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THE FUNDAMENTAL SURGERY THEOREM AND

THE CLASSIFICATION OF MANIFOLDS

by

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Abstract

The purpose of this paper is to present a survey of some important results in the classification of differentiable manifolds. We begin with the Poincaré conjecture and its partial solution using the h-cobordism theorem. We review next the work of Kervaire and Milnor, concerned with the diffeomorphism classes of homotopy spheres. The surgery problem developed from their work, and we present its solution in the simply-connected case, by Browder. This solution amounts to the surgery invariant theorem, the fundamental surgery theorem and associated results. We end our discussion with the plumbing theorem, and several important classification theorems of Browder, Novikov and Wall.

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Chapter I. The Poincaré Conjecture.

§1. The Poincaré Conjecture and the h-cobordism Theorem.

The original form of the Poincaré conjecture was the following:

1.1 If M is a closed 3-manifold such that $H_*(M)\cong H_*(S^3)$, then $M\cong S^3$.

This was shown to be false, through the following counter-example: The binary icosahedral group I^* is defined by the generators A, B, and C, and relations $A^3=B^2=C^5=ABC$ between them. I^* is perfect, and is a subgroup of S^3. Define a closed 3-manifold $M=S^3/I^*$. Then $\pi_1(M)=I^*$, and $H_1(M)=\pi_1(M)_{ab}=I^*_{ab}=1$. By Poincaré duality, $H_2(M)=1$. Thus, $H_*(M)=H_*(S^3)$, but M is not homeomorphic to S^3, because $\pi_1(M)=I^*$, whereas $\pi_1(S^3)=1$.

The failure of the original conjecture led to an amended formulation:

1.2 If M is a closed, simply-connected 3-manifold, then $M\cong S^3$.

Note that, by the Hurewicz isomorphism theorem, the Poincaré duality theorem, and the universal coëfficient theorem, the hypothesis that M is simply-connected implies that in fact $\pi_*(M)\cong\pi_*(S^3)$, and hence that $M\simeq S^3$.

Although there have been partial results concerning this conjecture, it has not yet been completely settled.

The Poincaré conjecture can be extended to dimensions other than 3:

1.3 If M is a closed n-manifold which is homotopically equivalent to S^n, it is homeomorphic to S^n.

This statement has been proved for $n\neq 3,4$. In fact, 1.3 can be stated in an apparently weaker form which is, by the Hurewicz isomorphism theorem, actually equivalent to 1.3:

1.4 If M is a closed, simply-connected n-manifold with the integral homology of S^n, then M is homeomorphic to S^n.

integral homology of S^n, then M is homeomorphic to S^n.

We will prove the generalized Poincaré conjecture in dimensions greater than 4 by means of the h-cobordism theorem.

A smooth manifold triad is defined to be a triple $(W;V,V')$, where W is a compact, smooth manifold, and the boundary of W is the disjoint union of two open and closed submanifolds V and V'.

1.5 Theorem (h-cobordism theorem): Suppose the triad $(W;V,V')$ has the properties: (1) W,V, and V' are simply-connected,

(2) $H_*(W,V)=0$,

(3) dim $W=n\geq 6$.

Then W is diffeomorphic to $V\times[0,1]$.

The following proposition is central to the proof of the generalized conjecture:

1.6 Proposition: Suppose W is a compact simply-connected smooth n-manifold, $n\geq 6$, with a simply-connected boundary V. Then the following four assertions are equivalent:

(1) W is diffeomorphic to D^n.

(2) W is homeomorphic to D^n.

(3) W is contractible.

(4) W has the integral homology of a point.

Proof: It is clear that $(1)\to(2)\to(3)\to(4)$, so that we need only prove $(4)\to(1)$. If D_0 is a smooth n-disc imbedded in intW, then $(W\backslash \mathrm{int}D_0;\partial D_0,V)$ satisfies the conditions of the h-cobordism theorem. In particular, by excision $H_*(W\backslash \mathrm{int}D_0,\partial D_0)\cong H_*(W,D_0)=0$.

Since the cobordism $(W;\phi,V)$ is the composition of $(D_0;\phi,\partial D_0)$ with a product cobordism $(W\backslash \mathrm{int}D_0;\partial D_0,V)$, W is homeomorphic to D_0. A theorem of Milnor shows that the composition preserves differentiable structures, so that W is in fact diffeomorphic to D_0. QED

We are now ready to prove the generalized conjecture.

Proof of 1.4: Case 1: $n>5$. If $D_0 \subseteq M$ is a smooth n-disc, then $M\backslash \mathrm{int} D_0$ satisfies the hypothesis of 1.6. In particular,

$$
\begin{aligned}
H_i(M\backslash \mathrm{int}D_0) &\cong H^{n-i}(M\backslash \mathrm{int}D_0, \partial D_0) && \text{by Poincaré duality} \\
&\cong H^{n-i}(M, D_0) && \text{by excision} \\
&\cong \begin{cases} 0 & \text{if } i>0 \\ \mathbb{Z} & \text{if } i=0 \end{cases} && \text{by the exact cohomology sequence.}
\end{aligned}
$$

Consequently, $M=(M\backslash \mathrm{int}D_0)\cup D_0$ is diffeomorphic to a union of two copies D_1^n, D_2^n of the n-disc with the boundaries identified under a diffeomorphism $h:\partial D_1^n \to \partial D_2^n$. Such a manifold is called a twisted sphere. The proof is completed by showing that any twisted sphere $M=D_1^n \cup_h D_2^n$ is homeomorphic to S^n. Let $g_1:D_1^n \to S^n$ be an embedding onto the southern hemisphere of $S^n \subseteq \mathbb{R}^{n+1}$. (I.e. the set $\{x \mid \|x\|=1,\ x_{n+1}\leq 0\}$.) Each point of D_2^n may be written tv, $0\leq t\leq 1$, $v\in \partial D_2^n$. Define $g:M\to S^n$ by $g(u)=g_1(u)$ for $u\in D_1^n$, $g(tv)= \sin(\frac{\pi t}{2})g_1(h^{-1}(v))+\cos(\frac{\pi t}{2})e_{n+1}$, where $e_{n+1}=(0,\ldots,0,1)\in \mathbb{R}^{n+1}$, for all points $tv\in D_2^n$. Then g is a well-defined injective continuous map onto S^n, and is hence a homeomorphism. This completes the proof for case 1.

Case 2: $n=5$. We use here:

1.7 Theorem: Suppose M^n is a closed, simply-connected smooth manifold with the homology of S^n. Then if $n=4,5$, or 6, M bounds a smooth, compact, contractible manifold.

Thus, 1.7 and 1.6 imply that M^5 bounds a manifold homeomorphic to D^6, so that M^5 is homeomorphic to S^5.

Remark: The generalized conjecture holds in dimensions 1 and 2 as well. The proof is trivial, because of the well-known classification of 1- and 2-manifolds.

By using 1.7 and 1.6 one can show that in fact a simply-connected homology n-sphere is diffeomorphic to S^n, for n=5,6. However, Milnor has proved that this is not true for n=7. The next section will be devoted to an examination of this result.

§2. Exotic Differential Structures on the 7-Sphere.

The invariant $\lambda(M^7)$

For every closed oriented smooth 7-manifold satisfying the hypothesis

2.1 $\qquad H^3(M)=H^4(M)=0$

we will define a residue class $\lambda(M)$ modulo 7. According to Thom every closed smooth 7-manifold M is the boundary of a smooth 8-manifold,B. The invariant $\lambda(M)$ will be defined as a function of the index τ and the Pontrjagin class p_1 of B^8.

If $\mu\in H_7(M^7)$ is the distinguished generator, then an orientation $\nu\in H_8(B^8,M^7)$ is determined by the relationship $\partial\nu=\mu$. Define a quadratic form over the group $H^4(B^8,M^7)$/torsion by the formula $\alpha\to\langle\nu,\alpha^2\rangle$. Let $\tau(B^8)$ be the index of this form (the number of positive terms minus the number of negative terms when the form is diagonalized over R).

Let $p_1\in H^4(B^8)$ be the first Pontrjagin class of the tangent bundle of B^8. (For the definition of Pontrjagin classes, see [Milnor 1974].) The hypothesis 2.1 (together with the long cohomology sequence of the pair (B^8,M^7)) implies that the inclusion homomorphism $\iota:H^4(B^8,M^7)\to H^4(B^8)$ is an isomorphism. Therefore, we can define a 'Pontrjagin number' $q(B^8)=\langle\nu,(\iota^{-1}p_1)^2\rangle$.

2.2 Theorem: The residue class of $2q(B^8)-\tau(B^8)$ modulo 7 does not depend on the choice of the manifold B^8.

Define $\lambda(M^7)$ as this residue class. As an immediate consequence, we have:

2.3 <u>Corollary</u>: If $\lambda(M^7)\neq 0$ then M is not the boundary of an 8-manifold with fourth Betti number zero.

<u>Proof of Theorem 2.2</u>: Let B_1^8, B_2^8 be manifolds both having boundary M^7. (We may assume they are disjoint.) Then $C^8=B_1^8\cup_{M^7}B_2^8$ is a closed 8-manifold which possesses a differentiable structure compatible with that of B_1^8 and B_2^8. Choose that orientation ν for C^8 which is consistent with the orientation ν_1 of B_1^8 (and therefore consistent with $-\nu_2$). Let $q(C^8)$ denote the Pontrjagin number $<\nu, p_1^2(C^8)>$.

According to [Thom 1954] we have

$$\tau(C^8)=<\nu, \frac{1}{45}(7p_2(C^8)-p_1^2(C^8)>,$$

and therefore

$$45\tau(C^8)+q(C^8)=7<\nu, p_2(C^8)>\equiv 0 \pmod 7$$

This implies

(1) $$2q(C^8)-\tau(C^8)\equiv 0 \pmod 7$$

2.4 <u>Lemma</u>: Under the above conditions we have

(2) $$\tau(C^8)=\tau(B_1^8)-\tau(B_2^8), \text{ and}$$

(3) $$q(C^8)=q(B_1^8)-q(B_2^8).$$

Formulae (1),(2), and (3) clearly imply that

$$2q(B_1^8)-\tau(B_1^8)\equiv 2q(B_2^8)-\tau(B_2^8), \pmod 7$$

which is just the statement of the theorem.

<u>Proof of Lemma 2.4</u>: Consider the diagram:

$$\begin{tikzcd} H^n(B_1,M)\oplus H^n(B_2,M) \arrow[d, "i_1\oplus i_2"] & H^n(C,M) \arrow[l, "h"', "\cong"] \arrow[d, "j"] \\ H^n(B_1)\oplus H^n(B_2) & H^n(C) \arrow[l, "k"'] \end{tikzcd}$$

Note that for n=4 these homomorphisms are all isomorphisms.

If $\alpha = jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C)$, then

(4) $$\langle \nu, \alpha^2 \rangle = \langle \nu, jh^{-1}(\alpha_1^2 \oplus \alpha_2^2) \rangle = \langle \nu_1 \oplus (-\nu_2), \alpha_1^2 \oplus \alpha_2^2 \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle$$

Thus the quadratic form of C is the 'direct sum' of the quadratic form of B_1 and the negative of the quadratic form of B_2. This clearly implies formula (2).

Define $\alpha_1 = i_1^{-1} p_1(B_1)$ and $\alpha_2 = i_2^{-1} p_1(B_2)$. Then the relation

$$k(p_1(C)) = p_1(B_1) \oplus p_1(B_2)$$

implies that $jh^{-1}(\alpha_1 \oplus \alpha_2) = p_1(C)$. The computation (4) now shows that

$$\langle \nu, p_1^2(C) \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle,$$

which is just formula (3). This completes the proof of the lemma and of the theorem.

The following property of the invariant λ is clear:

2.5 Lemma: If the orientation of M is reversed, then $\lambda(M)$ is multiplied by -1.

As a consequence we have:

2.6 Corollary: If $\lambda(M^7) \neq 0$ then M^7 possesses no orientation-reversing diffeomorphism onto itself.

A partial characterisation of the n-sphere

Consider the following hypothesis concerning a closed manifold M^n:

2.7 There exists a differentiable function $f: M \to R$ having only two critical points x_0, x_1. Furthermore, these critical points are non-degenerate.

(That is, if $u_1, \ldots, u_n$ are local coördinates in a neighbourhood of x_0 (or x_1) then the matrix $(\partial^2 f / \partial u_i \partial u_j)$ is nonsingular at x_0 (or x_1)).

2.8 Theorem: If M^n satisfies hypothesis 2.7 then there exists a homeomorphism of M onto S^n which is a diffeomorphism except possibly at a single point.

Proof: This result is entirely due to [Reeb 1952].

The proof will be based on the orthogonal trajectories of the manifolds f=constant. Normalise the function so that $f(x_0)=0, f(x_1)=1$. According to [Morse 1925, Lemma 4] there exist local coördinates $v_1,\ldots,v_n$ in a neighbourhood V of x_0 so that $f(x)=v_1+\ldots+v_n$ for $x \epsilon V$. (Morse assumes that f is of class C^3, and constructs coordinates of class C^1, but the same proof works in the C^∞ case.) The expression $ds^2=dv_1^2+\ldots+dv_n^2$ defines a Riemannian metric in the neighbourhood V. Choose a differentiable Riemannian metric for M^n which coïncides with this one in some neighbourhood V' of x_0. (This is possible by [Steenrod 1951, 6.7 and 12.2].) Now the gradient of f can be considered as a contravariant vector field.

Following Morse we consider the differential equation

$$\frac{dx}{dt} = \text{grad } f/|\text{grad } f|^2.$$

In the neighbourhood V' this equation has solutions

$$(v_1(t),\ldots,v_n(t))=(a_1\sqrt{t},\ldots,a_n\sqrt{t}) \qquad \text{for } 0 \le t < \varepsilon,$$

where $a=(a_1,\ldots,a_n)\epsilon R^n$ is any n-tuple with $\Sigma a_i^2=1$. These can be extended uniquely to solutions $x_a(t)$ for $0 \le t \le 1$. Note that these solutions satisfy the identity $f(x_a(t))=t$.

Map the interior of the unit sphere of R^n into M^n by the map

$$(a_1\sqrt{t},\ldots,a_n\sqrt{t}) \longrightarrow x_a(t).$$

It is easily verified that this defines a diffeomorphism of the open n-cell onto $M \setminus \{x_1\}$. The assertion of the theorem now follows.

Given any diffeomorphism $g:S^{n-1} \to S^{n-1}$, an n-manifold can be obtained as follows.

2.9 Construction: Let $M^n(g)$ be the manifold obtained from two copies of R^n by matching the subsets $R^n \setminus \{0\}$ under the diffeomorphism

$$u \longrightarrow v = \frac{1}{|u|} g\left(\frac{u}{|u|}\right).$$

(Such a manifold is clearly homeomorphic to S^n. If g is the identity map, then $M^n(g)$ is diffeomorphic to S^n.)

2.10 Corollary: A manifold M^n can be obtained by the construction 2.9 if and only if it satisfies the hypothesis 2.7.

Proof: If $M^n(g)$ is obtained by the construction 2.9, then the function

$$F(x) = \frac{|u|^2}{(1+|u|^2)} = \frac{1}{(1+|v|^2)}$$

will satisfy the hypothesis 2.7. The converse can be established by a slight modification of the proof of theorem 2.8.

Examples of 7-manifolds

Consider 3-sphere bundles over the 4-sphere, with the rotation group SO(4) as structural group. The equivalence classes of such bundles are in one-to-one correspondence (by [Steenrod, 1951, §18]) with the elements of the group $\pi_3(SO(4)) \cong Z \oplus Z$. A specific isomorphism between the groups is obtained as follows. For each $(h,j) \in Z \oplus Z$, let $f_{hj}: S \rightarrow SO(4)$ be defined by $f_{hj}(u) \cdot v = u^h \cdot v \cdot u^j$, for $v \in R$. (Quaternion multiplication is understood on the right of the equation.)

Let ι be the standard generator for $H^4(S^4)$. Let ξ_{hj} be the sphere bundle corresponding to $[f_{hj}] \in \pi_3(SO(4))$.

2.11 Lemma: The Pontrjagin class $p_1(\xi_{hj})$ equals $\pm 2(h-j)\iota$.

(The proof will be given later. One can show that the characteristic class $\bar{c}(\xi_{hj})$ (see [Steenrod 1951]) is equal to $(h+j)\iota$)

For each odd integer k let M_k^7 be the total space of the bundle ξ_{hj}, where h and j are determined by the equations $h+j=1$, $h-j=k$. This manifold has a natural differentiable structure and orientation, which will be described later.

2.12 Lemma: The invariant $\lambda(M_k^7)$ is the residue class modulo 7 of k^2-1.

2.13 Lemma: The manifold satisfies the hypothesis 2.7.

Combining these we have:

2.14 Theorem: For $k^2 \not\equiv 1 \bmod 7$ the manifold M_k^7 is homeomorphic, but not diffeomorphic, to S^7.

(For $k=\pm 1$ the manifold M_k^7 is diffeomorphic to S^7, but it is not known whether this is true for any other k with $k^2 \equiv 1 \bmod 7$.)

Clearly any differentiable structure on S^7 can be extended throughout $R^8\backslash\{0\}$. However:

2.15 Corollary: There exists a differentiable structure on S^7 which cannot be extended throughout R^8.

This follows immediately from the preceding assertions, together with corollary 2.3.

Proof of Lemma 2.11: It is clear that the Pontrjagin class $p_1(\xi_{hj})$ is a linear function of h and j. Furthermore it is known to be independent of the orientation of the fibre. But if the orientation of S^3 is reversed, then ξ_{hj} is replaced by $\xi_{-j,-h}$. This shows that $p_1(\xi_{hj})$ is given by an expression of the form $c(h-j)\iota$. Here c is a constant which will be evaluated later.

Proof of Lemma 2.12: Associated with each 3-sphere bundle $M_k^7 \to S^4$ there is a 4-cell bundle $\rho_k: B_k^8 \to S^4$. The total space B_k^8 of this bundle is a differentiable manifold with boundary M^7. The cohomology group $H^4(B_k^8)$ is generated by the element $\alpha=\rho_k^*(\iota)$. Choose orientations μ,ν for M_k^7 and B_k^8 so that $\langle\nu,(i^{-1}\alpha)^2\rangle=+1$. Then the index $\tau(B_k^8)$ will be 1.

The tangent bundle of B_k^8 is the Whitney sum of (1) the bundle of vectors tangent to the fibre, and (2) the bundle of vectors normal to the fibre. The first bundle (1) is induced (under ρ_k) from the bundle

ξ_{hj}, and therefore has Pontrjagin class $p_1=\rho_k^*(c(h-j)\iota)=ck\alpha$. The second is induced from the tangent bundle of S^4, and therefore has first Pontrjagin class zero. Now by the Whitney product theorem:

$$p_1(B_k^8)=ck\alpha+0.$$

For the special case k=1 it is easily verified that B_1^8 is the quaternion projective plane QP^2 with an 8-cell removed. But the Pontrjagin class $p_1(QP^2)$ is known to be twice a generator of $H^4(QP^2)$. Therefore the constant c must be ±2, which completes the proof of 2.11.

Now $q(B_k^8)=\langle\nu,(i^{-1}(\pm 2k\alpha))^2\rangle=4k^2$, and $2q-\tau=8k^2-1\equiv k^2-1 \pmod 7$. This completes the proof of Lemma 2.12.

<u>Proof of Lemma 2.13</u>: As coördinate neighbourhoods in the base space S^4 take the complement of the north pole, and the complement of the south pole. These can be identified with the Euclidean space R^4 under stereographic projection. Then a point which corresponds to $u\in R^4$ under one projection will correspond to $u'=\frac{u}{|u|^2}$ under the other.

The total space M_k^7 can now be obtained as follows (cf. [Steenrod 1951 §18]). Take two copies of $R^4\times S^3$ and identify the subsets $(R^4\backslash\{0\})\times S^3$ by the diffeomorphism $(u,v)\rightarrow(u',v')=\left(\frac{u}{|u|^2},\frac{u^h v u^j}{|u|}\right)$, (using quaternion multiplication). This makes the differentiable structure of M_k^7 precise.

Replace the coördinates (u',v') by (u'',v''), where $u''=u'(v')^{-1}$. Consider the function $f:M_k^7\rightarrow R$ defined by $f(x)=\frac{R(v)}{\sqrt{1+|u|^2}}=\frac{R(u'')}{\sqrt{1+|u''|^2}}$, where $R(v)$ denotes the real part of the quaternion v. It is easily verified that f has only two critical points (namely $(u,v)=(0,\pm1)$) and that these are non-degenerate. This completes the proof of Lemma 2.13.

§3. Groups of Homotopy Spheres.

The following results about homotopy n-spheres are proved in [Kervaire, Milnor 1963]:

(1) The h-cobordism classes of homotopy n-spheres form an abelian group Θ_n under the connected sum operation.

(2) The h-cobordism classes of homotopy n-spheres which bound parallelisable manifolds form a subgroup bP_{n+1} of Θ_n. (This will be proved below.)

(3) The quotient group Θ_n/bP_{n+1} is isomorphic to a subgroup of the cokernel of the Hopf-Whitehead homomorphism J_n (where $J_n:\pi_n(SO_k)\longrightarrow\pi_{n+k}(S^k)$), and is finite.

(4) The group bP_{n+1} is finite, for $n\neq 3$. (In particular, it is zero for n even, and finite cyclic for n odd, $n\neq 3$.)

(5) Thus, the group Θ_n of (h-cobordism classes of) homotopy n-spheres is finite, for $n\neq 3$.

We recall from above that every homotopy n-sphere, $n\neq 3,4$, is homeomorphic to S^n. [Smale 1962] has shown that two homotopy n-spheres, $n\neq 3,4$, are h-cobordant if and only if they are diffeomorphic. Thus (for $n\neq 3,4$ at least) the group Θ_n can be described as the set of diffeomorphism classes of differentiable structures on S^n, and the last result above can be interpreted as stating that there are only finitely many essentially different such structures, for each n, $n\neq 3,4$.

We will now prove assertions (2) and (3) above.

Let M be an s-parallelisable closed n-manifold. (I.e. $\tau_M\oplus\varepsilon^1$ is trivial, where τ_M is the tangent bundle of M, and ε^1 is the trivial line bundle.) Choose an embedding $i:M\longrightarrow S^{n+k}$, with $k>n+1$. Such an embedding exists and is unique up to differentiable isotopy.

3.1 Lemma (Kervaire, Milnor): An n-dimensional submanifold of S^{n+k}, $n<k$, is s-parallelisable if and only if its normal bundle is trivial.

Thus ν_M is trivial. Let ϕ be a trivialisation of ν_M. Then the Pontrjagin-Thom construction yields a map $p(M,\phi):S^{n+k}\longrightarrow S^k$. The homotopy class of $p(M,\phi)$ is a well-defined element of the stable homotopy group $\Pi_n=\pi_{n+k}(S^k)$. Allowing the trivialisation to vary, we obtain a set $p(M)=\{p(M,\phi)\}\subseteq\Pi_n$.

3.2 Lemma: $p(M)\subseteq\Pi_n$ contains the zero of Π_n if and only if M bounds a parallelisable manifold.

Proof: $\Leftarrow$. If $M=\partial W$ and W is a parallelisable manifold, then, because of dimensional considerations, the embedding $i:M\longrightarrow S^{n+k}$ can be extended to an embedding of W into D^{n+k+1}, and W will have trivial normal bundle. Choose a trivialisation ψ of ν_W and let $\phi=\psi|M$. The Pontrjagin-Thom map $p(M,\phi):S^{n+k}\longrightarrow S^k$ extends over D^{n+k+1}, and hence is null-homotopic.

$\Rightarrow$. If $p(M,\phi)\simeq 0$, we have a map $F:D^{n+k+1}\cong S^{n+k}\times[0,1]/S^{n+k}\times 1\longrightarrow S^k$ which satisfies $F|S^{n+k}\times 0=p(M,\phi)$, and $F|S^{n+k}\times 1=\varepsilon_*$, the constant map. F can be made regular at * (the base point), relative to $S^{n+k}\times 0$, so we shall assume, without loss of generality, that it is. Then $F^{-1}(*)\subseteq D^{n+k+1}$ is a submanifold W, and ϕ can be extended to a trivialisation ψ on W. By Lemma 3.1 above and the following lemma, W is parallelisable.

3.3 Lemma: A connected manifold with non-vacuous boundary is s-parallelisable if and only if it is parallelisable. [Kervaire, Milnor]

This completes the proof of Lemma 3.2.

3.4 Lemma: If M_0 is h-cobordant to M_1, then $p(M_0)=p(M_1)$.

Proof: If $M_0+(-M_1)=\partial W$, choose an embedding of W in $S^{n+k}\times[0,1]$ such that $M_q\longrightarrow S^{n+k}\times q$ for q=0,1. Then a trivialisation ϕ_q of ν_{M_q} extends to a trivialisation ψ on W, which restricts to a trivialisation ϕ_{1-q} on M_{1-q}. Clearly (W,ψ) gives rise to a homotopy between $p(M_0,\phi_0)$ and $p(M_1,\phi_1)$.

3.5 Lemma: If M and M' are s-parallelisable then $p(M)+p(M')\subset p(M\# M')\subset \Pi_n$.

Proof: Construct a manifold W with boundary $(-M)\sqcup(-M')\sqcup(M\# M')$ as follows: beginning with $M\times[0,1]\sqcup M'\times[0,1]$, join the boundary components $M\times 1$ and $M'\times 1$ by a smooth connected sum. This sum can be extended smoothly over neighbourhoods of the joined portions, in $M\times[0,1]$ and $M'\times[0,1]$. (The details of this construction are given in [Kervaire, Milnor 1963].) The manifold W has the homotopy type of the one-point union $M\vee M'$.

