INVOLUTIONS WITH 1- OR 2-DIMENSIONAL FIXED POINT SETS ON
ORIENTABLE TORUS BUNDELES OVER A 1-SPHERE AND ON UNIONS OF
ORIENTABLE TWISTED I-BUNDELES OVER A KLEIN BOTTLE

by

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Abstract

We obtain a complete equivariant torus theorem for involutions on 3-manifolds $M$. $M$ is not required to be orientable nor is $H_1(M)$ restricted to be infinite. The proof proceeds by a surgery argument.

Similar theorems are given for annuli and for discs. These are used to classify involutions on various spaces such as orientable twisted $I$-bundles over a Klein bottle.

Next we restrict our attention to orientable torus bundles over $S^1$ or unions of orientable twisted $I$-bundles over a Klein bottle. The equivariant torus theorem is applied to the problem of determining which of these spaces have involutions with 1-dimensional fixed point sets. It is shown that the fixed point set must be one, two, three, or four 1-spheres. Matrix conditions that determine which of these spaces have involutions with a given number of 1-spheres as the fixed point sets are obtained.

The involutions with 2-dimensional fixed point sets on orientable torus bundles over $S^1$ and on unions of orientable twisted $I$-bundles over a Klein bottle are classified. Only the orientable flat 3-space forms $M_1$, $M_2$, and $M_6$ have involutions with 2-dimensional fixed sets. Up to conjugacy, $M_1$ has two involutions, $M_2$ has four involutions, and $M_6$ has a unique involution.
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INTRODUCTION

We investigate the problem of classifying involutions with 1-dimensional or 2-dimensional fixed sets on orientable torus bundles over $S^1$ or on unions of orientable twisted $I$-bundles over a Klein bottle. Cutting these spaces on incompressible tori $T$ gives trivial $I$-bundles over tori or orientable twisted $I$-bundles over a Klein bottle.

In Chapter III we prove a complete equivariant torus theorem for involutions. This theorem allows the cutting to be done in a manner that respects both the involution $i$ and the fixed set $\text{Fix}$, i.e., such that either $iT \cap T = \emptyset$ or $iT = T$ and $T$ and $\text{Fix}$ are transversal. The problem then reduces to one of classifying involutions on trivial $I$-bundles over tori or orientable twisted $I$-bundles over a Klein bottle. Such theorems have been proved in [8] and [14] but under additional hypotheses, such as, the first homology being infinite and the manifold being orientable. In [11] an equivariant torus theorem was proved under the assumption that the fixed set is a number of isolated points. Our theorem extends these results. In the nonorientable case we may have to allow the cutting torus to be replaced by a Klein bottle. Even in the orientable case two types of exceptional cases are possible.

To prove the equivariant torus theorem cut and paste techniques are used. An equivariant transversality theorem
is also required. In the two dimensional case when \( M \) is nonorientable, transversality can not be guaranteed. Certain interesting exceptional points arise; these will be called saddle points. Saddle points must be treated separately in the surgery arguments.

Analogous to the torus theorem are the annulus and disc theorems. These theorems are used in Chapter IV to classify the involutions on the trivial \( I \)-bundle over a torus and on the orientable \( I \)-bundle over a Klein bottle.

Let \( M \) denote an orientable torus bundle over \( S^1 \) or a union of orientable twisted \( I \)-bundles over a Klein bottle. In [6] Kim and Sanderson have classified the orientation reversing involutions on orientable torus bundles over \( S^1 \). Our techniques allow us to classify the involutions with 2-dimensional fixed point set on all \( M \). A subclass of these spaces are the orientable flat 3-space forms \( M_1, M_2, \ldots M_6 \), see Wolf [15]. \( M_6 \) is not a torus bundle and \( H_1(M_6) \) is finite. We show in §10 that \( M_1, M_2 \) and \( M_6 \) are the only \( M \) having involutions with 2-dimensional fixed point sets. These involutions are determined up to conjugacy.

The case of 1-dimensional involutions on these spaces is far less restrictive. Each of the orientable space forms has involutions with 1-dimensional fixed point sets but these are not the only \( M \) with such involutions. We determine in §9 which spaces \( M \) have involutions with 1-dimensional
fixed point sets. We do not deal with the problem of uniqueness in this thesis.

Several topics for further research present themselves. For example, classify the involutions on the nonorientable torus bundles over $S^1$. Can an equivariant theorem be proved for surfaces of higher genus? One could also investigate how the results would generalize from involutions to $n$-cyclic actions and finite group actions.
I. EQUIVARIANT TRANSVERSALITY AND DISC THEOREMS

§1. Preliminaries

Use \( \cap \), \( \cup \) and \( \subseteq \) to denote set intersection, union and subset. \( \cup \) does not denote disjoint union. Use upper indices to indicate dimension.

Throughout we use the piecewise linear category. This is to avoid wild fixed sets which can arise in the topological category, see [1].

A piecewise linear homeomorphism will be called an isomorphism.

Definition 1.1

Let \( M \) be a manifold with boundary \( \partial M \) and \( F \) a submanifold of \( M \) of lower dimension. \( F \) is \textit{proper} if \( F \cap \partial M = \partial F \). In particular a point is proper in \( M \) only if it is in the interior of \( M \). We will assume that all submanifolds are proper.

\( F \) will usually denote a surface, a compact connected manifold. A surface \( F \) in a 3-manifold \( M \) is \textit{incompressible} if \( F \) is not a 2-sphere or 2-cell and if for each 2-cell \( B \) in \( M \) with \( B \cap F = \partial B \) there is a 2-cell \( D \subseteq F \) with \( \partial D = \partial B \). A manifold \( M \) is irreducible if each 2-sphere in \( M \) bounds a 3-cell in \( M \).

Let \( M \) be a connected compact 3-manifold. An \textit{involution} \( \iota \) is an isomorphism with \( \iota \neq \text{id} \) and \( \iota^2 = \text{id} \).
Let \( \text{Fix} \) denote the fixed set \( \text{Fix}=\text{fix}(\iota)=\{x: \iota(x)=x\} \). Let \( \iota \) be an involution on a manifold \( M \) and \( \iota' \) an involution on a manifold \( M' \). \( \iota \) and \( \iota' \) are conjugate if there is an isomorphism \( h:M\to M' \) with \( \iota'=h\cdot \iota \cdot h^{-1} \). Call \( h \) a conjugation between \( \iota \) and \( \iota' \).

\( \iota \) is conjugation extendable if given any conjugate \( \iota' \) of \( \iota \) and an isomorphism \( h_0:\partial M\to \partial M' \) with \( \iota'|\partial M'=h_0\cdot \iota|\partial M\cdot h_0^{-1} \) then there is a conjugation \( h:M\to M' \) extending the isomorphism \( h_0 \).

Note that if \( \iota \) is conjugation extendable then so is any conjugate \( \iota' \). Further, to show conjugation extendability it suffices to check the case \( \iota'=\iota \). \( \iota \) is conjugation extendable with respect to a class \( H \) of isomorphisms \( \partial M\to \partial M' \), depending on the choice of conjugate \( \iota' \) of \( \iota \), if at least for any \( h_0\in H \) with \( \iota'|\partial M'=h_0\cdot \iota|\partial M\cdot (h_0^{-1}) \) there is a conjugation \( h:M\to M' \) extending the isomorphism \( h_0 \).

The following construction will be used often. See Figure 1. Let \( M=B\sqcup \iota B \) with \( B\cap \iota B=\partial B\cap \partial \iota B \) and similarly for \( M' \) and \( B' \). Let \( h_0:B\to B' \) be an isomorphism such that \( h_0|:B\cap \iota B\to B'\cap \iota' B' \) is a conjugation between \( \iota|B\cap \iota B \) and \( \iota'|B'\cap \iota' B' \). \( h_0 \) is extended by equivariance to \( h:M\to M' \) if we define \( h|_B=h_0 \) and \( h|_{\iota B}=\iota'.h_0.(\iota^{-1}) \). Then \( h \) is a conjugation between \( \iota|M \) and \( \iota'|M' \).
Lemma 1.2

Given a simplicial subdivision $K$ of $M$ and an involution $\iota$ of $M$ there is a subdivision $L$ of $K$ with $\iota$ simplicial with respect to $L$.

Proposition 1.3

Let $\iota$ be an involution on a manifold $M$. Let $L$ be a subdivision of $M$ with $\iota : L \to L$ simplicial and let $L'$ be the first barycentric subdivision of $L$.

Then $\text{Fix} = \text{fix}(\iota)$ is a subcomplex of $L'$. $\text{Fix}$ is the union of disjoint 0-, 1- and 2-dimensional proper submanifolds. Write $\text{Fix}^0$, $\text{Fix}^1$ and $\text{Fix}^2$ respectively for the unions of the 0-, 1- and 2-dimensional components of $\text{Fix}$.

If $v \in \text{Fix}^0 \sqcup \text{Fix}^2$ then $\iota$ is locally orientation reversing at $v$. If $v \in \text{Fix}^1$ then $\iota$ is locally orientation preserving at $v$. In particular if $M$ is orientable then $\iota$ is orientation reversing if $\text{Fix}^0 \sqcup \text{Fix}^2 \neq \emptyset$ and $\iota$ is orientation preserving if $\text{Fix}^1 \neq \emptyset$. 

Figure 1.
Proof: Use the following:

1) Let \( \Delta \) be a standard \( m \)-simplex (with standard subdivision) invariant under \( \iota \). Then \( \text{Fix} \cap \Delta \) is a subcomplex of the first barycentric subdivision of \( \Delta \).

2) If \( \text{Fix} \) contains a 3-simplex then \( \iota = \text{id} \).

If \( v \in \text{Fix} \) is a vertex of \( \text{int}(L) \) consider the link \( L_k \) of \( v \).

3) If \( L_k \cap \text{Fix} \) contains a 1-cell then \( L_k \cap \text{Fix} \) is one 1-sphere. So \( v \in \text{Fix}^2 \).

4) If \( L_k \cap \text{Fix} \) consists of \( m \geq 0 \) vertices then
\[
\chi(L_k/\iota) - m = \frac{1}{2} (\chi(L_k) - m)
\]
Since \( L_k/\iota \) is a surface and \( L_k \) is a 2-sphere it follows \( m=2 \) and hence \( v \in \text{Fix}^1 \), or it follows \( m=0 \) and hence \( v \in \text{Fix}^0 \).

QED

Corollary 1.4

Let \( \iota \) be an involution on \( M \) with fixed set \( \text{Fix} \). Then \( M/\iota = M/(m \sim \iota(m) \) for all \( m \in M \) is a manifold with possible singularities. \( \text{Fix} = \text{Fix}/\iota \) a disjoint union of submanifolds \( \text{Fix}^0, \text{Fix}^1 \) and \( \text{Fix}^2 \) with \( \text{Fix}^0 \) and \( \text{Fix}^1 \) proper in \( M/\iota \) and with \( \text{Fix}^2 \) a submanifold of \( \partial(M/\iota) = (\partial M)/\iota \cup \text{Fix}^2 \).

Proof: Consider the link \( L_k \) of vertices of \( \text{Fix} \).

QED
Remark 1.5

The fixed point free involutions on a manifold $M$ correspond to 2-fold coverings by $M$. If $\iota$ is an involution then $M/\iota = M/(m\sim \iota(m))$ for all $m \in M$ is 2-fold covered by $M$. Conversely, if $p:M \rightarrow X$ is a 2-fold cover then define an involution $\iota$ by requiring $\iota$ to interchange the two points of $p^{-1}(x)$ for every $x \in X$. $\iota$ is the nontrivial deck transformation induced by $p$.

Involutions on $M$ with fixed set $\text{Fix}$ correspond to 2-fold coverings by $M$ branched on $\text{Fix}$. $\iota|_{M-\text{Fix}}$ is fixed point free and $p|_{M-\text{Fix}}$ is unbranched.

§2. Equivariant Transversality

In order to be able to perform surgeries on a surface $F_0$ in a 3-manifold $M$ we would like to perform an ambient isotopy on $F_0$ such that the isotopic surface $F$ has the property that $F$, $\iota F$ and $\text{Fix}$ are pairwise transversal. This can be done if the manifold is orientable. If $\text{Fix}^2 \neq \emptyset$ and $M$ is nonorientable, however pairwise transversality is not possible in general. This necessitates using a somewhat weaker form of transversality.
**Lemma 2.1**

Let $F$ be a proper surface in a 3-manifold with $F$, $\iota F$ and $\Fix$ pairwise transversal. Then the components of $F \cap \iota F$ are 1-spheres and proper 1-cells. If $C$ is a component of $F \cap \iota F$ with $C \cap \Fix^2 \neq \emptyset$ then $C \subset \Fix^2$.

**Proof:** The first statement follows by transversality of $F$ and $\iota F$. The second statement follows on considering the star of a point in $C \cap \Fix^2$.

QED

For the 3-cell $B^3 = \{(x,y,z): |x| \leq 1, |y| \leq 1, |z| \leq 1\}$ in $\mathbb{R}^3$, let $i:B^3 \rightarrow B^3$ be the map $i(x,y,z) = (-x,y,z)$. Then $\Fix(i)$ is the intersection of $B^3$ with the $yz$ plane. Let $S$ be the 1-sphere obtained as the join of $\{(1,1,1),(-1,-1,1)\}$ with $\{(-1,1,-1),(1,-1,-1)\}$ and let $D$ be the cone from $(0,0,0)$ on $S$. $D$ is a saddle shaped region. See Figure 2. (We could alternately take $D$ defined by $\{z=xy/\sqrt{x\cdot x+y\cdot y}\} \cup \{(0,0,0)\}$.)
Notice that \( D \cap iD \) is the part of the \( x \) and \( y \) axis in \( B^3 \) while \( D \cap \text{Fix}(i) \) is part of the \( y \) axis. \( D \) and \( \text{Fix}(i) \) are transversal and \( iD \) and \( \text{Fix}(i) \) are transversal, but \( D \) and \( iD \) are not transversal at \((0,0,0)\). There is a subdivision making these spaces simplicial with all the vertices on \( \partial B^3 \cup (0,0,0) \).

**Definition 2.2**

Let \( F \) be a proper surface in a 3-manifold \( M \) and \( i \) an involution on \( M \) with fixed set \( \text{Fix} \). Call a point \( v \) a saddle point if \( v \in F \cap \text{Fix}^2 \) and if \( (F, iF, \text{Fix}) \cap \text{star}(v) \) is isomorphic to \( (D, iD, \text{Fix}(i)) \).

**Remark 2.3**

Saddle points exist since \( i \) is an involution with fixed set \( \text{Fix}(i) \). Although \( \partial D \), \( \partial iD \), \( \partial \text{Fix}(i) \) are pairwise transversal there is no 2-cell \( E \) with \( \partial E = \partial D \) and \( E \), \( iE \), \( \text{Fix}(i) \) pairwise transversal. Otherwise, since \( \partial E \cap \partial iE - \text{Fix}(i) = (\pm 1,0,0) \) there is a 1-cell \( I \) of \( E \cap iE \) with \( (1,0,0) \in \partial I \) and this 1-cell must meet \( \text{Fix} \), contradicting the previous lemma.

Let \( d \) denote the identification \((x,1,z) \sim (x,-1,-z)\) for all \( x \) and \( z \). Then \( D/d \) is an annulus in a solid Klein bottle \( B^3/d \) and no isotopy of \( D/d \) moves it to an annulus with \( F \), \( iF \), and \( \text{Fix}(i)/d \) pairwise transversal.
Definition 2.4

Let $F$ be a proper surface in a 3-manifold and $\iota$ an involution on $M$ with fixed set $\text{Fix}$. Then $F$, $\iota F$, and $\text{Fix}$ are almost pairwise transversal if:

1) $F$, $\iota F$ and $\text{Fix}$ are pairwise transversal except at a finite number of saddle points, and
2) The only components of $F \cap \text{Fix}$ containing saddle points are 1-spheres and each such 1-sphere contains at most one saddle point.

Let $E$ be the closure of $(F \cap \iota F) - \text{Fix}^2$. $E$ consists of disjoint 1-spheres and proper 1-cells: in a neighborhood of a saddle point, $F \cap \iota F - \text{Fix}^2$ corresponds to $[-1,0) \times 0 \times 0 \sqcup (0,1] \times 0 \times 0$ in the $B^3$ model for saddle points.

Let $E$ be a component of $E$ that contains a saddle point $v$. Then $E$ has a fixed point and is invariant under $\iota$. Therefore, either $E$ is a 1-cell with no fixed points other than $v$ or $E$ is a 1-sphere with exactly two fixed points $v$ and $w$. By transversality $w$ is in $\text{Fix}^1$ or $\text{Fix}^2$. In the latter case $w$ is a saddle point. We obtain the following proposition:
Proposition 2.5

Let $F$, $iF$ and $\text{Fix}$ be almost pairwise transversal. Then the components of $F \cap iF$ are of one of the following forms:

1) Components with no saddle points (standard components):
   a) proper 1-cell $I$ with $I \cap \text{Fix}=\emptyset$ or $I \subset \text{Fix}^2$
   b) proper 1-cell $I$ with $I \cap \text{Fix}=I \cap \text{Fix}^1=v$, $v$ a point
   c) 1-sphere $S$ with $S \cap \text{Fix}=\emptyset$ or $S \subset \text{Fix}^2$
   d) 1-sphere $S$ with $S \cap \text{Fix}=S \cap \text{Fix}^1=v_1 \cup v_2$ where $v_1$ and $v_2$ are points.

2) Components with saddle points:
   Type I component: $S_1 \cup I$ with $S_1 \cap I=\text{Fix} \cap I=w$, $S_1 \subset \text{Fix}^2$ and $w$ is the only saddle point on $S_1 \cup I$.
   Type II component: $S_1 \cup S$ with $S_1 \cap S=w$, $S_1 \subset \text{Fix}^2$, $S \cap \text{Fix}=v \cup w$, $v \in \text{Fix}$ and $w$ is the only saddle point on $S_1 \cup S$.
   Type III component: $S_1 \cup S_2 \cup S$ with $S_1 \cap S_2=\emptyset$, $S_1 \cap S=w_1$, $S_1 \subset \text{Fix}^2$, $S \cap \text{Fix}=w_1 \cup w_2$ and $w_1$ and $w_2$ are the only saddle points on $S_1 \cup S_2 \cup S$.

   Here $S$, $S_1$, and $S_2$ are 1-spheres, $I$ are 1-cells and $w_i$ are points.

   Note a regular neighborhood of any of $S$, $S_1$, or $S_2$ is a solid Klein bottle in case 2). Thus case 2) does not occur if the manifold is orientable.

Proof: If the regular neighborhood $N$ of $S$ is a solid torus then $\text{Fix}\cap N$, $F \cap N$ and $iF \cap N$ are all annuli or all Möbius
bands. Consider the components of $F \cap \partial N$ and $iF \cap \partial N$ in $\partial N - \text{Fix}$ for a contradiction. For example, if they are all annuli then let $A$ be a component of $\partial N - \text{Fix}$. $A \cap \text{Fix}$ and $A \cap i\text{Fix}$ are two 1-spheres that intersect transversally at one point. This is not possible in an annulus $A$. Compare with the proof of case 3 and 4 in step 1 of the next theorem (transversality theorem).

QED

**Corollary 2.6**

If $F$, $iF$ and $\text{Fix}$ are almost pairwise transversal, then they are pairwise transversal if one of the following holds:

a) $M$ is orientable
b) $F$ is a 2-cell
c) $F$ is an annulus with $\partial F \cap i\partial F = \emptyset$.

**Proof:** In case a) regular neighborhoods of 1-spheres are solid tori.

In cases b) and c) Type II or III components are excluded since the 1-sphere $S$ is nonseparating. In case c) type I components are excluded a priori, while in case b) 1-sphere $S$, separates so a proper 1-cell $C$ cannot intersect $S$, transversally at one point.

QED
A proper 1-cell $I$ bounds a disc $D$ in a surface if
$I = \partial D - \partial F$.

**Corollary 2.7**

Let $F$, $iF$, and $\text{Fix}$ be almost pairwise transversal and $C$ a proper 1-cell or 1-sphere component of $F \cap iF$, that is, let $C$ be a standard component. Then any disc in $F$ or $iF$ bounded by $C$ contains only standard components.

**Proof:** As in case b) in previous corollary. \[QED\]

**Equivariant Transversality Theorem 2.8**

Let $\iota$ be an involution on a 3-manifold $M$ with $\text{Fix} = \text{fix(\iota)}$ and let $F_0$ be a proper surface in $M$. Then there is an ambient $\epsilon$-isotopy on $M$ taking $F_0$ to a proper surface $F$ such that $F$, $iF$ and $\text{Fix}$ are almost pairwise transversal. In $\partial M$, if $\partial F$, $i\partial F$ and $\text{Fix}$ are pairwise transversal then the isotopy may be taken to be the identity on $\partial M - N$ where $N$ is a given neighborhood of $\partial \text{Fix}^2 \cap \partial F$.

**Proof:** Let $F = F_0$ be a proper surface. By Proposition 1.3 and Lemma 1.2 subdivide $M$ so that $\iota$ is simplicial with respect to the subdivision, $\text{Fix}$ is a subcomplex of the subdivision and $\text{Fix}$ is a disjoint union of 0-, 1- and 2-dimensional
components $\text{Fix}^0$, $\text{Fix}^1$ and $\text{Fix}^2$. All isotopies performed in the construction will be done in the star neighborhoods of certain simplexes. By taking a sufficiently fine subdivision $\epsilon$-isotopies are obtained.

Step 1) Adjust $F$ near $\text{Fix}^2$.

By isotopies similar to those in the third step below we can assume $F$ and $\text{Fix}$ are transversal, the isotopy not moving $\partial F$ unless $\partial F$ and $\partial \text{Fix}$ are nontransversal. In particular $F \cap \text{Fix}^0 = \emptyset$. Then $F \cap \text{Fix}^2$ consists of disjoint 1-spheres and 1-cell components proper in $M$.

Let $S$ be a 1-sphere component of $F \cap \text{Fix}^2$. Let $N'$ be a regular neighborhood of $S$ with $N' \cap F$ and $N' \cap \text{Fix}$ transversal and each an annulus or Möbius band. $S$ has a regular neighborhood $N$ contained in $\text{int}(N')$ invariant under $\iota$ with no vertices on $\text{int}(N) - S$ such that $N \cap \text{Fix}$ is a regular neighborhood of $S$ and $\text{Fix} \cap \partial N$ has a regular neighborhood $Q$ in $\partial N$ which is invariant under $\iota$ and has no vertices except on $\text{Fix} \cup \partial Q$.

Case 1) $F \cap N$ and $\text{Fix} \cap N$ are annuli. Then $N$ is a solid torus, $\partial Q$ has four components and $N - \text{Fix}$ consists of two components $N_1$ and $N_2$ which are interchanged by $\iota$. Let $J_1$ and $J_2$ be components of $\partial Q$ with $J_1 = N_1$ and $\iota J_1 \neq J_2$. Let $A_1$ be the annulus with $\partial A_1 = J_1 \cup S$ having no vertices except on $\partial A_1$. $F$ is isotopic to a surface $F'$ by an ambient isotopy which is the identity on $M - N'$ and such that $F' \cap N' \cap \text{Fix}$ $\subseteq N$ and $F' \cap N$
= \mathbb{A}_1 \cup \mathbb{A}_2$. Since \( \iota J_1 \neq J_2 \) it follows \( F' \cap N \cap \iota(F' \cap N) = S \) and \( F' \cap N, \iota(F' \cap N), \text{Fix} \cap N \) are pairwise transversal.

**Case 2)** \( F \cap N \) and \( \text{Fix} \cap N \) are Möbius bands. Then \( N \) is a solid torus, and \( \partial Q \) has two components that are interchanged by \( \iota \). If \( J \) is one of these, then \( J \) and \( S \) determine a Möbius band \( A \) with \( \partial A = J \). Proceed as in case 1.

If \( M \) is orientable Case 3 and 4 do not arise. Only in these cases do saddle points arise.

