A SHARP INEQUALITY FOR POISSON'S EQUATION IN ARBITRARY DOMAINS AND ITS APPLICATIONS TO BURGERS' EQUATION

By

Wenzheng Xie

B. Sc., Zhongshan University, 1982

M. Sc., Fudan University, 1985

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

MATHEMATICS

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

March 1991

Department of Mathematics

The University of British Columbia
Vancouver, Canada

Date March 1, 1991

Abstract

Let Ω be an arbitrary open set in $\mathbb{R}^3$. Let $\|\cdot\|$ denote the $L^2(\Omega)$ norm, and let $\hat{H}^1_0(\Omega)$ denote the completion of $C^\infty_0(\Omega)$ in the Dirichlet norm $\|\nabla\cdot\|$. The pointwise bound

$$\sup_\Omega |u| \le \frac{1}{\sqrt{2\pi}} \|\nabla u\|^{1/2} \|\Delta u\|^{1/2}$$

is established for all functions $u \in \hat{H}^1_0(\Omega)$ with $\Delta u \in L^2(\Omega)$. The constant $1/\sqrt{2\pi}$ is shown to be the best possible.

Previously, inequalities of this type were proven only for bounded smooth domains or convex domains, with constants depending on the regularity of the boundary.

A new method is employed to obtain this sharp inequality. The key idea is to estimate the maximum value of the quotient $|u(x)|/\,\|\nabla u\|^{1/2}\,\|\Delta u\|^{1/2}$, where the point x is fixed, and the function u varies in the span of a finite number of eigenfunctions of the Laplacian. This method admits generalizations to other elliptic operators and other domains.

The inequality is applied to study the initial-boundary value problem for Burgers' equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \Delta\boldsymbol{u}\,, \qquad \boldsymbol{u} \in \hat{H}^1_0(\Omega)^3\,, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0\,,$$

in arbitrary domains, with initial data in $\hat{H}^1_0(\Omega)^3$. New a priori estimates are obtained. Adapting and refining known theory for Navier-Stokes equations, the existence and uniqueness of bounded smooth solutions are established.

As corollaries of the inequality and its proof, pointwise bounds are given for eigenfunctions of the Laplacian in terms of the corresponding eigenvalues in two- and three-dimensional domains.

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Acknowledgements

It is a pleasure to thank my advisor, Professor John G. Heywood, for suggesting the topic for this research, and for his very helpful advice and encouragement. I would also like to thank Professor Lon Rosen for helpful discussions about eigenfunction expansions.

Chapter 1

A sharp inequality for Poisson's equation

1.1 Introduction and the main result

In this chapter we establish the following

Theorem 1 *Let* Ω *be an arbitrary open set in* $\mathbb{R}^3$ *. For all* $u \in \hat{H}_0^1(\Omega)$ *with* $\Delta u \in L^2(\Omega)$ *, there holds*

$$\sup_{\Omega} |u| \leq \frac{1}{\sqrt{2\pi}} \|\nabla u\|^{1/2} \|\Delta u\|^{1/2}. \tag{1.1}$$

The constant $1/\sqrt{2\pi}$ *is the best possible.*

Throughout this thesis, $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. The gradient ∇ and the Laplacian Δ are understood in the distributional sense. The homogeneous Sobolev space $\hat{H}_0^1(\Omega)$ is defined to be the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\nabla \cdot\|$.

Inequalities of this type are used in the study of nonlinear partial differential equations (see [4, p. 299], [10, p. 12]). For bounded domains with smooth boundaries, an inequality of the form of (1.1), but with a constant depending on the domain, can be obtained by combining the Sobolev inequality

$$\sup_{\Omega} |u| \leq c \|u\|_{H^1(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2}, \tag{1.2}$$

with the Poincaré inequality

$$\|u\| \leq c \|\nabla u\|, \tag{1.3}$$

and the a priori estimate

$$\|u\|_{H^2(\Omega)} \leq c \|\Delta u\|. \tag{1.4}$$

The inequality (1.2) has been proven for domains that satisfy a weak cone condition [1]. The estimate (1.4) has been proven for domains with $C^{1,1}$ boundaries or convex domains (see [6]). However, simple examples given in [2] show that (1.4) fails to hold for domains with reentrant angles.

For bounded domains with possibly nonsmooth boundaries, the pointwise bound

$$\sup_{\Omega} |u| \leq c \, \|\Delta u\| \tag{1.5}$$

is known [8], with a constant c depending on the domain. In comparison, Inequality (1.1) has a smaller exponent on $\|\Delta u\|$. For some applications, it is crucial that this exponent is less than one (see the remark in Section 2.4). In Section 1.5, as a corollary of the inequality (1.1), we give a bound for the constant c.

Our proof of the inequality (1.1) is independent of such Sobolev inequalities and a priori estimates for elliptic equations, and of the various methods that are used in proving them. The key idea in the proof is to estimate the maximum value of the quotient $|u(x)| / \|\nabla u\|^{1/2} \|\Delta u\|^{1/2}$, where the point x is fixed, and the function u varies in the span of a finite number of eigenfunctions of the Laplacian. The method can be generalized to other elliptic operators and to other domains. Some of these generalizations will be given by the author in separate papers.

That an inequality of the form (1.1) should be valid for arbitrary open sets was suggested to the author by Professor J. G. Heywood. He conjectured that an analogous inequality also holds for the Stokes operator, and can be combined with the methods of [3] and [4] to obtain a regularity theory for the Navier-Stokes equations in arbitrary open sets. Partial results toward the proof of the analogous inequality for the Stokes operator have been obtained by the author. An existence theorem for smooth solutions of Burgers' equation based on (1.1) and the methods of [4] and [3] is given in Chapter 2.

1.2 Proof of the inequality for bounded smooth domains

In this section, we assume that Ω is bounded, with a C^∞ boundary $\partial\Omega$. It is well known that the eigenfunctions of $-\Delta$ can be chosen to form a complete orthonormal basis for $L^2(\Omega)$. Let ϕ_n denote the eigenfunctions, and λ_n the corresponding eigenvalues. Then $\phi_n \in C^\infty(\overline{\Omega})$, $\lambda_n > 0$, and they satisfy

$$-\Delta\phi_n = \lambda_n\phi_n\,, \qquad \phi_n|_{\partial\Omega} = 0\,, \qquad \|\phi_n\| = 1\,, \qquad (n = 1, 2, \cdots)\,,$$

$$\int_\Omega \phi_i\phi_j\,dx = 0\,, \qquad (i \neq j)\,.$$

Our proof of (1.1) has three steps.

Step 1. We first consider functions of the form

$$u(x) = \sum_{n=1}^{m} c_n\phi_n(x)\,,$$

where $c_1, \cdots, c_m$ are real numbers. We have

$$\|\nabla u\|^2 = \sum_{n=1}^{m} \lambda_n c_n^2\,, \qquad \|\Delta u\|^2 = \sum_{n=1}^{m} \lambda_n^2 c_n^2\,.$$

Hence, for any $y \in \Omega$,

$$\frac{u^2(y)}{\|\nabla u\|\,\|\Delta u\|} = \frac{\left(\sum\limits_{n=1}^{m} c_n\phi_n(y)\right)^2}{\left(\sum\limits_{n=1}^{m} \lambda_n c_n^2\right)^{1/2}\left(\sum\limits_{n=1}^{m} \lambda_n^2 c_n^2\right)^{1/2}}\,. \tag{1.6}$$