Embed W in $S^{n+k}\times[0,1]$ such that $(-M)$ and $(-M')$ are mapped into well-separated submanifolds of $S^{n+k}\times 0$, and such that the image of $M\# M'$ lies in $S^{n+k}\times 1$. Given trivialisations ϕ and ϕ' of the normal bundles of $(-M)$ and $(-M')$, it is not hard to see that there exists an extension defined throughout W. Let ψ denote the restriction to $M\# M'$ of this extension. Then clearly $p(M,\phi)+p(M',\phi')$ is homotopic to $p(M\# M',\psi)$. This completes the proof.

3.6 Lemma: The set $p(S^n)\subset\Pi_n$ is a subgroup of the stable homotopy group Π_n. For any homotopy sphere Σ the set $p(\Sigma)$ is a coset of this subgroup $p(S^n)$. Thus the correspondence $\Sigma\longrightarrow p(\Sigma)$ defines a homomorphism p' from Π_n to the quotient group $\Pi_n/p(S^n)$.

Proof: Combining the previous lemma with the identities

(1) $S^n\# S^n\cong S^n$ (2) $S^n\#\Sigma\cong\Sigma$ (3) $\Sigma\#(-\Sigma)\simeq S^n$, we obtain

(1) $p(S^n)+p(S^n)\subset p(S^n)$, which shows that $p(S^n)$ is a subgroup of Π_n,

(2) $p(S^n)+p(\Sigma)\subset p(\Sigma)$, which shows that $p(\Sigma)$ is a union of cosets of this subgroup, and

(3) $p(\Sigma)+p(-\Sigma)\subset p(S^n)$, which shows that $p(\Sigma)$ must be a single coset.

This completes the proof of Lemma 3.6.

By Lemma 3.2 the kernel of $p':\Theta_n \to \Pi_n/p(S^n)$ consists exactly of all h-cobordism classes of homotopy n-spheres which bound parallelisable manifolds. Thus, these elements form a group which we denote by $bP_{n+1} \subset \Theta_n$. It follows that bP_{n+1} is isomorphic to a subgroup of $\Pi_n/p(S^n)$. Since Π_n is finite [Serre 1951], this completes the proof of assertions (2) and (3). (The relationship with the Hopf-Whitehead homomorphism, mentioned in assertion (3), is established in [Kervaire 1959, p.349].)

Chapter II. The Fundamental Theorem of Surgery.

§4. The Surgery Problem.

The technique of surgery, which Kervaire and Milnor used to obtain their results on homotopy spheres, discussed above, was also a key element in Browder's solution of the surgery problem (which was based on work by Kervaire/Milnor, and Novikov).

Very informally, this problem can be stated as follows:

Given a map $f:M\to X$ between manifolds, when can f and M be modified to f' and M' such that $f':M'\to X$ is a homotopy equivalence?

To state a more precise version of this problem, we shall first need a few definitions.

A Poincaré pair (X,Y) of dimension m is a pair of CW complexes such that there is an element $[X]\in H_m(X,Y)$ of infinite order for which $[X]\cap\cdot:H^q(X)\to H_{m-q}(X,Y)$ is an isomorphism for all q. This property is called Poincaré duality, and $[X]$ is called the orientation class of (X,Y).

Let (X,Y) be a Poincaré pair of dimension m (Y may be empty), $(M,\partial M)$ a smooth compact oriented m-manifold with boundary, and $f:(M,\partial M)\to(X,Y)$ a map. A cobordism of f is a pair (W,F) where W is a smooth compact $(m+1)$-manifold, $\partial W=M\cup U^m\cup M'^m$, $\partial U=\partial M\cup\partial M'$, $F:(W,U)\to(X,Y)$, and $F|M=f$. If $U=\partial M\times I$ and $F(x,t)=f(x)$ for $x\in\partial M$, $t\in I$, then (W,F) will be called a cobordism of f rel Y.

Let us assume that $k>>m$ and that $(M,\partial M)$ is embedded in (D^{m+k},S^{m+k-1}) with normal bundle ν^k, so that $\nu|\partial M$ is equal to the normal bundle of ∂M in S^{m+k-1}. Let ξ^k be a k-plane bundle over X. A normal map is a map $f:(M,\partial M)\to(X,Y)$ of degree 1 together with a bundle map $b:\nu^k\to\xi^k$ covering f. A normal cobordism (W,F,B) of (f,b) is a cobordism (W,F) of f, together with an extension $B:\omega^k \to \xi^k$ of b, where ω^k is the normal bundle of W^{m+1} in $D^{m+k}\times I$, where the embedding is such that $(M,\partial M)\subset(D^{m+k}\times 0, S^{m+k-1}\times 0)$, $(M',\partial M')\subset(D^{m+k}\times 1, S^{m+k-1}\times 1)$ and $U\subset S^{m+k-1}\times I$.

A normal cobordism rel Y is a cobordism rel Y such that it is a normal cobordism and $B(v,t)=b(v)$ for $v\in\nu|\partial M$, $t\in I$.

The precise version of the surgery problem is:

Problem:Given a normal map (f,b), $f:(M,\partial M)\to(X,Y)$, $b:\nu^k\to\xi^k$, when is (f,b) normally cobordant to a homotopy equivalence of pairs?

A related question is the

Restricted Problem: Given a normal map (f,b), $f:(M,\partial M)\to(X,Y)$, $b:\nu\to\xi$, when is (f,b) normally cobordant rel Y to (f',b'), where $f':M'\to X$ is a homotopy equivalence?

The solution to the restricted problem is given by the following two theorems:

4.1 The Invariant Theorem: Let (f,b) be a normal map, as above, such that $f|\partial M$ induces an isomorphism in homology. Then there is an invariant $\sigma(f,b)$ defined, $\sigma=0$ if m is odd, $\sigma\in$ if $m\equiv 0 \pmod 4$ and $\sigma\in$ if $m\equiv 2 \pmod 4$ such that $\sigma(f,b)=0$ if (f,b) is normally cobordant to a map inducing a homology isomorphism.

4.2 The Fundamental Surgery Theorem: Let (f,b) be a normal map, as above, and suppose (1) $f|\partial M$ induces an isomorphism in homology, (2) X is simply-connected, and (3) $m\geq 5$. If m is odd then (f,b) is normally cobordant rel Y to a homotopy equivalence $f':M'\to X$. If m is even, then (f,b) is normally cobordant rel Y to (f',b') such that $f':M'\to X$ is a homotopy equivalence if and only if $\sigma(f,b)=0$.

Our discussion of surgery follows very closely the treatment of [Browder 1972], and consists of the definition of the invariant σ, the statement and proof of certain properties it has, the proof of the Invariant and Fundamental theorems, and the statement of certain consequences of the Fundamental theorem, particularly the technique of plumbing and the Plumbing Theorem. Finally we will use the latter to derive some classification results for manifolds.

§5. The Surgery Invariant.

Before defining σ we shall recall some pertinent facts about quadratic and bilinear forms over Z and Z_2.

A symmetric bilinear form $(\cdot,\cdot)$ on a Z-module V satisfies: (1) $(x,y)=(y,x)$ and (2) $(\lambda x+\lambda' x',y)=\lambda(x,y)+\lambda'(x',y)$ for $\lambda,\lambda'\in Z$, $x,x',y\in V$. If $\{b_i\}$ is a basis for V and $a_{ij}=(b_i,b_j)$, then the matrix $A=(a_{ij})$ represents $(\cdot,\cdot)$ in the sense that $(x,y)=xAy^t$ (where x and y on the right are representations of the elements in the basis $\{b_i\}$). If we pass to a new basis by an invertible matrix M, so that $b'=Mb$, then in terms of the new basis $(\cdot,\cdot)$ is represented by MAM^t.

The bilinear form $(\cdot,\cdot)$ defines a quadratic form $q:V\longrightarrow Z$ by $q(x)=(x,x)$. We have $(x,y)=\frac{1}{2}(q(x+y)-q(x)-q(y))$ so that $(\cdot,\cdot)$ is derivable from q. Each of q and $(\cdot,\cdot)$ is said to be associated to the other. The form $(\cdot,\cdot)$ also defines naturally a bilinear form $(\cdot,\cdot):V\times Q\longrightarrow Q$.

5.1 Proposition: If $(\cdot,\cdot)$ is a symmetric bilinear form on a finite dimensional vector space V over Q into Q, then there is a basis for V such that the matrix representing $(\cdot,\cdot)$ in that basis is diagonal.

Define the signature of a bilinear form (and hence of the associated quadratic form) to be the number of positive diagonal entries minus the number of negative diagonal entries, using a diagonal matrix representing the form. The signature is, in fact, invariant under a change of basis, and we shall think of it as an invariant of quadratic forms over Z, taking values in Z.

A quadratic (or bilinear) form over Z is called nonsingular if the determinant of the matrix A representing it is 1. Over a field it is called nonsingular if the determinant is nonzero.

5.2 <u>Proposition</u>: Let q be a nonsingular quadratic form on a finite dimensional vector space V over R. Then $\mathrm{sgn}(q)=0$ if and only if there is a subspace $U\subset V$ such that:

(1) $\dim_R U=\frac{1}{2}\dim_R V$ and (2) $(x,y)=0$ for $x,y\in U$.

Some results we will use follow.

5.3 <u>Proposition</u>: Let q be a nonsingular quadratic form $V\longrightarrow Z$ and suppose q is indefinite (i.e. neither positive nor negative definite). Then there is $x\in V$, $x\neq 0$ such that $q(x)=0$.

5.4 <u>Proposition</u>: Let q be a nonsingular quadratic form V and suppose $2|q(x,x)$ for all $x\in V$ (q is called even). Then $8|\mathrm{sgn}(q)$.

A quadratic form q on a Z_2-vector space V is a function $q:V\longrightarrow Z_2$ such that $q(0)=0$ and $q(x+y)-q(x)-q(y)=(x,y)$ is bilinear. Two quadratic forms q,q' on V are equivalent if there is an automorphism $\alpha:V\longrightarrow V$ such that $q=q'\circ\alpha$. Under this definition, it is clear that $(x,y)=(y,x)$ and $(x,x)=q(2x)-2q(x)=0$ so that $(\cdot,\cdot)$ is a symplectic bilinear form. If $(\cdot,\cdot)$ is nonsingular, it follows that V is of even dimension, and that we may find a basis $\{a_i,b_i\}$ for V such that $(a_i,b_j)=\delta_{ij}$, $(a_i,a_j)=(b_i,b_j)=0$. Such a basis is called symplectic. We shall now classify Z_2-vector spaces with nonsingular quadratic forms, and thereby define the Arf invariant of such forms.

Let U be the 2-dimensional Z_2-vector space, with basis a,b, such that $(a,a)=(b,b)=0$, $(a,b)=1$. There are two quadratic forms on U compatible with $(\cdot,\cdot)$: q_0 and q_1, defined by $q_1(a)=q_1(b)=1$, $q_0(a)=q_0(b)=0$. Note that for both $q_i(a+b)=1$. (The notations U, q_0 and q_1 will remain fixed throughout §5.)

5.5 Lemma: Any nonsingular quadratic form on a 2-dimensional Z_2-vector space is equivalent to q_0 or q_1.

Since such a space has only 4 elements, the isomorphism is easy to construct.

If q and q' are quadratic forms on spaces V and V', then $q\oplus q'$ is the quadratic form on $V\oplus V'$ given by $(q\oplus q')(v,v')=q(v)+q'(v')$.

5.6 Lemma: On $U\oplus U$, $q_0\oplus q_0$ is isomorphic to $q_1\oplus q_1$.

The proof consists of a simple rearrangement of bases.

Now we can begin classifying forms.

5.7 Proposition: A nonsingular quadratic form q on a Z_2-vector space (which must have even dimension $2m$) is equivalent either to

$$q_1\oplus(\oplus^{m-1}q_0) \quad \text{or to} \quad \oplus^{m}q_0.$$

Proof: Let $\{a_i,b_i\}$, $i=1,\ldots,m$ be a symplectic basis of V, and let V_i be the subspace spanned by a_i,b_i, and let $\psi_i=q|V_i$. Then by the nature of the basis, $q=\overset{m}{\underset{1}{\oplus}}\psi_i$, and by Lemma 5.5 ψ_i is equivalent to q_0 or q_1. By Lemma 5.6 $q_1\oplus q_1\cong q_0\oplus q_0$, so q is equivalent to either $\oplus^m q_0$ or $q_1\oplus(\oplus^{m-1}q_0)$. QED

To complete the classification, we must show that $\phi_0=\oplus^m q_0$ is not equivalent to $\phi_1=q_1\oplus(\oplus^{m-1}q_0)$. This is clear from the

5.8 Proposition: The quadratic form ϕ_1 on V sends a majority of elements of V to $1\in Z_2$, while ϕ_0 sends a majority of elements to $0\in Z_2$.

The proof is by induction on the dimension of V.

Using this notation, we define the Arf invariant of a nonsingular quadratic form q on V as follows:

$$\mathrm{Arf}(q)=\begin{cases} 0 \text{ if } q\cong\phi_0 \\ 1 \text{ if } q\cong\phi_1 \end{cases}.$$

Thus we have:

5.9 Theorem: (Arf) Two nonsingular quadratic forms on a finite dimensional Z_2-vector space are equivalent if and only if they have the same Arf invariant.

In analogy with a previous result concerning quadratic forms over Z, we have the

5.10 Proposition: Let q be a nonsingular quadratic form on the Z_2-vector space V. Then Arf(q)=0 if and only if there is a subspace $U\subset V$ such that

(1) $\mathrm{rank}_{Z_2}U=\frac{1}{2}\mathrm{rank}_{Z_2}V$, and (2) $q(x)=0$ for all $x\in U$.

Given a bilinear form $(\cdot,\cdot)$ on a vector space V, define R, the radical of V, to be $\{x\in V|\ (x,y)=0 \text{ for all } y\in V\}$.

If $q:V\to Z_2$ is a quadratic form with $(\cdot,\cdot)$ as associated bilinear form, we have defined Arf(q) only if R=0. If $q|R\equiv 0$, it is easily seen that q defines q' on V/R, and the radical of V/R is zero. In this case we may define Arf(q) to be Arf(q'). If $q|R\not\equiv 0$, it doesn't make sense to define the Arf invariant, and in fact the equivalence of the form is determined by rankV and rankR.

Thus we have:

5.11 Theorem: Let $q:V\to Z_2$ be a quadratic form over Z_2, R the radical of the associated bilinear form. Then the Arf invariant Arf(q) is defined if and only if $q|R\equiv 0$. In general, if $q|R\equiv 0$, then q is determined up to isomorphism by $\mathrm{rank}_{Z_2}V$, $\mathrm{rank}_{Z_2}R$, and Arf(q), while if $q|R\not\equiv 0$, then q is determined by $\mathrm{rank}_{Z_2}V$ and $\mathrm{rank}_{Z_2}R$.

Note: Browder uses the notation c(q) for the Arf invariant.

We will now define an invariant I which detects maps in the cobordism class of a homology isomorphism.

A map $f:(X,Y)\to(A,B)$ between Poincaré pairs of the same dimension is said to be of degree 1 if $f_*[X]=[A]$, where $f_*:H_*(X,Y)\to H_*(A,B)$ is the map in homology induced by f. We denote the map induced $H_*(X)\to H_*(A)$ by $\bar{f}_*$, and similar notation in cohomology.

5.12 Theorem: Maps of degree 1 split, i.e. with notation as above, there are

$$\alpha_*:H_*(A,B)\to H_*(X,Y),\quad \beta_*:H_*(A)\to H_*(X),$$
$$\alpha^*:H^*(X,Y)\to H^*(A,B),\quad \beta^*:H^*(X)\to H^*(A),$$

such that $f_*\alpha_*=1$, $\bar{f}_*\beta_*=1$, $\alpha^*f^*=1$, $\beta^*\bar{f}^*=1$.

The splittings are defined straightforwardly using the Poincaré duality isomorphisms, and their inverses.

It follows from this theorem that there are direct sum splittings

$$H_*(X,Y)=\ker f_*\oplus \operatorname{im}\alpha_*,\quad H_*(X)=\ker \bar{f}_*\oplus \operatorname{im}\beta_*,$$
$$H^*(X,Y)=\operatorname{im} f^*\oplus\ker\alpha^*,\quad H^*(X)=\operatorname{im}\bar{f}^*\oplus\ker\beta^*.$$

Thus we establish the following notation:

$$K_q(X,Y)=(\ker f_*)_q\subset H_q(X,Y),\quad K_q(X)=(\ker \bar{f}_*)_q\subset H_q(X),$$
$$K^q(X,Y)=(\ker\alpha^*)^q\subset H^q(X,Y),\quad K^q(X)=(\ker\beta^*)^q\subset H^q(X);$$

(and similarly for (co)homology with coefficients).

K^q and K_q have the following property:

In the exact homology and cohomology sequences of the pair (X,Y), all the maps preserve the direct sum splitting, so induce a diagram, commutative up to sign, with exact rows:

$$\begin{array}{ccccccccc}
\cdots\xrightarrow{i^*} & K^{q-1}(Y) & \xrightarrow{\delta} & K^q(X,Y) & \xrightarrow{j^*} & K^q(X) & \xrightarrow{i^*} & K^q(Y) & \xrightarrow{\delta}\cdots \\
 & \downarrow{\scriptstyle \partial[X]\cap\cdot} & & \downarrow{\scriptstyle [X]\cap\cdot} & & \downarrow{\scriptstyle [X]\cap\cdot} & & \downarrow{\scriptstyle \partial[X]\cap\cdot} & \\
\cdots\xrightarrow{\partial} & K_{m-q}(Y) & \xrightarrow{i_*} & K_{m-q}(X) & \xrightarrow{j_*} & K_{m-q}(X,Y) & \xrightarrow{\partial} & K_{m-q-1}(Y) & \xrightarrow{i_*}\cdots
\end{array}$$

The proof of this property consists of the proof that the direct sum splittings are preserved by the Poincaré duality map ($[X]\cap\cdot$) and the homology maps.

From this sequence, and using the definition of the K^q groups, we develop the following diagram, with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & 0 & \\
\cdots & K^q(Y) \arrow[l] \arrow[u] & K^q(X) \arrow[l] \arrow[u] & K^q(X,Y) \arrow[l, "j^*"'] \arrow[u] & K^{q-1}(Y) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
\cdots & H^q(Y) \arrow[l] \arrow[u] & H^q(X) \arrow[l] \arrow[u] & H^q(X,Y) \arrow[l, "j^*"'] \arrow[u] & H^{q-1}(Y) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
\cdots & H^q(B) \arrow[l] \arrow[u, "(f|Y)^*"] & H^q(A) \arrow[l] \arrow[u, "\bar{f}^*"'] & H^q(A,B) \arrow[l] \arrow[u, "f^*"'] & H^{q-1}(B) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Suppose $m=\dim(X,Y)=4k$ and consider the pairing

$K^{2k}(X,Y;\mathbb{Q})\otimes K^{2k}(X,Y;\mathbb{Q})\to\mathbb{Q}$ given by $(x,y)=(x\cup y)[X]$.

This is symmetric because the dimension is even.

Define $I(f)$ to be the signature of $(\cdot,\cdot)$ on $K^{2k}(X,Y;\mathbb{Q})$. Note that $(\cdot,\cdot)$ is the rational form of the integral form defined on $K^{2k}(X,Y)/\text{torsion}$ by the same formula. If $(f|Y)^*:H^*(B;\mathbb{Q})\to H^*(Y;\mathbb{Q})$ is an isomorphism, then so is $j^*:K^{2k}(X,Y;\mathbb{Q})\to K^{2k}(X;\mathbb{Q})$, and so $(x\cup y)[X]=((j^*x)\cup y)[X]$.

But we have the following property of the K^q groups:

5.13 Proposition: Under the pairing $H^q(X;F)\otimes H^{m-q}(X,Y;F)$ F given by $(x,y)=(x\cup y)[X]$, where F is a ring, $K^{m-q}(X,Y;F)$ is orthogonal to $\bar{f}^*(H^q(A;F))$, $K^q(X;F)$ is orthogonal to $f^*(H^{m-q}(A,B;F))$, and on $K^q(X;F)\otimes K^{m-q}(X,Y;F)$ the pairing is nonsingular if F is a field. If $F=\mathbb{Z}$, it is nonsingular on $K^q(X)/\text{torsion}\otimes K^{m-q}(X,Y)/\text{torsion}$.

The proof is straightforward verification, depending on certain elementary properties of the cup and cap products.

Taking $q=2k$ and $F=\mathbb{Q}$, we see that the pairing $(\cdot,\cdot)$ defined above is nonsingular. Similarly if $(f|Y)^*:H^*(B)\to H^*(Y)$ is an isomorphism, then the integral form is nonsingular. In particular this is the case if $Y=B=\phi$.

5.14 <u>Theorem</u>: Let $f:(X,Y)\to(A,B)$ be a map of degree 1 between Poincaré pairs of dimension $m=4k+1$. Then $I(f|Y)=0$.

<u>Proof</u>: The proof is an application of Proposition 5.2.

5.15 <u>Proposition</u>: Under the hypotheses of the theorem we have $\operatorname{rank}_{\mathbb{Q}}(\operatorname{im} i^*)^{2k}=\frac{1}{2}\operatorname{rank}_{\mathbb{Q}}K^{2k}(Y;\mathbb{Q})$, where $i^*:K^{2k}(X;\mathbb{Q})\to K^{2k}(Y;\mathbb{Q})$ is induced from the inclusion $i:Y\to X$.

<u>Proof</u>: We have a diagram, commutative up to sign:

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & K^{2k}(X;\mathbb{Q}) & \xrightarrow{i^*} & K^{2k}(Y;\mathbb{Q}) & \xrightarrow{\delta} & K^{2k+1}(X,Y;\mathbb{Q}) & \longrightarrow\cdots \\
& \downarrow{\scriptstyle [X]\cap\cdot} & & \downarrow{\scriptstyle [Y]\cap\cdot} & & \downarrow{\scriptstyle [X]\cap\cdot} & \\
\cdots\to & K_{2k+1}(X,Y;\mathbb{Q}) & \xrightarrow{\partial} & K_{2k}(Y;\mathbb{Q}) & \xrightarrow{i_*} & K_{2k}(X;\mathbb{Q}) & \longrightarrow\cdots
\end{array}$$

In this diagram the rows are exact and the vertical maps are isomorphisms. Hence $(\operatorname{im} i^*)^{2k}=(\ker i_*)_{2k}$. It is easily shown that the Universal Coëfficient Formulae hold for K^* and K_*, and thus, since $\mathbb{Q}$ is a field, $K^{2k}(Y;\mathbb{Q})\cong\operatorname{Hom}(K_{2k}(Y;\mathbb{Q}),\mathbb{Q})$, $K^{2k}(X;\mathbb{Q})\cong\operatorname{Hom}(K_{2k}(X;\mathbb{Q}),\mathbb{Q})$, and $i^*=\operatorname{Hom}(i_*,1)$. Hence $\operatorname{rank}_{\mathbb{Q}}(\operatorname{im} i^*)^{2k}=\operatorname{rank}_{\mathbb{Q}}(\operatorname{im} i_*)_{2k}$, and $\operatorname{rank}_{\mathbb{Q}}(\operatorname{im} i_*)_{2k}+\operatorname{rank}_{\mathbb{Q}}(\ker i_*)_{2k}=\operatorname{rank}_{\mathbb{Q}}K_{2k}(Y;\mathbb{Q})=\operatorname{rank}_{\mathbb{Q}}K^{2k}(Y;\mathbb{Q})$.
Hence, $\operatorname{rank}_{\mathbb{Q}}(\operatorname{im} i^*)^{2k}=\frac{1}{2}\operatorname{rank}_{\mathbb{Q}}K^{2k}(Y;\mathbb{Q})$. QED

5.16 <u>Lemma</u>: With the hypotheses of 5.15, $(\operatorname{im} i^*)^{2k}\subset K^{2k}(Y;\mathbb{Q})$ annihilates itself under the pairing $(\cdot,\cdot)$.

<u>Proof</u>: $(i^*x,i^*y)=((i^*x)\cup(i^*y))[Y]=(i^*(x\cup y))[Y]=(x\cup y)(i_*[Y])=0$ since $i_*[Y]=i_*\partial[X]=0$ in $H_{4k}(X)$.

<u>Proof of Theorem 5.14</u>: By 5.15, $(\operatorname{im} i^*)^{2k}\subset K^{2k}(Y;\mathbb{Q})$ is a subspace of rank $=\frac{1}{2}\operatorname{rank} K^{2k}(Y;\mathbb{Q})$, and by 5.16 it annihilates itself under the

pairing. Hence by Proposition 5.2, $\mathrm{sgn}(\cdot,\cdot)=0$ on $K^{2k}(Y;Q)$, so that $I(f|Y)=0$. QED

The sum of Poincaré pairs is defined as follows:
If $(X_i, X_0 \cup Y_i)$ $i=1,2$ are Poincaré pairs of dimension m, such that $X_1 \cap X_2 = X_0$, $Y_i \cap X_0 = Y_0$, and (X_0, Y_0) is a Poincaré pair of dimension m-1, then it follows [Browder 1972, p.13] that $(X_1 \cup X_2, Y_1 \cup Y_2)$ is a Poincaré pair of dimension m, called the sum of $(X_i, X_0 \cup Y_i)$ along (X_0, Y_0).