**Case 3)** \( F \cap N \) is an annulus and \( \text{Fix} \cap N \) is a Möbius band. Then \( N \) is a solid Klein bottle. Let \( A \) be one of the two (open) annuli components of \( \partial N - \text{Fix} \). There are two 1-spheres \( J_1 \) and \( J_2 \) which represent generators of \( H_1(A) = \mathbb{Z} \) with \( J_1 \) and \( J_2 \) intersecting transversally and at only one point \( x \). \( J_1 \) and \( S \) bound an annulus \( A_1 \) with \( A_1 \cap A_2 = S \cup I \) where \( I \) is a 1-cell with \( \partial I = x \cup y \) where \( y \in S \). Proceed as in Case 1 using \( F' \cap N = A_1 \cup \iota A_2 \). Then \( y \) is a saddle point and \( F' \cap N, \iota(F' \cap N) \) and \( \text{Fix} \cap N \) intersect pairwise transversally elsewhere in \( N \).

**Case 4)** \( F \cap N \) is a Möbius band and \( \text{Fix} \cap N \) is an annulus. This case is similar to case 3. Here \( A = \partial N - Q \) is an invariant annulus under \( \iota \). Find a curve \( J \) that bounds a Möbius band by lifting (from annulus \( A/\iota \)) a curve \( J' \) which represents twice a generator and which is embedded in \( A/\iota \) except for one transversal self intersection.
When $S$ is a 1-cell component of $F \cap \text{Fix}^2$, use an isotopy similar to the one of case 1 above. This isotopy may change $\partial F$ in $N \cap \partial M$.

**Step 2)** Adjust $F$ near $\text{Fix}^1$.

By step 1, $F \cap \text{Fix}^1$ consists of a number of vertices in $\text{int}(M)$. If $v \in F \cap \text{Fix}^1$ let $N'$ be a regular neighborhood of $v$ and let $N$ be the star neighborhood of $v$. Take the subdivision so that $N$ is in the interior of $N'$, $F \cap N$ is a proper 2-cell in $N$ and $\text{Fix} \cap N$ is a proper 1-cell. Since $F$ is transversal, $F \cap \partial N$ is a generator of $H_1(N - \text{Fix})$. Let $J'$ be a curve in the annulus $(\partial N - \text{Fix})/\sim$ representing twice a generator of this annulus. Take $J'$ embedded except for one transversal self intersection. $J'$ lifts to two 1-spheres $J$ and $\iota J$, which on coning to $v$ give 2-cells $D$ and $\iota D$. $D$, $\iota D$ and $\text{Fix}$ are pairwise transversal in $\text{int} N$. Proceed as in case 1 of step 1.

We obtain a surface $F$ and a neighborhood $N$ of $\text{Fix}$ such that $F$ has the required transversality properties in $N$. The following construction adjusts $F$ only on star neighborhoods of simplexes of $F \cap N$ where $F$ and $\iota F$ are not already pairwise transversal. By subdividing sufficiently we may assume without loss that $\text{Fix} = \emptyset$. For convenience assume also $\partial F = \emptyset$.

Let $K$ be a subdivision of $M$ with $\iota$ simplicial and $F$ a subcomplex of $K$. Let $\Delta$ be an $m$-simplex of $F$ in $K$ with $m = 0$, 1 or 2. Define $\text{St}(\Delta)$, the reduced star of $\Delta$ in $K$, to be all
3-simplexes $\sigma$ of $K$ with $\Delta \subseteq \sigma$ together with their faces. Let $\text{St}_F(\Delta)$, the reduced star of $\Delta$ in $F$, be all 2-simplexes $\sigma$ of $K$ with $\Delta \subseteq \sigma \subseteq F$ together with their faces. Let $p: M \longrightarrow M/\iota$ be the projection.

**Step 3)** There is a subdivision of $M$ and a proper surface $F'$ $\epsilon$-isotopic to $F$ such that for every simplex $\Delta$ of $F'$ either $p^{-1}(\Delta) \cap F' = \Delta$ or $\Delta$ is a 0- or 1-simplex with $\text{int}(\text{St}_{F'}(\Delta))$ and $\text{int}(\text{St}_{iF}(\Delta))$ transversal.

Call a simplex exceptional if it fails to satisfy these conditions and is of the highest possible dimension $m = 0, 1$ or 2. Induct on the number of such simplexes. If there are no exceptional simplexes the theorem is established.

Add all the vertices (and their translates under $\iota$) of form $(m+2)/(m+3) \ b + 1/(m+3) \ v$ where $b$ is the barycenter of $\Delta$ and $v$ is a vertex of $\text{St}(\Delta) - \Delta$. This determines a refinement $K'$ of $K$ with the same number of exceptional simplexes; no $m$-simplexes are subdivided for $m=1,2$ , while for $m=0$ transversality already holds away from vertices of $K$. Consider the reduced stars in $K'$. $\partial\text{St}'_F(\Delta)$ is a 1-sphere that decomposes $\partial\text{St}'(\Delta)$ into two components $D_+$ and $D_-$. There is an ambient isotopy taking $F$ to $F_1 = (F - \text{St}'_F(\Delta)) \cup D$. which is the identity except on $\text{St}'_F(\Delta)$. $F_1$ has fewer exceptional simplexes. When $m \neq 2$ this follows since $D_+ \cup D_-$ intersects the interior of any 2-simplex of $\text{St}(\Delta)$ transversally. 

QED
Regular neighborhoods of the standard components of $F \cap iF$ can be taken in a special form.

Definition 2.9

Let $F$, $iF$ and $\text{Fix}$ be almost pairwise transversal and $S$ a 1-sphere component of $F \cap iF$. Suppose, in addition, that the regular neighborhood of $S$ in $F$ and $iF$ is an annulus. Then there exists a regular neighborhood $V \subset \text{int}(M)$ of $S$ with the following properties:

1) $V \cap F$ and $V \cap iF$ are annuli. Since these intersect transversally, $V$ is a solid torus.
2) $\text{Fix}$ and $\partial V$ intersect transversally, $\text{Fix} \cap F \cap V \subset S$ and the closure of each component of $(\text{Fix} \cap V) - S$ meets $S$ and $\partial V$. In particular $\text{Fix}^0 \cap V = \emptyset$.
3) $\text{Fix} \cap V$ is an annulus, two proper 1-cells or empty.
4) If $iS=S$ then $iV=V$
5) If $iS\neq S$ then $iV \cap V = \emptyset$ and the above properties hold simultaneously for $iV$.

Property 3) can be arranged since if $\text{Fix} \cap S \neq \emptyset$ then $iS=S$. $i$ is an involution on a 1-sphere so either $i=\text{id}$ or $i$ has exactly two fixed points.

Call $V$ a standard neighborhood of $S$. The four 1-spheres $(F \sqcup iF) \cap \partial V$ decompose $\partial V$ into four (closed) annuli $a_1$, $a_2$, $\beta_1$, and $\beta_2$ with $a_1 \cap a_2 = \emptyset$ and $\beta_1 \cap \beta_2 = \emptyset$. Call these annuli the
standard annuli corresponding to the standard neighborhood of \( V \). Suppose \( \iota S = S \). Relabelling, if necessary, we may assume \( \iota (a_1 \cap \beta_1) = (a_1 \cap \beta_2) \). It follows that \( \iota a_1 = a_1 \). Then \( \iota \beta_1 = \beta_2 \) and \( \iota a_2 = a_2 \). When \( \text{Fix} \cap V \neq \emptyset \) we obtain \( \text{Fix} \cap a_1 \neq \emptyset, \text{Fix} \cap a_2 \neq \emptyset, \text{Fix} \cap \beta_1 = \emptyset = \text{Fix} \cap \beta_2 \), and each component of \( \text{Fix} \) meets both \( a_1 \) and \( a_2 \).

**Definition 2.10**

Let \( S \) be a 1-cell component of \( \text{Fix} \cap \text{Fix} \) where \( F, \iota F, \text{Fix} \) are pairwise transversal (near \( S \)). Then there exists a regular neighborhood \( V \) of \( S \) with \( V \cap \partial M \) a regular neighborhood of \( \partial S \), called a standard neighborhood of \( S \) with the following properties:

1) \( V \cap F \) and \( V \cap \iota F \) are 2-cells with \( \partial M \cap V \cap F \) and \( \partial M \cap V \cap \iota F \) each two 1-cells. Necessarily \( V \) is a 3-cell.

2), 4) and 5) as for 1-sphere standard neighborhoods.

3) \( \text{Fix} \cap V \) is a disc, one proper 1-cell or empty.

The four 1-cells \( (F \cup \iota F) \cap \partial V - \partial M \) subdivide \( \partial V - \partial M \) into four discs \( a_1, a_2, \beta_1 \) and \( \beta_2 \) with \( a_1 \cap a_2 = \emptyset, \beta_1 \cap \beta_2 = \emptyset \) and the properties as in the previous situation. Call these discs the standard discs corresponding to \( V \).
Remark 2.11

In the following theorem certain 1-sphere components $S$ of $F \cap iF$ have standard neighborhoods because $S$ bounds discs in $F$ and $iF$. In the disc theorem and partial annulus theorem, $F$ is orientable so again there are standard neighborhoods. In the torus theorem the construction will be made so as to keep $S$ in this form always. In the annulus theorem the case of a nonorientable $F$, a Möbius band, with 1-sphere components is treated separately.

Theorem 2.12

Let $M$ be a 3-manifold with involution $\iota$ and $F_0$ be an incompressible proper surface. Then there is an ambient isotopy of $M$ which is an $\iota$-isotopy on $\partial M$ taking $F_0$ to a proper surface $F$ such that $F$, $iF$, and $Fix$ are almost pairwise transversal and no 1-spheres in $F \cap iF$ bound 2-cells in $F$. If on $\partial M$, $\partial F$, $i\partial F$ and $\partial Fix$ are pairwise transversal then the isotopy may be taken to be the identity on $\partial M - N$ where $N$ is a given neighborhood of $\partial Fix \cap \partial F$.

Proof: By the preceding transversality theorem there is an $F$ with all the above properties except possibly 1-spheres in $F \cap iF$ bound 2-cells in $F$. By Corollary 2.7 those 2-cells contain no saddle components. Let $S$ be a 1-sphere of $F \cap iF$ innermost in $iF$, that is, there is a 2-cell $D \subset iFix$ with
D \cap F = \partial D = S. Since F is compressible, S bounds a 2-cell B in F. If \iota S = S then we may assume \iota B = D.

Let V be a standard neighborhood of S. Such a neighborhood exists since S bounds a disc in F and iF. Let a be the standard annulus meeting D but not B. Then \iota a \cap a = \emptyset. There is a bicollar D([-1, 1]) of D = D x 0 with

\partial D([-1, 1]) = D([-1, 1]) \cap F = S x [-1, 1]

and with D x 1 \cap a \neq \emptyset. Since D is innermost it follows that for a sufficiently thin collar (D x 1) \cap (D x 1) = \emptyset and F \cap (D x 1) = \emptyset. Consider F' = (F - (B \cup S x [-1, 1])) \cup D x 1. Then

F' \cap \iota F' \subseteq (F \cap \iota F) - S and F', \iota F' and Fix are almost pairwise transversal. Since M is irreducible and D \cup B is a 2-sphere, F' and F are ambient isotopic by an isotopy being the identity on \partial M. By induction, all 1-spheres bounding 2-cells can be removed.

QED

Definition 2.13

A 2-cell B in a 3-manifold is essential if it is proper and \partial B does not bound a 2-cell in \partial M. In an irreducible 3-manifold a nonseparating proper 2-cell is essential.

The following theorem also appears in [3].
Disc Theorem 2.14

Let $M$ be an irreducible 3-manifold with involution $\iota$. Suppose $M$ has an essential 2-cell $B_0$. Then there is an essential 2-cell $B \subset M$ such that $B$ and Fix are transversal and either $B \cap \iota B = \emptyset$ or $\iota B = B$. In the former case $B \cap \text{Fix} = \emptyset$ and in the latter case $B \cap \text{Fix}$ is a proper 1-cell of $B$ or one point in the interior of $B$. If $\partial B_0 \cap \partial B_0 = \emptyset$ then one can take $\partial B = \partial B_0$ and $B$ and $B_0$ are ambient isotopic by an isotopy that is the identity on $\partial M$.

Proof: By Theorem 2.12 and Corollary 2.6 there is an essential 2-cell $B$ with $B$, $\iota B$ and Fix pairwise transversal, $B$ and $B_0$ ambient isotopic and $B \cap \iota B$ is either empty or consists of proper 1-cells only. Assume $B \cap \iota B \neq \emptyset$ (in particular then $\partial B_0 \cap \iota B_0 \neq \emptyset$). By induction it suffices to show how to obtain a new 2-cell $B_i$ with fewer 1-cells in $B_i \cap \iota B_i$.

Let $D$ be an outermost disc of $B$: $D \subset B$ with $D \cap \iota B = \partial D \cap \iota B = I$ a proper 1-cell of $B$ and $\partial D - I \subset \partial B$.

If $\iota I = I$ define $D' = \iota B - \overline{D}$. If $\iota I \neq I$ define $D'$ to be the closure of the component of $\iota B - \overline{D}$ that does not contain $\iota I$. See Figure 3. Let $V$ be a standard neighborhood of $I$ and let $a_1$, $a_2$ and $\beta$ be standard discs of $V$ with $a_1 \cap a_2 = \emptyset$, $a_1 \cap \beta \cap \partial D = \emptyset$ and $\beta \cap D' \neq \emptyset$. Consider

$$B_i = (D \cup \beta \cup D') - \text{int}(V) \text{ and } B_2 = D \cup (\iota B - D').$$
Then $B_1 \cap iB_1 \subseteq (B \cap iB) - I$. If $B_1$ is essential we are done by induction or arrive at case $B_1 \cap iB_1 = \emptyset$. If $B_1$ is not essential then $\partial B_1$ bounds a 2-cell $E$ of $\partial M$. Since $M$ is irreducible the 2-sphere $B_1 \cup E$ bounds a 3-cell. This 3-cell does not meet $I$, otherwise $iB$ would not be essential. Using the 3-cell construct an ambient isotopy taking $B_2$ to $iB$.

So we may assume $B_2$ is essential. If $I = iI$ we have $iB_2 = B_2$ and note that $\text{Fix} \cap B_2 \subseteq \text{Fix} \cap I$ which is necessarily a point of $I$ or all of $I$. If $I \cap iI = \emptyset$ consider a sufficiently thin bicollar $Dx[-1,1]$ of $D = Dx0$ such that $Dx[-1,1] \cap I$ is a bicollar of $I$ in $iB$ and $Dx1$ meets $a_1$. Then $B_2' = (Dx1 \cup iB) - (Ix[-1,0] \cup D')$ is essential since it is isotopic to $B_2$ and $B_2' \cap iB_2' \subseteq B \cap iB - I$.

QED
II. INVOLUTIONS ON THE 3-CELL AND THE SOLID TORUS

§3. Some Involutions

The classification of involutions on a solid torus will be useful in the proof of theorems in the next chapter. The disc theorem will be used to reduce the problem of classifying involutions on a solid torus to one of classifying the involutions on a 3-cell.

Definition 3.1

Let \( C \) be the complex numbers. Let

- \( I = I^1 = [-1,1] \) be the standard 1-cell
- \( S^1 = \{ z \in C : |z| = 1 \} \) be the standard 1-sphere
- \( D^2 = \{ z \in C : |z| \leq 1 \} \) be the standard 2-cell
- \( T^2 = S^1 \times S^1 \) be the standard torus
- \( D_+ = \{ z \in D^2 : z = x + y \cdot i, y \geq 0 \} \), a 2-cell
- \( \text{Re} = \{ z \in D^2 : z = \bar{z} \} \subseteq D^2 \), a proper 1-cell
- \( \text{Im} = \{ z \in D^2 : z = -\bar{z} \} \subseteq D^2 \), a proper 1-cell

Define involutions on the above spaces as follows.

On \( S^1 : \kappa(z) = \overline{z} \) which is orientation reversing with fixed set two points \( \pm 1 \). \( a(z) = -z \) which is orientation preserving and fixed point free. Then \( a \cdot \kappa = \kappa \cdot a = -\kappa \) is conjugate to \( \kappa \) by a rotation by 90°.

On \( D^2 : \hat{\kappa}(z) = \overline{z} \) orientation reversing with fixed set one 1-cell \( \text{Re} \). \( \hat{a}(z) = -z \) which is orientation preserving with fixed set one point 0. Then \( -\hat{\kappa} \) is conjugate to \( \hat{\kappa} \) and has
fixed set Im.

On $I: r(t)=-t$ which is orientation reversing with fixed set one point $0$.

Define the map $\rho: S^1 \times S^1 \to S^1 \times S^1$ by $\rho(z, w) = (zw, w)$ and map $\tilde{\rho}: D^2 \times S^1 \to D^2 \times S^1$ similarly. Define involution $\omega: S^1 \times S^1 \to S^1 \times S^1$ by $\omega(z, w) = (w, z)$, which has fixed set one 1-sphere $\{(z, z): z\}$.

Lemma 3.2

There are five involutions up to conjugacy on an annulus $S^1 \times I$. They are: 1) $axid$ which is orientation preserving and fixed point free, 2) $axr$ which is orientation reversing and fixed point free, 3) $kxrt$ which is orientation preserving with fixed set two points, 4) $idxr$ which is orientation reversing with fixed set a 1-sphere, and 5) $kxid$ which is orientation reversing with fixed set two proper 1-cells.

Proof: When the dimension of the fixed set is one the fixed set separates. In the other case use the Euler characteristic argument given in part 4) of proof of Proposition 1.3.

QED
Definition 3.3

For the 3-cell $D^2 \times I$ define the following involutions (see Figure 4):

- $j_2 = \text{id}_{\mathbb{R}}$ having fixed set a proper 2-cell $D^2 \times 0$.
- $j_1 = \mathbb{R} \times \tau$ having fixed set an unkotted 1-cell $R \times 0$.
- $j_0 = \mathbb{R} \times \tau$ having fixed set one point $0 \times 0$.

$j_2$ and $j_0$ are orientation reversing while $j_1$ is orientation preserving. $j_1$ is conjugate to $j_1' = \mathbb{R} \times \text{id}$ which has fixed set $0 \times I$.

Theorem 3.4

An involution on a 3-cell is conjugate to $j_2$, $j_1$ or $j_0$.

All involutions on a 3-cell are conjugation extendable.

Proof: Let $\iota$ be an involution on 3-cell $E$. Apply Lemma 1.2 and Proposition 1.3.

Suppose $\text{Fix}^2 \neq \emptyset$. Since $\text{Fix}^2$ is proper and $\pi_1 E = 1$, $\text{Fix}^2$ separates. Let $E_1$ and $E_2$ be the components, $E = E_1 \sqcup E_2$ with

![Figure 4](image-url)
$\text{Fix}^2 = E_1 \cap E_2$. If $\text{Fix}^2$ were compressible then let $B$ be a compressing disc in $E_1$, say. Then $\partial B$ compresses $\text{Fix}^2$ in $E_2$. Using a Mayer-Vietoris sequence we see $[\partial B]$ must be trivial in $H_1(E_1) \oplus H_1(E_2) \cong H_1(\text{Fix}^2)$. Hence $\text{Fix}^2$ is a proper 2-cell.

We show $i$ is conjugate to $j$. Let $D_1 = D^2 \times [0,1]$. Construct an isomorphism $h_0$ from the disc which is the closure of $\partial E_1 - \text{Fix}$ to the closure of $\partial D_1 - D^2 \times 0$. In the conjugation extendable case we may assume this isomorphism is given. Extend $h_0$ over the fixed set $\text{Fix}$ and then cone to a point to obtain an isomorphism $h : E_1 \to D^2 \times [0,1]$. Extend this isomorphism by equivariance to get a conjugation.

Suppose $\text{Fix}^2 = \emptyset$. The Euler characteristic argument referred to in part 4) of proof of Proposition 1.3 applied to $i \mid \partial E$ shows $\text{Fix} \cap \partial E$ has 0 or 2 fixed points. In the former case $i$ is orientation reversing so $\text{Fix}^1 \cup \text{Fix}^2 = \emptyset$ and in the latter case $\text{Fix}^1 \neq \emptyset$.

Suppose $\text{Fix}^1 \cup \text{Fix}^2 = \emptyset$. By a Lefshetz number argument $i$ has one fixed point only, call it $v$. $\partial E / i$ is a projective plane so there is a conjugation $h_0 : \partial E \to \partial (D^2 \times I)$ between $i \mid \partial E$ and $j_0 \mid \partial E$. In the conjugation extendable case $h_0$ is given. By subdividing we may assume $\text{star}(v) \cap \partial E = \emptyset$ and $E - \text{int} (\text{star}(v)) \cong S^2 \times I$. Extend $h$ to a conjugation $E - \text{star}(v) \to D^2 \times I - \text{star}(0,0)$. This can be done by Theorem 1 in [9] since $i$ has no fixed point on $\partial E$. Finally cone to $v$. 
Suppose $\text{Fix}^1 \neq \emptyset$. By the above $\text{Fix}^1 \cap \partial E$ is two points. Consider the double $E \sqcup E'$ of $E$. It is a 3-sphere with involution $i \sqcup i'$ induced by $i$. By a result of Waldhausen [16], this involution has fixed set one unknotted 1-sphere.

Let $B$ be a 2-cell with $\partial B$ the fixed set and such that $B$ is in general position with respect to $\partial E$. $B \cap E$ is a punctured 3-cell and all but one component of $B \cap \partial E$ is a 1-sphere in the interior of $B$. By standard arguments these can be removed giving a 1-cell $B' \subseteq E$ with $\partial B' \subseteq \partial E \sqcup \text{Fix}^1$. This shows $\text{Fix}^1$ is one unknotted proper 1-cell.

We claim there is a (nonproper) disc $B$ embedded in $E$ with $\text{Fix} \subseteq \partial B \subseteq \text{Fix} \sqcup \partial E$ such that $B \cap iB = \text{Fix}$. Moreover, if $C \subseteq \partial E$ is a 1-cell with $C \cap iC = \partial C = \partial \text{Fix}$ then we may assume $\partial B \cap \partial E = C$. To establish this claim note that if $N$ is a star neighborhood of $\text{Fix}$ then the closure $V$ of $E - N$ is a solid torus with $i|$ fixed point free and orientation preserving. $V \cap N$ is an annulus and $i|(V \cap N)$ is also fixed point free. Using these facts and Disc Theorem 2.14 it is possible to construct a disc as required in the claim.

Construct a conjugation to $j_1$ as follows. Construct an isomorphism $B \longrightarrow D \times 0$ and extend by equivariance to $B \sqcup iB \longrightarrow D \times 0$. In the conjugation extendable case we set $C = h^{-1}( (\partial D \cap D_*) \times 0)$. Then $B \sqcup iB$ separates $E$ into two 3-cell components. We extend to an isomorphism over one of these components and then by equivariance to the other, giving a
Lemma 3.5

Let F be a 2-sided surface in a 3-manifold M and let \( \iota \) be an involution on M with \( \iota F = F \) and such that \( \iota \) interchanges sides of F. Then F is ambient isotopic to a surface \( F' \) with \( F' \cap \iota F' = \emptyset \).

Proof: Construct an \( \iota \) invariant bicollar \( F \times [-1, 1] \) of \( F = F \times 0 \) by using a star neighborhood of F. Then consider \( F' = F \times 1 \).

QED

Remark 3.6

Suppose \( \iota F = F \) for a 2-sided surface F. Let N be the 3-manifold obtained by cutting M along F. That is, replace \( F \times [-1, 1] \subseteq M \) by distinct copies \( F_1 \times [-1, 0] = F \times [-1, 0] \) and \( F_2 \times [0, 1] = F \times [0, 1] \). N has a subdivision induced from M. Let \( d: F_1 = F_1 \times 0 \rightarrow F_2 = F_2 \times 0 \) be the canonical identification. Then \( M \cong N/d \). Since \( \iota \) is simplicial there is a canonical involution k on N with \( \iota = k/d \). Note k.d = d.k and k(\( F_1 \)) = F_1 iff \( \iota \) does not interchange the bicollar. Conversely if k.d = d.k for an involution k then k induces an involution \( \iota \) in M with \( \iota F = F \).
4. Involutions on the Solid Torus

Definition 4.1

Let $V$ be the solid torus

$V = D^2 \times S^1 = \{(z,w) : |z| \leq 1, |w| = 1, \ z, w \in \mathbb{C}\}$. Recall Definition 3.1.