Let y and m be fixed. Then this quotient is a smooth and homogeneous function of $(c_1, \cdots, c_m)$ in $I\!R^m\backslash\{0\}$. Hence it attains its maximum value at some point $(\tilde{c}_1, \cdots, \tilde{c}_m)$, i.e., when the function is $\tilde{u} = \sum_{n=1}^{m} \tilde{c}_n\phi_n$. This maximum value is greater than zero, by the well-known fact that $\phi_1(y) \neq 0$. Differentiating

$$\log\frac{u^2(y)}{\|\nabla u\|\,\|\Delta u\|} \equiv \log u^2(y) - \frac{1}{2}\log\,\|\nabla u\|^2 - \frac{1}{2}\log\,\|\Delta u\|^2$$

with respect to c_n at the critical point, we get

$$\frac{2\phi_n(y)}{\tilde{u}(y)} - \frac{\lambda_n \tilde{c}_n}{\|\nabla \tilde{u}\|^2} - \frac{\lambda_n^2 \tilde{c}_n}{\|\Delta \tilde{u}\|^2} = 0,$$

for $n = 1, \cdots, m$. Letting $\mu = \|\Delta \tilde{u}\|^2 / \|\nabla \tilde{u}\|^2$, we obtain

$$\frac{2\phi_n(y)}{\mu + \lambda_n} = \frac{\tilde{u}(y)}{\|\Delta \tilde{u}\|^2} \lambda_n \tilde{c}_n .$$

Hence

$$\sum_{n=1}^{m} \left(\frac{2\phi_n(y)}{\mu + \lambda_n} \right)^2 = \left(\frac{\tilde{u}(y)}{\|\Delta \tilde{u}\|^2} \right)^2 \sum_{n=1}^{m} \lambda_n^2 \tilde{c}_n^2 = \frac{\tilde{u}^2(y)}{\|\Delta \tilde{u}\|^2} .$$

Therefore the maximum value of the quotient (1.6) is

$$\frac{\tilde{u}^2(y)}{\|\nabla \tilde{u}\| \|\Delta \tilde{u}\|} = \frac{\|\Delta \tilde{u}\|}{\|\nabla \tilde{u}\|} \cdot \frac{\tilde{u}^2(y)}{\|\Delta \tilde{u}\|^2} = 4\sqrt{\mu} \sum_{n=1}^{m} \left(\frac{\phi_n(y)}{\mu + \lambda_n} \right)^2 . \tag{1.7}$$

Step 2. To bound the right hand side of (1.7), we use the Green function for the Helmholtz equation and its eigenfunction expansion. Let y and μ be fixed as above, and let

$$g(x) = \frac{e^{-\sqrt{\mu}|x-y|}}{4\pi |x-y|} . \tag{1.8}$$

It is easy to verify that $\Delta g(x) = \mu g(x)$ for all $x \neq y$. Let $h(x)$ satisfy

$$\Delta h = \mu h , \qquad h|_{\partial\Omega} = g|_{\partial\Omega} .$$

There is a unique solution $h \in C^\infty(\overline{\Omega})$, since $\mu > 0$ and the domain is bounded and smooth. The function $h(x)$ attains its minimum value at some point $x_1 \in \overline{\Omega}$. If $x_1 \in \partial\Omega$, then $h(x_1) = g(x_1) > 0$. If $x_1 \in \Omega$, then $\Delta h(x_1) \geq 0$, and hence $h(x_1) = (1/\mu)\Delta h(x_1) \geq 0$. Therefore, we always have $h(x) \geq 0$ in Ω. Let $G = g - h$. Then

$$\Delta G = \mu G \quad \text{in } \Omega \backslash \{y\} , \qquad G|_{\partial\Omega} = 0 .$$

We obtain $G(x) \geq 0$ in $\Omega \backslash \{y\}$ similarly. Hence we have $0 \leq G(x) \leq g(x)$ for all $x \in \Omega \backslash \{y\}$, and therefore

$$\int_\Omega G^2 \, dx \leq \int_{\mathbb{R}^3} g^2 \, dx = \int_0^\infty \left(\frac{e^{-\sqrt{\mu} r}}{4\pi r} \right)^2 4\pi r^2 dr = \frac{1}{8\pi\sqrt{\mu}} . \tag{1.9}$$

Let $\Omega_\epsilon \subset \Omega$ be a ball centered at y with radius ϵ. For any $n \geq 1$, we have

$$\int_{\Omega\setminus\Omega_\epsilon} (\phi_n \Delta G - G\Delta\phi_n)\, dx = \int_{\partial(\Omega\setminus\Omega_\epsilon)} \left(\phi_n \frac{\partial G}{\partial \nu} - G\frac{\partial \phi_n}{\partial \nu}\right) dS\,,$$

by Green's formula, where ν is the outward normal to the boundary. Hence

$$\int_{\Omega\setminus\Omega_\epsilon} (\mu + \lambda_n) G\phi_n\, dx = -\int_{r=\epsilon} \left(\phi_n \frac{\partial G}{\partial r} - G\frac{\partial \phi_n}{\partial r}\right) dS\,,$$

where $r = |x - y|$. Since G has the same type of singularity as $1/4\pi r$, by letting $\epsilon \longrightarrow 0$, we obtain

$$(\mu + \lambda_n) \int_\Omega G\phi_n\, dx = \phi_n(y)\,. \tag{1.10}$$

Therefore, by Parseval's equality,

$$\int_\Omega G^2\, dx = \sum_{n=1}^{\infty} \left(\int_\Omega G\phi_n\, dx\right)^2 = \sum_{n=1}^{\infty} \left(\frac{\phi_n(y)}{\mu + \lambda_n}\right)^2.$$

This, together with (1.9), provide a bound for the maximum value (1.7):

$$\frac{\tilde{u}^2(y)}{\|\nabla\tilde{u}\|\|\Delta\tilde{u}\|} \leq 4\sqrt{\mu} \int_\Omega G^2\, dx \leq 4\sqrt{\mu}\, \frac{1}{8\pi\sqrt{\mu}} = \frac{1}{2\pi}\,.$$

Thus we have proven Inequality (1.1) for all functions of the form $u = \sum_{n=1}^{m} c_n\phi_n$.

Step 3. Now, let u be any function in $\hat{H}_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$. Since Ω is bounded, we have $\hat{H}_0^1(\Omega) \subset L^2(\Omega)$ by virtue of the Poincaré inequality. Hence we have the expansion $u = \sum_{n=1}^{\infty} c_n\phi_n$ in $L^2(\Omega)$, where $c_n = \int_\Omega u\phi_n\, dx$. Let $u_m = \sum_{n=1}^{m} c_n\phi_n$. Integrating by parts, we have

$$\int_\Omega \nabla u \cdot \nabla u_m\, dx = -\int_\Omega u\Delta u_m\, dx = \sum_{n=1}^{m} \lambda_n c_n \int_\Omega u\phi_n\, dx = \sum_{n=1}^{m} \lambda_n c_n^2 = \|\nabla u_m\|^2\,.$$

Hence we get $\|\nabla u_m\| \leq \|\nabla u\|$ by using the Schwarz inequality. Similarly, from

$$\begin{aligned}\int_\Omega \Delta u_m \Delta u\, dx &= -\sum_{n=1}^{m} \lambda_n c_n \int_\Omega \phi_n \Delta u\, dx = -\sum_{n=1}^{m} \lambda_n c_n \int_\Omega u\Delta\phi_n\, dx \\ &= \sum_{n=1}^{m} \lambda_n^2 c_n^2 = \|\Delta u_m\|^2\,,\end{aligned}$$

we get $\|\Delta u_m\| \leq \|\Delta u\|$. Therefore, by the result of Step 2, we obtain

$$\sup_{\Omega} |u_m|^2 \leq \frac{1}{2\pi} \|\nabla u_m\| \, \|\Delta u_m\| \leq \frac{1}{2\pi} \|\nabla u\| \, \|\Delta u\| .$$

By a well-known interior regularity theorem for elliptic equations, $\Delta u \in L^2(\Omega)$ implies $u \in H^2_{loc}(\Omega)$, which in turn implies that $u \in C(\Omega)$, by a well-known Sobolev imbedding theorem. Now, if (1.1) were not true, then there would be some $x_0 \in \Omega$ such that

$$|u(x_0)|^2 > \frac{1}{2\pi} \|\nabla u\| \, \|\Delta u\| \geq \sup_{\Omega} |u_m|^2 ,$$

which is obviously contradictory to the fact that $\lim_{m\to\infty} \|u_m - u\| = 0$. This completes the proof of (1.1) for bounded smooth domains.