If (X,Y) and (A,B) are the sums, respectively, of $(X_i, Y_i \cup X_0)$ and $(A_i, B_i \cup A_0)$, and $f:(X,Y) \to (A,B)$ with $f(X_i) \subset A_i$, then the following are equivalent:

(1) f has degree 1

(2) $f_0 = f|(X_0, Y_0)$ has degree 1

(3) $f_i = f|(X_i, Y_i \cup X_0)$ have degree 1

(all with appropriate orientations).

We say that f is the sum of f_1 and f_2.

5.17 Theorem: Suppose $f:(X,Y) \to (A,B)$, a degree 1 map, is the sum of two maps $f_i:(X_i, X_0 \cup Y_i) \to (A_i, A_0 \cup B_i)$, $i=1,2$, and suppose that the map on the intersection $f_0^*: H^*(A_0, B_0; Q) \to H^*(X_0, Y_0; Q)$ is an isomorphism. Then $I(f) = I(f_1) + I(f_2)$.

If (X,Y) is a Poincaré pair of dimension $m=4k$ we may consider the symmetric pairing $H^{2k}(X,Y;Q) \otimes H^{2k}(X,Y;Q) \to Q$ given by $(x,y)=(x \cup y)[X]$, and we define $I(X,Y)$ to be the signature of $(\cdot,\cdot)$ on $H^{2k}(X,Y;Q)$.

5.18 Theorem: $I(f) = I(X,Y) - I(A,B)$.

Thus we have the important theorem

5.19 Theorem: Let $f:(X,Y) \to (A,B)$ be a map of degree 1 between Poincaré pairs of dimension $m=4k$. Suppose $(f|Y)^*: H^*(B;Q) \to H^*(Y;Q)$ is an isomorphism and that f is cobordant rel Y to $f':(X',Y) \to (A,B)$ such that

$f'^*: H^*(A;Q) \longrightarrow H^*(X';Q)$ is an isomorphism. Then $I(f)=0$.

Proof: Let U be the cobordism rel Y between X and X', so that $\partial U = X \cup X'$, $X \cap X' = Y$, $(U,\partial U)$ is a Poincaré pair of dimension m+1, compatibly oriented, and F is the map $(U,Y) \longrightarrow (A,B)$ such that $F|X=f$, $F|X'=f'$. We may consider F as a map of degree 1 $G:(U, X\cup X') \longrightarrow (A\times I, A\times 0 \cup B\times I \cup A\times 1)$. By Theorem 5.14, $I(G|X\cup X')=0$, and by Theorem 5.17 $I(G|X\cup X')=I(f)-I(f')$. Now $I(f')=0$ since f'^* is an isomorphism, and hence $I(f)=0$. QED

Let (X,Y) be a Z_2-Poincaré pair of dimension m (i.e. (X,Y) satisfies Poincaré duality for homology with coefficients in Z_2). Define a linear map $\ell_i: H^{m-i}(X,Y;Z_2) \longrightarrow Z_2$ by $\ell_i(x)=(Sq^i x)[X]$, where Sq^i is the i^{th} Steenrod square (see [Steenrod 1962]) and $[X] \in H_m(X,Y;Z_2)$ is the orientation class. By Poincaré duality, $H^i(X;Z_2) \otimes H^{m-i}(X,Y;Z_2) \longrightarrow Z_2$ given by $(x,y)=(x\cup y)[X]$ is a nonsingular pairing, so that $H^i(X;Z_2)$ is isomorphic, using this pairing to $\mathrm{Hom}(H^{m-i}(X,Y;Z_2),Z_2)$, and hence $\ell_i(x)=(x,v_i)$ for a unique $v_i \in H^i(X;Z_2)$, for all $x \in H^{m-i}(X,Y;Z_2)$.

Define the Wu class of X to be $V=1+v+v+\ldots$, $v_i \in H^i(X;Z_2)$ as above.

5.20 Proposition: Let (X,Y) and (A,B) be Z_2-Poincaré pairs of dimension m, $f:(X,Y) \longrightarrow (A,B)$ a map of degree 1 (mod 2) (i.e. $f_*[X]=[A]$ for f_* defined on homology with Z_2 coëfficients). Then $v_i(X)=\bar{v}_i+f^*(v_i(A))$, where $\bar{v}_i \in K^i(X)$.

The proof consists of a calculation to show that $v_i(X)-f^*(v_i(A)) \in K^i(X)$

5.21 Proposition: With notation as in 5.20, suppose m=2q. Then the pairing $(\cdot,\cdot)$ on $K^q(X,Y;Z_2)$ is symplectic (i.e. $(x,x)=0$ for all x) if and only if $f^*v_q(A)=v_q(X)$.

Proof: $(x,x)=x^2[X]=(Sq^q x)[X]=(x\cup v_q(X))[X]=(x,v_q(X))$ for $x \in H^q(X,Y;Z_2)$, and since $K^q(X,Y;Z_2)$ and $(\mathrm{im}\, f^*)^q$ are orthogonal by Proposition 5.13, $(x,f^*v_i(A))=0$ for $x \in K^q(X,Y;Z_2)$. Hence for $x \in K^q(X,Y;Z_2)$, $(x,x)=(x,\bar{v}_q)$

by Proposition 5.20. Then $(x,x)=0$ if and only if $\bar{v}_q=v_q(X)-f^*v_q(A)=0$.

5.22 Corollary: Let (X,Y) and (A,B) be oriented Poincaré duality pairs of dimension $m=4\ell$, and let $f:(X,Y)\to(A,B)$ be of degree 1. If $f^*v_{2\ell}(A)$ $=v_{2\ell}(X)$, then the pairing $(x,y)=(x\cup y)[X]$ (for $x,y\in K^*(X,Y)/\text{torsion}$) is even (i.e. $2|(x,x)$ for all x).

This follows from the fact that (x,x) reduced mod 2 is zero by 5.21 and thus (x,x) must be even.

5.23 Corollary: Let (X,Y) and (A,B) be oriented Poincaré pairs of dimension $m=4\ell$, $f:(X,Y)\to(A,B)$ of degree 1 such that $(f|Y)_*:H_*(Y)\to H_*(B)$ is an isomorphism. If $f^*(v_{2\ell}(A))=v_{2\ell}(X)$, then $I(f)$ is divisible by 8.

This follows directly from 5.22 and Proposition 5.4.

Let us now investigate the Wu class, with the aim of showing that it is preserved by normal maps.

Let (X,Y) be a pair of spaces, and ξ^k a fibre bundle over X with fibre F such that $H_*(F;Z_2)=H_*(S^{k-1};Z_2)$. Then we may define the Thom space $T(\xi)=X\cup cE(\xi)$ using the projection of ξ as attaching map. There is a Thom class $U\in H^k(T(\xi);Z_2)$ such that

$$\cdot\cup U:H^q(X;Z_2)\to H^{q+k}(T(\xi);Z_2)$$

$$\cdot\cup U:H^q(X,Y;Z_2)\to H^{q+k}(T(\xi),T(\xi|Y);Z_2)$$

$$\cdot\cap U:H_s(T(\xi),T(\xi|Y);Z_2)\to H_{s-k}(X,Y;Z_2)$$

$$\cdot\cap U:H_s(T(\xi);Z_2)\to H_{s-k}(X;Z_2)$$

are isomorphisms. Let $h:\pi_r(A,B)\to H_r(A,B;Z_2)$ be the Hurewicz homomorphism mod 2. We have the following important theorem of Spivak:

5.24 Theorem: Let (X,Y) be an n-dimensional Poincaré pair, with X simply-connected and Y a finite complex up to homotopy type. Then there is a spherical fibre space ξ with X as base space, its fibre a homotopy $(k-1)$-sphere, and an element $\alpha\in\pi_{n+k}(T(\xi),T(\xi|Y))$ such that $h(\alpha)\cap U=[X]$.

The fibre bundle ξ is called the Spivak normal fibre space of X, and can also be defined for homology with coëfficients.

5.25 <u>Proposition</u>: Let (X,Y) be a Z_2-Poincaré pair of dimension m, ξ^k a Z_2 Spivak normal fibre space over X (i.e. the fibre of ξ is a Z_2 homology $(k-1)$-sphere), $\alpha\in\pi_{m+k}(T(\xi),T(\xi|Y))$ such that $h(\alpha)\cap U=[X]$ in $H_m(X,Y;Z_2)$. Then $V(X)\cup U=Sq^{-1}(U)$.

We recall the fact that the Thom class $U\in H^k(T(\xi);Z_2)$ is characterised by the fact that $j^*(U)$ generates $H^k(\Sigma F;Z_2)=Z_2$, where $j:\Sigma F\to T(\xi)$ is the inclusion of the Thom complex over a point into the whole Thom complex.

5.26 <u>Proposition</u>: Let $b:\xi\to\xi'$ be a map of fibre spaces covering $f:X\to X'$, where ξ and ξ' have fibre F, $H_*(F;Z_2)=H_*(S^{k-1};Z_2)$. Then b induces a map of Thom complexes $T(b):T(\xi)\to T(\xi')$, and $T(b)^*U'=U$, where U and U' are the Thom classes of ξ and ξ'.

<u>Proof</u>: Let E,E' be the total spaces of ξ,ξ' resp., so that the following diagram commutes:

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{\pi} & X \\ {\scriptstyle 1}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ F & \longrightarrow & E' & \xrightarrow{\pi'} & X' \end{array}$$

Hence, f,b induce $T(b):X\cup_\pi cE\to X'\cup_{\pi'}cE'$, and the diagram

$$\begin{array}{ccc} \Sigma F & \xrightarrow{j} & T(\xi) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle T(b)} \\ \Sigma F & \xrightarrow{j'} & T(\xi') \end{array}$$

commutes. Hence $j^*T(b)^*U'=j'^*U'$, so that $j^*T(b)^*U'$ generates $H^k(\Sigma F;Z_2)$, and thus $T(b)^*U'=U$. QED

5.27 <u>Corollary</u>: Let (X,Y) and (A,B) be Z_2 Poincaré pairs of dimension m, ξ' a fibre space over A with fibre F a $(k-1)$-dimensional Z_2 homology sphere. Let $f:(X,Y)\to(A,B)$ be of degree 1 in Z_2 homology, and let $\xi=f^*(\xi')$. Suppose there is an element $\alpha\in\pi_{m+k}(T(\xi),T(\xi|Y))$ such that $h(\alpha)\cap U=[X]$. Then $f^*(V(A))=V(X)$, in particular $f^*v_q(A)=v_q(X)$ for all q.

<u>Proof</u>: By 5.26, if $b:\xi\to\xi'$ is the natural map, $T(b)^*U'=U$. Setting $V(X)=V$,

$V(A)=V'$, we have, using 5.25, $T(b)^*(V'\cup U')=f^*V'\cup T(b)^*U'=f^*(V')\cup U$ $=T(b)^*(Sq^{-1}U')=Sq^{-1}T(b)^*U'=Sq^{-1}U=V\cup U$. Hence $f^*V'=V$.

5.28 Theorem: Let (X,Y) and (A,B) be oriented Poincaré pairs of dimension $m=4\ell$, $f:(X,Y)\to(A,B)$ of degree 1 such that $(f|Y)_*$ is an isomorphism, and ξ' a fibre space over A with fibre F a Z_2 homology $(k-1)$-sphere. Set $\xi=f^*\xi'$ and suppose there is $\alpha\in\pi_{m+k}(T(\xi),T(\xi|Y))$ such that $h(\alpha)\cap U$ equals the orientation class of (X,Y) reduced mod 2. Then $I(f)$ is divisible by 8.

Proof: By 5.27, $f^*v_{2\ell}(A)=v_{2\ell}(X)$, so by 5.23 $I(f)$ is divisible by 8.

Let (f,b) be a normal map, $f:(M,\partial M)\to(A,B)$ of degree 1, M a smooth oriented m-manifold with boundary, (A,B) an oriented Poincaré pair of dimension m, $m=4\ell$, and $b:\nu\to\eta$ a linear bundle map covering f, ν the normal bundle of $(M,\partial M)\subset(D^{m+k},S^{m+k-1})$, η a k-plane bundle over A.

5.29 Corollary: If (f,b) is a normal map with $(f|\partial M)_*$ an isomorphism, then $I(f)$ is divisible by 8.

Proof: The pair (f,b) satisfies the conditions of 5.28, where $\xi'=\eta$ is a linear bundle over (A,B).

Thus, we may make the following definition:

Let (f,b) be a normal map $f:(M,\partial M)\to(A,B)$, etc. with $(f|\partial M)_*$ an isomorphism, $m=4\ell$ the dimension of M. Define $\sigma(f,b)=\frac{1}{8}I(f)$. Then the Invariant Theorem for $m=4\ell$ follows from Theorem 5.19.

Let (X,Y) and (A,B) be oriented Poincaré pairs of dimension $m=2q$, and let $f:(X,Y)\to(A,B)$ be a map of degree 1. Let ξ be the Spivak normal fibre space of (X,Y), and η that of (A,B), and let $\alpha\in\pi_{m+k}(T(\xi),T(\xi|Y))$, $\beta\in\pi_{m+k}(T(\eta),T(\eta|B))$ be the elements defined such that $h(\alpha)\cap U_\xi=[X]$, $h(\beta)\cap U_\eta=[A]$, where U_ξ,U_η are the Thom classes of ξ,η and h is the Hurewicz homomorphism. Let $b:\xi\to\eta$ be a map of fibre spaces covering f. We shall call the pair (f,b) a normal map of Poincaré pairs. Note that this definition is analogous to that of a normal map given above. We also define normal cobordism and normal cobordism rel B of Poincaré pairs by the same analogy.

Browder [1972, III.4] defines, using Spanier and Whitehead's S-theory, a quadratic form $\psi: K^q(X,Y;Z_2) \to Z_2$ with associated bilinear form $(\cdot,\cdot)$, where $(x,y)=(x\cup y)[X]$ for x,y $K^q(X,Y;Z_2)$. If $(f|Y)^*: H^*(B;Z_2) \to H^*(Y;Z_2)$ is an isomorphism, it follows from Proposition 5.13 that $(\cdot,\cdot)$ is nonsingular on $K^q(X,Y;Z_2)$ $(\cong K^q(X;Z_2))$. Then the Arf invariant of ψ is defined.

Let (f,b) be a normal map of Poincaré complexes, $f:(X,Y) \to (A,B)$, and suppose that $(f|Y)^*: H^*(B;Z_2) \to H^*(Y;Z_2)$ is an isomorphism. Then define the Kervaire invariant $c(f,b)=\mathrm{Arf}(\psi)$.

Now we will develop some properties of the Kervaire invariant.

Let (f,b) be a normal map, $f:(X,Y) \to (A,B)$, etc. and suppose in addition that Y and B are sums of Poincaré pairs along the boundaries, and that f sends summands into summands. In particular, suppose that $Y=Y_1 \cup Y_2$, $Y_0=Y_1 \cap Y_2$, $B=B_1 \cup B_2$, $B_0=B_1 \cap B_2$, $f(Y_i) \subseteq B_i$, and that (B_i,B_0) and (Y_i,Y_0) are Poincaré pairs compatibly oriented with (X,Y) and (A,B). If ξ,η are the Spivak normal fibre spaces of (X,Y) and (A,B), then $\xi|Y_i$, $\eta|B_i$ are the corresponding Spivak normal fibre spaces, so that if $f_i=f|Y_i$, $b_i=b|(\xi|Y_i)$, then (f_i,b_i) are all normal maps, $i=0,1,2$.

Note that if $f_2^*: H^*(B_2;Z_2) \to H^*(Y_2;Z_2)$ is an isomorphism then it follows that $f_0^*: H^*(B_0;Z_2) \to H^*(Y_0;Z_2)$ is also an isomorphism.

5.30 <u>Theorem:</u> Let (f,b) be a normal map as above, so that $f|Y$ is the sum of f_1 and f_2 on Y_1 and Y_2, etc. Suppose $f_2^*: H^*(B_2;Z_2) \to H^*(Y_2;Z_2)$ is an isomorphism. Then $c(f_1,b_1)=0$.

This theorem has the following corollaries:

5.31 Corollary: If (f,b) is a normal map and is normally cobordant rel Y to (f',b'), $f'^{*}:H^{*}(A,B;Z_2)\to H^{*}(X',Y;Z_2)$ an isomorphism, then $c(f,b)=0$.

5.32 Corollary: If (f,b) is a normal map, $f:(X,Y)\to(A,B)$, then $c(f|Y,b|(\xi|Y))=0$.

The first corollary is derived from the theorem by using the normal cobordism as a normal map, the second by taking $Y_2=\phi$.

The proof of Theorem 5.30 relies on the definition of ψ, and is given in [Browder 1972, III.4].

Let (f,b), $f:(X,Y)\to(A,B)$ be a normal map of Poincaré pairs, and suppose (X,Y) and (A,B) are sums of Poincaré pairs, i.e. $X=X_1\cup X_2$, $A=A_1\cup A_2$, $X_0=X_1\cap X_2$, $A_0=A_1\cap A_2$, $Y_i=X_i\cap Y$, $B_i=A_i\cap B$, $f(X_i)\subseteq A_i$, and $(X_i,X_0\cup Y_i)$, $(A_i,A_0\cup B_i)$ are Poincaré pairs oriented compatibly with (X,Y) and (A,B).

Set $f_i=f|X_i:(X_i,X_0\cup Y_i)\to(A_i,A_0\cup B_i)$, $f_0=f|X_0:(X_0,Y_0)\to(A_0,B_0)$, and b_i the appropriate restriction of b.

Now suppose that $(f|Y)^{*}:H^{*}(B;Z_2)\to H^{*}(Y;Z_2)$ and $f_0^{*}:H^{*}(A_0;Z_2)\to H^{*}(X_0;Z_2)$ are isomorphisms. It follows easily from arguments with the Mayer-Vietoris sequence that $(f_i|X_0\cup Y_i)^{*}$ are isomorphisms, so $c(f,b)$, $c(f_1,b_1)$, and $c(f_2,b_2)$ are all defined.

5.33 Theorem: $c(f,b)=c(f_1,b_1)+c(f_2,b_2)$.

Proof: We shall present a partial proof here; the balance is to be found in [Browder 1972].

Let ψ,ψ_1 and ψ_2 be the quadratic forms defined on $K^q(X,Y)$, $K^q(X_1,X_0\cup Y_1)$ and $K^q(X_2,X_0\cup Y_2)$ respectively. An argument with the Mayer-Vietoris sequence (which is really the exact sequence of the triple of pairs $(X_0,Y_0)\subset(X,Y)\subset(X,Y\cup X_0)$, where the last pair is replaced by the excisive pair $(X_1,X_0\cup Y_1)\cup(X_2,X_0\cup Y_2)$) gives an isomorphism

$\rho_1 \oplus \rho_2 : K^q(X_1, X_0 \cup Y_1) \oplus K^q(X_2, X_0 \cup Y_2) \to K^q(X,Y)$,

where ρ_1 is defined by the diagram

$$\begin{tikzcd} K^q(X_1, X_0 \cup Y_1) \arrow[dr, "\rho_1"'] & K^q(X, X_2 \cup Y) \arrow[l, "\cong"'] \arrow[d] \\ & K^q(X,Y) \end{tikzcd}$$

where the isomorphism comes from an excision, and the vertical arrow is induced by inclusion (similarly for ρ_2).

It remains to show $\psi(\rho_i x)=\psi_i(x)$, $x \in K^q(X_i, X_0 \cup Y_i)$. Then ψ is isomorphic to the direct sum $\psi_1 \oplus \psi_2$, so that $\mathrm{Arf}(\psi)=\mathrm{Arf}(\psi_1)+\mathrm{Arf}(\psi_2)$.

The remainder of the proof is given on pp. 72-73 of Browder.

Now suppose (A,B) is a Poincaré complex of dimension m, and ξ is a linear bundle over A, $f:(M,\partial M) \to (A,B)$ is of degree 1,and $b:\nu \to \xi$ is a linear bundle map covering f, ν is the normal bundle of $(M,\partial M)$ in (D^{m+k}, S^{m+k-1}); i.e. (f,b) is a normal map in the original sense. Then by Theorem I.4.19 of Browder, there is a fibre homotopy equivalence (unique up to homotopy) $b':\xi \to \eta$ such that $T(b')_*(T(b)_*(\alpha))=\beta$, where $\alpha \in \pi_{m+k}(T(\nu), T(\nu|\partial M))$ and $\beta \in \pi_{m+k}(T(\eta), T(\eta|B))$ are the elements such that $h(\alpha) \cap U_\nu = [M]$ and $h(\beta) \cap U_\eta = [A]$. Then $b'b:\nu \to \eta$, and $(f,b'b)$ is a normal map of Poincaré pairs, and we define $\sigma(f,b)=c(f,b'b) \in Z_2$ if $m=4k+2$ and $(f|\partial M)^*$ on Z_2 cohomology is an isomorphism.

5.34 Proposition: The value of (f,b) is independent of the choice of $\beta \in \pi_{m+k}(T(\eta), T(\eta|B))$, and thus depends only on the normal map (f,b).

The proof of 5.34 is provided in [Browder 1972].

With this definition of $\sigma(f,b)$ for $m \equiv 2 \pmod 4$, we see that Corollary 5.31 provides the proof of the Invariant Theorem for $m \equiv 2 \pmod 4$ and thus completes the proof of that theorem.

We have also proved the following two properties of the invariant σ:

5.35 <u>Proposition</u>: (Addition Property) Suppose (f,b) is a normal map which is the sum of two normal maps (f_1,b_1) and (f_2,b_2), and such that $f|\partial M$, $f|\partial M_i$ $i=1,2$, and $f|M_0$ induce isomorphisms in homology. Then $\sigma(f,b)=\sigma(f_1,b_1)+\sigma(f_2,b_2)$.

This property is proved for $m=4\ell$ by Theorem 5.17, and for $m=4\ell+2$ by Theorem 5.33. It is vacuously true for $m=2q+1$.

5.36 <u>Proposition</u>: (Cobordism Property) Let (f,b) be a normal map, $f:(M,\partial M)\to(X,Y)$, $b:\nu\longrightarrow\xi$, and set $f'=f|\partial M:\partial M\longrightarrow Y$, $b'=b|(\nu|\partial M):\nu|\partial M\longrightarrow\xi|Y$. If $m=2k+1$ then $(f',b')=0$.

This property follows from Theorem 5.14 for the case $m=4\ell+1$ and from Corollary 5.32 for the case $m=4\ell+3$.

Let us call the quantity $I(X,Y)$ defined above the index of X. Then by the Hirzebruch Index Theorem [Hirzebruch 1966], we have Index $M=L_k(p_1(\xi^{-1}),\ldots,p_k(\xi^{-1}))[X]$, and Theorem 5.18 gives us directly the following

5.37 <u>Proposition</u>: (Index Property) If $Y=\phi$, $m=4k$, (f,b) a normal map, then $8\sigma(f,b)=$index $M-$index X, and index X equals the signature of the quadratic form on $H^{2k}(X;Q)$ given by $\langle x\cup x,[X]\rangle$, where $[X]$ is the orientation class in $H_{4k}(X;Q)$.

Finally we state without proof the

5.38 <u>Proposition</u>: (Product Formulae) Let (f_1,b_1), (f_2,b_2) be normal maps $f_i:(M_i,\partial M_i)\longrightarrow(X_i,\partial X_i)$. Suppose $\sigma(f_1\times f_2,b_1\times b_2)$, $\sigma(f_1,b_1)=\sigma_1$, and $\sigma(f_2,b_2)=\sigma_2$ are all defined (i.e. $f_1\times f_2|\partial(M_1\times M_2)$, $f_i|\partial M_i$ are all homology isomorphisms with appropriate coëfficients).

Then (1) $\sigma(f_1\times f_2,b_1\times b_2)=I(X_1)\sigma_2+I(X_2)\sigma_1+8\sigma_1\sigma_2$ when $M_1\times M_2$ is of dimension $4k$, where $I(X_i)$ is the index of X_i,

(2) $\sigma(f_1\times f_2,b_1\times b_2)=\chi(X_1)\sigma_2+\chi(X_2)\sigma_1$ when $M_1\times M_2$ is of dimension $4k+2$

where $\chi(X_i)$ is the Euler characteristic of X_i.

Note that $I(X)=0$ by definition if $\dim X \not\equiv 0 \pmod 4$.

§6. Surgery below the Middle Dimension.

We will now describe the technique of surgery, the use of which will enable us to solve the surgery problem.

Suppose that $\phi: S^p \times D^{q+1} \longrightarrow M^m$, $p+q+1=m$, is a differentiable embedding, into the interior of M if $\partial M \neq \phi$. Let $M_0 = M \setminus \operatorname{int}(\operatorname{im} \phi)$. Then $\partial M_0 = \partial M \cup \phi(S^p \times S^q)$. Define $M' = M_0 \cup_\phi D^{p+1} \times S^q$, with $\phi(x,y)$ identified to $(x,y) \in S^p \times S^q = \partial(D^{p+1} \times S^q)$. Then M' is a manifold, $\partial M' = \partial M$, and M' is said to be the result of surgery using ϕ, on M. It is sometimes denoted by $\chi(M,\phi)$ (e.g. by Milnor). We may define a cobordism W_ϕ^{m+1} between M and M' as follows:

$W_\phi = M \times [0,1] \cup (D^{p+1} \times D^{q+1})$ such that $(x,y) \in S^p \times D^{q+1} \subset \partial(D^{p+1} \times D^{q+1})$ is identified with $(\phi(x,y),1) \in M \times I$. Clearly $\partial W_\phi = M \cup (\partial M \times I) \cup M'$, and W_ϕ is called the trace of the surgery. As we have defined it, W_ϕ is not a smooth manifold with boundary. However, it has a canonical smooth structure (i.e. it is PL-homeomorphic to a smooth manifold) which is described in [Milnor 1961]. (Milnor calls W_ϕ $\omega(M,\phi)$.)

