INVOLUTIONS WITH 1- OR 2-DIMENSIONAL FIXED POINT SETS ON ORIENTABLE TORUS BUNDLES OVER A 1-SPHERE AND ON UNIONS OF ORIENTABLE TWISTED I-BUNDLES OVER A KLEIN BOTTLE

by

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Abstract

We obtain a complete equivariant torus theorem for involutions on 3-manifolds M. M is not required to be orientable nor is $H_1(M)$ restricted to be infinite. The proof proceeds by a surgery argument.

Similar theorems are given for annuli and for discs. These are used to classify involutions on various spaces such as orientable twisted I-bundles over a Klein bottle.

Next we restrict our attention to orientable torus bundles over S^1 or unions of orientable twisted I-bundles over a Klein bottle. The equivariant torus theorem is applied to the problem of determining which of these spaces have involutions with 1-dimensional fixed point sets. It is shown that the fixed point set must be one, two, three, or four 1-spheres. Matrix conditions that determine which of these spaces have involutions with a given number of 1-spheres as the fixed point sets are obtained.

The involutions with 2-dimensional fixed point sets on orientable torus bundles over S^1 and on unions of orientable twisted I-bundles over a Klein bottle are classified. Only the orientable flat 3-space forms M_1, M_2 and M_6 have involutions with 2-dimensional fixed sets. Up to conjugacy, M_1 has two involutions, M_2 has four involutions, and M_6 has a unique involution.

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INTRODUCTION

We investigate the problem of classifying involutions with 1-dimensional or 2-dimensional fixed sets on orientable torus bundles over S^1 or on unions of orientable twisted I-bundles over a Klein bottle. Cutting these spaces on incompressible tori T gives trivial I-bundles over tori or orientable twisted I-bundles over a Klein bottle.

In Chapter III we prove a complete equivariant torus theorem for involutions. This theorem allows the cutting to be done in a manner that respects both the involution ι and the fixed set **Fix**, i.e., such that either $\iota T \cap T = \emptyset$ or $\iota T = T$ and T and **Fix** are transversal. The problem then reduces to one of classifying involutions on trivial I-bundles over tori or orientable twisted I-bundles over a Klein bottle. Such theorems have been proved in [8] and [14] but under additional hypotheses, such as, the first homology being infinite and the manifold being orientable. In [11] an equivariant torus theorem was proved under the assumption that the fixed set is a number of isolated points. Our theorem extends these results. In the nonorientable case we may have to allow the cutting torus to be replaced by a Klein bottle. Even in the orientable case two types of exceptional cases are possible.

To prove the equivariant torus theorem cut and paste techniques are used. An equivariant transversality theorem is also required. In the two dimensional case when M is nonorientable transversality can not be guaranteed. Certain interesting exceptional points arise; these will be called saddle points. Saddle points must be treated separately in the surgery arguments.

Analogous to the torus theorem are the annulus and disc theorems. These theorems are used in Chapter IV to classify the involutions on the trivial I-bundle over a torus and on the orientable I-bundle over a Klein bottle.

Let **M** denote an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle. In [6] Kim and Sanderson have classified the orientation reversing involutions on orientable torus bundles over S^1. Our techniques allow us to classify the involutions with 2-dimensional fixed point set on all **M**. A subclass of these spaces are the orientable flat 3-space forms M_1, $M_2, \cdots M_6$, see Wolf [15]. M_6 is not a torus bundle and $H_1(M_6)$ is finite. We show in §10 that M_1, M_2 and M_6 are the only **M** having involutions with 2-dimensional fixed point sets. These involutions are determined up to conjugacy.

The case of 1-dimensional involutions on these spaces is far less restrictive. Each of the orientable space forms has involutions with 1-dimensional fixed point sets but these are not the only **M** with such involutions. We determine in §9 which spaces **M** have involutions with 1-dimensional fixed point sets. We do not deal with the problem of uniqueness in this thesis.

Several topics for further research present themselves. For example, classify the involutions on the nonorientable torus bundles over S^1. Can an equivariant theorem be proved for surfaces of higher genus? One could also investigate how the results would generalize from involutions to n-cyclic actions and finite group actions.

I. EQUIVARIANT TRANSVERSALITY AND DISC THEOREMS

§1. Preliminaries

Use $\sqcap$, $\sqcup$ and $\sqsubset$ to denote set intersection, union and subset. $\sqcup$ does not denote disjoint union. Use upper indices to indicate dimension.

Throughout we use the **piecewise linear category.** This is to avoid wild fixed sets which can arise in the topological category, see [1].

A piecewise linear homeomorphism will be called an **isomorphism.**

Definition 1.1

Let M be a manifold with boundary ∂M and F a submanifold of M of lower dimension. F is **proper** if $F \sqcap \partial M = \partial F$. In particular a point is proper in M only if it is in the interior of M. We will assume that all submanifolds are proper.

F will usually denote a surface, a compact connected manifold. A surface F in a 3-manifold M is **incompressible** if F is not a 2-sphere or 2-cell and if for each 2-cell B in M with $B \sqcap F = \partial B$ there is a 2-cell $D \sqsubset F$ with $\partial D = \partial B$. A manifold M is **irreducible** if each 2-sphere in M bounds a 3-cell in M.

Let M be a connected compact 3-manifold. An **involution** ι is an isomorphism with $\iota \neq id$ and $\iota^2 = id$.

Let **Fix** denote the **fixed set** **Fix**$=\mathrm{fix}(\iota)=\{x:\iota(x)=x\}$. Let ι be an involution on a manifold M and ι' an involution on a manifold M'. ι and ι' are **conjugate** if there is an isomorphism $h:M \longrightarrow M'$ with $\iota'=h\circ\iota\circ h^{-1}$. Call h a **conjugation** between ι and ι'.

ι is **conjugation extendable** if given any conjugate ι' of ι and an isomorphism $h_0:\partial M \longrightarrow \partial M'$ with $\iota'|_{\partial M'}=h_0\circ\iota|_{\partial M}\circ h_0^{-1}$ then there is a conjugation $h:M \longrightarrow M'$ extending the isomorphism h_0.

Note that if ι is conjugation extendable then so is any conjugate ι'. Further, to show conjugation extendability it suffices to check the case $\iota'=\iota$. ι is **conjugation extendable with respect to a class** H of isomorphisms $\partial M \longrightarrow \partial M'$, depending on the choice of conjugate ι' of ι, if at least for any $h_0 \in H$ with $\iota'|=h_0\circ\iota|\circ(h_0^{-1})$ there is a conjugation $h:M \longrightarrow M'$ extending the isomorphism h_0.

The following construction will be used often. See Figure 1. Let $M=B\sqcup\iota B$ with $B\sqcap\iota B=\partial B\sqcap\partial\iota B$ and similarly for M' and B'. Let $h_0:B \longrightarrow B'$ be an isomorphism such that $h_0|:B\sqcap\iota B \longrightarrow B'\sqcap\iota'B'$ is a conjugation between $\iota|B\sqcap\iota B$ and $\iota'|B'\sqcap\iota'B'$. h_0 is **extended by equivariance** to $h:M \longrightarrow M'$ if we define $h|_B=h_0$ and $h|_{\iota B}=\iota'\circ h_0\circ(\iota^{-1})$. Then h is a conjugation between $\iota|M$ and $\iota'|M'$.

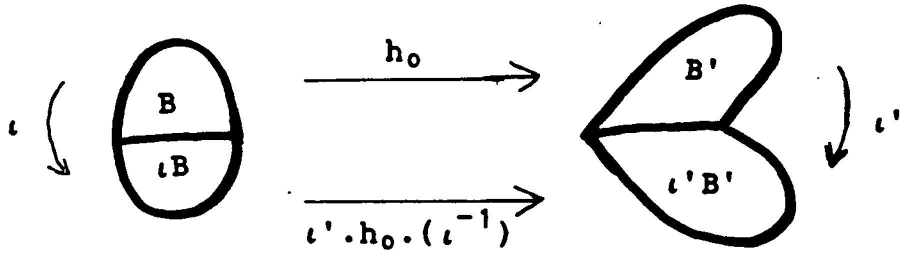


Figure 1.

Lemma 1.2

Given a simplicial subdivision K of M and an involution ι of M there is a subdivision L of K with ι simplicial with respect to L.

Proposition 1.3

Let ι be an involution on a manifold M. Let L be a subdivision of M with ι:L——>L simplicial and let L' be the first barycentric subdivision of L.

Then **Fix**=fix(ι) is a subcomplex of L'. **Fix** is the union of disjoint 0-, 1- and 2-dimensional proper submanifolds. Write $\mathbf{Fix}^0$, $\mathbf{Fix}^1$ and $\mathbf{Fix}^2$ respectively for the unions of the 0-, 1- and 2-dimensional components of **Fix**.

If $v \in \mathbf{Fix}^0 \sqcup \mathbf{Fix}^2$ then ι is locally orientation reversing at v. If $v \in \mathbf{Fix}^1$ then ι is locally orientation preserving at v. In particular if M is orientable then ι is orientation reversing if $\mathbf{Fix}^0 \sqcup \mathbf{Fix}^2 \neq \emptyset$ and ι is orientation preserving if $\mathbf{Fix}^1 \neq \emptyset$.

Proof: Use the following:

1) Let Δ be a standard m-simplex (with standard subdivision) invariant under ι. Then $\mathbf{Fix} \sqcap \Delta$ is a subcomplex of the first barycentric subdivision of Δ.

2) If **Fix** contains a 3-simplex then ι=id.

If $v \in \mathbf{Fix}$ is a vertex of int(L) consider the link *Lk* of v.

3) If $Lk \sqcap \mathbf{Fix}$ contains a 1-cell then $Lk \sqcap \mathbf{Fix}$ is one 1-sphere. So $v \in \mathbf{Fix}^2$.

4) If $Lk \sqcap \mathbf{Fix}$ consists of m≥0 vertices then

$$\chi(Lk/\iota) - m = \frac{1}{2}\,(\chi(Lk) - m)$$

Since Lk/ι is a surface and *Lk* is a 2-sphere it follows m=2 and hence $v \in \mathbf{Fix}^1$, or it follows m=0 and hence $v \in \mathbf{Fix}^0$.

QED

<u>Corollary 1.4</u>

Let ι be an involution on M with fixed set **Fix**. Then $M/\iota = M/(m \sim \iota(m)$ for all $m \in M)$ is a manifold with possible singularities. **Fix** $\cong$ $\mathbf{Fix}/\iota$ a disjoint union of submanifolds $\mathbf{Fix}^0$, $\mathbf{Fix}^1$ and $\mathbf{Fix}^2$ with $\mathbf{Fix}^0$ and $\mathbf{Fix}^1$ proper in M/ι and with $\mathbf{Fix}^2$ a submanifold of $\partial(M/\iota) = (\partial M)/\iota \sqcup \mathbf{Fix}^2$.

Proof: Consider the link *Lk* of vertices of **Fix**.

QED

Remark 1.5

The fixed point free involutions on a manifold M correspond to 2-fold coverings by M. If ι is an involution then $M/\iota = M/(m \sim \iota(m)$ for all $m \in M)$ is 2-fold covered by M. Conversely, if $p: M \longrightarrow X$ is a 2-fold cover then define an involution ι by requiring ι to interchange the two points of $p^{-1}(x)$ for every $x \in X$. ι is the nontrivial deck transformation induced by p.