Define the following involutions on $V$ (see Figure 5):

1. $j_A = \hat{\pi} \pi d$ having fixed set the annulus $R \times S^1$.
2. $j_M = p \cdot (\hat{\pi} \pi d)$ having fixed set the Möbius band
   \[ \{(s \cdot e^{\pi it}, e^{2 \pi it}) : 0 \leq s \leq 1, -1 \leq t \leq 1\} \]
3. $j_{2D} = \pi \pi d$ having fixed set two 2-cells $D^2 \times \pm 1$.
4. $j_{DP} = p \cdot (\pi \pi d)$ having fixed set a 2-cell and a point
   \[ D^2 \times \pm 1 \cup 0 \times -1. \]
5. $j_S = \hat{\pi} \pi d$ having fixed set one 1-sphere $0 \times S^1$.
6. $j_{2C} = \hat{\pi} \pi d$ having fixed set two 1-cells $R \times \pm 1$.
7. $j_{2P} = \hat{\pi} \pi d$ having fixed set two points $0 \times \pm 1$.
8. $j_N = \hat{\pi} \pi d$ fixed point free and orientation reversing.
9. $j_O = \pi \pi d$ fixed point free and orientation preserving.

So $j_M(z,w) = (\bar{z}w, w)$ and $j_{DP}(z,w) = (z\bar{w}, \bar{w})$. The subscript describes the fixed point set or for the fixed point free involutions the orientability type. Recall by Proposition 1.3, since $V$ is orientable, involutions with 0- or 2-dimensional fixed sets are orientation reversing. Those with 1-dimensional fixed sets are orientation preserving. None of the above involutions are conjugate since all have different fixed sets or orientation type. Using $V = D^2 \times I / d$
Figure 5.

*Fixed point sets for the standard involutions.*

- \( j_A \)
- \( j_S \)
- \( j_M \)
- \( j_{2C} \)
- \( j_{2D} \)
- \( j_{2P} \)
- \( j_{DP} \)
- \( j'_M \)
- \( j'_{DP} \)
where \( d = \hat{a}x(\tau|\alpha I) \), involutions conjugate to \( j_M \) and \( j_{DP} \) can be defined as follows:

\[
\begin{align*}
  j_M' = \hat{a}xid/d & \text{ having fixed set Möbius band } \text{Re}xI/d \\
  j_{DP}' = idxr/d & \text{ having fixed set } D^2x0 \sqcup 0x1/d.
\end{align*}
\]

**Theorem 4.2**

If \( i \) and \( i' \) are involutions on \( V = D^2xS^1 \) with nonempty isomorphic fixed point sets or if \( i \) and \( i' \) are fixed point free and of the same orientation type, then \( i \) and \( i' \) are conjugate. An involution on \( V \) is conjugate to one of the nine involutions listed above.

**Proof:** Let \( i \) be an involution on \( V \). We show it is conjugate to a standard involution. For any given essential 2-cell \( B_0 \) of \( V \) there is an isomorphism \( h \) of \( V \) which takes \( B_0 \) to \( B = D^2x-1 \). So by applying the Disc Theorem 2.14, and replacing \( i \) by the conjugate involution \( h^{-1}i \cdot h \) for a suitable \( h \), we may assume \( i \) satisfies:

**Case 1)** \( iB = B \) and \( B \) intersects \( \text{Fix} \) transversally at \( \text{Re}x-1 \)

**Case 2)** \( iB = B \) and \( B \) intersects \( \text{Fix} \) transversally at \( 0x-1 \)

**or 3')** \( iB \cap B = \emptyset \)

There is an isomorphism \( (D^2x1)/d \rightarrow D^2xS^1 \) where \( d = idx(\tau|\alpha I) \) is given by \( d(z,t) = (z; e^{i\pi t}) \). The isomorphism takes \( D^2x-1/d \) to \( B \). Write also \( B = D^2x-1 \). In case 3') by adjusting the isomorphism \( h \) we may assume that \( iB = D^2x1 = (D^2x0)/d \). Call
C. = \(D^2 \times [-1,0]\) and \(C. = D^2 \times [0,1]\). The case 3') splits into two cases:

Case 3) \(\iota B \cap B = \emptyset\) and \(\iota C. = C.\).

Case 4) \(\iota B \cap B = \emptyset\) and \(\iota C. = C.\).

We show first, if the involution \(\iota\) falls into:

- case 1) it has fixed set that of \(j_A\) or \(j_M\),
- case 2) it has fixed set that of \(j_S\),
- case 3) it is fixed point free as \(j_N\) and \(j_Q\) are,
- case 4) it has fixed set that of \(j_{2D}, j_{DP}, j_{2C} \text{ or } j_{2P}\),

and we show second, if \(\iota\) and \(\iota'\) fall into the same case 1) - 4) then \(\iota\) and \(\iota'\) are conjugate. This will complete the proof because the nine standard involutions cover all possible fixed sets that can arise and none occurs in more than one case 1) - 4).

All constructions done for \(\iota\) are to be performed for \(\iota'\) also even if not explicitly stated. Use a prime ' to denote the corresponding construct.

In Case 1): The involution \(\iota\) on \(D^2 \times I/d\) induces an involution \(\lambda\) on \(D^2 \times I\) with the property \(\lambda.d = d.\lambda\) when restricted to \(D^2 \times \partial I\). \(\text{Fix}(\lambda)\) is proper and 2-dimensional since \(\text{Fix} = \text{fix}(\iota)\) is transversal to \(B\). So \(\lambda\) is conjugate to the standard involution \(j_{2}\) of the 3-cell. In particular \(\text{Fix}(\lambda)\) is a 2-cell. \(\text{Fix}\) is obtained by identifying two disjoint 1-cells in the boundary of the two cell so \(\text{Fix}\) is
an annulus or Möbius band.

Suppose \( i' \) is given. \( \lambda|B \) and \( \lambda|B' \) are conjugate so there is a conjugation \( h:B \to B' \). Using \( d \) and \( d' \) we may extend \( h \) to a conjugation \( D^2 \times \partial I \to D^2 \times \partial I \). \( \partial \text{Fix}(\lambda) \) decomposes \( \partial D^2 \times I \) into two 2-cells which are interchanged under \( \lambda \). Let \( J \) be an (open) component of \( (\partial D^2 \times I) - \text{Fix}(\lambda) \) selected so that \( h(J) = J' \). Then \( \text{Fix} \) is an annulus if \( J \) and \( d(J) \) are in the same 2-cell determined by \( \partial \text{Fix}(\lambda) \) and \( \text{Fix} \) is a Möbius band if \( J \) and \( id(J) \) are in the same 2-cell determined by \( \partial \text{Fix}(\lambda) \). Therefore, \( h \) can be extended to a conjugation

\[
h: D^2 \times \partial I \sqcup \text{Fix}(\lambda) \to D^2 \times \partial I \sqcup \text{Fix}(\lambda').
\]

Extend \( h \) over one of the 2-cells that \( \partial \text{Fix}(\lambda) \) decomposes \( D^2 \times I \) into. Then extend to the other cell by equivariance. This gives a conjugation \( h \) defined on \( \partial(D^2 \times I) \). By the conjugation extendable property for the 3-cell, \( h \) extends to all of \( D^2 \times I \) and hence induces a conjugation on \( D^2 \times I / \partial \) between \( \iota \) and \( \iota' \).

In case 2): As in case 1) the involution \( \iota \) on \( D^2 \times I / \partial \) induces an involution \( \lambda \) on \( D^2 \times I \) with \( \lambda . d = d . \lambda \) on \( D^2 \times \partial I \). \( \text{Fix}(\lambda) \) is proper and 1-dimensional so \( \lambda \) is conjugate to the standard involution \( j \), of the 3-cell. So \( \text{Fix} \) is a 1-sphere.

Suppose \( \iota' \) is given. \( \lambda|B \) and \( \lambda|B' \) are conjugate so there is a conjugation \( h: D^2 \times \partial I \to D^2 \times \partial I \) with \( h . d = d' . h \). Since \( \lambda|\partial B \) is orientation preserving, \( h \) extends to \( \partial(D^2 \times I) \).
By the conjugation extendable property for 3-cells, h extends to all of $D^2 \times I$ and hence induces a conjugation on $D^2 \times I/d$.

In case 3): The involution must be fixed point free. Let $\iota$ and $\iota'$ be of same orientation type. Construct an isomorphism $h:B \to B'$ and extend to a conjugation $h:B \sqcup \iota B \to B' \sqcup \iota'B'$ by equivariance. Since the orientation type is the same, $h$ extends to all of $\partial C$, and then to an isomorphism $h:C \to C'$ by coning. Finally extend to $D^2 \times S^1 = C \sqcup C'$ by equivariance.

In case 4): $\iota|C$ and $\iota|C$ are involutions on 3-cells so each has fixed set a point, a proper 1-cell or a proper 2-cell. Since $(B \sqcup \iota B) \cap \text{Fix} = \emptyset$ it follows $\text{Fix} = \text{Fix}(\iota|C) \sqcup \text{Fix}(\iota|C)$. Moreover $\iota|C$ and $\iota|C$ must have the same orientation type so $\text{Fix}$ is one of: two 2-cells, two points, a 2-cell union a point, or two 1-cells. Suppose $\iota$ and $\iota'$ have isomorphic fixed sets. Arrange notation so that $\iota|C$ and $\iota'|C'$ have isomorphic fixed sets. Construct a conjugation $h:B \sqcup \iota B \to B' \sqcup \iota'B'$ as in case 3). In view of the conjugation extendable property of 3-cells, it suffices to show $h$ extends to a conjugation $\partial C \to \partial C'$. Let $G = \partial C$, and let $\text{Fix}$ now denote $\text{Fix}((\iota|C)$.

Case 4.1) $\iota|C$ is conjugate to $j_2$. Fix decomposes $G$ into two components one of which, $E$, contains $B$. Extend $h$ to an isomorphism $h:E \to E'$ and by equivariance to a
conjugation $h: G \longrightarrow G'$.

**Case 4.2** $i|C, \text{ is conjugate to } j_1$. $G/\iota$ is a 2-sphere such that $\text{Fix}=\text{Fix}/\iota$ misses $B/\iota$. So lifting an appropriate 1-cell $J_1$ in $(G/\iota) -(B/\iota)$ gives a 1-cell $J$ of $G$ with $J \cap J = \text{Fix}$. In addition, $J \sqcup \iota J$ determines a component $E$ of $\partial G$ containing $B$ but not $\iota B$. Define an isomorphism $h_0: J \longrightarrow J'$ and extend by equivariance to $J \sqcup \iota J$. Notice we could have selected $h_0: J \longrightarrow \iota'J'$ instead, so if orientations are fixed for $J \sqcup \iota J$ and $J' \sqcup \iota'J'$, we may select $h_0$ to be either orientation preserving or orientation reversing. Hence $h|_{B \sqcup h_0}$ extends over the (open) annulus $E -(B \sqcup J \sqcup \iota J)$ to an isomorphism $h_1: E \longrightarrow E'$. Extend by equivariance to $G$.

**Case 4.3** $i|C, \text{ is conjugate to } j_0$. Then $E=(G-\text{int}(B \sqcup \iota B))/\iota$ is a Möbius band and $h$ induces an isomorphism $\partial E \longrightarrow \partial E'$. This isomorphism extends to all of $E$. $G$ double covers $G/\iota$ and the isomorphism lifts to an isomorphism $h': G \longrightarrow G'$ extending $h$. By construction $h'$ is a conjugation.

QED

Let $C_iP$ be the space obtained by coning a real projective space to a point $v_i$. Then $(C_iP, v_i)$ is isomorphic to $(B^3, 0)/j_0$. The descriptions of the standard involutions $j$ on a solid torus $V$ can be used to compute $V/j$. Use $\text{Fix}$ to denote $\text{Fix}/j$ and recall Corollary 1.4 in this connection.
$V/j_A$ is a solid torus $D^2 \times S^1$ with $\text{Fix}$ an annulus $\text{Re}(\partial D^2) \times S^1$.

$V/j_{M'}$ is a solid Klein bottle $D^2 \times I / \approx x(\tau \mid \partial I)$ with $\text{Fix}$ the Möbius band $\text{Re}I$.

$V/j_{2D}$ is a 3-cell with $\text{Fix}$ two 2-cells.

$V/j_{DP'}$ is $C_1P$ with $\text{Fix}$ the point $v_1$ and a 2-cell.

$V/j_S$ is a solid torus $D^2 \times S^1$ with $\text{Fix}$ the 1-sphere $0 \times S^1$.

$V/j_{2C}$ is a 3-cell with $\text{Fix}$ two proper unknotted 1-cells.

$V/j_{2P}$ is a boundary connected sum of $C_1P$ and $C_2P$ with $\text{Fix} = v_1 \sqcup v_2$

(i.e.) $V/j_{2P} = D^2 \times S^i \sqcup ((z,1) \sim (-z,1), (z,-1) \sim (-z,-1))$ where $S^4 = D^4 \sqcup S^1$, $v_1 = (0,1)$, $v_2 = (0,-1)$ and $D^2 \times I$ is the connected sum disc.

$V/j_N$ is a solid Klein bottle $D^2 \times I / \approx x(\tau \mid \partial I)$.

$V/j_O$ is a solid torus.

**Corollary 4.3**

If $\iota$ is an involution on a solid torus $V$ then $V/\iota$ is isomorphic to one of the spaces $V/j$ above. The isomorphism type of the fixed set and orientability type of $\iota$ determine $V/\iota$ up to isomorphism.
**Example 4.4**

\[ j_0 \] is not conjugation extendable because \( s(z,w) = (z,zw) \) determines a conjugation \( s: \partial V \to \partial V \) for \( j_0 | \partial V \) that does not extend to \( V \).

**Corollary 4.5**

The orientation reversing involutions are conjugation extendable. If \( V \) and \( V' \) are solid tori with conjugate orientation preserving involutions \( \iota \) and \( \iota' \) respectively then the involutions are conjugation extendable with respect to the class of isomorphisms \( \partial V \to \partial V' \) that:

1) extend to isomorphisms \( V \to V' \), for the case \( \iota \) conjugate to \( j_0 \) or \( j_S \).

2) extend to isomorphisms \( V_{\text{Fix}} \to V'_{\text{Fix}} \), for the case \( \iota \) conjugate to \( j_{2c} \).

**Proof:** It suffices to show conjugation extendable for the standard involutions \( j \) only. In fact it suffices to show given \( h': \partial V \to \partial V \) an isomorphism with \( h'.j. (h'^{-1}) = j \), that \( h' \) extends to an isomorphism \( H': V \to V \) with \( H'.j. (H'^{-1}) = j \). Now \( h' \) induces an isomorphism \( h: \partial V/j \to \partial V/j \). We show, for each \( j \), \( h \) extends to an isomorphism \( H: V/j \to V/j \). Let \( p: V \to V/j \) be induced by inclusion. \( p|(V-\text{Fix}) \) is a double cover and \( p|\text{Fix} \) is an isomorphism. Check that \( H.(p|): V-\text{Fix} \to (V-\text{Fix})/j \) lifts to \( H': V-\text{Fix} \to V-\text{Fix} \) and thus
obtain a conjugation.

Let $W = V/j$. When $Fix^2 \neq \emptyset$ notice that $h$ is only defined on a proper submanifold $(\partial V)/j$ of $\partial (V/j)$.

The extensions $H$ of $h$ are clear for $j_A$, $j_{2D}$, $j_S$, $j_O$, $j_{DP}$ and $j_M$. For $j_N$ and $j_{2P}$, $\partial W$ is a Klein bottle. For $j_N$, the boundary of an essential proper disc $D$ represents the unique element of order two in $H_1(\partial W) = \mathbb{Z}_2 \oplus \mathbb{Z}$. Since $h(\partial D)$ is also of order 2, $h(\partial D)$ bounds an essential proper disc $D'$. Extend by coning over $D$ and then over the 3-cell $W-(\partial W \cup D)$. For $j_{2P}$, $\partial D^2 \times S^1$ is a 1-sphere that separates $\partial W$ and is 2-sided in $\partial W$. Such 1-spheres represent the element $(0,2) \in H_1(\partial W)$. Proceed as above. For $j_{2C}$, let $h: \partial W \cup Fix \longrightarrow \partial W' \cup Fix'$ be given. A 1-sphere in $\partial W-Fix$ that decomposes $\partial W \cup Fix$ into two components each containing one component of $Fix$ has the property that it bounds a proper 2-cell $D$ in $W$ that misses $Fix$. Use this 2-cell to extend $h$.

QED

Remark 4.6

Consider the solid Klein bottle $V = D^2 \times I/d$ where $d = \mathbb{R}x(\tau \mid \partial I)$. Then $j_A = \mathbb{R}xid/d$ with fixed set an annulus, $j_M = -\mathbb{R}xid/d$ with fixed set a Möbius band, $j_{DC} = idx\tau/d$ with fixed set a 2-cell and a 1-cell, $j_{CP} = -\mathbb{R}x\tau/d$ with fixed set a point and a 1-cell, and $j_S = \mathbb{R}xid/d$ with fixed set a 1-sphere are the only five involutions on a solid Klein
bottle, up to conjugacy. The proof is very similar to the one given for the solid torus. Since \(d\) is orientation reversing, however, case 3) does not arise and in case 4) only the combinations that were disallowed previously can occur.
III. EQUIVARIANT ANNULUS AND TORUS THEOREMS

§5. Annulus Theorems

Definition 5.1

A proper annulus A in a 3-manifold M is trivial, if A decomposes M into a solid torus \( V = \mathbb{D}^2 \times \mathbb{S}^1 \) and a submanifold \( M_0 \) such that:

\[
M = M_0 \cup V, \quad M_0 \cap V = \partial M_0 \cap \partial V = A
\]

and there exists a nonseparating proper 2-cell \( B \subset V \) with \( B \cap A = \partial B \cap A \) a nonseparating 1-cell in A.

Otherwise call A nontrivial. Call V a solid torus that trivializes A.

Note that if A does not separate M or if \( \partial A \) is in different boundary components of M then A is nontrivial.

Definition 5.2

Call a nontrivial incompressible proper annulus an essential annulus. Call an incompressible proper Möbius band an essential Möbius band.

Let F be a surface and S a component of \( F \cap \iota F \). In some surgeries performed later we will wish to replace F by \( F' = F \times [-1, 1] \) where \( F \times [-1, 1] \) is a bicollar of F. To insure that \( F' \), \( \iota F' \) and Fix are transversal at least in standard neighborhoods (see Definitions 2.9-.10), the following lemma
Lemma 5.3

Let $V$ be a solid torus and $\iota: V \to V$ an involution. Let $A_0$ and $A_1$ be annuli in $\partial V$ with $\partial A_0 = S_0 \cup S_1$, $\partial A_1 = S_1 \cup S_2$, $A_0 \cap A_1 = S_1$, $\iota S_0 = S_0$, $(A_0 \cup A_1) \cap \iota(A_0 \cup A_1) = S_0$ and $\text{Fix} \cap (A_0 \cup A_1) \subseteq S_0$. Then there is a proper annulus $A \subset V$ such that $A$, $\iota A$ and $\text{Fix}$ intersect transversally with $A \cap \iota A = S$ a 1-sphere and $\partial A$ having one component in $\text{int}(\iota A_0)$ and the other component is $S_2$.

A similar statement holds if $V$ is a 3-cell and $A_0$, $A_1$ are 2-cells and $S_1$ are 1-cells. See Figure 6.

Proof: By transversality of $\text{Fix}$, by taking a sufficiently small regular neighborhood $N$ of $A_0 \cup A_1 \cup \iota A_0 \cup \iota A_1$, we may assume one of the following holds:

1) $\text{Fix} \cap N = \emptyset$

2) $\text{Fix} \cap N$ consists of two disjoint 1-cells $I_1$ with

![Diagram](https://via.placeholder.com/150)

Figure 6.
exactly one point of $\partial I^1$ in $S_0$ and the other in $\text{int}(V) \cap \partial N$
or 3) $\text{Fix}_N$ is an annulus such that one boundary component is $S_0$ and the other in $\text{int}(V) \cap \partial N$.

Further, there is a regular neighborhood $N'$ of $S_0 \subset N$ such that $iN'=N'$, $A_0'=N' \cap A_0$ is an annulus, $A_0' \cup iA_0'=N' \cap \partial V$ and properties 1)-3) hold with respect to $N'$.

Let $B$ be the annulus which is the closure of $N' - A_0 \cup iA_0$. Then there is an $S$ in $\text{int}(B)$ with $iS=S$ and $\text{Fix}_B \subset S$. There is an annulus $A''$ in $\overline{V-\partial N'}$ such that $\partial A''=S \cup S$ and $A'' \cap iA''=S$. Let $A'$ be the component of $B-S$ that meets $iA_0$. Then $A=A' \cup A''$ is the desired annulus.

QED

Remark 5.4

A solid Klein bottle is a twisted $I$-bundle over an annulus. The annulus is essential but it does not separate the boundary.

Lemma 5.5

If $U$ is a solid torus then $U$ has no essential annuli. If $U$ is a solid Klein bottle then $U$ has no essential annuli that separate $\partial U$.

Moreover, suppose $A'$ is an annulus contained in $\partial U$ such that a nonseparating proper disc $D$ of $U$ intersects $A'$ in exactly one nonseparating $I$-cell of $A'$. If $A$ is an
incompressible proper annulus disjoint from $A'$ then the solid torus which trivializes $A$ may be taken to be disjoint from $A'$.

**Proof:** Suppose $A$ is an essential annulus. Then let $D$ be any proper nonseparating 2-cell of $U$. (When $A'$ is given, take $D$ as in the statement.) Make $A$ and $D$ transversal. Since $A$ is incompressible adjust $D$ so that $A \cap D$ consists of 1-cells only.

If $A \cap D = \emptyset$ then $A$ is contained in a 3-cell obtained by removing a sufficiently small regular neighborhood of $D$ from $U$. This contradicts incompressibility.

If $A \cap D \neq \emptyset$ let $B$ be an outermost 2-cell of $D$ (and disjoint from $A'$ if $A'$ is given): so $B \cap A = \partial B \cap A = I$ is a 1-cell and $B \cap \partial U = \partial B - I$. If $I$ bounds a 2-cell in $A$, then by an isotopy moving $B$, obtain a disc $D'$ with fewer 1-cells in $A \cap D'$. Assume now that $I$ does not bound a 2-cell in $A$. Then $I$ separates $A$. Let $V$ be the closure of the component of $U - A$ that meets int($B$). $\partial A$ decomposes $\partial U$ into two annuli or possibly, in the case where $U$ is a solid Klein bottle, into an annulus and two Möbius bands. However, in the latter case $\partial B \cap \partial U$ must meet the annulus. It follows that $\partial V \cap \partial U$ is an annulus and $V$ is a solid torus with the properties making $A$ trivial.

QED
We next state the partial annulus theorem and the annulus theorem. The proofs are omitted. They are similar in spirit to the proof of the torus theorem.

**Partial Annulus Theorem 5.6**

Let $M$ be an irreducible 3-manifold with involution $\iota$. Let $A_0$ be an essential annulus with $\partial A_0 \cap \iota \partial A_0 = \emptyset$. Then:

1) there is an essential annulus $A$ with $A \cap \iota A = \emptyset$ and $\partial A \cup \iota \partial A = \partial A_0 \cup \iota \partial A_0$

or 2) there are two disjoint essential annuli $A_1, A_2$ with $\iota A_1 = A_1$, $\iota A_2 = A_2$, and $\partial (A_1 \cup A_2) = \partial A_0 \cup \iota \partial A_0$ and $\text{Fix}$ is transversal to $A_1$ and $A_2$.

**Example 5.7**

The involution $\mathbb{R} \times \text{id}$ on $D^2 \times I$ induces an involution $\iota$ on $\mathbb{R}P^2 \times I = D^2 \times I / d$ where $d = \mathbb{R} \times \text{id}$ is an identification defined on $\partial D^2 \times I = S^1 \times I$. $\text{Fix} = (\text{Re} \cup \{i\}) \times I$. No essential annulus or Möbius band satisfies $A \cap \iota A = \emptyset$ or $\iota A = A$ and $A$ and $\text{Fix}$ transversal. There is an annulus, however, with $\iota A = A$ but it is not transversal to $\text{Fix}$. 
Example 5.8

Consider the nonorientable twisted I-bundle $I \times I / \partial I$ over a torus, where $d = (\tau|\partial I) \times I \partial I \tau \sqcup \partial I \tau$. The involution $\iota = \partial I \times \partial I \tau / \partial I \tau$ has Möbius bands but no annuli $A$ with $A \cap \iota A = \emptyset$ or $\iota A = A$ and $A$ and $\text{Fix}$ transversal.