1.3 Proof of the inequality for arbitrary domains

Let Ω be an arbitrary open set in $\mathbb{R}^3$, and suppose that $u \in \hat{H}^1_0(\Omega)$ and $\Delta u \in L^2(\Omega)$. We can choose a sequence of bounded domains Ω_n with smooth boundaries, such that $\Omega_1 \subset \Omega_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. For each $n \geq 1$, there exists a unique $u_n \in \hat{H}^1_0(\Omega_n)$ such that

$$\int_{\Omega_n} \nabla u_n \cdot \nabla v \, dx = \int_{\Omega_n} \nabla u \cdot \nabla v \, dx , \qquad \forall\, v \in \hat{H}^1_0(\Omega_n) , \tag{1.11}$$

by the Riesz theorem. We get $\|\nabla u_n\|_{L^2(\Omega_n)} \leq \|\nabla u\|$ by letting $v = u_n$ and using the Schwarz inequality. Integrating by parts on the right hand side of (1.11), we obtain

$$\int_{\Omega_n} \nabla u_n \cdot \nabla v \, dx = -\int_{\Omega_n} (\Delta u) v \, dx , \qquad \forall\, v \in \hat{H}^1_0(\Omega_n) ,$$

and hence $\Delta u_n = \Delta u|_{\Omega_n}$. Therefore, by the result of Step 3, we have

$$\sup_{\Omega_n} |u_n|^2 \leq \frac{1}{2\pi} \|\nabla u_n\|_{L^2(\Omega_n)} \, \|\Delta u_n\|_{L^2(\Omega_n)} \leq \frac{1}{2\pi} \|\nabla u\| \, \|\Delta u\| . \tag{1.12}$$

Setting u_n equal to zero in $\Omega \backslash \Omega_n$, we get $u_n \in \hat{H}^1_0(\Omega)$. From (1.11) we have

$$\lim_{n\to\infty} \int_{\Omega} \nabla u_n \cdot \nabla v \, dx = \int_{\Omega} \nabla u \cdot \nabla v \, dx , \qquad \forall\, v \in C_0^\infty(\Omega) .$$

This and $\|\nabla u_n\| \leq \|\nabla u\|$ imply that $\lim_{n\to\infty} u_n = u$ in $\hat{H}_0^1(\Omega)$. Therefore, by the inequality $\|v\|_{L^6(\Omega)} \leq c\|\nabla v\|$ (see [6, p. 10] for a simple proof giving $c = \sqrt[6]{48}$), we have $\lim_{n\to\infty} \|u_n - u\|_{L^6(\Omega)} = 0$. This and (1.12) imply (1.1) by reasoning similarly as in Step 3 above.

1.4 Proof that the constant in the inequality is optimal

We first consider the case $\Omega = I\!R^3$. Define

$$u(x) = f(r) = \begin{cases} 1, & r = 0, \\ \dfrac{1 - e^{-r}}{r}, & r > 0, \end{cases}$$

where $r = |x|$. The function u is continuous, with a maximum value $u(0) = 1$. We notice that $u/4\pi$ is equal to the difference between the fundamental solution for the Laplace equation, and that of the Helmholtz equation ((1.8), with $\mu = 1$). Hence we immediately obtain $\Delta u = -e^{-r}/r$ in the distributional sense. We have $\Delta u \in L^2(\Omega)$ since

$$\int_{I\!R^3} |\Delta u|^2\, dx = \int_0^\infty \left(-\frac{e^{-r}}{r}\right)^2 4\pi r^2\, dr = 2\pi\,.$$

Integrating by parts (which is easily justified), we obtain

$$\int_{I\!R^3} |\nabla u|^2\, dx = -\int_{I\!R^3} u\Delta u\, dx = \int_0^\infty \frac{1 - e^{-r}}{r}\,\frac{e^{-r}}{r}\, 4\pi r^2\, dr = 2\pi\,.$$

Hence, the equality in (1.1) actually occurs for the function u.

To show that $u \in \hat{H}_0^1(I\!R^3)$, we modify the function u to define a sequence of functions. Let f' denote df/dr. For each $n \geq 1$, let

$$u_n(x) = \begin{cases} f(1/n) + \dfrac{f'(1/n)}{2n}\left(n^2 r^2 - 1\right), & 0 \leq r < 1/n, \\ f(r), & 1/n \leq r < n, \\ \dfrac{f'(n)^2}{4f(n)}\left(n - \dfrac{2f(n)}{f'(n)} - r\right)^2, & n \leq r < r_n \equiv n - \dfrac{2f(n)}{f'(n)}, \\ 0, & r_n \leq r < \infty. \end{cases}$$

It is easily seen that $u_n \in C_0^1(I\!R^3)$ and that u_n is piecewise C^2. By explicit calculation, we find that

$$\lim_{n\to\infty} \|\nabla(u_n - u)\|_{L^2(I\!R^3)} = 0\,.$$

Hence $u \in \hat{H}_0^1(I\!R^3)$. (It is interesting to note that $u \notin H_0^1(I\!R^3)$ since $u \notin L^2(I\!R^3)$).

By explicit calculation, we also find that

$$\lim_{n\to\infty} u_n(0) = 1\,, \qquad \lim_{n\to\infty} \|\Delta(u_n - u)\|_{L^2(I\!R^3)} = 0\,.$$

Hence

$$\lim_{n\to\infty} \frac{u_n(0)}{\|\nabla u_n\|_{L^2(I\!R^3)}^{1/2}\, \|\Delta u_n\|_{L^2(I\!R^3)}^{1/2}} = \frac{1}{\|\nabla u\|_{L^2(I\!R^3)}^{1/2}\, \|\Delta u\|_{L^2(I\!R^3)}^{1/2}} = \frac{1}{\sqrt{2\pi}}\,. \tag{1.13}$$

We now consider an arbitrary open set Ω. Let $|x - x_0| \le \epsilon$ be a ball contained in it. For each $n \ge 1$, define $v_n(x) = u_n(\epsilon\, r_n^{-1}(x - x_0))$. Then v_n vanishes outside the ball. Clearly, we have $v_n \in C_0^1(\Omega)$ and $\Delta v_n \in L^2(\Omega)$, for all $n = 1, 2, \cdots$. It is easy to verify that

$$\frac{v_n(x_0)}{\|\nabla v_n\|^{1/2}\, \|\Delta v_n\|^{1/2}} = \frac{u_n(0)}{\|\nabla u_n\|_{L^2(I\!R^3)}^{1/2}\, \|\Delta u_n\|_{L^2(I\!R^3)}^{1/2}}\,.$$

Noticing (1.13), we conclude that the constant $1/\sqrt{2\pi}$ in Inequality (1.1) cannot be improved, for any given domain. This completes the proof of our theorem.

1.5 Corollaries

In this section, we give several immediate corollaries of Theorem 1. Let Ω denote an arbitrary open set in $I\!R^3$, except in Corollary 4. Note that the constants in the corollaries are not claimed to be the best possible, except for the special case stated in Corollary 1.

Corollary 1 *If $u \in \hat{H}_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, then*

$$\sup_\Omega |u| \le \frac{1}{2\sqrt{2\pi}}(\,\|\nabla u\| + \|\Delta u\|).$$

The equality occurs for some functions in the case $\Omega = I\!R^3$.