Involutions on M with fixed set **Fix** correspond to 2-fold coverings by M branched on **Fix**. $\iota|$ M-**Fix** is fixed point free and $p|$ M-**Fix** is unbranched.

§2. Equivariant Transversality

In order to be able to perform surgeries on a surface F_0 in a 3-manifold M we would like to perform an ambient isotopy on F_0 such that the isotopic surface F has the property that F, ιF and **Fix** are pairwise transversal. This can be done if the manifold is orientable. If $\mathbf{Fix}^2 \neq \emptyset$ and M is nonorientable, however pairwise transversality is not possible in general. This necessitates using a somewhat weaker form of transversality.

Lemma 2.1

Let F be a proper surface in a 3-manifold with F, ιF and **Fix** pairwise transversal. Then the components of $F \sqcap \iota F$ are 1-spheres and proper 1-cells. If C is a component of $F \sqcap \iota F$ with $C \sqcap \mathbf{Fix}^2 \neq \emptyset$ then $C \sqsubset \mathbf{Fix}^2$.

Proof: The first statement follows by transversality of F and ιF. The second statement follows on considering the star of a point in $C \sqcap \mathbf{Fix}^2$.

QED

For the 3-cell $B^3 = \{(x,y,z): |x| \leq 1, |y| \leq 1, |z| \leq 1\}$ in $\mathbf{R}^3$ let $i: B^3 \longrightarrow B^3$ be the map $i(x,y,z) = (-x,y,z)$. Then **Fix**(i) is the intersection of B^3 with the y z plane. Let S be the 1-sphere obtained as the join of $\{(1,1,1),(-1,-1,1)\}$ with $\{(-1,1,-1),(1,-1,-1)\}$ and let D be the cone from $(0,0,0)$ on S. D is a saddle shaped region. See Figure 2. (We could alternately take D defined by $\{z = xy/\sqrt{x \cdot x + y \cdot y}\} \sqcup \{(0,0,0)\}$.)

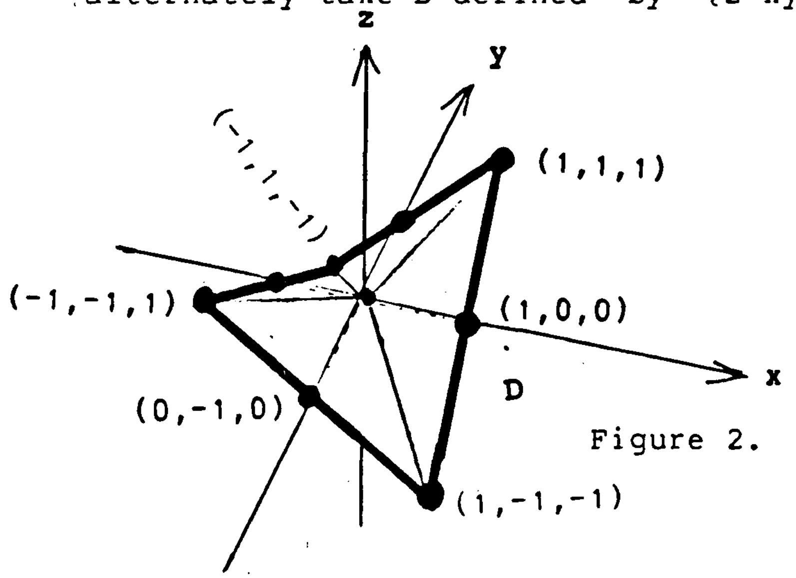


Figure 2.

Notice that $D \sqcap iD$ is the part of the x and y axis in B^3 while $D \sqcap$ **Fix**(i) is part of the y axis. D and **Fix**(i) are transversal and iD and **Fix**(i) are transversal, but D and iD are not transversal at (0,0,0). There is a subdivision making these spaces simplicial with all the vertices on $\partial B^3 \sqcup (0,0,0)$.

Definition 2.2

Let F be a proper surface in a 3-manifold M and ι an involution on M with fixed set **Fix**. Call a point v **a saddle point** if $v \epsilon F \sqcap \mathbf{Fix}^2$ and if $(F, \iota F, \mathbf{Fix}) \sqcap \mathrm{star}(v)$ is isomorphic to (D,iD,**Fix**(i)).

Remark 2.3

Saddle points exist since i is an involution with fixed set **Fix**(i). Although ∂D, ∂iD, ∂**Fix**(i) are pairwise transversal there is no 2-cell E with $\partial E = \partial D$ and E, iE, **Fix**(i) pairwise transversal. Otherwise, since $\partial E \sqcap \partial iE$ - **Fix**(i) = $(\pm 1,0,0)$ there is a 1-cell I of $E \sqcap iE$ with $(1,0,0) \epsilon \partial I$ and this 1-cell must meet **Fix**, contradicting the previous lemma.

Let d denote the identification $(x,1,z) \sim (x,-1,-z)$ for all x and z. Then D/d is an annulus in a solid Klein bottle B^3/d and no isotopy of D/d moves it to an annulus with F, iF, and **Fix**(i)/d pairwise transversal.

<u>Definition 2.4</u>

Let F be a proper surface in a 3-manifold and ι an involution on M with fixed set **Fix**. Then F, ιF, and **Fix** are **almost pairwise transversal** if:

1) F, ιF and **Fix** are pairwise transversal except at a finite number of saddle points,

and 2) The only components of $F \sqcap \mathbf{Fix}$ containing saddle points are 1-spheres and each such 1-sphere contains at most one saddle point.

Let E be the closure of $(F \sqcap \iota F) - \mathbf{Fix}^2$. E consists of disjoint 1-spheres and proper 1-cells: in a neighborhood of a saddle point, $F \sqcap \iota F - \mathbf{Fix}^2$ corresponds to $[-1,0)\times0\times0 \sqcup (0,1]\times0\times0$ in the B^3 model for saddle points.

Let E be a component of E that contains a saddle point v. Then E has a fixed point and is invariant under ι. Therefore, either E is a 1-cell with no fixed points other than v or E is a 1-sphere with exactly two fixed points v and w. By transversality w is in $\mathbf{Fix}^1$ or $\mathbf{Fix}^2$. In the latter case w is a saddle point. We obtain the following proposition:

Proposition 2.5

Let F, ιF and **Fix** be almost pairwise transversal. Then the components of $F \cap \iota F$ are of one of the following forms:

1) Components with no saddle points (standard components):

a) proper 1-cell I with $I \cap \mathbf{Fix} = \emptyset$ or $I \subset \mathbf{Fix}^2$

b) proper 1-cell I with $I \cap \mathbf{Fix} = I \cap \mathbf{Fix}^1 = v$, v a point

c) 1-sphere S with $S \cap \mathbf{Fix} = \emptyset$ or $S \subset \mathbf{Fix}^2$

d) 1-sphere S with $S \cap \mathbf{Fix} = S \cap \mathbf{Fix}^1 = v_1 \cup v_2$ where v_1 and v_2 are points.

2) Components with saddle points:

Type I component: $S_1 \cup I$ with $S_1 \cap I = \mathbf{Fix} \cap I = w$, $S_1 \subset \mathbf{Fix}^2$ and w is the only saddle point on $S_1 \cup I$.

Type II component: $S_1 \cup S$ with $S_1 \cap S = w$, $S_1 \subset \mathbf{Fix}^2$, $S \cap \mathbf{Fix} = v \cup w$, $v \in \mathbf{Fix}$ and w is the only saddle point on $S_1 \cup S$.

Type III component: $S_1 \cup S_2 \cup S$ with $S_1 \cap S_2 = \emptyset$, $S_i \cap S = w_i$, $S_i \subset \mathbf{Fix}^2$, $S \cap \mathbf{Fix} = w_1 \cup w_2$ and w_1 and w_2 are the only saddle points on $S_1 \cup S_2 \cup S$.

Here S, S_1 and S_2 are 1-spheres, I are 1-cells and w_i are points.

<u>Note</u> a regular neighborhood of any of S, S_1, or S_2 is a solid Klein bottle in case 2). Thus case 2) does not occur if the manifold is orientable.

Proof: If the regular neighborhood N of S is a solid torus then $\mathbf{Fix} \cap N$, $F \cap N$ and $\iota F \cap N$ are all annuli or all Möbius bands. Consider the components of $F \sqcap \partial N$ and $\iota F \sqcap \partial N$ in ∂N - **Fix** for a contradiction. For example, if they are all annuli then let A be a component of ∂N-**Fix**. $A \sqcap$ **Fix** and $A \sqcap \iota$**Fix** are two 1-spheres that intersect transversally at one point. This is not possible in an annulus A. Compare with the proof of case 3 and 4 in step 1 of the next theorem (transversality theorem).

QED

Corollary 2.6

If F, ιF and **Fix** are almost pairwise transversal, then they are pairwise transversal if one of the following holds:

a) M is orientable

b) F is a 2-cell

c) F is a annulus with $\partial F \sqcap \iota \partial F = \emptyset$.

Proof: In case a) regular neighborhoods of 1-spheres are solid tori.

In cases b) and c) Type II or III components are excluded since the 1-sphere S is nonseparating. In case c) type I components are excluded a priori, while in case b) 1-sphere S_1 separates so a proper 1-cell C cannot intersect S_1 transversally at one point.

QED

A proper 1-cell I **bounds** a disc D in a surface if $I=\overline{\partial D-\partial F}$.

Corollary 2.7

Let F, ιF, and **Fix** be almost pairwise transversal and C a proper 1-cell or 1-sphere component of $F \sqcap \iota F$, that is, let C be a standard component. Then any disc in F or ιF bounded by C contains only standard components.

Proof: As in case b) in previous corollary.

QED

Equivariant Transversality Theorem 2.8

Let ι be an involution on a 3-manifold M with **Fix**=fix(ι) and let F_0 be a proper surface in M. Then there is an ambient ϵ-isotopy on M taking F_0 to a proper surface F such that F, ιF and **Fix** are almost pairwise transversal. In ∂M, if ∂F, $\iota\partial F$ and **Fix** are pairwise transversal then the isotopy may be taken to be the identity on ∂M-N where N is a given neighborhood of $\partial \mathbf{Fix}^2 \sqcap \partial F$.

Proof: Let $F=F_0$ be a proper surface. By Proposition 1.3 and Lemma 1.2 subdivide M so that ι is simplicial with respect to the subdivision, **Fix** is a subcomplex of the subdivision and **Fix** is a disjoint union of 0-, 1- and 2-dimensional components $\mathbf{Fix}^0$, $\mathbf{Fix}^1$ and $\mathbf{Fix}^2$. All isotopies performed in the construction will be done in the star neighborhoods of certain simplexes. By taking a sufficiently fine subdivision ϵ-isotopies are obtained.

Step 1) Adjust F near $\mathbf{Fix}^2$.

By isotopies similar to those in the third step below we can assume F and **Fix** are transversal, the isotopy not moving ∂F unless ∂F and $\partial\mathbf{Fix}$ are nontransversal. In particular $F \sqcap \mathbf{Fix}^0 = \emptyset$. Then $F \sqcap \mathbf{Fix}^2$ consists of disjoint 1-spheres and 1-cell components proper in M.