Annulus Theorem 5.9

Let $M$ be an irreducible 3-manifold with involution $\iota$. Suppose $A_0$ is an essential annulus or Möbius band in $M$ with $\partial A_0 = S_0 \sqcup S_2$ where, if $A_0$ is a Möbius band, $S_0 = S_2$. Let $R_1$, respectively $R_2$, be the component of $\partial M$ with $S_0 \subseteq R_1$, respectively $S_2 \subseteq R_2$. Assume $R_1$ is incompressible and if $R_1 \neq R_2$ assume also that $R_1$ is not a projective space.

Then there is an essential annulus or Möbius band $A$ with either $A \cap \iota A = \emptyset$, or $\iota A = A$ and $A$ and $\text{Fix}$ transversal and in both cases $\partial A \sqcup \iota \partial A \subseteq R_1 \sqcup R_2 \sqcup \iota R_1 \sqcup \iota R_2$.

If $M$ is orientable $A$ may be taken to be an annulus.
§6. Equivariant Torus Theorem

Lemma 6.1

Let M be an irreducible 3-manifold containing an incompressible torus. Let F be a 1-sided Klein bottle in the interior of M and W a regular neighborhood of F in M with ∂W a torus. Then ∂W is an incompressible torus.

Proof: If not then M=W∪U with U a solid torus. Necessarily W is an orientable twisted I-bundle over T and M is orientable. The inclusion of U in M determines an index two subgroup of π₁(M). Consider p:M→M, the 2-sheeted covering corresponding to that subgroup. Then p⁻¹(W)=Tx[-1,1] where T is a torus with p(Tx0)=F. p⁻¹(U)=V₁∪V₂ is two disjoint solid tori. M is a lens space. But M and hence M contains a 2-sided incompressible torus.

QED

Equivariant Torus Theorem 6.2

Let M be an irreducible 3-manifold with involution ρ. Suppose M contains an incompressible torus. Then one of the following holds:

(I) There is a 2-sided incompressible torus or Klein bottle T in int(M) transversal to Fix with T∩ρT=∅ or ρT=T.
(II) \( M = V_1 \cup V_1 \cup U_1 \cup U_1 \), where \( V_1 \) and \( U_1 \) are solid tori and \( \partial V_1 = V_1 \) and \( \partial U_1 = U_1 \).

There are annuli \( A_i, i = \pm 1 \), with
\[
A_i \cap A_1 = A_i \cap V_1 = \partial A_i = \partial V_1 = V_1 \cap V_1 = U_1 \cap U_1
\]
and \( V_1 \cap U_1 = A_1, V_1 \cap U_{-1} = \partial A_1, \partial V_1 = A_1 \cup \partial A_{-1}, \partial U_1 = A_1 \cup \partial A_{-1} \). See Figure 7.

\( A_1 \cup A_{-1} \) is a 2-sided incompressible torus or Klein bottle transversal to \( \operatorname{Fix} \). \( \iota | V_1 \) is orientation preserving.

(III) \( M = V_1 \cup V_2 \cup V \), where \( V_1, V_2 \) and \( V \) are solid tori each invariant under \( \iota \) such that \( \iota \) is orientation preserving when restricted to any of \( V_1, V_2 \) and \( V \). There is a 1-sided Klein bottle \( T \) with \( T \cap \iota T = S \cap \text{int}(V) \), a generator of \( \pi_1(V) \).

\( V_1 \cap V_2 = (T \cap \iota T) - \text{int}(V) \) are two annuli. \( T, \iota T \) and \( \operatorname{Fix} \) are pairwise transversal and \( \operatorname{Fix} \cap \partial V_2 = \emptyset \) and \( \operatorname{Fix} \cap S \neq \emptyset \). \( V \) is a standard neighborhood of \( S \). See Figure 7.

(IV) \( M = W \cup V \), where \( W \) is a twisted I-bundle over a torus \( T \subseteq W \) and \( V \) is a solid torus with \( \partial W = \partial V = W \cap V \) and \( \iota W = W, \iota T = T \) and \( \iota V = V \). \( \operatorname{Fix} \) is transversal to \( \partial W \) and \( T \) except for a possible 1-sphere component \( S \) of \( \operatorname{Fix} \) contained in \( T \).

**Proof:** Let \( T_0 \) be an incompressible torus in \( \text{int}(M) \). By Theorem 2.12 assume \( T_0, \iota T_0 \) and \( \operatorname{Fix} \) are almost pairwise transversal and that no 1-spheres in \( T_0 \cap \iota T_0 \) bound 2-cells in \( T_0 \).
As a first step we handle the cases where saddle components arise. Only Type III and Type II components are possible. In both cases since $S$ and $S_1$ intersect transversally at one point, there can be only one component in $T_0 \cap \iota T_0$.

Suppose $T_0 \cap \iota T_0$ is a Type III component $S \sqcup S_1 \sqcup S_2$. Then $S_1$ and $S_2$ bound an annulus $A$ in $T_0$ since $S_1 \cap S_2 = \emptyset$ and both intersect $S$ transversally once. Let $T = A \sqcup \iota A$. Then $\iota T = T$ and $T$ and $\text{Fix}$ are transversal. $T$ is 1-sided since a regular neighborhood of $S_1$ is a solid Klein bottle. Let $N$ be a regular neighborhood of $T$ invariant under $\iota$. If $\partial N$ is incompressible then it is a 2-sided torus satisfying (I). If $\partial N$ is compressible we arrive at (IV).

Suppose $T_0 \cap \iota T_0$ is a Type II component $S \sqcup S_1$. First we construct a torus $T'$ isotopic to $T = T_0$ with $\iota T' = T'$. Let $N(S)$ and $N(S_1)$ be regular neighborhoods of $S$ and $S_1$, respectively, both invariant under $\iota$ such that $N = N(S) \sqcup N(S_1)$ is a regular neighborhood of $S \sqcup S_1$ and such that $T \cap N(S)$ and $T \cap N(S_1)$ are
annuli, $N(S) \cap \text{Fix}$ is a Möbius band, $N(S_1) \cap \text{Fix}^2$ is a proper 2-cell and $N(S_1) \cap \text{Fix}^1$ is a proper 1-cell. Both $N(S)$ and $N(S_1)$ are Klein bottles. By transversality there are two disjoint open 2-cell components $K_1$ and $K_2$ of $N(S)-(T_\cup i^*T)$ that meet $\text{Fix}^1$ and there are two disjoint open 2-cell components $L_1$ and $L_2$ of $N(S)-(T_\cup i^*T)$ that do not meet $\text{Fix}^2$. By considering the effect of $i$ near saddle points we see $A=(K_1 \cup K_2 \cup L_1 \cup L_2) \cap \partial N$ is an annulus with $\partial A = C \cup i^* C$ where $C=\partial N \cap T$. The closure of $A \cup (T-N) \cup i^*(T-N)$ is a 2-sphere which by the irreducibility of $M$ bounds a 3-cell $E$. $E$ cannot contain the proper punctured torus $T \cap N$ so $E \cap \text{int}(N)=\emptyset$. Since $\text{Fix}^1$ is transversal to $\partial E$ and $i^*\partial E = \partial E$ it follows $i^*E=E$. In particular $i|E$ is conjugate to $j_1$, the standard involution of a 3-cell with fixed set one 1-cell. $A$ is invariant and contains $\text{Fix}^1 \cap \partial E$. Hence one shows there is a proper 2-cell $D$ with $\partial D$ a generator of $H_1(A)$ such that $\text{Fix}^1 \cap E$ is a proper 1-cell of $D$ and $iD=D$. Since $i\partial D=\partial D$, by taking $N$ sufficiently small we can construct a proper punctured torus $P$ in $N$ with $\partial P=D$ and $iP=P$ (namely isotope $T \cap N$). Consider the torus $T'=P \cup D$. $\text{Fix}^2$ intersects $T'$ transversally at $S$, and $\text{Fix}^1$ is contained in $T'$. $T'$ is 1-sided. Let $W$ be a regular neighborhood of $T'$ invariant under $i$. If $\partial W$ is incompressible then it is a 2-sided torus satisfying (I). If $\partial W$ is compressible we arrive at case (IV).
We may now assume $T_0 \cap \iota T_0$ has no saddle components. $T_0$, $\iota T_0$ and $\text{Fix}$ are pairwise transversal and $T_0 \cap \iota T_0$ consists of disjoint 1-spheres bounding annuli in $T_0$ and $\iota T_0$. We successively construct incompressible tori or Klein bottles $T$ with fewer 1-spheres in $T \cap \iota T$, but always keep $T \cap \iota T$ consisting of 1-spheres bounding annuli in $T$ and $\iota T$. Therefore any 1-sphere of $T \cap \iota T$ will always have a standard neighborhood. See Definitions 2.9-.10. It also follows then that any 1-sided Klein bottle arising from such a construction has a regular neighborhood $W$ with $\partial W$ a torus. So Lemma 6.1 is applicable.

Note: Suppose $T$ satisfies all the conditions of (I) except that $T$ is 1-sided instead of 2-sided. Let $W$ be a regular neighborhood of $T$. We can take $W$ so that $\partial W$ and $\text{Fix}$ are transversal and $\iota W=W$ or $W \cap \iota W=\emptyset$. $\partial W$ is 2-sided. If $\partial W$ is incompressible, $\partial W$ satisfies (I). If $\partial W$ is compressible, by Lemma 6.1 $T$ is a torus. Now $V=M-W$ is a solid torus. If $\iota T=T$ we have (IV). If $\iota T \cap T=\emptyset$ then the solid torus $V$ contains an embedded 1-sided torus $\iota T$, a contradiction.

There are four main cases now depending on the number of 1-spheres of $T \cap \iota T$ and the compressibility of certain surfaces.

Assume $T \cap \iota T$ consists of at least two 1-spheres. Let $A \subset \iota T$ be an innermost annulus: $A \cap T=\partial A$. $\partial A$ decomposes $T$ into two annuli $A'$ and $A''$ with $T=A' \cup A''$ and
\[ A = A' = A'' = A' \cap A''. \] \( T' = A' \cup A \) and \( T'' = A'' \cup A \) are tori or Klein bottles. See Figure 8.

**Case 1)** \( T' \) is incompressible.

**Case 1.1)** \( \iota \partial A = \partial A \) and \( \iota A = A' \).

Then \( \iota T' = T' \). One sees Fix is transversal to \( T' \) by considering the standard neighborhoods of \( \partial A \). We arrive at case (I) or (IV).

**Case 1.2)** Either \( \iota \partial A = \partial A \) and \( \iota A = A'' \) or \( \iota \partial A \cap \partial A \) is a single 1-sphere \( S \) and \( \iota A \subseteq A'' \).

In the latter case \( \iota S = S \). Let \( V_1 \) and \( V_2 \) be distinct standard neighborhoods of \( \partial A \) and let \( \gamma_1 \) and \( \gamma_2 \) be the two distinct standard annuli that meet both \( A \) and \( A' \). Let \( T_1 = (A' \cup A \cup \gamma_1 \cup \gamma_2) - \text{int}(V_1 \cup V_2) \). Then \( T_1 \cap \iota T_1 \subseteq (T_1 \cap \iota T) - \partial A \) because \( (\gamma_1 \cup \gamma_2) \cap \iota (\gamma_1 \cup \gamma_2) = \emptyset \). \( T' \) and \( T_1 \) are ambient isotopic so \( T_1 \) is incompressible. \( \text{Fix} \cap (\gamma_1 \cup \gamma_2) = \emptyset \) and \( A \) is innermost so \( T_1, \iota T_1 \), and \( \text{Fix} \) are pairwise transversal. Proceed with \( T_1 \).
Case 1.3) Either $\partial A \cap \partial A$ is a single 1-sphere $S$ and $A \subseteq A'$ or $\partial A \cap \partial A = \emptyset$.

Let $\partial A = S \cup S'$. Let $V$ be a standard neighborhood of $S$ and let $\gamma$ be the standard annulus that meets both $A$ and $A'$. Let $Ax[0, \varepsilon]$ be a sufficiently thin collar of $A = Ax0$ in $M$ such that

$$S' \times [0, \varepsilon] \subseteq A', \ S \times [0, \varepsilon] \subseteq T \text{ and } (A \cap \partial V) \times [0, \varepsilon] = (Ax[0, \varepsilon]) \cap \partial V.$$  

The collar exists since $V$ is a solid torus. In the first case, $\iota S = S$ and $\iota \gamma = \gamma$. By Lemma 5.3 if $(Ax \varepsilon) \cap \gamma \neq \emptyset$ we may assume $(Ax \varepsilon) \cap V$ and $\iota ((Ax \varepsilon) \cap V)$ intersect transversally in a 1-sphere $S_1$ and that both are transversal to $\text{Fix}$. In all other cases set $S_1 = S$. Define

$$T_1 = (Ax \varepsilon) \cup \overline{A'} - ((S' \cup S) \times [0, \varepsilon]) \cup (S \times [0, \varepsilon]) - A'$$

Then $T_1 \cap \iota T_1 \subseteq ((T \cap \iota T) - \partial A) \cup S_1$. $T_1$ is incompressible since it is ambient isotopic to $T'$. $T_1$, $\iota T_1$ and $\text{Fix}$ are pairwise transversal. Proceed with $T_1$.

By case 1) we may now assume $T'$ and $T''$ are compressible.

Case 2) For every annulus $A \subseteq \iota T$ with $A \cap T = \partial A$, both corresponding surfaces $T'$ and $T''$ are compressible and $T \cap \iota T$ contains more than two 1-spheres.

Then let $A_1$ and $A_2$ in $\iota T$ be annuli with $A_1 \cap T = \partial A_1$ and $A_2 \cap T = \partial A_2$ with $\partial A_1 = S_0 \cup S_1$ where $S$, $S_1$, and $S_2$ are 1-spheres with
S₁≠S₂. Let A, A₁', and A₂' be the three annuli of T that these 1-spheres decompose T into: ∂A=S₁∪S₂ and ∂A₁'=S₀∪S₁, i=1,2. See Figure 9.

Define T₁=A∪A₁∪A₂. T₁ is incompressible. Otherwise T₁ bounds a solid torus or a Klein bottle U. Say A₁' ⊆ U. A₁' is trivial in U by Lemma 5.5. If A₁'∪A₁ bounds the trivializing torus then the incompressible T=A₁'∪A₂'∪A is ambient isotopic to A₁∪A₂'∪A which was compressible by hypothesis. If A₁'∪A₂∪A bounds the trivializing torus then, since A₁' and A₂ meet on S₀, A₂ must also be trivial in A₁'∪A₂∪A. So T is ambient isotopic to A₂'∪A₂ which was assumed compressible.

We have five cases:

Case 2.1) i(S₁∪S₂)=S₁∪S₂ and iS₀ ⊆ A. Then i(A₁∪A₂)=A.

Case 2.2) i(S₁∪S₂)=S₁∪S₂ and iS₀ ⊆ A'.

Then i(A₁∪A₂)=A₁'∪A₂' and iS₀=S₀.

Case 2.3) iS₁=S₁ and iS₀=S₀. Then iA₁=A₁' and iS₂ ⊆ A₂'.

Case 2.4) i(S₁∪S₂)∩(S₁∪S₂) = ∅.

Figure 9.
Case 2.5) \((S_1 \cup S_2) \cap (S_1 \cup S_2)\) is a one 1-sphere.

These cases cover all possibilities. In each case we find a

\(T_i\) with fewer 1-spheres.

In case 2.5) this follows from the other cases. After relabelling assume \(S_1\) is the 1-sphere in the intersection. Then \((S_1=S_1\). By case 2.3) we assume \((S_0 \neq S_0\). Let \(A_3\) be the innermost annulus adjacent to \(A_1\): \(A_3 \subseteq T_1\) with \(A_3 \cap T_1 = \partial A_2 = S_1 \cup S_3\) where \(S_3 \neq S_0\). By case 2.3) again we may assume \((S_3 \neq S_3\). By case 2.1) and 2.2) we may assume \((S_0 \neq S_3\).

So we have \((S_0 \cup S_3) \cap (S_0 \cup S_3) = \emptyset\) and case 2.4) gives the reduction.

In case 2.1) use \(T_1 = T_1\). Fix is transversal to \(T_1\) since the standard annulus meeting \(A_1\) and \(A\) is invariant.

In case 2.2) and case 2.3): For \(i=1,2\) let \(V_i\) be the standard neighborhoods of \(S_i\) with \(\gamma_i\) the standard annuli that meet both \(A\) and \(A_i\). In all cases \((\gamma_1 \cup \gamma_2) \cap \gamma_1 \cup \gamma_2) = \emptyset\). Define \(T_2\) to be the incompressible surface ambient isotopic to \(T_1\) given by

\[T_2 = T_1 \cup \gamma_1 \cup \gamma_2 - \text{int}(V_1 \cup V_2)\]

Then \(T_2\), \(T_2\) and \(\text{Fix}\) are pairwise transversal and \(T_2 \cap T_2 \subseteq T_1 \cap T_1 - (S_1 \cup S_2)\).

In case 2.4): First assume \((S_0 \neq S_0\). By symmetry assume \((S_0 \neq S_1\). Let \((A_1 \cup A_2) \times [0,\epsilon]\) be a sufficiently thin collar of \(A_1 \cup A_2 = (A_1 \cup A_1) \times 0\) in \(M\) such that \((S_1 \cup S_2) \times [0,\epsilon] \subseteq T\) and \(S_0 \times [0,\epsilon] \subseteq A_1\). Define \(T_2\) as
\[(A_1 \cup A_2) \times \varepsilon \cup (S_1 \cup S_2 \times [0, \varepsilon]) \cup ((S_1 \cup S_2) \times [0, \varepsilon] - A).\]

Then \(T_2 \cap \iota T_2 \subseteq (T \cap \iota T) - S_0\). \(T_2\) is ambient isotopic to incompressible \(T_1\). \(T_2\), \(\iota T_2\) and \(\text{Fix}\) are pairwise transversal.

If \(\iota S_0 = S_0\), proceed as above but replace the condition \(S_0 \times [0, \varepsilon] \subseteq A\) by \(S_1 \times [0, \varepsilon] \subseteq A\). Use Lemma 5.3 on a standard neighborhood of \(S_0\) to adjust the collar so that \((A_1 \cup A_2) \times \varepsilon\) and \(\iota (A_1 \cup A_2) \times \varepsilon\) intersect transversally in one 1-sphere \(S_3\).

Then \(T_2 \cap \iota T_2 \subseteq (\iota (T \cap \iota T) - (S_0 \cup S_1)) \cup S_3\).

**Case 3)** For each annulus \(A \subseteq \iota T\) with \(A \cap T = \partial A\), both corresponding surfaces \(T'\) and \(T''\) are not incompressible and \(T \cap \iota T\) is exactly two 1-spheres.

Set \(\iota T = A_{1,1} \cup A_{1,1} \cup A_{1,-1} \cup A_{1,-1}\) with \(A_{1,1} \cap A_{1,1} = \partial A_{1,1} = \partial A_{1,-1} = T \cap \iota T = S_1 \cup S_2\). Then \(T = \iota A_{1,1} \cup A_{1,1}\). There are solid tori or Klein bottles \(U_i\) and \(V_i\) \((i = \pm 1)\) pairwise disjoint on their interiors with \(\partial V_i = A_{1,1} \cup \iota A_{1,1}\) and \(\partial U_i = A_{1,1} \cup \iota A_{1,-1}\). None of \(U_i\) or \(V_i\) are solid Klein bottles. Otherwise, if say \(V_1\) is a solid Klein bottle, then since \(S_1\) decomposes \(\partial V_1\) into two annuli it follows that \(S_1\) bounds a disc in \(V_1\). This contradicts the incompressibility of \(T\). By considering the standard annuli of a standard neighborhood of \(S_1\), we see \(\iota V_i = V_i\) and \(\iota U_i = U_{-i}\).

Next we show \(\iota | V_i\) and \(\iota | V_{-i}\) are orientation preserving. If not then by Section 4, \(\iota | V_i\) is conjugate to \(j_A, j_{2D}, j_N, j_M\) or \(j_{DP}\), the standard involutions on a solid torus. \(j_{2D}\) and \(j_{DP}\) are not possible since \(S_1\) or \(S_2\) would bound a disc
contradicting the incompressibility of $T$.

If $\iota|V_1$ is conjugate to $j_M$ then say $S_1 \subset \text{Fix}$ and $S_2 \cap \text{Fix} = \emptyset$. Then $\iota|V_2$ has a 2-dimensional fixed set component that has only one boundary component. It follows $\iota|V_2$ is also conjugate to $j_M$. So $\text{Fix}$ contains a Klein bottle $K$. There is a regular neighborhood $W$ of $K$ with $\partial W = \partial W$ and $W \cap \text{Fix} = \emptyset$. Since $V_i$ are solid tori and $K \cap V_i$ is a Möbius band, $\partial W$ is a torus. By Lemma 6.1, $\partial W$ is incompressible. We arrive at (I).

If $\iota|V_1$ is conjugate to $j_A$, then $[S_1]$ represents a generator of $H_1(V_1)$ and hence there is an ambient isotopy taking $\iota T$ to $\partial U_1$ (move $A_1$ to $\iota A_1$). This contradicts that $\iota T$ is incompressible.

Finally suppose $\iota|V_1$ is conjugate to the involution $j_N = \hat{x}a$ on $D^2 \times S^1$. If $\iota S_1 = S_1$ then $S_1' = 1 \times S^1$ and $S_2' = -1 \times S^1$ determine annuli $A'$ and $\iota A'$ of $\partial D^2 \times S^1$. It is possible to construct a conjugation $\partial V_1 \rightarrow \partial D^2 \times S^1$ taking $A_1$ to $A'$. This conjugation extends to a conjugation $V_1 \rightarrow D^2 \times S^1$. But $[S_1']$ is a generator of $H_1(D^2 \times S^1)$ and we get a contradiction as for the $j_A$ case above. If $\iota S_1 = S_2$ then use $S_1' = \partial D^2 \times 1$ and $S_2' = \partial D^2 \times -1$ and proceed as above but this time obtaining a contradiction as for $j_{2D}$ above.
Case 4) $T \cap iT$ is a single 1-sphere $S$.

Then $iS=\S$. Let $V$ be a standard neighborhood of $S$ and let $a_1$, $a_2$, $\beta_1$, and $\beta_2$ be the standard annuli with $a_1 \cap a_2 = \emptyset$, $\beta_1 \cap \beta_2 = \emptyset$, $ia_1 = a_1$, $ia_2 = a_2$ and $i\beta_1 = \beta_2$.

Define $T_1 = (T \cup iT \cup a_1 \cup a_2) - \text{int}(V)$ and $T_2 = (T \cup iT \cup \beta_1 \cup \beta_2) - \text{int}(V)$.

If $T$ is 2-sided then $T_1$ is 2-sided. Also $iT_1 = T_1$. Since $T$ is 2-sided it follows that a sufficiently thin collar $Tx[0, \epsilon]$ of $T = Tx0$ can intersect only one of $\text{int}(a_1)$ and $\text{int}(a_2)$. Hence $T_1$ cannot separate and therefore $T_1$ is incompressible. We arrive at (I).

From now on assume $T$ is 1-sided. $T_1$ and $T_2$ are tori. This follows since $V$ is a solid torus and either both of the annuli $T - \text{int}(V)$ and $iT - \text{int}(V)$ are "twisted" relative to $V$ (if $T$ is a Klein bottle) or neither is (if $T$ is a torus).

If either of $T_1$ or $T_2$ is incompressible we arrive at (I). Assume then that $T_1$ and $T_2$ are compressible. Then $T_1$ bounds a solid torus $V_1$. If $S \cap V_1$ then $V_1$ contains a 1-sided torus or Klein bottle, a contradiction. So $M = V \cup V_1 \cup V_2$ with $\text{int}(V)$, $\text{int}(V_1)$ and $\text{int}(V_2)$ pairwise disjoint.

By choice of $a_1$ and $a_2$, $i$ interchanges the components of $\partial a_1$. Therefore $i|a_1$ is conjugate to one of $\text{id}x\tau$, $\kappa x\tau$ or $ax\tau$, the standard involutions of $S'xI$. Let $S_1$ be a 1-sphere of $a_1$ that is the image of $S'x0$ under some conjugation.
Note that $S^1$ does not bound a disc $D$ in $V_1$, otherwise $T$ would be compressible.

