra $T_R(M)$</u> of M over R is defined to be

$$T_R(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \cdots .$$

It is clear that $\Lambda * \Gamma$ contains as a subring the tensor algebra $T_R(\overline{\Lambda} \otimes_R \overline{\Gamma})$

of the bimodule $\overline{\Lambda} \otimes_R \overline{\Gamma}$. Moreover, we have :

<u>Theorem 1.3.1</u>. ([18], §5) The group $\overline{K}_1(\Lambda * \Gamma)$ is generated by the images, under the obvious maps, of $\overline{K}_1\Lambda$, $\overline{K}_1\Gamma$ and $\overline{K}_1 T_R(\overline{\Lambda} *_R \overline{\Gamma})$.

The following result is due to Gersten (cf. [2], p.646). If $\overline{K}_1 R[t] = 0$ where $R[t]$ is the polynomial extension of R, (in other words, if $k_* : K_1 R \longrightarrow K_1 R[t]$ is an isomorphism or $\tilde{C}(R,id) = 0$ by Theorem 1.2.1), then $\overline{K}_1 T_R(M) = 0$ for any free R-bimodule M. Therefore :

<u>Theorem 1.3.2</u>. ([18], Theorem 6.2) If $\overline{\Lambda} \otimes_R \overline{\Gamma}$ is a free R-bimodule and if $\tilde{C}(R,id) = 0$, then

$$\overline{K}_1(\Lambda * \Gamma) \cong \overline{K}_1\Lambda \oplus \overline{K}_1\Gamma.$$

Finally, let G be a group and $R(G)$ the group ring with augmentation ε_G. It is known that $\operatorname{Ker} \varepsilon_g$ is a free R-bimodule. We close our review by the following theorem, which is a direct consequence of Theorem 1.3.2.

<u>Theorem 1.3.3</u>. Let R be a ring such that $\tilde{C}(R,id) = 0$ and let G_1 and G_2 be groups. Then

$$\overline{K}_1 R(G_1 * G_2) \cong \overline{K}_1 R(G_1) \oplus \overline{K}_1 R(G_2).$$

In particular, if F is a free group of rank m, we write $F = T_1 * \ldots * T_m$

as a free product of m infinite cyclic groups, so that

$$\overline{K}_1 R(F) \cong \bigoplus_{j=1}^{m} \overline{K}_1 R(T_j).$$

As an application, Stallings has shown that

$$Wh\ (G_1 * G_2) \cong Wh\ G_1 \oplus Wh\ G_2$$

and so $Wh\ F = 0$ for a free group F.

CHAPTER 2

COHERENT RINGS

§2.1. Introduction

The present chapter is devoted to the study of a special class of rings, called coherent rings, which are of importance in Algebraic K-theory. We first recall the definition of such rings.

Let R be a ring. A right R-module M is said to be _finitely presented_ if there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ of right R-modules, where F is free and both F and K are finitely generated. Notice that if M is finitely presented and if there is another exact sequence $0 \longrightarrow K' \longrightarrow F' \longrightarrow M \longrightarrow 0$ of right R-modules, with F' free and finitely generated, then K' is necessarily finitely generated. For a proof, see ([2]).

Definition 2.1.1. A ring R is called _right coherent_ if any finitely generated submodule of a free right R-module is finitely presented. An equivalent property is : Any homomorphism $f : R^n \longrightarrow R^m$ of right R-modules R^n and R^m has finitely generated kernel.

A general reference for coherent rings is Chase [7], Bourbaki [6] and Soublin [16]. Of course, any right Noetherian ring is right coherent.

However, there are important examples of coherent rings, which are not Noetherian.

Theorem 2.1.2. Let F be a free group. Then the integral group ring $Z(F)$ is right coherent.

The proof of this theorem is implicitly contained in the argument of ([8], Theorem 31). We do not reproduce the argument in [8], since the result will follow from our main theorem in the next section. Note that $Z(F)$ is not right Noetherian unless F is cyclic.

We caution that there is no "Hilbert basis theorem" for coherent rings. Indeed, Soublin in [17] gave an example of a coherent ring whose polynomial extension is not coherent. Nevertheless, we will see in section 2 that the polynomial extension of $Z(F)$ is right coherent.

We have the following result, the proof of which is contained in [9] (also cf. proof of Theorem 31 in [8]) :

Theorem 2.1.3. If R is right coherent and has finite right global dimension, then $k_* : K_1R \longrightarrow K_1R_\alpha[t]$ is an isomorphism for any automorphism α of R. In other words $\tilde{C}(R,\alpha) = 0$.

This theorem implies, for example, that for a group G with $Z(G)$ right coherent and of finite right global dimension, the exotic summand $\tilde{C}(Z(G),\alpha)$ in the Farrell-Hsiang decomposition formula for $Wh\ (G \times_\alpha T)$

becomes zero. This greatly simplifies the determination of $Wh(G \times_{\alpha} T)$.

Before proving the coherence of $Z(G)$ for certain classes of groups G, we close this section with the following result on direct limit of coherent rings.

Lemma 2.1.4. ([6], p.63) Let $\{R_\lambda\}$ be a directed system of rings R_λ and let R be their direct limit. Suppose that R is flat as a left R_λ-module for each λ. If each R_λ is right coherent, then R is right coherent.

§2.2. The Group Ring of a Free Group over a Noetherian Ring

Let R be a right Noetherian ring and F a free group. The main purpose of this section is to show that the group ring $R(F)$ is right coherent. In fact, we will prove the following more general result.

Theorem 2.2.1. Let R be a right Noetherian ring and let G_1 and G_2 be groups such that $R(G_1)$ and $R(G_2)$ are right coherent. Let $G = G_1 * G_2$ be the free product of G_1 and G_2. Then $R(G)$ is right coherent.

Before proving the theorem, we introduce some relevant terminology. Let R be a ring and $N = (r_{ij})$ an $m \times n$ matrix over R. If $f : R^n \longrightarrow R^m$ is the homomorphism (of right R-modules) associated to N, then the kernel of f is precisely the solution space of N. We call N a (right) coherent matrix if its solution space is finitely generated as a right R-module. It follows that a ring R is right coherent if and only if all $m \times n$ matrices over R are (right) coherent.

Now, let N be an $m \times n$ matrix over R. Let N_1 (resp. N_2) be the $m \times n$ matrix over R obtained from N by an elementary row operation (resp. elementary column operation) and let N_3 be the $(m + 1) \times (n + 1)$ matrix $\left(\begin{array}{c|c} N & 0 \\ \hline 0 & 1 \end{array}\right)$. Then the following lemma is trivial:

<u>Lemma 2.2.2</u>. For each i, N_i is coherent if and only if N is coherent.

Let R be a ring. Let G_1 and G_2 be groups and $G = G_1 * G_2$ their free product so that there are natural inclusions

$$\begin{tikzcd} & R(G_1) \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & R(G) \\ & R(G_2) \arrow[ur] & \end{tikzcd}$$

Recall that each element $1 \neq g \in G$ can be uniquely expressed as a product

(1) $$g = g_1 g_2 \cdots g_n$$

where $g_i \neq 1$, g_i is in G_1 or G_2 and g_i, g_{i+1} are not in the same free factor G_1 or G_2. If $g \in G$ is expressed in the form (1), then we call n the <u>syllable length</u> $|g|$ of g (cf. [13], p.182).

Our next lemma is a key step towards the proof of Theorem 2.2.1.

Lemma 2.2.3. Let M_1 be a submodule of $R(G)^m$ (as right $R(G)$-module) generated by certain elements in $R(G_1)^m$ and M_2 a submodule of $R(G)^m$ generated by certain elements in $R(G_2)^m$. Let $K = (M_1 + M_2) \cap R^m$. Then

$$(M_1 + K\cdot R(G)) \cap (M_2 + K\cdot R(G)) = K\cdot R(G). \tag{2}$$

Here $K\cdot R(G)$ denotes the right $R(G)$-module generated by K.

Proof : Let M_i^o be the $R(G_i)$-submodule of $R(G_i)^m$ generated by the same set of elements which generate M_i $(i = 1,2)$. Then $M_i^o \subset M_i$ $(i = 1,2)$. Let $K\cdot R(G_i)$ be the right $R(G_i)$-module generated by K.

One direction of inclusions in (2) is obvious. So, let $x \in (M_1 + K\cdot R(G)) \cap (M_2 + K\cdot R(G))$. Then, considering x as an element in $R(G)^m$, we can express x uniquely as

$$x = \sum_i c_i w_i \tag{3}$$

with $c_i \in R^m$ and $w_i \in G$ such that $|w_1| \geq |w_2| \geq \cdots$. Also, x can be expressed uniquely as

$$x = \sum_j a_j u_j , \tag{4}$$

and

$$(5) \qquad x = \sum_k b_k v_k ,$$

where a_j (resp. b_k) is in $M_1^o + K \cdot R(G_1)$ (resp. $M_2^o + K \cdot R(G_2)$) and u_j (resp. v_k) is an element in G starting with a nontrivial element in G_2 (resp. G_1) for each j (resp. k) except that one of them may be trivial. We assert that $c_i \in K$ for each i. Since $c_i \in R^m$, it suffices to prove that $c_i \in M_1 + M_2$ for each i.

Without loss of generality, we can assume that w_1 is an element in G starting with a nontrivial element in G_2. Then, in the expression (4), there is a j (say $j = 1$), such that $u_1 = w_1$. We claim that $a_1 = c_1$. For this purpose, write

$$a_1 = c_1' + \sum_\ell d_\ell \overline{u}_\ell,$$

where c_1', $d_\ell \in R^m$ and $\overline{u}_\ell$ is a nontrivial element in G_1, for each ℓ. If $d_\ell \neq 0$ for some ℓ, then $d_\ell \overline{u}_\ell w_1$ must appear in the expression (3), which contradicts the fact that w_1 is of maximum syllable length. Hence, all $d_\ell = 0$ so that $a_1 = c_1' = c_1$. Therefore

$$c_1 \in M_1^o + K \cdot R(G_1) \subset M_1 + M_2.$$

(Note that, if w_1 is an element starting with a nontrivial element in G_1, then we will consider (5) and it will lead to the conclusion that $c_1 \varepsilon M_2^o + K\cdot R(G_2) \subset M_1 + M_2$.) Consequently $c_1 \varepsilon K$.

Having proved $c_1 \varepsilon K$, we can apply the same procedure to $x - c_1w_1$ to conclude inductively that $c_i \varepsilon K$ for all i. Hence $x \varepsilon K\cdot R(G)$.

This completes the proof.

For convenience, we state the following result, the proof of which is trivial.

Lemma 2.2.4. Let N be an $m \times n$ matrix over $R(G_1)$ (resp. $R(G_2)$). Let $f : R(G_1)^n \longrightarrow R(G_1)^m$ (resp. $f : R(G_2)^n \longrightarrow R(G_2)^m$) be the homomorphism associated with N and $g : R(G)^n \longrightarrow R(G)^m$ the homomorphism associated with N (considered as a matrix over $R(G)$). If Ker f is a finitely generated right $R(G_1)$-module (resp. $R(G_2)$-module), then Ker g is a finitely generated right $R(G)$-module.

