THE ALGEBRA OF CERTAIN
ONE- AND TWO- DIMENSIONAL
PRIMITIVE FORMS.

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THE ALGEBRA OF CERTAIN ONE- AND TWO-DIMENSIONAL

PRIMITIVE FORMS.

by

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PRIMITIVE FORMS.

Introduction.

The purpose of the study of geometry is to discover the properties of figures in space. The properties of any figure are determined by the relations existing between the elements which make up the figure. Hence, in order to avoid a vicious circle, we must regard certain of the elements composing the figure as fundamental and these must remain undefined. Likewise we must consider certain of the propositions stating relations between these elements as fundamental and these must remain unproved.

From these undefined elements and unproved propositions it is possible, by methods of formal logic, to build up a set of propositions stating further properties of the figure. Any such set of propositions, arranged according to a sequence of logical deduction, is called a mathematical science. Such a science is itself purely abstract, its structure depending wholly upon the undefined elements and unproved propositions with which we commence. Hence the value of any mathematical science is dependent on the value of its possible concrete applications.

The aim of the following paper is to set up a concrete
II.

representation of a mathematical science. A method is defined whereby the four fundamental, arithmetical operations, addition, multiplication, subtraction, and division may be performed on lines, points, and planes. The operations are defined in such a manner that the associative and commutative laws hold for multiplication and addition. Multiplication is also distributive with respect to addition. The set of all lines on a point is shown to form a number system. The latter part of the paper deals with projectivities in primitive one-dimensional forms; projectivities between two sets of elements of the same type; and an extension of the former work to the set of all planes on a point and the set of all planes on a line.

The work is presented in a purely abstract form. Concrete applications arise from an interpretation of the abstract work with reference to the figures given, in accordance with the notation employed.

A knowledge of the principles of general projective geometry is presupposed throughout. Unless otherwise stated all work has reference to figures in a single plane.
The theory of complete quadrilaterals, complete quadrangles, and quadrangular sets is fundamental in the following work. Accordingly definitions of these and certain important theorems involving them are here noted.

A complete plane four-line is called a complete quadrilateral. A complete quadrilateral, therefore, has four sides and six vertices formed by the intersection of pairs of these sides. Two vertices are called opposite if they are not on the same side of the quadrilateral.

Thus fig. 1 is a complete quadrilateral composed of the four sides \(a, b, c, d\) and the six vertices \(A, B, C, D, E, F\). The vertices \(A\) and \(D\), \(B\) and \(E\), \(C\) and \(F\), form pairs of opposite vertices.

A complete quadrangle, as the name implies, is a complete plane four-point. It consists of four vertices and six sides formed by the lines which join the vertices in pairs. Two sides are called opposite if they are not on the same vertex.
Fig. 2

Thus fig. 2 is a complete quadrangle consisting of the four vertices $A, B, C, D$ and the six sides $a, b, c, d, e, f$. The sides $a$ and $c$, $b$ and $d$, $e$ and $f$, form pairs of opposite sides.

A quadrilateral set of lines is the set of lines formed by the projection of the vertices of a complete quadrilateral from a point not on any of its sides.

Fig. 3

In fig. 3, $a, b, c, d, e, f$ form a quadrilateral set of lines.
Any three vertices of a complete quadrilateral either form a triangle or are collinear. In a quadrilateral set of lines on a point we call the three lines, which are the projections of three collinear vertices, a line triple of lines. The projections of three vertices forming a triangle are called a triangle triple of lines. We denote a quadrilateral set of lines by

\[ Q(abc, def) \]

where \( abc \) is a line triple of lines and \( def \) is a triangle triple of lines, and where \( a \) and \( d \), \( b \) and \( e \), \( c \) and \( f \), are each the projections of a pair of opposite vertices. We write the quadrilateral set of lines in fig. 3

\[ Q(abc, def) \]

A quadrangular set of points is the section by a transversal of the sides of a complete quadrangle. The section, by the transversal, of three lines of the complete quadrangle which are on the same vertex is called a point triple of points. The section, by the transversal, of three sides of the quadrangle which form a triangle is called a triangle triple of points. We denote a quadrangular set of points by

\[ Q(ABC, DEF) \]

where \( ABC \) is a point triple of points and \( DEF \) is a triangle triple of points and where \( A \) and \( D \), \( B \) and \( E \), \( C \) and \( F \), are each the section by the transversal of a pair of opposite sides.
In fig. 4, E, F, G, H, I, J form a quadrangular set of points and we write it

\[ Q(\{E, F, G, H, I, J\}) \]

An important property of quadrilateral and quadrangular sets is given by the following theorem:

**Theorem 1.** The section, by a transversal, of a quadrilateral set of lines is a quadrangular set of points.

For this reason it is customary to call a quadrilateral set of lines a quadrangular set of lines. This term has the advantage of uniformity and will be used throughout the remainder of this paper.

It is convenient to note, at this point, that whenever the word *quadrilateral* appears in the following work it has reference to the first of the two figures. With reference to the second figure the reader is asked to substitute the word *quadrangle*.
The following theorem gives another important property of quadrangular sets.

**Theorem 2.** If any set of elements is projective with a quadrangular set it is itself a quadrangular set.

The most important property of quadrangular sets in relation to subsequent work is given by

**Theorem 3.** If all but one of the elements of a quadrangular set are given the remaining one is uniquely determined.

The proofs of these theorems will be found in any work on complete quadrilaterals and complete quadrangles.

2. **Terminology.**

The entire work is presented in the on terminology. That is, instead of saying, *a point lies on a line*, we say, *a point is on a line*. The statement, *a line passes through a point*, becomes, in this terminology, *a line is on a point*. As a consequence of the theorem of duality and the symmetry established by this terminology, any proposition concerning lines, which is established by logical deduction from our undefined elements and unproved propositions is equally true when line is replaced by point; point by plane; or line by plane. That is, by a judicious choice of terminology, the one proof can be made to hold in all four cases.

3. **Notation.**

Although definitions and proofs are given in what is essentially the abstract form, they refer primarily,
in each case, to the first figure given. With reference to this figure capital letters denote points and small letters denote lines. The notation $X, Y$ refers to the point on the lines $X$ and $Y$. The symbol $X, Y$ refers to the line on the points $X$ and $Y$. By duality the same proof holds for the second figure in reference to which capital letters denote lines and small letters, points. The line on the points $X$ and $Y$ is then written $X, Y$, while $X, Y$ is the point on the lines $X$ and $Y$.

Thus for example, the statement, the element $\lambda$ is on the elements $P_1$ and $P_2$, frequently written, $\lambda$ is on $P_1$ and $P_2$, means with reference to fig. 5(a) that the line $\lambda$ is on the points $P_1$ and $P_2$. With reference to fig. 5(b) however, it means that the point $\lambda$ is on the lines $P_1$ and $P_2$. In reference to any figure, therefore, small letters and capital letters refer to two distinct types of elements.