Note also that if there is an annulus $A \subset V_1$ with $\partial A = S_1 \cup S_2$ and $\iota A = A$ then we arrive at property (I) and (IV) as follows. Torus $V_1$ is separated by $A$. Since $\iota$ interchanges the components of $\partial a_1$, $\iota$ interchanges the components of $A - V_1$. $A$ is trivial in $V_1$, so it follows $V_1$ can be given a trivial $I$-bundle structure over $A$. There is an annulus $B \subset V$ with $\partial B = S_1 \cup S_2$ and $\iota B = B$. $V$ is an $I$-bundle over $B$. Consider $T_3 = A \cup B$. It follows $V \cup V_1$ is an $I$-bundle over $T_3$ with $\partial (V \cup V_1) = T_2$ a torus. Moreover $T_3$ does not separate $T$ so $T_3$ is 1-sided. If $T_3$ is a torus we arrive at (IV). If $T_3$ is a Klein bottle, Lemma 6.1 gives (I).

Since $V$ is a standard neighborhood, $\text{Fix } \cap a_1 = \emptyset$ if and only if $\text{Fix } \cap a_2 = \emptyset$. Therefore $\iota| a_1$ and $\iota| a_2$ are conjugate.

Case 4.1) $\iota| a_1$ is conjugate to $\text{id}_{\mathbb{T}}$. Then $\iota| V_1$ has a 2-dimensional fixed set that meets $\partial V_1$ in two fixed 1-spheres. By Section 4 it follows that $S_1$ bounds a disc or $S_1 \cup S_2$ bound an annulus $A$ fixed by $\iota| V_1$. By the above comments, we arrive at (I) or (IV).

Case 4.2) $\iota| a_1$ is conjugate to $a \mathbb{R}$. Then $\iota| V_1$ is orientation reversing. $\iota| \partial V_1$ is conjugate to $a \alpha \pi$ on $S^1 \times S^1$ by a conjugation taking $S_1$ to $S^1 \times (-1)^i$. By Section 4 $\iota| V_1$ is conjugate to $a \alpha \pi$ or $a \alpha \tilde{\pi}$ by a conjugation extending the one given on the boundaries. In the first case $S_1$ bounds a disc
and in the second case $S_1 \cup S_2$ bound an annulus with $\iota A = A$. Again by the above comments we arrive at (I) or (IV).

Case 4.3) $\iota | a_1$ is conjugate to $\kappa \kappa \tau$. Then $\iota | V_1$ and $\iota | V$ are orientation preserving. Now $\iota | V_2$ is orientation reversing if and only if $T$ is a torus. To see this let $S_1 = a_1 \cap \beta_1$, and without loss say $S_1 \subset T$. Orient $S_1$, $S_1$, and $\iota S_1$ bound two annuli $A_1$ and $A_2$ of $\partial V_2$ with $\iota A_1 = A_2$. Consider the ways of inducing an orientation on $\iota S_1$. The orientation induced by $A_1$ and the orientation induced by $a_1$ are the same if and only if $T$ is a torus. Since $\iota | a_1$ is orientation reversing the orientation induced by $a_1$ and the orientation induced by $\iota$ are opposite. So $\iota$ and $A_1$ induce opposite orientations on $\iota S_1$, if and only if $T$ is a torus. Since $\iota A_1 = A_2$ the claim follows.

If $T$ is a torus then $\iota | V_2$ is orientation reversing so $\iota | \partial V_2$ is conjugate to the involution $\alpha \kappa \kappa$ on $S^1 \times S^1$ by a conjugation taking $S_1$ to $\iota x S^1$. As in case 4.2) we arrive at (I) or (IV).

If $T$ is a Klein bottle then we arrive at (III). $\iota | \partial V_2$ is fixed point free so $\iota | V_2$ is conjugate to $j_S$ or $j_O$ while $\iota | V_1$ is conjugate to $j_{2C}$.

QED
IV. INVOLUTIONS ON ORIENTABLE I-BUNDLES OVER TORI AND KLEIN BOTTLES

§7. Involutions on the Trivial I-Bundle Over a Torus

As an application of the annulus theorem we classify the involutions on various I-bundles.

Definition 7.1

Let \( W = S^1 \times S^1 \times I \) be the trivial I-bundle over the torus \( T = S^1 \times S^1 \).

Define the following involutions on \( W \) (see Figure 10):

- \( k_T = \text{id} \times \text{id} \times \tau \) having fixed set the torus \( S^1 \times S^1 \times 0 \)
- \( k_{2A} = \text{id} \times \text{id} \times \text{id} \) having fixed set two annuli \( S^1 \times \pm 1 \times I \)
- \( k_{2S} = \text{id} \times \text{id} \times \tau \) having fixed set two 1-spheres \( S^1 \times \pm 1 \times 0 \)
- \( k_A = (\rho \cdot (\text{id} \times \kappa)) \times \text{id} \) having fixed set the annulus \( S^1 \times 1 \times I \)
- \( k_S = (\rho \cdot (\text{id} \times \kappa)) \times \tau \) having fixed set the 1-sphere \( S^1 \times 1 \times 0 \)
- \( k_{4C} = \kappa \times \kappa \times \text{id} \) having fixed set four 1-cells \( \pm 1 \times \pm 1 \times I \)
- \( k_{4P} = \kappa \times \kappa \times \tau \) having fixed set four points \( \pm 1 \times \pm 1 \times 0 \)
- \( k_{OF} = \text{ax} \times \text{id} \times \text{id} \)
- \( k_{NI} = \text{ax} \times \text{id} \times \tau \)
- \( k_{NF} = \text{ax} \times \kappa \times \text{id} \)
- \( k_{OI} = \text{ax} \times \kappa \times \tau \)

Here \( \rho \cdot (\text{id} \times \kappa)(z, w) = (z\bar{w}, \bar{w}) \). The last four involutions are fixed point free. The subscript \( O \) means the involution is orientation preserving, \( N \) means it is orientation reversing, \( F \) means it keeps the boundary components fixed (as sets),
Figure 10.

*Fixed point sets for the standard involutions.*

\[ W = S_{12}^1 \times S_{12}^1 \times I \]

\[ k_T \]

\[ k_A \]

\[ k_{2A} \]

\[ k_S \]

\[ k_{2S} \]

\[ k_{4C} \]

\[ k_{4P} \]
and I means it interchanges the two boundary components. The orientation type of the other involutions is determined by the dimensions of their fixed point sets. These eleven involutions are not conjugate, because fixed point sets, orientability type and F/I. properties are conjugacy class invariants.

\(k_A\) and \(k_S\) are conjugate to the following. On \(S'\times I\times I\) define the identification \(d=xr|_{\partial I}\times id\). Then \(W \cong S'\times I\times I/d\).

Let \(k'_A=(id\times r\times id)/d\) with fix set \((S'\times 0\times I)/d\) and \(k'_S=(id\times r\times r)/d\) with fix set \((S'\times 0\times 0)/d\). The other involutions can be given similar alternate conjugate representations using \(d=xr|\times id\) or \(d=id\times r|\times id\).

**Lemma 7.2**

Let \(A\) be an essential annulus in \(T_x[-1,1]\). Then \((T_x[-1,1],A) \cong (S'\times S'\times [-1,1], S'\times 1\times [-1,1])\).

**Proof:** By an isomorphism take one of the boundary components of \(A\) as \(S'\times 1\times [-1,1]\).

First show \(A\) meets both boundary components of \(T_x[-1,1]\). If not then let \(A'\) be the annulus \(S'\times J\times [-1,1]\) where \(J\) is an interval chosen sufficiently close to \(1\times S'\) and small enough so that \(A'\cap \partial A = \emptyset\) and \(A'\) is in the component of \(T_x[-1,1]\) - \(A\) that meets \(T_x-1\). Then, in the solid torus \(T_x[-1,1]/(z,w,-1)\sim(z,w',-1)\), the disc \(1\times S'\times [-1,1]/\sim\) is a
nonseparating 2-cell meeting $A'$ in one nonseparating 1-cell of $C$. By Lemma 5.5 there is a solid torus $V$ that trivializes $A$ and does not meet $A'$. Necessarily $V$ does not meet $Tx^{-1}$ so $A$ is also trivial in $Tx[-1,1]$. This is a contradiction.

Next adjust by an isomorphism so that $A$ and $1 \times S^1 \times [-1,1]$ meet in a single proper 1-cell. Then the isomorphism can be constructed.

QED

**Theorem 7.3**

Let $\iota$ and $\iota'$ be involutions on $W = S^1 \times S^1 \times I$ with isomorphic fixed point sets. If $\iota$ and $\iota'$ are fixed point free, assume, in addition, that both have the same orientation type and that $\iota$ interchanges boundary components of $\partial W$ if and only if $\iota'$ interchanges boundary components of $\partial W$.

Then $\iota$ and $\iota'$ are conjugate. An involution on $W$ is conjugate to one of the eleven involutions listed above.

**Proof:** Let $\iota$ be an involution on $W = S^1 \times S^1 \times I$. We show it is conjugate to a standard involution. By the Annulus Theorem 5.9 there is an essential annulus $A$ with either $\iota A \cap A = \emptyset$ or $\iota A = A$ and $A$ and $\text{Fix}$ transversal. In the latter case, by Lemma 3.5 assume the collar of $A$ is not interchanged. By the previous lemma take $A$ of form $S^1 \times 1 \times [-1,1]$. With further
adjustment take $iA=S'x-1x[-1,1]$ if $iA \neq A$. Let $W_=S'x{x+y+yi:y \geq 0}x[-1,1]$ and $W_=S'x{x+y+yi:y \leq 0}x[-1,1]$.

There are three cases:

Case 1) $iA=A$, $A$ and $Fix$ are transversal and the collar of $A$ is not interchanged.
Case 2) $iA \cap A= \emptyset$ and $iW_=W_-$.
Case 3) $iA \cap A= \emptyset$ and $iW_=W_-$.

We show that $i$ is conjugate to:

in case 1) $k_{OF}$, $k_{NI}$, $k_T$, $k_{2A}$, $k_{2S}$, $k_A$ or $k_S$
in case 2) $k_{NF}$, $k_{OI}$, $k_{2A}$, $k_{2S}$, $k_A$, $k_S$, $k_{4C}$ or $k_{4P}$.
in case 3) $k_{OF}$, $k_{NI}$, $k_{NF}$ or $k_{OI}$.

Several of the standard involutions are listed in more than one case. Each standard involution (or at least a conjugate of one) can in fact arise in the case it has been listed under. To see this it suffices to display an annulus $A'$ in $W$, not necessarily of form $S'x1x1$, with properties analogous to those of $A$. Consider $A'$ as follows: in case 3) take $1xS'x1$; in case 2) take $S'x1x1$; in case 1) take $S'x1x1$ for $k_{OF}$, $k_{NI}$, $k_T$, take $1xS'x1$ for $k_{2A}$, $k_{2S}$ and take \{$(z,z^2,t):z,t$\} for $k_A$, $k_S$.

Call two involutions on $W$ of same type if they have isomorphic fixed sets and, in addition, when they are fixed point free, if they have the same orientation type and simultaneously interchange or do not interchange boundary
components. It suffices to show, first, that \( t \) has the same type as a standard involution listed under a corresponding case, and second, if \( t \) and \( t' \) have the same type and fall into the same case 1) - 3) then they are conjugate.

Constant use is made of Section 4. Reserve \( j \) to denote standard involutions on the solid torus. All constructions done for \( t \) are to be performed for \( t' \), even if not explicitly stated.

Case 1) \( tA=A \), \( A \) and Fix are transversal and the collar of \( A \) is not interchanged.

Then \( W=S'xIx[-1,1]/d \) where \( d=idx(\tau|\partial I)xid. \) The involution \( t \) induces an involution \( \lambda \) on the solid torus \( V=S'xIx[-1,1] \) with the property \( \lambda .d=d.\lambda \) when restricted to \( S'x\partial Ix[-1,1] \). Fix(\( \lambda \)) is proper since Fix is transversal to \( A \). Let \( A \) also denote the copy \( S'xIx1 \) in \( V \). Since the collar is not interchanged \( \lambda (A)=A \). See Remark 3.6. By adjusting \( t \) in a collar of \( A \) we may assume \( t|A \) and \( \lambda |A \) are one of the five standard involutions on an annulus (Lemma 3.2).

Let \( S \) be a fixed component of \( \partial A \). \( [S]\in H_1(V)=\mathbb{Z} \) is a generator. Write \( V=D^2xS' \). Let \( M=S'x1 \) and \( L=1xS' \). Then \([M]\) and \([L]\) generate \( H_1(\partial V)=\mathbb{Z}\oplus\mathbb{Z} \) and with a proper choice of orientations \( [S]=[L]+a[M] \) where \( a\in\mathbb{Z} \).

\( \lambda \) is not conjugate to \( j_S, j_M, j_{DP} \) or \( j_{2p} \): If \( \lambda \) were conjugate to \( j_S \) then, since \( j_S \) is orientation preserving and \( \partial \text{Fix}(j'S)=\emptyset \), \( \lambda|A=axid. \) Therefore \( S \) is kept setwise fixed by
\( \lambda \). So \([S/\lambda]\) represents twice the generator of \( H_1(V/\lambda) = \mathbb{Z} \). However \([\text{Fix}(\lambda)]\) also represents a generator of \( H_1(V) \) and \([\text{Fix}(\lambda)/\iota]\) is a generator of \( H_1(V/\lambda) \). If \( \lambda \) were conjugate to \( j_{\mathbb{M}} \) then \( \iota_*(S) = \iota_*([L] + a\iota_*[M]) = [L] + [M] + a[M] \neq \pm[S] \) contradicting \( \iota A = A \). If \( \lambda \) were conjugate to \( j_{\mathbb{DP}} \) then \( \iota_*(S) = -(L) + [M] + a[M] \neq \pm[S] \) contradicting \( \iota A = A \). If \( \lambda \) were conjugate to \( j_{2\mathbb{P}} \) then since \( j_{2\mathbb{P}} \) is orientation reversing and \( \partial \text{Fix}(j_{2\mathbb{P}}) = \emptyset \), \( \lambda|A = axr \). Therefore \( [S] = \iota_*(S) \). This implies \( [L] + a[M] = -(L) + a[M] \), a contradiction.

Hence \( \lambda \) is conjugate to \( j_\mathbb{A}, j_{2\mathbb{D}}, j_{2\mathbb{C}}, j_0 \) or \( j_N \). Let \( B \) be a component of \( V - \text{int}(A \cup d(A)) \) that meets \( S \). We investigate the five possibilities for \( \lambda|A \). Since we will see these give rise to involutions of different types, select a conjugation \( h: A \to A' \) between \( \lambda|A \) and \( \lambda'|A' \) and choose \( S' = h(S) \). This conjugation extends to a conjugation \( h: A \cup d(A) \to A' \cup d'(A') \).

Case 1.1) \( \lambda|A = axr \). Let \( \text{Fix}(\lambda|A) = \{x, y\} \). Necessarily \( \lambda \) is conjugate to \( j_{2\mathbb{C}} \). If \( x \) and \( d(x) \) are in the same component of \( \text{Fix}(\lambda) \) then \( \text{Fix} \) is two 1-spheres. Otherwise \( \text{Fix} \) is one 1-sphere. So \( \iota \) has the type of \( k_S \) or \( k_{2S} \). Now \( x' \) and \( d'(x') \) are in the same component of \( \text{Fix}(\lambda') \) iff \( x \) and \( d(x) \) are in the same component of \( \text{Fix}(\lambda) \). The conjugation extends over \( \text{Fix} \) so extend it to a conjugation \( h: V \to V' \) between \( \lambda \) and \( \lambda' \). A conjugation \( h: W \to W' \) between \( \iota \) and \( \iota' \) is induced.
Case 1.2) \( \lambda|A=\kappa \times d \). Proceed as in case 1.1) except now \( \text{Fix}(\lambda|A) \) is two 2-cells, so \( \text{Fix} \) is either two or one annuli. Thus \( \iota \) has type of \( k_{2A} \) or \( k_{A} \). Now \( \iota B=B \). \( h \) extends over \( B \) since \( \text{Fix} \) separates \( B \) into two components. Extend similarly over the annulus \( \partial V-(B \sqcup A \sqcup \iota A) \).

Case 1.3) \( \lambda|A=\text{id} x \tau \). Then \( \lambda \) is conjugate to \( j_{A} \) so \( \text{Fix} \) is a torus and \( \iota \) has the type of \( k_{T} \). Proceed as in 1.1).

Case 1.4) \( \lambda|A=\alpha x \tau \). It follows \( \lambda B \cap B=\emptyset \). So \( \lambda \) is conjugate to \( j_{N} \) and \( \iota \) is of type \( k_{NI} \). Proceed as in 1.1).

Case 1.5) \( \lambda|A=\text{id} x d \). Then \( \lambda \) is conjugate to \( j_{2C} \) or \( j_{O} \). \( j_{2C} \) is not possible since \( [S]=\lambda_{*}[S]=j_{2C*}([L]+a[M])=-[L]-a[M]=-S \). So \( \lambda \) is conjugate to \( j_{O} \) and \( \iota \) is of type \( k_{OF} \). Let \( B_{1}=B \) and \( B_{2}=\partial V-(A \sqcup d(A) \sqcup B) \). Let \( J \) be a nonseparating 1-cell of \( A \) with \( J \cap \lambda J=\emptyset \) and let \( I_{i} \) be any path in \( B_{i} \) from \( \partial J \) to \( d(\partial J) \). \( B_{i} / \iota \) is an annulus so by lifting an embedded path that is path homotopic to \( I_{i} / \iota \) we may also assume that \( I_{i} \cap I_{i}=\emptyset \). By making proper choices, we arrange that \( I_{1} \sqcup I_{2} \sqcup J \sqcup d(J) \) bounds a 2-cell in \( V \). A similar property holds for \( \lambda' \) for \( J'=h(J) \). Use \( I_{i} \) and \( I_{i}' \) to extend \( h \) to a conjugation \( \partial V \rightarrow \partial V' \) and complete the argument as before.

Case 2) \( \iota A \cap A=\emptyset \) and \( \iota W.=W \). Let \( S \) be a fixed component of \( \partial A \). Let \( B \) be a component of \( W \cap \partial W \) that meets \( S \). Let \( \lambda=\iota|W \). There are two possibilities:

2a) \( \lambda B=B \)
2b) \(\lambda B \cap B = \emptyset\).

In 2a) \(i\) interchanges boundary components of \(\partial W\), while in 2b) it does not. \(W\) is a solid torus, so \(\lambda\) is conjugate to one of the standard involution of the solid torus.

This splits the present case into four subcases. In fact

- in case 2a) if \(i\) is orientation preserving then \(\lambda\) is conjugate to \(j_{2C}\).
  - if \(i\) is orientation reversing then \(\lambda\) is conjugate to \(j_A\) or \(j_N\).
- in case 2b) if \(i\) is orientation preserving then \(\lambda\) is conjugate to \(j_S\) or \(j_0\).
  - if \(i\) is orientation reversing then \(\lambda\) is conjugate to \(j_{2P}\).

To show this note that in \(H_1(W) = \mathbb{Z}, [S]\) is a generator. Since \(S \subset \partial W - \text{Fix}\), \(\lambda\) cannot be conjugate to \(j_{2D}, j_M\) or \(j_{DP}\).

In case 2a) \(\lambda_{*}[S] = -\mu(\lambda)[S]\), while in case 2b) \(\lambda_{*}[S] = \mu(\lambda)[S]\), where \(\mu(\lambda)\) is +1 if \(\lambda\) is orientation preserving and -1 if \(\lambda\) is orientation reversing. If \(\lambda\) is conjugate to \(j_{2C}\) or \(j_{2P}\) then \(\lambda[S] = -[S]\). In all other cases \(\lambda[S] = [S]\). This establishes the claim.

\(\lambda = i|W.\) must satisfy the (similar) case 2a) - 2b).

Combining \(\lambda\) and \(\lambda_{*}\) in all the different possible ways gives involutions of types as listed previously. For example, combining a \(j_A\) with a \(j_N\) gives an involution of type \(k_A\).
Let \( \iota' \) be of same type as \( \iota \). It remains to show they are conjugate. Find an isomorphism \( h : A \rightarrow A' \) and extend by equivariance to \( h : A \sqcup \iota A \rightarrow A' \sqcup \iota' A' \). It suffices to show that \( h \) extends to \( W \), when \( \lambda = \iota | W^+ \) and \( \lambda' = \iota' | W \) are conjugate. Take \( B \) and \( B' \) as above.

Case 2.1) \( \lambda \) is conjugate to \( j_A \). \( \text{Fix} \cap B \) is a 1-sphere and the components of \( B - \text{Fix} \) are interchanged. Extend \( h \) over one of these components and then extend over all of \( B \) by equivariance. Similarly for the other annulus of \( W \cap \partial W \). By the conjugation extendable property of \( j_A \) this conjugation extends to all of \( W \).

Case 2.2) \( \lambda \) is conjugate to \( j_N \). Then \( B/\iota \) is a Möbius band. The isomorphism \( h \) extends to \( B/\iota \). Lift to \( B \) and proceed as in 2.1.

Case 2.3) \( \lambda \) is conjugate to \( j_{2C} \). \( B/\iota \) is a 2-cell with \( \text{Fix} \cap B/\iota \) being two points. There is an isomorphism which extends the given induced one on \( \partial B/\iota \) and takes the two points of \( \text{Fix} \cap B/\iota \) to \( \text{Fix}' \cap B'/\iota' \) in either of the two possible ways. This isomorphism lifts and with correct choices \( h \) extends as in 2.1.

Case 2.4) \( \lambda \) is conjugate to \( j_{2P} \). Extend \( h \) in any way to \( B \) and then extend by equivariance to \( \partial W \).

Case 2.5) \( \lambda \) is conjugate to \( j_0 \) or \( j_S \). Then \( B \cap \lambda B = \emptyset \). Let \( J \) be a proper 1-cell of \( A \) and let \( J' = h(J) \). Select a proper 1-cell \( I \) of \( B \) with \( \partial I = \partial (J \cup \lambda J) \cap A \) and consider
C=I \cup \lambda I \cup J \cup \lambda J. C cannot be used to extend h since even if C bounds a disc in W, C' may not bound a disc in W'. As before let W_+ \cong D^2 \times S^1, M=S'^x1 and L=1xS'. Then [M] and [L] generate \( H_1(\partial W_+)=\mathbb{Z} \oplus \mathbb{Z} \) and with correct choices \([S]=[L]+a[M]\) as classes in \( H_1(\partial W_+)\), for some \( a \in \mathbb{Z} \). By changing \( I \) assume \([C]=\mu[L]+b[M]\) where \( \mu \) is 0 or 1 and \( b \in \mathbb{Z} \). Achieve this by altering the path class of \( I \) by concatenating with \( S \) and by using the fact that \( \lambda_*[S]=[S] \). Now \( W_+/\lambda \) is a torus. Let \( p: H_1(\partial W_+) \to H_1(\partial W_+/\lambda) \) be the obvious homomorphism. Let \([M_1]\) and \([L_1]\) be generators for \( H_1(\partial W_+/\lambda) \) defined as for \( \partial W_+ \). Without loss for \( j_0 \),
\[
p[M]=[M_1] \quad \text{and} \quad p[L]=2[L_1]
\]
and for \( j_S \),
\[
p[M]=2[M_1] \quad \text{and} \quad p[L]=[L_1].
\]
For \( j_S \), \( p[C]=\mu[L_1]+2b[M_1] \). Since \( C/\lambda \) is double covered by \( C \) it follows \( \mu \) is even. So \( \mu=0 \). Let \( C_1=C \).

For \( j_0 \), suppose \( \mu \) is 0. Then \( p[C]=b[M_1] \) so \( b \) is even since again \( C/\lambda \) is double covered by \( C \). Then \([\{I \cup J\}/\lambda]=(b/2)[M_1]\) and it follows \( \{I \cup J\}/\lambda \) lifts to a 1-sphere. This is a contradiction since \( \partial(I \cup J) \neq \emptyset \). So \( \mu=1 \). Then let \( I_1 \) be a proper 1-cell in \( B \) with \( I_1 \cap I=\partial I=\partial I_1 \) such that \( C_1=I_1 \cup \lambda I_1 \cup J \cup \lambda J \) has class \([C_1]=d[M]\) for some \( d \in \mathbb{Z} \).