Let S be a **1-sphere component** of $F \sqcap \mathbf{Fix}^2$. Let N' be a regular neighborhood of S with $N' \sqcap F$ and $N' \sqcap \mathbf{Fix}$ transversal and each an annulus or Möbius band. S has a regular neighborhood N contained in int(N') invariant under ι with no vertices on int(N) - S such that $N \sqcap \mathbf{Fix}$ is a regular neighborhood of S and $\mathbf{Fix} \sqcap \partial N$ has a regular neighborhood Q in ∂N which is invariant under ι and has no vertices except on $\mathbf{Fix} \sqcup \partial Q$.

Case 1) $F \sqcap N$ and $\mathbf{Fix} \sqcap N$ are annuli. Then N is a solid torus, ∂Q has four components and N-**Fix** consists of two components N_1 and N_2 which are interchanged by ι. Let J_1 and J_2 be components of ∂Q with $J_i = N_i$ and $\iota J_1 \neq J_2$. Let A_i be the annulus with $\partial A_i = J_i \sqcup S$ having no vertices except on ∂A_i. F is isotopic to a surface F' by an ambient isotopy which is the identity on M-N' and such that $F' \sqcap N' \sqcap \mathbf{Fix} \sqsubset N$ and $F' \sqcap N$

$= A_1 \sqcup A_2$. Since $\iota J_1 \neq J_2$ it follows $F' \sqcap N \sqcap \iota(F' \sqcap N) = S$ and $F' \sqcap N$, $\iota(F' \sqcap N)$, $\mathbf{Fix} \sqcap N$ are pairwise transversal.

Case 2) $F \sqcap N$ and $\mathbf{Fix} \sqcap N$ are Möbius bands. Then N is a solid torus, and ∂Q has two components that are interchanged by ι. If J is one of these, then J and S determine a Möbius band A with $\partial A = J$. Proceed as in case 1.

If M is orientable Case 3 and 4 do not arise. Only in these cases do saddle points arise.

Case 3) $F \sqcap N$ is an annulus and $\mathbf{Fix} \sqcap N$ is a Möbius band. Then N is a solid Klein bottle. Let A be one of the two (open) annuli components of $\partial N - \mathbf{Fix}$. There are two 1-spheres J_1 and J_2 which represent generators of $H_1(A) = \mathbf{Z}$ with J_1 and J_2 intersecting transversally and at only one point x. J_i and S bound an annulus A_i with $A_1 \sqcap A_2 = S \sqcup I$ where I is a 1-cell with $\partial I = x \sqcup y$ where $y \in S$. Proceed as in Case 1 using $F' \sqcap N = A_1 \sqcup \iota A_2$. Then y is a saddle point and $F' \sqcap N$, $\iota(F' \sqcap N)$ and $\mathbf{Fix} \sqcap N$ intersect pairwise transversally elsewhere in N.

Case 4) $F \sqcap N$ is a Möbius band and $\mathbf{Fix} \sqcap N$ is an annulus. This case is similar to case 3. Here $A = \partial N - Q$ is an invariant annulus under ι. Find a curve J that bounds a Möbius band by lifting (from annulus A/ι) a curve J' which represents twice a generator and which is embedded in A/ι except for one transversal self intersection.

When S is a **1-cell component** of $F \sqcap \mathbf{Fix}^2$, use an isotopy similar to the one of case 1 above. This isotopy may change ∂F in $N \sqcap \partial M$.

Step 2) Adjust F near $\mathbf{Fix}^1$.
By step 1, $F \sqcap \mathbf{Fix}^1$ consists of a number of vertices in int(M). If $v \in F \sqcap \mathbf{Fix}^1$ let N' be a regular neighborhood of v and let N be the star neighborhood of v. Take the subdivision so that N is in the interior of N', $F \sqcap N$ is a proper 2-cell in N and $\mathbf{Fix} \sqcap N$ is a proper 1-cell. Since F is transversal, $F \sqcap \partial N$ is a generator of $H_1(N - \mathbf{Fix})$. Let J' be a curve in the annulus $(\partial N - \mathbf{Fix})/\iota$ representing twice a generator of this annulus. Take J' embedded except for one transversal self intersection. J' lifts to two 1-spheres J and ιJ, which on coning to v give 2-cells D and ιD. D, ιD and **Fix** are pairwise transversal in int N. Proceed as in case 1 of step 1.

We obtain a surface F and a neighborhood N of **Fix** such that F has the required transversality properties in N. The following construction adjusts F only on star neighborhoods of simplexes of $\overline{F-N}$ where F and ιF are not already pairwise transversal. By subdividing sufficiently we may assume without loss that $\mathbf{Fix}=\emptyset$. For convenience assume also $\partial F=\emptyset$.

Let K be a subdivision of M with ι simplicial and F a subcomplex of K. Let Δ be an m-simplex of F in K with m = 0, 1 or 2. Define $St(\Delta)$, the reduced star of Δ in K, to be all

3-simplexes σ of K with $\Delta \subset \sigma$ together with their faces. Let $St_F(\Delta)$, the reduced star of Δ in F, be all 2-simplexes σ of K with $\Delta \subset \sigma \subset F$ together with their faces. Let $p:M \longrightarrow M/\iota$ be the projection.

Step 3) There is a subdivision of M and a proper surface F' ϵ-isotopic to F such that for every simplex Δ of F' either $p^{-1}p(\Delta) \cap F' = \Delta$ or Δ is a 0- or 1-simplex with $int(St_{F'}(\Delta))$ and $int(St_{\iota F'}(\Delta))$ transversal.

Call a simplex exceptional if it fails to satisfy these conditions and is of the highest possible dimension m = 0, 1 or 2. Induct on the number of such simplexes. If there are no exceptional simplexes the theorem is established.

Add all the vertices (and their translates under ι) of form $(m+2)/(m+3)\ b + 1/(m+3)\ v$ where b is the barycenter of Δ and v is a vertex of $St(\Delta)-\Delta$. This determines a refinement K' of K with the same number of exceptional simplexes; no m-simplexes are subdivided for m=1,2 , while for m=0 transversality already holds away from vertices of K. Consider the reduced stars in K'. $\partial St'_F(\Delta)$ is a 1-sphere that decomposes $\partial St'(\Delta)$ into two components D_+ and D_-. There is an ambient isotopy taking F to $F_1=(F - St'_F(\Delta)) \sqcup D_+$ which is the identity except on $St'_F(\Delta)$. F_1 has fewer exceptional simplexes. When $m \neq 2$ this follows since $D_+ \sqcup D_-$ intersects the interior of any 2-simplex of $St(\Delta)$ transversally.

QED

Regular neighborhoods of the standard components of $F \sqcap \iota F$ can be taken in a special form.

Definition 2.9

Let F, ιF and **Fix** be almost pairwise transversal and S a 1-sphere component of $F \sqcap \iota F$. Suppose, in addition, that the regular neighborhood of S in F and ιF is an annulus. Then there exists a regular neighborhood $V \sqsubset \mathrm{int}(M)$ of S with the following properties:

1) $V \sqcap F$ and $V \sqcap \iota F$ are annuli. Since these intersect transversally, V is a solid torus.

2) **Fix** and ∂V intersect transversally, $\mathbf{Fix} \sqcap F \sqcap V \sqsubset S$ and the closure of each component of $(\mathbf{Fix} \sqcap V) - S$ meets S and ∂V. In particular $\mathbf{Fix}^0 \sqcap V = \emptyset$.

3) $\mathbf{Fix} \sqcap V$ is an annulus, two proper 1-cells or empty.

4) If $\iota S = S$ then $\iota V = V$

5) If $\iota S \neq S$ then $\iota V \sqcap V = \emptyset$ and the above properties hold simultaneously for ιV.

Property 3) can be arranged since if $\mathbf{Fix} \sqcap S \neq \emptyset$ then $\iota S = S$. ι is an involution on a 1-sphere so either $\iota = \mathrm{id}$ or ι has exactly two fixed points.

Call V a **standard neighborhood** of S. The four 1-spheres $(F \sqcup \iota F) \sqcap \partial V$ decompose ∂V into four (closed) annuli α_1, α_2, β_1 and β_2 with $\alpha_1 \sqcap \alpha_2 = \emptyset$ and $\beta_1 \sqcap \beta_2 = \emptyset$. Call these annuli the

standard annuli corresponding to the standard neighborhood of V. Suppose $\iota S=S$. Relabelling, if necessary, we may assume $\iota(\alpha_1 \cap \beta_1)=(\alpha_1 \cap \beta_2)$. It follows that $\iota\alpha_1=\alpha_1$. Then $\iota\beta_1=\beta_2$ and $\iota\alpha_2=\alpha_2$. When $\mathbf{Fix} \cap V\neq\emptyset$ we obtain $\mathbf{Fix} \cap \alpha_1\neq\emptyset$, $\mathbf{Fix} \cap \alpha_2\neq\emptyset$, $\mathbf{Fix} \cap \beta_1=\emptyset=\mathbf{Fix} \cap \beta_2$, and each component of **Fix** meets both α_1 and α_2.

Definition 2.10

Let S be a 1-cell component of $\mathbf{Fix} \cap \iota\mathbf{Fix}$ where F, ιF, **Fix** are pairwise transversal (near S). Then there exists a regular neighborhood V of S with $V \cap \partial M$ a regular neighborhood of ∂S, called a **standard neighborhood** of S with the following properties:

1) $V \cap F$ and $V \cap \iota F$ are 2-cells with $\partial M \cap V \cap F$ and $\partial M \cap V \cap \iota F$ each two 1-cells. Necessarily V is a 3-cell.

2), 4) and 5) as for 1-sphere standard neighborhoods.

3) $\mathbf{Fix} \cap V$ is a disc, one proper 1-cell or empty.

The four 1-cells $(F \cup \iota F) \cap \overline{\partial V-\partial M}$ subdivide $\overline{\partial V-\partial M}$ into four discs α_1, α_2, β_1 and β_2 with $\alpha_1 \cap \alpha_2=\emptyset$, $\beta_1 \cap \beta_2=\emptyset$ and the properties as in the previous situation. Call these discs the **standard discs** corresponding to V.

Remark 2.11

In the following theorem certain 1-sphere components S of $F \sqcap \iota F$ have standard neighborhoods because S bounds discs in F and ιF. In the disc theorem and partial annulus theorem, F is orientable so again there are standard neighborhoods. In the torus theorem the construction will be made so as to keep S in this form always. In the annulus theorem the case of a nonorientable F, a Möbius band, with 1-sphere components is treated separately.

Theorem 2.12

Let M be a 3-manifold with involution ι and F_0 be an incompressible proper surface. Then there is an ambient isotopy of M which is an ϵ-isotopy on ∂M taking F_0 to a proper surface F such that F, ιF, and **Fix** are almost pairwise transversal and no 1-spheres in $F \sqcap \iota F$ bound 2-cells in F. If on ∂M, ∂F, $\iota \partial F$ and ∂**Fix** are pairwise transversal then the isotopy may be taken to be the identity on $\partial M - N$ where N is a given neighborhood of $\partial \mathbf{Fix}^2 \sqcap \partial F$.