4. Assumptions.

The mathematical science here presented is based on:
(a) the undefined elements point, line, and plane.

(b) the following unproved propositions hereafter called assumptions:

A. The Assumptions of Alignment:

1. If \( a \) and \( b \) are distinct elements there is at least one element \( P \) on both \( a \) and \( b \).

2. If \( a \) and \( b \) are distinct elements there is not more than one element \( P \) on both \( a \) and \( b \).

3.

Fig. 6

If \( a, b, \) and \( c \) (fig. 6) are distinct elements not all on a common element \( P \), and if \( d \) and \( e \) are any other two distinct elements such that \( b, c, d \) are on a common element \( R \), and \( c, a, e \) are on a common element \( S \) then there exists an element \( f \) such that \( a, b, f \) are on the common element \( T \) and \( d, e, f \) are on the common element \( U \).

E. The Assumptions of Extension:

1. There exists at least one element \( P \).

2. All elements \( a \) are not on the same element \( P \).
3. There are at least three elements \(a, b,\) and \(c\) on every element \(P\).

**P. The Assumption of Projection:**

If a projectivity leaves each of three distinct elements \(a, b,\) and \(c\) on an element \(P\) invariant, it leaves every other element \(d\) on \(P\) invariant.

5. **Scale on an element \(P\).**

We choose three elements \(\ell_\alpha, \ell_\beta,\) and \(\ell_\gamma\) on an element \(P\) and call them fundamental elements. With reference to these fundamental elements, we are able to define addition, multiplication, subtraction, and division by means of our undefined elements and assumptions. We make the definition in such a manner that the operations are single-valued for every pair of elements \(\ell_\alpha\) and \(\ell_\beta\) on \(P\) for which the operations are defined. The fundamental elements \(\ell_\alpha, \ell_\beta,\) and \(\ell_\gamma\) are said to determine a scale on \(P\).

6. **Definition of Addition.**

We proceed to define addition and to prove the theorems which follow as a logical consequence of the definition. The definition interpreted with reference to fig. 7(a) gives the addition of lines. Addition of points follows from the interpretation of the definition with reference to fig. 7(b).

Consider a scale on \(P\) determined by means of the elements \(\ell_\alpha, \ell_\beta,\) and \(\ell_\gamma\) (fig. 7). Let \(\ell_\alpha\) and \(\ell_\gamma\) be any two
Fig. 7(a)

Fig. 7(b)

Fig. 7
distinct elements on \( P \). We indicate the construction for an
element which we define as the sum of \( l_x \) and \( l_\eta \).

Choose \( P_0 \) any element on \( l_\omega \), and \( P_{\alpha} \) and \( P_{\alpha'} \) any two
elements on \( l_\omega \). Let \( a \) denote the element \( \overline{P_0P_{\alpha}} \), and \( b \) the
element \( \overline{P_0P_{\alpha'}} \). Let \( x \) denote the element on \( \overline{l_x, a} \) and \( \overline{l_\omega, \alpha} \), and
\( y \) the element on \( \overline{l_\eta, b} \) and \( \overline{l_\omega, \alpha} \). The element \( l_{x+y} \) which is on
\( \overline{l_x, \eta} \) and \( P \), we define as the sum of \( l_x \) and \( l_\eta \) and write
\[
l_{x} + l_{y} = l_{x+y} \]

7. **Theorems on addition**.

**Theorem 4.** If \( l_x \) and \( l_\eta \) are distinct from \( l_\omega \) and \( l_\omega' \), then
the necessary and sufficient condition for the equality
\[
l_{x} + l_{\eta} = l_{x+y} \]
is
\[
Q(l_x, l_\omega, l_\omega', l_{x+y})
\]

**Proof.** If the equality \( l_{x} + l_{\eta} = l_{x+y} \) holds then from the de­
finition and with reference to fig. 7 the quadrilateral \( axby \)
exists. This quadrilateral determines the quadrangular set
\[
Q(l_x, l_\omega, l_\omega', l_{x+y})
\]
which is therefore a necessary condition for the equality \( l_{x} + l_{\eta} = l_{x+y} \). But \( Q(l_x, l_\omega, l_\omega', l_{x+y}) \) is also
a sufficient condition for the equality to hold. For if we
have \( Q(l_x, l_\omega, l_\omega', l_{x+y}) \) then the quadrilateral \( axby \) exists and
hence we have the construction for \( l_{x+y} \).

**Corollary 1**.

(a) \( l_{x} + l_{\omega} = l_{x} \)

(b) \( l_{x} + l_{\omega'} = l_{\omega} + l_{x} = l_{\omega} \) for \( l_{x} \neq l_{\omega} \).

These relations are readily obtained from the definition.
The element \( \overline{x, y} \) of our definition becomes (fig. 8), in this case, \( \rho \) since \( \ell_y = \ell_o \). The element \( y \) becomes \( \overline{\rho, \rho_0} \) or \( \rho \) and hence \( \overline{x, y} \) becomes \( \overline{x, a} \). The element common to \( x, a \) and \( \rho \) is \( \ell_x \). Therefore by definition \( \ell_x + \ell_0 = \ell_\infty \).

Similarly \( \ell_0 + \ell_\infty = \ell_\infty \).
In this case (fig. 9) \( l_x^* b \) becomes \( P_\infty \) since \( l_x = l_\infty \). The element \( y \) of the definition is \( \overline{P_\infty l_\infty} \) or \( l_\infty \) and hence \( x, q \) becomes \( P_\infty \). The element common to \( \overline{a, x} \) and \( P \) is \( \overline{P_\infty} = l_\infty \) which from definition is \( l_x + l_\infty = l_\infty \) for \( l_x \neq l_\infty \). Similarly \( l_\infty + l_x = l_\infty \).
Corollary 2. The operation of addition is single-valued for every pair of elements \( l_x \) and \( l_y \) on \( \mathcal{P} \) except for the pair \( l_o, l_o \).

**Proof.** By Theorem 1. \( \mathcal{Q}(l_o, l_x l_0, l_0 l_y l_{x+y}) \) is a necessary and sufficient condition for the equality \( l_x + l_y = l_{x+y} \). In case \( l_x \) and \( l_y \) are respectively different from \( l_o \) and \( l_o \), five of the elements of this quadrangular set are given. Hence the remaining element \( l_{x+y} \) is uniquely determined.

The operation of addition is accordingly single-valued for every pair of elements \( l_x \) and \( l_y \) neither of which equals \( l_o \) or \( l_o \). If \( l_x \) or \( l_y \) coincide with \( l_o \), Corollary 1. gives the values of \( l_{x+y} \) and these are in each case unique.

**Theorem 5.** The operation of addition is associative, that is,

\[
(l_x + l_y) + l_z = l_x + (l_y + l_z)
\]

for any three elements \( l_x, l_y, l_z \) for which the above expressions are defined.

**Proof.** Determine \( l_{x+y} \) (fig. 10) as in definition. Denote \( l_{x+y} \) by \( P_{x+y} \). Construct \( l_{x+y} + l_z \) by means of \( l_o, l_x, P_{x+y}, l_z \).