In any event we obtain a curve \( C_1 \) with \([C_1]=0 \in H_1(W_+)\). Since \( j_S \) and \( j_0 \) determine different \( \mu \) it follows that for \( \iota \) we can define \( C_1' \) in the same way as \( C_1 \) (i.e.) using \( I_1' \) if
C_1 uses I_1. Extend h over I (and I, for j_0 case). This h then extends to a conjugation by construction.

Case 3) \( iA \cap A = \emptyset \) and \( iW = W \). Then \( i \) is fixed point free. Suppose \( i' \) is of same type. Let \( h:A \rightarrow A' \) be any isomorphism and extend by equivariance to a conjugation \( h:A \cup iA \rightarrow A' \cup i'A' \). Fix a component \( S \) of \( \partial A \) and let \( S' = h(S) \). Let \( B \) be the component of \( W + \partial W \) that meets \( S \). \( B \) is an annulus. Similarly define \( B' \). Since \( i \) and \( i' \) have the same interchange type (F/I property) we have \( h(\partial B - S) = \partial B' - S' \). Since they have the same orientation type \( h|_{\partial B} \) extends to \( h:B \rightarrow B' \). The isomorphism determined on the annulus \( A \cup B \cup iA \) necessarily extends to an isomorphism of the solid tori \( W \rightarrow W' \). Extend to \( W = W \cup W' \) by equivariance.

QED

**Corollary 7.4**

Let \( W \) and \( W' \) be trivial I-bundles over a torus. Involutions conjugate to \( k_T \) are conjugation extendable. If \( i \) on \( W \) is conjugate to \( k_A \) or \( k_{2A} \) and \( i' \) on \( W' \) is conjugate to \( i \) then a conjugation \( h: \partial W \rightarrow \partial W' \) is conjugation extendable if it satisfies the following condition: Let \( \text{Fix}_i \) be a component of \( \text{Fix} = \text{Fix}(i) \) and let \( \text{Fix}_i x [-1,1] \) be a bicollar of \( \text{Fix}_i = \text{Fix}_i x 0 \) such that \( \partial \text{Fix}_i x [-1,1] \) bicollars \( \partial \text{Fix}_i \). Similarly for \( \text{Fix}_i' \), where \( \text{Fix}_i' \) is a component of \( \text{Fix}' = \text{Fix}(i') \) meeting \( h(\partial \text{Fix}_i) \). Then require that \( h \) extends to an
There are conjugations of $\partial W$ that are not extendable!

Proof:

For $k_T$, the fixed set separates. Let $\iota$ on $W$ be conjugate to $k_T$. Let $W_*$ be the closure of one of the components. Then $W_*$ is isomorphic to $T\mathbb{I}[0,1]$ by an isomorphism taking $\partial W_* - \text{Fix}$ to $T\mathbb{I}0$ and $\text{Fix}$ to $T\mathbb{I}1$ where $T=S'\times S'$. Clearly the isomorphism $h:T\mathbb{I}0 \to T'\times 0$ extends to an isomorphism on $W_*$ taking $T\mathbb{I}1$ to $T'\times 1$. Extend by equivariance.

For $k_{A'}$, let $\iota$, $\iota'$ and $h$ be as in statement of corollary. It follows $h$ extends to a conjugation $h: \partial W \cup \text{Fix} \to \partial W' \cup \text{Fix}'$. Cutting $W$ open along $\text{Fix}$ gives a solid torus $V$ having two copies of $\text{Fix}$ in its boundary. The involution $\iota$ on $W$ is induced by an involution $\lambda$ on $V$ which interchanges these copies of $\text{Fix}$. Similarly for $W'$. By the condition on the bicollar, $h|_{(\partial W \cup \text{Fix})}$ is induced from a conjugation $h: \partial V \to \partial V'$. Now $\lambda$ is conjugate to $j_N$ so by the conjugation extendable property for $j_N$, $h_1$ extends over $V$ and hence induces a conjugation on $W$ extending $h$.

For $k_{2A'}$, let $\iota$, $\iota'$ and $h$ be as in statement of corollary. It follows $h$ extends to a conjugation $h: \partial W \cup \text{Fix}, \to \partial W \cup \text{Fix},'$. Since all components of $\text{Fix}$ and
Let $\partial W \cap \text{Fix}$ be annuli. $h$ extends to a conjugation $h: \partial W \cap \text{Fix} \rightarrow \partial W \cap \text{Fix}'$. Let $C$ and $\iota C$ be the two 3-cells that Fix decomposes $W$ into. By the bicollar condition $h(\partial C)$ is contained in one of the two 3-cells that Fix' decomposes $W'$ into. Say $h(\partial C) \subset \partial C'$. Then extend $h$ to an isomorphism $h: W \sqcup C \rightarrow W' \sqcup C'$ by coning to a vertex and extend by equivariance to the desired conjugation.

QED

**Corollary 7.5**

If $\iota$ is an orientation preserving involution on $W=S^1 \times S^1 \times I$ then $W/\iota$ is isomorphic to one of the following spaces:

- $W/k_{2S} \cong D^2 \times S^1$ a solid torus with $\text{Fix}/k_{2S}$ two unknotted 1-spheres $(\pm 1/2) \times S^1$,
- $W/k_S \cong D^2 \times S^1$ a solid torus with $\text{Fix}/k_S$ one unknotted 1-sphere $\{(e^{\pi it}/2, e^{2\pi it}) : -1 \leq t \leq 1\}$ representing twice a generator of $H_1(D^2 \times S^1)$,
- $W/k_{O1}$ an orientable twisted I-bundle over a Klein bottle,
- $W/k_{4C} \cong S^2 \times I$ with $\text{Fix}/k_{4C} = \{\text{four points}\} \times I$.
- $W/k_{Og} \cong W$.

**Proof:** $\iota$ is conjugate to a standard involution $k$. Use the representations for the standard involutions. In all cases except for $k_{Og}$, $W/k \cong S^1 \times \{x+y \cdot i : 0 \leq y\} \times I/(g \sqcup g')$ where $g$ is an
identification of $S'xI$ and $g'$ is an identification of $S'x-IxI$ depending on $k$. For $k_S$ note that $S'x\{x+y\cdot i: 0 \leq y\}x0/(g \cup g')$ is a Möbius band with boundary $\text{Fix}/k_S$.

QED

8. Involutions on the Orientable I-Bundle Over a Klein Bottle

Definition 8.1

Let $W=S'xIxI/d$ be the orientable twisted I-bundle over the Klein bottle $S'xIx0/d$, where $d=\tau x(\tau | \partial I)\tau$. More explicitly, $W=S'x[-1,1]x[-1,1]/(z,-1,t)\sim (z,1,-t)$. $z=\pm i$ is a separating annulus, whereas $z=1$ is a nonseparating Möbius band. See Figure 11. The I-fibers are $zxsIxI$. An involution $\lambda$ on $S'xIxI$ with $\lambda | .d=d.| \lambda$ where $\lambda |$ denotes $\lambda |(S'x \partial IxI)$ induces an involution $k=\lambda/d$ on $W$. $\text{Fix}=\text{Fix}(k) = (\text{Fix}(\lambda) \cup \text{Fix}(d^{-1}.\lambda |))/d$.

Define the following involutions on $W$ (see Figure 11):

- $k_K=\text{id}x\text{id}x\tau/d$ having fixed set a Klein bottle $S'xIx0/d$
- $k_{2M}=\text{xxid}x\text{id}/d$ having fixed set two Möbius bands

$\pm 1xIxI/d$
\[ k_{2S} = \kappa xidxr/d \text{ having fixed set two 1-spheres } \pm 1x1x0/d \]
\[ k_A = (-\kappa) xidxid/d \text{ having fixed set an annulus } \pm ix1x1xI/d \]
\[ k_S = (-\kappa) xidxr/d \text{ having fixed set a 1-sphere } \pm ix1x0/d \]
\[ k_{A2P} = idxTxid/d \text{ having fixed set an annulus and two points } (S'x0xI \cup \pm 1x-1x0)/d \]
\[ k_{S2C} = idxTxr/d \text{ having fixed set a 1-sphere and two 1-cells } (S'x0x0 \cup \pm 1x-1xI)/d \]
\[ k_{2C} = (-\kappa) xTxid/d \text{ having fixed set two 1-cells } \pm ix0xI/d \]
\[ k_{2p} = (-\kappa) xTxr/d \text{ having fixed set two points } \pm ix0x0/d \]
\[ k_0 = axidxid/d \text{ fixed point free and orientation preserving} \]
\[ k_N = axidxr/d \text{ fixed point free and orientation reversing} \]

These eleven involutions are not conjugate because their fixed point sets or orientability types are different. Recall that the involutions with even dimensional fixed point sets are orientation reversing and that the ones with odd dimensional fixed sets are orientation preserving since \( W \) is orientable.

**Lemma 8.2**

Let \( W = S'x1xI/d \) be the orientable \( I \)-bundle over a Klein bottle. Then if \( A \) is an essential annulus there is an ambient isotopy moving \( A \) so that \( A \) is of form \( S'x-1xI/d \) if \( A \) is nonseparating or of form \( \pm ix1xI/d \) if \( A \) is separating.
Figure 11.

Fixed point sets for the standard involutions.
Proof: Remove components of $A \cap (S' \times x \times 0)/d$. For details see [11].

QED

Theorem 8.3

Let $\iota$ and $\iota'$ be involutions on the orientable $I$-bundle over a Klein bottle, $W=S' \times x I \times I/d$ where $d=\kappa \times (\tau|\partial I) \times \tau$. Suppose $\iota$ and $\iota'$ have isomorphic fixed sets and if $\iota$ and $\iota'$ are fixed point free assume, in addition, that they have the same orientation type.

Then $\iota$ and $\iota'$ are conjugate. An involution on $W$ is conjugate to one of the eleven involutions listed above.

Proof: The proof is similar to the proof of Theorem 7.3. Let $\iota$ be an involution on $W=S' \times x I \times I/d$. We show it is conjugate to a standard involution. By the Annulus Theorem 5.9 there is an essential annulus $A$ with either $\iota A \cap A = \emptyset$ or $\iota A = A$ and $A$ and $\text{Fix}$ transversal. In the latter case by Lemma 3.5, assume the collar of $A$ is not interchanged. By the previous lemma take $A$ to be nonseparating of form $S' \times x I \times I/d$ or separating of form $\pm x I \times I/d$. In the case where $A$ is nonseparating and $\iota A \cap A = \emptyset$ make $\iota A = S' \times x \times 0 \times I$. Let $W_+ = S' \times [0,1] \times I$ and $W_- = S' \times [-1,0] \times I$. 
There are five cases:

Case 1) \( iA = A, \) \( A \) is nonseparating, \( A \) and \( \text{Fix} \) are transversal and the collar of \( A \) is not interchanged.

Case 2) \( iA \cap A = \emptyset, \) \( A \) is nonseparating and \( iW = W \).

Case 3) \( iA \cap A = \emptyset, \) \( A \) is nonseparating and \( iW = W \).

Case 4) \( iA = A, \) \( A \) is separating, \( A \) and \( \text{Fix} \) are transversal and the collar of \( A \) is not interchanged.

Case 5) \( iA \cap A = \emptyset \) and \( A \) is separating.

We show that \( i \) is conjugate to:

- in case 1) \( k,K, k_{2M}, k_{2S}, k_A, k_{S}, k_0 \) or \( k_N \)
- in case 2) \( k_{A2P}, k_{S2C}, k_{2C} \) or \( k_{2P} \)
- in case 4) \( k,K, k_{2M}, k_{2S}, k_{A2P} \) or \( k_{S2C} \)
- in case 5) \( k_A, k_S, k_{2C}, k_{2P}, k_0 \) or \( k_N \)

and that case 3) does not arise.

Several of the standard involutions are listed in more than one case. Each standard involution (or at least a conjugate of one) can in fact arise in the case it has been listed under. To see this it suffices to display an annulus \( A' \) in \( W \) with properties analogous to those of \( A \). Consider \( A' \) as follows: in case 1) take \( S'x-1xI/d \); in case 2) take \( S'x(-1/2)xI/d \); in case 4) take \( \pm ixIxI/d \); in case 5) take \( e^{\pm i\pi/4}xIxI/d \).

It suffices to show, first, that \( i \) has the same (fixed set) type as a standard involution listed under a
corresponding case, and second, if \( i \) and \( i' \) have the same type and fall into the same case 1) - 5) then they are conjugate.

Constant use is made of Section 4. Reserve \( j \) to denote standard involutions on the solid torus. All constructions done for \( i \) are to be performed for \( i' \), even if not explicitly stated.

**Case 1)** Proceed as for Case 1) of Theorem 7.3. The identification is now \( d=\kappa x(\tau |)x\tau \) instead of \( d=\text{id}x(\tau |)\text{id} \). Thus two of the five possibilities for \( \lambda |A \) give different fixed sets. When \( \lambda |A=\kappa \text{id} \) we have \( \text{Fix}(\lambda) \) is two 2-cells. Then \( \text{Fix} \) is either two Möbius bands or one annulus. When \( \lambda |A=\text{id} \tau \) we have \( \text{Fix}(\lambda) \) is an annulus. Then \( \text{Fix} \) is a Klein bottle.

**Case 2)** \( iA \cap A=\emptyset \), \( A \) nonseparating and \( iW_-W_- \). Select the component \( S=S'=x-1x-1 \) of \( \partial A \). Let \( B_+ \) be the component of \( W_+ \cap \partial W \) that meets \( S \). Let \( \lambda_+ = \iota |W_+ \). There are two possibilities:

2a) \( \lambda_+ B_+=B_+ \)

2b) \( \lambda_+ B_+ \cap B_+=\emptyset \).

Similarly for \( B_- \) and \( \lambda_- = \iota |W_- \). Suppose \( \lambda_+ \) satisfies case 2a). Then \( \lambda_+(S)=S'x0x-1 \). Since \( (\lambda_+ |A).d=\lambda_+ |A \) evaluating at \( S \) gives \( \lambda_+(S'x1x1)=S'x0x-1 \). Therefore \( \lambda_+ \) satisfies case 2b). Similarly if \( \lambda_- \) satisfies case 2b) then \( \lambda_- \) satisfies case 2a). The conjugacy class of \( \lambda \) is also restricted by the
orientation type of \( i \).

Up to symmetry there are four cases:

Case 2.1) \( \lambda \) is conjugate to \( j_{2C} \) and
\( \lambda \) is conjugate to \( j_{s} \).

Case 2.2) \( \lambda \) is conjugate to \( j_{2C} \) and
\( \lambda \) is conjugate to \( j_{0} \).

Case 2.3) \( \lambda \) is conjugate to \( j_{A} \) and
\( \lambda \) is conjugate to \( j_{2p} \).

Case 2.4) \( \lambda \) is conjugate to \( j_{N} \) and
\( \lambda \) is conjugate to \( j_{2p} \).

These give rise to involutions with fixed sets as claimed for this case. The case is completed as case 2) in Theorem 7.3.

Case 3) \( iA \cap A = \emptyset \), \( A \) nonseparating and \( iW = W \). Take \( S = S' \times I \times 0 \) with some choice of orientation. Then \( S \) is a generator of \( H_1(W) \) and \( H_1(W) \). Now \( i|W \cap S| = d \cdot i|W \cap S| \) but \( d*|S| = -|S| \). This is a contradiction so this case cannot arise.

Case 5) \( iA \cap A = \emptyset \) and \( A \) separates. It follows that \( iA \) also separates and that \( iA \) is contained in one of the two components that \( A \) decomposes \( W \) into. By a suitable isomorphism, we may assume \( A = e^{\pm i\pi/4} \times I \times I/d \) and \( iA = e^{\pm i3\pi/4} \times I \times I/d \). \( A \) and \( iA \) decompose \( W \) into three solid tori components \( U_0, U_1, U_2 \) with \( U_1 \cap U_2 = \emptyset, U_0 \cap U_1 = A, U_0 \cap U_2 = iA \) and \( iU_1 = U_2 \) and \( iU_0 = U_0 \). Moreover, if \( S \) is a component of \( \partial A \) then
\([s] \in H_1(U_0)\) is a generator and \([s] \in H_1(U_1)\) is twice a generator. \(\lambda = \iota|U_0\) is an involution on a solid torus that interchanges the disjoint annuli \(A\) and \(\iota A\) and both annuli have boundaries representing a generator of \(H_1(U_0)\). This is the same situation as for \(\lambda\) in case 2) of Theorem 7.3. That argument showed \(\lambda\) is conjugate to \(j_{2C}, j_A, j_N, j_S, j_0\) or \(j_{2P}\). Since \(\iota\) has the same fixed set as \(\iota|U_0\) we obtain fixed sets as listed above. Suppose \(\iota'\) also falls into this case. Then select an isomorphism \(h:A \to A'\) which we extend by equivariance to a conjugation \(h:A \cup \iota A \to A' \cup \iota A'\). The arguments for case 2) in Theorem 7.3 show \(h\) extends to a conjugation \(h:U_0 \to U_0'\). The following claim shows \(h|A\) extends to an isomorphism \(h:U_1 \to U_1'\). Extend to \(U_2\) by equivariance obtaining a conjugation \(h:W \to W'\) between \(\iota\) and \(\iota'\) and concluding this case.

Claim: Let \(U\) be a solid torus and \(A\) an annulus in \(\partial U\). Suppose a component \(S\) of \(\partial A\) represents twice the generator of \(H_1(U)\). Similarly for \(A'\) in \(U'\). Then an isomorphism \(h:A \to A'\) extends to an isomorphism \(h:U \to U'\).

To prove this, choose 1-spheres \(M\) and \(L\) in \(\partial U\) so that \([M]\) and \([L]\) generate \(H_1(\partial U)\) and \([M]\) is trivial in \(H_1(U)\). The choice can be made so that \([S]\) = \(a[M]+2[L]\) where "a" is an odd integer. Let \(I\) be a proper 1-cell in \(A\) meeting both boundary components of \(A\). Similarly for \(U'\), using \(I'=h(I)\). \(B=\partial V-A\) is an annulus. There is a proper 1-cell \(J\) of \(B\) with \(\partial J=\partial I\). Let
$S_i = I \sqcup J$. Then $[S_i] = b[M] + c[L]$ where "c" is an odd integer since $S_i \cap S_i$ is a point. There is an isomorphism of $\partial U$ to itself leaving $A$ fixed which changes $[S_i]$ by a given multiple of $[S]$. So we may choose $J$ so that $[S_i] = d[M] + [L]$ for some $d \in \mathbb{Z}$. Choose $J'$ similarly. Extend $h|A$ to an isomorphism $h: A \sqcup J \to A \sqcup J'$. Since $\text{int}(B-J)$ is an open 2-cell, $h$ can be extended by coning to an isomorphism $h: \partial U \to \partial U'$. Then $(2d-a)h_*(M) = h_*(2[S_i]-[S]) = 2[S_i']-[S_i'] = (2d'-a')[M']$. Since $a$ and $a'$ are odd and $[M]$ and $[M']$ are generators we get $h^*_M = \pm [M']$. Hence $h$ extends to $h: U \to U'$.

**Case 4)** $\iota A = A$, $A$ is separating, $A$ and $\text{Fix}$ are transversal and the collar of $A$ is not interchanged.

Let $W = \{z: z = x + yi, x > 0\} \times I/d$ and $W' = \{z: z = x + yi, x < 0\} \times I/d$. Then $\iota W = W$, and $W$ and $W'$ are solid tori. Let $S$ be a component of $\partial A$. $\lambda_\iota = \iota|W_\iota$ is conjugate to a standard involution $j$ of a torus $D^2 \times S'$. Let $M = S' \times 1$ and $L = 1 \times S'$. Then on choosing correct orientations $[S] = a[M] + 2[L] \in H_1(\partial(D^2 \times S'))$ where $a$ is odd. Since $\iota A = A$ it follows $\lambda_\iota^* [S] = \mu[S]$ where $\mu = \pm 1$ and $\mu$ depends only on $\iota|A$. Checking these conditions for the standard involutions on a solid torus gives:

- $\mu = 1$ and $\lambda_\iota$ is conjugate to $j_0$, $j_S$ or $j_M$.
- $\mu = -1$ and $\lambda_\iota$ is conjugate to $j_{2C}$ or $j_{DP}$.

Similarly for $\lambda_\iota = \iota|W'$. The collar is not interchanged so $\iota$
and $\iota | A$ have the same orientability type. We obtain four cases:

Case 4.1) $\lambda_1$ and $\lambda_2$ are conjugate to $j_M$ and $\iota | A$ is conjugate to $\text{id}_x r$ or $\text{ax}_r$.

Case 4.2) $\lambda_1$ and $\lambda_2$ are conjugate to $j_{DP}$ and $\iota | A$ is conjugate to $\text{axid}$.

Case 4.3) $\lambda_1$ and $\lambda_2$ are conjugate to $j_S$ or $j_O$ and $\iota | A$ is conjugate to $\text{axid}$.

Case 4.4) $\lambda_1$ and $\lambda_2$ are conjugate to $j_{2C}$ and $\iota | A$ is conjugate to $\text{ax}_r$.

For case 4.1) we will show the isomorphism class of the fixed set determines $\iota | A$. We will also show the different cases 4.1) - 4.4) have nonisomorphic fixed sets. Therefore given $\iota'$ with fixed set isomorphic to that of $\iota$, there is a conjugation $h: A \to A'$. We show $h$ extends to a conjugation $h: W_+ \to W_+'$. Similarly $h$ extends over $W$ and the proof will be complete. Let $B = \overline{\partial W} - A$. $B$ is an annulus.

In case 4.1) $\iota | A$ is conjugate to $\text{id}_x r$ or $\text{ax}_r$. In the first case $\text{Fix}_\cap A = S^1$ so $\text{Fix}$ is a Klein bottle while in the second case $\text{Fix}_\cap A = \emptyset$ so $\text{Fix}$ is two Möbius bands. In the first case $B/\iota$ is a Möbius band. It follows $h: A \to A'$ extends to a conjugation $h: \partial W \to \partial W,'$. In the second case $B/\iota$ is an annulus with $\partial (B/\iota) = (\partial A/\iota) \sqcup (\partial \text{Fix}_\cap W_+)/\iota$. Extend $h/\iota$ and lift to a conjugation $h: \partial W \to \partial W,'$. In both cases the conjugation extendable property of $j_M$ shows this
conjunct extension extends to $h: W \rightarrow W'$.

In case 4.2) let $\text{Fix}_i$ denote the fixed 2-disc component of $\lambda$. Let $\text{Fix}_i \times [-1,1]$ be a bicollar of $\text{Fix}_i$. Since a component of $\partial A$ has intersection number $\pm 2$ with $\text{Fix}_i$, it follows $\text{Fix}_i \times 1$ meets both components of $A - \text{Fix}_i$. Similarly for $\text{Fix}_i \times (-1)$ Hence $\text{Fix}_i \bigcup \partial \text{Fix}_i$ is bicollared so it must be an annulus. Thus the fixed set of $\iota$ is an annulus and two points. Extend the conjugation to $A \bigcup \partial \text{Fix}_i \rightarrow A' \bigcup \partial \text{Fix}_i'$. Since $\partial W_i - (\partial \text{Fix}_i \bigcup A)$ is two open 2-cells that are interchanged under $\iota$ we can extend to a conjugation $\partial W_i \rightarrow \partial W_i'$ and so by the conjugation extendable property of $j_{DP}$ to $W_i$.

In case 4.3) let $S$ be a fixed component of $A$. $W_i/\iota$ is a solid torus. Let $p: H_1(W_i) \rightarrow H_1(W_i/\iota)$ be the obvious homomorphism. $[S] = a[M] + 2[L]$ where $a$ is an odd integer. First we show $j_Q$ is not possible. Compare with case 2.5) of Theorem 7.3. $p[S] = a[M] + 4[L]$. However, $S$ double covers $S/\iota$ so it follows $a$ is even, a contradiction.

So only $j_S$ occurs and $\text{Fix}_i$ is two 1-spheres. Then $p[S] = 2a[M] + 2[L]$ and $[S/\iota] = a[M] + [L]$. Let $I$ be a proper 1-cell in $A$ that meets both boundary components of $A$ such that $I \cap I' = \emptyset$. Similarly for $I' = h(I)$. There is a proper 1-cell $J_1$ in $W_i/\iota$ with $\partial J_1 = \partial (I/\iota)$ and $[J_1 \cup I] = \pm [M]$. Let $S_0 = J_1 \bigcup I \cup \iota I$ where $J$ is a lift of $J_1$ by $p^{-1}$. Then $[S_0] = \pm [M]$. Similarly for $\iota'$. Extend $h: A \rightarrow A'$ to
h:A ∪ J → A ∪ J' and then by equivariance to h:A ∪ J ∪ tJ → A' ∪ J' ∪ tJ'. Now ∂W_2 -(A ∪ J ∪ tJ) consists of two 2-cells that are interchanged under t. So h extends to h:∂W → ∂W'. The condition on [S_0] and the conjugation extendable property of jS show h extends to a conjugation on W.