Proof: By the preceding transversality theorem there is an F with all the above properties except possibly 1-spheres in $F \sqcap \iota F$ bound 2-cells in F. By Corollary 2.7 those 2-cells contain no saddle components. Let S be a 1-sphere of $F \sqcap \iota F$ innermost in ιF, that is, there is a 2-cell $D \sqsubset \iota$**Fix** with

$D \cap F = \partial D = S$. Since F is compressible, S bounds a 2-cell B in F. If $\iota S = S$ then we may assume $\iota B = D$.

Let V be a standard neighborhood of S. Such a neighborhood exists since S bounds a disc in F and ιF. Let a be the standard annulus meeting D but not B. Then $\iota a \cap a = \emptyset$. There is a bicollar $D \times [-1,1]$ of $D = D \times 0$ with

$$\partial D \times [-1,1] = D \times [-1,1] \cap F = S \times [-1,1]$$

and with $D \times 1 \cap a \neq \emptyset$. Since D is innermost it follows that for a sufficiently thin collar $(D \times 1) \cap \iota(D \times 1) = \emptyset$ and $F \cap \iota(D \times 1) = \emptyset$. Consider $F' = (F - (B \cup S \times [-1,1])) \cup D \times 1$. Then $F' \cap \iota F' \subset (F \cap \iota F) - S$ and F', $\iota F'$ and **Fix** are almost pairwise transversal. Since M is irreducible and $D \cup B$ is a 2-sphere, F' and F are ambient isotopic by an isotopy being the identity on ∂M. By induction, all 1-spheres bounding 2-cells can be removed.

QED

<u>Definition 2.13</u>

A 2-cell B in a 3-manifold is **essential** if it is proper and ∂B does not bound a 2-cell in ∂M. In an irreducible 3-manifold a nonseparating proper 2-cell is essential.

The following theorem also appears in [3].

Disc Theorem 2.14

Let M be an irreducible 3-manifold with involution ι. Suppose M has an essential 2-cell B_0. Then there is an essential 2-cell $B \subset M$ such that B and **Fix** are transversal and either $B \cap \iota B = \emptyset$ or $\iota B = B$. In the former case $B \cap \mathbf{Fix} = \emptyset$ and in the latter case $B \cap \mathbf{Fix}$ is a proper 1-cell of B or one point in the interior of B. If $\partial B_0 \cap \iota \partial B_0 = \emptyset$ then one can take $\partial B = \partial B_0$ and B and B_0 are ambient isotopic by an isotopy that is the identity on ∂M.

Proof: By Theorem 2.12 and Corollary 2.6 there is an essential 2-cell B with B, ιB and **Fix** pairwise transversal, B and B_0 ambient isotopic and $B \cap \iota B$ is either empty or consists of proper 1-cells only. Assume $B \cap \iota B \neq \emptyset$ (in particular then $\partial B_0 \cap \iota B_0 \neq \emptyset$). By induction it suffices to show how to obtain a new 2-cell B_i with fewer 1-cells in $B_i \cap \iota B_i$.

Let D be an outermost disc of B: $D \subset B$ with $D \cap \iota B = \partial D \cap \iota B = I$ a proper 1-cell of B and $\partial D - I \subset \partial B$.
If $\iota I = I$ define $D' = \overline{\iota B - \iota D}$. If $\iota I \neq I$ define D' to be the closure of the component of $\iota B - \iota D$ that does not contain ιI. See Figure 3. Let V be a standard neighborhood of I and let α_1, α_2 and β be standard discs of V with $\alpha_1 \cap \alpha_2 = \emptyset$, $\alpha_1 \cap \beta \cap D \neq \emptyset$ and $\beta \cap D' \neq \emptyset$. Consider

$$B_1 = (D \cup \beta \cup D') - \mathrm{int}(V) \text{ and } B_2 = D \cup (\iota B - D').$$

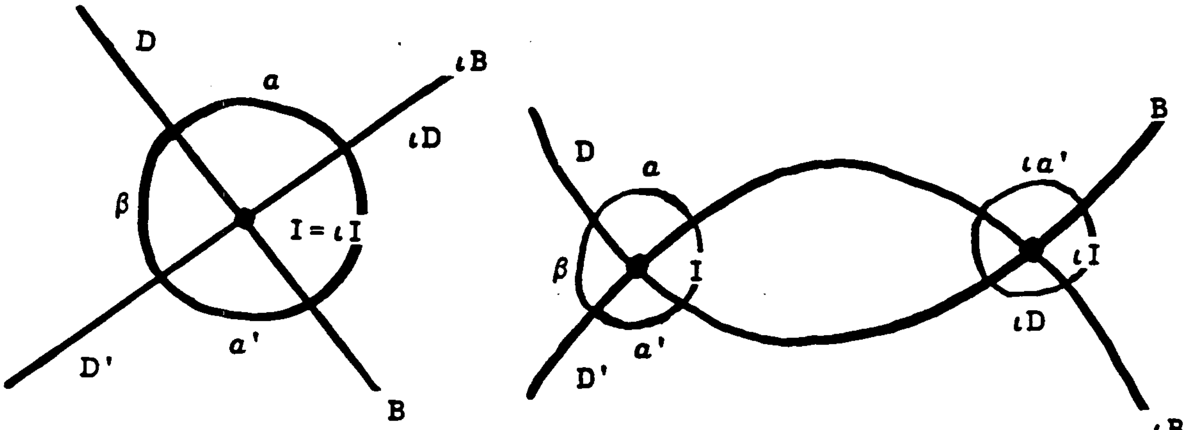


Figure 3.

Then $B_1 \cap \iota B_1 \subset (B \cap \iota B) - I$. If B_1 is essential we are done by induction or arrive at case $B_1 \cap \iota B_1 = \emptyset$. If B_1 is not essential then ∂B_1 bounds a 2-cell E of ∂M. Since M is irreducible the 2-sphere $B_1 \cup E$ bounds a 3-cell. This 3-cell does not meet I, otherwise ιB would not be essential. Using the 3-cell construct an ambient isotopy taking B_2 to ιB.

So we may assume B_2 is essential. If $I = \iota I$ we have $\iota B_2 = B_2$ and note that $\mathbf{Fix} \cap B_2 \subset \mathbf{Fix} \cap I$ which is necessarily a point of I or all of I. If $I \cap \iota I = \emptyset$ consider a sufficiently thin bicollar $D \times [-1,-1]$ of $D = D \times 0$ such that $D \times [-1,-1] \cap I$ is a bicollar of I in ιB and $D \times 1$ meets a_1. Then $B_2' = (D \times 1 \cup \iota B) - (I \times [-1,0) \cup D')$ is essential since it is isotopic to B_2 and $B_2' \cap \iota B_2' \subset B \cap \iota B - I$.

QED

II. INVOLUTIONS ON THE 3-CELL AND THE SOLID TORUS

§3. Some Involutions

The classification of involutions on a solid torus will be useful in the proof of theorems in the next chapter. The disc theorem will be used to reduce the problem of classifying involutions on a solid torus to one of classifying the involutions on a 3-cell.

Definition 3.1

Let C be the complex numbers. Let

$I=I^1=[-1,1]$ be the standard 1-cell

$S^1=\{z\in C : |z|=1\}$ be the standard 1-sphere

$D^2=\{z\in C : |z|\leq 1\}$ be the standard 2-cell

$T^2=S^1 x S^1$ be the standard torus

$D_+=\{z\in D^2 : z=x+y\cdot i,\ y\geq 0\}$, a 2-cell

$Re=\{z\in D^2 : z=\bar{z}\} \subset D^2$, a proper 1-cell

$Im=\{z\in D^2 : z=-\bar{z}\} \subset D^2$, a proper 1-cell

Define involutions on the above spaces as follows.

On S^1: $\kappa(z)=\bar{z}$ which is orientation reversing with fixed set two points ± 1. $a(z)=-z$ which is orientation preserving and fixed point free. Then $a\circ\kappa=\kappa\circ a=-\kappa$ is conjugate to κ by a rotation by 90°.

On D^2: $\hat{\kappa}(z)=\bar{z}$ orientation reversing with fixed set one 1-cell Re. $\hat{a}(z)=-z$ which is orientation preserving with fixed set one point 0. Then $-\hat{\kappa}$ is conjugate to $\hat{\kappa}$ and has fixed set Im.

On I:$\tau(t)=-t$ which is orientation reversing with fixed set one point 0.

Define the map $\rho:S^1xS^1 \longrightarrow S^1xS^1$ by $\rho(z,w)=(zw,w)$ and map $\hat{\rho}:D^2xS^1 \longrightarrow D^2xS^1$ similarly. Define involution $\omega:S^1xS^1 \longrightarrow S^1xS^1$ by $\omega(z,w)=(w,z)$, which has fixed set one 1-sphere $\{(z,z):z\}$.

Lemma 3.2

There are five involutions up to conjugacy on an annulus S^1xI. They are: 1) αxid which is orientation preserving and fixed point free, 2) $\alpha x\tau$ which is orientation reversing and fixed point free, 3) $\kappa x\tau$ which is orientation preserving with fixed set two points, 4) id$x\tau$ which is orientation reversing with fixed set a 1-sphere, and 5) κxid which is orientation reversing with fixed set two proper 1-cells.

Proof: When the dimension of the fixed set is one the fixed set separates. In the other case use the Euler characteristic argument given in part 4) of proof of Proposition 1.3.

QED

Definition 3.3

For the 3-cell D^2xI define the following involutions (see Figure 4):

$j_2=idx\tau$ having fixed set a proper 2-cell D^2x0.

$j_1=\hat{\kappa}x\tau$ having fixed set an unkotted 1-cell Rex0.

$j_0=\hat{\alpha}x\tau$ having fixed set one point 0x0.

j_2 and j_0 are orientation reversing while j_1 is orientation preserving. j_1 is conjugate to $j_1'=\hat{\alpha}xid$ which has fixed set 0xI.

Theorem 3.4

An involution on a 3-cell is conjugate to j_2, j_1 or j_0. All involutions on a 3-cell are conjugation extendable.

Proof: Let ι be an involution on 3-cell E. Apply Lemma 1.2 and Proposition 1.3.

Suppose $\mathbf{Fix}^2\neq\emptyset$. Since $\mathbf{Fix}^2$ is proper and $\pi_1E=1$, $\mathbf{Fix}^2$ separates. Let E_1 and E_2 be the components, $E=E_1\sqcup E_2$ with

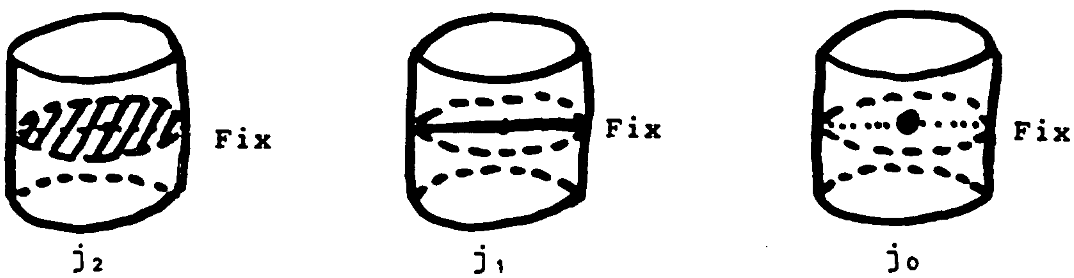


Figure 4.