As a consequence, we have :

Corollary 2.2.5. If $R(G_1)$ (resp. $R(G_2)$) is a right coherent ring and if N is a matrix over $R(G)$ with entries from $R(G_1)$ (resp. $R(G_2)$), then the solution space of N is finitely generated as a right $R(G)$-module.

Now, we make the following important remark :

Remark 2.2.6. (Modified Higman's trick)

Let N' be an $m' \times n'$ matrix over $R(G)$. Each entry of N' is a finite linear combination of the form $\sum_g r_g g$ with $r_g \in R$ and $g \in G$. Note that g is of the form (1). We perform successive simplifications of N' by changing it to

$$\left[\begin{array}{c|c} N' & 0 \\ \hline 0 & 1 \end{array}\right],$$

an $(m' + 1) \times (n' + 1)$ matrix over $R(G)$, and reduce this matrix by elementary row and column operations (where we multiply row from the left and column from the right) so as to (i) make linear combinations shorter and (ii) reduce the syllable lengths of g. This reduction process can be illustrated by :

$$\begin{bmatrix} * & * & * \\ * & \sum +g_1g_2 & * \end{bmatrix} \to \begin{bmatrix} * & * & * & 0 \\ * & \sum +g_1g_2 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} * & * & * & 0 \\ * & \sum +g_1g_2 & * & 0 \\ 0 & g_2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} * & * & * & 0 \\ * & \sum & * & -g_1 \\ 0 & g_2 & 0 & 1 \end{bmatrix}.$$

Finally, we reduce N' to the matrix of the form

(6) $$N = [\, N_{G_1} \mid N_{G_2} \,],$$

where

$$N_{G_1} = \begin{bmatrix} a_{11} & \cdots & a_{1\lambda} \\ \vdots & & \vdots \\ a_{m1} & & a_{m\lambda} \end{bmatrix} \quad \text{and} \quad N_{G_2} = \begin{bmatrix} b_{11} & \cdots & b_{1\mu} \\ \vdots & & \vdots \\ b_{m1} & & b_{m\mu} \end{bmatrix}$$

are $m \times \lambda$ matrix and $m \times \mu$ matrix over $R(G_1)$ and $R(G_2)$ respectively, and $\lambda + \mu = n$, for some m and n.

Proof of Theorem 2.2.1 : We need to show that any $m \times n$ matrix over $R(G)$ is right coherent.

By Lemma 2.2.2 and Remark 2.2.6, we can assume without loss of generality that the given $m \times n$ matrix N over $R(G)$ is of the form (6).

Now, let $a_i = (a_{1i}, \cdots, a_{mi})$ $(i = 1, \cdots, \lambda)$ and $b_j = (b_{1j}, \cdots, b_{mj})$ $(j = 1, \cdots, \mu)$. Then let M_1 be the submodule of $R(G)^m$ generated by the elements $a_1, \cdots, a_\lambda$ in $R(G_1)^m$ and M_2 the submodule of $R(G)^m$ generated by the elements $b_1, \cdots, b_\mu$ in $R(G_2)^m$. If $f : R(G)^n \longrightarrow R(G)^m$ is the homomorphism associated with N, we have the following presentation for $M_1 + M_2$:

$$(7) \qquad 0 \longrightarrow \ker f \longrightarrow R(G)^n \xrightarrow{f} M_1 + M_2 \longrightarrow 0.$$

Next, let $K = (M_1 + M_2) \cap R^m$. Then K is a submodule of R^m (as right R-module) and so K is finitely generated since R is right Noetherian. Suppose that K is generated by $c_k = (c_{1k}, \cdots, c_{mk})$ $(k = 1, \cdots, \sigma)$ in R^m, and let

$$N_R = \begin{bmatrix} c_{11} & \cdots & c_{1\sigma} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{m\sigma} \end{bmatrix}$$

be the $m \times \sigma$ matrix over R determined by the generators of K. Consider the new $m \times (n + \sigma)$ matrix

$$\overline{N} = [\, N_{G_1} \mid N_R \mid N_{G_2} \,]$$

over $R(G)$, and let $\overline{N}_G = [\, N_{G_1} \mid N_R \,]$ and $\overline{N}_{G_2} = [\, N_R \mid N_{G_2} \,]$. Since $K \subset M_1 + M_2$, it follows that $a_1, \cdots, a_\lambda$, $c_1, \cdots, c_\sigma$, $b_1, \cdots, b_\mu$ still generate $M_1 + M_2$. If $g : R(G)^{n+\sigma} \longrightarrow R(G)^m$ is the homomorphism associated with $\overline{N}$, we have another presentation for $M_1 + M_2$:

$$(8) \qquad 0 \longrightarrow \text{Ker } g \longrightarrow R(G)^{n+\sigma} \xrightarrow{g} M_1 + M_2 \longrightarrow 0$$

To prove that Ker f in (7) is finitely generated, we only need to show that Ker g in (8) is finitely generated (compare remark in §2.1). For this purpose, let $(x_1, \cdots, x_\lambda,\ z_1, \cdots, z_\sigma,\ y_1, \cdots, y_\mu) \in \text{Ker } g$. Then

$$a_1x_1 + \cdots + a_\lambda x_\lambda + c_1z_1 + \cdots + c_\sigma z_\sigma + b_1y_1 + \cdots + b_\mu y_\mu = 0$$

so that

$$(9) \qquad a_1x_1 + \cdots + a_\lambda x_\lambda + c_1z_1 + \cdots + c_\sigma z_\sigma = -\,(b_1y_1 + \cdots + b_\mu y_\mu)\ .$$

Let $\overline{x}$ be the element of the left hand side of (9). Then (9) tells us that $\overline{x} \in (M_1 + K\cdot R(G)) \cap (M_2 + K\cdot R(G))$ and so $\overline{x} \in K\cdot R(G)$, by Lemma 2.2.3. Thus

$$\overline{x} = c_1 z_1' + \cdots + c_\sigma z_\sigma' \tag{10}$$

for some $z_1', \cdots, z_\sigma' \in R(G)$. By (9) and (10), we have

$$a_1 x_1 + \cdots + a_\lambda x_\lambda + c_1 z_1 + \cdots + c_\sigma z_\sigma = c_1 z_1' + \cdots + c_\sigma z_\sigma'$$

and

$$c_1 z_1' + \cdots + c_\sigma z_\sigma' = -(b_1 y_1 + \cdots + b_\mu y_\mu).$$

That is,

$$a_1 x_1 + \cdots + a_\lambda x_\lambda + c_1(z_1 - z_1') + \cdots + c_\sigma(z_\sigma - z_\sigma') = 0$$

and

$$c_1 z_1' + \cdots + c_\sigma z_\sigma' + b_1 y_1 + \cdots + b_\mu y_\mu = 0.$$

These mean that $(x_1, \cdots, x_\lambda, z_1 - z_1', \cdots, z_\sigma - z_\sigma')$ is in the solution space of $\overline{N}_{G_1}$ and $(z_1', \cdots, z_\sigma', y_1, \cdots, y_\mu)$ is in that of $\overline{N}_{G_2}$. Since $R(G_1)$ and $R(G_2)$ are right coherent, the solution spaces of $\overline{N}_{G_1}$ and $\overline{N}_{G_2}$ are finitely generated right $R(G)$-modules (Corollary 2.2.5).

Finally, note that if $(x_1', \cdots, x_\lambda', z_1'', \cdots, z_\sigma'')$ is in the solution space of $\overline{N}_{G_1}$, then $(x_1', \cdots, x_\lambda', z_1'', \cdots, z_\sigma'', \overbrace{0, \cdots, 0}^{\mu \text{ terms}})$ is in that of $\overline{N}$ and if $(z_1'', \cdots, z_\sigma'', y_1', \cdots, y_\mu')$ is in the solution space of $\overline{N}_{G_2}$, then $(\overbrace{0, \cdots, 0}^{\lambda \text{ terms}}, z_1'', \cdots, z_\sigma'', y_1', \cdots, y_\mu')$ is in that of $\overline{N}$. Since

$$\begin{aligned}
&(x_1, \cdots, x_\lambda, z_1, \cdots, z_\sigma, y_1, \cdots, y_\mu) \\
&= (x_1, \cdots, x_\lambda, z_1 - z_1', \cdots, z_\sigma - z_\sigma', \overbrace{0, \cdots, 0}^{\mu \text{ terms}}) + (\overbrace{0, \cdots, 0}^{\lambda \text{ terms}}, z_1', \cdots, z_\sigma', y_1, \cdots, y_\mu),
\end{aligned}$$

and since the solution spaces of $\overline{N}_{G_1}$ and $\overline{N}_{G_2}$ are finitely generated, it follows from the above observation that Ker g is finitely generated.

This completes the proof.

Remark 2.2.7. Waldhausen in [19] considers the question of the coherence of a group ring $Z(G)$ when G is an amalgamated product. The proof of Theorem 2.2.1 is partly inspired by his arguments. When suitably adapted, our present proof will also show the coherence of the free product of two rings, under appropriate hypothesis.

Corollary 2.2.8. Let R be a right Noetherian ring and F_n a free group of finite rank. Then $R(F_n)$ is right coherent. In particular, if A is a finitely generated abelian group, then $Z(A \times F_n)$ is right coherent.

Proof : Since F_n is the free product of a free group F_{n-1} of rank $n - 1$ and an infinite cyclic group T, the first assertion follows from Theorem 2.2.1 by induction on n. The second assertion follows from the first since $Z(A \times F) = Z(A)(F)$ and $Z(A)$ is Noetherian.

As a consequence of Lemma 2.1.4 and Corollary 2.2.8, we have :

Corollary 2.2.9. Let R be a right Noetherian ring and F a free group Then $R(F)$ is right coherent. Moreover, if A is a abelian group, then $Z(A \times F)$ is right coherent.

We will use the following result to prove the triviality of $Wh(G \times_\alpha T)$ for a certain class of groups G.

Corollary 2.2.10. Let F be a free group and A a free abelian group of finite rank. Then

$$\tilde{C}(Z(A \times F),\alpha) = 0$$

for any automorphism α of $A \times F$.

Proof : Since rt. gl. dim $Z(F) \leq 2$ (cf. [8], Lemma 33), it follows that rt. gl. dim $Z(A \times F)$ = rt. gl. dim $Z(F)$ + rank of $A < \infty$ (cf. [1], Lemma 2). The assertion now follows from Theorem 2.1.3, since $Z(A \times F)$ is right coherent.

We close this chapter by the following remark :

Remark 2.2.11. Let R be a ring and G be a group. Let $\alpha : G \longrightarrow \text{Aut } R$ be a homomorphism of G into the automorphism group Aut R of R. The twisted group ring $R_\alpha(G)$ is defined as follows : additively $R_\alpha(G) = R(G)$ and multiplication is given by

$$(rg)(r'g') = r(\alpha(g))(r')gg'$$

for any $rg, r'g' \in R_\alpha(G)$. Then the results in Theorem 2.2.1 and Lemma 2.2.3 remain true if we replace group rings by twisted group rings. As a consequence, we see that the group ring $Z(A \times_\alpha F)$, of a semi-direct product $A \times_\alpha F$, is right coherent.