The element \( l_{x+y+z} = l_{x+y} + l_z \) is determined by \( P_{x+y} \) and \( l_z \).

Construct \( l_y + l_z \) by means of \( P_{x+y}, P_{x+y}, l_z \). If then, \( l_{x+y+z} = l_{y+z} \), it will be seen that \( l_{x+(y+z)} \) is determined by \( P_{x+y}, P_z \), and \( l_{x+y+z} \) must be the same element. Hence the operation of addition is associative.

**Theorem 6.** The operation of addition is commutative for
Fig. 11(a)

Fig. 11(b)

Fig. 11
Fig. 12(a)

Fig. 12(b)

Fig. 12
every pair of elements $l_x$ and $l_y$ on $P$ for which addition is defined; that is,

$$l_x + l_y = l_y + l_x.$$  

**Proof.** Construct $l_x + l_y = l_x + l_y$ (fig. 11). From the complete quadrilateral $a x b y$ we have $Q(l_x, l_y, l_a, l_x + l_y)$. But by Theorem 4 this is a necessary and sufficient condition for

$$l_y + l_x = l_x + l_y$$  

But $l_x + l_y = l_x + l_y$ by construction. Therefore $l_x + l_y = l_y + l_x$, or the operation of addition is commutative.

**Theorem 7.** Any three elements $l_x, l_y, l_c (l_c \neq l_a)$ satisfy the relationship

$$l_x, l_y, l_c, l_{x+c}, l_{x+c}$$

that is, if we make every element $l_x$ on $P$ correspond to $l_x'$ where $l_x' = l_x + l_c$, $l_c$ being any fixed element on $P \neq l_a$, the correspondence so established is projective.

**Proof.** With reference to Fig. 12

$$[l_x] = [l_x'] = [l_{x+c}]$$

that is,

$$l_x = l_{x+c}$$

$$l_y = l_{y+c}$$

$$l_c = l_{o+c}$$

$$l_o = l_{o+c}$$

since the result of two or more perspectivities is a projectivity. From Corollary: Theorem 3

$$l_o + l_c = l_o + l_c = l_x$$

$$l_o \sim l_c.$$
Also from Corollary 1, Theorem 3,

\[ l_\infty + l_x = l_\infty + l_y = l_\infty \]
\[ l_\infty \sim l_\infty \]
\[ l_\infty \approx l_\infty \]
\[ l_\infty \approx l_\infty \]...

8. Subtraction.

Subtraction is the inverse operation to addition. In other words, given two elements \( l_x \) and \( l_y \) on \( P \), we seek to find a third element \( l_z \), say, such that \( l_z \) is the sum of \( l_x \) and \( l_y \). That is,

\[ l_x = l_z + l_y \]

We call \( l_z \) the difference of \( l_x \) and \( l_y \) and write

\[ l_z = l_x - l_y = l_{x-y} \]

9. Definition of Subtraction.

As in addition consider a scale on \( P \) (fig. 13). Let \( l_x \) and \( l_y \) be any two elements on \( P \). We indicate the construction of an element \( l_z \) such that it is the difference of \( l_x \) and \( l_y \). Let \( P_0 \) be any element on \( l_0 \), and \( P_0, P_\infty \) be any two elements on \( l_\infty \). Denote \( P_0, P_\infty \) by \( \alpha \), \( P_0, P_\infty \) by \( \beta \). Let the element common to \( l_\alpha, l_\beta \) and \( P_\infty \) be \( l_y \). Denote the element common to \( l_\alpha, l_\beta \) and \( P_\infty \) by \( l_x \). The element common to \( \alpha, \beta \) and \( P \) is defined as the difference of \( l_x \) and \( l_y \) viz.,

\[ l_z = l_x - l_y = l_{x-y} \]

10. Theorems on Subtraction.

Theorem 8. The necessary and sufficient condition for the equality \( l_x - l_y = l_{x-y} \) is \( \langle l_\infty, l_x, l_y, l_\infty, l_\infty, l_\infty \rangle \).
Proof. If the equality \( l_x - l_y = l_{x-y} \) holds, by reference to fig. 13 the quadrilateral \( a, b, c, \) exists and this determines the quadrilateral set \( Q(l_x, l_y, l_{x-y}) \) which is therefore a necessary condition for the equality \( l_x - l_y = l_{x-y} \). But \( Q(l_x, l_y, l_{x-y}) \) implies the existence of the quadrilateral \( a, b, c, \) ... exists and this determines the quadrangular set \( Q(U^x, \ldots) \) which is therefore a necessary condition for the equality \( l_x - l_y = l_{x-y} \). But \( Q(U^x, \ldots) \) implies the existence of the quadrilateral \( a, b, c, \) ... We have, therefore, the construction for \( l_x, l_y \) and hence \( Q(l_x, l_y, l_{x-y}) \) is a sufficient condition for the equality \( l_x - l_y = l_{x-y} \) to hold.

Corollary 1.

(a) \( l_x - l_x = l_0 \)
(b) \( l_x - l_0 = l_x \)
(c) \( l_x - l_{x'} = l_{x'} \quad l_x \neq l_{x'} \)

Proof. These follow directly from the definition as did similar properties in connection with addition.

Corollary 2. The operation of subtraction is single-valued for every pair of elements \( l_x \) and \( l_y \) on \( P \) except the pair \( l_0, l_0 \).

Proof. By Theorem 8, the necessary and sufficient condition for the equality \( l_x - l_y = l_{x-y} \) is \( Q(l_x, l_y, l_{x-y}) \). If both \( l_x \) and \( l_y \) are different from \( l_0 \) and \( l_0 \), five of the elements in this set are given and hence (Theorem 3) the sixth, \( l_{x-y} \) is uniquely determined. Thus if \( l_x \) and \( l_y \) are distinct from \( l_0 \) and \( l_0 \), the operation of subtraction is single-valued. If \( l_x \) or \( l_y \) coincide with \( l_0 \) or \( l_0 \) the values of \( l_{x-y} \) are given by Corollary 1 and are in each case unique. Hence, except for the pair \( l_0, l_0 \), subtraction is a single-valued operation for every pair of elements \( l_x \) and \( l_y \) on \( P \).
11. **Definition of Multiplication.**

Consider a scale on \( P \) determined (fig. 14) by means of \( l_x, l_y, \) and \( l_0 \). Let \( l_x \) and \( l_y \) be any other two elements on \( P \). We indicate the construction for an element \( l_{x,y} \) which we shall call the product of the elements \( l_x \) and \( l_y \) in this order. Let \( P_x, P_y, \) and \( P_0 \) be any elements on \( l_x, l_y, \) and \( l_0 \) respectively. Denote \( P_x, P_y \) by \( \alpha \), and \( P_0, P_0 \) by \( \beta \). Let \( \alpha \) be the element common to \( l_x, l_y \) and \( \beta \) the element common to \( l_x, l_y \) and \( P_0 \). We define the element \( l_{x,y} \), common to \( \alpha \) and \( P_0 \) as the product of \( \alpha \) and \( P_0 \) in this order and write

\[ l_x l_y = l_{x,y} \]

12. **Theorems on Multiplication.**

**Theorem 9.** The necessary and sufficient condition for the equality \( l_x l_y = l_{x,y} \) is

\[ Q( l_0 l_x l_y, l_0 l_x l_{x,y} ) \]

for every pair of elements \( l_x \) and \( l_y \) for which multiplication is defined.