If W^{m+1} is a manifold with $\partial W = M \cup (\partial M \times I) \cup M'$ and W' has $\partial W' = M' \cup (\partial M' \times I) \cup M''$, then we may define the sum of the two cobordisms by taking $\bar{W} = W \cup W'$ and identifying $M' \subset \partial W$ with $M' \subset \partial W'$. Then it is clear that $\partial \bar{W} = M \cup (\partial M \times I) \cup M''$.

6.1 Theorem: Let W be a cobordism with $\partial W = M \cup (\partial M \times I) \cup M'$. Then there is a sequence of surgeries based on embeddings ϕ_i, $i=1,\ldots,k$, each surgery being on the manifold which results from the previous surgery, and such that W is the sum of $W_{\phi_1}, \ldots, W_{\phi_k}$.

The proof is an immediate consequence of the Morse Lemma, and a lucid proof may be found in [Milnor 1961].

6.2 <u>Proposition</u>: If M' is the result of surgery on M based on an embedding $\phi:S^p\times D^{q+1}\to M$, then M is the result of surgery on M' based on an embedding $\psi:S^q\times D^{p+1}\to M'$ such that the traces of the two surgeries are the same.

6.3 <u>Proposition</u>: Let $\phi:S^p\times D^{q+1}\to M^m$ be a smooth embedding in the interior of M, $p+q+1=m$, and let W_ϕ be the trace of the surgery based on ϕ. Then W_ϕ has $M\cup_{\bar{\phi}}D^{p+1}$ as a deformation retract, where $\bar{\phi}=\phi|S^p\times 0$.

<u>Proof</u>: $W_\phi\cong(M\times I)\cup_\phi(D^{p+1}\times D^{q+1})$, image $\phi\subset M\times I$, so we may deform $M\times I$ to $M\times 1$ leaving $M\times 1\cup_\phi(D^{p+1}\times D^{q+1})$ fixed. Then $D^{p+1}\times D^{q+1}$ may be deformed onto $(D^{p+1}\times 0)\cup(S^p\times D^{q+1})$, leaving this latter subspace fixed. This then yields the deformation retraction of W_ϕ to $M\cup_{\bar{\phi}}D^{p+1}$.

6.4 <u>Proposition</u>: (a) Let $f:(M,\partial M)\to(A,B)$ be a map, M an oriented smooth m-manifold, (A,B) a pair of spaces, and let $\phi:S^p\times D^{q+1}\to\text{int } M$ be a smooth embedding, $p+q+1=m$. Then f extends to $F:(W_\phi,\partial M\times I)\to(A\times I,B\times I)$ to get a cobordism of f if and only if $f\circ\bar{\phi}$ is homotopic to the constant map $S^p\to A$.

(b) Suppose in addition that η^k is a linear k-plane bundle over A, $b:\nu^k\to\eta^k$ is a linear bundle map covering f, ν the normal bundle of $(M,\partial M)\subset(D^{m+k},S^{m+k-1})$, $k>>m$. Then b extends to $\bar{b}:\omega\to\eta$ covering F, where ω is the normal bundle of $W_\phi\subset D^{m+k}\times I$, if and only if $b|(\nu|\phi(S^p))$ extends to $\omega|D^{p+1}\times 0$, covering $F|D^{p+1}\times 0$.

<u>Proof</u>: Since $M\cup_{\bar{\phi}}D^{p+1}$ is a deformation retract of W_ϕ, it follows that f extends to W_ϕ if and only if f extends to $M\cup_{\bar{\phi}}D^{p+1}$. But the latter is true if and only if $f\circ\bar{\phi}$ is null-homotopic, which proves (a).

For (b), it follows from the bundle covering homotopy property, and the fact that $M\cup_{\bar{\phi}}D^{p+1}$ is a deformation retract of W_ϕ, that b extends to ω if and only if b extends to $\omega|D^{p+1}\times 0$. QED

If (f,b) is a normal map, $\phi:S^p\times D^{q+1}\longrightarrow \text{int } M^m$, $p+q+1=m$, $f:(M,\partial M)\longrightarrow(A,B)$, and if the trace of ϕ can be made a normal cobordism by extending f and b over W_ϕ, we will say that the surgery based on ϕ is a normal surgery on (f,b).

From Theorem 6.1, it follows easily that any normal cobordism rel B is the composite of normal surgeries.

Let $\phi:S^p\times D^{q+1}\longrightarrow \text{int } M^m$ be an embedding, with $p+q+1=m$. W_ϕ is the trace, and M' the result of the corresponding surgery. We will investigate the effect of surgery on the homotopy of M; in particular, we will examine the relation between the homotopy groups of M and M', below the 'middle dimension'.

6.5 <u>Theorem</u>: If $p<\frac{m-1}{2}$ then $\pi_i(M')\cong\pi_i(M)$ for $i<p$, and

$$\pi_p(M')\cong\pi_p(M)/\{\bar{\phi}_{\#}\pi_p(S^p)\},$$

where {G} denotes the $Z[\pi_1(M)]$ submodule of $\pi_p(M)$ generated by G.

<u>Proof</u>: By 6.3, W_ϕ is of the same homotopy type as $M\cup_{\bar{\phi}}D^{p+1}$. Hence $\pi_i(W_\phi)\cong\pi_i(M)$ for $i<p$, and $\pi_p(W_\phi)\cong\pi_p(M)/\{\bar{\phi}_{\#}\pi_p(S^p)\}$. By 6.2 and 6.3, we have also that $W_\phi\cong W_\psi\simeq M'\cup_{\bar{\psi}}D^{q+1}$, where $\psi:S^q\times D^{p+1}\longrightarrow M'$ gives the surgery which reverses the effect of surgery base on ϕ. Hence $\pi_i(W_\phi)\cong\pi_i(M')$ for $i<q$, $\pi_q(W_\phi)\cong\pi_q(M')/\{\bar{\psi}_{\#}\pi_q(S^q)\}$. Since $p<\frac{m-1}{2}$, $q>p$, so $\pi_i(M')\cong\pi_i(W_\phi)$ for $i\leq p$ and the result follows. QED

Let (f,b) be such that $f:(M,\partial M)\longrightarrow(A,B)$, $b:\nu^k\longrightarrow\eta^k$, $k>>m$, η a linear bundle over A, ν the normal bundle of $(M,\partial M)\subset(D^{m+k},S^{m+k-1})$, and let $\bar{\phi}:S^p\longrightarrow\text{int } M$ be a smooth embedding. Suppose that f extends to $\bar{F}:\bar{M}\longrightarrow A$, where $\bar{M}=M\cup_{\bar{\phi}}D^{p+1}$. We consider the problem of 'thickening $\bar{M}$ to a normal cobordism', i.e. of extending $\bar{\phi}$ to a smooth embedding $\phi:S^p\times D^{q+1}\longrightarrow\text{int } M^m$, $p+q+1=m$ such that $\bar{\phi}=\phi|S^p\times 0$, and so that $F:(W_\phi,\partial M\times I)\longrightarrow(A\times I,B\times I)$ can be covered by a bundle map $\bar{b}:\omega\longrightarrow\eta$ extending b, where ω is the normal bundle of W_ϕ in $D^{m+k}\times I$, and F is the extension of $\bar{F}$, unique up to homotopy. (When this is possible, normal surgery based on $\iota\phi$ will kill the class of $\bar{\phi}$ in $\pi_p(M)$.) Let $V_{k,q+1}$ be the space of orthonormal k-frames in R^{k+q+1}.

6.6 Theorem: There is an obstruction $O\in\pi_p(V_{k,q+1})$ such that $O=0$ if and only if $\bar{\phi}$ extends to ϕ such that $F:W_{\bar{\phi}}\longrightarrow A$ can be covered by $\bar{b}:\omega\longrightarrow\eta$ extending b as above.

Proof: Since k is very large, we may extend the embedding $M\subset D^{m+k}$ to $M\cup_{\bar{\phi}}D^{p+1}\subset D^{m+k}\times I$, with D^{p+1} smoothly embedded and meeting $D^{m+k}\times 0$ perpendicularly. The normal bundle γ of $D^{p+1}\subset D^{m+k}\times I$ is trivial. $\bar{F}$ defines a homotopy of $f\circ\bar{\phi}$ to a point, which is covered by a bundle homotopy b on $\nu|\bar{\phi}(S^p)$, ending with a map of $\nu|\bar{\phi}(S^p)$ into a single fibre of η, i.e. a trivialisation of $\nu|\bar{\phi}(S^p)$, which is well-defined up to homotopy. This trivialisation of $\nu|\bar{\phi}(S^p)$, which is a subbundle of $\gamma|\bar{\phi}(S^p)$, which is also trivial, therefore defines a map α of S^p into the k-frames of R^{q+k+1}, $\alpha:S^p\longrightarrow V_{k,q+1}$, which gives an element $\alpha\in\pi_p(V_{k,q+1})$. Now if $\bar{\phi}$ extends to ϕ and b extends to $\bar{b}$ as above, then the normal bundle ω of W_ϕ restricted to D^{p+1}, $\omega|D^{p+1}$ is a subbundle of γ extending $\nu|\bar{\phi}(S^p)$, and $\bar{b}$ defines an extension of α to $\alpha':D^{p+1}\longrightarrow V_{k,q+1}$. Hence $O=0$ in $\pi_p(V_{k,q+1})$.

Conversely, if $O=0$, then α extends to $\alpha':D^{p+1}\longrightarrow V_{k,q+1}$, and α' defines a trivial subbundle ω' of dimension k in γ, extending $\nu|\bar{\phi}(S^p)$. The subbundle ω'' orthogonal to ω' in γ is trivial (being a bundle over D^{p+1}) and the total space of ω'' is $D^{p+1}\times R^{q+1}\subset D^{p+1}\times R^{q+k+1}$, the total space of γ (all up to homeomorphism). Since $\omega''|\bar{\phi}(S^p)$

equals the normal bundle of $\bar{\phi}(S^p)$ in M, this embedding defines $\phi:S^p\times{}^{q+1}\longrightarrow M$, and α' defines the extension of b to $\bar{b}:\omega\longrightarrow\eta$, where $\omega|D^{p+1}=\omega'$ by construction. QED

We shall now study $V_{k,q+1}$ in order to analyse the obstruction $\mathcal{O}$. ($\mathcal{O}$ will often be referred to as 'the obstruction to thickening $(\bar{M},\bar{F})$ to a normal cobordism'.)

Recall that the group SO(k+q+1) acts transitively on the set of orthonormal k-frames in R^{k+q+1} and SO(q+1) is the subgroup leaving a given frame fixed. Hence $V_{k,q+1}=SO(k+q+1)/SO(q+1)$, and $V_{k,q+1}$ is topologised to make this a homeomorphism. Further, we recall that $SO(n)\xrightarrow{i}SO(n+1)\xrightarrow{p}S^n$ is a fibre bundle map, where p is the map which evaluates an orthogonal transformation on the unit vector $v_0=(1,0,\ldots,0)\in S^n\subset R^{n+1}$, i.e. $p(T)=T(v_0)$. (For this material, reference may be made to [Husemoller 1966].)

6.7 Lemma: $i_*:\pi_i(SO(n))\longrightarrow\pi_i(SO(n+1))$ is an isomorphism for $i<n-1$, and a surjection for $i\leq n-1$.

Proof: $\pi_i(S^n)=0$ for $i<n$, so the result follows from the exact homotopy sequence of the fibration $SO(n+1)\xrightarrow{p}S^n$:

$$\ldots\longrightarrow\pi_{i+1}(S^n)\xrightarrow{\partial}\pi_i(SO(n))\xrightarrow{i\#}\pi_i(SO(n+1))\xrightarrow{p\#}\pi_i(S^n)\longrightarrow\ldots \quad \text{QED}$$

6.8 Lemma: The map $p:SO(n+1)\longrightarrow S^n$ is the projection of the principal SO(n) bundle associated with the oriented tangent bundle of S^n.

Proof: Let $f=(f_1,\ldots,f_n)$ be a tangent frame to S^n at $v_0=(1,0,\ldots,0)$. Define a map $e:SO(n+1)\longrightarrow F$, the bundle of frames of S^n, by $e(T)$ is the frame $(T(f_1),\ldots,T(f_n))$ at $T(v_0)\in S^n$. Then e is surjective, and injective. Hence e is a homeomorphism, and the lemma follows.

6.9 Lemma: The composite $\pi_n(S^n)\xrightarrow{\partial}\pi_{n-1}(SO(n))\xrightarrow{p\#}\pi_{n-1}(S^{n-1})$ is the boundary in the exact sequence of the tangent S^{n-1} bundle to S^n, and is 0 if n is odd, and multiplication by 2 if n is even.

Proof: The tangent S^{n-1} bundle is obtained from the bundle of frames by taking the quotient by SO(n-1) SO(n), the structure group of the bundle. Hence we have the commutative diagram:

$$\begin{array}{ccc}
SO(n) & \xrightarrow{p} & SO(n)/SO(n-1)=S^{n-1} \\
{\scriptstyle i}\downarrow & & \downarrow \\
SO(n+1) & \longrightarrow & SO(n+1)/SO(n-1) \\
\downarrow & & \downarrow{\scriptstyle \bar{p}} \\
S^n & \xrightarrow{1} & S^n
\end{array}$$

It follows that in the exact sequence for the right hand bundle, $\bar{\partial}=p_{\#}\partial:\pi_i(S^n)\longrightarrow\pi_{i-1}(S^{n-1})$. Now by the Euler-Poincaré Theorem the tangent sphere bundle has a cross-section (there is a nonsingular tangent vector field) if and only if the Euler characteristic $\chi(M)$ is zero. More precisely, the only obstruction to a cross-section to the tangent sphere bundle of a manifold M^m is $\chi(M)g$, where $g\in H^m(M;Z)$ is the class dual to the orientation class of M. Now if $M=S^n$, the obstruction to a cross-section can also be identified with the characteristic map (see [Steenrod 1951, 23.4]) $\bar{\partial}:\pi_n(S^n)\to\pi_{n-1}(S^{n-1})$. Hence $\bar{\partial}=0$ if n is odd, multiplication by 2 if n is even.

6.10 Theorem: $p_{\#}:\pi_n(SO(n+1))\longrightarrow\pi_n(S^n)$ is surjective if and only if n=1,3, or 7.

Proof: If $p_{\#}$ is surjective, then there is a map $\alpha:S^n\longrightarrow SO(n+1)$ such that $p\circ\alpha\simeq 1$, and hence the principal bundle of τ_{S^n} has a section and is therefore trivial, i.e. S^n is parallelisable. But it is known that S^n is parallelisable if and only if n=1,3 or 7.

6.11 Corollary: ker $i_{\#}:\pi_{n-1}(SO(n))\longrightarrow\pi_{n-1}(SO(n+1))$ is Z if n is even, Z_2 if n is odd and $n\neq 1,3,7$, and 0 if n=1,3,7.

Proof: $\ker i_{\#}=\partial\pi_n(S^n)\cong\pi_n(S^n)/p_{\#}\pi_n(SO(n+1))$. If n is odd, by 6.9 $p_{\#}\pi_n(SO(n+1))\supseteq 2\pi_n(S^n)$, and by 6.10 the inclusion is strict, if $n\neq 1,3,7$, hence $\pi_n(S^n)/p_{\#}\pi_n(SO(n+1))\cong Z_2$ if n is odd, $n\neq 1,3,7$. If $n=1,3$, or 7, $p_{\#}$ is surjective, so $\ker i_{\#}=0$.

If n is even, by 6.9 $p_{\#}\circ\partial$ is a monomorphism, so $\partial:\pi_n(S^n)\rightarrow\pi_{n-1}(SO(n))$ is a monomorphism, so $\ker i_{\#}\cong Z$. QED

6.12 Theorem: $\pi_i(V_{k,m})=0$ for $i<m$, $\pi_m(V_{k,m})=Z_2$ if m is odd, Z if m is even, $k\geq 2$. Further $j_{\#}:\pi_i(V_{k,m})\rightarrow\pi_i(V_{k+1,m})$ is an isomorphism for $i\leq m$, $k\geq 2$, and $j_{\#}:\pi_m(V_{1,m})=\pi_m(S^m)\rightarrow\pi_m(V_{k,m})$ is surjective, and an isomorphism if m is even, where j is inclusion.

Proof: First, take $k=2$, so that $V_{2,m}=SO(m+2)/SO(m)$ and we have a natural fibration over $S^{m+1}=SO(m+2)/SO(m+1)$ with fibre $S^m=SO(m+1)/SO(m)$. Also we have a commutative diagram of fibre bundles:

$$\begin{tikzcd} SO(m+1) \arrow[r, "p"] \arrow[d] & S^m \arrow[d, "j"] \\ SO(m+2) \arrow[r] \arrow[d, "p"'] & V_{2,m} \arrow[d] \\ S^{m+1} \arrow[r, "1"] & S^{m+1} \end{tikzcd}$$

By the naturality of the homotopy exact sequences we have:

$$\begin{tikzcd} \pi_{m+1}(S^{m+1}) \arrow[r, "1"] \arrow[d, "\partial"'] & \pi_{m+1}(S^{m+1}) \arrow[d, "\partial'"] \\ \pi_m(SO(m+1)) \arrow[r, "p_{\#}"] & \pi_m(S^m) \end{tikzcd}$$

By 6.9 $p_{\#}\circ\partial=0$ if m is even, $p_{\#}\circ\partial$ is multiplication by 2 if m is odd. Hence $\partial'=p_{\#}\circ\partial$, and from the exact homotopy sequence of the fibre bundle,

$$\pi_{i+1}(S^{m+1})\xrightarrow{\partial'}\pi_i(S^m)\xrightarrow{j_{\#}}\pi_i(V_{2,m})\rightarrow\pi_i(S^{m+1})=0 \quad \text{for } i\leq m,$$

we deduce that $j_{\#}$ is surjective for $i\leq m$, and $\pi_i(V_{2,m})=0$ for $i<m$, $\pi_m(V_{2,m})=Z$ if m is even, $\pi_m(V_{2,m})=Z_2$ if m is odd.

Consider next the natural inclusion $V_{k,m}\rightarrow V_{k+1,m}$ given by including $SO(m+k) \to SO(m+k+1)$ in such a way that the subgroup $SO(m)$ is preserved. We have the commutative diagram:

$$\begin{array}{ccc} SO(m) & \xrightarrow{1} & SO(m) \\ \downarrow & & \downarrow \\ SO(m+k) & \xrightarrow{i} & SO(m+k+1) \\ \downarrow & & \downarrow \\ V_{k,m} & \xrightarrow{j} & V_{k+1,m} \end{array}$$

and a corresponding diagram incorporating the exact sequences,

$$\begin{array}{ccccccccc} \cdots \to \pi_i(SO(m)) & \to & \pi_i(SO(m+k)) & \to & \pi_i(V_{k,m}) & \to & \pi_{i-1}(SO(m)) \to \cdots \\ \downarrow 1 & & \downarrow i_{\#} & & \downarrow j_{\#} & & \downarrow 1 \\ \cdots \to \pi_i(SO(m)) & \to & \pi_i(SO(m+k+1)) & \to & \pi_i(V_{k+1,m}) & \to & \pi_{i-1}(SO(m)) \to \cdots \end{array}$$

By Lemma 6.7, $i_{\#}$ is an isomorphism for $i<m+k-1$, and, since $k\geq 2$, it follows that $j_{\#}$ is an isomorphism for $i\leq m$. QED

The following theorem describes what can be accomplished toward solution of the surgery problem, by the use of surgery below the middle dimension.

6.13 Theorem: Let $(M,\partial M)$ be a smooth compact m-manifold with boundary, $m\geq 4$, ν^k the normal bundle for $(M,\partial M)\subset(D^{m+k},S^{m+k-1})$, $k>>m$. Let A be a finite complex, $B\subseteq A$, η^k a k-plane bundle over A, let $f:(M,\partial M)\to(A,B)$, and let $b:\nu\to\eta$ be a linear bundle map covering f.

Then there is a cobordism W of M, with $\partial W=M\cup(\partial M\times I)\cup M'$, $\partial M'=\partial M\times 1$, an extension F of f, $F:(W,\partial M\times I)\to(A,B)$ with $F|\partial M\times t=f|\partial M$ for each $t\in I$, and an extension $\bar{b}$ of b, $b:\omega\to\eta$, where ω is the normal bundle of W in $D^{m+k}\times I$, such that $f'=F|M':M'\to A$ is $[\frac{m}{2}]$-connected ($[a]$ is the greatest integer not larger than a).

Proof: The proof is by induction: we shall assume that $f:M\to A$ is n-connected, $n+1\leq[\frac{m}{2}]$, and show how to construct W,F, etc. as above, with $f':M'\to A$ $(n+1)$-connected ($n+1$ is any nonnegative integer).

If $n+1=0$, we need only show how to make the map induced on π_0 surjective. Since A is a finite complex, A has only a finite number of components, $A=A_1\sqcup A_2\sqcup\ldots\sqcup A_r$. Let $a_i\in A_i$, and take $M'=M\sqcup S_1^m\sqcup\ldots\sqcup S_r^m$, where S_i^m is the m-sphere. Let $W=M\times I\sqcup D_1^{m+1}\sqcup\ldots\sqcup D_r^{m+1}$ and let $F:W\to A$ be defined by $F|M\times t=f$ for each $t\in I$, $F(D_i^{m+1})=a_i$. Since the normal bundle of D_i^{m+1} is trivial, and the extension condition on the bundle map is easy to fulfill on the D_i^{m+1}, it follows directly that b extends to $\bar{b}$ over W. Clearly the map induced by $f'=F|M'$ is onto $\pi_0(A)$, which proves the initial step of our induction.

Now assume $n+1=1$, $f:M\to A$ is 0-connected. Let M_1 and M_2 be two components of M such that $f(M_1)$ and $f(M_2)$ are in the same component of A. Take two points $x_i\in\text{int } M_i$, $i=1,2$, and define $\bar{\phi}:S^0\to M$ by $\bar{\phi}(1)=x_1$, $\bar{\phi}(-1)=x_2$. Since $f(\bar{\phi}(S^0))$ lies in a single component of A, it follows that $f:M\to A$ extends to $\bar{f}:M\cup_{\bar{\phi}}D^1\to A$. Then, since $m\geq 4$, it follows from Theorems 6.6 and 6.12 that $\bar{\phi}$ extends to $\phi:S^0\times D^m\to M$ defining a normal cobordism of f to f' and reducing the number of components of M. Using this argument repeatedly, we arrive at a 1-to-1 correspondence of components.

Now we consider the fundamental groups. Let $\{a_1,\ldots,a_s;r_1,\ldots,r_t\}$ and $\{x_1,\ldots,x_k;y_1,\ldots,y_\ell\}$ be presentations of $\pi_1(A)$ and $\pi_1(M)$, resp. Let s copies of S^0 be embedded disjointly in an m-cell D^m int M, $\phi':{}_sS^0\to M$, and assume the base point of M is in D^m and $f(D^m)=*$, the base point of A. Let $\bar{M}=M\cup_{\phi'}(\underset{s}{\cup}D^1)$. Then $\pi_1(\bar{M})\cong\pi_1(M)*F$, where F is a free group on s generators $g_1,\ldots,g_s$, where each g_i is the homotopy class of a loop in $D^m\cup(\underset{s}{\cup}D^1)$ consisting of a path in D^m, one of the D^1's, and another path in D^m. Hence $\pi_1(\bar{M})=\{x_1,\ldots,x_k,g_1,\ldots,g_s;y_1,\ldots,y_\ell\}$.

Define $\bar{f}:\bar{M}\to A$ extending f by letting the image of the i^{th} D^1

traverse a loop representing the generator a_i. Then $\bar{f}_\#:\pi_1(\bar{M})\to\pi_1(A)$ is surjective, and furthermore we may represent $\bar{f}_\#$ on the free groups $\{x_1,\ldots,x_k,g_1,\ldots,g_s\}$ and $\{a_1,\ldots,a_s\}$ by a function α, with $\alpha(x_i)=x_i'$, x_i' a word in the a_j, and $\alpha(g_i)=a_i$. Then as above, we may extend ϕ' to $\phi:(\underset{s}{\cup}S^0)\times D^m\to M$ to define a normal cobordism of f, and with $W_\phi=\bar{M}$, and $F:W_\phi\to A$ homotopic to $\bar{f}:\bar{M}\to A$. (Here W_ϕ is the trace of the simultaneous surgeries.) By Proposition 6.2, $\pi_1(M')\cong\pi_1(W_\phi)$, where M' is defined by $\partial W_\phi=M\cup(\partial M\times I)\cup M'$, and hence $f_\#':\pi_1(M')\to\pi_1(A)$ is surjective, $\pi_1(M')$ has the same presentation as $\pi_1(\bar{M})$, and $f_\#'$ is also represented by α on the free groups. In particular, f' is 1-connected.