CHAPTER 3

WHITEHEAD GROUPS OF SOME SEMI-DIRECT PRODUCTS OF FREE GROUPS

§3.1. The Whitehead Group of a Direct Product $\prod_{i=1}^{n} F_i$ of Free Groups F_i

Bass, Heller and Swan ([4]) proved that $\mathrm{Wh}\, A = 0$ for a free abelian group A, and Stallings ([18]) and Gersten ([10]) have shown that $\mathrm{Wh}\, F = 0$ for a free group F. The main purpose of this section is to generalize these results to the following :

Theorem 3.1.1. Let $D = \prod_{i=1}^{n} F_i$ be a direct product of n free groups F_i. Then $\mathrm{Wh}\, D = 0$.

Proof : Let the number of noncyclic factors in D be k. We will prove the theorem by induction on k.

For $k = 0$, D is just a free abelian group so that $\mathrm{Wh}\, D = 0$. This starts the induction.

Now, suppose inductively that the theorem holds for any such group with $k - 1$ noncyclic factors. To show $\mathrm{Wh}\, D = 0$, write $D = D' \times F$ where F is noncyclic, and the number of noncyclic factors in D' is $k - 1$. Then, for an infinite cyclic group T, $\mathrm{Wh}\,(D' \times T) = 0$ by induction hypothesis. It follows from the Bass-Heller-Swan decomposition formula for

Wh $(D' \times T)$ (cf. Theorem 1.2.4) that

(1) $$\tilde{C}(Z(D'),\mathrm{id}) = 0 \quad \text{and} \quad \tilde{K}_0 Z(D') = 0.$$

Next, suppose that F is of finite rank m and write $F = T_1 * \cdots * T_m$ as a free product of m infinite cyclic groups. Since $\tilde{C}(Z(D'),\mathrm{id}) = 0$, it follows from Theorem 1.3.3 that

(2) $$\overline{K}_1 Z(D')(F) \cong \bigoplus_{j=1}^{m} \overline{K}_1 Z(D')(T_j).$$

Again, using $\tilde{C}(Z(D'),\mathrm{id}) = 0$, we deduce from Theorem 1.2.2 that the sequence

$$0 \longrightarrow K_1 Z(D') \longrightarrow K_1 Z(D')(T_j) \longrightarrow K_0 Z(D') \longrightarrow 0$$

is short exact for each j ; i.e. $\overline{K}_1 Z(D')(T_j) \cong K_0 Z(D')$ for each j. Then it follows from (2) that

$$\overline{K}_1 Z(D')(F) \cong K_0 Z(D') \oplus \cdots \oplus K_0 Z(D') \qquad (m \text{ copies}) ;$$

i.e., the sequence

(3) $$0 \longrightarrow K_1 Z(D') \longrightarrow K_1 Z(D')(F) \xrightarrow{\psi} K_0 Z(D') \oplus \cdots \oplus K_0 Z(D') \longrightarrow 0$$

is short exact. Passing to Whitehead groups, we have (cf. Remark 1.2.3)

$$0 \longrightarrow \mathrm{Wh}\ D' \longrightarrow \mathrm{Wh}\ (D' \times F) \xrightarrow{\tilde{\psi}} \tilde{K}_0 Z(D') \oplus \cdots \oplus \tilde{K}_0 Z(D') \longrightarrow 0$$

Hence, by (1) and the induction hypothesis for D', we have

$$\text{Wh } D = \text{Wh } (D' \times F) = 0.$$

The case when F has infinite rank does not need to worry us since a matrix over $Z(D)(F)$ involves entries which are sums of words involving only a finite number of free generators of F.

This completes the proof.

In addition to the vanishing of Wh D, we have the following vanishing results, as in (1) :

Corollary 3.1.2. $\tilde{C}(Z(D),\text{id}) = 0$ and $\tilde{K}_0 Z(D) = 0$.

§3.2. The Whitehead Group of $\left(\prod_{i=1}^{n} F_i\right) \times_\alpha T$

Let F be a free group, α an automorphism of F and T an infinite cyclic group. Then Farrell and Hsiang ([8]) have shown that Wh $(F \times_\alpha T) = 0$. This section is devoted to the following generalized result :

Theorem 3.2.1. Let $D = \prod_{i=1}^{n} F_i$ be a direct product of n free groups F_i. Let α be an automorphism of D which leaves all but one of the noncyclic factors in D pointwise fixed. Then Wh $(D \times_\alpha T) = 0$.

Proof : Let k be the number of noncyclic factors in D. We prove by induction on k.

For $k = 0$, D is just a free abelian group A and so, by the Farrell-Hsiang decomposition formula for $\mathrm{Wh}\,(D \times_\alpha T)$ (cf. Theorem 1.2.4), $\mathrm{Wh}\,(D \times_\alpha T) = 0$ since $\tilde{C}(Z(A),\alpha) = 0$, $\mathrm{Wh}\,A = 0$ and $\tilde{K}_o Z(A) = 0$.

For $k = 1$, D is of the form $A \times F$ with A free abelian of finite rank and F noncyclic. Then, in the Farrell-Hsiang decomposition formula for $\mathrm{Wh}\,((A \times F) \times_\alpha T)$, the term $\mathrm{Wh}\,(A \times F) = 0$ by Theorem 3.1.1 and $\tilde{K}_o Z(A \times F) = 0$ by Corollary 3.1.2. Also, thanks to the coherence property of $Z(A \times F)$ and the fact that $Z(A \times F)$ is of finite right global dimension, $\tilde{C}(Z(A \times F),\alpha) = 0$ (Corollary 2.2.10). Hence $\mathrm{Wh}((A \times F) \times_\alpha T) = 0$.

Now, suppose inductively that the theorem holds for any such group with $k - 1$ noncyclic factors. Let $D = H \times F$ with F noncyclic and α fixed on F while H has $k - 1$ noncyclic factors. To show that $\mathrm{Wh}\,(D \times_\alpha T) = 0$, write $D \times_\alpha T = D' \times F$ with $D' = H \times_\alpha T$. The situation is now completely analogous to Theorem 3.1.1 and the same argument as there gives $\mathrm{Wh}\,(D \times_\alpha T) = 0$.

This completes the proof.

In addition to the triviality of $\mathrm{Wh}\,(D \times_\alpha T)$, we have, by the Farrell-Hsiang decomposition formula for $\mathrm{Wh}\,(D \times_\alpha T)$ that $\tilde{C}(Z(D),\alpha) = 0$. Moreover, by considering $\mathrm{Wh}\,((D \times_\alpha T) \times T_1)$ which is just $\mathrm{Wh}\,((D \times T_1) \times_{\alpha \times \mathrm{id}_{T_1}} T)$, we have the following vanishing results :

<u>Corollary 3.2.2.</u> (i) $\tilde{C}(Z(D \times_\alpha T),\mathrm{id}) = 0$.

(ii) $\tilde{K}_o Z(D \times_\alpha T) = 0$.

The result (ii) of Corollary 3.2.2 implies, in particular, that for the fundamental group $\pi_1(M)$ of a closed surface M (other than the real projective plane), the projective class group of $Z(\pi_1(M))$ is trivial.

We also need the following slight generalization of Corollary 3.2.2 (i) in §3.4.

<u>Corollary 3.2.3.</u> Let D and α be as given in Theorem 3.2.1. Then

$$\tilde{C}(Z(D \times_\alpha T), \alpha^\mu) = 0$$

for any integer μ, where α^μ denotes the automorphism $\alpha^\mu \times id_T$ of $D \times_\alpha T$ induced by α^μ of D.

<u>Proof</u> : Let $T_1 = \langle t_1 \rangle$ be another infinite cyclic group generated by t_1. Consider the semi-direct product $(D \times_\alpha T) \times_{\alpha^\mu} T_1$. Then, by change of generators in $T \times T_1$, $(D \times_\alpha T) \times_{\alpha^\mu} T_1$ can be seen to be isomorphic to $(D \times S) \times_{\alpha \times id_S} T$, where $S = \langle t^{-\mu} t_1 \rangle$ is an infinite cyclic group generated by $t^{-\mu} t_1$. By Theorem 3.2.1, $Wh((D \times S) \times_{\alpha \times id_S} T) = 0$ and so $Wh((D \times_\alpha T) \times_{\alpha^\mu} T_1) = 0$. Hence $\tilde{C}(Z(D \times_\alpha T), \alpha^\mu) = 0$.

This completes the proof.

There is a topological application of Theorem 3.2.1. If M is a closed surface (other than the real projective plane) and $(S^1)^k$ is the k-dimensional torus, then the fundamental group of $M \times (S^1)^k$ is of the form $D \times_\alpha T$. Hence Theorem 3.2.1 implies the following (cf. [14], p.393) :

Corollary 3.2.4. If N is a differentiable or PL manifold of $\dim \geq 5$ which is h-cobordant to $M \times (S^1)^k$, then N is actually diffeomorphic or PL-homeomorphic to $M \times (S^1)^k$ respectively.

§3.3. K_1 of Twisted Free Associative Algebras

Let R be a ring and X a set of non-commuting variables $\{x_\lambda\}_{\lambda\varepsilon\Lambda}$. Let $R\{X\}$ be the free associative algebra on X over R. Gersten has shown that if $K_1R \longrightarrow K_1R[t]$ is an isomorphism, where $R[t]$ is the polynomial extension of R, then $K_1R \longrightarrow K_1R\{X\}$ is an isomorphism (cf. [10] and [2], p.646).

This section presents a generalization of Gersten's result to twisted free associative algebras which we will apply in §3.4.

Let X be a set of non-commuting variables $\{x_\lambda\}_{\lambda\varepsilon\Lambda}$ and let $a = \{\alpha_\lambda\}_{\lambda\varepsilon\Lambda}$ be a set of automorphisms α_λ of R. The a-twisted free associative algebra on X over R, denoted by $R_a\{X\}$, is defined as follows :

Additively, $R_a\{X\} = R\{X\}$ so that its elements are finite linear combinations of words $w(x_\lambda)$ in x_λ with coefficients in R.

If $w(x_\lambda) = x_{\lambda_1} \cdots x_{\lambda_k}$ is a word in x_λ, we denote the automorphism $\alpha_{\lambda_1} \cdots \alpha_{\lambda_k}$ by $w(\alpha_\lambda)$.

Multiplication in $R_\alpha\{X\}$ is given by :

$$(rw(x_\lambda))(r'w'(x_\lambda)) = rw(\alpha_\lambda)^{-1}(r')w(x_\lambda)w'(x_\lambda),$$

for any $rw(x_\lambda)$, $r'w'(x_\lambda) \in R_\alpha\{X\}$.

We shall consider $R_\alpha\{X\}$ as an R-ring with augmentation $\varepsilon_X : R_\alpha\{X\} \longrightarrow R$ defined by $\varepsilon_X(x_\lambda) = 0$ for each $x_\lambda \in X$. Denote by $\overline{K}_1R_\alpha\{X\}$ the cokernel of the homomorphism $i_* : K_1R \longrightarrow K_1R_\alpha\{X\}$ induced by the inclusion $i : R \longrightarrow R_\alpha\{X\}$. Note that the augmentation ε_X induces a homomorphism $\varepsilon_{X*} : K_1R_\alpha\{X\} \longrightarrow K_1R$ which splits i_* .