**Proof.** By definition and with reference to fig. 14, if the equality \( l_x l_y = l_{x,y} \) holds we have the quadrilateral \( axby \) which gives rise to the quadrangular set \( Q( l_0 l_x l_y, l_0 l_y l_{x,y} ) \).

This is therefore a necessary condition. But if we have

\[ Q( l_0 l_x l_y, l_0 l_y l_{x,y} ) \]

the existence of the quadrilateral \( axby \) is implied and hence we have the construction for \( l_{x,y} = l_x l_y \).

\[ Q( l_0 l_x l_y, l_0 l_y l_{x,y} ) \]

is therefore a sufficient as well as a necessary condition for the equality \( l_x l_y = l_{x,y} \).

**Corollary 1.** (a) \( l_x l_y = l_{x,y} \) \( \iff \) \( l_x \neq l_0 \)
These relations are immediate consequences of the definition. 

**Proof.** (a) \( l_x, l = l, l_x = l_x, l_x \neq l_\infty \)

**Proof.** The element \( l_{y, b} \) of our definition becomes (fig. 15) \( l_{y, b} = P \), since \( l_y = l_x \). The element \( y \) becomes \( \overline{P, P} \), which is \( a \). The element common to \( \overline{x, a} \) and \( P \) is, in accordance with our definition, the product of \( l_x \) and \( l \), and this is \( l_x \). Similarly \( l, l_x = l_x \).
Proof. The element $\overrightarrow{y, b}$ of our definition becomes (fig. 16) $l_0, b$ since $l_y = l_o$. The element $y$ becomes $\overrightarrow{l_0, b, P_0}$ which is $l_0$. Hence $\overrightarrow{x, y}$ becomes $\overrightarrow{x, l_0}$ and therefore, in accordance with our definition, $\overrightarrow{x, l_0} = l_0$. Similarly $\overrightarrow{l_0, l_\infty} = l_\alpha$. 
Fig. 17

Proof. The element $\overline{l_\eta, b}$ of our definition becomes $\overline{l_\omega, b} = P_\omega$. Accordingly the element $\eta$ becomes $P_\omega, P_\infty$ and $\overline{x, y}$ becomes $P_\infty$. The element common to $P$ and $P_\infty$ is $\omega$ and hence $l_\omega = l_\infty$. Similarly $l_\infty l_\omega = l_\infty$. 
Fig. 18(a)

Fig. 18(b)

Fig. 18
Corollary 2. Multiplication is a single-valued operation for every pair of elements $l_x$ and $l_y$ on $P$ except the pairs $l_0$, $l_\omega$, and $l_\omega l_x$.

Proof. If $l_x$ and $l_y$ are both different from $l_0$, $l_\omega$, and $l_\omega$, five of the elements in the quadrangular set $Q(l_0$, $l_x$, $l_\omega$, $l_y$, $l_x y)$ are given. The sixth element $l_{x y}$ is then uniquely determined (Theorem 3). But $Q(l_0$, $l_x$, $l_\omega$, $l_y$, $l_x y)$ is the necessary and sufficient condition for the equality $l_x l_y = l_{x y}$ (Theorem 8). Hence multiplication is a single-valued operation except for the cases excluded above. If $l_x$ or $l_y$ coincide with $l_0$, $l_\omega$, we have the value of $l_{x y}$ given by Corollary 1. These values are all unique. Therefore multiplication is a single-valued operation for every pair of elements on $P$ except $l_0$, $l_\omega$, and $l_\omega l_x$.

Theorem 10. The operation of multiplication is associative. That is,

$$(l_x l_y) l_z = l_x (l_y l_z),$$

for every three elements $(l_x$, $l_y$, $l_z)$ on $P$ for which the above expressions are defined. (fig. 18).

Proof. Consider a scale on an element $P$. Construct $l_{x y} = l_x l_y$ by means of $P_0$, $P_\omega$ as in definition. Let $l_{x y}$ be $P_{x' y'}$. Determine $l_{x y} l_z = l_{x y z}$ by means of $P_0$, $P_{x'}$ and $P_\omega$. The element $l_{x y z}$ is by definition the element common to $x', z$ and $P$. Construct the element $l_{y z} = l_{y z}$ by means of $P_0$, $P'$ and $P_\omega$. Let the element $l_{x z} = l_{x z}$ be determined by means of $P_0$, $P'$, and $P_\omega$. The element $l_{x l_y l_z}$ is the element common to $x', z$ and $P$. 
and therefore in accordance with assumption A must be the same as \( (\alpha, \eta, \lambda) \).

**Theorem 11.** If \( l_x, l_y, l_z \) are any three elements on \( P \) such that the equality \( l_x l_y = l_x \eta \) holds we have

\[
(1) \quad l_\infty l_o l_x = l_\infty l_o l_y l_x^y
\]
\[
(2) \quad l_\infty l_o l_x l_y = l_\infty l_o l_x l_y
\]

that is, if we make every element \( l_x \) on \( P \) correspond to \( l_x' \) where \( l_x' = l_x l_c \), \( c \in \lambda \), \( c \) being any fixed element on \( P \neq l_o \), we set up a projective correspondence.

**Proof.** With reference to Fig. 19

\[
\left[ l_x \right] \frac{\alpha}{\beta} \left[ x \right] \frac{\alpha}{\lambda} \left[ l_x \lambda \right]
\]

that is

\[
l_x \frac{\alpha}{\lambda} l_x \lambda
\]
\[
l_y \frac{\alpha}{\lambda} l_y \lambda
\]
\[
l_o \frac{\alpha}{\lambda} l_o \lambda
\]
\[
l_x' \frac{\alpha}{\lambda} l_x' \lambda
\]

since the result of two or more perspectivities is a projectivity. But by Corollary 1 Theorem 9

\[
l_o' l_c = l_o l_c = l_o
\]
\[
l_\infty' l_c = l_\infty l_c = l_\infty
\]

Therefore

\[
l_\infty l_o l_x = l_\infty l_o l_y l_x^y \quad \text{if} \quad l_c = l_y
\]

Similarly

\[
\left[ l_\infty l_o \right], l_y \frac{\alpha}{\lambda} \left[ l_\infty l_o l_x \right] l_x l_y
\]

follows as the result of two perspectivities.