$\mathbf{Fix}^2 = E_1 \sqcap E_2$. If $\mathbf{Fix}^2$ were compressible then let B be a compressing disc in E_1, say. Then ιB compresses $\mathbf{Fix}^2$ in E_2. Using a Mayer-Vietoris sequence we see $[\partial B]$ must be trivial in $H_1(E_1) \oplus H_1(E_2) \cong H_1(\mathbf{Fix}^2)$. Hence $\mathbf{Fix}^2$ is a proper 2-cell.

We show ι is conjugate to j_2. Let $D_1 = D^2 \mathbf{x} [0,1]$. Construct an isomorphism h_0 from the disc which is the closure of $\partial E_1 - \mathbf{Fix}$ to the closure of $\partial D_1 - D^2 \mathbf{x} 0$. In the conjugation extendable case we may assume this isomorphism is given. Extend h_0 over the fixed set $\mathbf{Fix}$ and then cone to a point to obtain an isomorphism $h: E_1 \longrightarrow D^2 \mathbf{x} [0,1]$. Extend this isomorphism by equivariance to get a conjugation.

Suppose $\mathbf{Fix}^2 = \emptyset$. The Euler characteristic argument referred to in part 4) of proof of Proposition 1.3 applied to $\iota | \partial E$ shows $\mathbf{Fix} \sqcap \partial E$ has 0 or 2 fixed points. In the former case ι is orientation reversing so $\mathbf{Fix}^1 \sqcup \mathbf{Fix}^2 = \emptyset$ and in the latter case $\mathbf{Fix}^1 \neq \emptyset$.

Suppose $\mathbf{Fix}^1 \sqcup \mathbf{Fix}^2 = \emptyset$. By a Lefshetz number argument ι has one fixed point only, call it v. $\partial E / \iota$ is a projective plane so there is a conjugation $h_0 : \partial E \longrightarrow \partial (D^2 \mathbf{x} I)$ between $\iota|$ and $j_0|$. In the conjugation extendable case h_0 is given. By subdividing we may assume $\mathrm{star}(v) \sqcap \partial E = \emptyset$ and $E - \mathrm{int}(\mathrm{star}(v)) \cong S^2 \mathbf{x} I$. Extend h to a conjugation $E - \mathrm{star}(v) \longrightarrow D^2 \mathbf{x} I - \mathrm{star}(0,0)$. This can be done by Theorem 1 in [9] since ι has no fixed point on ∂E. Finally cone to v.

Suppose $\mathbf{Fix}^1 \neq \emptyset$. By the above $\mathbf{Fix}^1 \cap \partial E$ is two points. Consider the double $E \cup E'$ of E. It is a 3-sphere with involution $\iota \cup \iota'$ induced by ι. By a result of Waldhausen [16], this involution has fixed set one unknotted 1-sphere. Let B be a 2-cell with ∂B the fixed set and such that B is in general position with respect to ∂E. $B \cap E$ is a punctured 3-cell and all but one component of $B \cap \partial E$ is a 1-sphere in the interior of B. By standard arguments these can be removed giving a 1-cell $B' \subset E$ with $\partial B' \subset \partial E \cup \mathbf{Fix}^1$. This shows $\mathbf{Fix}^1$ is one unknotted proper 1-cell.

We claim there is a (nonproper) disc B embedded in E with $\mathbf{Fix} \subset \partial B \subset \mathbf{Fix} \cup \partial E$ such that $B \cap \iota B = \mathbf{Fix}$. Moreover, if $C \subset \partial E$ is a 1-cell with $C \cap \iota C = \partial C = \partial \mathbf{Fix}$ then we may assume $\partial B \cap \partial E = C$. To establish this claim note that if N is a star neighborhood of $\mathbf{Fix}$ then the closure V of E-N is a solid torus with $\iota|$ fixed point free and orientation preserving. $V \cap N$ is an annulus and $\iota|(V \cap N)$ is also fixed point free. Using these facts and Disc Theorem 2.14 it is possible to construct a disc as required in the claim.

Construct a conjugation to j_1 as follows. Construct an isomorphism $B \longrightarrow D_+ x0$ and extend by equivariance to $B \cup \iota B \longrightarrow Dx0$. In the conjugation extendable case we set $C = h^{-1}((\partial D \cap D_+)x0)$. Then $B \cup \iota B$ separates E into two 3-cell components. We extend to an isomorphism over one of these components and then by equivariance to the other, giving a conjugation $E \longrightarrow D\times I$.

QED

Lemma 3.5

Let F be a 2 sided surface in a 3-manifold M and let ι be an involution on M with $\iota F=F$ and such that ι interchanges sides of F. Then F is ambient isotopic to a surface F' with $F' \cap \iota F'=\emptyset$.

Proof: Construct an ι invariant bicollar $F\times[-1,1]$ of $F=F\times 0$ by using a star neighborhood of F. Then consider $F'=F\times 1$.

QED

Remark 3.6

Suppose $\iota F=F$ for a 2-sided surface F. Let N be the 3-manifold obtained by cutting M along F. That is, replace $F\times[-1,1] \subset M$ by distinct copies $F_1\times[-1,0]=F\times[-1,0]$ and $F_2\times[0,1]=F\times[0,1]$. N has a subdivision induced from M. Let $d:F_1=F_1\times 0 \longrightarrow F_2=F_2\times 0$ be the canonical identification. Then $M \cong N/d$. Since ι is simplicial there is a canonical involution k on N with $\iota=k/d$. Note $k\circ d=d\circ k$ and $k(F_1)=F_1$ iff ι does not interchange the bicollar. Conversely if $k\circ d=d\circ k$ for an involution k then k induces an involution ι in M with $\iota F=F$.

§4. Involutions on the Solid Torus

Definition 4.1

Let V be the solid torus $V=D^2xS^1=\{(z,w):|z|\leq 1,|w|=1,\ z,w\epsilon C\}$. Recall Definition 3.1.

Define the following involutions on V (see Figure 5):

$j_A=\hat{\kappa}xid$ having fixed set the annulus $RexS^1$.

$j_M=\hat{\rho}\circ(\hat{\kappa}xid)$ having fixed set the Möbius band $\{(s\cdot e^{\pi it},e^{2\pi it}):0\leq s\leq 1,-1\leq t\leq 1\}$

$j_{2D}=idx\kappa$ having fixed set two 2-cells $D^2x\pm 1$.

$j_{DP}=\hat{\rho}\circ(idx\kappa)$ having fixed set a 2-cell and a point $D^2x1\sqcup 0x-1$.

$j_S=\hat{\alpha}xid$ having fixed set one 1-sphere $0xS^1$.

$j_{2C}=\hat{\kappa}x\kappa$ having fixed set two 1-cells $Rex\pm 1$.

$j_{2P}=\hat{\alpha}x\kappa$ having fixed set two points $0x\pm 1$.

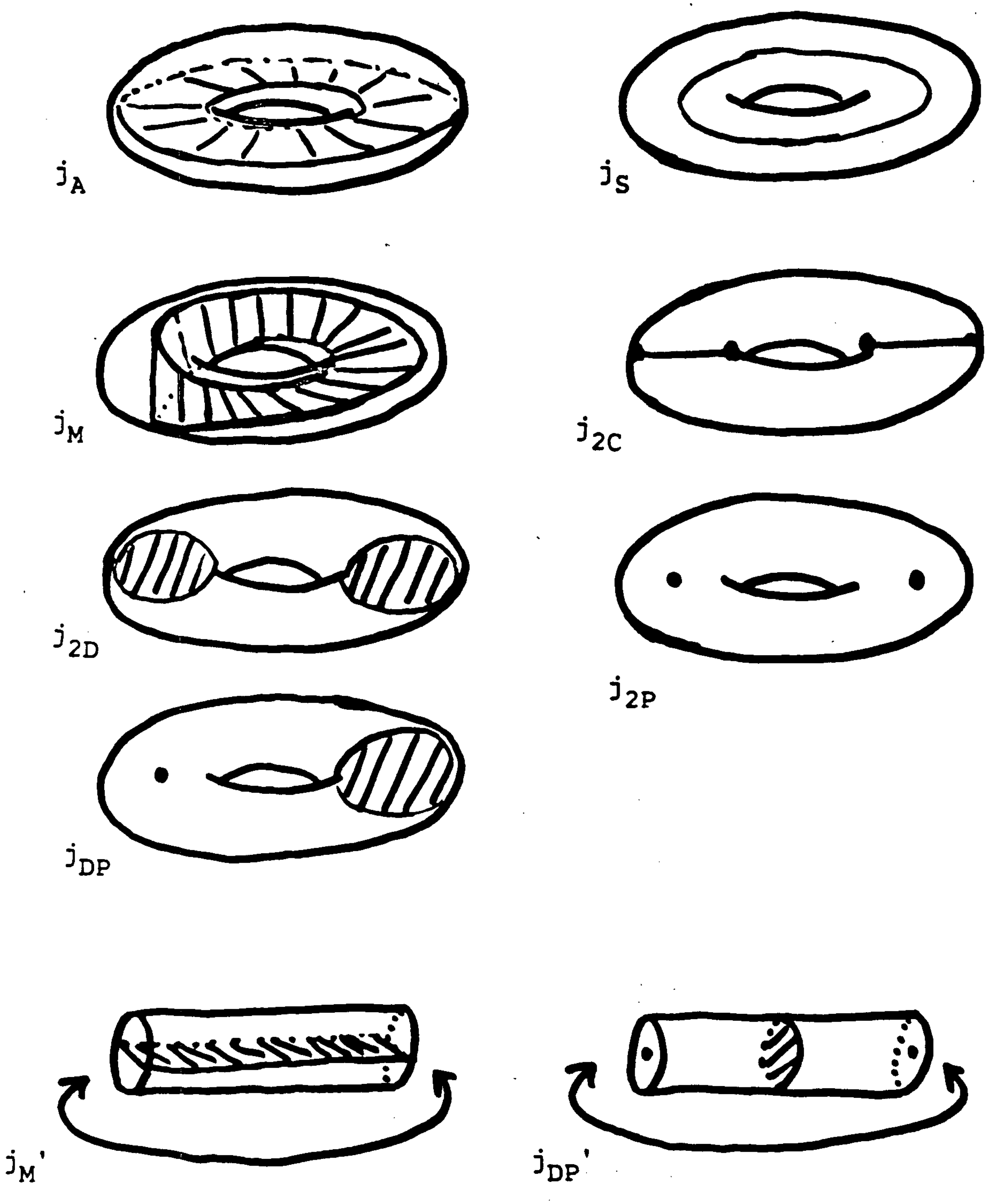
$j_N=\hat{\kappa}x\alpha$ fixed point free and orientation reversing.

$j_O=idx\alpha$ fixed point free and orientation preserving.