Now, let N'' be an invertible matrix over $R_\alpha\{X\}$. By Higman's trick, we can make N'' equivalent in $K_1R_\alpha\{X\}$ to

$$N' = N'_o + N'_1x_1 + \cdots + N'_nx_n ,$$

where $x_1, \cdots, x_n$ are distinct elements of X and N'_i $(i = 0,1,\cdots,n) \in \mathfrak{m}_m(R)$ for some integer m. (Here $\mathfrak{m}_m(R)$ denotes the ring of all $m \times m$ matrices over R). By applying the homomorphism ε_{X*} to N', it follows that N'_o is invertible. Hence N'' can be made equivalent in $\overline{K}_1R_\alpha\{X\}$ to

$$N = I + N_1x_1 + \cdots + N_nx_n , \tag{1}$$

where $N = N_o'^{-1}N'$ and $N_i = N_o'^{-1}N'_i$ $(i = 1,\cdots,n)$.

The inverse of this matrix N exists and can be written explicitly in the ring of formal power series. Since this inverse exists in $R_\alpha\{X\}$, all but a finite number of its coefficients are zero. That is, if

$$M = M_o + M_1x_1 + \cdots + M_nx_n + \sum_{i,j=1}^{n} M_{i,j}x_ix_j + \cdots$$

is a matrix over $R_\alpha\{X\}$ where all M_i, $M_{i,j}$, $\cdots$ are matrices over R, such that $MN = NM = I$, then there is an integer $K > 0$ such that $M_{i_1,i_2,\ldots,i_k} = 0$ for all $k > K$, where $i_1, i_2, \cdots, i_k$ run over $1, \cdots, n$ respectively. From $NM = I$, we get, by equating coefficients of monomials in the x's, the following relations :

$$M_o = I \;;$$

$$M_i = -N_i \quad (i = 1, \cdots, n) \;;$$

$$M_{i,j} = N_i\alpha_i^{-1}(N_j) \quad (i, j = 1, \cdots, n) \;;$$

$$\vdots$$

$$M_{i_1,i_2,\cdots i_\ell} = (-1)^\ell N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{\ell-1}}^{-1})(N_{i_\ell})$$
$$(i_1, i_2, \cdots, i_\ell = 1, \cdots, n).$$

Hence, for all $k > K$,

$$N_{i_1}\alpha_{i_1}^{-1}(N_{i_1}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{k-1}}^{-1})(N_{i_k}) = 0. \tag{2}$$

Let us call an element $P \in \mathcal{m}_m(R)$ <u>β-twisted nilpotent</u> (β is any automorphism of R) if there exists an integer $k > 0$ such that

$$P\beta^{-1}(P) \cdots \beta^{-(k-1)}(P) = 0.$$

Hence, it follows from (2) that each N_i $(i = 1,\cdots,n)$ in (1) is α_i-twisted nilpotent.

Our next lemma is the key to the main result:

<u>Lemma 3.3.1.</u> The matrix N in (1) is a product of matrices of the form $I + Pw(x_1,\cdots,x_n)$, where P is an $w(\alpha_1,\cdots,\alpha_n)$-twisted nilpotent matrix over R. ($w(x_1,\cdots,x_n)$ denotes a word in $x_1,\cdots,x_n$).

<u>Proof</u> : Recall from (1) and (2) that each N_i $(i = 1,\cdots,n)$ in (1) is α_i-twisted nilpotent. Consider

$$I + Q = (I - N_1x_1) \cdots (I - N_nx_n)N \ .$$

Then Q is of the form $\sum\limits_j Q_js_j$, where each s_j is a monomial of degree at least two in the $x_1,\cdots,x_n$ and

$$Q_j = \pm N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1} \cdots \alpha_{i_{\ell-1}}^{-1})(N_{i_\ell})$$

$(i_1, i_2, \cdots, i_\ell = 1, \cdots, n)$ where $\ell \geq 2$. Hence, for $k > K/2$,

$$Q_j\beta^{-1}(Q_j) \cdots \beta^{-(k-1)}(Q_j) = 0,$$

for each j, where β is an automorphism obtained in replacing the x_i in s_j by α_i respectively. That is, Q_j is $s_j(\alpha_1,\cdots,\alpha_n)$-twisted nilpotent for each j. Now, consider

$$I + Q' = \prod_j (I - Q_j s_j)(I + Q).$$

Then Q' is of the form $\sum_\sigma Q'_\sigma y_\sigma$, where each y_σ is a monomial of degree at least four in the $x_1, \cdots, x_n$ and for $k > K/4$,

$$Q'_\sigma \gamma^{-1}(Q'_\sigma) \cdots \gamma^{-(k-1)}(Q'_\sigma) = 0,$$

for each σ, where γ is an automorphism obtained in replacing the x_i in y_σ by α_i respectively. That is, Q'_σ is $y_\sigma(\alpha_1,\cdots,\alpha_n)$-twisted nilpotent for each σ.

Left multiplying $I + Q'$ by $\prod_\sigma(I - Q'_\sigma y_\sigma)$, and repeating the above argument, we will finally arrive at the conclusion that

$$\prod(I + Pw(x_1,\cdots,x_n))\cdot N = I$$

where P is an $w(\alpha_1,\cdots,\alpha_n)$-twisted nilpotent matrix over R and $w(x_1,\cdots,x_n)$ is a word in $x_1,\cdots,x_n$.

This completes the proof.

The above discussions are modifications of those given in [10] and ([2], p.647) for (untwisted) free associative algebras ; and the following result is already contained in the above proof (also, cf [4]).

Lemma 3.3.2. For any automorphism β of R, $\overline{K}_1R_\beta[t]$ is generated by the elements of the form $I + Pt$ where P is an β-twisted nilpotent matrix over R.

The following main theorem then follows immediately from Lemma 3.3.1 and Lemma 3.3.2 :

Theorem 3.3.3. The group $\overline{K}_1R_\alpha\{X\}$ is generated by the homomorphic images of $\overline{K}_1R_\beta[t]$ under the homomorphisms

$$\overline{K}_1R_\beta[t] \longrightarrow \overline{K}_1R_\alpha\{X\}$$

induced by the homomorphism $R_\beta[t] \longrightarrow R_\alpha\{X\}$ which maps t into a word $w(x_\lambda)$ in x_λ with $\beta = w(\alpha_\lambda)$ and $w(x_\lambda)$ runs over all the words in x_λ.

Corollary 3.3.4. If R is a ring such that $\tilde{C}(R,\beta) = 0$ for any automorphism β of R, then $\overline{K}_1R_\alpha\{X\} = 0$; in other words, if $K_1R \longrightarrow K_1R_\beta[t]$ is an isomorphism, then $K_1R \longrightarrow K_1R_\alpha\{X\}$ is an isomorphism.

Now, let α be an automorphism of R and for each $\lambda \in \Lambda$, let

$$\alpha_\lambda = \alpha^{m_\lambda}$$

for some integer m_λ. In this case, we denote $R_\alpha\{X\}$ by $R_\alpha\{X\}$. Thus, we have the following corollaries which will be needed in §3.4.

Corollary 3.3.5. The group $\overline{K}_1R_\alpha\{X\}$ is generated by the homomorphic images of $\overline{K}_1R_{\alpha^\mu}[t]$ (μ any integer) under the homomorphisms

$$\overline{K}_1R_{\alpha^\mu}[t] \longrightarrow \overline{K}_1R_\alpha\{X\}$$

induced by the homomorphisms $R_{\alpha^\mu}[t] \longrightarrow R_\alpha\{X\}$ which map t into a word $w(x_\lambda)$ such that μ is the total exponent sum of α appearing in $w(\alpha_\lambda)$, and $w(x_\lambda)$ runs over all the words in x_λ.

Corollary 3.3.6. If R is a ring such that $\tilde{C}(R,\alpha^\mu) = 0$ for any integer μ, then $\overline{K}_1R_\alpha\{X\} = 0$.

§3.4. The Whitehead Group of $\left(\left(\prod_{i=1}^{n} F_i\right) \times_\alpha T\right) \times_{\alpha\times id_T} F$.

Let $D = \prod_{i=1}^{n} F_i$ be a direct product of n free groups F_i and α an automorphism of D which leaves all but one of the noncyclic factors in D pointwise fixed. In Theorem 3.2.1, we proved that $Wh(D \times_\alpha T) = 0$, where T is an infinite cyclic group. Let F be another free group. The purpose of this section is to generalize Theorem 3.2.1 to :

Theorem 3.4.1. $Wh((D \times_\alpha T) \times_{\alpha\times id_T} F) = 0$.

The factor T in the theorem can presumably be dropped, although this is not entirely obvious. Compare the results at the end of this section.

From now on, we denote also by α, the automorphism $\alpha \times id_T$ of $D \times_\alpha T$ induced by the automorphism α on D.

Suppose that F is generated by $\{t_\lambda\}$ and let $R_\alpha[F]$ be the α-twisted group ring of F over a ring R with automorphism α. Notice that, in general, there is no augmentation from $R_\alpha[F]$ into R. Now, let $R = Z(D \times_\alpha T)$. Then $R_\alpha[F]$ is canonically isomorphic to $Z((D \times_\alpha T) \times_\alpha F)$ (cf. §1.1). Define a mapping $\varepsilon_F : R_\alpha[F] \longrightarrow R$ by

$$\varepsilon_F(rt_\lambda) = rt$$

for all $r \in R$ and $t_\lambda \in F$, where t is the generator of T. Then it is clear that ε_F is a homomorphism of $R_\alpha[F]$ onto R with $\varepsilon_F(r) = r$ for all $r \in R$, i.e., we can consider $R_\alpha[F]$ as an R-ring with augmentation ε_F. Note that the homomorphism $i_* : K_1R \longrightarrow K_1R_\alpha[F]$ induced by the inclusion $R \longrightarrow R_\alpha[F]$ is one-to-one, since the homomorphism $\varepsilon_{F*} : K_1R_\alpha[F] \longrightarrow K_1R$, induced by ε_F, splits i_*. Moreover, the kernel of ε_F, denoted by $\overline{R}_\alpha[F]$, is a free R-bimodule generated by

$$w - t^{|w|}$$

where w runs over all the words in t_λ and $|w|$ is the total exponent sum of t_λ appearing in w, and as bimodules,

$$R_\alpha[F] = R \oplus \overline{R}_\alpha[F].$$

Also, we have $R_\alpha[F * F'] = R_\alpha[F] * R_\alpha[F']$ where F' is another free group.