Corollary 2 *If $u \in H_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, then u satisfies*

$$\sup_\Omega |u| \le \frac{1}{\sqrt{2\pi}} \|u\|^{1/4} \|\Delta u\|^{3/4}.$$

Corollary 3 *If $u \in H_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, then*

$$\sup_\Omega |u| \le \frac{1}{4\sqrt{2\pi}}(\|u\| + 3 \|\Delta u\|) .$$

Corollary 4 *Let Ω be an open set in $I\!R^3$ such that the Poincaré inequality*

$$\|u\| \le \gamma \|\nabla u\| , \qquad \forall\, u \in H_0^1(\Omega) \tag{1.14}$$

holds. Then, for all $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, there holds

$$\sup_\Omega |u| \le \sqrt{\frac{\gamma}{2\pi}} \|\Delta u\| .$$

Proof of the corollaries. Corollary 1 follows from Inequality (1.1) directly and the example given in Section 1.4. Corollary 2 follows from (1.1) by using

$$\|\nabla u\|^2 = - \int_\Omega u \Delta u \, dx \le \|u\| \, \|\Delta u\| . \tag{1.15}$$

Corollary 3 follows from Corollary 2 and Young's inequality. Corollary 4 follows from (1.1), since we have $\|\nabla u\| \le \gamma \|\Delta u\|$ from (1.15) and (1.14).

It is easy to show that Theorem 1 and the corollaries are also valid for vector-valued or complex-valued functions.

1.6 Pointwise bounds for eigenfunctions of the Laplacian

As a special case of Corollary 2, we have

Theorem 2 *Let Ω be an arbitrary open set in $I\!R^3$. If $\lambda > 0$ and ϕ satisfy*

$$-\Delta \phi = \lambda \phi , \qquad \phi \in H_0^1(\Omega) , \qquad \|\phi\| = 1 ,$$

then

$$\sup_\Omega |\phi| \le \frac{\lambda^{3/4}}{\sqrt{2\pi}} .$$

That is, we have a pointwise bound for any eigenfunction ϕ of the Laplacian, depending only on the corresponding eigenvalue λ. As in Corollary 2, the constant here is not optimal. We give a better constant in the theorem below.

Theorem 3 *Let Ω be a bounded open set in $\mathbb{R}^3$ with a smooth boundary. If $\lambda > 0$ and ϕ satisfy*

$$-\Delta\phi = \lambda\phi, \qquad \phi \in H_0^1(\Omega), \qquad \|\phi\| = 1,$$

then

$$\sup_\Omega |\phi| \le \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{3}\right)^{3/4}. \tag{1.16}$$

Proof. For any $y \in \Omega$ and any $\mu > 0$, as the equality (1.10), we have

$$\phi(y) = (\mu + \lambda) \int_\Omega G\phi \, dx .$$

By the Schwarz inequality and (1.9), we have

$$\begin{aligned} |\phi(y)| &= (\mu+\lambda) \left| \int_\Omega G\phi \, dx \right| \\ &\le (\mu+\lambda) \left(\int_\Omega G^2 \, dx \right)^{1/2} \left(\int_\Omega \phi^2 \, dx \right)^{1/2} \\ &\le (\mu+\lambda) \left(\int_{\mathbb{R}^3} g^2 \, dx \right)^{1/2} \\ &= (\mu+\lambda) \left(\frac{1}{8\pi\sqrt{\mu}} \right)^{1/2} . \end{aligned}$$

The right hand side attains a minimum value when $\mu = \lambda/3$. Letting $\mu = \lambda/3$, we obtain (1.16).

The following theorem is an analogue of Theorem 3 in two dimensions.

Theorem 4 *Let Ω be a bounded open set in $\mathbb{R}^2$ with a smooth boundary. If $\lambda > 0$ and ϕ satisfy*

$$-\Delta\phi = \lambda\phi, \qquad \phi \in H_0^1(\Omega), \qquad \|\phi\| = 1,$$

then

$$\sup_\Omega |\phi| \le \sqrt{\frac{\lambda}{\pi}}. \tag{1.17}$$

Proof. In two dimensions, the fundamental solution corresponding to (1.8) is

$$g(x) = \frac{1}{2\pi} K_0(\sqrt{\mu}|x-y|),$$

where K_0 is a modified Bessel function. We have

$$\int_{\mathbb{R}^2} g^2\,dx = \int_0^\infty \left(\frac{1}{2\pi} K_0(\sqrt{\mu} r)\right)^2 2\pi r\,dr = \frac{1}{4\pi\mu}.$$

Hence, similar to the proof of Theorem 3, we have

$$\begin{aligned} |\phi(y)| &\leq (\mu+\lambda)\left(\int_{\mathbb{R}^2} g^2\,dx\right)^{1/2} \\ &= (\mu+\lambda)\left(\frac{1}{4\pi\mu}\right)^{1/2}. \end{aligned}$$

The right hand side attains a minimum value when $\mu = \lambda$. Letting $\mu = \lambda$, we obtain (1.17).

Since for special domains, the eigenvalues and eigenfunctions are explicitly known in terms of special functions, these pointwise bounds can be used to derive inequalities for the special functions.

Chapter 2

Application to Burgers' equation

2.1 The main result

In this chapter, we apply the inequality (1.1) to study the following problem for the three-dimensional Burgers' equation:

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \Delta \boldsymbol{u}\,, \\ &\boldsymbol{u}(t) \in \hat{\boldsymbol{H}}_0^1(\Omega)\,, \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0\,. \end{aligned} \tag{2.1}$$

Here, the spatial domain Ω is an arbitrary open set in $I\!R^3$, and the initial vector field $\boldsymbol{u}_0$ is given in $\hat{\boldsymbol{H}}_0^1(\Omega) \equiv \hat{H}_0^1(\Omega)^3$. The Burgers' equation is studied for its analogy with the Navier-Stokes equations. We establish the following theorem by methods which will carry over immediately to the Navier-Stokes equations, if we are successful in proving the analogue of (1.1) for the Stokes operator. Hereafter, we use D_t to denote the partial derivative with respect to the time variable (t or s).

Theorem 5 *Let Ω be an arbitrary open set in $I\!R^3$. Let $\boldsymbol{u}_0 \in \hat{\boldsymbol{H}}_0^1(\Omega)$ be given. Let*

$$T = \frac{256\pi^2}{27\|\nabla \boldsymbol{u}_0\|^4}\,.$$

Then there exists a unique vector field $\boldsymbol{u}$ such that

$$\begin{aligned} \boldsymbol{u} &\in \boldsymbol{C}^\infty(\Omega \times (0,T)) \cap C^\infty((0,T), \boldsymbol{L}_\infty(\Omega))\,, \\ \boldsymbol{u} - \boldsymbol{u}_0 &\in C\left([0,T), \boldsymbol{H}_0^1(\Omega)\right) \cap C^\infty\left((0,T), \boldsymbol{H}_0^1(\Omega)\right)\,, \\ \Delta \boldsymbol{u} &\in C^\infty((0,T), \boldsymbol{L}^2(\Omega))\,, \end{aligned}$$

and satisfies Problem (2.1). The solution also satisfies the estimates:

$$\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|^2 \le tF(t)\,,$$

$$t^{k+1/2}\,\|D_t^{k+1}\boldsymbol{u}\| + t^{k+1/4}\,\|D_t^k\boldsymbol{u}\|_\infty + t^k\,\|\nabla D_t^k\boldsymbol{u}\| + t^{k+1/2}\,\|\Delta D_t^k\boldsymbol{u}\| \le F(t)\,,\quad k\ge 0\,,$$

$$\int_0^t \left(\|D_t\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|_\infty^4 + \|\Delta\boldsymbol{u}\|^2\right)\,ds \le F(t)\,,$$

$$\int_0^t s^{2k}\left(\|D_t^{k+1}\boldsymbol{u}\|^2 + s^{-1/2}\,\|D_t^k\boldsymbol{u}\|_\infty^2 + s^{-1}\,\|\nabla D_t^k\boldsymbol{u}\|^2 + \|\Delta D_t^k\boldsymbol{u}\|^2\right)\,ds \le F(t)\,,\quad k\ge 1\,,$$

for all $t\in(0,\,T)$, where the $F(t)$ denotes appropriate continuous functions on $[0,\,T)$ that can be obtained explicitly in terms of k and $\|\nabla\boldsymbol{u}_0\|$, independently of Ω.