Let us consider the exact sequence of the map $f:M\to A$ in homotopy,

$$\cdots\to\pi_{n+1}(f)\longrightarrow\pi_n(M)\longrightarrow\pi_n(A)\longrightarrow\pi_n(f)\to\cdots$$

Recall that the elements of the groups $\pi_{n+1}(f)$ are defined by commutative diagrams:

$$\begin{array}{ccc} S^n & \xrightarrow{\alpha} & M \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ D^{n+1} & \xrightarrow{\beta} & A \end{array} \qquad (*)$$

where k is the inclusion of the boundary, and all maps and homotopies preserve base points. Thus β defines a map $\bar{f}:M\cup_\alpha D^{n+1}\to A$ extending f.

6.14 <u>Lemma</u>: Let $f:M\to A$ be n-connected, $n>0$, and let $(\beta,\alpha)\in\pi_{n+1}(f)$ be the element represented by the above diagram (*). If $\bar{f}:M\cup_\alpha D^{n+1}\to A$ is defined by β as above, then $\pi_i(\bar{f})=\pi_i(f)=0$ for $i\leq n$, and $\pi_{n+1}(\bar{f})\cong\pi_{n+1}(f)/K$, where K is a normal subgroup containing the $\pi_1(M)$ module generated by the element (β,α) in $\pi_{n+1}(f)$.

<u>Proof of Lemma 6.14</u>: Consider the commutative diagram:

$$\begin{array}{ccccccc} \cdots\to\pi_{\ell+1}(f) & \longrightarrow & \pi_\ell(M) & \xrightarrow{f_\#} & \pi_\ell(A) & \to\cdots \\ \downarrow{\scriptstyle j_\#} & & \downarrow{\scriptstyle i_\#} & & \downarrow{\scriptstyle 1} & \\ \cdots\to\pi_{\ell+1}(\bar{f}) & \longrightarrow & \pi_\ell(M\cup_\alpha D^{n+1}) & \xrightarrow{F_\#} & \pi_\ell(A) & \to\cdots \end{array}$$

Here $i: M \to M \cup_\alpha D^{n+1}$ is inclusion, and $j_\#$ is induced by $(1,i)$ on the diagram (*) (i.e. $j_\#[\beta',\alpha'] = [\beta', i\circ\alpha']$). Clearly, $i_\#$ is an isomorphism for $\ell<n$, and surjective for $\ell=n$, so it follows easily that $\pi_\ell(\bar{f}) = \pi_\ell(f) = 0$ for ℓ n (by the Five Lemma).

Clearly any map of S^n into $M \cup_\alpha D^{n+1}$ is homotopic to a map into M, so that any pair (β',α'):

$$\begin{array}{ccc} S^n & \xrightarrow{\alpha'} & M \cup_\alpha D^{n+1} \\ k\downarrow & & \downarrow \bar{f} \\ D^{n+1} & \xrightarrow{\beta'} & A \end{array}$$

is homotopic to a pair of the form $(\beta'', i\circ\alpha'')$:

$$\begin{array}{ccccc} S^n & \xrightarrow{\alpha''} & M & \xrightarrow{i} & M \cup_\alpha D^{n+1} \\ \downarrow & & \downarrow f & & \downarrow \bar{f} \\ D^{n+1} & \xrightarrow{\beta''} & A & \xrightarrow{1} & A \end{array}$$

Hence $j_\#: \pi_{n+1}(f) \to \pi_{n+1}(\bar{f})$ is surjective.

Clearly (β,α) is in the kernel of $j_\#$ and hence everything obtained from (β,α) by the action of $\pi_1(M)$ is also in ker $j_\#$, which proves the lemma. QED

We have already shown that we may assume, without loss of generality, that $f: M \to A$ is 1-connected, and that the fundamental groups have presentations $\pi_1(M) = \{x_1,\ldots,x_k, g_1,\ldots,g_s; y_1,\ldots,y_\ell\}$, y_i words in $x_1,\ldots,x_k$ only, $\pi_1(A) = \{a_1,\ldots,a_s; r_1,\ldots,r_t\}$, with $f_\#: \pi_1(M) \to \pi_1(A)$ presented by the function (on the free groups) $\alpha(x_j) = x_j'(a_1,\ldots,a_s)$ a word in $a_1,\ldots,a_s$, $j=1,\ldots,k$, and $\alpha(g_i) = a_i$, $i=1,\ldots,s$.

6.15 <u>Lemma</u>: ker $f_\#$ is the smallest normal subgroup containing the words $x_j^{-1}(x_j'(g_1,\ldots,g_s))$, $j=1,\ldots,k$ and $r_i(g_1,\ldots,g_s)$, $i=1,\ldots,t$.

<u>Proof of Lemma 6.15</u>: Adding the relations $x_j^{-1}(x_j'(\bar{g}))$ makes $g_1,\ldots,g_s$ into a set of generators. Adding the $r_i(\bar{g})$ makes the group into $\pi_1(A)$, with α defining the isomorphism. The map α annihilates $x_j^{-1}(x_j'(\bar{g}))$

and $r_i(\bar{g})$, so that these elements generate ker $f_{\#}$ as a normal subgroup.

For each element $x_j^{-1}(x_j'(\bar{g}))$ and $r_i(\bar{g})$ choose an element $\bar{x}_j, \bar{r}_i \in \pi_2(f)$ such that $\bar{x}_j = x_j^{-1}(x_j'(\bar{g}))$, $\bar{r}_i = r_i(\bar{g})$, and choose embeddings $S^1 \to M$ to represent the $\bar{x}_j$ and $\bar{r}_i$ (also denoted by $\bar{x}_j$ and $\bar{r}_i$) such that their images are all disjoint, which is possible by general position, since m 4. Let $\bar{M} = M \cup (\bigcup_{k+t} D_\gamma^2)$, with the 2-discs attached by these embeddings. It follows from Lemma 6.14 that $\bar{f}_{\#}: \pi_1(\bar{M}) \to \pi_1(A)$ is an isomorphism. Using again Theorems 6.6 and 6.12, it follows that there is a normal cobordism W, and a map $F: W \to A$ such that $\bar{M} \subset W$ is a deformation retract and $F|\bar{M} = \bar{f}$, so that $F_{\#}: \pi_1(W) \to \pi_1(A)$ is an isomorphism. By Propositions 6.2 and 6.3, it follows that if M' is the result of surgery, then $f_{\#}': \pi_1(M') \to \pi_1(A)$ is an isomorphism, and hence $\pi_2(A) \to \pi_2(f)$ is surjective, and thus $\pi_2(f)$ is abelian.

We now proceed to the induction step. Suppose $f: M \to A$ is n-connected, $n > 0$, and if $n=1$ suppose $\pi_1(M) \to \pi_1(A)$ is an isomorphism, so that $\pi_2(f)$ is abelian.

6.16 <u>Lemma</u>: $\pi_{n+1}(f)$ is a finitely generated module over $\pi_1(M)$.

This lemma is proved using universal covering spaces [Browder 1972].

Now we may represent each of this finite number of generators in $\pi_{n+1}(f)$ by a diagram

$$\begin{tikzcd} S^n \arrow[r, "\alpha_i"] \arrow[d] & M \arrow[d, "f"] \\ D^{n+1} \arrow[r, "\beta_i"] & A \end{tikzcd}$$

If $n+1 \leq [\frac{m}{2}]$, then $n < \frac{m}{2}$ and it follows from Whitney's embedding theorem ('general position') that we may choose (β_i, α_i) so that the α_i have disjoint images. Setting $\bar{M} = M \cup (\bigcup_i D_i^{n+1})$, D_i^{n+1} attached by α_i, $\bar{f}: \bar{M} \to A$ defined by the β_i, we may apply Theorems 6.6 and 6.12 to thicken $\bar{M}$ to a normal cobordism W of M, and using 6.14, $\pi_\ell(\bar{f}) = 0$ for $\ell \leq n+1$. If M' is the result of the surgeries (i.e. $\partial W = M \cup (\partial M \times I) \cup M'$), from Propositions 6.2 and 6.3 it follows that $\pi_i(f') \cong \pi_i(\bar{f}) = 0$ for $i \leq n+1$. This completes the proof of Theorem 6.13. QED

Note that we have always used the low dimensionality of the groups involved to ensure that $\mathcal{O}$ was zero (by Theorem 6.12) and to find representatives of elements of $\pi_{n+1}(f)$ which were embeddings. To derive results in higher dimensions, we shall have to find other means of dealing with these obstacles.

§7. Initial Results in the Middle Dimension.

Let (A,B) be an oriented Poincaré pair of dimension m, let M be an oriented smooth compact m-manifold with boundary ∂M, and let $f:(M,\partial M) \to (A,B)$ be a map of degree 1. Let η^k be a linear k-plane bundle over A, $k \gg m$, and let ν^k be the normal bundle of $(M,\partial M)$ in (D^{m+k}, S^{m+k-1}). Suppose $b:\nu \to \eta$ is a linear bundle map covering f. Then (f,b) is what we have called a normal map. (Recall that we defined a normal cobordism of (f,b) rel B to be an $(m+1)$-manifold W with $\partial W = M \cup (\partial M \times I) \cup M'$, together with an extension of f, $F:(W,\partial M \times I) \to (A,B)$ for which $F|\partial M \times t = f|\partial M$ for each $t \in I$, and an extension $\bar{b}$ of b to the normal bundle ω of W in $D^{m+k} \times I$.)

Suppose further that A is a simply-connected CW complex, $m \geq 5$, and that $(f|\partial M)_*: H_*(\partial M) \to H_*(B)$ is an isomorphism.

7.1 Theorem: There is a normal cobordism rel B of (f,b) to (f',b') such that $f':M' \to A$ is $[\frac{m}{2}]+1$-connected if and only if $\sigma(f,b)=0$.

In particular, this is true if m is odd.

The proof of this theorem will occupy the balance of the present chapter. First note the ultimate corollary.

7.2 Corollary: (Fundamental Theorem of Surgery) The map f' above is a homotopy equivalence. Hence, (f,b) is normally cobordant rel B to a homotopy equivalence if and only if $\sigma(f,b)=0$. In particular, there is such a normal cobordism if m is odd.

Proof of Corollary 7.2: By the naturality of the exact homology sequence of pairs, we have

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(\partial M') \arrow[r] \arrow[d, "(f'|\partial M')_*"'] & H_i(M') \arrow[r] \arrow[d, "f'_*"] & H_i(M',\partial M') \arrow[r] \arrow[d, "\bar{f}'_*"] & H_{i-1}(\partial M') \arrow[r] \arrow[d, "(f'|\partial M')_*"] & H_{i-1}(M') \arrow[r] \arrow[d, "f'_*"] & \cdots \\
\cdots \arrow[r] & H_i(B) \arrow[r] & H_i(A) \arrow[r] & H_i(A,B) \arrow[r] & H_{i-1}(B) \arrow[r] & H_{i-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Since $(f|\partial M)_*:H_*(\partial M)\to H_*(B)$ is an isomorphism, and $\partial M'=\partial M$, $f'|\partial M'=f|\partial M$, we see that $(f'|\partial M')_*$ is an isomorphism in each dimension. By 7.1, $f':M'\to A$ is $[\frac{m}{2}]+1$-connected, so that $f'_*:H_i(M')\to H_i(A)$ is an isomorphism for $i\leq\frac{m}{2}$. Thus by the Five Lemma, $\bar{f}'_*:H_i(M',\partial M')\to H_i(A,B)$ is an isomorphism for $i\leq\frac{m}{2}$. Since f' is a map of degree 1, it follows from Poincaré duality that $f'^*:H^j(A)\to H^j(M')$ is an isomorphism for $j\geq m-\frac{m}{2}=\frac{m}{2}$. Now $f'^{*j}:H^j(A)\to H^j(M')$ is given by

$f'^{*j}=\mathrm{Hom}(f'_{*j},Z)+\mathrm{Ext}(f'_{*j-1},Z)$, according to the Universal Coëfficient Theorem, where $f'_{*j}:H_j(M')\to H_j(A)$, etc.

Since f'_{*i} is an isomorphism for $i\leq\frac{m}{2}$, it follows that f'^{*j} is an isomorphism for $j\leq\frac{m}{2}$, and hence $f'^*:H^j(A)\to H^j(M')$ is an isomorphism for all j. Thus, $H^*(f')=0$, and the Universal Coëfficient Theorem implies that $H_*(f')=0$. But M' and A are simply-connected, so that by the Relative Hurewicz Theorem and the Theorem of Whitehead we have the result: $f':M'\to A$ is a homotopy equivalence. This establishes the corollary.

We shall develop certain preliminary results before proceeding with the proof of Theorem 7.1.

By Theorem 6.13, we may assume that $f:M\to A$ is $[\frac{m}{2}]$-connected, i.e. $\pi_i(f)=0$ for $i\le[\frac{m}{2}]$. Set $\ell=[\frac{m}{2}]$. Since A and M are simply-connected, it follows from the Relative Hurewicz Theorem that $\pi_{\ell+1}(f)\cong H_{\ell+1}(f)$. This gives a commutative diagram:

$$\begin{array}{ccccccc}
\cdots\to\pi_{\ell+1}(f) & \longrightarrow & \pi_\ell(M) & \xrightarrow{f_\#} & \pi_\ell(A) & \to & 0\\
h\big\downarrow\cong & & h\big\downarrow & & h\big\downarrow & & \\
\cdots\to H_{\ell+1}(f) & \longrightarrow & H_\ell(M) & \xrightarrow{f_*} & H_\ell(A) & \to & 0
\end{array}$$

where h is the Hurewicz homomorphism, and $f_\#$ is the map induced by f in homotopy. Recall that f_* is surjective, and splits by Theorem 5.12. It follows that $(\ker f_*)_\ell=h(\ker f_\#)_\ell$.

Whitney's embedding theorem states: 'Let $c:V^n\to M^m$ be a continuous map of smooth manifolds, $m\ge 2n$, $m-n>2$, M simply-connected, V connected. Then c is homotopic to a smooth embedding.' (A proof can be found in [Milnor 1965].)

Since $\ell\le\frac{m}{2}$, it follows from Whitney's embedding theorem that any element $x\in\pi_{\ell+1}(f)$ may be represented by $(\beta,\bar{\phi})$, where $\bar{\phi}:S^\ell\to\text{int } M$ is a smooth embedding, and $\beta:D^{\ell+1}\to A$, $\beta\circ i=f\circ\bar{\phi}$. Set $\bar{M}=M\cup_{\bar{\phi}}D^{\ell+1}$, $\bar{f}:\bar{M}\to A$ the extension of f defined using β.

We should like to thicken $(\bar{M},\bar{f})$ to a normal cobordism; i.e. to perform normal surgery using $\bar{\phi}$, and to examine $\pi_{\ell+1}(f')$, where f' is the map on the result of the surgery, with the hope of having killed the homotopy class of $\bar{\phi}$. However, there are two difficulties we must face:

First, if $m=2\ell$, then according to Theorems 6.6 and 6.12, there is an obstruction $\mathcal{O}$ to thickening $(\bar{M},\bar{f})$ to a normal cobordism, which lies in a nontrivial group $\pi_\ell(V_{k,\ell})$.

Second, although we may compute $\pi_{\ell+1}(\bar{f})$ using Lemma 6.14, it is no longer clear how this group is related to $\pi_{\ell+1}(f')$, if $\ell=[\frac{m}{2}]$.

We shall first direct our attention toward the second difficulty.

Unless stated otherwise, we shall assume henceforth that (f,b) is a normal map stisfying the hypotheses of Theorem 7.1, and $f:M\to A$ is q-connected, where $q=[\frac{m}{2}]$, i.e. $m=2q$ or $2q+1$.

7.3 <u>Lemma</u>: f is (q+1)-connected if and only if $f_*:H_q(M)\to H_q(A)$ is an isomorphism, i.e. if and only if $K_q(M)=0$.

<u>Proof</u>: By the Relative Hurewicz Theorem, $\pi_{q+1}(f)\cong H_{q+1}(f)$, and by Theorem 5.12, $f_*:H_{q+1}(M)\to H_{q+1}(A)$ is surjective, so that

$$H_{q+1}(f)\cong(\ker f_*)_q\cong K_q(M). \qquad \text{QED}$$

Thus we need not examine homotopy, but will study the effect of surgery on homology. The following lemma will allow us to simplify our arguments by considering only the case of closed manifolds.

Let (f_i,b_i), $i=1,2$, be two disjoint copies of the normal map (f,b), so that $f_i:(M_i,\partial M_i)\to(A_i,B_i)$, $i=1,2$, is just f renamed. Then by the Sum Theorem for Poincare pairs [Browder 1972, I.3.2], $A_3=A_1\cup A_2$ with B_1 identified to B_2 is a Poincaré complex (called the double of A), $M_3=M_1\cup M_2$, united along $\partial M_1=\partial M_2$, is a smooth closed oriented manifold, and $f_3=f_1\cup f_2$, $b_3=b_1\cup b_2$ define a normal map $(f_3,b_3):M_3\to A_3$. Since $(f|\partial M)_*$ is an isomorphism, the Mayer-Vietoris sequences imply that $H_i(f_3)=0$ for $i<q+1$, and

$$H_{q+1}(f_3)\cong K_q(M_3)\cong K_q(M_1)\oplus K_q(M_2).$$

Now suppose $\phi:S^q\times D^{m-q}\to \text{int } M_1$ is a smooth embedding such that $f_1\circ\phi\simeq\varepsilon_*$ (the constant map), and such that ϕ defines a normal surgery on M_1 and, by inclusion, on M_3 (with respect to (f_1,b_1) and (f_3,b_3)). If a prime denotes the result of surgery, we have $M_3'=M_1'\cup M_2$ and $K_q(M_3')\cong K_q(M_1')\oplus K_q(M_2)$. This follows from the fact that the surgery has not affected the factor M_2 in the decomposition of M_3.

Thus we have:

7.4 <u>Proposition</u>: The effect of normal surgery on $K_q(M)$ is the same as the effect of the induced surgery on $K_q(M_3)$, and hence to compute its effect, we may assume $\partial M=B=\phi$.

This construction will simplify the algebra in our discussion.

Let $\phi:S^q\times D^{m-q}\to\text{int } M$ be a smooth embedding which defines a normal surgery on M (with respect to (f,b)). Set $M_0=M\backslash\text{int im }\phi$, and let $M'=M_0\cup D^{q+1}\times S^{m-q-1}$, so that $\phi(S^q\times S^{m-q-1})$ is identified with $S^q\times S^{m-q-1}=\partial(D^{q+1}\times S^{m-q-1})$. Then M' is the result of the surgery on M. Since ϕ defines a normal surgery, $H_q(M')\cong H_q(A)\oplus K_q(M')$, and we wish to determine how $K_q(M)$ changes to $K_q(M')$ (which is the same as the change of $H_q(M)$ to $H_q(M')$).

We formulate some useful results concerning the relation between Poincare duality in manifolds and submanifolds.

7.5 <u>Proposition</u>: Let U and W be compact m-manifolds with boundary, $f:U\to\text{int } W$, $g:(W,\partial W)\to(W,W\backslash\text{int } U)$ embeddings, with orientations compatible. Then the following diagram commutes:

$$\begin{tikzcd}
H^q(W,\partial W) \arrow[d, "{[W]\cap\cdot}"'] & H^q(W,W\backslash\text{int } U) \arrow[l, "g^*"'] \arrow[r, "\cong"] \arrow[d, "{(g_*[W])\cap\cdot}"] & H^q(U,\partial U) \arrow[d, "{[U]\cap\cdot}"] \\
H_{m-q}(W) \arrow[r, "1"] & H_{m-q}(W) & H_{m-q}(U) \arrow[l, "f_*"']
\end{tikzcd}$$

so that for $x\in H^q(U/\partial U)$, $f_*([U]\cap x)=[W]\cap\bar{g}^*(x)$, where $\bar{g}:W/\partial W\to U/\partial U$.

<u>Proof</u>: If $\bar{f}:(U,\partial U)\to(W,W\backslash\text{int } U)$, then $\bar{f}_*[U]=g_*[W]$, since we have oriented U and W compatibly. Then the commutativity follows from the naturality of the cap product. QED

7.6 <u>Corollary</u>: Set E equal to the normal tube of $f:N^n\to W^m$, N closed and oriented, and let $\bar{g}:W/\partial W\to E/\partial E=T(\nu)$, where ν is the normal bundle of $N^n\subset W^m$. Let $U\in H^{m-n}(T(\nu))$ be the Thom class. Then $[W]\cap\bar{g}^*U=f_*[N]$.

Proof: Since $[E]\cap U=[N]$ by 7.5, $f_*([E]\cap U)=f_*[N]=[W]\cap(\bar{g}^*U)$. QED

The intersection pairing in homology, $\cdot:H_q(M)\otimes H_{m-q}(M,\partial M)\to Z$ is defined by $x\cdot y=(x',y')=(x'\cup y')[M]$, where $x'\in H^{m-q}(M,\partial M)$, $y'\in H^q(M)$ are dual to x,y, i.e. $[M]\cap x'=x$, $[M]\cap y'=y$. This induces an intersection product $\cdot:H_q(M)\otimes H_{m-q}(M)\to Z$ by $x\cdot y=x\cdot j_*(y)$, where $j:M\to(M,\partial M)$ is inclusion.

The properties of the bilinear form $(\cdot,\cdot)$ on cohomology induce analogous properties for the intersection pairing, such as

(a) With coëfficients in a field F, $H_q(M;F)\otimes H_{m-q}(M,\partial M;F)\to F$ is a nonsingular pairing. (This also holds over Z, modulo torsion.)

(b) If $x\in H_q(M)$, $y\in H_{m-q}(M)$, $x\cdot y=(-1)^{q(m-q)}y\cdot x$.

7.7 Proposition: Let $x\in H_q(M)$, $y\in H_{m-q}(M,\partial M)$, $x'\in H^{m-q}(M,\partial M)$, $y'\in H^q(M)$ be such that $[M]\cap x'=x$, $[M]\cap y'=y$. Then $x\cdot y=x'(y)$.

Proof: $x\cdot y=(x'\cup y')[M]=x'([M]\cap y')=x'(y)$, using elementary properties of the cup and cap products.

Now let $\phi:S^q\times D^{m-q}\to\operatorname{int} M$ be a smooth embedding. Set $E=S^q\times D^{m-q}$, $M_0=M\backslash\phi(\operatorname{int} E)$, $M'=M_0\cup(D^{q+1}\times S^{m-q-1})$, the result of surgery based on ϕ.

Following [Kervaire, Milnor 1963] we will consider the exact sequences of the pairs (M,M_0) and (M',M_0).

As usual, we have the excision $\phi:(E,\partial E)\to(M,M_0)$ which induces isomorphisms on the relative homology and cohomology groups. Thinking of E as the normal tube of $S^q\subset M$, let $U\in H^{m-q}(E,\partial E)=Z$ be the Thom class, a generator (cf. 7.6). If $\mu=[E]\cap U$, then $\mu=i_*[S^q]$, $i:S^q\to E$, and $\mu\cdot x=U(x)$ for any $x\in H_{m-q}(E,\partial E)$ by 7.7. This induces an isomorphism $H_{m-q}(E,\partial E)\to Z$ by property (a) above. Let $j:M\to(M,M_0)$ be the inclusion.

7.8 Proposition: $\mu\cdot(j_*(y))=(\phi_*(\mu))\cdot y$.

Proof: $\mu\cdot(j_*(y))=U(j_*(y))=(j^*U)(y)=(\phi_*(\mu))\cdot y$, using 7.7 and 7.6, and identifying $j_*:H_*(M)\to H_*(M,M_0)$ with the collapsing map

$$\bar{j}_*:H_*(M)\to H_*(M/M_0)\cong H_*(E/\partial E).$$

7.9 Corollary: The following sequence is exact:

$$0\to H_{m-q}(M_0)\to H_{m-q}(M)\xrightarrow{x\cdot} Z\xrightarrow{d} H_{m-q-1}(M_0)\to H_{m-q-1}(M)\to 0,$$

where $x=\phi_*(\mu)$, $\mu\in H_q(S^q\times D^{m-q})$ is the image of $[S^q]$, the orientation class of S^q.

Proof: The sequence is that of (M,M_0), replacing $H_{m-q}(M,M_0)$ by using the diagram

$$\begin{tikzcd} H_{m-q}(E,\partial E) \arrow[r, "\cong"] \arrow[d, "\mu\cdot"] & H_{m-q}(M,M_0) \\ Z & \end{tikzcd}$$

and using 7.8 to identify $x\cdot$.

Thus there is an exact sequence

$$0\to H_{q+1}(M_0)\to H_{q+1}(M')\xrightarrow{y\cdot} Z\xrightarrow{d'} H_q(M_0)\xrightarrow{i'_*} H_q(M')\to 0$$

where $y=\psi_*(\mu')$, $\mu'=k'_*[S^{m-q-1}]$ generates $H_{m-q-1}(D^{q+1}\times S^{m-q-1})$, $\psi:D^{q+1}\times S^{m-q-1}\to M'$ is the natural embedding, and $k':S^{m-q-1}\to D^{q+1}\times S^{m-q-1}$ is inclusion.

Let $\lambda\in H_{r+1}(S^q\times D^{r+1},S^q\times S^r)=Z$ be the generator such that $U(\lambda)=1$, and similarly for λ'. (We shall allow λ and $\mu\cdot\lambda$, λ' and $\mu'\cdot\lambda'$ to be confused.)