Let $T_R(\overline{R}_\alpha[F] \otimes_R \overline{R}_\alpha[F'])$ be the tensor algebra of $\overline{R}_\alpha[F] \otimes_R \overline{R}_\alpha[F']$ over R. Then it is easy to see that this is nothing but the twisted free associative algebra $R_\alpha\{X\}$ over a set X of non-commuting variables x given by

$$x = (w - t^{|w|}) \otimes (w' - t^{|w'|})$$

where $w - t^{|w|}$ (resp. $|w'| - t^{|w'|}$) runs over the generators of $\overline{R}_\alpha[F]$ (resp. $\overline{R}_\alpha[F']$). Since $\tilde{C}(R,\alpha^\mu) = \tilde{C}(Z(D \times_\alpha T),\alpha^\mu) = 0$ for any integer μ (Corollary 3.2.3), it follows from Corollary 3.3.6 that

$$\overline{K}_1 T_R(\overline{R}_\alpha[F] \otimes_R \overline{R}_\alpha[F']) = 0.$$

Hence, by Theorem 1.3.1, we have

$$\overline{K}_1 R_\alpha[F * F'] \cong \overline{K}_1 R_\alpha[F] \oplus \overline{K}_1 R_\alpha[F'].$$

That is,

(1) $$\overline{K}_1 Z((D \times_\alpha T) \times_\alpha (F * F')) \cong \overline{K}_1 Z((D \times_\alpha T) \times_\alpha F) \oplus \overline{K}_1 Z((D \times_\alpha T) \times_\alpha F').$$

Finally we give the proof of Theorem 3.4.1. It is similar to that of Theorem 3.1.1.

Proof of Theorem 3.4.1 :

First, suppose that F is of finite rank m and write $F = T_1 * \cdots * T_m$ as a free product of m infinite cyclic groups. Then, it follows from (1) that

$$(2) \qquad \overline{K}_1 Z((D \times_\alpha T) \times_\alpha F) \cong \bigoplus_{j=1}^{m} \overline{K}_1 Z((D \times_\alpha T) \times_\alpha T_j).$$

Using $\check{C}(Z(D \times_\alpha T), \alpha^\mu) = 0$ for any integer μ, we deduce, from Theorem 1.2.2, that the sequence

$$0 \longrightarrow K_1 Z(D \times_\alpha T) \underset{\varepsilon_{T_j *}}{\overset{i_*}{\rightleftarrows}} K_1 Z((D \times_\alpha T) \times_\alpha T_j) \longrightarrow (K_o Z(D \times_\alpha T))^{\alpha_*} \longrightarrow 0$$

is split short exact, for each j. That is

$$\overline{K}_1 Z((D \times_\alpha T) \times_\alpha T_j) \cong (K_o Z(D \times_\alpha T))^{\alpha_*}$$

for each j, and so, it follows from (2) that

$$\overline{K}_1 Z((D \times_\alpha T) \times_\alpha F) \cong (K_o Z(D \times_\alpha T))^{\alpha_*} \oplus \cdots \oplus (K_o Z(D \times_\alpha T))^{\alpha_*} \quad (m \text{ copies}).$$

In other words, the sequence

$$0 \longrightarrow K_1 Z(D \times_\alpha T) \underset{\varepsilon_{F*}}{\overset{i_*}{\rightleftarrows}} K_1 Z((D \times_\alpha T) \times_\alpha F) \longrightarrow$$
$$(K_o Z(D \times_\alpha T))^{\alpha_*} \oplus \cdots \oplus (K_o Z(D \times_\alpha T))^{\alpha_*} \longrightarrow 0$$

is short exact. Passing to Whitehead groups (cf. Remark 1.2.3), we have the short exact sequence

$$0 \longrightarrow Wh(D \times_\alpha T) \underset{\dashleftarrow}{\longrightarrow} Wh((D \times_\alpha T) \times_\alpha F) \longrightarrow$$

$$(\tilde{K}_o Z(D \times_\alpha T))^{\alpha_*} \oplus \cdots \oplus (\tilde{K}_o Z(D \times_\alpha T))^{\alpha_*} \longrightarrow 0.$$

But $Wh(D \times_\alpha T) = 0$ (Theorem 3.2.1) and $\tilde{K}_o Z(D \times_\alpha T) = 0$ (Corollary 3.2.2). Hence $Wh((D \times_\alpha T) \times_\alpha F) = 0$.

This completes the proof.

Finally, since $(D \times_\alpha T) \times_{\alpha \times id_T} F$ and $(D \times_\alpha F) \times_{\alpha \times id_F} T$ are isomorphic, we have, in addition to Theorem 3.4.1, that

Corollary 3.4.2. (i) $\tilde{C}(Z(D \times_\alpha F), \alpha \times id_F) = 0$.

(ii) $(\tilde{K}_o Z(D \times_\alpha F))^{(\alpha \times id_F)_*} = 0$.

(iii) $Wh(D \times_\alpha F)/_{I((\alpha \times id_F)_*)} = 0$.

§3.5. Concluding Remarks

Let D be the direct product of n free groups. We observe that the condition we impose on α in §3.2 and §3.4 can be dropped if one can prove that the integral group ring $Z(D)$ is right coherent. In fact, the coherent property of $Z(D)$ will follow from the following conjectural result : Let R be a ring such that $R[T]$, the Laurent series extension of R, is right coherent (this implies that R is coherent). Let G_1 and G_2 be groups such that $R(G_1)$ and $R(G_2)$ are right coherent. Then

$R(G_1 * G_2)$ is right coherent.

Next, let R be a ring and α an automorphism of R. Then it is an interesting and important question to ask whether the coherence of $R[T]$ will imply that of $R_\alpha[T]$. This result, if true, will give the triviality of $\mathrm{Wh}\, G$ for another class of groups G.

Finally, let $F_1, F_2, \cdots, F_n$ be free groups. Let α_1 be an automorphism of F_1. Form the semi-direct product $F_1 \times_{\alpha_1} F_2$. Then, let α_2 be an automorphism of $F_1 \times_{\alpha_1} F_2$ and again form the semi-direct product $(F_1 \times_{\alpha_1} F_2) \times_{\alpha_2} F$. Repeating the same procedure, we arrive at the group

$$G = (\cdots((F_1 \times_{\alpha_1} F_2) \times_{\alpha_2} F_3) \times \cdots) \times_{\alpha_{n-1}} F_n .$$

It would be useful to have a method of computing $\mathrm{Wh}\, G$, $\tilde{K}_0 Z(G)$ and $\tilde{C}(Z(G),\alpha)$. We have proved that $\mathrm{Wh}\, G = 0$ and $\tilde{K}_0 Z(G) = 0$ when $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} =$ identity and for some other special cases (cf. Theorem 3.1.1, Theorem 3.2.1 and Theorem 3.4.1), but the general case remains open.

CHAPTER 4

GROUPS $F \times_\alpha T$ WITH ONE DEFINING RELATOR

§4.1. Introduction

In [8], Farrell and Hsiang have shown that the fundamental group $\pi_1(M)$ of a closed surface M (not the sphere or the projective plane) is of the form $F \times_\alpha T$, where F is a free group and T an infinite cyclic group, and so $\mathrm{Wh}\ \pi_1(M) = 0$. Their proof is topological. Also, it is well known that such a group $\pi_1(M)$ can be presented as a group with one defining relator.

The purpose of this chapter is to obtain certain necessary and sufficient conditions for a group with one defining relator to be of the form $F \times_\alpha T$. This will give us an algebraic proof of the result for $\pi_1(M)$ previously mentioned.

Now, let us recall those definitions, terminology and results from Combinatorial Group Theory which will be subsequently used. For more details and undefined terms, we refer to Karrass, Magnus and Solitar ([13]).

Let G be a group with generators $a, b, c, \cdots$. A word $R(a,b,c,\cdots)$ which defines the identity element 1 in G is called a <u>relator</u>. Let $P, Q, R, \cdots$ be any relators of G. If every relator in G is derivable from $P, Q, R, \cdots$, we call $P, Q, R, \cdots$ <u>a set of defining relators</u> for G on $a, b, c, \cdots$. If $P, Q, R, \cdots$ is a set of defining relators for G, we call

$$< a,b,c,\cdots ; P(a,b,c,\cdots), Q(a,b,c,\cdots), R(a,b,c,\cdots), \cdots >$$

a presentation of G and write

$$G = < a,b,c, \cdots ; P,Q,R, \cdots > . \tag{1}$$

The free group F_n on the n free generators $x_1, x_2, \cdots, x_n$ is the group with generators $x_1, x_2, \cdots, x_n$ and the empty set of defining relators. A cyclically reduced word in $x_1, \cdots, x_n$ is a word in which the symbols $x_i^{\varepsilon}, x_i^{-\varepsilon}$ ($\varepsilon = \pm 1$, $i = 1,\cdots,n$) do not occur consecutively and it does not simultaneously begin with x_i^{ε} and end with $x_i^{-\varepsilon}$ ($\varepsilon = \pm 1$, $i = 1,\cdots,n$).

If $w(x_1, \cdots, x_n)$ is a word in $x_1, \cdots, x_n$, denote by $\sigma_w(x_i)$ the exponent sum of w on x_i.

Theorem 4.1.1. ([13], Theorem 1.3) Let F_n be a free group on the free generators $x_1, \cdots, x_n$ and let w_1, w_2 be two cyclically reduced words. Then w_1, w_2 define conjugate elements of F_n if and only if w_1 is a cyclic permutation of w_2.

Theorem 4.1.2. ([13], Theorem 4.10) (Freiheitssatz Theorem) Let G be a group presented by

$$G = < x_1, \cdots, x_n ; R(x_1, \cdots, x_n) >$$

where $R(x_1, \cdots, x_n)$ is a cyclically reduced word in x_i $(i = 1, \cdots, n)$, which involves x_n. Then the subgroup of G generated by $x_1, \cdots, x_{n-1}$ is freely generated by them.

Theorem 4.1.3. ([13], Theorem 4.11) (Conjugacy Theorem for Groups with One Defining relator) Let $G = < x_1, \cdots, x_n ; R(x_1, \cdots, x_n) >$ and $H = < x_1, \cdots, x_n ; S(x_1, \cdots, x_n) >$. Then G is isomorphic to H under the mapping $x_i \longrightarrow x_i$ if and only if $R(x_1, \cdots, x_n)$ and $S^{\varepsilon}(x_1, \cdots, x_n)$ are conjugate in the free group on $x_1, \cdots, x_n$, for $\varepsilon = 1$ or -1.

We close this section by recalling the Tietze transformations ([13], §1.5).

Let G be a group presented by (1). Then H. Tietze has shown that any other presentation of G can be obtained by a repeated application of the following transformations to (1) :

(T1) If the words $S, U, \cdots$ are derivable from $P, Q, R, \cdots,$ then add $S, U, \cdots$ to the defining relators in (1).

(T2) If some of the relators, say, $S, U, \cdots,$ listed among the defining relators $P, Q, R, \cdots,$ are derivable from the others, delete $S, U, \cdots$ from the defining relators in (1).

(T3) If $K, M, \cdots$ are any words in $a, b, c, \cdots,$ then adjoin the symbols $x, y, \cdots$ to the generators in (1) and adjoin the relations $x = K$, $y = M, \cdots$ to the defining relators in (1).

(T4) If some of the defining relators in (1) take the form $p = V$, $q = W, \cdots$ where $p, q, \cdots$ are generators in (1) and V, W are words in the generators other than $p, q, \cdots,$ then delete $p, q, \cdots$ from the generators, delete $p = V$, $q = W$, $\cdots$ from the defining relations, and replace $p, q, \cdots$ by $V, W, \cdots$ respectively, in the remaining defining relators in (1).