Theorem 5 is an adaptation and refinement of known existence theorems for the Navier-Stokes equations, based on a differential inequality for $\|\nabla u(t)\|$, and its analogue for Galerkin approximations. The method originated with Prodi [9], who used it to prove the existence of generalized solutions in bounded domains. Heywood [3] introduced a further infinite sequence of differential inequalities, to obtain classically smooth solutions. He also extended the method to unbounded domains. Heywood and Rannacher [4] developed the method further through use of weight functions dependent on the time variable to give more precise estimates as $t\to 0^+$. All of these developments are incorporated into the existence theorems given here.

The principal innovation here is that the nonlinear term is now estimated in a new way, using the inequality (1.1), to give results that are not only sharper but also valid in arbitrary domains.

The energy estimate basic to many works on the Navier-Stokes equations is not valid for the three-dimensional Burgers' equation. Theorem 5 is independent of it. Observe that the solution can have an infinite $\boldsymbol{L}^2(\Omega)$-norm in unbounded domains.

We point out that unlike the Navier-Stokes equations, there is a maximum principle for solutions of Burgers' equation. An existence theorem for Burgers' equation based on the maximum principle was given by Kiselev and Ladyzhenskaya [5]. Incorporating it with Theorem 5, the solution can be continued globally in time.

2.2 Preliminaries

In this section we list some lemmas that will be used later.

Notations. We use boldface symbols to denote three-dimensional vector-valued functions and their spaces. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$ or $\boldsymbol{L}^p(\Omega)$ norm. When $p = 2$, we simply use $\|\cdot\|$ to denote the norm, and use $(\cdot,\cdot)$ to denote the inner product. We use $\|\cdot\|_\infty$ to denote the $\sup_\Omega|\cdot|$ norm.

The vector version of Theorem 1. Let Ω be an arbitrary open set in $\mathbb{R}^3$. If $\boldsymbol{u} \in \hat{\boldsymbol{H}}_0^1(\Omega)$ and $\Delta\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, then

$$\|\boldsymbol{u}\|_\infty \le \frac{1}{\sqrt{2\pi}}\|\nabla\boldsymbol{u}\|^{1/2}\|\Delta\boldsymbol{u}\|^{1/2}. \tag{2.2}$$

The constant $1/\sqrt{2\pi}$ is the best possible.

Proof. The inequality (2.2) is obtained by simply applying (1.1) to each component:

$$\begin{aligned}
\|\boldsymbol{u}\|_\infty^2 &\le \sum_{i=1}^3 \|u_i\|_\infty^2 \\
&\le \frac{1}{2\pi}\sum_{i=1}^3 \|\nabla u_i\|\,\|\Delta u_i\| \\
&\le \frac{1}{2\pi}\left(\sum_{i=1}^3 \|\nabla u_i\|^2\right)^{1/2}\left(\sum_{i=1}^3 \|\Delta u_i\|^2\right)^{1/2} \\
&= \frac{1}{2\pi}\|\nabla\boldsymbol{u}\|\,\|\Delta\boldsymbol{u}\|.
\end{aligned}$$

It is obvious that the constant remains optimal.

Hölder's inequalities. If $p, q > 1$ and $1/p + 1/q = 1$, then

$$\left|\int fg\,dx\right| \le \left(\int |f|^p\,dx\right)^{1/p}\left(\int |g|^q\,dx\right)^{1/q}. \tag{2.3}$$

If $p, q, r > 1$ and $1/p + 1/q + 1/r = 1$, then

$$\left|\int fgh\,dx\right| \le \left(\int |f|^p\,dx\right)^{1/p}\left(\int |g|^q\,dx\right)^{1/q}\left(\int |h|^r\,dx\right)^{1/r}.$$

These are well known. We will use the case $p = 6$, $q = 2$, $r = 3$:

$$\left|\int_\Omega fgh\,dx\right| \le \|f\|_6\,\|g\|\,\|h\|_3. \tag{2.4}$$

Sobolev inequalities. For all $\boldsymbol{u} \in C_0^\infty(\mathbb{R}^3)$, there hold

$$\|\boldsymbol{u}\|_6 \leq c\|\nabla \boldsymbol{u}\|, \tag{2.5}$$

$$\|\boldsymbol{u}\|_3 \leq c\|\boldsymbol{u}\|^{1/2}\|\nabla \boldsymbol{u}\|^{1/2}. \tag{2.6}$$

A simple proof of (2.5) can be found in [6, p. 10], with $c = \sqrt[6]{48}$. Letting $p = 4/3, q = 4$ and $f = g = |\boldsymbol{u}|^{3/2}$ in (2.3), we obtain

$$\int |\boldsymbol{u}|^3\,dx \leq \left(\int |\boldsymbol{u}|^2\,dx\right)^{3/4}\left(\int |\boldsymbol{u}|^6\,dx\right)^{1/4}.$$

Combining this with (2.5), we obtain (2.6).

Young's inequality. If $a, b, p, q > 0$ and $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.7}$$

2.3 Galerkin approximations

Similar to Section 1.2, we first assume that Ω is a bounded open set in $\mathbb{R}^3$, with a C^∞ boundary $\partial\Omega$. The vector-valued eigenfunctions of $-\Delta$ can be chosen to form a complete orthonormal basis for $\boldsymbol{L}^2(\Omega)$. Let $\boldsymbol{\phi}_n$ denote the eigenfunctions, and λ_n the corresponding eigenvalues. Then $\boldsymbol{\phi}_n \in C^\infty(\overline{\Omega})$, $\lambda_n > 0$, and they satisfy

$$-\Delta\boldsymbol{\phi}_n = \lambda_n\boldsymbol{\phi}_n, \qquad \boldsymbol{\phi}_n|_{\partial\Omega} = 0, \qquad \|\boldsymbol{\phi}_n\| = 1, \qquad (n = 1, 2, \cdots),$$

$$(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j) = 0, \qquad (i \neq j).$$

These eigenfunctions and eigenvalues can be obtained immediately from their scalar counterparts.

We seek Galerkin approximations in the form

$$\boldsymbol{u}^m(x,t) = \sum_{n=1}^{m} c_n^m(t)\boldsymbol{\phi}_n(x)$$

where the $c_n^m(t)$ are smooth functions of t. The advantage of using this form is that $\partial\boldsymbol{u}^m/\partial t$ and $\Delta\boldsymbol{u}^m$ are also linear combinations of the first m eigenfunctions. Let

$\boldsymbol{u}^m$ satisfy

$$\left(\frac{\partial \boldsymbol{u}^m}{\partial t} - \Delta \boldsymbol{u}^m + \boldsymbol{u}^m \cdot \nabla \boldsymbol{u}^m,\, \boldsymbol{\phi}_n\right) = 0\,, \tag{2.8}$$

$$(\boldsymbol{u}^m(\cdot, 0) - \boldsymbol{u}_0,\, \boldsymbol{\phi}_n) = 0, \quad (n = 1, 2, \cdots, m), \tag{2.9}$$

i.e.,

$$\begin{aligned} \frac{d}{dt} c_n^m(t) &= -\lambda_n c_n^m(t) - \sum_{i,j=1}^{m} (\boldsymbol{\phi}_i \cdot \nabla \boldsymbol{\phi}_j, \boldsymbol{\phi}_n) c_i^m(t) c_j^m(t)\,, \\ c_n^m(0) &= (\boldsymbol{u}_0\,,\, \boldsymbol{\phi}_n)\,, \qquad\qquad (n = 1, 2, \cdots, m). \end{aligned} \tag{2.10}$$

To find a time interval on which the solution exists, we need a priori estimates. Hereafter we suppress the superscript m. From (2.8) we obtain

$$\left(D_t^{k+1}\boldsymbol{u} - \Delta D_t^k \boldsymbol{u} + D_t^k(\boldsymbol{u} \cdot \nabla \boldsymbol{u}),\, \boldsymbol{v}\right) = 0\,, \qquad \forall\, \boldsymbol{v} \in \operatorname{span}\{\boldsymbol{\phi}_1, \cdots, \boldsymbol{\phi}_m\}\,. \tag{2.11}$$

In particular, we can take $\boldsymbol{v} = D_t^i \Delta^j \boldsymbol{u}$. Here $i, j, k \geq 0$.