**Theorem 12** The operation of multiplication is distributive.
with respect to addition. That is,

\[ l_x (l_x + l_y) = l_x l_x + l_x l_y = l_{x+y} \]

\[ (l_x + l_y) l_z = l_x l_z + l_y l_z \]

for \( l_x, l_y \) and \( l_z \) any three elements on \( P \) for which the above expressions are defined.

**Proof.** Let \( l_x + l_y = l_{x+y} \), \( l_x (l_x + l_y) = l_x (l_{x+y}) = l_{x+y} \).

\( (l_x + l_y) l_z = l_x + l_y l_z = l_{x+y} l_z \), etc. From Theorem 11 we have

\[ l_x l_y l_z l_{x+y} = l_x l_y l_{x+y} \]

But since the equality \( l_x + l_y = l_{x+y} \) holds we have by Theorem 4

\[ Q(l_x, l_y, l_z, l_{x+y}) \]. The above projectivity implies therefore, the existence of \( Q(l_x, l_y, l_z, l_{x+y}) \) (Theorem 2).

But by Theorem 4 this is a necessary and sufficient condition for the equality \( l_{x+y} + l_{x+y} = l_{x+y} \) to hold

\[ l_x (l_{x+y}) = l_{x+y} \]

In a similar manner we obtain the result

\[ (l_x + l_y) l_z = l_x l_z + l_y l_z \]

**Theorem 13.** The operation of multiplication is commutative.

That is \( l_x l_y = l_y l_x \) for any two elements \( l_x, l_y \) on \( P \) for which the expressions above are defined.

**Proof.** Let \( l_x l_y = l_y l_x \). By Theorem 11

\[ l_x l_y l_z = l_x l_y l_z \]

and also by the same theorem

\[ l_x l_y l_z = l_x l_y l_z \]

But the fundamental theorem of projective geometry states: If \( 1, 2, 3, 4 \) are four elements of a one-dimensional primitive form, and \( 1, 2, 3 \) are any three elements of another or the same
Fig. 20(a)

Fig. 20(b)

Fig. 20
23. primitive form, then, for any projectivity giving
\[ 1\ 2\ 3\ 4 \rightarrow 1'\ 2'\ 3'\ 4' \]
and
\[ 1\ 2\ 3\ 4 \rightarrow 1\ 2\ 3'\ 4' \]
we have \( 4' = 4 \), or the fourth element is uniquely determine.
Applying this theorem to the above relations we have,
\[ l_x y = l_y x, \]
or the operation of multiplication is commutative.

13. **Division**

Division is the inverse operation to multiplication. In other words, given any two elements \( l_x \) and \( l_y \) on \( P \), we seek a third element \( l_z \), say, such that \( l_x \) is the product of \( l_z \) and \( l_y \).
That is
\[ l_x = l_z l_y. \]

We call \( l_z \) the quotient of \( l_x \) by \( l_y \) and write
\[ l_z = \frac{l_x}{l_y}. \]

14. **Definition of Division.**

As in multiplication let \( l_o, l, l_\infty \) be any three elements on \( P \) and let \( R_o, R, R_\infty \) be any elements on \( l_o, l, l_\infty \) respectively. Denote \( \overline{R_o, R}, \) by \( a \), and \( \overline{R, R_\infty} \) by \( b \). Let \( y \) be the element common to \( l_y a \) and \( R_\infty \). Let \( x \) be the element common to \( x b \) and \( R \). The element common to \( x b \) and \( p \) is defined as the quotient of \( l_x \) by \( l_y \) viz.,
\[ l_x y = \frac{l_x}{l_y}. \]
15. Theorems on Division.

Theorem 14. The necessary and sufficient condition for the equality \( I_x = \frac{l_x}{l_y} \) is

\[ Q(l_\alpha, l_\gamma, l_\omega, l_\alpha, l_\gamma) \]

where \( l_\alpha \) and \( l_\gamma \) are any two elements on \( P \) and \( l_\gamma \neq l_\alpha \).

Proof. With reference to fig. 20 we have the quadrilateral \( a_\gamma b_\alpha x_\gamma \) which gives rise to the quadrilateral set \( Q(l_\alpha, l_\gamma, l_\omega, l_\alpha, l_\gamma) \) which is therefore a necessary condition for \( l_\gamma = \frac{l_x}{l_\gamma} \). But if we have \( Q(l_\alpha, l_\gamma, l_\omega, l_\alpha, l_\gamma) \) this implies the existence of the quadrilateral \( a_\gamma b_\alpha x_\gamma \) which gives the construction for \( l_\gamma = \frac{l_x}{l_\gamma} \). Hence \( Q(l_\alpha, l_\gamma, l_\omega, l_\alpha, l_\gamma) \) is a sufficient as well as necessary condition for the equality \( l_\gamma = \frac{l_x}{l_\gamma} \) to hold.

Corollary 1. If \( l_\alpha \), any element on \( P \), is distinct from \( l_\alpha \) and \( l_\omega \) the following results hold:

\[(a) \quad \frac{l_x}{l_\alpha} = l_1
\]
\[(b) \quad \frac{l_x}{l_\omega} = l_\alpha
\]
\[(c) \quad \frac{l_x}{l_\omega} = l_\infty
\]
\[(d) \quad \frac{l_\alpha}{l_x} = \frac{l_\omega}{l_{\omega}}
\]
\[(e) \quad \frac{l_\omega}{l_{\omega}} = l_0
\]
\[(f) \quad \frac{l_{\omega}}{l_{\omega}} = l_0
\]
Proof. These results follow directly from the definition as did similar results in the case of multiplication.

Corollary 2. The operation of division is single-valued for every pair of elements \( l_x \) and \( l_y \) on \( P \) except the pairs \( l_\alpha, l_\beta \) and \( l_\alpha', l_\beta' \).

Proof. If \( l_x \) and \( l_y \) are not equal to \( l_\alpha, l_\beta \), or \( l_\alpha', l_\beta' \), we are given five of the elements in the quadrilateral set

\[
Q(l_\alpha, l_\beta, l_x, l_y, l_\alpha', l_\beta')
\]

The remaining element \( l_{x/\beta} \) is, therefore, uniquely determined (Theorem 3). But by Theorem 14 \( Q(l_\alpha, l_\beta, l_x, l_y, l_\alpha', l_\beta') \) is a necessary and sufficient condition for the equality \( l_y = l_{x/\beta} \) to hold. The quotient of \( l_x \) by \( l_y \), namely \( l_{x/\beta} \) is then uniquely determined except in cases noted above. But values for these cases are given by Corollary 1. except for pairs \( l_\alpha, l_\beta \) and \( l_{\alpha'}, l_{\beta'} \) and these values are in each case unique.

This completes the discussion of the four operations addition, multiplication, subtraction, and division. These operations are known as rational operations.


The notions of "Group of Operations," "Number Systems," and "Isomorphism" are fundamental in Mathematics. They are defined on pp. 67, 149, and 150 of Veblen and Young's Projective Geometry, Volume I.