The transformations (T1), (T2), (T3), and (T4) are called <u>Tietze transformations</u>.

Let G be a group presented by

$$(2) \qquad G = \langle x_1, x_2, \cdots, x_n ; R(x_1, x_2, \cdots, x_n) \rangle \qquad (n \geq 2)$$

where we assume that R is cyclically reduced and involves all the generators $x_1, x_2, \cdots, x_n$.

Recall that if there is a split short exact sequence

$$(3) \qquad 1 \longrightarrow N \longrightarrow G \underset{\longleftarrow}{\longrightarrow} T \longrightarrow 1 ,$$

where $T = \langle t \rangle$ is an infinite cyclic group, then G is the semi-direct product $N \times_\alpha T$ of N and T with respect to the automorphism $\alpha : N \to N$ defined by $\alpha(g) = tgt^{-1}$ for all $g \in N$. If G is a group given by (2), it is easy to find a homomorphism $\phi : G \longrightarrow T$ from G onto T. Let N be the kernel of ϕ. Then G satisfies (3) and so is of the form $N \times_\alpha T$. Of course, in general there exist many such homomorphisms ϕ and therefore many splittings.

§4.2. Groups with Two Generators and One Defining Relator

Let G be a group presented by

(1) $$G = \langle a,b \; ; \; R(a,b) \rangle$$

where R is cyclically reduced and involves both a and b. Then we know that G is of the form $N \times_\alpha T$.

The purpose of this section is to obtain certain necessary and sufficient conditions for the factor N in $N \times_\alpha T$ to be free corresponding to certain natural choices of the epimorphism $\phi : G \longrightarrow T$.

We distinguish the following three cases.

Case 1 : $\sigma_R(a) \neq 0$ and $\sigma_R(b) = 0$ or vice versa ;

Case 2 : $\sigma_R(a) \neq 0$ and $\sigma_R(b) \neq 0$;

Case 3 : $\sigma_R(a) = \sigma_R(b) = 0$.

We are able to settle cases 1 and 2, but not quite case 3.

First, let us consider case 1. In this case, the epimorphism $\phi : G \longrightarrow T$ is uniquely defined up to sign by

(2) $$\phi(a) = 1, \quad \phi(b) = t^{\varepsilon} \quad (\varepsilon = 1 \text{ or } -1).$$

Notice that in case 3, the above ϕ (given by (2)) is one of the choices of the homomorphisms from G onto T. Of course, there are many other homomorphisms from G onto T ; for example, one of them will be the homomorphism $\phi_1 : G \longrightarrow T$ defined by $\phi_1(a) = t$, $\phi_1(b) = 1$. We will see later, by an example, that in case 3, G may be of the form $N \times_\alpha T$ and $N_1 \times_\alpha T$ with N free and N_1 not free.

From now on, we will consider case 1 or case 3 with the above homomorphism ϕ. Thus, we may assume that the factor T, in $N \times_\alpha T$, is just the infinite cyclic group generated by b in G and then N is nothing but the normal subgroup of G generated by a.

To obtain a presentation of N, we make use of a Reidemeister-Schreier rewriting process, and as Schreier representatives for G mod N we choose t^i, where i runs over all integers ([13], §2.3 and §4.4). We find that N is generated by the elements a_i defined by

$$a_i = b^i a b^{-i} \quad (i, \text{ any integer}).$$

Now, we rewrite $R(a,b)$ in terms of a_i as follows : Every symbol a^ε ($\varepsilon = \pm 1$) in $R(a,b)$ is replaced by a_s^ε where s is the sum of the exponents of the b-symbols preceding the particular a^ε in $R(a,b)$. Thus $R(a,b)$ can be expressed in terms of a_i as :

(3) $$R(a,b) = R_o(a_\lambda, a_{\lambda+1}, \cdots, a_\mu)$$

with $\lambda < \cdots < \mu$. Then N is generated by a_i and has as defining relators

(4) $$P_i = b^i R(a,b) b^{-i}$$

$$= R(a_{\lambda+i}, a_{\lambda+1+i}, \cdots, a_{\mu+i})$$

(i, any integer), and so N can be presented as

(5) $$N = < \cdots, a_{-1}, a_o, a_1, \cdots ; \cdots, P_{-1}, P_o, P_1, \cdots > \quad .$$

The following result, mentioned in ([5], §3) gives sufficient conditions for N to be free.

Lemma 4.2.1. If a_λ and a_μ each appears just once in P_o (= R_o) with exponent 1 or -1, then N is freely generated by $a_\lambda, a_{\lambda+1}, \cdots, a_{\mu-1}$.

Proof : Since a_λ and a_μ each appears just once in P_o with exponent 1 or -1, it follows from (5) that $a_{\lambda+i}$ and $a_{\mu+i}$ each appears just once in P_i with exponent 1 or -1, for each i. Thus, from (5), we get

(6) $$a^{\varepsilon}_{\lambda+i} = w(a_{\lambda+i+1}, \cdots, a_{\mu+i})$$

for $i < 0$, where $\varepsilon = 1$ or -1, and

(7) $$a^{\eta}_{\mu+j} = w'(a_{\lambda+j}, \cdots, a_{\mu+j-1})$$

for $j \geq 0$, where $\eta = 1$ or -1.

Then, by applying Tietze transformation (T4) repeatedly, we can delete P_i and the corresponding generators $a_{\lambda+i}$ for $i < 0$ in N, and we can delete P_j and the corresponding generators $a_{\mu+j}$ for $j \geq 0$ in N. Hence $N = < a_\lambda, a_{\lambda+1}, \cdots, a_{\mu-1} >$, i.e. N is freely generated by $a_\lambda, a_{\lambda+1}, \cdots, a_{\mu-1}$.

This completes the proof.

We will show that the conditions given in Lemma 4.2.1 are also necessary for N to be free. The next lemma is the key to our main result.

Lemma 4.2.2. Let H be a group presented by

$$(8) \qquad H = < y_1, y_2, \cdots, y_n \; ; \; S(y_1, y_2, \cdots, y_n) >$$

where S is cyclically reduced and suppose that y_n appears at least twice in S. Then y_n cannot be expressed as a word in terms of $y_1, y_2, \cdots, y_{n-1}$ in H. (Note that if y_n^2 appears in S, then y_n is considered to appear twice in S).

Proof : Suppose that $y_n = V(y_1, \cdots, y_{n-1})$, a word in $y_1, \cdots, y_{n-1}$. Then $Q = y_n V^{-1}$ is a relator in H so that, by Tietze transformation (T1),

$$H = < y_1, \cdots, y_n \; ; \; S, Q > .$$

Therefore, by Tietze transformation (T4)

$$H = < y_1, \cdots, y_{n-1} ; S(y_1, \cdots, y_{n-1}, V(y_1, \cdots, y_{n-1})) > .$$

Thus, by Freiheitssatz theorem (Theorem 4.1.2), $S(y_1,\cdots,y_{n-1}, V(y_1,\cdots,y_{n-1}))$ must reduce to the empty word and so S is derivable from Q. By Tietze transformation (T2), we have

$$H = < y_1, \cdots, y_n ; Q > .$$

Hence, by Theorem 4.1.3, Q^{ε} ($\varepsilon = 1$ or -1) and S are conjugate in the free group on $y_1, \cdots, y_n$. Therefore, by Theorem 4.1.1, Q^{ε} ($\varepsilon = 1$ or -1) must be a cyclic permutation of S, which is impossible since y_n appears at least twice in S.

This completes the proof.

Now, without loss of generality, we can assume, in (3), that

$$\lambda = 1 \quad \text{and} \quad \mu = k$$

for some integer $k > 1$. Let H and H' be the groups presented respectively by

(9) $$H = < a_1, a_2, \cdots ; P_o, P_1, P_2, \cdots >$$

and

(10) $$H' = < a_{k-1}, \cdots, a_1, a_o, a_{-1}, \cdots ; P_{-1}, P_{-2}, \cdots > .$$

Then N is the free product of H and H' with amalgamation over the common subgroup freely generated by $a_1, \cdots, a_{k-1}$ (cf. [13], §4.2). Thus, H and H' are subgroups of N. Let

$$\lambda_j = \sigma_{R_o}(a_j) \qquad (j = 1, \cdots, k). \tag{11}$$

Let

$$F_o = H_o = < a_1, a_2, \cdots, a_k ; P_o (= R_o) > ,$$

$$F_m = < a_{1+m}, a_{2+m}, \cdots, a_{k+m} ; P_m > , \qquad (m > 0)$$

$$H_m = < a_1, a_2, \cdots, a_{k+m} ; P_o, P_1, \cdots, P_m > , \qquad (m > 0).$$

Then H_m is the free product of H_{m-1} and F_m with amalgamation over the common subgroup freely generated by $a_{1+m}, \cdots, a_{k+m-1}$. Hence, we have an ascending chain of groups

$$H_o \subset H_1 \subset \cdots \subset H_m \subset \cdots$$

such that $H = \bigcup_j H_j$ ([13], p.33). In this way, we regard H_m as subgroup of H for each integer $m \geq 0$.

We will use the following lemma, due to E.S. Rapaport ([15]).

<u>Lemma 4.2.3.</u> Let E be a group presented by

$$E = < x_1, x_2, \cdots, x_m ; Q > .$$

Then $m - 1$ is maximal for all presentations of E, that is, the difference between the number of generators and the number of relators is always $\leq m-1$.

Next, we are going to determine a necessary condition for H to be free. The following lemma may relate to case 3 with the homomorphism ϕ, but not to case 1.

<u>Lemma 4.2.4.</u> If all the exponent sums λ_j $(j = 1,\cdots,k)$ given in (11) are zero, then H given by (9) cannot be a free group (and so N is not free).

<u>Proof</u> : Consider the subgroup $H_o = < a_1, a_2, \cdots, a_k ; P_o >$ of H. Suppose that H_o is free. Then, by Lemma 4.2.3, H_o is free of rank at most $k - 1$. Now, since $\lambda_j = 0$ $(j = 1, \cdots, k)$, the abelianization of H_o can be presented by

$$< a_1, a_2, \cdots, a_k ; a_j a_\ell = a_\ell a_j \quad (j, \ell = 1, \cdots, k), \quad P_o >$$

which is $< a_1, a_2, \cdots, a_k ; a_j a_\ell = a_\ell a_j \quad (j, \ell = 1, \cdots, k) >$ by Tietze transformation (T2). Hence this abelianization is free abelian of rank k which contradicts the fact that H_o is free of rank at most $k - 1$. Hence H_o is not free and so H is not free. This completes the proof.

Thus, we can, in addition, assume that some of the λ_j $(j = 1,\cdots,k)$ in (11) are not zero. Let $\overline{H}$ be the abelianization of H. Then $\overline{H}$ has generators

$$a_1, a_2, \cdots ;$$

and has a set of defining relators :

$$a_i a_j = a_j a_i \qquad (i, j > 0),$$

and

(12) $$P_i' = \sum_{j=1}^{k} \lambda_j a_{i+j} ,$$

where $i \geq 0$ and λ_j $(j = 1, \cdots, k)$ are given in (11). (Here, we use additive notations just for convenience).