7.10 Lemma: $i_*d'(\lambda')=\phi_*(\mu)=x$ and $i'_*d(\lambda)=\psi_*(\mu')=y$.

Proof: Let $m=q+r+1$. We have a commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & H_{r+1}(S^q\times D^{r+1},S^q\times S^r) \arrow[r, "\partial_1"] \arrow[d, "\phi_*"] & H_r(S^q\times S^r) \arrow[r, "i_{1*}"] \arrow[d, "\phi_{0*}"] & H_r(S^q\times D^{r+1}) \arrow[r] \arrow[d, "\phi_*"] & \cdots \\ \cdots \arrow[r] & H_{r+1}(M,M_0) \arrow[r, "\partial"] & H_r(M_0) \arrow[r, "i_*"] & H_r(M) \arrow[r] & \cdots \end{tikzcd}$$

Clearly, if $\lambda\in H_{r+1}(S^q\times D^{r+1},S^q\times S^r)$ such that $U(\lambda)=1$, then $\partial_1\lambda=1\otimes[S^r]\in H_r(S^q\times S^r)$. We also have the commutative diagram

$$\begin{tikzcd} H_*(S^q\times S^r) \arrow[r, "i_{2*}"] \arrow[d, "\phi_{0*}"] & H_*(D^{q+1}\times S^r) \arrow[d, "\psi_*"] \\ H_*(M_0) \arrow[r, "i'_*"] & H_*(M') \end{tikzcd}$$

and $i_{2*}(1\otimes[S^r])=\mu'$.

Hence $i_*^! d(\lambda)=i_*^!\partial\phi_*(\lambda)=i_*^!\phi_{0*}\partial_1(\lambda)=\psi_* i_{2*}(1\otimes[S^r])=\psi_*(\mu')=y$.

A similar argument proves the other assertion. QED

7.11 Theorem: Let $\phi:S^q\times D^{r+1}\longrightarrow M$ be an embedding, M a closed m-manifold, $m=q+r+1$, $q\leq r+1$. Suppose $\bar{\phi}_*[S^q]=\phi(\mu)=\dot{x}$ generates an infinite cyclic direct summand of $H_q(M)$. Then rank $H_q(M')<$rank $H_q(M)$, and torsion $H_q(M')\cong$ torsion $H_q(M)$, i.e. the free part of $H_q(M)$ is reduced and the torsion part is not increased. Further $H_i(M')\cong H_i(M)$ for $i<q$.

7.12 Corollary: Let (f,b) be a normal map, $f:(M,\partial M)\longrightarrow(A,B)$, $(f|\partial M)_*$ an isomorphism, and let $\phi:S^q\times D^{r+1}\longrightarrow \mathrm{int}\, M$ be an embedding which defines a normal cobordism of (f,b), $q\leq r+1$. Suppose $\phi_*(\mu)=x$ generates an infinite cyclic direct summand of $K_q(M)$. Then rank $K_q(M')<$rank $K_q(M)$, and torsion $K_q(M')\cong$torsion $K_q(M)$, while $K_i(M')\cong K_i(M)$ for $i<q$.

The corollary follows directly from 7.11 and Proposition 7.4.

With a field of coefficients we have analogous results:

7.13 Theorem: Let ϕ,M be as in 7.11, and suppose $\phi_*(\mu)=x\neq 0$ in $H_q(M;F)$. Then $\mathrm{rank}_F H_q(M';F)<\mathrm{rank}_F H_q(M;F)$, and $H_i(M';F)\cong H_i(M;F)$ for $i<q$.

7.14 Corollary: With the hypotheses of 7.12, suppose only that $\phi_*(\mu)=x\neq 0$ in $K_q(M;F)$. Then $\mathrm{rank}_F K_q(M';F)<\mathrm{rank}_F K_q(M;F)$ and $K_i(M';F)\cong K_i(M;F)$ for $i<q$.

The proof of 7.14 is similar to that of 7.12.

Proof of Theorem 7.11: Consider the exact sequence of Corollary 7.9:

$$0\longrightarrow H_{r+1}(M_0)\xrightarrow{i_*}H_{r+1}(M)\xrightarrow{x\cdot}Z\xrightarrow{d}H_r(M_0)\longrightarrow H_r(M)\longrightarrow 0.$$

Since x generates an infinite cyclic direct summand, it follows from property (a) of the intersection pairing that there is an element $y\in H_{r+1}(M)$ such that $x\cdot y=1$ (since $\partial M=\phi$).

Hence $x\cdot$ is surjective and we get

$$i_*:H_r(M_0)\cong H_r(M) \qquad 0\longrightarrow H_{r+1}(M_0)\xrightarrow{i_*}H_{r+1}(M)\longrightarrow Z\longrightarrow 0 \qquad (1)$$

Consider the exact sequence of Corollary 7.9 for (M',M_0) and the diagram from Lemma 7.10:

$$\begin{tikzcd} 0 \arrow[r] & H_{q+1}(M_0) \arrow[r] & H_{q+1}(M') \arrow[r, "y\cdot"] & Z \arrow[r, "d'"] \arrow[dr] & H_q(M_0) \arrow[r, "i_*'"] \arrow[d, "i_*"] & H_q(M') \arrow[r] & 0 \\ & & & & H_q(M) & & \end{tikzcd} \quad (2)$$

where $i_*d'(\lambda')=x$. Since x generates an infinite cyclic direct summand, it follows that i_*d' splits, so that d' splits, and

$$H_q(M_0)\cong Z\oplus H_q(M') \qquad i_*':H_{q+1}(M_0)=H_{q+1}(M') \qquad (3)$$

From (3) it follows that rank $H_q(M')$=rank $H_q(M_0)-1$, and since $q=r$ or $r+1$, from (1) it follows that rank $H_q(M)\geq$rank $H_q(M_0)$, so that rank $H_q(M')<$rank $H_q(M)$ (the difference being 1 if $q=r$, 2 if $q=r+1$). From (1) it follows that torsion $H_q(M_0)$ is isomorphic to torsion $H_q(M)$, and from (3) it follows that torsion $H_q(M_0)\cong$torsion $H_q(M')$. Hence torsion $H_q(M')\cong$torsion $H_q(M)$. QED

The proof of 7.13 is almost identical, using (1), (2), and (3) with coëfficients in F, and using property (a) of intersection with coëfficients in F. The details are omitted.

To proceed further in the proof of the Fundamental Theorem, we must consider different dimensions separately; in particular, we must distinguish 3 cases: m odd, $m\equiv 0 \pmod 4$, and $m\equiv 2 \pmod 4$.

§8. The Proof of the Fundamental Theorem for m odd.

From Corollary 7.12 we may deduce the following theorem.

8.1 Theorem: Let (f,b) be a normal map, $f:(M,\partial M)\to(A,B)$, A simply-connected, $(f|\partial M)_*$ an isomorphism, $m=2q+1\geq 5$. There is a normal cobordism rel B of (f,b) to (f',b'), such that $f':M'\to A$ is q-connected, and $K_q(M')\cong$ torsion $K_q(M)$.

Proof: By Theorem 7.11, we may first find a normal cobordism rel B to (f_1,b_1), such that $f_1:M_1\to A$ is q-connected. We note that the surgeries used in 7.11 are on embedded spheres of dimension less than q, so that it follows from Propositions 6.2 and 6.3 that $K_q(M_1)\cong K_q(M)\oplus F$, where F is the free abelian group produced by killing torsion classes in $K_{q-1}(M)$. Thus we may assume without loss of generality that f is q-connected.

Let $x\in K_q(M)$ be a generator of an infinite cyclic direct summand. Since f is q-connected, it follows from the Relative Hurewicz Theorem that $\pi_{q+1}(f)\cong H_{q+1}(f)$, and $H_{q+1}(f)\cong K_q(M)$ by Theorem 5.12. Since $q<\frac{m}{2}$, it follows from the Whitney Embedding Theorem that we may represent $x'\in\pi_{q+1}(f)$ by (β,α),
$$\begin{tikzcd} S^q \arrow[r, "\alpha"] \arrow[d] & M \arrow[d, "f"] \\ D^{q+1} \arrow[r, "\beta"'] & A \end{tikzcd}$$
such that α is a smooth embedding. Then β defines a map $\bar{f}:\bar{M}\to A$ where $\bar{M}=M\cup_\alpha D^{q+1}$, and by Theorem 6.12, since $q<m-q$, the obstruction to thickening $\bar{M}$ to a normal cobordism is zero. If $x'\in\pi_{q+1}(f)$ is such that α represents $x\in K_q(M)$, then by Corollary 7.11, $K_q(M')$ has rank one less than $K_q(M)$, and the same torsion subgroup. Iterating this procedure until the rank is zero proves the theorem. QED

We derive an important diagram by uniting the two exact sequences of Corollary 7.9.

8.2 Lemma: We have a diagram:

$$\begin{tikzcd} & & & H_{q+1}(M') \arrow[d, "y\cdot"] & & \\ & & & Z \arrow[d, "d'"] & & \\ 0 \arrow[r] & H_{q+1}(M) \arrow[r, "x\cdot"] & Z \arrow[r, "d"] & H_q(M_0) \arrow[r, "i_*"] \arrow[d, "i'_*"] & H_q(M) \arrow[r] & 0 \\ & & & H_q(M') & & \end{tikzcd}$$

where $i_*d'(\lambda')=x=\phi_*(\mu)$, $i_*^!d(\lambda)=y=\psi_*(\mu')$, μ is a generator of $H_q(S^q\times D^{q+1})$, μ' of $H_q(D^{q+1}\times S^q)$, etc.

Hence, $H_q(M')/(i_*^!dZ)\cong H_q(M)/(i_*d'Z)$.

Proof: This follows directly from Corollary 7.9 and Lemma 7.10, and the fact that $H_q(M)/(i_*d'Z)\cong H_q(M_0)/(d'Z\oplus dZ)\cong H_q(M')/(i_*^!dZ)$. QED

If $x=i_*d'(\lambda')$ is a torsion element of order s, then $x\cdot$ is the zero map, so that part of the diagram of 8.2 becomes the short exact sequence:

$$0\longrightarrow Z\xrightarrow{d}H_q(M_0)\xrightarrow{i_*}H_q(M)\longrightarrow 0 \qquad (1)$$

Since i_* is a homomorphism, $sd'(\lambda')\in\ker i_*=\operatorname{im} d$, so we have:

$$sd'(\lambda')=d(n)=d((-t)\lambda)=-td(\lambda),\text{ and } sd'(\lambda')+td(\lambda)=0 \qquad (2)$$

in $H_q(M_0)$, for some $t\in Z$.

8.3 Lemma: Suppose x is a torsion element of finite order s in $H_q(M)$. Then y is of infinite order if t=0, and of (finite) order t if $t\neq 0$.

Proof: Since $d(\lambda)$ is of infinite order by (1) (which implies that d is injective), (2) shows that $d'(\lambda')$ is also of infinite order if $t\neq 0$ (since $s\neq 0$). Clearly $ty=ti_*^!d(\lambda)=i_*^!(-sd'(\lambda'))=0$, since $i_*^!\circ d'=0$, and using (2). Hence (order y)$|$t.

If $t'y=0$, then $t'i_*^!d(\lambda)=i_*^!(t'd(\lambda))=0$, so $t'd(\lambda)\in\ker i_*^!=\operatorname{im} d'$, and $t'd(\lambda)=-s'd'(\lambda')$ for some $s\in Z$, or $s'd'(\lambda')+t'd(\lambda)=0$ in $H_q(M_0)$. Applying i_*, we get $s'i_*d'(\lambda')=s'x=0$, so $s'=\ell\cdot s$. Subtracting ℓ times (2) from $s'd'(\lambda')+t'd(\lambda)=0$ we get $(t'-\ell t)d(\lambda)=0$. But $d(\lambda)$ is of infinite order, so $t'-\ell t=0$, or $t'=\ell t$. Hence $t|t'$, and t=order y.

Suppose t=0 so that $sd'(\lambda')=0$. Then $\ker i_*^!\subseteq$torsion $H_q(M_0)$, so $i_*^!$ is injective on dZ, and hence $y=i_*^!d(\lambda)$ is of infinite order in $H_q(M')$.

Consider the commutative diagram on the next page, in which d and d' are from the exact sequences of Corollary 7.9.

$$\begin{tikzcd} H_q(S^q\times S^q) & H_{q+1}(D^{q+1}\times S^q, S^q\times S^q)=Z \arrow[l, "\partial'"'] \arrow[d, "d'"] \\ Z=H_{q+1}(S^q\times D^{q+1}, S^q\times S^q) \arrow[u, "\partial"] \arrow[r, "d"] & H_q(M) \end{tikzcd} \tag{3}$$

Recall that $\lambda\in H_{q+1}(S^q\times D^{q+1},S^q\times S^q)$ is such that $\partial\lambda=1\otimes[S^q]$, and $\lambda'\in H_{q+1}(D^{q+1}\times S^q,S^q\times S^q)$ is such that $\partial'\lambda'=[S^q]\otimes 1$.

Suppose M is closed, so that $\partial M_0=S^q\times S^q$, and $\phi_0:S^q\times S^q\to M$ is the inclusion of the boundary. Then we have the exact sequence diagram of Poincaré duality:

$$\begin{tikzcd} \cdots \arrow[r] & H^q(M_0) \arrow[r, "\phi_0"] \arrow[d, "{[M_0]\cap\cdot}"] & H^q(S^q\times S^q) \arrow[r, "\delta"] \arrow[d, "{[S^q\times S^q]\cap\cdot}"] & H^{q+1}(M_0,S^q\times S^q) \arrow[r] \arrow[d, "{[M_0]\cap\cdot}"] & \cdots \\ \cdots \arrow[r] & H_{q+1}(M_0,S^q\times S^q) \arrow[r, "\partial_0"] & H_q(S^q\times S^q) \arrow[r, "\phi_{0*}"] & H_q(M_0) \arrow[r] & \cdots \end{tikzcd}$$

Thus, $[S^q\times S^q]\cap(\text{im }\phi_0^*)=\ker\phi_{0*}$. (4)

By (3), $d'(\lambda')=\phi_{0*}\partial'(\lambda')=\phi_{0*}([S^q]\otimes 1)$, and

$$d(\lambda)=\phi_{0*}\partial(\lambda)=\phi_{0*}(1\otimes[S^q]),$$

so that (2) can be rewritten as $\phi_{0*}(s([S^q]\otimes 1)+t(1\otimes[S^q]))=0$.

8.4 <u>Lemma</u>: Let q be even. Then $\phi_{0*}(s([S^q]\otimes 1)+t(1\otimes[S^q]))=0$ implies either $s=0$ or $t=0$.

<u>Proof</u>: Let $U\in H^q(S^q)$ be such that $U[S^q]=1$. Then $[S^q\times S^q]\cap(U\otimes 1)=1\otimes[S^q]$ and $[S^q\times S^q]\cap(1\otimes U)=[S^q]\otimes 1$, in $H_q(S^q\times S^q)$. Hence

$$[S^q\times S^q]\cap(s(1\otimes U)+t(U\otimes 1))=s([S^q]\otimes 1)+t(1\otimes[S^q]),$$

and by (4) it follows that $s(1\otimes U)+t(U\otimes 1)=\phi_0^*(z)$ for some $z\in H^q(M_0)$. But $\phi_0^*:H^{2q}(M_0)\to H^{2q}(S^q\times S^q)$ is zero, as ϕ_0 is the inclusion of the (connected) boundary of M_0. Hence $(s(1\otimes U)+t(U\otimes 1))^2=\phi_0^*(z^2)=0$. But $(s(1\otimes U)+t(U\otimes 1))^2=2st(U\otimes U)$ if $q=\dim U$ is even. Hence it is zero if and only if $s=0$ or $t=0$. QED

<u>Proof of Theorem 7.1 for m=2q+1, q even:</u> By Theorem 8.1, we may assume $f:M\to A$ is q-connected and $K_q(M)$ is a torsion group. Let $x\in K_q(M)$ be the

generator of a cyclic summand of order s. Let $\phi:S^q\times D^{q+1}\to M$ be an embedding with $\phi_*(\mu)=x$, and defining a normal cobordism of (f,b). Assume M is closed, using Proposition 7.4. Consider the diagram of Lemma 8.2. By Lemma 7.10, $i_*d'(\lambda')=x$, a generator of a summand $Z_s\subseteq H_q(M)$. By (2) and Lemma 8.4, $sd'(\lambda')=0$, so $d'(\lambda')$ generates a cyclic direct summand $Z_s\subseteq H_q(M\;)$.

From (1) it follows that torsion $H_q(M_0)$ is isomorphic to a subgroup of torsion $H_q(M)$, and since $H_q(M')\cong H_q(M_0)/d'Z$, it follows that torsion $H_q(M')$ is isomorphic to a subgroup of torsion $H_q(M)$ with at least one cyclic summand Z_s missing, so the same is true for $K_q(M')$. (It follows also that rank $H_q(M')$=rank $H_q(M)+1$.)

By Theorem 8.1 we may find a normal cobordism of (f',b') to (f",b") with $K_q(M'')$=torsion $K_q(M')$.

Iterating these constructions a finite number of times (since $K_q(M)$ is finitely generated) will produce an (f_1,b_1) normally cobordant to (f,b) with $K_q(M_1)=0$, and f_1 (q+1)-connected. This completes the proof for $m\equiv 1 \pmod 4$. QED

Proof of Theorem 7.1 for m=2q+1, q odd: Let $\phi:S^q\times D^{q+1}\to M$ be an embedding which defines a normal cobordism, i.e. so that (f,b) extend over the trace of the surgery based on ϕ, W_ϕ. Let $\omega:S^q\to SO(q+1)$, with SO(q+1) acting on D^{q+1} from the right, and define a new embedding $\phi_\omega:S^q\times D^{q+1}\to M$ by $\phi_\omega(x,t)=\phi(x,t\omega(x))$. Then ϕ_ω defines a surgery with the result $M'=M_0\cup_{\omega'}D^{q+1}\times S^q$, where M_0 comes from surgery using ϕ, and ω' is the diffeomorphism $S^q\times S^q\to S^q\times S^q$ given by $\omega'(x,y)=(x,y\omega(x))$.

8.5 Lemma: The trace of the surgery based on ϕ_ω also defines a normal cobordism if and only if the homotopy class $[\omega]$ goes to zero in $\pi_q(SO(q+k+1))$, i.e. $i_\#[\omega]=0$ where $i:SO(q+1)\to SO(q+k+1)$ is inclusion.

Proof: The map $\phi_{i\omega}:S^q\times D^{q+1}\times R^k\to M\times R^k$ given by $\phi_{i\omega}(x,t,r)=(\phi(x,t\omega(x)),r)$ $=(\phi_\omega(x,t),r)$ defines a new framing of the normal bundle to S^q in D^{m+k}, i.e. of $\nu|S^q\oplus\nu'$, where ν is the normal bundle of $M\subset D^{m+k}$, ν' the normal bundle of $S^q\subset M$. Then ϕ_ω defines a normal cobordism if and only if the framing extends to a framing of the normal bundle of D^{q+1} in $D^{m+k}\times I$, so that the first part of the frame defines an embedding of $D^{q+1}\times D^{q+1}$ in $D^{m+k}\times I$ extending $\phi_\omega:S^q\times D^{q+1}\subseteq M\subset D^{m+k}$, and the second part of the frame extends the trivialisation of $\nu|\phi(S^q\times D^{q+1})$ defined by $b:\nu\to\eta$, to a trivialisation of the normal bundle of $D^{q+1}\times D^{q+1}$, and hence that of $M\times I\cup D^{q+1}\times D^{q+1}$.

Now $S^q=\partial D^{q+1}$, $D^{q+1}\subset D^{m+k}\times I$ such that the normal bundle of S^q in $D^{m+k}\times 0$ is the restriction to S^q of γ, the normal bundle of D^{q+1} in $D^{m+k}\times I$. Now γ has a framing defined on S^q by the map $\hat{\phi}:S^q\times D^{q+1}\times R^k\to E(\nu)$, $\hat{\phi}(x,t,r)=(\phi(x,t),r)$ since ϕ defined a normal cobordism. The difference of these two framings is a map of S^q into $SO(q+k+1)$ which is obviously $i\omega$.

Hence the frame $\phi_{i\omega}$ extends over D^{q+1} if and only if $i\omega$ is homotopic to zero in $SO(q+k+1)$. QED

By Lemma 6.7, $\pi_q(SO(q+r))\to\pi_q(SO(q+r+1))$ is an isomorphism for $r>1$, so that $\ker i_\#$, $i_\#:\pi_q(SO(q+1))\to\pi_q(SO(q+k+1))$, is the same for all $k\geq 1$. For $k=1$, the exact homotopy sequence of the fibre space

$$SO(q+1)\xrightarrow{i}SO(q+2)\longrightarrow S^{q+1}$$

gives the result that $(\ker i_\#)_q=\partial_0\pi_{q+1}(S^{q+1})$, where $\partial_0:\pi_{q+1}(S^{q+1})\to\pi_q(SO(q+1))$ is the boundary of the exact sequence. Hence from Lemma 8.5, if $\phi:S^q\times D^{q+1}\to M$ defines a normal cobordism, then we may change ϕ by $\omega:S^q\to SO(q+1)$ if $[\omega]\in\partial_0\pi_{q+1}(S^{q+1})$, and ϕ_ω will still define a normal cobordism.

Now we will compare the effect of the surgeries based on ϕ and ϕ_ω.

Let $g_1'=[S^q]\otimes 1$, $g_2'=1\otimes[S^q]\in H^q(S^q\times S^q)$.

8.6 Lemma: Let $\bar{g}$ be a generator of $\pi_{q+1}(S^{q+1})$, and let $[\omega]=m\partial_0(\bar{g})$, $\phi'=\phi_\omega$. Then $\phi_{0*}'(g_1')=\phi_{0*}(g_1')+2m\phi_{0*}(g_2')$, $\phi_{0*}'(g_2')=\phi_{0*}(g_2')$.

Proof: Recall that Lemma 6.19 says that the composition

$$\pi_{q+1}(S^{q+1})\xrightarrow{\partial_0}\pi_q(SO(q+1))\xrightarrow{p_\#}\pi_q(S^q)$$

is multiplication by 2, if q is odd. Now, ϕ_0' is represented by the composition $S^q\times S^q\xrightarrow{\omega'}S^q\times S^q\xrightarrow{\phi_0}M_0$, where ω' is given by $(x,y)\to(x,y\omega(x))$. If y is taken to be the base point $y_0\in S^q$, then by definition $y_0\omega(x)=p\omega(x)$, where $p:SO(q+1)\to S^q$ is the bundle projection. Hence on $S^q\times y_0$, $\phi'(x,y_0)$ $=\phi_0(x,p\omega(x))$, so $\phi_0'=\phi_0(1\times p\omega)\Delta$ on $S^q\times y_0$, where $\Delta:S^q\to S^q\times S^q$ is given by $x\to(x,x)$.

If $g\in\pi_q(S^q)$ is the generator, $i_1(x)=(x,y_0)$, $i_2(x)=(y_0,x)$, $g_j=(i_j)_\# g$, then $\Delta_\# g=g_1+g_2$, and $h(g_j)=g_j'$, where h is the Hurewicz homomorphism.

Thus, $\phi_{0\#}'(g_1)=\phi_{0\#}(1\times p\omega)_\#\Delta_\#(g)=\phi_{0\#}(1\times p\omega)_\#(g_1+g_2)=\phi_{0\#}(g_1+2mg_2)$

$$=\phi_{0\#}(g_1)+2m\phi_{0\#}(g_2).$$

Since $\omega(y_0)$ is the identity of $SO(q+1)$, we have $\phi_0'|y_0\times S^q=\phi_0|y_0\times S^q$, so $\phi_{0\#}'(g_2)=\phi_{0\#}(g_2)$. The result in homology follows by applying h. QED

Returning to the diagram of Lemma 8.2, where $d(\lambda)=\phi_{0*}(1\otimes[S^q])$ $=h\phi_{0\#}(g_2)$, and $d'(\lambda')=h\phi_{0\#}(g_1)$, if we costruct the analogous diagram using ϕ_ω instead of ϕ, we find $d_\omega(\lambda)=h\phi_{\omega 0\#}(g_2)=d(\lambda)$, and $d_\omega'(\lambda')=h\phi_{\omega 0\#}(g_1)$ $=d'(\lambda')+2md(\lambda)$, or $d(\lambda)=d_\omega(\lambda)$, $d'(\lambda')=d_\omega'(\lambda')-2md_\omega(\lambda)$. Hence (2) becomes

$$s(d_\omega'(\lambda')-2md_\omega(\lambda))+td_\omega(\lambda)=0,\quad\text{or}\quad sd_\omega'(\lambda')+(t-2ms)d_\omega(\lambda)=0. \quad (5)$$

8.7 Proposition: Let p be a prime and let $x\in K_q(M)$ be an element of finite order such that $(x)_p\neq 0$ in $K_q(M;Z_p)$, where $(\cdot)_p$ denotes reduction mod p. Let $\phi:S^q\times D^{q+1}\to \text{int } M$ be an embedding which represents x, i.e. $\phi_*(\mu)=x$, and which defines a normal surgery of (f,b). Then one may choose $\omega:S^q\to SO(q+1)$ so that $\phi_\omega:S^q\times D^{q+1}\to \text{int } M$ also defines a normal surgery of (f,b), order(torsion $K_q(M'_\omega)$)$\leq$order(torsion $K_q(M)$), and $\mathrm{rank}_{Z_p} K_q(M'_\omega;Z_p)<\mathrm{rank}_{Z_p} K_q(M;Z_p)$. (The order of a torsion group T is the smallest positive integer n such that $nx=0$ in T for all $x\in T$.)