To provide for future reference we summarize the
17. **Definition of a Group.**

A set or class \( G \), of elements is said to form a group with respect to an operation, which we designate by \( \circ \), if:

\( G_1 \): the result \( a \circ b \) of performing the operation \( \circ \) in the order given, upon \( a \) and \( b \), where \( a \) and \( b \) are any pair of equal or distinct elements of the class \( G \), is a uniquely determined element of \( G \).

\( G_2 \): the relation \((a \circ b) \circ c = a \circ (b \circ c)\) holds for \( a \), \( b \), and \( c \) any three equal or distinct elements of \( G \).

\( G_3 \): there exists an element \( i \) of \( G \) such that the relation \( a \circ i = a \) holds for every element \( a \) of \( G \). This element \( i \) is called the identity element.

\( G_4 \): there exists for every element \( a \) of \( G \) another element \( a' \) such that the relation \( a \circ a' = i \) holds. The element \( a' \) is called the inverse element of \( a \).

A set of elements \( G \) which, in addition to satisfying the conditions \( G_i, i = 1, 2, 3, 4 \) above, satisfies the relation \( a \circ b = b \circ a \).
is said to form a **commutative** or **abelian** group with respect to the operation .

18. **Definition of a Number System.**

A set \( N \) of elements is said to form a number system if there exists two distinct operations, which we designate by \( \oplus \) and \( \odot \), and if these operations operate, in the prescribed order, upon pairs of elements of \( N \) in such a manner that the following conditions are fulfilled:

\( N_1 \): The set \( N \) of elements forms a group with respect to the operation \( \oplus \).

\( N_2 \): The set \( N \) of elements forms a group with respect to the operation \( \odot \) except, if the identity element with respect to operation \( \oplus \) is \( i_+ \) then no inverse exists for the element \( i_+ \) with respect to the operation \( \odot \).

\( N_3 \): For \( a, b, c \) any three elements of \( N \) the relation

\[
\Sigma (b \odot c) = (a \odot b) \oplus (a \oplus c)
\]

and

\[
(b \oplus c) \odot a = (b \odot c) \oplus (b \oplus a)
\]

hold.

The elements which compose a number system are called numbers.

A number system whose elements form abelian groups with respect to both multiplication and addition is said to form a **field**.

"Group" as defined on page 26.
From the above definition all the theory of number systems with reference to the rational operations, may be developed. Our ordinary algebra of rational or real numbers is merely a special case of this theory. It will be shown that the algebra of points and the algebra of lines are also cases of the same theory.

19. **Isomorphism.**

If a one-to-one reciprocal correspondence exists between the numbers of two systems such that the sum of any two numbers of the one system corresponds to the sum of the two corresponding numbers of the other system; and such that the product of any pair of numbers of the one system corresponds to the product of the corresponding numbers of the other system, there is said to be an abstract equivalence or an isomorphism between the two systems. The two systems are called isomorphic systems.

20. **Nonhomogeneous coördinates.**

**Theorem 15.** The set of all elements \( \lambda_a \) on an element \( P \), from which the element \( \lambda \) has been excluded, forms a number system.

**Proof.** If we excluded \( \lambda_a \) from the set of all elements \( \lambda_a \) on the element \( P \), the set forms a group with respect to addition for:
$G_i$ is satisfied by Theorem 3 Cor. 2

$G_2$ is satisfied by Theorem 4

$G_3$ is satisfied by Theorem 3 Cor. 1(a)

where $1_e$ is the element $e$.

$G_4$ is satisfied, for by the definition and theorems on subtraction, the existence of an element $'a'$ for every element $1_a$ is assured.

The above set, therefore, fulfils condition $N'$ for the existence of a number system. The same set also forms a group with respect to multiplication, except that for $1_e$, the identity element with respect to addition, there exists no inverse.

$G_5$ holds by Theorem 8 Cor. 2

$G_6$ holds by Theorem 9

$G_7$ holds by Theorem 8 Cor. 1, $1$, being the identity element with respect to multiplication.

$G_8$ is satisfied since by the definition of, and theorem on, division the existence of an inverse $'a'$ for every element $1_a$ other than $1_e$, is assured.

Hence the above set fulfils $N'_2$. But this set also satisfies $N'_3$ for by Theorem 11, for the above set, multiplication is distributive with respect to addition. The set of all
elements $\ell_a$ on $P$ from which $\ell_\alpha$ has been excluded forms, therefore, a number system.

Corollary 1. The set of all elements $\ell_a$ on $P$ from which $\ell_\alpha$ has been excluded, forms a field with respect to multiplication and addition.

This follows readily since the number system of Theorem 15 forms an abelian group with respect to addition for by Theorem 5 this operation was proved commutative. It is also an abelian group with respect to multiplication which was proved commutative by Theorem 12.

We can, therefore, regard an element $\ell_a$ on $P, (\ell_\alpha \neq \ell_\alpha)$ as a number of a number system meaning by this nothing more than is implied in the definition of a number system. The element $\ell_\alpha$, however, exists and for this reason we keep the idea of number distinct from the idea of the element $\ell_a$. To accomplish this we introduce a field of numbers $\alpha, b, c$. which is isomorphic with the field elements $\ell_a$ on $P$.

In this new number system $\ell_\alpha$, the identity element with respect to addition, is called zero and is denoted by the symbol 0. The identity element with respect to multiplication is called unity and is denoted by 1. Then

$\ell_0$ corresponds to 0
$\ell_1$ corresponds to 1

and, in general, every element $\ell_a$, $\ell_\alpha$ excepted, corresponds to a number $a$ of the new system and conversely to every number $a$ of this system there corresponds an element $\ell_a$. 
of the system on $P$.

To correspond to the exceptional element $l_0$, we introduce a symbol $\infty$ which as yet has no property other than that of being the correspondent of $l_0$. The symbol $\infty$ is not a number of the new field.

The number $a$ of the new field, to which the element $l_a$ corresponds, is called the non-homogeneous coordinate of $l_a$. In the future the term "the element $a$" will mean the element whose non-homogeneous coordinate is $a$. Relations between elements on $P$ can, therefore, be expressed by means of equations involving their coordinates. For example the equality

$$l_x + l_y = l_z$$

can be expressed as

$$x + y = z$$

21. **Definition of an One-dimensional Primitive Form.**

The pencil of points, the pencil of lines and the pencil of planes are called the primitive geometric forms of the first grade or one-dimensional primitive forms.