Lemma 4.2.5. If $\lambda_j = 0$ for all $j = 1, \cdots, k$ except $j = j_o$, then $\overline{H}$ is not free abelian (and so H is not free) when $j_o \neq 1$ or -1.

Proof : Since $\lambda_j = 0$ except $j = j_o$, it follows from (12) that

$$\lambda_{j_o} a_{i+j_o} = 0$$

for $i \geq 0$. Since $\lambda_{j_o} \neq 1$ or -1, the subgroup $< a_{j_o} ; \lambda_{j_o} a_{j_o} >$ of $\overline{H}$, is clearly not free abelian. Hence $\overline{H}$ is not free abelian.

Lemma 4.2.6. If $\lambda_j = 0$ for all $j = 1, \cdots, k$ except $j = j_o$ and $\lambda_{j_o} = 1$ or -1, then $\overline{H} = <0>$ or $\overline{H}$ is free abelian on the generators $a_1, \cdots, a_{j_o-1}$ according as $j_o = 1$ or $j_o \neq 1$.

Proof : It follows from (12) that

$$\varepsilon a_{i+j_o} = 0 \quad (\varepsilon = 1 \text{ or } -1, \quad i \geq 0).$$

If $j_o = 1$, then $a_i = 0$ for all $i \geq 1$ and so $\overline{H} = \langle 0 \rangle$. If $j_o \neq 1$, then $a_i = 0$ for all $i \geq j_o$ so that $\overline{H}$ is free abelian on the generators $a_1, \cdots, a_{j_o-1}$.

This completes the proof.

Next, suppose that at least two λ_j's are not equal to zero. Without loss of generality, we can assume $\lambda_1 \neq 0$.

Lemma 4.2.7. If $\overline{H}$ is free abelian, then its rank is at most $k - 1$.

Proof : Suppose that $\overline{H}$ is free abelian with generators $f_1, f_2, \cdots, f_m, \cdots$. If any ℓ elements $(\ell \leq k - 1)$ in this set of generators are linearly dependent, then there is nothing to prove.

Thus, suppose that there are $k - 1$ elements, say

$$f_1, f_2, \cdots, f_{k-1}$$

in this set, which are linearly independent. Since $\lambda_1 \neq 0$, by applying the relators (12) repeatedly, we see that there exists some integer $j_o > 0$ such that

$$(13)\quad \begin{cases} \lambda_1^{m_1} f_1 = m_{j_o,1} a_{j_o} + m_{j_o+1,1} a_{j_o+1} + \cdots + m_{j_o+k-2,1} a_{j_o+k-2} \\ \quad\vdots \\ \lambda_1^{m_{k-1}} f_{k-1} = m_{j_o,k-1} a_{j_o} + m_{j_o+1,k-1} a_{j_o+1} + \cdots + m_{j_o+k-2,k-1} a_{j_o+k-2} \end{cases}$$

and

$$\lambda_1^{m_k} f_k = m_{j_o,k} a_{j_o} + m_{j_o+1,k} a_{j_o+1} + \cdots + m_{j_o+k-2,k} a_{j_o+k-2}$$

where m_i $(i = 1,\cdots,k)$ are some integers > 0 and all $m_{\rho,\rho'}$ are integers. Since $f_1, \cdots, f_{k-1}$ are assumed to be linearly independent, the determinant

$$D = \begin{vmatrix} m_{j_o,1} & m_{j_o+1,1} & \cdots & m_{j_o+k-2,1} \\ \vdots & & & \\ m_{j_o,k-1} & m_{j_o+1,k-1} & \cdots & m_{j_o+k-2,k-1} \end{vmatrix}$$

is different from zero. Hence we can express each of the

$$Da_{j_o}, Da_{j_o+1}, \cdots, Da_{j_o+k-2}$$

in terms of a linear combination of $f_1, \cdots, f_{k-1}$ and so $D\lambda_1^{m_k} f_k$ can be expressed as a linear combination of $f_1, \cdots, f_{k-1}$ with $D\lambda_1^{m_k} \neq 0$. Therefore $f_1, \cdots, f_k$ are linearly dependent and hence $\overline{H}$ is free of rank at most $k - 1$.

<u>Lemma 4.2.8.</u> If a_k appears at least twice in R_o, then the group H given by (9) is not free (and so N is not free).

<u>Proof</u> : Suppose that H is free. Then, by Lemma 4.2.6 and Lemma 4.2.7, it follows that H is free of rank at most $k - 1$. Let $f_1, \cdots, f_\ell$ ($\ell \leq k - 1$) be a set of free generators for H ; i.e.

$$(14) \qquad H = < f_1, \cdots, f_\ell > .$$

Note that each f_j ($j = 1, \cdots, \ell$) is a word in a_i ($i = 1, 2, \cdots$). Let m be the maximal subscript of all the generators a_i that are involved in $f_1, \cdots, f_\ell$. Then H is finitely generated by $a_1, \cdots, a_m$ so that

$$(15) \qquad H = H_{m-k} = < a_1, \cdots, a_m ; P_o, \cdots, P_{m-k} >$$

Thus each $a_i = U_i(f_1, \cdots, f_\ell)$ ($i = 1, \cdots, m$) is a word in $f_1, \cdots, f_\ell$ and the relators $P_o, \cdots, P_{m-k}$ all reduce to the empty word upon replacing a_i by U_i ($i = 1, \cdots, m$) since (15) can be transformed into (14). Now consider

$$H_{m-k+1} = < a_1, \cdots, a_m, a_{m+1} ; P_o, \cdots, P_{m-k}, P_{m-k+1} >.$$

Then $H = H_{m-k+1}$ so that H can be presented also by

$$H = < f_1, \cdots, f_\ell , a_{m+1} ; P_{m-k+1} > ,$$

where $P_{m-k+1} = Q(f_1, \cdots, f_\ell, a_{m+1})$ with a_{m+1} appearing at least twice in Q. Since H is generated by $f_1, \cdots, f_\ell$,

$$a_{m+1} = V(f_1, \cdots, f_\ell) ,$$

a word in $f_1, \cdots, f_\ell$. But this is impossible by Lemma 4.2.2. Hence H is not free.

This completes the proof.

Similarly, we can prove that :

<u>Lemma 4.2.9.</u> If a_1 appears at least twice in R_o, then the group H' given by (10) is not free (and so N is not free).

Hence, we have proved :

<u>Corollary 4.2.10.</u> If the normal subgroup N given by (5) is free, then each of a_1 and a_k must appear just once in R_o, with exponent 1 or -1.

Combining Lemma 4.2.1 and Corollary 4.2.10, we obtain the following :

<u>Theorem 4.2.11.</u> Let G, R_o and N be as given by (1), (3) and (5). Then N is free if and only if each of a_λ and a_μ appears just once in R_o with exponent 1 or -1.

The following example indicates that in case 3, the suitation is much less definite.

<u>Example 4.2.12.</u> Let G be a group presented by

$$G = < a, b \; ; \; R = ab^2a^{-1}b^{-2} > .$$

Then $\sigma_R(a) = 0$, $\sigma_R(b) = 0$.

Choose $\phi : G \longrightarrow T$ to be the epimorphism defined by

$$\phi(a) = 1, \qquad \phi(b) = t.$$

Then $G = N \times_\alpha T$ with N the normal subgroup of G generated by a. Using the above notations, we see that

$$N = < \cdots, a_{-1}, a_o, a_1, \cdots ; \cdots P_{-1}, P_o, P_1, \cdots > .$$

But $P_o = R_o = a_o a_2^{-1}$ so that the conditions in Theorem 4.2.11 are satisfied. Hence N is free.

Next, choose $\phi_1 : G \longrightarrow T$ to be the epimorphism defined by

$$\phi_1(a) = t, \qquad \phi_1(b) = 1.$$

Then $G = N_1 \times_\alpha T$ with N the normal subgroup of G generated by b. Let $b_i = a^i b a^{-i}$ (i, any integer). Then N_1 has generators b_i. Rewrite $R = ab^2a^{-1}b^{-2}$ in terms of b_i, we have

$$R_o = b_1^2 b_o^{-2} .$$

Thus the conditions in Theorem 4.2.11 are not satisfied. Hence N_1 is not free.

Finally, we return to case 2. That is, let

(16) $$G = < a, b \ ; R(a,b) >$$

with $\sigma_R(a) \neq 0$ and $\sigma_R(b) \neq 0$. Let $\lambda_1 = \sigma_R(a)$ and $\lambda_2 = \sigma_R(b)$. Then we can write $\lambda_1 = k\mu_1$ and $\lambda_2 = k\mu_2$ with μ_1 and μ_2 relatively prime. Then $\lambda_1\mu_2 - \lambda_2\mu_1 = 0$. Choose ν_1 and ν_2 with ν_1 the minimum positive integer such that

(17) $$\nu_1\mu_2 - \nu_2\mu_1 = 0.$$

In fact, we can just take $\nu_1 = |\mu_1|$. (If $\lambda_2 = k'\lambda_1$, then $\mu_1 = 1$, $\mu_2 = k'$ so that $\mu_2 - k'\mu_1 = 0$. In this case, $\nu_1 = 1$). Next, since μ_1 and μ_2 are relatively prime, we can choose m_1 and m_2 with m_1 the minimum positive integer such that

$$m_1\mu_1 + m_2\mu_2 = 1.$$

In this case, the homomorphism $\phi : G \longrightarrow T$ is again uniquely defined up to exponent sign by

$$\phi(a) = t^{\mu_2} \quad \text{and} \quad \phi(b) = t^{-\mu_1} .$$

It is easy to verify that the elements

$$a,\ \cdots,\ a^{\nu_1-1},\ b^\ell,\ ab^\ell,\ \cdots,\ a^{\nu_1-1}b^\ell \quad (\ell,\ \text{any integer}) \tag{18}$$

form a Schreier representatives for G mod N. To obtain a presentation of N, we make use of a Reidemeister-Schreier rewriting process and as Schreier representatives for G mod N, we choose (18) ([13],§2.3). We find that N is generated by the elements a_i defined by

$$a_i = (a^j b^\ell)a(a^{j+1}b^\ell)^{-1} \quad \text{or} \quad (a^j b^\ell)a(b^{\ell-\nu_2})^{-1} \tag{19}$$

where i is any integer and $0 \le j < \nu_1 - 1$ (or $j = \nu_1 - 1$), ℓ any integer such that

$$i = j\mu_2 - \ell\mu_1 \tag{20}$$

Note that $a_{j\mu_2} = (a^j)a(a^{j+1})^{-1} = 1$ for $0 \le j < \nu_1 - 1$.