2.4 Main estimates

Let $k = 0$ and let $\boldsymbol{v} = \Delta \boldsymbol{u}$ in (2.11). We get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2 + \|\Delta \boldsymbol{u}\|^2 &= (\boldsymbol{u} \cdot \nabla \boldsymbol{u}\,, \Delta \boldsymbol{u}) && (2.12) \\ &\leq \|\boldsymbol{u}\|_\infty \|\nabla \boldsymbol{u}\| \, \|\Delta \boldsymbol{u}\| && (2.13) \\ &\leq \frac{1}{\sqrt{2\pi}} \|\nabla \boldsymbol{u}\|^{3/2} \|\Delta \boldsymbol{u}\|^{3/2} && (2.14) \\ &\leq \frac{1}{4\sqrt{2\pi}} \left(\alpha^{-6} \|\nabla \boldsymbol{u}\|^6 + 3\alpha^2 \|\Delta \boldsymbol{u}\|^2\right) && (2.15) \\ &= C_\epsilon \|\nabla \boldsymbol{u}\|^6 + \epsilon \|\Delta \boldsymbol{u}\|^2\,. && (2.16) \end{aligned}$$

We obtain (2.12) by integration by parts; (2.13) by using the Schwarz inequality; (2.14) by using (2.2); (2.15) for any $\alpha > 0$, by using Young's inequality (2.7); and (2.16) by letting

$$\epsilon = \frac{3\alpha^2}{4\sqrt{2\pi}}\,, \qquad C_\epsilon = \frac{\alpha^{-6}}{4\sqrt{2\pi}} = \frac{27}{1024\pi^2\epsilon^3}\,.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2 + (1 - \epsilon)\|\Delta u\|^2 \leq C_\epsilon \|\nabla \boldsymbol{u}\|^6\,.$$

Let

$$\varphi(t) = \|\nabla \boldsymbol{u}(t)\|^2 + 2(1-\epsilon)\int_0^t \|\Delta \boldsymbol{u}(s)\|^2\, ds\,.$$

Then, when $0 < \epsilon \le 1$, we have

$$\frac{d}{dt}\varphi(t) \le 2C_\epsilon \varphi^3(t)\,, \qquad \varphi(0) = \|\nabla \boldsymbol{u}(0)\|^2\,.$$

Comparing this with

$$\frac{d}{dt}\Phi(t) = 2C_\epsilon \Phi^3(t)\,, \qquad \Phi(0) = \|\nabla \boldsymbol{u}_0\|^2\,,$$

and noticing that we have $\|\nabla \boldsymbol{u}(0)\|^2 \le \|\nabla \boldsymbol{u}_0\|^2$ from (2.9), we obtain

$$\varphi(t) \le \Phi(t) \equiv (\|\nabla \boldsymbol{u}_0\|^{-4} - 4C_\epsilon t)^{-1/2}\,,$$

i.e.,

$$\|\nabla \boldsymbol{u}(t)\|^2 + 2(1-\epsilon)\int_0^t \|\Delta \boldsymbol{u}(s)\|^2 ds \le \frac{\|\nabla \boldsymbol{u}_0\|^2}{\sqrt{1-t/\epsilon^3 T}}\,, \qquad (0 \le t < \epsilon^3 T)\,,$$

where

$$T = \frac{256\pi^2}{27\|\nabla \boldsymbol{u}_0\|^4}\,.$$

Letting $\epsilon = 1$ we obtain

$$\|\nabla \boldsymbol{u}(t)\|^2 \le \frac{\|\nabla \boldsymbol{u}_0\|^2}{\sqrt{1-t/T}}\,, \qquad (0 \le t < T)\,. \tag{2.17}$$

Letting $\epsilon = \sqrt[6]{t/T}$, we obtain

$$\int_0^t \|\Delta \boldsymbol{u}\|^2 ds \le \frac{\|\nabla \boldsymbol{u}_0\|^2}{2\left(1-\sqrt[6]{t/T}\right)\sqrt{1-\sqrt{t/T}}}\,, \qquad (0 \le t < T)\,. \tag{2.18}$$

Since $\|\nabla \boldsymbol{u}(t)\|^2 = \sum_{n=1}^m \lambda_n |c_n^m(t)|^2$, the a priori estimate (2.17) ensures that the solution of the o.d.e. system (2.10), hence the Galerkin approximation, exists on the interval $[0, T)$, with T independent of m and Ω.

Remark. If the inequality (1.5) is used instead of (1.1), one would obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2 + \|\Delta \boldsymbol{u}\|^2 \le c\,\|\nabla \boldsymbol{u}\|\,\|\Delta \boldsymbol{u}\|^2\,,$$

which does not lead to a bound for $\|\nabla \boldsymbol{u}\|$, unless one restricts the initial values to those satisfying $c\|\nabla \boldsymbol{u}_0\| \leq 1$.

2.5 Further estimates

To prove the smoothness of the solution, we need further a priori estimates. We first prove some differential inequalities.

We use c to denote a constant that does not depend on m or Ω, but possibly depends on k. The actual value of c may change at each occurrence.

Lemma 1 *For* $0 < t < \infty$, *there hold*

$$\begin{aligned}
\|D_t^k \boldsymbol{u}\|^2 &\leq c\|\Delta D_t^{k-1}\boldsymbol{u}\|^2 \\
&\quad + c\sum_{i=0}^{k-1} \|D_t^i \boldsymbol{u}\|_\infty^2 \, \|\nabla D_t^{k-1-i}\boldsymbol{u}\|^2, \qquad k \geq 1, \quad (2.19)\\
\frac{d}{dt}\|D_t^k \boldsymbol{u}\|^2 + \|\nabla D_t^k \boldsymbol{u}\|^2 &\leq c(\|\nabla \boldsymbol{u}\|^4 + 1)\,\|D_t^k \boldsymbol{u}\|^2 \\
&\quad + c\sum_{i=1}^{k-1} \|\nabla D_t^i \boldsymbol{u}\|^2\,\|\nabla D_t^{k-i}\boldsymbol{u}\|^2, \qquad k \geq 1, \quad (2.20)\\
\frac{d}{dt}\|\nabla D_t^k \boldsymbol{u}\|^2 + \|\Delta D_t^k \boldsymbol{u}\|^2 &\leq c\left(\|\nabla \boldsymbol{u}\|^4 + \|\boldsymbol{u}\|_\infty^2\right)\,\|\nabla D_t^k \boldsymbol{u}\|^2 \\
&\quad + c\sum_{i=1}^{k-1} \|D_t^i \boldsymbol{u}\|_\infty^2\,\|\nabla D_t^{k-i}\boldsymbol{u}\|^2, \qquad k \geq 1, \quad (2.21)\\
\|D_t^k \boldsymbol{u}\|_\infty^2 &\leq c\|\nabla D_t^k \boldsymbol{u}\|\,\|\Delta D_t^k \boldsymbol{u}\|, \qquad k \geq 0. \quad (2.22)
\end{aligned}$$

Proof of (2.19). Letting $\boldsymbol{v} = D_t^{k+1}\boldsymbol{u}$ in (2.11) and using the Schwarz inequality, we get

$$\|D_t^{k+1}\boldsymbol{u}\| \leq \|\Delta D_t^k \boldsymbol{u} - D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|.$$

Hence

$$\|D_t^{k+1}\boldsymbol{u}\|^2 \leq 2\,\|\Delta D_t^k \boldsymbol{u}\|^2 + 2\,\|D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|^2. \tag{2.23}$$

We have

$$D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = \sum_{i=0}^{k} cD_t^i\boldsymbol{u}\cdot\nabla D_t^{k-i}\boldsymbol{u}, \tag{2.24}$$

by the Leibniz formula, hence

$$\|D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\| \le c\sum_{i=0}^{k}\|D_t^i\boldsymbol{u}\|_\infty\,\|\nabla D_t^{k-i}\boldsymbol{u}\|\,. \tag{2.25}$$

Using this in (2.23) and replacing k by $k-1$, (2.19) is obtained.