So $j_M(z,w)=(\bar{z}w,w)$ and $j_{DP}(z,w)=(z\bar{w},\bar{w})$. The subscript describes the fixed point set or for the fixed point free involutions the orientability type. Recall by Proposition 1.3, since V is orientable, involutions with 0- or 2-dimensional fixed sets are orientation reversing. Those with 1-dimensional fixed sets are orientation preserving. None of the above involutions are conjugate since all have different fixed sets or orientation type. Using $V=D^2xI/d$

Figure 5.

Fixed point sets for the standard involutions.



where $d=\hat{a}\mathbf{x}(\tau|\partial I)$, involutions conjugate to j_M and j_{DP} can be defined as follows:

$j_M'=\hat{\kappa}\mathrm{x}id/d$ having fixed set Möbius band $\mathrm{Rex}I/d$

$j_{DP}'=id\mathrm{x}\tau/d$ having fixed set $D^2\mathrm{x}0 \sqcup 0\mathrm{x}1/d$.

<u>Theorem 4.2</u>

If ι and ι' are involutions on $V=D^2\mathbf{x}S^1$ with nonempty isomorphic fixed point sets or if ι and ι' are fixed point free and of the same orientation type, then ι and ι' are conjugate. An involution on V is conjugate to one of the nine involutions listed above.

Proof: Let ι be an involution on V. We show it is conjugate to a standard involution. For any given essential 2-cell B_0 of V there is an isomorphism h of V which takes B_0 to $B=D^2\mathbf{x}-1$. So by applying the Disc Theorem 2.14, and replacing ι by the conjugate involution $h^{-1}\circ\iota\circ h$ for a suitable h, we may assume ι satisfies:

Case 1) $\iota B=B$ and B intersects **Fix** transversally at $\mathrm{Re}\mathbf{x}-1$

Case 2) $\iota B=B$ and B intersects **Fix** transversally at $0\mathbf{x}-1$

or 3') $\iota B \sqcap B=\emptyset$

There is an isomorphism $(D^2\mathbf{x}I)/d \longrightarrow D^2\mathbf{x}S^1$ where $d=id\mathrm{x}(\tau|_{\partial I})$ is given by $d(z,t)=(z;e^{i\pi t})$. The isomorphism takes $D^2\mathbf{x}-1/d$ to B. Write also $B=D^2\mathbf{x}-1$. In case 3') by adjusting the isomorphism h we may assume that $\iota B=D^2\mathrm{x}1=(D^2\mathbf{x}0)/d$. Call

$C_- = D^2 x[-1,0]$ and $C_+ = D^2 x[0,1]$. The case 3') splits into two cases:

Case 3) $\iota B \sqcap B = \emptyset$ and $\iota C_+ = C_-$

Case 4) $\iota B \sqcap B = \emptyset$ and $\iota C_+ = C_+$.

We show **first**, if the involution ι falls into:

case 1) it has fixed set that of j_A or j_M,

case 2) it has fixed set that of j_S,

case 3) it is fixed point free as j_N and j_O are,

case 4) it has fixed set that of j_{2D}, j_{DP}, j_{2C} or j_{2P},

and we show **second**, if ι and ι' fall into the same case 1) - 4) then ι and ι' are conjugate. This will complete the proof because the nine standard involutions cover all possible fixed sets that can arise and none occurs in more than one case 1) - 4).

All constructions done for ι are to be performed for ι' also even if not explicitly stated. Use a prime ' to denote the corresponding construct.

In Case 1): The involution ι on $D^2 xI/d$ induces an involution λ on $D^2 xI$ with the property $\lambda \circ d = d \circ \lambda$ when restricted to $D^2 x\partial I$. $Fix(\lambda)$ is proper and 2-dimensional since **Fix**$=fix(\iota)$ is transversal to B. So λ is conjugate to the standard involution j_2 of the 3-cell. In particular $Fix(\lambda)$ is a 2-cell. **Fix** is obtained by identifying two disjoint 1-cells in the boundary of the two cell so **Fix** is an annulus or Möbius band.

Suppose ι' is given. $\lambda|B$ and $\lambda|B'$ are conjugate so there is a conjugation $h:B \longrightarrow B'$. Using d and d' we may extend h to a conjugation $D^2 x \partial I \longrightarrow D^{2\prime} x \partial I$. $\partial Fix(\lambda)$ decomposes $\partial D^2 xI$ into two 2-cells which are interchanged under λ. Let J be an (open) component of $(\partial D^2 x-1) - Fix(\lambda)$ selected so that $h(J)=J'$. Then **Fix** is an annulus if J and d(J) are in the same 2-cell determined by $\partial Fix(\lambda)$ and **Fix** is a Möbius band if J and $\iota d(J)$ are in the same 2-cell determined by $\partial Fix(\lambda)$. Therefore, h can be extended to a conjugation

$$h:D^2 x \partial I \sqcup Fix(\lambda) \longrightarrow D^{2\prime} x \partial I \sqcup Fix(\lambda').$$

Extend h over one of the 2-cells that $\partial Fix(\lambda)$ decomposes $D^2 xI$ into. Then extend to the other cell by equivariance. This gives a conjugation h defined on $\partial(D^2 xI)$. By the conjugation extendable property for the 3-cell, h extends to all of $D^2 xI$ and hence induces a conjugation on $D^2 xI/d$ between ι and ι'.

In case 2): As in case 1) the involution ι on $D^2 xI/d$ induces an involution λ on $D^2 xI$ with $\lambda \circ d = d \circ \lambda$ on $D^2 x \partial I$. $Fix(\lambda)$ is proper and 1-dimensional so λ is conjugate to the standard involution j_1 of the 3-cell. So **Fix** is a 1-sphere.

Suppose ι' is given. $\lambda|B$ and $\lambda|B'$ are conjugate so there is a conjugation $h:D^2 x \partial I \longrightarrow D^{2\prime} x \partial I$ with $h \circ d = d' . h$. Since $\lambda|\partial B$ is orientation preserving, h extends to $\partial(D^2 xI)$.

By the conjugation extendable property for 3-cells, h extends to all of D^2xI and hence induces a conjugation on D^2xI/d.

In case 3): The involution must be fixed point free. Let ι and ι' be of same orientation type. Construct an isomorphism $h:B \longrightarrow B'$ and extend to a conjugation $h:B \sqcup \iota B \longrightarrow B' \sqcup \iota' B'$ by equivariance. Since the orientation type is the same, h extends to all of ∂C_+ and then to an isomorphism $h:C_+ \longrightarrow C'_+$ by coning. Finally extend to $D^2xS^1=C_+ \sqcup C_-$ by equivariance.

In case 4): $\iota|C_+$ and $\iota|C_-$ are involutions on 3-cells so each has fixed set a point, a proper 1-cell or a proper 2-cell. Since $(B \sqcup \iota B) \sqcap$ **Fix**$=\emptyset$ it follows **Fix**$=\mathrm{Fix}(\iota|C_+) \sqcup \mathrm{Fix}(\iota|C_-)$. Moreover $\iota|C_+$ and $\iota|C_-$ must have the same orientation type so **Fix** is one of: two 2-cells, two points, a 2-cell union a point, or two 1-cells. Suppose ι and ι' have isomorphic fixed sets. Arrange notation so that $\iota|C_+$ and $\iota'|C_+'$ have isomorphic fixed sets. Construct a conjugation $h:B \sqcup \iota B \longrightarrow B' \sqcup \iota' B'$ as in case 3). In view of the conjugation extendable property of 3-cells, it suffices to show h extends to a conjugation $\partial C_+ \longrightarrow \partial C_+'$. Let $G=\partial C_+$ and let **Fix** now denote $\mathrm{Fix}(\iota|C_+)$.

Case 4.1) $\iota|C_+$ is conjugate to j_2. **Fix** decomposes G into two components one of which, E, contains B. Extend h to an isomorphism $h:E \longrightarrow E'$ and by equivariance to a conjugation $h:G \longrightarrow G'$.

Case 4.2) $\iota|C_+$ is conjugate to j_1. G/ι is a 2-sphere such that **Fix**=**Fix**$/\iota$ misses B/ι. So lifting an appropriate 1-cell J_1 in $(G/\iota)-(B/\iota)$ gives a 1-cell J of G with $J \sqcap \iota J$=**Fix**. In addition, $J \sqcup \iota J$ determines a component E of ∂G containing B but not ιB. Define an isomorphism $h_0:J \longrightarrow J'$ and extend by equivariance to $J \sqcup \iota J$. Notice we could have selected $h_0:J \longrightarrow \iota' J'$ instead, so if orientations are fixed for $J \sqcup \iota J$ and $J' \sqcup \iota' J'$, we may select h_0 to be either orientation preserving or orientation reversing. Hence $h|_B \sqcup h_0$ extends over the (open) annulus $E-(B \sqcup J \sqcup \iota J)$ to an isomorphism $h_1:E \longrightarrow E'$. Extend by equivariance to G.

Case 4.3) $\iota|C_+$ is conjugate to j_0. Then $E=(G-\mathrm{int}(B \sqcup \iota B))/\iota$ is a Möbius band and h induces an isomorphism $\partial E \longrightarrow \partial E'$. This isomorphism extends to all of E. G double covers G/ι and the isomorphism lifts to an isomorphism $h':G \longrightarrow G'$ extending h. By construction h' is a conjugation.

QED

Let C_iP be the space obtained by coning a real projective space to a point v_i. Then (C_iP,v_i) is isomorphic to $(B^3,0)/j_0$. The descriptions of the standard involutions j on a solid torus V can be used to compute V/j. Use **Fix** to denote **Fix**$/j$ and recall Corollary 1.4 in this connection.

V/j_A is a solid torus $D^2 x S^1$ with **Fix** an annulus $Re(\partial D^2) x S^1$.

V/j_M' is a solid Klein bottle $D^2{}_{+} xI/-\hat{\kappa}x(\tau|\partial I)$ with **Fix** the Möbius band RexI.

V/j_{2D} is a 3-cell with **Fix** two 2-cells.

V/j_{DP}' is C_1P with **Fix** the point v_1 and a 2-cell.

V/j_S is a solid torus $D^2 x S^1$ with **Fix** the 1-sphere $0xS^1$.

V/j_{2C} is a 3-cell with **Fix** two proper unknotted 1-cells.

V/j_{2P} is a boundary connected sum of C_1P and C_2P with **Fix**$=v_1 \sqcup v_2$

(i.e.) $V/j_{2P}=D^2 x S_+ \ / \ ((z,1)\sim(-z,1) \ , \ (z,-1)\sim(-z,-1))$ where $S_+=D_+ \sqcap S$, $v_1=(0,1)$, $v_2=(0,-1)$ and $D^2 xi$ is the connected sum disc.

V/j_N is a solid Klein bottle $D^2 xI/\hat{\kappa}x(\tau|\partial I)$.

V/j_O is a solid torus.

<u>Corollary 4.3</u>

If ι is an involution on a solid torus V then V/ι is isomorphic to one of the spaces V/j above. The isomorphism type of the fixed set and orientability type of ι determine V/ι up to isomorphism.

Example 4.4

j_O is not conjugation extendable because $s(z,w)=(z,zw)$ determines a conjugation $s:\partial V \longrightarrow \partial V$ for $j_o|\partial V$ that does not extend to V.