Now, we rewrite $R(a,b)$ in terms of a_i as :

$$R(a,b) = R_o(a_\lambda,\ a_{\lambda+1},\ \cdots,\ a_\mu) \tag{21}$$

with $\lambda < \cdots < \mu$ (cf. Example 4.2.14). Then N is generated by a_i and has as defining relators $a_{j\mu_2}$ $(0 \le j < \nu_1 - 1)$ and

$$\begin{aligned} P_i &= a^j b^\ell R(a,b)(a^j b^\ell)^{-1} \\ &= R_o(a_{\lambda+i},\ a_{\lambda+1+i},\ \cdots,\ a_{\mu+i}) \quad (i,\ \text{any integer}) \end{aligned}$$

where i, j, ℓ satisfy (20), and so

$$(22) \quad N = \langle \cdots, a_{-1}, a_o, a_1, \cdots ; \; a_o, a_{\mu_2}, \cdots, a_{(\nu_1-1)\mu_2}, P_i \quad (i, \text{ any integer}) \rangle .$$

Hence, by the same arguments as for case 1, we have :

<u>Theorem 4.2.13.</u> Let G, R_o and N be given by (16), (21) and (22). Then N is free if and only if each of a_λ and a_μ appears just once in R_o with exponent 1 or -1.

We close this section with the following example.

<u>Example 4.2.14.</u> Let G be the group presented by

$$G = \langle \; a, b \; ; \; R = a^{\lambda_1} b^{\lambda_2} \; \rangle$$

with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. Then G is of the form $N \times_\alpha T$ with N free. Withour loss of generality, we can assume $\lambda_1 > 0$. Let $\lambda_1 = k\mu_1$ and $\lambda_2 = k\mu_2$ $(k > 0)$. Then $\nu_1 = \mu_1$ and $\nu_2 = \mu_2$ in (17). We see that N can be presented by (22). Now, we rewrite $R = a^{\lambda_1} b^{\lambda_2}$ in terms of a_i as follows :

$$R_o = \overbrace{(b^o)a(ab^o)^{-1}(ab^o)a(a^2b^o)^{-1} \cdots (a^{\nu_1-1}b^o)^{-1}(a^{\nu_1-1}b^o)a}^{\nu_1 \text{ appearances of } a}(b^{-\nu_2})^{-1} \times$$

$$\times \overbrace{(b^{-\nu_2})a(ab^{-\nu_2})^{-1}(ab^{-\nu_2})a(a^2b^{-\nu_2})^{-1} \cdots (a^{\nu_1-1}b^{-\nu_2})^{-1}(a^{\nu_1-1}b^{-\nu_2})a}^{\nu_1 \text{ appearances of } a}(b^{-2\nu_2})^{-1}$$

$\times \cdots \times$

$$\times\ (b^{-(k-1)\nu_2})a\overbrace{(ab^{-(k-1)\nu_2})^{-1}(ab^{-(k-1)\nu_2})a(a^2b^{-(k-1)\nu_2})^{-1}\cdots(a^{\nu_1-1}b^{-(k-1)\nu_2})}^{\nu_1 \text{ appearances of } a}a(b^{-k\nu_2})^{-1}b^{-k\nu_2}\times b^{\lambda_2}$$

$$= a_o a_{\mu_2} \cdots a_{(\nu_1-1)\mu_2} a_{+\nu_2\mu_1} a_{\mu_2+\nu_2\mu_1} \cdots a_{(\nu_1-1)\mu_2+\nu_2\mu_1}$$

$$a_{+2\nu_2\mu_1} \cdots a_{(\nu_1-1)\mu_2+(k-1)\nu_2\mu_1}$$

(Note that $b^{-k\nu_2}b^{\lambda_2} = 1$ since $\lambda_2 = k\nu_2$). It is clear that all the subscripts are different and hence we can apply Theorem 4.2.13 to conclude that N is free.

§4.3. Groups with n $(n > 2)$ Generators and One Defining Relators

Let G be a group presented by

(1) $$G = \langle x_1, \cdots, x_n ; R(x_1, \cdots, x_n) \rangle$$

where R is cyclically reduced and involves all the generators. Then $G = N \times_\alpha T$. In this section, we will obtain certain necessary and sufficient conditions for the factor N to be free corresponding to certain natural choices of the epimorphism $\phi : G \longrightarrow T$.

If all $\sigma_R(x_i) \neq 0$, then the suitation is basically similar to case 2 of §4.2, but the number of cases to be treated are so numerous that we will not detail them here. Rather, we shall take up the case where, say, $\sigma_R(x_n) = 0$. In this case, we can choose $\phi : G \longrightarrow T$ by defining

$$\phi(x_j) = 0 \quad (j = 1,2,\cdots,n-1), \quad \phi(x_n) = t.$$

(Note that there are many other choices of ϕ). Thus we may assume that the factor T, in $N \times_\alpha T$, is just the infinite cyclic group generated by x_n in G and N is nothing but the normal subgroup of G generated by $x_1, x_2, \cdots, x_{n-1}$.

To obtain a presentation of N, we make use of a Reidemeister-Schreier rewriting process and as Schreier representatives for G mod N, we choose x_n^i, where i runs over all integers ([13], §2.3). We find that N is generated by the elements $x_{i,j}$ defined by

$$x_{i,j} = x_n^i x_j x_n^{-i}$$

where $j = 1, \cdots, n-1$ and i any integer. Now, we rewrite $R(x_1,\cdots,x_n)$ in terms of $x_{i,j}$ as follows : Every symbol x_j^ε $(j = 1,\cdots,n-1;\ \varepsilon = 1$ or $-1)$ in $R(x_1, \cdots, x_n)$ is replaced by $x_{s,j}^\varepsilon$ where s is the sum of the exponents of the x_n-symbols preceding the particular x_j^ε in $R(x_1, \cdots, x_n)$. Thus $R(x_1, \cdots, x_n)$ can be expressed in terms of $x_{i,j}$ as :

(2) $R(x_1,\cdots,x_n)$

$$= R_o(x_{\lambda_1,1},\cdots,x_{\mu_1,1},x_{\lambda_2,2},\cdots,x_{\mu_2,2},\cdots,x_{\lambda_{n-1},n-1},\cdots,x_{\mu_{n-1},n-1})$$

with $\lambda_j \leq \cdots \leq \mu_j$ $(j = 1,\cdots,n-1)$. Then N is generated by $x_{i,j}$ (i runs over all integers and j runs over $1, \cdots, n-1$) and has a set of defining relators

$$P_i = x_n^i R_o x_n^{-i}$$

$$= R_o(x_{\lambda_1+i,1},\cdots,x_{\mu_1+i,1},\cdots,x_{\lambda_2+i,2},\cdots,x_{\mu_2+i,2},\cdots,x_{\lambda_{n-1}+i,n-1},\cdots,x_{\mu_{n-1}+i,n-1})$$

(i any integer) and so

$$N = < x_{i,j}, \text{ (} i \text{ runs over all integers and } j \text{ runs over } 1,\cdots,n-1) \text{ ; } \cdots P_{-1}, P_o, P_1, \cdots > .$$

By adapting the arguments for the case of two generators, we can prove :

Theorem 4.3.1. The normal subgroup N is free if and only if one of the $x_{\lambda_1,1}, x_{\lambda_2,2}, \cdots, x_{\lambda_{n-1},n-1}$ and one of the $x_{\mu_1,1}, x_{\mu_2,2}, \cdots, x_{\mu_{n-1},n-1}$ each appears just once in R_o given by (2), with exponent 1 or -1.

Corollary 4.3.2. ([8]) Let G be the fundamental group of a closed orientable surface. Then G is of the form $N \times_\alpha T$ with N free.

Proof : Recall that G can be presented by

$$G = < x_1,\cdots,x_n,\ y_1,\cdots,y_n\ ;\ R = x_1y_1x_1^{-1}y_1^{-1}\cdots\ x_ny_nx_n^{-1}y_n^{-1} >.$$

Then $\sigma_R(y_n) = 0$, so that as discussed above, we can take $T = <y_n>$ and N the normal subgroup of G generated by $x_1,\cdots,x_n,\ y_1,\cdots,y_{n-1}$. The generators for N are given by

$$x_{i,j} = y_n^i x_j y_n^{-i} \qquad (i,\ \text{any integer}\ ;\ j = 1,\ \cdots,\ n)$$

$$y_{i,\ell} = y_n^i y_\ell y_n^{-i} \qquad (i,\ \text{any integer}\ ;\ \ell = 1,\ \cdots,\ n-1).$$

By rewriting R in terms of $x_{i,j}$ and $y_{i,\ell}$, we get

$$R_o = x_{o,1}y_{o,1}x_{o,1}^{-1}y_{o,1}^{-1}\cdots x_{o,n-1}y_{o,n-1}x_{o,n-1}^{-1}y_{o,n-1}^{-1}x_{o,n}x_{1,n}^{-1}\ .$$

Since $x_{o,n}$ and $x_{1,n}$ each appears just once in R_o with exponent 1 or -1, the conditions in Theorem 4.3.1 are satisfied. Hence N is free.

Corollary 4.3.3. Let G be the group presented by

$$G = < y_1,\ y_2,\cdots,y_n;\ R = \underbrace{y_{n-1}S_1y_{n-1}S_2\cdots\ S_{m-1}y_{n-1}}_{m \text{ appearances of } y_{n-1}}S_m,\ (m \neq 1) >$$

where $S_1,\ S_2,\ \cdots,\ S_{m-1}$ are words in $y_1,\ \cdots,\ y_{n-2}$ (some of them may be trivial), and S_m is a word in $y_1, \cdots, y_{n-2}, y_n$ with the condition that $\sigma_{S_m}(y_n) = km$ for some integer $k \neq 0$. Then G is of the form $N \times_\alpha T$, with N free.

<u>Proof</u> : Let $x_j = y_j$ $(j = 1, \cdots, n-2, n)$ and $x_{n-1} = y_{n-1}y_n^k$. Then, by applying Tietze transformations repeatedly, we see that G can be presented by

$$G = \langle x_1, x_2, \cdots, x_n \; ; \; R' = x_{n-1}x_n^{-k}S_1' \cdots S_{m-1}'x_{n-1}x_n^{-k}S_m' \rangle$$

where $S_1', \cdots, S_{m-1}'$ are words in $x_1, \cdots, x_{n-2}$ and S_m' is a word in $x_1, \cdots, x_{n-2}, x_n$ with $\sigma_{S_m'}(x_n) = km$. Hence $\sigma_{R'}(x_n) = 0$. Let N be the normal subgroup generated by $x_1, \cdots, x_{n-1}$. Then the generators for N are

$$x_{i,j} = x_n^i x_j x_n^{-i}$$

(i, any integer ; $j = 1, \cdots, n-1$). By rewriting R' in terms of $x_{i,j}$, we get

$$R' = R_o = x_{o,n-1}S_1'x_{-k,n-1}S_2'x_{-2k,n-1} \cdots S_{m-1}'x_{-(m-1)k,n-1}S_m'$$

where $S_1', S_2', \cdots, S_{m-1}', S_m'$ are words in $x_{i,\ell}$ (i, any integer ; $\ell = 1, \cdots, n-2$). Since $x_{o,n-1}$ and $x_{-(m-1)k,n-1}$ each appears in R_o just once with exponent 1, the conditions in Theorem 4.3.1 are satisfied. Hence N is free.

Corollary 4.3.4. ([8]) Let G be the fundamental group of a closed nonorientable surface. Then G is of the form $N \times_\alpha T$ with N free.

Proof : Recall that G can be presented by

$$G = \langle y_1, \cdots, y_n \; ; \; y_1^2 y_2^2 \cdots y_n^2 \rangle \; .$$

Then apply Corollary 4.3.3.

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