In case 4.4) λ, has fixed set Fix_1 ∪ Fix_2, where Fix_1 and Fix_2 are proper 1-cells of W. Arguments similar to those given already show that Fix_1 ∩ A cannot be exactly one point, i=1,2. Say then that ∂Fix_2 ⊆ A. Similarly for λ. Then the fixed set of t is two 1-cells Fix_1, Fix_1 and one 1-sphere Fix_2 ∪ Fix_2. (∂W_2 - A)/t is a 2-cell and h/t is given on the boundary. Clearly h/t can be extended over the 2-cell. On lifting obtain a conjugation h:∂W → ∂W'. Since Fix_1 ∩ A are two points in the same component of Fix_1 ∩ W and since also h(Fix_1 ∩ A)=Fix' ∩ A' are in the same component of Fix' ∩ W, the conjugation extendable property of j_{2C} gives a conjugation h:W → W'.

QED

Corollary 8.4

On the orientable I-bundle over a Klein bottle involutions with 2-dimensional fixed sets are conjugation extendable.
Proof: For $k_{A2P}$ the conjugation extends over the fixed set. Then cut open on the fixed set and use the conjugation extendable property of the solid torus involution $j_{2P}$. The other cases are similar to those for the trivial I-bundle over a torus.

QED

Corollary 8.5

If $\iota$ is an orientation preserving involution on an orientable twisted I-bundle $W=S^1\times I/I/\delta$ over a Klein bottle then $W/\iota$ is isomorphic to one of the following spaces:

1. $W/k_{2S} \cong D^2\times S^1$ a solid torus with $\text{Fix}/k_{2S}$ two unknotted 1-spheres $\pm 1/2\times S^1$,
2. $W/k_S \cong D^2\times S^1$ a solid torus with $\text{Fix}/k_S$ one unknotted 1-sphere $\{(e^{\pi it}/2, e^{2\pi it}) : -1 \leq t \leq 1\}$ representing twice a generator of $H_1(D^2\times S^1)$,
3. $W/k_{2C} \cong D^2\times I/(a|\partial D^2)\times I$ a 3-cell with $\text{Fix}/k_{2C} = \pm 1/2\times I$ and $((3/4)(z/|z|))\times 0$ two 1-cells and one linked 1-sphere,
4. $W/k_{2C} \cong D^2\times I/(a|\partial D^2)\times I$ an orientable I-bundle over a projective plane with $\text{Fix}/k_{2C} = (\pm 1/2)\times I$ two 1-cell fibers,
5. $W/k_0 \cong W$.

Proof: $\iota$ is conjugate to a standard involution $k$. Use the representations for the standard involutions. $W/k$ arises from the following subspaces of $W$ by identifications on
their boundaries: \( \{x + y \cdot i : 0 \leq y\} \times [0,1] \times [0,1] \) for \( k_{2S} \) and \( k_0 \), \( S' \times [0,1] \) for \( k_S \), and \( S' \times [0,1] \times [0,1] \) for \( k_{S2C} \) and \( k_{2C} \).

QED
V. INVOLUTIONS ON ORIENTABLE TORUS BUNDLES OVER A 1-SPHERE 
AND ON UNIONS OF ORIENTABLE TWISTED I-BUNDLES OVER KLEIN BOTTLES

§9. Involutions With 1-Dimensional Fixed Sets

Let \( g : T^2 \longrightarrow T^2 \) be an isomorphism where \( T^2 = S'^1 \times S^1 \) and let \( d : T^2 \times -1 \longrightarrow T^2 \times 1 \) be defined by \( d(x, -1) = (g(x), 1) \). Define the torus bundle \( M_g \) by \( M_g = T^2 \times I / d \). Then \( M_g \) is irreducible and \( T^2 \times 1 \) is a nonseparating incompressible 2-sided torus. Up to isotopy \( g : T^2 \longrightarrow T^2 \) is uniquely determined by \( g_* : H_1(T^2) \longrightarrow H_1(T^2) \). Let \( S_1 = S'^1 \times 1 \) and \( S_2 = 1 \times S^1 \). Then with respect to the basis \([S_1], [S_2] \) of \( H_1(T^2) \), \( g_* \) is given by a matrix \( M(g) \) of \( GL_2(Z) \). The matrix with respect to a different basis of \( H_1(T^2) \) is a conjugate \( Q^{-1} M(g) Q \) of \( M(g) \), where \( Q \in GL_2(Z) \). \( M_g \) is orientable if and only if \( g \) is orientation preserving, \( g \) is orientation preserving when and only when the \( \det(M(g)) = 1 \).
Of interest are the orientable flat space forms \( M_1, \ldots, M_5 \). See [15]. These are determined by \( g \) as follows:

\( M_1 = S^1 \times S^1 \times S^1 \): \( g = \text{id} \). Then \( M(g) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

\( M_2 \): \( g = \kappa \kappa \kappa \) (i.e.) \( g(x, y) = (\overline{x}, \overline{y}) \). Then \( M(g) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \).

\( M_3 \): \( g = \omega \cdot (\kappa \times \text{id}) \cdot \rho \) (i.e.) \( g(x, y) = (y, \overline{xy}) \). Then \( M(g) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

\( M_4 \): \( g = \omega \cdot (\kappa \times \text{id}) \) (i.e.) \( g(x, y) = (y, \overline{x}) \). Then \( M(g) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

\( M_5 \): \( g = \omega \cdot \rho \cdot (\kappa \times \text{id}) \) (i.e.) \( g(x, y) = (y, \overline{xy}) \). Then \( M(g) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \).

Each of these spaces has involutions with 1-dimensional fixed sets.

Let \( W_1 \) and \( W_2 \) be two orientable twisted \( I \)-bundles over a Klein bottle. An isomorphism \( d : \partial W_1 \rightarrow \partial W_2 \) determines a union along the boundaries of two orientable twisted \( I \)-bundles over a Klein bottle \( M_\delta = (W_1 \cup W_2) / d \). \( M_\delta \) is double covered by an orientable torus bundle over \( S^1 \). \( M_\delta \) is irreducible. Let \( T^2 = \partial W_1 \). The nonseparating annuli of \( W_1 \) determine a canonical generator \((1, 0)\) of \( H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \) up to sign. Separating (nontrivial) annuli or Möbius bands of \( W_1 \) determine a generator \((0, 1)\) up to sign. Note that the involutions \( k_{2M} \) and \( k_{A2P} \) on \( W_1 \) are isomorphisms of \( W_1 \) that reverse the signs of these generators. As before, the isomorphism \( d \) determines a matrix of \( \text{GL}_2(\mathbb{Z}) \).
An alternate description for these spaces is

\[ M = T^2 \times I / (d(x),-1) \sim (x,-1), (d_x(x),1) \sim (x,1) \]

where \( d \) and \( d_x \) are fixed point free orientation reversing isomorphisms. Then \( T^2 \times O \) decomposes \( M \) into two orientable twisted \( I \)-bundles over a Klein bottle.

Note that \( M_2 \) is a union of orientable twisted \( I \)-bundles over a Klein bottle: \( S' \times \pm 1 \times I \) are two Klein bottles.

The orientable flat 3-dimensional space forms have been classified (see Wolf [15]). Up to affine equivalence there are only six such space forms \( M_1, \ldots, M_6 \) and \( M_6 \). Define \( M_6 \) by

\[ M_6 = S' \times S' \times I / (x,y,-1) \sim (-x,-y,-1), (x,y,1) \sim (-x,-y,1) \]

\( M_6 \) is a union of orientable twisted \( I \)-bundles over a Klein bottle but is not a torus bundle since \( H_1(M_6) = \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) is finite. \( M_6 \) is also known as the Hantzsche-Wendt manifold (see [4]).

We need two lemmas which describe the position of incompressible tori in \( M \). For details see [11].

**Lemma 9.1**

Let \( M \) be an orientable torus bundle over \( S' \) and let \( T \) be an incompressible torus in \( M \).

If \( T \) is nonseparating then \( M \cong T^2 \times [-1,1]/d \) where \( d: T^2 \times [-1,1] \rightarrow T^2 \times I \) is an isomorphism.

If \( T \) is separating then \( T \) decomposes \( M \) into \( W_1 \) and \( W_2 \), two orientable twisted \( I \)-bundles over a Klein bottle.
Lemma 9.2

Let $M$ be the union of orientable twisted $I$-bundles over a Klein bottle and let $T$ be an incompressible torus in $M$.

If $T$ is nonseparating then $M$ is an orientable torus bundle over $S^1$.

If $T$ is separating then $T$ decomposes $M$ into $W_1$ and $W_2$, two orientable twisted $I$-bundles over a Klein bottle. $M = W_1 \sqcup W_2$ and $T = \partial W_1 = \partial W_2 = W_1 \cap W_2$.

Proof: Two fold cover $M$ by a torus bundle $\tilde{M}$. Argue by cases depending on whether $p^{-1}(T)$ is one or two tori. Note the deck transformation of $\tilde{M}$ is a fixed point free involution. For details see [11].

QED

Lemma 9.3

Let $M$ be an orientable torus bundle over $S^1$. Suppose $M$ is also a union of orientable twisted $I$-bundles over a Klein bottle. Then $M$ has a basis as an orientable torus bundle over $S^1$ so that its matrix is $\begin{bmatrix} -i & a \\ 0 & -1 \end{bmatrix}$ and a (canonical) basis as a orientable twisted $I$-bundle over a Klein bottle so that its matrix is $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.
Proof: $M = T' \times I / d$ for some identification $d : T' \times I \to T \times 1$. Let $p : T' \times I \to M$ be the induced projection. $M$ contains a Klein bottle $K$. Isotope $K$ so that $p^{-1}(K)$ consists of annuli $A_i$ meeting both components of $\partial (T' \times I)$. Since $K$ is a Klein bottle one of the $A_i$ must have boundary components representing opposite elements $\gamma$ and $-\gamma$ of $H_1(T')$. Since $M$ is orientable it follows $M$ has a matrix of the form $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$.

On the other hand, $M$ is a union of orientable twisted I-bundles $W_1$ and $W_2$ over a Klein bottle. $W_1 \cap W_2 = T$. $T'$ determines a nonseparating annulus in each of $W_1$ and $W_2$. Hence by choosing appropriate generators we may assume the matrix of $M$, as a union of orientable twisted I-bundles over a Klein bottle, is $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. $M$ has a two fold covering $q : M \to M$ by an orientable torus bundle over $S^1$, $M = U_1 \sqcup U_2$ such that $U_i \cong T^2 \times I$ double covers $W_i$. The deck transformation restricted to $U_i$ is a fixed point free involution $k_i$ on $U_i$ that interchanges boundary components of $W_i$. Hence $k_i$ is conjugate to $k_{01} = \alpha x \alpha x r$. Using this involution one sees that the matrix for the torus bundle $M$ is $\begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix}$.

The torus $T'$ which determines nonseparating annuli of $W_1$ and $W_2$ must lift to two tori and therefore a matrix for $M$ is also

$\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & -2a \\ 0 & 1 \end{pmatrix}$

Abelianizing the fundamental group of a torus bundle $M$,
with matrix \[
\begin{bmatrix}
1 & c \\
0 & 1
\end{bmatrix}
\]
, one sees that \(H_1(M_c) = \mathbb{Z} \oplus \mathbb{Z}(\mathbb{Z}/c\mathbb{Z})\). Also
the matrix \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]
conjugates \[
\begin{bmatrix}
1 & c \\
0 & 1
\end{bmatrix}
\]
to \[
\begin{bmatrix}
1 & -c \\
0 & 1
\end{bmatrix}
\]
and \(M_c\) and \(M_{-c}\) are isomorphic.

It follows \(b = \pm a\) and so \(b = a\) for a suitable choice of generators.

QED

**Proposition 9.4**

Let \(M\) be an orientable torus bundle over \(S^1\) or a union of orientable twisted \(I\)-bundles over a Klein bottle. Let \(\iota\) be an involution on \(M\) and let \(T\) be a separating incompressible torus with \(T \cap \iota T = \emptyset\). Then there is a nonseparating incompressible torus with \(\iota T = T\) and \(\text{Fix } T\) transversal.

**Proof:** By Lemmas 9.1 and 9.2, \(M = W_1 \sqcup W_2'\) with \(W_1 \sqcup W_2' = T\) where \(W_1\) and \(W_2'\) are orientable twisted \(I\)-bundles over a Klein bottle. Without loss say \(\iota T\) is in \(W_2'\). Using the lemmas again we see \(M = W_1 \sqcup (T'x[-1,1]) \sqcup W_2\) with (for \(i=1,2\))
\[
\partial W_i = T'x(-1)^i
\]
\(\iota W_1 = W_2\), \((T'x[-1,1]) = T'x[-1,1]\) and with \(W_i\) orientable twisted \(I\)-bundles over a Klein bottle.

Let \(A\) be a nonseparating proper annulus in \(W_1\). Write \(\partial A = S_1 \sqcup S_2\). There is an essential annulus \(A_0\) in \(T'x[0,1]\) with \(\partial A_0 = S_1 \sqcup \iota S_2\). Then \(\partial A_0 \cap \iota \partial A_0 = \emptyset\). By the Partial Annulus
Theorem 5.6 there are disjoint essential annuli $A_1$ and $A_2$ transversal to $\text{Fix}$ with $\partial A_0 \cap \partial A_0 = \partial A_1 \cup \partial A_2$ and with either $\iota A_1 = A_1$ $(i=1,2)$ or $\iota A_2 = A_2$.

Let $T_1 = A_1 \cup A_2$. Essential annuli of $T'x[-1,1]$ must meet both boundary components so $T_1$ is connected. $T_1$ is a torus. This follows since $A$ is nonseparating in $W$, so $A$ is "twisted" relative to $T'x[-1,1]$ as is $\iota A$.

$\iota T_1 = T_1$ and $T_1$ is transversal to $\text{Fix}$. $T_1$ is nonseparating hence incompressible.

QED

Recall that up to conjugacy there are three orientation preserving involutions $j_{2C}$, $j_S$ and $j_0$ on a solid torus $V$, five orientation preserving involutions $k_{2S}$, $k_S$, $k_{4C}$, $k_{OF}$ and $k_{OI}$ on a trivial $I$-bundle $W$ over a torus and five orientation preserving involutions $k_{2S'}$, $k_{S'}$, $k_{S2C'}$, $k_{2C'}$ and $k_{0'}$ on an orientable twisted $I$-bundle over a Klein bottle.

By applying the Torus Theorem 6.2 to $M$ we obtain the following theorem.
Theorem 9.5

Let $M$ be an orientable torus bundle over $S^1$ or a union of orientable twisted $I$-bundles over a Klein bottle. Let $\iota$ be an involution with a 1-dimensional fixed set component. Then one of the following holds:

1) There is a nonseparating (incompressible) torus $T$ with $\iota T \cap T = \emptyset$. $T$ and $\iota T$ decompose $M$ into two trivial $I$-bundles $W_1$ and $W_2$ over a torus with $\iota|_{W_1}$ conjugate to $k_{2S}$, $k_S$ or $k_{OI}$.

2) There is a nonseparating (incompressible) torus $T$ with $\iota T = T$ and its collar is not interchanged. $M \cong W/d$ where $W$ is a trivial $I$-bundle over a torus and $d$ is an isomorphism between the boundary components of $W$. $\iota$ is induced from an involution on $W$ that is conjugate to $k_{4C}$.

3) There is a separating incompressible torus $T$ with $\iota T = T$. $M$ is the union of orientable $I$-bundles $W'_1$ and $W'_2$ over a Klein bottle with $W'_1 \cap W'_2 = T$. $\iota|_{W'_1}$ and $\iota|_{W'_2}$ are both conjugate to $k_{S2C}$ or $k_{2C}$ or are both conjugate to $k_{2S}'$, $k_S'$ or $k_{O}'$.

4) $M$ satisfies case (III) of Torus Theorem 6.2 and $\text{Fix}(\iota)$ is exactly one $I$-sphere. $\iota|_{V_1}$ is conjugate to $j_{2C}$ and $\iota|_{V_2}$ is conjugate to $j_{0}$. $M$ is $M_6$.

Proof: Apply the Torus Theorem 6.2. Assume for the moment that case (III) does not occur. Then since $M$ is orientable there is a torus $T$ satisfying case (I) or (II) of that
theorem. By Proposition 9.4, if $iT = T$ then $T$ separates. If $iT = T$ and the collar is interchanged then we arrive at case (1) by considering the boundary of a collar of $T$. We show that case (II) can be eliminated. Then to complete the proof, case (III) is handled.

Suppose case (II) occurs and suppose $iS_1 = S_1$. Since $A_1 \sqcup A_i$ is a separating incompressible torus it follows by Lemmas 9.1 and 9.2 that $U_1 \sqcup V_1$ is an orientable twisted I-bundle over a Klein bottle. $iA$, is a separating annulus so $S^1$ bounds a proper Möbius band $A$ of $U_1$. $K = A \sqcup iA$ is a Klein bottle with $iK = K$. Then by Lemma 6.1 the boundary of a regular neighborhood of $K$ is an incompressible torus invariant under $i$ giving case (I).

Suppose $iS_1 = S_2$. Since $i$ has a 1-dimensional fixed set and $\text{Fix}_{i} S_1 = \emptyset$, assume $i|V_1$ is conjugate to $j_{S}$. As above $iA_i$ is a separating annulus in the orientable twisted I-bundle over a Klein bottle $U_1 \sqcup V_1$. So $S_1$ represents twice a generator of $H_1(V_1)$. Selecting appropriate generators $[M]$ and $[L]$ of $H_1(\partial V_1) = \mathbb{Z} \oplus \mathbb{Z}$ where $[M]$ is trivial in $H_1(V_1)$, we may assume $[S_1] = a[M] + 2[L]$ for some integer $a$. Let $p : H_1(\partial V_1) \longrightarrow H_1(\partial V_1/i)$ be the obvious homomorphism. Then $p[M] = 2[M_i]$ and $p[L] = [L_i]$ for suitable generators $[M_i]$ and $[L_i]$ of $H_1(\partial V_1/i)$. Then $p[S_1] = 2a[M_i] + 2[L_i]$ but this contradicts that $S_1 \longrightarrow S_1/i$ is an isomorphism.
Now consider case (III). Let $V_1''$ be the closure of the component of $M-(T \cup iT)$ that contains $V_1$. Let $V_1'$ be the solid torus obtained from $V_1''$ by replacing $S$ by two copies $S_1$ and $S_2$. Then $V_1' \cong V_1''/(S_1 \sim S_2)$. Similarly for $V_2'$ and $V_2''$. Let $W_1$ be a regular neighborhood of $T$. $\partial W_1$ is an incompressible torus. By Lemmas 9.1 and 9.2, $W_2=M-W_1$ is an orientable twisted $I$-bundle over a Klein bottle. $\iota T \cap \text{int}(W_1)$ is a separating annulus. Let $\partial(\iota T-W_1)=C_1 \sqcup C_2$. See Figure 12. It follows $C_1$, $C_2$, $S_1$ and $S_2$ represent twice a generator in both $H_1(V_1')$ and $H_1(V_2')$. Let $\iota$, be the involution on $V_1'$ induced by $\iota$. Similarly for $\iota_2$. From the Torus Theorem 6.2, $\iota_1$ is conjugate to $j_2^C$ and $\iota_2$ is conjugate to $j_S$ or $j_0$. The argument given above for case (II) shows $\iota_2$ is not conjugate to $j_S$ since $\iota_2$ interchanges components of $\partial V_2'-(S_1 \sqcup S_2)$. Also the 1-cell $S_1/\iota_1$ in $V_1'/\iota_1$ cannot meet both components of $\text{Fix}(\iota_1)$ because otherwise its lift $S_1$ would be a 1-sphere representing an odd multiple of a generator of $H_1(V_1')$. Hence $S_1$ meets only one component of $\text{Fix}(\iota_1)$. Therefore

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure12.png}
\caption{Figure 12.}
\end{figure}
Fix(ι) is one 1-sphere.

Since $C_1$ is a boundary component of the nonseparating annulus $\iota T \cap W_1$ of $W_1$ and of the separating annulus $\iota T \cap W_2$ of $W_2$ we see that a $(1,0)$-generator of $W_1$ and a $(0,1)$-generator of $W_2$ correspond.

Let $C_2$ be a 1-sphere in $T$ meeting $S$ transversally in one point. Let $A$ be the annulus of $V_2'$ determined by $T$. Since $\iota_2$ is conjugate to $j_0$ and $\partial A$ represents twice a generator of $H_1(V_2')$, there is a proper 2-cell $D$ of $V_2'$ with $\iota_2 D \cap D = \emptyset$. Also arrange that $D$ meets $A$ in two proper nonseparating 1-cells of $A$, one of which is $C_1$, and that $D$ meets $\iota_2 A$ in two proper nonseparating 1-cells. Then $\iota_2(\partial D \cap \iota_2 A)$ and $\partial D \cap A$ are four disjoint nonseparating proper 1-cells of $A$. Now consider $A$ as an annulus in $V_2'$. Since $\iota_1$ is conjugate to $j_{2c}$ and $(j_{2c})^*$ on $H_1(V)$ is multiplication by $-1$, it follows that $\iota_1 \iota_2(\partial D \cap \iota_2 A) \sqcup (\partial D \cup A)$ bounds a disc in $V_2$. But $(\iota_1 \iota_2)|_{\iota T} = \text{id}$ so $\partial D$ bounds a disc $D_1$ in $V_1'$. By choice of $C_1$, $(D \cup D_1) \cap W_1$ is two Möbius bands so its...
boundary is a \((0,1)\)-generator of \(W_1\). On the other hand \((D \cup D_1) \cap W_2\) is a nonseparating annulus of \(W_2\) since it has two boundary components and does not separate \(V_1\). This gives \(M_6\).

To show how \(M_6\) arises in this manner we give the following construction. See Figure 13. Let \(V_1'=D^2 \times S^1\) and

\[
A_j = \{(e^{2\pi i(t+v)}, e^{4\pi it}) : t \in \mathbb{R}, 0 \leq v \leq 1/4\}
\]

Let \(\iota_1 = j_0 = \text{id}\) and \(\iota_2 = j_2 = \hat{\kappa}x\). Let \(d\) be the identification

\[
(e^{2\pi it}, e^{4\pi it}) \sim (e^{2\pi i(1-t)}, e^{4\pi i((3/4)-t)})
\]

on \(\partial A_j\). Let \(V_1'' = V_1'/d\). Define \(h : A_2 \rightarrow A_1\) to be induced from the identity and define

\[
h|_{\iota_2 A_2} = \iota_1 \cdot h \cdot \iota_2 | A_2 = \hat{\kappa}x(-\kappa)
\]

(ie) \(h: \partial V_2'' \rightarrow \partial V_1\) is \(h(x) = x\) if \(x \in A_1\) and \(h(x) = (\hat{\kappa}x(-\kappa))(x)\) otherwise. Then \(M_6 = V_1' \cup V_2'' / h \cup d\) and involution \(\iota_1 \cup \iota_2\) has fixed set one 1-sphere. (In the previous construction one can take \(D \cup D' = \{t = -1/8\}\))

QED

Corollary 9.6

Let \(\iota\) be an involution on an orientable torus bundle over \(S^1\) or a union of orientable twisted \(I\)-bundles over a Klein bottle with a 1-dimensional fixed set. Then the fixed set is one, two, three or four 1-spheres.
Corollary 9.7

Let $M$ an orientable torus bundle over $S^1$ or a union of orientable twisted $I$-bundles over a Klein bottle. Let $\iota$ be an involution on $M$ with a 1-dimensional fixed set.

Then $M/\iota$ is a lens space, $P^3$, $P^3\#P^3$, or the boundary union of a solid torus with an orientable twisted $I$-bundle over a Klein bottle.

Proof: Use Corollary 7.5 and Corollary 8.5. Consider the cases of Theorem 9.5.