<u>Proof</u>: By Lemma 8.2, $H_q(M)/(x)\cong H_q(M')/(y)$, where (x) indicates the subgroup generated by x. If the order of x is s, then (2) gives $sd'(\lambda')+td(\lambda)=0$, and Lemma 8.3 states that the order of y is t if $t\neq 0$, and is infinite if $t=0$. By Lemma 8.5 we may change ϕ so that (2) becomes (5): $sd'_\omega(\lambda')+(t-2ms)d_\omega(\lambda)=0$, so that $H_q(M)/(x)\cong H_q(M'_\omega)/(y_\omega)$ with order $y_\omega=t-2ms$ if $t-2ms$ 0, and y_ω of infinite order if $t-2ms=0$. Choose m so that $-s\leq(t-2ms)\leq s$, which guarantees that order $y_\omega\leq$order x or y_ω is of infinite order. Hence, order(torsion $H_q(M'_\omega)$) is not larger than order(torsion $H_q(M)$), and so order(torsion $K_q(M'_\omega)$) is less than or equal to order(torsion $K_q(M)$). But if $(x)_p\neq 0$, then by Corollary 7.14, $\mathrm{rank}_{Z_p} K_q(M'_\omega;Z_p)<\mathrm{rank}_{Z_p} K_q(M;Z_p)$. QED

We are now able to complete the proof of Theorem 7.1 for $m\equiv 3 \pmod 4$.

Let (f,b) be a normal map, and by Theorem 8.1 we may assume f is q-connected, and $K_q(M)$ is a torsion group. Let p be the largest prime dividing order $K_q(M)$, and let $x\in K_q(M)$ be an element such that $(x)_p\neq 0$ in $K_q(M;Z_p)$. By Whitney's embedding theorem we may find an embedded S^q int M^{2q+1} representing x, and by Theorems 6.6 and 6.12, we may extend this embedding to an embedding $\phi:S^q\times D^{q+1}\rightarrow \mathrm{int}\ M$ such that ϕ defines a normal surgery on (f,b).

By Proposition 8.7, ϕ may be chosen so that order(torsion $K_q(M')$) $\leq$order(torsion $K_q(M)$), and $\mathrm{rank}_{Z_p} K_q(M';Z_p)<\mathrm{rank}_{Z_p} K_q(M;Z_p)$.

Proceeding in this fashion step by step, we will find after a finite number of such surgeries, a normal cobordism of (f,b) to (f_1,b_1) such that f_1 is q-connected, order(torsion $K_q(M_1)$)$\leq$order(torsion $K_q(M)$), and $\text{rank}_{Z_p} K_q(M_1;Z_p)=0$. Since the Universal Coëfficient Theorem holds for the K_*, K^* groups, $K_q(M_1;Z_p)\cong K_q(M_1)\otimes Z_p$, because $K_i(M_1)=0$ for $i<q$, and it follows that $K_q(M_1)$ is a torsion group of order prime to p, and order $K_q(M_1)\leq$order $K_q(M)$. Since $K_q(M)$ has p-torsion, it follows that, in fact, order $K_q(M_1)<$order $K_q(M)$. Hence we have reduced the order of the kernel, and so a finite number of iterations will make the order of the kernal zero, thus producing a normal cobordism of (f,b) with some $(\bar{f},\bar{b})$, where $\bar{f}$ is q-connected and $K_q(\bar{M})=0$. Hence $\bar{f}$ is actually (q+1)-connected, which proves Theorem 7.1 for $m\equiv 3 \pmod 4$. This also completes the proof of Theorem 7.1 for m odd.

§9. The Proof of the Fundamental Theorem for m even.

Set $m=2q$. Let (f,b) be a normal map with $f:(M,\partial M)\longrightarrow(A,B)$ such that $(f|\partial M)_*:H_*(\partial M)\longrightarrow H_*(B)$ is an isomorphism, and f is q-connected. Then $K_i(M)=0$ for $i<q$, and by Poincaré duality $K^{m-i}(M,\partial M)\cong K^{m-i}(M)=0$ for $i<q$. Since the K_* and K^* groups satisfy the Universal Coëfficient Theorem, it follows that $K_i(M)=0$ for $i>q$, and $K_q(M)$ is free. Let $x\in K_q(M)$ be represented by an embedding $\alpha:S^q\longrightarrow \text{int } M$, so that $(\beta,\alpha)\in\pi_{q+1}(f)$, and define $\bar{M}=M\cup_\alpha D^{q+1}$, $\bar{f}:\bar{M}\longrightarrow A$ extending f, defined using $\beta:D^{q+1}\longrightarrow A$. By Theorem 6.6, there is an obstruction $\mathcal{O}\in\pi_q(V_{k,q})$ (which is Z if q is even, Z_2 if q is odd) such that $\mathcal{O}=0$ if and only if $\bar{f}:\bar{M}\longrightarrow A$ can be thickened to a normal cobordism. Let $x'\in K^q(M,\partial M)$ be defined by $[M]\cap x'=x\in K_q(M)$. Recall that, as part of our definition above of the surgery invariant $\sigma(f,b)$, we defined a bilinear pairing $(\cdot,\cdot)$ on $K^q(M,\partial M)$, and made use of a quadratic form $\psi:K^q(M,\partial M;Z_2)\longrightarrow Z_2$.

9.1 Theorem: The obstruction $\mathcal{O}$ to thickening $\bar{f}:\bar{M}\to A$ to a normal cobordism is given by

$$\mathcal{O}=(x',x') \text{ if } q \text{ is even}, \qquad \mathcal{O}=\psi((x')_2) \text{ if } q \text{ is odd},$$

where $(\cdot)_2$ denotes reduction mod 2.

Before proving Theorem 9.1, we shall use it to complete the proof of Theorem 7.1.

Theorem 4.1 states that if (f,b) is normally cobordant rel B to a homotopy equivalence, then $\sigma(f,b)=0$. Thus, our intent is to assume that $\sigma(f,b)=0$, and then to construct a normal cobordism of (f,b) to a homotopy equivalence.

First, suppose q is even. Then $(f,b)=\frac{1}{8}I(f)$, so that if $(f,b)=0$, it follows that I(f), the signature of $(\cdot,\cdot)$ on $K^q(M,\partial M)$, is zero. By Theorem 6.13 we may assume that $K^i(M)\cong K^i(M,\partial M)=0$ for $i<q$, and is free for $i=q$. By Proposition 5.3, there is an $x'\in K^q(M,\partial M)$ such that $(x',x')=0$, so by 9.1, $[M]\cap x'=x\in K_q(M)$ can be represented by $\phi:S^q\times D^q\to \text{int } M$, (i.e. $\phi_*(\mu)=x$, μ the generator of $H_q(S^q\times D^q)$), such that the surgery based on ϕ defines a normal cobordism of (f,b). But we may choose x' to be indivisible (for otherwise, $x'=kx''$, where x'' is indivisible, and $(x',x')=0=(kx'',kx'')=k^2(x'',x'')$, so $(x'',x'')=0$), so the generator of a direct summand of $K^q(M,\partial M)$. Hence, by Corollary 7.12, rank $K_q(M')<$ rank $K_q(M)$, and f' is still q-connected, where $f':(M',\partial M')\to(A,B)$ results from normal surgery based on ϕ (in fact, the rank decreases by 2: see Lemma 8.2). Since (f,b) and (f',b') are normally cobordant, $I(f')=I(f)=0$ (see Theorem 5.14), and we may repeat the procedure. In fact, if we iterate the process until K_q is reduced to zero, the resulting map is (q+1)-connected, as desired.

Now take q odd. Then $\sigma(f,b)=c(f,b)$ is the Arf invariant of on $K^q(M,\partial M;Z_2)$. If $\sigma(f,b)=0$, then there is certainly some $y\in K^q(M,\partial M;Z_2)$ for which $\psi(y)=0$ (see for example Propsition 5.8 or 5.10). If f is q-connected, then $K^q(M,\partial M;Z_2)\cong K^q(M,\partial M)\otimes Z_2$, and $y=(x')_2$ for some indivisible $x'\in K^q(M,\partial M)$. By 9.1, $x=[M]\cap x'$ is represented by $\phi:S^q\times D^q\longrightarrow \text{int } M$ such that ϕ defines a normal cobordism, and by Corollary 7.12, rank $K_q(M')<$rank $K_q(M)$, with f' still q-connected. But $\sigma(f',b')=\sigma(f,b)=0$, since (f',b') is normally cobordant to (f,b), so we may proceed as above to produce a (q+1)-connected map. This completes the proof of Theorem 7.1, and hence of the Fundamental Theorem.

The balance of this section will be taken up by the proof of Theorem 9.1.

Let (f,b) be a normal map, $f:(M,\partial M)\longrightarrow(A,B)$, M is of dimension $m=2q$, and f is q-connected. Choose an $x\in K_q(M)$, and let it be represented by an embedding $\alpha:S^q\longrightarrow \text{int } M$. Let ζ^q denote the normal bundle of the image of α in M, and set $\bar{M}=M\cup_\alpha D^{q+1}$. Then f may be extended to $\bar{f}:\bar{M}\longrightarrow A$. Let $\mathcal{O}\in\pi_q(V_{k,q})$ be the obstruction to thickening $\bar{M}$ and $\bar{f}$ to a normal cobordism (cf. Theorem 6.6), and let $\partial:\pi_q(V_{k,q})\longrightarrow\pi_{q-1}(SO(q))$ be the connecting homomorphism in the exact sequence of the fibre bundle $p:SO(k+q)\longrightarrow V_{k,q}=SO(k+q)/SO(q)$, with fibre $SO(q)$.

We define the characteristic map of a k-plane bundle over a sphere as follows: let $\xi^k=(E,S^n,\pi)$ be a k-dimensional orientable vector bundle. If $S^n=\{(x_i)\in R^{n+1}\,|\,x_0^2+x_1^2+\ldots+x_n^2=1\}$, then we may define two subsets D^n_+ and D^n_- such that D^n_+ (resp. D^n_-) is the hemisphere centred on the N (resp. S) pole of S^n, i.e. $D^n_+=\{(x_i)\in S^n\,|\,x_n\geq 0\}$, and similarly for D^n_-. Clearly $S^n=D^n_+\cup D^n_-$, and it is easy to show that $D^n_+\cap D^n_-\cong S^{n-1}$ (the 'equator' of S^n).

Since the restrictions of ξ to D^n_+ and D^n_- are both trivial, we may choose trivialisations τ_+ and τ_- such that $\tau_+ : E|D^n_+ \to D^n_+ \times R^k$ (similarly for τ_-). Since τ_+ and τ_- are fibre isomorphic, the map $s_x : R^k \to R^k$ defined for each $x \in S^{n-1} \cong D^n_+ \cap D^n_-$ by $\tau_- \circ \tau_+^{-1}(x,y) = (x, s_x(y))$ is in fact an orientation-preserving linear transformation of R^k, i.e. $s_x \in SO(k)$. Thus, we have defined a map $c(\xi) : S^{n-1} \to SO(k)$ given by $c(\xi)(x) = s_x$. This is called the characteristic map of ξ, and although it is not unique, it is well-defined up to homotopy. (Thus it can be held that the characteristic 'map' is not really a map, but only an element of $\pi_{n-1}(SO(k))$.)

With ζ and $\mathcal{O}$ defined as above, we have

9.2 Proposition: $\partial\mathcal{O}$ is the characteristic map of ζ, an element of $\pi_{q-1}(SO(q))$.

Proof: Choose a base point $\mathcal{J}_0 \in SO(q+k)$, a $(q+k)$-frame in R^{q+k}. Let $p : SO(q+k) \to V_{k,q} \cong SO(q+k)/SO(q)$ be the projection, given by selecting the first k elements of a $(k+q)$-frame. Let $x_0 \in S^q$ be a base point such that, if $h : S^q \to SO(q+k)/SO(q) = V_{k,q}$ represents $\mathcal{O}$, then $h(x_0) = p(\mathcal{J}_0)$. Divide S^q into two cells, $S^q = D^q_+ \cup D^q_-$, so that $x_0 \in D^q_+ \cap D^q_- \cong S^{q-1} \cong \partial D^q_+ = \partial D^q_-$. Without loss of generality, we may assume that $h(D^q_-) = p(\mathcal{J}_0)$, since D^q_- is contractible. Let $\hat{h} : D^q_+ \to SO(q+k)$ be such that $\hat{h}(x_0) = \mathcal{J}_0$ and $p \circ \hat{h} = h$ on D^q_+. Then $p\hat{h}(S^{q-1}) = h(S^{q-1}) = p(\mathcal{J}_0)$, so that the first k elements of $\hat{h}(y)$ for $y \in S^{q-1}$ make up the base frame of $V_{k,q}$. Let $i : SO(q) \to SO(q+k)$ be the representation of $SO(q)$ acting on the subspace of R^{q+k} orthogonal to the space spanned by $p(\mathcal{J}_0)$. Then there is a map $\gamma : S^{q-1} \to SO(q)$ such that $\hat{h}(y) = \mathcal{J}_0(i \circ \gamma(y))$. By the definition of ∂, γ represents $\partial\mathcal{O} \in \pi_{q-1}(SO(q))$ (see [Steenrod 1951]).

Now ζ is the orhtogonal bundle to the trivial bundle spanned by $h(x)$, for $x \in S^q$. Since $h(D^q_-) = p(\mathcal{J}_0)$, the last q vectors in $\mathcal{J}_0$ give a trivialisation of ζ over D^q_-, and since $p \circ \hat{h} = h$, the last q vectors of $h(x)$, for $x \in D^q_+$, give a trivialisation of ζ over D^q_+. Since $\gamma(y)$, for $y \in S^{q-1}$, sends the last part of $\mathcal{J}_0$ into the last part of $\hat{h}(y)$, it follows that γ is $c(\zeta)$, the characteristic map of ζ (see [Steenrod 1951, (18.1)]). QED

From our discussion above of the homotopy properties of $SO(n)$, we derive the following

9.3 <u>Proposition</u>: The boundary $\partial : \pi_q(V_{k,q}) \to \pi_{q-1}(SO(q))$ is a monomorphism for $q \neq 1, 3$, or 7.

<u>Proof</u>: By comparing various related fibre bundles, we produce the following commutative diagram:

$$\begin{tikzcd}
SO(q) \arrow[r] \arrow[d, "i_1"'] & SO(q) \arrow[r, "p'"] \arrow[d, "i_2"'] & SO(q)/SO(q-1) = S^{q-1} \arrow[d, "i_3"'] \\
SO(q+1) \arrow[r, "j"] \arrow[d, "p_1"'] & SO(q+k) \arrow[r] \arrow[d, "p_2"'] & V_{k+1,q-1} \arrow[d, "p_3"'] \\
S^q = V_{1,q} \arrow[r, "j'"] & V_{k,q} \arrow[r] & V_{k,q}
\end{tikzcd}$$

where the p_j are the projections of fibre bundles, and i_j are inclusions of fibres. Let ∂_j be the connecting homomorphism in the homotopy exact sequence of the bundle with projection p_j. By Lemma 6.9, if q is even, $p'_{\#}\partial_1 : \pi_q(S^q) \to \pi_{q-1}(S^{q-1})$ is multiplication by two, and is thus injective. But by the commutativity of the diagram, $p'_{\#} \circ \partial_1 = \partial_3 \circ j'_{\#}$. Hence $j'_{\#}$ is a monomorphism, and since by Theorem 6.12 $\pi_q(V_{k,q}) = Z$ if q is even, it follows that $\partial_3 = \partial$ is a monomorphism if q is even.

If $q \neq 1, 3$, or 7, and q is odd, then by Corollary 6.11 $\ker i_{\#} = Z_2$, where $i_{\#} : \pi_{q-1}(SO(q)) \to \pi_{q-1}(SO(q+1))$. Hence ∂_1 is onto $Z_2 \subset \pi_{q-1}(SO(q))$, and since $j'_{\#}:\pi_q(S^q)\to\pi_q(V_{k,q})$ is surjective by Theorem 6.12, $\partial_1=\partial_3\circ j'_{\#}$, it follows that $\partial_3(\pi_q(V_{k,q}))\supseteq Z_2$. Since $\pi_q(V_{k,q})=Z_2$ for q odd (by 6.12), we have $\partial_3=\partial$ a monomorphism for $q\neq 1,3$, or 7. QED

Thus for $q\neq 1,3$, or 7, the obstruction $\mathcal{O}$ to doing normal surgery on a particular S^q embedded in M^{2q} can be identified with the characteristic map of ζ, the normal bundle of the chosen S^q in M, $\mathcal{O}\in\ker i_{\#}\subseteq\pi_{q-1}(SO(q))$, and is therefore zero if ζ is trivial. Now $\ker i_{\#}$ is generated by $\partial_1(\iota)$, where $\iota\in\pi_q(S^q)$ is the class of the identity, so that $\partial_1(\iota)$ is the characteristic map for the tangent bundle τ of S^q. It follows that $\mathcal{O}=\lambda(\partial_1(\iota))$ for some $\lambda\in Z$.

If q is even, the Euler class $\chi(\tau)=2g\in H^q(S^q)$, where g is the generator for which $g[S^q]=1$. This follows from the general formula $\chi(\tau_M)=\chi(M)g$; or may be deduced for $M=S^q$, q even, using the fact that τ_M is equivalent to the normal bundle of the diagonal M in $M\times M$. For if $U\in H^q(E,E_0)$ is the Thom class, it follows from Corollary 7.6 that $[S^q\times S^q]\cap\eta^*U=[S^q]\otimes 1+1\otimes[S^q]$, the homology class of the diagonal, where $\eta:S^q\times S^q\to E/E_0$ is the natural collapsing map. Hence $\eta^*U=g\otimes 1+1\otimes g$, and $\eta^*(U^2)=(\eta^*U)^2=(g\otimes 1+1\otimes g)^2=2g\otimes g$, if q is even. Since η^* is an isomorphism on H^{2q}, it follows that $U^2=2gU$, so $\chi(\tau)=2g$, since by definition $\chi(\xi)U_\xi=(U_\xi)^2$ for a bundle ξ.

The Euler class is represented by the universal Euler class $\chi\in H^q(BSO(q))$, where $BSO(q)$ is the classifying space for oriented q-plane bundles (see [Husemoller 1966] or [Steenrod 1951]). That is, if $c:X\to BSO(q)$ is the classifying map of a q-plane bundle ξ over X, $c^*(\gamma)=\xi$, where γ is the universal q-plane bundle over $BSO(q)$, then $\chi(\xi)=c^*(\chi)$. If $c:S^q\to BSO(q)$ represents τ_{S^q}, then $c^*(\chi)=2g$ as above, but if $c':S^q\to BSO(q)$ represents $\lambda(\tau_{S^q})$ in the homotopy group $\pi_{q-1}(SO(q))$, then λc and c' are homotopic, i.e. $[\lambda c]=[c']$ in $\pi_q(BSO(q))$. Hence $c'^*=\lambda c^*$, so we have:

9.4 Lemma: If q is even and $\partial_2 \mathcal{O}=\lambda\partial_1(\iota)$, then $\chi(\zeta)=2\lambda g$, where ζ is the normal bundle of $\alpha(S^q)$ in M^{2q}, representing an element in $K_q(M)$, $\mathcal{O}$ the obstruction to doing a normal surgery on this S^q.

9.5 Lemma: $\chi(\zeta)[S^q]=(x',x')$, where $[M]\cap x'=x$, $\alpha:S^q\to M^{2q}$ is an embedding representing $x\in K_q(M)$, ζ the normal bundle of $\alpha(S^q)$, as above.

Proof: $\chi(\zeta)U=U^2$ by definition of χ, where $U\in H^q(E(\zeta)/E_0(\zeta))$ is the Thom class. Clearly $(\chi(\zeta))[S^q]=(\chi(\zeta)U)[E]=U^2[E]=(\eta^*U)^2[M]$, where $[E]\in H_{2q}(E(\zeta)/E_0(\zeta))$ is the orientation class, so $[E]=\eta_*[M]$, where $\eta:M/\partial M\to E/E_0$ is the natural collapsing map.

By Corollary 7.6, $[M]\cap\eta^*U=x$, so that $\eta^*U=x'$. Hence

$$\chi(\zeta)[S^q]=(\eta^*U)^2[M]=(x')^2[M]=(x',x'). \qquad \text{QED}$$

By 9.4 and 9.5 for q even, $(x',x')=2\lambda$ where $\partial_2\mathcal{O}=\lambda\partial_1(\iota)$. By 9.3 ∂_2 is a monomorphism for q even, so we may identify $\mathcal{O}$ with (x',x'), which proves Theorem 9.1 for q even.

Finally, we turn our attention to the case of q odd.

Let $\alpha_i:S^q\to M^{2q}$, $i=1,2$, be embeddings representing $x_i\in K_q(M)$, where, as usual, $K_q(M)$ is defined using a normal map (f,b), $f:(M,\partial M)\to(A,B)$, $(f|\partial M)_*:H_*(\partial M)\to H_*(B)$ an isomorphism. Suppose the α_i have disjoint images, and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the obstructions to doing normal surgery on $\alpha_1(S^q)$ and $\alpha_2(S^q)$ respectively. Join $\alpha_1(S^q)$ to $\alpha_2(S^q)$ by an arc, disjoint (except, of course, at its endpoints) from both images. By thickening this to a tube $T\cong D^q\times[1,2]$ we may take

$$(\alpha_1(S^q)\setminus(D^q\times 1))\cup\partial_0 T\cup(\alpha_2(S^q)\setminus(D^q\times 2)),$$

where $\partial_0 T=\partial D^q\times[1,2]$, $D^q\times i=T\cap\alpha_i(S^q)$. This subset of M is homeomorphic to S^q, and so gives us an embedding $\alpha:S^q\to M$ representing x_1+x_2, which can be made differentiable by 'rounding the corners'.

9.6 Lemma: $O=O_1+O_2$ in $\pi_q(V_{k,q})$, where O is the obstruction to doing surgery on $\alpha(S^q)$.

Proof: Since $T\subset M$, we may multiply T by $[0,\varepsilon]$ to obtain $T\times[0,\varepsilon]\subset M\times I$. If we have $M\subset D^{m+k}$, then $M\times I\subset D^{m+k}\times I$, and by composing embeddings we produce $T\times[0,\varepsilon]\subset D^{m+k}\times I$. Choose $D_i^{q+1}\subset D^{m+k}\times I$ such that $\alpha_i(S^q)=\partial D_i^{q+1}$, and D_i^{q+1} meets $D^{m+k}\times 0$ transversally in $\alpha_i(S^q)$. Then we may assume that a neighbourhood of $\alpha_i(S^q)$ in D_i^{q+1} is given by $\alpha_i(S^q)\times[0,\varepsilon]$. Set $D^{q+1}=\{D_2^{q+1}\setminus(D^q\times 1\times[0,\varepsilon])\}\cup\{(\partial D^q\times[1,2]\times[0,\varepsilon])\cup(D^q\times[1,2]\times\varepsilon)\}$

$$\cup\{D_2^{q+1}\setminus(D^q\times 2\times[0,\varepsilon])\}.$$

This is a (q+1)-cell meeting $D^{m+k}\times 0$ transversally in $\alpha(S^q)$, and we may smooth this D^{q+1}, together with $\alpha(S^q)$, by 'rounding corners'. The smoothed D^{q+1} is the union of three cells, $D^{q+1}=A_1\cup B\cup A_2$, which correspond to the three expressions in braces, in the expression for D^{q+1} above, after closure and smoothing. Assume $A_i\subset D_i^{q+1}$. Then $C_i=D_i\setminus\operatorname{int} A_i$ is a (q+1)-cell, $\partial C_i\cap\partial D_i=F_i$, F_i a q-cell in ∂D_i, $B\cap A_i=\partial C_i\cap A_i\subset\partial B$ and $\partial B\setminus((\partial C_1\cap A_1)\cup(\partial C_2\cap A_2))=S^{q-1}\times I$.

Since the definition of the obstruction O doesn't depend on the choice of the framing of the normal bundle γ of D^{q+1}, we may assume that the framings over D^{q+1}, D_1^{q+1}, and D_2^{q+1} have been chosen so that the framings over D^{q+1} and D_i^{q+1} coincide over A_i. Further we may assume that the framings of ν, the normal bundle of M in D^{m+k}, over $\alpha(S^q)$, $\alpha_1(S^q)$, and $\alpha_2(S^q)$, induced by b, have been chosen so that over F_i they are all the same, coming from a framing of $\nu|T$ (note that T is a cell), and the framings of γ, γ_1, and γ_2 may be assumed to extend that of ν over $T\cap\alpha(S^q)$, $T\cap\alpha_i(S^q)$ (as is appropriate). Thus the three maps β,β_i, $i=1,2$, $\beta:\alpha(S^q)\to V_{k,q}$, $\beta_i:\alpha_i(S^q)\to V_{k,q}$ defining O and O_i, may be taken to be the base k-frame over $T\cap\alpha(S^q)$,

$T\cap\alpha_i(S^q)$, and $\beta|(\alpha_i(S^q)\cap\alpha(S^q))=\beta_i|(\alpha_i(S^q)\cap\alpha(S^q))$. It follows that for the homotopy classes, $[\beta]=[\beta_1]+[\beta_2]$ in $\pi_q(V_{k,q})$, or $\mathcal{O}=\mathcal{O}_1+\mathcal{O}_2$. QED

9.7 Lemma: If $\mathcal{O}=0$, then $\psi((x')_2)=0$, with notation as above.

Proof: Since $\mathcal{O}=0$, we can perform normal surgery based on $\alpha:S^q\to M^{2q}$, so that the trace is a normal cobordism W^{2q+1}, $W=M\cup(\partial M\times I)\cup M'$, and if $i:\partial W\to W$ and $k:M\to\partial W$ are inclusions, $i_*k_*x=0$. It follows from elementary results about K_* and K^* (see p. 21 above), that $x''=i^*z$, $z\in K^q(W)$, where $x''\in K^q(\partial W)$ is defined by $[\partial W]\cap x''=k_*x$, and $K^q(W)$ comes from the map $F:W\to A\times I$ extending f on M.