Proof of (2.20). Letting $\boldsymbol{v} = D_t^k\boldsymbol{u}$ in (2.11) we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|D_t^k\boldsymbol{u}\|^2 \;+\; \|\nabla D_t^k\boldsymbol{u}\|^2 &= -\left(D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u}), D_t^k\boldsymbol{u}\right) && (2.26)\\
&\le c\sum_{i=0}^{k}\|D_t^i\boldsymbol{u}\|_6\,\|\nabla D_t^{k-i}\boldsymbol{u}\|\,\|D_t^k\boldsymbol{u}\|_3 && (2.27)\\
&\le c\sum_{i=0}^{k}\|\nabla D_t^i\boldsymbol{u}\|\,\|\nabla D_t^{k-i}\boldsymbol{u}\|\,\|D_t^k\boldsymbol{u}\|^{1/2}\,\|\nabla D_t^k\boldsymbol{u}\|^{1/2} && (2.28)\\
&\le \frac{1}{2}\|\nabla D_t^k\boldsymbol{u}\|^2 + c\left(\|\nabla\boldsymbol{u}\|^4+1\right)\|D_t^k\boldsymbol{u}\|^2\\
&\quad + c\sum_{i=1}^{k-1}\|\nabla D_t^i\boldsymbol{u}\|^2\,\|\nabla D_t^{k-i}\boldsymbol{u}\|^2\,. && (2.29)
\end{aligned}$$

We obtain (2.26) by integration by parts; (2.27) by using Leibniz's formula (2.24) and Hölder's inequality (2.4); (2.28) by using Sobolev inequalities (2.5) and (2.6); (2.29) by using Young's inequalities. Hence (2.20) is proved.

Proof of (2.21). Letting $\boldsymbol{v} = \Delta D_t^k\boldsymbol{u}$ in (2.11) we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla D_t^k\boldsymbol{u}\|^2 + \|\Delta D_t^k\boldsymbol{u}\|^2 &= \left(D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u}), \Delta D_t^k\boldsymbol{u}\right)\\
&\le \frac{1}{4}\|\Delta D_t^k\boldsymbol{u}\|^2 + c\|D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|^2. && (2.30)
\end{aligned}$$

From (2.25), we have

$$\begin{aligned}
\|D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|^2 &\le \frac{1}{4}\|\Delta D_t^k\boldsymbol{u}\|^2 + c\,\|\nabla\boldsymbol{u}\|^4\,\|\nabla D_t^k\boldsymbol{u}\|^2\\
&\quad + c\sum_{i=0}^{k-1}\|D_t^i\boldsymbol{u}\|_\infty^2\,\|\nabla D_t^{k-i}\boldsymbol{u}\|^2\,, && (2.31)
\end{aligned}$$

since

$$\begin{aligned}
\|D_t^k\boldsymbol{u}\|_\infty^2\,\|\nabla\boldsymbol{u}\|^2 &\le \frac{1}{2\pi}\|\nabla D_t^k\boldsymbol{u}\|\,\|\Delta D_t^k\boldsymbol{u}\|\,\|\nabla\boldsymbol{u}\|^2\\
&\le \frac{1}{4}\|\Delta D_t^k\boldsymbol{u}\|^2 + c\,\|\nabla\boldsymbol{u}\|^4\,\|\nabla D_t^k\boldsymbol{u}\|^2\,,
\end{aligned}$$

by using (2.2) and (2.7). The inequality (2.21) follows from (2.30) and (2.31).

Proof of (2.22). This is directly obtained by applying (2.2) to the function $D_t^k \boldsymbol{u}$.

Now, differential inequalities (2.20) and (2.21) cannot be integrated directly, because we do not have initial values at $t = 0$. To overcome this difficulty, we will follow the method of Heywood and Rannacher [4], introducing the weight functions t^k.

Lemma 2 *Suppose φ, ψ, α, $\beta \in C^1(0,T)$ are all non-negative and satisfy*

$$\frac{d\varphi}{dt} + \psi \le \alpha\varphi + \beta\,, \qquad (0 < t < T)\,.$$

Suppose also that

$$\int_0^t s^{n-1}\varphi(s)\,ds \le F_1(t)\,, \quad \int_0^t \alpha(s)\,ds \le F_2(t)\,, \quad \int_0^t s^n\beta(s)\,ds \le F_3(t)\,,$$

where n is a positive integer and $F_1, F_2, F_3 \in C[0,T)$. Then we have

$$t^n\varphi(t) + \int_0^t s^n\psi(s)\,ds \le F_4(t)\,, \qquad (0 < t < T)\,,$$

where $F_4 = (F_3 + nF_1)\exp F_2 \in C[0,T)$.

Proof. We have

$$\frac{d}{dt}(t^n\varphi) + t^n\psi \le \alpha t^n\varphi + t^n\beta + nt^{n-1}\varphi.$$

For $0 < \epsilon < t < T$, let $\Phi_\epsilon(t) = t^n\varphi(t) + \int_\epsilon^t s^n\psi(s)\,ds$. Then

$$\frac{d\Phi_\epsilon}{dt} \le \alpha\Phi_\epsilon + t^n\beta + nt^{n-1}\varphi\,, \qquad \Phi_\epsilon(\epsilon) = \epsilon^n\varphi(\epsilon)\,.$$

Hence

$$\begin{aligned}
\Phi_\epsilon(t) &\le \left[\epsilon^n\varphi(\epsilon) + \int_\epsilon^t \exp\left(-\int_\epsilon^s \alpha\,dr\right)\left(s^n\beta + ns^{n-1}\varphi\right)ds\right]\exp\int_\epsilon^t \alpha\,ds \\
&\le \left[\epsilon^n\varphi(\epsilon) + \int_\epsilon^t \left(s^n\beta + ns^{n-1}\varphi\right)ds\right]\exp\int_\epsilon^t \alpha\,ds \\
&\le [\epsilon^n\varphi(\epsilon) + F_3(t) + nF_1(t)]\exp F_2(t)
\end{aligned}$$

for $\epsilon < t < T$. The existence of $\int_0^t s^{n-1}\varphi(s)\,ds$ implies that

$$\liminf_{\epsilon\to 0+} \epsilon^n \varphi(\epsilon) = 0\,,$$

completing the proof of the lemma.

Now, corresponding to Lemma 1, we prove the following estimates.