22. **Types of Projectivities.**

Let $l_x$ be any element on $P$, and $l_a$ a fixed element on $P, (l_a \neq l_x)$. If $l_x$ is made to correspond to $l_x'$ where

$$l_x' = l_x + l_a$$

the correspondence so established is projec-
tive by Theorem 6. This can also be written
\[ x' = x + a \]
where \( x', x, a \) are respectively the non-homogeneous coordinates of \( l_x', l_x \), and \( l_a \). We call a projectivity of form
\[ x' = x + a \]
a projectivity of Type I. If \( l_x \) is made to correspond to \( l_x' \) where \( l_x' = l_x l_a \) or \( x' = x a \) we again set up a projective correspondence by Theorem 10. We call a projectivity of form \( x' = x a \) a projectivity of Type II. If we make \( l_x \) correspond to \( l_x' \) where \( l_x' = l_x l_1 \) or \( x' = x \), we also set up a projectivity. For we can construct \( l_x' \), the quotient of \( l_1 \) by \( l_x \), if \( l_x \neq 0 \), and \( l_1 \neq \infty \). (fig. 21). With reference to this figure we have
\[
\left[ \begin{array}{c} l_x \\ \frac{d}{a} \\ [y] \frac{l_y}{a} \\ [z] \frac{b}{a} \end{array} \right] = \left[ \begin{array}{c} l_x' \\ \frac{d}{a} \\ [y] \frac{l_y}{a} \\ [z] \frac{b}{a} \end{array} \right]
\]
But the result of two or more perspectivities is a projectivity. Hence \( l_x \sim l_x' \), or the correspondence \( x' = \frac{l_x}{x} \) is projective. We call a projectivity of the form \( x' = \frac{l_x}{x} \) a projectivity of Type III. We have then three types of projectivity.

I \[ x' = x + a \]

II \[ x' = x a \]

III \[ x' = \frac{l_x}{x} \]

23. **Properties of the Symbol \( \infty \)**

We now discuss certain of the properties that must be assigned to the symbol \( \infty \) which we arbitrarily introduced
into our system. By Theorem 3 Cor. 1. \( l_\infty + l_\theta = l_\infty \)

Then the projectivity \( \mathcal{M}' = \mathcal{M} + \alpha \) leaves the element \( l_\infty \) invariant. But \( l_\infty \) corresponds to the symbol \( \infty \). We can write therefore as one property of the symbol

\[ \infty + \alpha = \infty \]

The equality

\[ l_\infty l_\infty = l_\infty \]

is given by Theorem 8 Cor. 1. That is the projectivity \( \mathcal{M}' = \mathcal{M} + \alpha \) leaves invariant the element \( l_\infty \), to which \( \infty \) corresponds. As a further property of the symbol \( \infty \) we write

\[ \infty \cdot \alpha = \infty \]

By Theorem 13 Cor. 1

\[ \frac{l_\infty}{l_0} = l_\infty \]

and

\[ \frac{l_\infty}{l_\infty} = l_\infty \]

That is a projectivity of form \( \mathcal{M}' = \frac{1}{x} \) makes \( l_\infty \) correspond to \( l_0 \) and \( l_0 \) correspond to \( l_\infty \). We can therefore assign further properties to the symbol \( \infty \), viz.,

\[ \infty = \frac{1}{l_0} \]

\[ l_0 = 0 \]

24. Properties of Projectivities of Types (I), (II), and (III)

The properties of these three projectivities or transformations are of prime importance in the development of the
algebra of the elements \( a \) on the element \( P \). They are, therefore, noted here.

A projectivity of type I, \( x' = x + a \), has been shown to leave invariant the element \( x_0 \) to which the symbol \( \infty \) corresponds in the isomorphism established between the elements \( x_0 \) on \( P \) and the numbers of the number system \( a, b, c \) \( \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \) (section 23). Furthermore the above transformation leaves no other element on \( P \) invariant. For suppose it does leave another element on \( P \), \( x_2 \) say, invariant. Then \( x_2 \) corresponds to \( x_2 \) under transformation I. But \( x' = x + a \) that is, \( x_2 \) corresponds to \( x_2 + a \) or \( x + a = x \) from which \( a = 0 \). The transformation I becomes then
\[
x' = x
\]
or the identity transformation. We can make any element \( x \) on \( P \) correspond to any other element \( x_2 \) on \( P \) by a proper choice of \( a \). For if \( x \) correspond to \( x_2 \) under I then
\[
\begin{align*}
  y &= x + a \\
  a &= y - x
\end{align*}
\]
that is by choosing \( a = y - x \), the element \( x \) is made to correspond to \( y \). The definition of subtraction guarantees the existence of \( a \). Having once chosen \( a \), however, the projectivity is fully determined for we have
\[
\begin{align*}
  0 &\sim a \\
  x &\sim y \\
  y &\sim a
\end{align*}
\]
and hence, by the fundamental theorem of projective geometry, our projectivity is fully determined.

A projectivity of type II, \( x' = x \cdot a \), leaves invariant \( a \) to which corresponds the symbol \( \infty \) (section 23). It also leaves \( l_0 \), to which \( o \) corresponds, invariant since, by Theorem 9, Corollary 1,

\[
l_\infty l_0 = l_0.
\]

Furthermore no other element on \( P \) is invariant under transformation II. For suppose \( l_2 \) were invariant under II. Then \( l_2 \) would correspond to \( l' \) that is,

\[
z = z \cdot a
\]

\[
\sigma \cdot a = \frac{z}{-z}
\]

Transformation II becomes then,

\[
x' = x.
\]

or the identity transformation. We can make \( l_x \) correspond to any other element \( l_2 \) on \( P \) by properly choosing \( a \).

For if under II, \( l_x \) corresponds to \( l_2 \) then

\[
z = x \cdot a
\]

\[
\sigma \cdot a = \frac{z}{-z}
\]

The existence of \( a \) is guaranteed by the definition of division. Once \( a \) is chosen, however, as in the case of type I the projectivity is fully determined.

Under a projectivity of type III, \( \infty \) corresponds to \( o \) and \( o \) corresponds to \( \infty \) (section 23). Furthermore two applications of the projectivity bring us back to the element
with we started. For if \( \pi \) denote the transformation

\[
\pi(o \circ \gamma) = \pi \circ \frac{1}{\gamma}
\]

\[
\pi(\infty \circ \gamma) = \pi \circ \gamma
\]

Such a transformation is called an involution.

25. **Products of Projectivities.**

The product of two projectivities of type I is a projectivity of type I.

\[
x' = x + a
\]

\[
x'' = x' + b
\]

\[
x''' = x'' + c = x + c
\]

where \( c = a + b \).

The product of two projectivities of type I is also commutative since the operation of addition was shown to be commutative.

The product of two projectivities of type II is a projectivity of type II.

\[
x' = xa
\]

\[
x'' = x'c
\]

\[
x''' = x'c = x + c
\]

where \( c = ab \).

The product of two such projectivities is commutative since the operation of multiplication was shown to be commutative.

The product of two projectivities of type III has been shown to be an involution.
26. The Algebra of a Primitive One-dimensional Form.

The complete algebra of a set of elements \(\mathcal{A}\) on an element \(P\) may be developed by employing the definitions of the four rational operations, the conception of an element \(a\) on \(P\) \((a \neq \omega)\), as a number of a number system, and the properties of projectivities (I), (II), and (III).