Corollary 4.5

The orientation reversing involutions are conjugation extendable. If V and V' are solid tori with conjugate orientation preserving involutions ι and ι' respectively then the involutions are conjugation extendable with respect to the class of isomorphisms $\partial V \longrightarrow \partial V'$ that:

1) extend to isomorphisms $V \longrightarrow V'$, for the case ι conjugate to j_O or j_S.

2) extend to isomorphisms $V \sqcup \mathbf{Fix} \longrightarrow V' \sqcup \mathbf{Fix}'$, for the case ι conjugate to j_{2C}.

Proof: It suffices to show conjugation extendable for the standard involutions j only. In fact it suffices to show given $h':\partial V \longrightarrow \partial V$ an isomorphism with $h' \circ j \circ (h'^{-1})=j$, that h' extends to an isomorphism $H':V \longrightarrow V$ with $H' \circ j \circ (H'^{-1})=j$. Now h' induces an isomorphism $h:\partial V/j \longrightarrow \partial V/j$. We show, for each j, h extends to an isomorphism $H:V/j \longrightarrow V/j$. Let $p:V \longrightarrow V/j$ be induced by inclusion. $p|(V-\mathbf{Fix})$ is a double cover and $p|\mathbf{Fix}$ is an isomorphism. Check that $H \circ (p|)$: $V-\mathbf{Fix} \longrightarrow (V-\mathbf{Fix})/j$ lifts to $H':V-\mathbf{Fix} \longrightarrow V-\mathbf{Fix}$ and thus obtain a conjugation.

Let $W=V/j$. When $\mathbf{Fix}^2 \neq \emptyset$ notice that h is only defined on a proper submanifold $(\partial V)/j$ of $\partial(V/j)$.

The extensions H of h are clear for j_A, j_{2D}, j_S, j_O, j_{DP} and j_M. For j_N and j_{2P}, ∂W is a Klein bottle. For j_N, the boundary of an essential proper disc D represents the unique element of order two in $H_1(\partial W)=\mathbf{Z}_2 \oplus \mathbf{Z}$. Since $h(\partial D)$ is also of order 2, $h(\partial D)$ bounds an essential proper disc D'. Extend by coning over D and then over the 3-cell $W-(\partial W \sqcup D)$. For j_{2P}, $\partial D^2 \mathbf{x} i$ is a 1-sphere that separates ∂W and is 2-sided in ∂W. Such 1-spheres represent the element $(0,2) \epsilon H_1(\partial W)$. Proceed as above. For j_{2C}, let $h: \partial W \sqcup \mathbf{Fix} \longrightarrow \partial W' \sqcup \mathbf{Fix}'$ be given. A 1-sphere in $\partial W - \mathbf{Fix}$ that decomposes $\partial W \sqcup \mathbf{Fix}$ into two components each containing one component of **Fix** has the property that it bounds a proper 2-cell D in W that misses **Fix**. Use this 2-cell to extend h.

QED

Remark 4.6

Consider the solid Klein bottle $V=D^2 \mathbf{x} I/d$ where $d=\hat{\kappa}\mathbf{x}(\tau|\partial I)$. Then $j_A = \hat{\kappa}\mathbf{x}id/d$ with fixed set an annulus, $j_M = -\hat{\kappa}\mathbf{x}id/d$ with fixed set a Möbius band, $j_{DC} = id\mathbf{x}\tau/d$ with fixed set a 2-cell and a 1-cell, $j_{CP} = -\hat{\kappa}\mathbf{x}\tau/d$ with fixed set a point and a 1-cell, and $j_S = \hat{\alpha}\mathbf{x}id/d$ with fixed set a 1-sphere are the only five involutions on a solid Klein bottle, up to conjugacy. The proof is very similar to the one given for the solid torus. Since d is orientation reversing, however, case 3) does not arise and in case 4) only the combinations that were disallowed previously can occur.

III. EQUIVARIANT ANNULUS AND TORUS THEOREMS

§5. Annulus Theorems

Definition 5.1

A proper annulus A in a 3-manifold M is **trivial**, if A decomposes M into a solid torus $V=D^2 \times S^1$ and a submanifold M_0 such that:

$$M = M_0 \sqcup V, \quad M_0 \sqcap V = \partial M_0 \sqcap \partial V = A$$

and there exists a nonseparating proper 2-cell $B \subset V$ with $B \sqcap A = \partial B \sqcap A$ a nonseparating 1-cell in A.

Otherwise call A **nontrivial.** Call V a solid torus that trivializes A.

Note that if A does not separate M or if ∂A is in different boundary components of M then A is nontrivial.

Definition 5.2

Call a nontrivial incompressible proper annulus an **essential annulus.** Call an incompressible proper Möbius band an essential Möbius band.

Let F be a surface and S a component of $F \sqcap \iota F$. In some surgeries performed later we will wish to replace F by $F'=F \times 1$ where $F \times [-1,1]$ is a bicollar of F. To insure that F', $\iota F'$ and **Fix** are transversal at least in standard neighborhoods (see Definitions 2.9-.10), the following lemma is used.

Lemma 5.3

Let V be a solid torus and $\iota:V \longrightarrow V$ an involution. Let A_0 and A_1 be annuli in ∂V with $\partial A_0=S_0 \sqcup S_1$, $\partial A_1=S_1 \sqcup S_2$, $A_0 \sqcap A_1=S_1$, $\iota S_0=S_0$, $(A_0 \sqcup A_1) \sqcap \iota(A_0 \sqcup A_1)=S_0$ and **Fix** $\sqcap (A_0 \sqcup A_1) \sqsubset S_0$. Then there is a proper annulus $A \sqsubset V$ such that A, ιA and **Fix** intersect transversally with $A \sqcap \iota A=S$ a 1-sphere and ∂A having one component in $\mathrm{int}(\iota A_0)$ and the other component is S_2.

A similar statement holds if V is a 3-cell and A_0, A_1 are 2-cells and S_i are 1-cells. See Figure 6.

Proof: By transversality of **Fix**, by taking a sufficiently small regular neighborhood N of $A_0 \sqcup A_1 \sqcup \iota A_0 \sqcup \iota A_1$ we may assume one of the following holds:

1) **Fix** $\sqcap N=\emptyset$

2) **Fix** $\sqcap N$ consists of two disjoint 1-cells I_i with

$A=A' \sqcup A''$

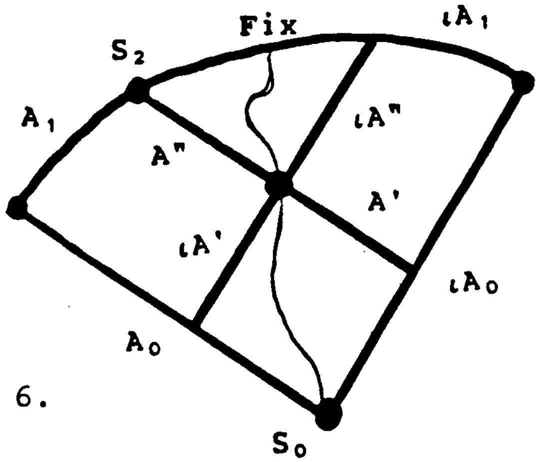


Figure 6.

exactly one point of ∂I_i in S_0 and the other in $\mathrm{int}(V) \cap \partial N$ or 3) $\mathbf{Fix} \cap N$ is an annulus such that one boundary component is S_0 and the other in $\mathrm{int}(V) \cap \partial N$.

Further, there is a regular neighborhood N' of $S_0 \subset N$ such that $\iota N'=N'$, $A_0'=N' \cap A_0$ is an annulus, $A_0' \sqcup \iota A_0'=N' \cap \partial V$ and properties 1)-3) hold with respect to N'.

Let B be the annulus which is the closure of $N' - A_0 \sqcup \iota A_0$. Then there is an S in $\mathrm{int}(B)$ with $\iota S=S$ and $\mathbf{Fix} \cap B \subset S$. There is an annulus A'' in $\overline{V-N'}$ such that $\partial A''=S_2 \sqcup S$ and $A'' \cap \iota A''=S$. Let A' be the component of $B-S$ that meets ιA_0. Then $A=A' \sqcup A''$ is the desired annulus.

QED

Remark 5.4

A solid Klein bottle is a twisted I-bundle over an annulus. The annulus is essential but it does not separate the boundary.

Lemma 5.5

If U is a solid torus then U has no essential annuli. If U is a solid Klein bottle then U has no essential annuli that separate ∂U.

Moreover, suppose A' is an annulus contained in ∂U such that a nonseparating proper disc D of U intersects A' in exactly one nonseparating 1-cell of A'. If A is an incompressible proper annulus disjoint from A' then the solid torus which trivializes A may be taken to be disjoint from A'.

Proof: Suppose A is an essential annulus. Then let D be any proper nonseparating 2-cell of U. (When A' is given, take D as in the statement.) Make A and D transversal. Since A is incompressible adjust D so that $A \cap D$ consists of 1-cells only.

If $A \cap D = \emptyset$ then A is contained in a 3-cell obtained by removing a sufficiently small regular neighborhood of D from U. This contradicts incompressibility.

If $A \cap D \neq \emptyset$ let B be an outermost 2-cell of D (and disjoint from A' if A' is given): so $B \cap A = \partial B \cap A = I$ is a 1-cell and $B \cap \partial U = \overline{\partial B - I}$. If I bounds a 2-cell in A, then by an isotopy moving B, obtain a disc D' with fewer 1-cells in $A \cap D'$. Assume now that I does not bound a 2-cell in A. Then I separates A. Let V be the closure of the component of $U-A$ that meets $\mathrm{int}(B)$. ∂A decomposes ∂U into two annuli or possibly, in the case where U is a solid Klein bottle, into an annulus and two Möbius bands. However, in the latter case $\partial B \cap \partial U$ must meet the annulus. It follows that $\partial V \cap \partial U$ is an annulus and V is a solid torus with the properties making A trivial.

QED

We next state the partial annulus theorem and the annulus theorem. The proofs are omitted. They are similar in spirit to the proof of the torus theorem.

Partial Annulus Theorem 5.6

Let M be an irreducible 3-manifold with involution ι. Let A_0 be an essential annulus with $\partial A_0 \sqcap \iota\partial A_0=\emptyset$. Then:

1) there is an essential annulus A with $A \sqcap \iota A=\emptyset$ and $\partial A \sqcup \iota\partial A=\partial A_0 \sqcup \iota\partial A_0$

or 2) there are two disjoint essential annuli A_1, A_2 with $\iota A_1=A_1$, $\iota A_2=A_2$, and $\partial(A_1 \sqcup A_2)=\partial A_0 \sqcup \iota\partial A_0$ and **Fix** is transversal to A_1 and A_2.

Example 5.7

The involution $\hat{\kappa}$xid on D^2x I induces an involution ι on $RP^2xI=D^2xI/d$ where $d=\alpha xid$ is an identification defined on $\partial D^2xI=S^1xI$. **Fix**=$(Re \sqcup \{i\})xI$. No essential annulus or Möbius band satisfies $A \sqcap \iota A=\emptyset$ or $\iota A=A$ and A and **Fix** transversal. There is an annulus, however, with $\iota A=A$ but it is not transversal to **Fix**.