Lemma 3 *For $0 < t < T$, there hold*

$$\begin{aligned}
\int_0^t s^{2i-2}\,\|D_t^i \boldsymbol{u}\|^2 ds &\le F(t)\,, \quad i\ge 1\,, &(2.32)\\
t^{2i-1}\,\|D_t^i \boldsymbol{u}\|^2 + \int_0^t s^{2i-1}\,\|\nabla D_t^i \boldsymbol{u}\|^2 ds &\le F(t)\,, \quad i\ge 1\,, &(2.33)\\
t^{2i}\,\|\nabla D_t^i \boldsymbol{u}\|^2 + \int_0^t s^{2i}\,\|\Delta D_t^i \boldsymbol{u}\|^2 ds &\le F(t)\,, \quad i\ge 0\,, &(2.34)\\
\int_0^t s^{2i}\|D_t^i \boldsymbol{u}\|_\infty^2 ds &\le F(t)\,, \quad i\ge 0\,, &(2.35)
\end{aligned}$$

where the $F(t)$ denotes generically a continuous and increasing function on $[0,\,T)$ depending only on i and $\|\nabla \boldsymbol{u}_0\|$, independent of Ω and m.

Proof. We obtain (2.34-0) from (2.17) and (2.18), and then obtain (2.35-0) by using (2.22-0).

From (2.34-0) and (2.35-0), we see that the coefficients in the differential inequalities (2.20) and (2.21) are integrable, i.e., they satisfy the condition set for α in Lemma 2.

We proceed by mathematical induction. Let $k\ge 1$ and assume that (2.32)-(2.35) are true for $i \le k-1$.

From the assumption and (2.19) we have

$$\begin{aligned}
\int_0^t s^{2k-2}\,\|D_t^k \boldsymbol{u}\|^2 ds &\le c\int_0^t s^{2k-2}\,\|\Delta D_t^{k-1}\boldsymbol{u}\|^2 ds\\
&\quad + c\sum_{i=0}^{k-1}\int_0^t \left(s^{2k-2-2i}\,\|\nabla D_t^{k-1-i}\boldsymbol{u}\|^2\right)\left(s^{2i}\,\|D_t^i \boldsymbol{u}\|^2\right) ds\\
&\le F(t)\,.
\end{aligned}$$

Hence we obtain (2.32-k). This technique of appropriately distributing the weight functions will be repeatedly used below without further comment.

With (2.32-k) and the assumption, we can apply Lemma 2 to the differential inequality (2.20) to obtain (2.33-k), which in turn enables us to apply Lemma 2 again to the differential inequality (2.21) and obtain (2.34-k). Finally, we obtain (2.35-k) from (2.22). This completes the proof.

Lemma 4 *For $0 < t < T$, there holds*

$$t^{2i+1}\,\|\Delta D_t^i \boldsymbol{u}\|^2 \;\le\; F(t)\,, \quad i \ge 0\,, \tag{2.36}$$

$$t^{2i+1/2}\|D_t^i \boldsymbol{u}\|_\infty^2 \;\le\; F(t)\,, \quad i \ge 0\,, \tag{2.37}$$

Proof. Letting $\boldsymbol{v} = \Delta D_t^k \boldsymbol{u}$ in (2.11) we get

$$\|\Delta D_t^k \boldsymbol{u}\|^2 \le 2\,\|D_t^{k+1}\boldsymbol{u}\|^2 + 2\|D_t^k(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|^2.$$

Using (2.31) we obtain

$$\begin{aligned}\|\Delta D_t^k \boldsymbol{u}\|^2 \;\le\; & c\,\|D_t^{k+1}\boldsymbol{u}\|^2 + c\,\|\nabla\boldsymbol{u}\|^4\,\|\nabla D_t^k \boldsymbol{u}\|^2 \\ & + c\sum_{i=0}^{k-1}\|D_t^i\boldsymbol{u}\|_\infty^2\,\|\nabla D_t^{k-i}\boldsymbol{u}\|^2\,.\end{aligned} \tag{2.38}$$

Using the estimates obtained in Lemma 3, by mathematical induction on (2.38) and (2.22), the proof is completed.

2.6 Proof of the theorem

With the estimates given in Lemma 3 and Lemma 4, we can follow the argument in [3] to prove the existence and regularity of the solution of Problem (2.1), as asserted in Theorem 5. The solution in the bounded smooth domain is obtained as a limit of a subsequence of the Galerkin approximations. Given an arbitrary domain, we can choose a sequence of bounded smooth subdomains expanding to the domain, and solve the problem in each subdomain with properly chosen initial data, and take a subsequence of these solutions which converge to the solution in the given domain. The estimates carry over in the above mentioned processes of taking limits.

Thus, we need only prove the uniqueness of the solution.

Suppose $\boldsymbol{v}$ is another solution. Let $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$. Then

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u} + \boldsymbol{w} \cdot \nabla \boldsymbol{w} = \Delta \boldsymbol{w}\,, \tag{2.39}$$

and

$$\lim_{t \to 0} \|\nabla \boldsymbol{w}(t)\| = 0\,. \tag{2.40}$$

Multiplying (2.39) with $\Delta \boldsymbol{w}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2 + \|\Delta \boldsymbol{w}\|^2 = (\boldsymbol{u} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u} + \boldsymbol{w} \cdot \nabla \boldsymbol{w}\,,\, \Delta \boldsymbol{w})\,.$$

We have

$$\begin{aligned}
(\boldsymbol{u} \cdot \nabla \boldsymbol{w}\,,\, \Delta \boldsymbol{w}) &\le \|\boldsymbol{u}\|_\infty \|\nabla \boldsymbol{w}\|\, \|\Delta \boldsymbol{w}\| \\
&\le \frac{1}{3}\|\Delta \boldsymbol{w}\|^2 + c\,\|\boldsymbol{u}\|_\infty^2\, \|\nabla \boldsymbol{w}\|^2\,, \\
(\boldsymbol{w} \cdot \nabla \boldsymbol{u}\,,\, \Delta \boldsymbol{w}) &\le \|\boldsymbol{w}\|_\infty \|\nabla \boldsymbol{u}\|\, \|\Delta \boldsymbol{w}\| \\
&\le c\,\|\nabla \boldsymbol{w}\|^{1/2}\, \|\nabla \boldsymbol{u}\|\, \|\Delta \boldsymbol{w}\|^{3/2} \\
&\le \frac{1}{3}\|\Delta \boldsymbol{w}\|^2 + c\,\|\nabla \boldsymbol{u}\|^4 \|\nabla \boldsymbol{w}\|^2\,, \\
(\boldsymbol{w} \cdot \nabla \boldsymbol{w}\,,\, \Delta \boldsymbol{w}) &\le \|\boldsymbol{w}\|_\infty \|\nabla \boldsymbol{w}\|\, \|\Delta \boldsymbol{w}\| \\
&\le c\,\|\nabla \boldsymbol{w}\|^{3/2}\, \|\Delta \boldsymbol{w}\|^{3/2} \\
&\le \frac{1}{3}\|\Delta \boldsymbol{w}\|^2 + c\,\|\nabla \boldsymbol{w}\|^6\,,
\end{aligned}$$

using the inequality (2.2) and Young's inequality. Hence

$$\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2 \le c\left(\|\boldsymbol{u}\|_\infty^2 + \|\nabla \boldsymbol{u}\|^4\right) \|\nabla \boldsymbol{w}\|^2 + c\|\nabla \boldsymbol{w}\|^6$$

Since

$$\int_0^t \left(\|\boldsymbol{u}\|_\infty^2 + \|\nabla \boldsymbol{u}\|^4\right) ds \le F(t)\,,$$

and we have (2.40), it is easy to prove that $\|\nabla \boldsymbol{w}(t)\| \equiv 0$ for all $0 < t < T$. Hence $\boldsymbol{w}(t) \equiv 0$.

Remark 1: If a bounded portion of the boundary of Ω is C^m, then $D_t^k \boldsymbol{u}(x,t)$ is uniformly C^m up to that portion of the boundary, for all k.

Remark 2: If we consider Burgers' equation with a "viscosity coefficient" ν:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \Delta \boldsymbol{u},$$

then T should be multiplied by ν^3. If we consider nonhomogeneous boundary values and an external force term, then we can prove a similar theorem of local existence and uniqueness of the solution, with T depending on the given data.

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