If $K^q(\partial W;Z_2)$ is defined for the map $\partial F:\partial W\to A\times 0\cup B\times I\cup A\times 1$, and ψ_0 is the quadratic form $K^q(\partial W;Z_2)\to Z_2$ used in the definition of the Kervaire invariant, it follows from a lemma in [Browder 1972, III.4.13] that $\psi_0((i^*z)_2)=\psi((x'')_2)=0$. Now ∂F is clearly the sum of (f,b) on M and (f',b') on M' (the result of surgery). By an intermediate result in the proof of Theorem 5.12, $\psi_0(\eta^*(x')_2)=\psi((x')_2)$, $x'\in K^q(M,\partial M)$, so it remains to show that $\eta^*(x')_2=(x'')_2$ (where $\eta:\partial W\to M/\partial M$).

Consider $k_*x=k_*([M]\cap x')=k_*(\eta_*[\partial W]\cap x')=[\partial W]\cap\eta^*x'$, using identities of the cap product (cf. Corollary 7.6), so that since $[\partial W]\cap x''=k_*x$, it follows that $x''=\eta^*x'$, and hence $\psi((x')_2)=0$. QED

Now we prove that $\mathcal{O}=\psi((x')_2)$. If $\mathcal{O}=0$, then $\psi((x')_2)=0$ by 9.7, so it remains to show that if $\mathcal{O}=1$ then $\psi((x')_2)=1$.

By taking the connected sum with the map $S^q\times S^q\to S^{2q}$, or alternately doing a normal surgery on a $S^{q-1}\subset D^{2q}\subset M^{2q}$, we may add to $K_q(M)$ the free module on two generators a_1 and a_2, corresponding to $[S^q]\otimes 1$ and $1\otimes[S^q]$ in $H^q(S^q\times S^q)$, and add to $K^q(M,\partial M)$ the elements g_1,g_2 such that $[M\#(S^q\times S^q)]\cap g_i=a_i$, with $(g_1,g_2)=1$, $(g_i,g_i)=0$, $i=1,2$, orthogonal to the original $K^q(M,\partial M)$, and such that $\psi(g_1)=\psi(g_2)=0$. Hence

Hence $\psi(g_1+g_2)=\psi(g_1)+\psi(g_2)+(g_1,g_2)=1$.

If $\beta:S^q\hookrightarrow M\#(S^q\times S^q)$ represents the diagonal class a_1+a_2, it follows from 9.7 that the obstruction $\mathcal{O}$ to surgery on β is 1, since if it were zero, then $\psi(g_1+g_2)$ would be zero. Then on the sum embedding $\alpha+\beta$ representing $x+(a_1+a_2)$, the obstruction $\mathcal{O}''=\mathcal{O}+\mathcal{O}'$ by Lemma 9.6, so that $\mathcal{O}''=1+1=0$. Hence $\psi((x')_2+(g_1+g_2))=0$ by 9.7. But since $((x')_2,(g_1+g_2))=0$,

$$\psi((x')_2+(g_1+g_2))=\psi((x')_2)+\psi(g_1+g_2)=\psi((x')_2)+1=0,$$

we see that $\psi((x')_2)=1$. QED

This completes the proof of Theorems 9.1 and 7.1, and thus of the Fundamental Theorem.

Chapter III. Plumbing and the Classification of Manifolds.

§10. Intersection and Plumbing.

Let N_1 and N_2 be smooth submanifolds of dimension p (resp. q) of a smooth m-manifold M, such that p+q=m. A point $x \in N_1 \cap N_2$ will be called discrete if there is an open neighbourhood V of x in M such that $V \cap N_1 \cap N_2 = \{x\}$. Note that if every point in $N_1 \cap N_2$ is discrete, then $N_1 \cap N_2$ is a discrete subset of M.

If $x \in N_1 \cap N_2$ is discrete and V is as above (i.e. V is open in M and $V \cap N_1 \cap N_2 = \{x\}$) then $(V \backslash N_1) \cup (V \backslash N_2) = V \backslash \{x\}$. Thus we have a pairing $H^q(V, V\backslash N_1) \otimes H^p(V, V\backslash N_2) \to H^{p+q}(V, V\backslash\{x\})$ given by the relative cup product.

Suppose that M, N_1 and N_2 are oriented, and let $[M]_x \in H_m(M, M\backslash\{x\})$, $[N_1]_y \in H_p(N_1, N_1\backslash\{y\})$ and $[N_2]_z \in H_q(N_2, N_2\backslash\{z\})$ be the generators compatible with the orientations. Let E_i, i=1,2, be a tubular neighbourhood of N_i in M, $E_i^0 = E_i \backslash N_i$. Then the inclusion $(E_i, E_i^0) \subset (M, M\backslash N_i)$ is an excision, so $H^*(M, M\backslash N_i) \cong H^*(E_i, E_i^0)$. If the E_i are oriented, and r_i denotes the inclusion $(V, V\backslash N_i) \subset (E_i, E_i^0)$, then by the Thom Isomorphism Theorem there is an element $U_1 \in H^q(E_1, E_1^0)$ such that $r_1^* U_1 \in H^q(V, V\ N_1)$ is a generator, and $\cdot \cup U_1$, $\cdot \cap U_1$ are isomorphisms (similarly for N_2). We shall also assume that the orientations are compatible, i.e. so that $[M]_x \cap r_i^* U_i = [N_i]_x$ for $x \in N_i$.

Under the preceding conditions we may define the sign or orientation of a discrete point $x \in N_1 \cap N_2$ by $\operatorname{sgn}(x) = (r_1^* U_1 \cup r_2^* U_2)[M]_x$, using the pairing above. We shall call x a (homologically) transverse point of intersection if $\operatorname{sgn}(x) = \pm 1$. Note that geometrically transverse points are also homologically transverse. (A point $x \in N_1 \cap N_2$ is geometrically transverse if x has an open neighbourhood V in M such that there is a diffeomorphism

$(V, V\cap N_1, V\cap N_2) \to (R^m, R^p\times 0, 0\times R^q)$.)

If N_1 is compact and $N_1\cap N_2\cap\partial M$ is empty, it has been shown that given an $\varepsilon>0$ there is a diffeomorphism $h:M\to M$, which is the identity on ∂M, and is ε-isotopic to 1_M, such that $h(N_1)\cap N_2$ consists solely of (geometrically) transverse points.

On p.50 above we defined a pairing $\cdot : H_p(M)\otimes H_q(M)\to Z$ by

$x\cdot y=(x'\cup y')[M]$, where $x'\in H^q(M,\partial M)$, $y'\in H^p(M)$ are defined by $[M]\cap x'=x$, $[M]\cap y'=j_*y$, and j is inclusion.

Let N_1^p, N_2^q be compact oriented submanifolds of M^m, a compact oriented manifold with boundary, $m=p+q$, and suppose N_1 is closed in M, $\partial M\cap N_1=\phi$, and $\partial M\cap N_2=\partial N_2$. Assume further that N_1 and N_2 intersect (homologically) transversally. Let $i_j:N_j\to M$ denote the inclusions. We state without proof the following theorem from [Browder 1972].

10.1 <u>Theorem</u>: $(i_{1*}[N_1])\cdot(i_{2*}[N_2])=\Sigma\,\mathrm{sgn}(x)$, where the sum is taken over all points $x\in N_1\cap N_2$.

Thus, the intersection of the orientation classes counts the number of intersection points, with sign.

If N^q is a closed submanifold lying in the interior of M^{2q}, with normal bundle ζ^q, then we may consider how N intersects itself. It is possible (see above) to change N by an ε-isotopy so that it intersects itself transversally. Then Theorem 10.1 gives us: $i_*[N]\cdot i_*[N]=\Sigma\,\mathrm{sgn}(x)$, the sum running over the points of self-intersection. However, we can also interpret this result using the normal bundle ζ:

10.2 <u>Proposition</u>: $i_*[N]\cdot i_*[N]=\chi(\zeta)[N]$, where $\chi(\zeta)$ is the Euler class of ζ.

We are now prepared to describe the construction known as plumbing disc bundles.

Let ζ_i be a q-plane bundle over a smooth q-manifold N_i, and let E_i

be the total space of the closed disc bundle associated to ζ_i. Suppose that ζ_i, E_i and N_i are oriented compatibly for $i=1,2$.

Choose $x_i \in N_i$ and $B_i \subset N_i$ a q-cell with $x_i \in \operatorname{int} B_i$. Since B_i is contractible, $\zeta_i | B_i$ is trivial, and that part of E_i lying over B_i is diffeomorphic to $B_i \times D_i$, where D_i is a q-disc, such that the fibres are mapped to $x \times D_i$. We may choose diffeomorphisms

$$h_-, h_+ : B_1 \to D_2, \qquad k_-, k_+ : D_1 \to B_2,$$

where a subscripted + indicates orientation-preserving, and a − indicates orientation-reversing.

We plumb E_1 with E_2 at x_1 and x_2 by identifying the subsets of the disjoint union $E_1 \sqcup E_2$ given by $B_1 \times D_1$ and $B_2 \times D_2$ using the map $I_+(x,y) = (k_+ y, h_+ x)$ or the map $I_-(x,y) = (k_- y, h_- x)$. We shall say that the plumbing is with sign +1 if I_+ is used, and with sign −1 if I_- is used. The resulting manifold is denoted by $E_1 \square E_2$, and it can be smoothed in a canonical way.

Since both of I_+ and I_- preserve orientation if q is even, and reverse it if q is odd, $E_1 \square E_2$ can be oriented compatibly with N_1, ζ_1, N_2, and ζ_2 if q is even, and with $N_1, \zeta_1, -N_2$, and ζ_2 if q is odd. Note that $N_i \subset E_i \subset E_1 \square E_2$, where the inclusions are obvious, and that $N_1 \cap N_2 = \{x_1\} = \{x_2\}$ (in $E_1 \square E_2$), which is a transversal intersection, and that the sign of x is the same as the sign of the plumbing. (Of course, all of this discussion can be applied to the case of plumbing one manifold with itself, if we choose two distinct points in it and take $E_1 = E_2$.)

If we choose several pairs of points in N_1 and N_2, we may plumb E_1 and E_2 together repeatedly, choosing the sign of each plumbing. We will still denote the result by $E_1 \square E_2$, and we see from 10.1 that

$i_{1*}[N_1]\cdot i_{2*}[N_2]$ is determined by the way we choose the sign of the plumbings. Thus, if we choose a number n_{12}, and plumb E_1 with E_2 at n_{12} points, always with sign +1, then we have $i_{1*}[N_1]\cdot i_{2*}[N_2]=n_{12}$. We may go on to plumb with other disc bundles, by making sure that the points in $N_1 \cup N_2$ we choose to plumb at are well away from the finite number of points in $N_1 \cap N_2$, and by choosing the signs of the plumbings, we may cause $i_*[N_j]\cdot i_*[N_k]=n_{jk}$, $j\neq k$, to take on any value we like. (Note that we must have $n_{kj}=(-1)^q n_{jk}$.) The self-intersections are determined by the Euler class $\chi(\zeta_i)$, according to Proposition 10.2. Thus, we arrive at the remarkable

10.3 Theorem: Let M be a symmetric n×n matrix with integer entries, and with even diagonal entries. Then for k>1 there is a manifold W^{4k} with boundary such that W is (2k-1)-connected, ∂W is (2k-2)-connected, $H_{2k}(W)$ is free abelian, the matrix of the intersection pairing $H_{2k}(W)\otimes H_{2k}(W)\to Z$ is given by M (or equivalently, M is the matrix of the bilinear form $(\cdot,\cdot)$ on $H^k(W,\partial W)$), and there is a normal map (f,b), with $f:(W,\partial W)\to(D^{4k},S^{4k-1})$ for which M is the intersection matrix on $K_{2k}(W)$.

The proof is provided in detail in [Browder 1972].

We have from the same source the

10.4 Lemma: In the construction of 10.3, ∂W is a homotopy sphere if and only if the determinant of M is ±1.

Consider the following 8×8 matrix due to Hirzebruch:

$$M_0=\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & 0 & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & 0 & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{pmatrix}$$

This matrix is, as required, symmetric and even on the diagonal. Simple computation shows that $|M_0|=1$ and that the signature of M_0 is 8.

We may quickly prove the following theorem of Milnor.

10.5 <u>Theorem</u>: Let $k>1$. There is a manifold W and a normal map (f,b), $f:(W,\partial W)\to(D^{4k},S^{4k-1})$ such that $(f|\partial W)$ is a homotopy equivalence, and $\sigma(f,b)=1$.

<u>Proof</u>: Let W be the $4k$-manifold with boundary constructed in Theorem 10.3 using the matrix M_0. Since $|M_0|=1$, we have by 10.4 that ∂W is a homotopy sphere. By 10.3, the bilinear form $(\cdot,\cdot)$ on $K^{2k}(W,\partial W)$ has matrix M_0, and $\operatorname{sgn} M_0=8$. Thus, if (f,b) is the normal map of 10.3, it follows that $\sigma(f,b)=\frac{1}{8}I(f)=\frac{1}{8}\operatorname{sgn} M_0=1$. QED

A somewhat different construction in dimensions congruent to 2 mod 4 gives us the following theorem of Kervaire.

10.6 <u>Theorem</u>: For q odd there is a manifold U and a normal map (g,c) such that $g:(U,\partial U)\to(D^{2q},S^{2q-1})$ with $\sigma(g,c)=1$.

Taking Theorems 10.5 and 10.6 together with Proposition 5.35 (the Addition Property of σ), we derive immediately the Plumbing Theorem:

10.7 <u>Theorem</u>: If $m=2k>4$, then there is an m-manifold M with boundary, and a normal map (g,c), $g:(M,\partial M)\to(D^m,S^{m-1})$, $c:\nu^k\to\varepsilon^k$ (where ε^k is the trivial bundle over D^m), with $g|\partial M$ a homotopy equivalence and with $\sigma(g,c)$ taking on any desired value.

§11. <u>The Homotopy Types of Smooth Manifolds and Classification.</u>

It has been shown by [Browder 1962] and [Novikov 1964] that certain necessary conditions for a space to be of the homotopy type

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It has been shown by [Browder 1962] and [Novikov 1964] that certain necessary conditions for a space to be of the homotopy type of a smooth manifold are sometimes also sufficient.

In the theorem we will use the following notation: $h:\pi_i \to H_i$ is the Hurewicz homomorphism, ξ is an oriented k-plane bundle over a space X, $U \in H^k(T(\xi))$ is its Thom class, p_i are its Pontrjagin classes, and L_k are the Hirzebruch polynomials.

11.1 Theorem: Let X be a simply-connected Poincaré complex of dimension $m \geq 5$, ξ an oriented k-plane bundle over X, $k > m+1$, $\alpha \in \pi_{m+k}(T(\xi))$ such that $h(\alpha) \cap U = [X]$. If (1) m is odd, or

(2) $m=4k$ and index $X = (L_k(p_1, p_2, \ldots, p_k))[X]$,

then there is a homotopy equivalence $f:M \to X$, for some smooth m-manifold M, such that $\nu = f^*(\xi)$ is the normal bundle of an embedding $M \subset S^{m+k}$, and f can be found in the normal cobordism class represented by α.

Outline of Proof: A representative $\bar{f}:S^{m+k} \to T(\xi)$ of α is chosen, and the manifold M is defined by pulling X back to a submanifold of S^{m+k} via $\bar{f}$ (after some modifications). The map $\bar{f}$ induces a normal map (f,b) with $f:M \to X$, $b:\nu \to \xi$. Then by the Fundamental Theorem of Surgery (4.2), (f,b) is normally cobordant to a homotopy equivalence if m is odd, and if $m=2q$ then (f,b) is normally cobordant to a homotopy equivalence if and only if $\sigma(f,b)=0$. But if $m=4k$, then by the Index Property (Proposition 5.35), $\sigma(f,b) = (L_k(p_1, \ldots, p_k))[X]$-index X, which is zero when (2) holds. QED

Remark: If m=6,14,30, or 62 (none of which are covered by 11.1), then with the above hypotheses there is a homotopy equivalence $f:M \to X$ with $f^*(\xi) = \nu$, but f may not be normally cobordant to a map representing α.

We have defined above the connected sum of Poincaré complexes for the purpose of the Addition Property. Given Poincaré pairs (X_i, Y_i), k-plane bundles ξ_i over X_i, smooth manifolds $(M_i, \partial M_i)$, and normal maps (f_i, b_i) such that $f_i:(M_i, \partial M_i) \to (X_i, Y_i)$, we have the Poincaré pair $(X_1 \# X_2, Y_1 \amalg Y_2)$, the smooth manifold $M_1 \# M_2$ with boundary $\partial M_1 \amalg \partial M_2$, and the normal map $(f_1 \# f_2, b_1 \# b_2)$ such that $f_1 \# f_2:(M_1 \# M_2, \partial M_1 \amalg \partial M_2) \to (X_1 \# X_2, Y_1 \amalg Y_2)$ and $b_1 \# b_2: \nu_{\#} \to \xi_1 \# \xi_2$, where $\nu_{\#}$ is the normal bundle of $M_1 \# M_2$ in D^{m+k}.

If M_i and Y_i are all nonempty, we may define the connected sum along (components of) the boundary. See [Browder 1972] for details. We produce analogous constructs: $M_1 \natural M_2$, $X_1 \natural X_2$, and maps $f_1 \natural f_2$, $b_1 \natural b_2$. Note that $\partial(M_1 \natural M_2) = \partial M_1 \# \partial M_2$, and that $(X_1 \natural X_2, Y_1 \# Y_2)$ form a Poincaré pair. Then $(f_1 \natural f_2, b_1 \natural b_2)$ is a normal map.

11.2 <u>Proposition</u>: Let $(f,b),(g,c)$ be normal maps with $f:(M,\partial M) \to (X,Y)$, $g:(N,\partial N) \to (D^m, S^{m-1})$. Then $(f \natural g, b \natural c)$ is normally cobordant to (f,b).

This proposition together with previous results leads to the

11.3 <u>Theorem</u>: Let (X,Y) be a m-dimensional Poincaré pair with X simply-connecetd and Y nonempty, $m \geq 5$, and let (f,b) be a normal map with $f:(M,\partial M) \to (X,Y)$ and $(f|\partial M)_*$ an isomorphism. Then there is a normal map (g,c), $g:(U,\partial U) \to (D^m, S^{m-1})$ with $g|\partial U$ a homotopy equivalence, such that $(f \natural g, b \natural c)$ is normally cobordant rel Y to a homotopy equivalence. In particular, (f,b) is normally cobordant to a homotopy equivalence.

<u>Proof</u>: By the Plumbing Theorem (10.7) there is a (g,c) as above with $\sigma(g,c) = -\sigma(f,b)$. By the Addition Property, Proposition 5.35, $\sigma(f \natural g, b \natural c) = \sigma(f,b) + \sigma(g,c) = 0$, so by the Fundamental Theorem (4.2) $(f \natural g, b \natural c)$ is normally cobordant rel Y to (f',b'), where $f':M' \to X$ is a homotopy equivalence. (Note that $(X \natural D^m, Y \# S^{m-1}) \cong (X,Y)$). Then 11.2 shows that (f,b) is normally cobordant to (f',b'). QED

Recall that a cobordism W between M and M' (i.e. $\partial W = M \cup U \cup M'$, $\partial M \subset U$, $\partial M' \subset U$) is an h-cobordism if the inclusions $M \subset W$, $M' \subset W$, $\partial M \subset U$, and $\partial M' \subset U$ are all homotopy equivalences.

With this definition we can state the classification theorem of Novikov, and its corollary.

11.4 <u>Theorem</u>: Let X be a simply-connected Poincaré complex of dimension $m \geq 4$, and (f_i, b_i) for $i=0,1$, be normal maps with $f_i : M_i \longrightarrow X$, where M_i is a smooth m-manifold. Suppose that f_0 and f_1 are homotopy equivalences. If f_0 is normally cobordant to f_1, then there is a normal map (g,c) with $g:(U, \partial U) \longrightarrow (D^{m+1}, S^m)$, where $g|\partial U$ is a homotopy equivalence, such that (f_0, b_0) is h-cobordant to $(f_1 \; g|\partial U, b_1 \; c|\partial U)$. In particular, M_0 is h-cobordant to M_1 if m is even, and to $M_1 \# (\partial U)$ if m is odd.

11.5 <u>Corollary</u>: Let M and M' be closed smooth simply-connected manifolds of dimension not less that 5. A homotopy equivalence $f:M \longrightarrow M'$ is homotopic to a diffeomorphism $f':M\#\Sigma \longrightarrow M'$ for some homotopy sphere $\Sigma = \partial U$, U parallelisable (thus M is homeomorphic to $M\#\Sigma$) if and only if there is a bundle map $b: \nu \longrightarrow \nu'$ covering f such that $T(b)_*(\alpha) = \alpha'$, where α, α' are the natural collapsing maps $\alpha \in \pi_{m+k}(T(\nu))$, $\alpha' \in \pi_{m+k}(T(\nu'))$.

Finally we have a theorem of Wall and its corollary.

11.6 <u>Theorem</u>: Let (X,Y) be a Poincaré pair of dimension $m \geq 6$, with both X and Y simply-connected, Y nonempty. Let ξ be a k-plane bundle over X, and choose $\alpha \in \pi_{m+k}(T(\xi), T(\xi|Y))$ such that $h(\alpha) \cap U = [X]$. Then the normal map represented by α is normally cobordant to a homotopy equivalence (f,b), $f:(M, \partial M) \longrightarrow (X,Y)$, which is unique up to h-cobordism. In particular, (X,Y) has the homotopy type of a differentiable manifold, unique up to h-cobordism in the given normal cobordism class.

We will prove the existence part of this theorem. The proof of uniqueness (as well as the other proofs omitted from this section) is to be found in [Browder 1972, II.3].

Proof: Let (f',b') with $f':(M',\partial M')\to(X,Y)$ be a normal map representing α. By the Cobordism Property, 5.36, $\sigma(f'|\partial M',b'|\partial M')=0$, so that by the Fundamental Theorem (4.2) $(f'|\partial M',b'|\partial M')$ is normally cobordant to a homotopy equivalence. This normal cobordism extends to a normal cobordism of (f',b') to some (f'',b'') such that $f''|\partial M''$ is a homotopy equivalence. By Theorem 11.3, (f'',b'') is normally cobordant to a homotopy equivalence, (f,b).

11.7 Corollary: Let M and M' be compact smooth simply-connected manifolds of dimension $m\geq 6$, with ∂M and $\partial M'$ simply-connected and nonempty. Then a homotopy equivalence $f:(M,\partial M)\to(M',\partial M')$ is isotopic to a diffeomorphism $f':M\to M'$ if and only if there is a bundle map $b:\nu\to\nu'$ covering f such that $T(b)_*(\alpha)=\alpha'$, where ν,ν' are the normal bundles, and $\alpha\in\pi_{m+k}(T(\nu),T(\nu|\partial M))$, $\alpha'\in\pi_{m+k}(T(\nu'),T(\nu'|\partial M'))$ are the collapsing maps.

Bibliography

Browder,W.: Homotopy type of differentiable manifolds. Proceedings of the Aarhus Symposium, 1962, 42-46.

——— Surgery on simply-connected manifolds. Berlin-Heidelberg-New York: Springer 1972.

Hirzebruch,F.: New topological methods in algebraic geometry. 3rd Ed. Berlin-Heidelberg-New York: Springer 1966.

Husemoller,D.: Fibre bundles. New York: McGraw Hill 1966.

Kervaire,M.: An interpretation of G. Whitehead's generalisation of the Hopf invariant. Ann. Math. 69 (1959), 345-364.

——— Milnor,J.: Groups of homotopy spheres I. Ann. Math. 77 (1963), 504-537.

Milnor,J.: On manifolds homeomorphic to the 7-sphere. Ann. Math. 64 (1956), 399-405.

——— A procedure for killing the homotopy groups of differentiable manifolds. Symposia in Pure Math., Amer. Math. Soc. 3 (1961), 39-55.

——— Lectures on the h-cobordism theorem, notes by L. Siebenmann and J. Sondow. Princeton: University Press 1965.

——— Characteristic classes. Princeton: University Press 1974.

Morse,M.: Relations between the numbers of critical points of a real function of n independent variables. Trans. Amer. Math. Soc. 27 (1925), 345-396.

Novikov,S.P.: Homotopy equivalent smooth manifolds I. AMS Tranlations 48 (1965), 271-396.

Reeb,G.: Sur certain propriétés topologiques des variétés feuilletées, Actual. sci. industr. 1183, Paris, 1952, 91-154.

Serre,J.-P.: Homologie singulière des espaces fibrés. Applications, Ann. Math. 54 (1951), 425-505.

Smale,S.: Generalized Poincaré conjecture in dimensions greater than four, Ann. Math. 74 (1961), 391-406.

Steenrod,N.: The topology of fibre bundles. Princeton Math. Series 14. Princeton: University Press 1951.

——— Epstein,D.B.A.: Cohomology operations. Annals of Math. Studies No. 50, Princeton Univ. Press 1962.

Thom,R.: Quelques propriétés globales des variétés differentiables. Comment. Math. Helv. 28 (1954), 17-86.