For the special case of the set of points on a line, such an algebra is developed in Veblen and Young's Projective Geometry, Vol. I, pp. 154-----. By duality the same argument holds for either of the primitive one-dimensional forms: the pencil of lines on a point, or the pencil of planes on a line, and also for the primitive two-dimensional form, the set of planes on a point. No proof will, therefore, be given for the following theorems which apply in the general case.

**Theorem 16.** Any projectivity on an element \(P\) is the product of projectivities of the three types (I), (II), and (III) and may be expressed in the form

\[
x' = \frac{ax + b}{cx + d}, \quad ad - be \neq 0
\]

**Theorem 17.** Every equation of the form

\[
x' = \frac{ax + b}{cx + d}
\]

represents a projectivity on an element \(P\) provided \(ad - be \neq 0\).

**Theorem 18.** Any projective correspondence between the elements
x on P and the elements y on Q, P and Q being distinct elements, may, by a proper choice of coordinates on the two elements P and Q, be represented analytically by the relation
\[ y = x \]
If the coordinates on P and Q are so related that the relation
\[ y = x \]
represents a projective correspondence, then any projective correspondence between the elements x and the elements y is given by
\[ y = \frac{ax + b}{cx + d} \quad \text{and} \quad ad - bc \neq 0 \]

The introduction of homogeneous coordinates removes the need of using the symbol \( \infty \) from which arose so much difficulty. We have then

Theorem 19. In homogeneous coordinates a projectivity on an element P is represented by a linear homogeneous transformation in two variables
\[ \rho x_i' = ax_i + bx_i, \quad \rho x_2' = cx_2 + dx_2 \quad \text{and} \quad ad - bc \neq 0 \]
where \( \rho \) is an arbitrary factor of proportionality and where \( x_i' \) and \( x_2' \) are the homogeneous coordinates of the element x.
27. The Set of Planes on a Point and the Set of Planes on a Line.

As was stated in section 26, Theorems 1–19 are also true by duality for the set of all planes on a point and the set of all planes on a line. A suitable interpretation of the notation employed must be agreed upon, however, before the foregoing proofs will read in terms of planes and points, or planes and lines. Some additional assumptions will also be necessary. As this would involve the keeping in mind of a great number of possible interpretations of the notation something more compact and suggestive is desirable. We therefore give the following notation by means of which the previous proofs and definitions might have been given.

Let $S_o$ represent a point, $S$, a line, and $S, \alpha$ a plane. A set of points is represented by $S_{i,\alpha}$, where $i$ takes on different values for each point. Similarly a set of lines is represented by $S_{\alpha, j}$ and a set of planes by $S_{i, \alpha, \beta}$. The symbol $S_{ij, \alpha, \beta}$, therefore, represents a point, line, or plane according as $j = 0, 1, 2$. The line on the two points $S_{0, \alpha}$ and $S_{0, \beta}$ is denoted by $S_{0, \alpha}, S_{0, \beta}$. Similarly $S_{i, \alpha}, S_{i, \beta}$ denotes the plane on the lines $S_{i, \alpha}$ and $S_{i, \beta}$, and, in general, $S_{i, \alpha}, S_{i, \beta}$ represents the element common to the elements $S_{i, \alpha}$ and $S_{i, \beta}$.

The assumptions upon which the mathematical science, developed in this paper, has been based are given in terms of this new notation with such additions as are necessary.
because of the extension of the work to include planes.

A. The Assumptions of Alignment:

1. If \( S_{i,a} \) and \( S_{i,b} \) are distinct elements on \( S_j \) there is at least one element \( S_{i,h} \) on both \( S_{i,a} \) and \( S_{i,b} \).

\[ i, j, h = 0, 1, 2 \]

2. If \( S_{i,a} \) and \( S_{i,b} \) are distinct elements on \( S_j \) there is not more than one element \( S_{i,h} \) on both \( S_{i,a} \) and \( S_{i,b} \).

\[ i, j, h = 0, 1, 2 \]

3. If \( S_{i,a} \), \( S_{i,b} \), \( S_{i,c} \) are distinct elements on \( S_j \) and not all on the common element \( S_{k,p} \) and if \( S_{i,d} \), \( S_{i,e} \) are any other two elements on \( S_j \) such that \( S_{i,b} \), \( S_{i,c} \), \( S_{i,d} \) are on the common element \( S_{k,a} \) and \( S_{i,c} \), \( S_{i,a} \), \( S_{i,d} \) are on the common element \( S_{k,b} \), then there exists an element \( S_{i,f} \) on such that \( S_{i,a} \), \( S_{i,b} \), \( S_{i,c} \) are on the common element \( S_{k,a} \) and \( S_{i,d} \), \( S_{i,e} \), \( S_{i,f} \) are on the common element \( S_{k,b} \).

\[ i, j, k = 0, 1, 2 \]

E. The Assumptions of Extension:

1. There exists at least one element \( S_j \).

2. All elements \( S_{i,a} \) are not on the same \( S_j \).

3. All elements \( S_{i,c} \) are not on the same \( S_k \).

4. There are at least three elements \( S_{i,a} \), \( S_{i,b} \), \( S_{i,c} \) on every \( S_j \).

\[ i, j, k = 0, 1, 2 \]

P. The Assumption of Projection:

If a projectivity in a \( S_k \) leaves each of three distinct elements \( S_{i,a} \), \( S_{i,b} \), \( S_{j} \) on \( S_k \) invariant it leaves
every element on $S_j$ invariant.

$$i, j, k = o, r, s \quad i \neq j \neq k$$

Theorems proved by use of this notation can be readily interpreted to refer to any one of the four cases:

1. the set of all points on a line
2. the set of all lines on a point
3. the set of all planes on a point
4. the set of all planes on a line

To illustrate this, the definition of addition given in section 6 will be repeated.

Consider a scale on $S_j$ determined by means of $s_{i, o}$, $s_{i, r}$, and $s_{i, s}$. Let $s_{i, x}$ and $s_{i, y}$ be any two distinct elements on $S_j$. We indicate the construction for an element $s_{i, x+y}$ which we define as the sum of $s_{i, x}$ and $s_{i, y}$.

Choose $s_{j, o}$ any element on $s_{i, o}$, and $s_{j, r}$ and $s_{j, s}$ any two elements on $s_{i, s}$. Let $a$ denote the element $s_{j, o} + s_{j, r}$ and $b$ the element $s_{j, s} + s_{j, r}$. Let $x$ denote the element on $s_{i, x}$ and $s_{j, r}$ and $y$ the element on $s_{i, y}$ and $s_{j, s}$. The element $s_{i, x+y}$ which is on $x, y$ and $S_j$ we define as the sum of $s_{i, x}$ and $s_{i, y}$ and write

$$s_{i, x+y} = s_{i, x} + s_{i, y}$$

The above construction with $i = o$, $j = r$, gives the addition of points on a line. When $i = r$, $j = o$ we have the construction for the sum of two lines on a point. If $i = s$, $j = o$ we have the addition of planes on a point and if $i = r$, $j = s$, the addition of planes on a line.
Bibliography.

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