In 1) when each of $\iota|W_i$ is conjugate to $k_2s$ or $k_s$, $M/\iota$ is a union along the boundaries of two solid tori. When one of $\iota|W_i$ is conjugate to $k_0I$ then $M/\iota$ is the boundary union of a solid torus with an orientable twisted $I$-bundle over a Klein bottle.

In 2) $W/k_{2c}$ is $S^2xI$. The identification of $S^2x-1$ with $S^2xI$ gives the lens space $S^2xS^1$.

In 3) for $k_{2s'}$, $k_s'$ and $k_o'$ we get the same spaces as in 1). For $k_{s2c'}$ and $k_{2c'}$, capping $W/k_{s2c'}$ gives a 3-sphere and capping $W/k_{2c'}$ gives $P^3$.

In 4) $V/\iota$ is a 3-cell and $V_1V_1/\iota$ is two 2-cells. $V_1/\iota$ is a 3-cell so $V_1U_2V_1/\iota$ is a solid torus. Since $V_2$ is also a solid torus $M/\iota$ is a lens space.

QED
Note that torus bundles may also be unions of twisted I-bundles over a Klein bottle. (See Lemma 9.3)

In the following let $Q \in \text{GL}_2(\mathbb{Z})$. Let $E \in \text{GL}_2(\mathbb{Z})$ be in the subgroup $K$ of $\text{GL}_2(\mathbb{Z})$ generated by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

Note $\mathbb{Z} \to \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ induces an exact sequence:

$$1 \to K \to \text{GL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}_2) \to 1$$

**Theorem 9.8**

Let $M$ be an orientable torus bundle over $S^1$ or a union of orientable twisted I-bundles over a Klein bottle. Then $M$ has an involution $\iota$ with fixed set exactly $n$ 1-spheres ($n > 0$) if and only if one of a), b) or c) below holds.

a) $M$ is a torus bundle with matrix conjugate to one of the following for some $Q$ or $E$:

1. $Q^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Q \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $n = 2$ or $4$

2. $Q^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Q \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ $n = 1$ or $3$

3. $Q^{-1} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} Q \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ $n = 2$

4. $EP$ where $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $n = 4$

5. $EP$ where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $n = 1$ or $3$

6. $EP$ where $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $n = 2$
b) M is a union of orientable twisted I-bundles over a Klein bottle with matrix (with respect to some set of canonical generators) one of the following for some Q or E:

In (7) - (9) the inverses of matrices are to be taken over the field of rationals but the product matrix is required to be integral.

(7) \[
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}^{-1} Q \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

n = 2, 3 or 4

(8) \[
\begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix}^{-1} Q \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

n = 1 or 2

(9) \[
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}^{-1} Q \begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix}
\]

n = 1 or 2

(10) EP where \( P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) or \( P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \)

n = 2 or 4

(11) EP where \( P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \) or \( P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \)

n = 1 or 3

(12) EP where \( P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \) or \( P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \)

n = 3

(13) EP where \( P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \) or \( P = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \)

n = 2

c) M is \( M_6 \) and \( n = 1 \)

Remark 9.9

The proof shows how in each case (1) - (13) the involution on M arises from involutions on orientable torus bundles over \( S^1 \) or orientable twisted I-bundles over a Klein bottle.
Proof: Apply Theorem 9.5.

Suppose case (1) of Theorem 9.5 occurs: There is a nonseparating torus with \( \iota T \cap T = \emptyset \).

Let \( W_1 \) and \( W_2 \) be the two components of \( M \) determined by \( T \) and \( \iota T \). Let \( h_1 : W_1 \to T^2 \times I \) and \( h_2 : W_2 \to T^2 \times I \) be conjugations between \( \iota|W_i \) and the standard involutions \( k_{2S} \), \( k_S \) or \( k_{OI} \) with \( h_1(T) = T^2 \times 1 \). Let \( d : T^2 \times I \to T^2 \times I \) be the identification induced by \( T \). Then the involutions induce an identification \( \iota_2 . d . \iota_1 : T^2 \times 1 \to T^2 \times 1 \). Consider the effect of these isomorphisms in \( H_i(T^2) = H_i(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \) with respect to basis \([ S^1 x 1] \) and \([ 1 x S^1] \) where orientations are induced from \( S^1 \subseteq \mathbb{C} \). Then \( M \cong T^2 \times I / q \) where the matrix of \( q \) is \( M(d)^{-1} M(\iota_2) M(d) M(\iota_1) \). The matrix of \( k_{2S} \) and \( k_{OI} \) is 
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]
and of \( k_S \) is 
\[
\begin{bmatrix}
1 & -1 \\
0 & -1
\end{bmatrix}
\]. We obtain (1) - (3) above.

Suppose case (2) of Theorem 9.5 occurs: There is a nonseparating torus with \( \iota T = T \) and the collar of \( T \) is not interchanged.

Cut \( M \) open on \( T \). Then \( M \cong T \times I / d' \) where \( d' : T^2 \times I \to T^2 \times I \) is an isomorphism. \( \iota \) is induced from an involution \( \lambda \) on \( T \times I \) that satisfies \( d'(\lambda|T \times 1) = (\lambda|T \times 1).d' \). Let \( h : T \times I \to T^2 \times I \) be a conjugation between \( \lambda \) and \( k_{4C} \) (with \( h(T \times 1) = T^2 \times 1 \)). Define \( d = h| . d'.(h|T^2 \times 1)^{-1} \). Then \( M \cong T^2 \times I / d \) and \( \iota \) is conjugate to the involution on \( T^2 \times I / d \) induced from the involution \( k_{4C} = k \times k \times \text{id} \) on \( T^2 \times I \). The matrix of \( M \) is conjugate to the matrix of \( d : T^2 = T^2 \times 1 \to T^2 \times 1 = T^2 \). Note \( d \) satisfies
d.(\kappa \kappa \kappa) = (\kappa \kappa \kappa).d.

Now \(T^2/\kappa \kappa \kappa\) is a 2-sphere. The class of all isomorphisms of \(T^2/\kappa \kappa \kappa\), up to isotopy, that keep the four points \(\pm 1 \pm 1\) fixed is generated by two Dehn twists: a twist on \(S'_x i/\kappa \kappa \kappa\) and a twist on \(i x S'/\kappa \kappa \kappa\), see [2]. A twist on \(S'_x i/\kappa \kappa \kappa\) lifts to a twist on \(S'_x i\) together with a twist on \(S'_x i\) in the "same" direction. With respect to the basis \([S'_x i], [1 x S']\) of \(H_1(T^2)\) the lifted twist has matrix \[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\] (or \[
\begin{bmatrix}
1 & -2 \\
0 & 1
\end{bmatrix}
\]). Hence if \(d\) keeps all four points fixed (and therefore \(\text{Fix}\) is four 1-spheres) then the matrix of \(d\) is in the subgroup \(K\) of \(\text{GL}_2(\mathbb{Z})\).

In general \(d\) induces a permutation on the four points \(\pm 1 \pm 1\). Label these points as \(1=1 x 1\), \(2=-1 x 1\), \(3=1 x -1\) and \(4=-1 x -1\). The number of 1-sphere fixed components of \(t\) is the number of orbits of the permutation induced by \(d\) on \(\{1, 2, 3, 4\}\). Note that \(\rho_1(z, w)=(zw, w)\) commutes with \(\kappa \kappa \kappa\), induces the permutation \((34)\) and has matrix \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]. \(\rho_1\) is in fact a Dehn twist. \(\rho_2(z, w)=(-zw, w)\) has the same matrix and induces the permutation \((12)\). \(\rho_3(z, w)=(z, zw)\) commutes with \(\kappa \kappa \kappa\), induces the permutation \((24)\) and has matrix \[
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\]. \(\rho_4(z, w)=(z, -zw)\) has the same matrix and induces the permutation \((13)\). Also note that the following isomorphisms commute with \(\kappa \kappa \kappa\) and have matrix the identity: \(axid\) inducing permutation \((12)(34)\), \(idxa\) inducing \((13)(24)\) and \(axa\) inducing \((14)(23)\).
There is a composition $r$ of these isomorphisms such that $d=d'.r$ and such that $d'$ induces the identity permutation: Use $axi$, $idxa$ and $axa$ to reduce the possibilities to permutations $(34)$, $(24)$, $(14)$, $(12)(34)$, $(243)$, $(234)$, $(1423)$, $(1234)$ and $(1243)$. Then use the $p$'s to generate these permutations. So the matrix of $d$ is of form $EP$ where $E$ is the matrix of $d'$ and hence is in $K$ and $P$ is of form listed in (4) - (6).

Suppose case (3) of Theorem 9.5 occurs: There is a separating torus with $\iota T=T$.

Then $M$ is a union of two orientable twisted $I$-bundles $W_1$ and $W_2$ over Klein bottles. Let $d:\partial W_1\longrightarrow \partial W_2$ be the identification. As above, by changing $\iota|W_1$ by a conjugation on $W_1$, we may assume that $\iota_1=\iota|W_1$ and $\iota_2=\iota|W_2$ are standard involutions. Then $\iota_2d=\iota_1$. Select representatives of $H_1(\partial W_1)=\mathbb{Z}\oplus\mathbb{Z}$ in the canonical way: $(1,0)$ arising from a nonseparating annulus of $W$, and $(0,1)$ from a separating annulus.

Suppose $\iota|\partial W_1$ is fixed point free. $T_1=\partial W_1/\iota_1$ is a torus. We can select a basis for $T_1$ such that $p_1:\partial W_1\longrightarrow T_1$ has matrix $M(p_1)=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ if $\iota_1$ is conjugate to $k_2s'$ or $k_5'$ and matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ if $\iota_1$ is conjugate to $k_0'$. Similarly for $T_2$. $d:\partial W_1\longrightarrow \partial W_2$ induces an isomorphism $q:T_1\longrightarrow T_2$ with $p_2.d=q.p_1$. Then the matrix of $d$ is (as a product in $\text{GL}_2(\mathbb{Q})$) the matrix $M(p_2)^{-1}QM(p_1)$ where $Q$ is the matrix of $q$. 
Conversely, up to isotopy a matrix $Q \in \text{GL}_2(\mathbb{Z})$ determines an isomorphism $q: T_1 \rightarrow T_2$ and this isomorphism lifts if $M(p_2)^{-1}QM(p_1) \in \text{GL}_2(\mathbb{Z})$. This gives (7) - (9).

Suppose $t|\partial W_1$ is not fixed point free. Then $t|\partial W_1$ is conjugate to $\kappa x \kappa$. $t_1$ and $t_2$ are conjugate to $k_{2C}' = (-\kappa)x\tau x\tau/d$ or to $k_{S2C}' = \text{id}x\tau x\tau/d$. For convenience use the representation $\kappa x\tau x\text{id}/d$ for $k_{S2C}'$ instead.

For both $k_{2C}'$ and $k_{S2C}'$: $S_1 = S' \times 0 \times 1$ is invariant, meets both fixed 1-cells and represents $(1,0) \in H_1(\partial W_1)$, that is, is a boundary component of a nonseparating annulus. For $k_{2C}'$: $S_2 = 1 \times 1 \times 1 \sqcup 1 \times -1 \times 1$ is invariant, meets both fixed 1-cells and represents $(0,1)$. For $k_{S2C}'$: $S_2 = i \times 1 \times 1 \sqcup -i \times -1 \times 1$ is invariant, meets only one fixed 1-cell and represents $(0,1)$. The curves $S_1$ and $S_2$ give a way to assign labels 1, 2, 3 and 4 to the four points of $\text{Fix} \cap \partial W_1$ (e.g. 1 to $S_1 \cap S_2$). The fixed sets of $k_{2C}'$ and $k_{S2C}'$ match these labels in a different way. The matching can be arranged to occur in the same way if a twist on $S_2$ is performed in the $k_{S2C}'$ case. The matching and $d$ determine the number of components of $\text{Fix}$.

Proceed as in the case above where $T$ was a nonseparating torus with $tT = T$. In the above listing (12) and (13) arise from combining a $k_{2C}'$ and a $k_{S2C}'$. (11) and (12) arise from combining a $k_{2C}'$ and a $k_{2C}'$ or from combining a $k_{S2C}'$ and a $k_{S2C}'$.

QED
Let $M$ be an orientable torus bundle over $S^1$ with involution $\iota$. Call $\iota$ fiber preserving if there is a fibration $p: M \to S^1$ such that $\iota(p^{-1}(x)) = p^{-1}(\iota(x))$ for all $x \in S^1$ and if $\text{Fix} = \text{Fix}(\iota)$ is transversal to each fiber $p^{-1}(x)$.

Note the involution $\iota(x,y,z) = (y,x,-z)$ on $S^1 \times S^1 \times S^1$ is not fiber preserving, in this sense, with respect to fibration obtained by projection to the last coordinate.

**Corollary 9.10**

Let $M$ be an orientable torus bundle over $S^1$. Then $M$ has a fiber preserving involution with fixed set exactly $n$ 1-spheres ($n > 0$) if and only if the matrix of $M$ is conjugate to one of (4), (5) or (6) of Theorem 9.8.

**Proof:** Let $x \in S^1$ be such that the fiber $T = p^{-1}(x)$ meets the fixed set. Then $\iota T = T$. Since $T$ and $\text{Fix}$ are transversal the collar of $T$ is not interchanged. So $T$ satisfies case (2) of Theorem 9.5. Now follow the proof of Theorem 9.8.

QED
§10. Involutions With 2-Dimensional Fixed Sets

In the previous section the space forms $M_1$, $M_2$ and $M_6$ were defined. Recall also Definition 3.1. Define the following involutions on these spaces.

On $M_1$:

$a_{2T}=\text{id}xx\text{id}/d$ having fixed set two tori $S'x\pm1xI/d$

$a_T=\omega\text{id}/d$ having fixed set the nonseparating torus

\{$(z,z)|z\in\mathbb{C})xI/d$\

On $M_2$:

$\beta_{2K}=\text{id}xx\text{id}/d$ having fixed set two Klein bottles $S'x\pm1xI/d$

$\beta_K=\omega\text{id}/d$ having fixed set the Klein bottle $\{(z,z)|z\in\mathbb{C})xI/d$

$\beta_T=\text{id}(-\kappa)x\text{id}/d$ having fixed set separating torus $S'x\pm1xI/d$

$\beta_{T4P}=\kappa xx\tau/d$ having fixed set a torus and four points

$(S'xS'x-1\cup\pm1x1x0)/d$

On $M_6$:

$\gamma_{K2P}=(-\kappa)x\text{id}/d$ having fixed set a Klein bottle and

two points $(S'xS'x1\cup1x1x-1)/d$

Here $d$ denotes the identification for the corresponding space form $M_1$. 

Lemma 10.1

Let $M$ be an orientable torus bundle over $S^1$ or a union of orientable twisted $I$-bundles over a Klein bottle. Let $\iota$ be an involution with fixed set containing a Klein bottle. Then there is a separating incompressible torus $T$ with $\iota T = T$ and $T \cap \text{Fix} = \emptyset$.

Proof: By Lemma 6.1 the boundary of a regular neighborhood of the Klein bottle $K$ is an incompressible torus. Since $K$ is fixed under $\iota$, arrange that the regular neighborhood is invariant under $\iota$ and that it meets $\text{Fix}$ only at $K$. Then let $T$ be the boundary of this regular neighborhood.

QED

Theorem 10.2

Let $M$ be an orientable torus bundle over $S^1$ or a union of orientable twisted $I$-bundles over a Klein bottle. Let $\iota$ be an involution on $M$ with a 2-dimensional fixed set component. Then $M$ is isomorphic to $M_1$, $M_2$ or $M_6$ and $\iota$ is conjugate to one of the seven involutions defined above.

Proof: We apply the Torus Theorem 6.2.

Some economy could be achieved by showing that the fixed set contains an incompressible torus or Klein bottle $F$. When $F$ is a torus, $F$ could then be used to construct a
torus $T$ with $\iota_T \cap T = \emptyset$. The present approach has the advantage that it generalized to the case of orientation reversing involutions. It also parallels the proof of Theorem 7.3.

$M$ is orientable so case (IV) of the Torus Theorem 6.2 does not arise and 2-sided Klein bottles do not occur. Cases (II) and (III) do not arise since $\iota$ is orientation reversing. So there is an incompressible torus $T$ with either $T \cap T = \emptyset$ or $T = T$ and $T$ and $\text{Fix}$ transversal.

By Proposition 9.4 there are three cases:

Case 1) $\iota_T \cap T = \emptyset$ and $T$ does not separate.

Case 2) $\iota_T = T$, $T$ does not separate and the bicollar of $T$ is not interchanged.

Case 3) $\iota_T = T$ and $T$ separates.

By the last lemma we may assume $T$ does not meet any Klein bottle components of $\text{Fix}$.

We show $\iota$ is conjugate to:

in case 1) $a_{2T}$, $a_T$, or $\beta_{T4P}$.

in case 2) $a_T$ if $T \cap \text{Fix}$ is one 1-sphere,

$a_{2T}$, $a_T$, or $\beta_T$ otherwise.

in case 3) $\beta_{2K}$, $\beta_K$, $\beta_T$, $\beta_{T4P}$ or $\gamma_{K2P}$.

Several of the standard involutions are listed in more than one case. Each standard involution can arise in the case it is listed under. Namely consider $T =: S'xixI$, $\{(x,ix,t):x,t\}$, $S'xS'x(1/2)$, $S'xS'x0$, $S'xS'x0$, $S'xixI$, $\{(x,ix,t):x,t\}$, $S'xS'x(1/2)$, $S'xS'x0$, $S'xS'x0$. 

It suffices to show \( \iota \) has the same fixed set as a standard involution listed under the corresponding case, and that if \( \iota \) and \( \iota' \) have isomorphic fixed sets and fall into the same case 1)-3) then they are conjugate.

We make use of Lemmas 9.1 and 9.2, Corollary 7.4 and Corollary 8.4.

Case 1) \( \iota T \cap T = \emptyset \) and \( T \) does not separate.

\( \iota T \cup T \) decomposes \( M \) into two components \( W_1 \cong T^2 \times I \) with \( \partial W_1 = T \cup iT \). Since there is a fixed set, \( \iota W_1 = W_1 \). The boundary components of \( W_1 \) are interchanged so \( \iota |(W_1) \) is conjugate to \( k_T, k_{4p} \) or \( k_{N1} \). Since \( \iota \) has a 2-dimensional fixed set assume \( \iota |(W_1) \) is conjugate to \( k_T \). Fixed sets of the same type as for the standard involutions \( a_{2T}, a_T, \) or \( \beta_{T4p} \) are obtained. Given \( \iota' \) with isomorphic fixed set, let \( h:W_2 \rightarrow W_2' \) be a conjugation between \( \iota |(W_2) \) and \( \iota |(W_2') \). Then \( (h|):\partial W_1 = \partial W_2 \rightarrow \partial W_2' = \partial W_1' \) is a conjugation which by the conjugation extendable property for \( k_T \) extends to a conjugation \( h:W_1 \rightarrow W_1' \). Hence \( \iota \) and \( \iota' \) are conjugate.

Case 2) \( \iota T = T \), \( T \) does not separate and \( \iota \) does not interchange the collar.

Then \( M \) is isomorphic to \( W/\partial \) where \( W = T^2 \times I \) and \( \partial: T^2 \times I \rightarrow T^2 \times -1 \) is an isomorphism. \( \iota \) is induced by an involution \( \lambda \) on \( W \) that does not interchange the components.
of $\partial W$. Since $\lambda$ has a two dimensional fixed component, $\lambda$ is conjugate to $k_A$ or $k_{2A}$.

Case 2.1) $\lambda$ is conjugate to $k_A$. Then $\partial \text{Fix}(\lambda) = S_1 \cup S_2$ has two components so $T \cap \text{Fix}$ has one component. Orient $S_1$. Then annulus $\text{Fix}(\lambda)$ induces an orientation on $S_2$. If $d_*(S_1) = -[S_2]$ then $\text{Fix}$ is a Klein bottle meeting $T$. So $d_*(S_1) = [S_2]$ and $\text{Fix}$ is a torus. Let $\iota'$ be conjugate to $\iota$ and assume $\iota'$ falls into this case also. Construct $W'/d' \cong M'$ as above. Construct a conjugation $T^2 \times 1 \rightarrow (T^2 \times 1)'$ between $\lambda | (T^2 \times 1)$ and $\lambda' | (T^2 \times 1)'$. Extend to a conjugation $h: \partial W \rightarrow \partial W'$ by defining $h(T^2 \times 1) = d''. h.(d^{-1})$. Since $\text{Fix}$ is a torus in an orientable manifold, $h$ extends over a bicollar of $\text{Fix}$. Then the conjugation extendable property of $k_A$ shows $h$ extends to a conjugation $h: W \rightarrow W'$ between $\lambda$ and $\lambda'$. $h$ induces a conjugation between $\iota$ and $\iota'$.

Case 2.2) $\lambda$ is conjugate to $k_{2A}$. Then $\text{Fix}(\lambda)$ has two annular components $A_1$ and $A_2$ and $\text{Fix}$ meets $T$ in two 1-spheres. Let $S_{ij} = T^2 \times (-1)^i \cap A_j$. Pick orientations so that each represents the same element of $H_1(W)$. $d: T^2 \times 1 \rightarrow T^2 \times -1$ is orientation preserving and must take $S_{11}$ to $S_{21}$ or $S_{22}$. There are four subcases:

1) $d(S_{11}) = S_{21}$ and $d_*(S_{11}) = [S_{21}]$. Then $\text{Fix}$ is two tori.

2) $d(S_{11}) = S_{21}$ and $d_*(S_{11}) = -[S_{21}]$. Then $\text{Fix}$ is two Klein bottles. These meet $T$ so this case does not occur.

3) $d(S_{11}) = S_{22}$ and $d_*(S_{11}) = [S_{22}]$. Then $\text{Fix}$ is a
nonseparating torus since $d$ must interchange the components of $W$-Fix.

4) $d(S_{1,1}) = S_{2,2}$ and $d_{*}(S_{1,1}) = -[S_{2,2}]$. Then Fix is one separating torus.

If $i'$ is also given then it will fall into the same one of these subcases. So a conjugation can be constructed between $\lambda$ and $\lambda'$ as in Case 2.1.

Case 3) $iT = T$, $T$ separates. Then $M = W_1 \sqcup W_2$ with $T = W_1 \cap W_2$ where $W_i$ are orientable twisted 1-bundles over a Klein bottle. We have $iW_i = W_i$ and $i|(W_i)$ is orientation reversing. So $\lambda_1 = i|W_1$ is conjugate to $k_K$, $k_{2M}$, $k_A$, $k_{A2P}$, $k_{2P}$ or $k_N$. Since $i|\partial W_1 = i|\partial W_2$ it follows:

Case 3.1) Both $\lambda_1$ and $\lambda_2$ are conjugate to $k_K$, $k_{2P}$ or $k_N$.

Case 3.2) Both $\lambda_1$ and $\lambda_2$ are conjugate to $k_{2M}$, $k_A$ or $k_{A2P}$.

In case 3.1) by symmetry assume $\lambda_1$ is conjugate to $k_K$. Fix is two Klein bottles, a Klein bottle and two points, or just one Klein bottle. If $i'$ is also given construct a conjugation by taking any conjugation between $\lambda_2$ and $\lambda_2'$ and extending to $W_1$ using the conjugation extendable property of $k_K$.

In case 3.2) the fixed set always intersects $T$. By construction we avoided Klein bottles meeting $T$. Therefore $\lambda_1$ is not conjugate to $k_{2M}$.

Next, it is possible that $\lambda_1$ is conjugate to $k_A$ and $\lambda_2$ is conjugate to $k_{A2P}$. Let $\text{Fix} \cap T = S_1 \sqcup S_2$. Torus $T$ induces
an orientation on $S_2$ once an orientation on $S_1$ is fixed. In $k_A$ the annular fixed set induces the same orientation on $S_2$ as $T$ does. In $k_{A2P}$ the annular fixed set induces an orientation on $S_2$ opposite to the one $T$ induces. Therefore $T$ would be a Klein bottle.

Hence we have $\lambda_1$ and $\lambda_2$ are conjugate to $k_A$ and the fixed set is a torus or $\lambda_1$ and $\lambda_2$ are conjugate to $k_{A2P}$ and the fixed set is a torus and four points. If $\iota'$ is given it is easy to construct a conjugation between $\iota'$ and $\iota$ since $k_A$ and $k_{A2P}$ have the conjugation extendable property.

QED
Bibliography