Example 5.8

Consider the nonorientable twisted I-bundle $I\times I\times I/d$ over a torus, where $d=(\tau|\partial I)\times id\times\tau \sqcup id\times(\tau|\partial I)\times\tau$. The involution $\iota=id\times\tau\times\tau/d$ has Möbius bands but no annuli A with $A\sqcap\iota A=\emptyset$ or $\iota A=A$ and A and **Fix** transversal.

Annulus Theorem 5.9

Let M be an irreducible 3-manifold with involution ι. Suppose A_0 is an essential annulus or Möbius band in M with $\partial A_0=S_{01}\sqcup S_{02}$ where, if A_0 is a Möbius band, $S_{01}=S_{02}$. Let R_1, respectively R_2, be the component of ∂M with $S_{01}\sqsubset R_1$, respectively $S_{02}\sqsubset R_2$. Assume R_1 is incompressible and if $R_1\neq R_2$ assume also that R_1 is not a projective space.

Then there is an essential annulus or Möbius band A with either $A\sqcap\iota A=\emptyset$, or $\iota A=A$ and A and **Fix** transversal and in both cases $\partial A\sqcup\iota\partial A\sqsubset R_1\sqcup R_2\sqcup\iota R_1\sqcup\iota R_2$.

If M is orientable A may be taken to be an annulus.

§6. Equivariant Torus Theorem

Lemma 6.1

Let M be an irreducible 3-manifold containing an incompressible torus. Let F be a 1-sided Klein bottle in the interior of M and W a regular neighborhood of F in M with ∂W a torus. Then ∂W is an incompressible torus.

Proof: If not then $M=W\sqcup U$ with U a solid torus. Necessarily W is an orientable twisted I-bundle over T and M is orientable. The inclusion of U in M determines an index two subgroup of $\pi_1(M)$. Consider $p:\tilde{M} \longrightarrow M$, the 2-sheeted covering corresponding to that subgroup. Then $p^{-1}(W)=T\times[-1,1]$ where T is a torus with $p(T\times 0)=F$. $p^{-1}(U)=V_1\sqcup V_2$ is two disjoint solid tori. $\tilde{M}$ is a lens space. But M and hence $\tilde{M}$ contains a 2-sided incompressible torus.

QED

Equivariant Torus Theorem 6.2

Let M be an irreducible 3-manifold with involution ι. Suppose M contains an incompressible torus. Then one of the following holds:

(I) There is a 2-sided incompressible torus or Klein bottle T in int(M) transversal to **Fix** with $T\cap\iota T=\emptyset$ or $\iota T=T$.

(II) $M=V_{-1}\sqcup V_1\sqcup U_{-1}\sqcup U_1$ where V_i and U_i are solid tori and $\iota V_i=V_i$ and $\iota U_{-1}=U_1$.

There are annuli A_i, $i=\pm 1$, with

$A_1\sqcap A_{-1} = A_i\sqcap \iota A_i = \partial A_i = \partial\iota A_i = V_1\sqcap V_{-1} = U_1\sqcap U_{-1}$ and $V_i\sqcap U_i=A_i$, $V_i\sqcap U_{-i}=\iota A_i$, $\partial V_i=A_i\sqcup \iota A_i$, $\partial U_i=A_i\sqcup \iota A_{-i}$. See Figure 7.

$A_1\sqcup A_{-1}$ is a 2-sided incompressible torus or Klein bottle transversal to **Fix**. $\iota|V_i$ is orientation preserving.

(III) $M=V_1\sqcup V_2\sqcup V$ where V_1, V_2 and V are solid tori each invariant under ι such that ι is orientation preserving when restricted to any of V_1, V_2 and V. There is a 1-sided Klein bottle T with $T\sqcap \iota T = S \sqsubset \text{int}(V)$ a generator of $\pi_1(V)$. $V_1\sqcap V_2 = (T\sqcap \iota T) - \text{int}(V)$ are two annuli. T, ιT and **Fix** are pairwise transversal and $\mathbf{Fix}\sqcap\partial V_2=\emptyset$ and $\mathbf{Fix}\sqcap S\neq\emptyset$. V is a standard neighborhood of S. See Figure 7.

(IV) $M=W\sqcup V$ where W is a twisted I-bundle over a torus $T \sqsubset W$ and V is a solid torus with $\partial W=\partial V=W\sqcap V$ and $\iota W=W$, $\iota T=T$ and $\iota V=V$. **Fix** is transversal to ∂W and T except for a possible 1-sphere component S of $\mathbf{Fix}^1$ contained in T.

Proof: Let T_0 be an incompressible torus in $\text{int}(M)$. By Theorem 2.12 assume T_0, ιT_0 and **Fix** are almost pairwise transversal and that no 1-spheres in $T_0\sqcap \iota T_0$ bound 2-cells in T_0.

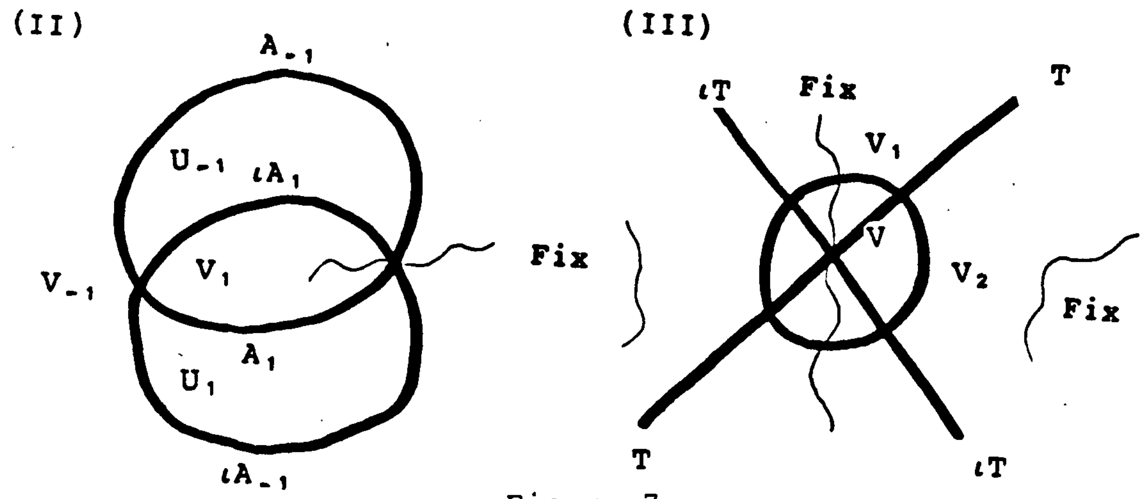


Figure 7.

As a first step we handle the cases where saddle components arise. Only Type III and Type II components are possible. In both cases since S and S_1 intersect transversally at one point, there can be only one component in $T_0 \sqcap \iota T_0$.

Suppose $T_0 \sqcap \iota T_0$ is a Type III component $S \sqcup S_1 \sqcup S_2$. Then S_1 and S_2 bound an annulus A in T_0 since $S_1 \sqcap S_2 = \emptyset$ and both intersect S transversally once. Let $T = A \sqcup \iota A$. Then $\iota T = T$ and T and **Fix** are transversal. T is 1-sided since a regular neighborhood of S_1 is a solid Klein bottle. Let N be a regular neighborhood of T invariant under N. If ∂N is incompressible then it is a 2-sided torus satisfying (I). If ∂N is compressible we arrive at (IV).

Suppose $T_0 \sqcap \iota T_0$ is a Type II component $S \sqcup S_1$. First we construct a torus T' isotopic to $T = T_0$ with $\iota T' = T'$. Let N(S) and $N(S_1)$ be regular neighborhoods of S and S_1 respectively, both invariant under ι such that $N = N(S) \sqcup N(S_1)$ is a regular neighborhood of $S \sqcup S_1$ and such that $T \sqcap N(S)$ and $T \sqcap N(S_1)$ are annuli, $N(S)\cap\mathbf{Fix}$ is a Möbius band, $N(S_1)\cap\mathbf{Fix}^2$ is a proper 2-cell and $N(S_1)\cap\mathbf{Fix}^1$ is a proper 1-cell. Both $N(S)$ and $N(S_1)$ are Klein bottles. By transversality there are two disjoint open 2-cell components K_1 and K_2 of $N(S)-(T\cup\iota T)$ that meet $\mathbf{Fix}^1$ and there are two disjoint open 2-cell components L_1 and L_2 of $N(S_1)-(T\cup\iota T)$ that do not meet $\mathbf{Fix}^2$. By considering the effect of ι near saddle points we see $A=(K_1\cup K_2\cup L_1\cup L_2)\cap\partial N$ is an annulus with $\partial A=C\cup\iota C$ where $C=\partial N\cap T$. The closure of $A\cup(T-N)\cup\iota(T-N)$ is a 2-sphere which by the irreducibility of M bounds a 3-cell E. E cannot contain the proper punctured torus $T\cap N$ so $E\cap\operatorname{int}(N)=\emptyset$. Since $\mathbf{Fix}^1$ is transversal to ∂E and $\iota\partial E=\partial E$ it follows $\iota E=E$. In particular $\iota|E$ is conjugate to j_1, the standard involution of a 3-cell with fixed set one 1-cell. A is invariant and contains $\mathbf{Fix}^1\cap\partial E$. Hence one shows there is a proper 2-cell D with ∂D a generator of $H_1(A)$ such that $\mathbf{Fix}^1\cap E$ is a proper 1-cell of D and $\iota D=D$. Since $\iota\partial D=\partial D$, by taking N sufficiently small we can construct a proper punctured torus P in N with $\partial P=D$ and $\iota P=P$ (namely isotope $T\cap N$). Consider the torus $T'=P\cup D$. $\mathbf{Fix}^2$ intersects T' transversally at S_1 and $\mathbf{Fix}^1$ is contained in T'. T' is 1-sided. Let W be a regular neighborhood of T' invariant under ι. If ∂W is incompressible then it is a 2-sided torus satisfying (I). If ∂W is compressible we arrive at case (IV).

We may now assume $T_0 \sqcap \iota T_0$ has no saddle components. T_0, ιT_0 and **Fix** are pairwise transversal and $T_0 \sqcap \iota T_0$ consists of disjoint 1-spheres bounding annuli in T_0 and ιT_0. <u>We successively construct incompressible tori or Klein bottles T with fewer 1-spheres in</u> $T \sqcap \iota T$, but always keep $T \sqcap \iota T$ consisting of 1-spheres bounding annuli in T and ιT. Therefore any 1-sphere of $T \sqcap \iota T$ will always have a standard neighborhood. See Definitions 2.9-.10. It also follows then that any 1-sided Klein bottle arising from such a construction has a regular neighborhood W with ∂W a torus. So Lemma 6.1 is applicable.

Note: Suppose T satisfies all the conditions of (I) except that T is 1-sided instead of 2-sided. Let W be a regular neighborhood of T. We can take W so that ∂W and **Fix** are transversal and $\iota W=W$ or $W \sqcap \iota W=\emptyset$. ∂W is 2-sided. If ∂W is incompressible, ∂W satisfies (I). If ∂W is compressible, by Lemma 6.1 T is a torus. Now $V=\overline{M-W}$ is a solid torus. If $\iota T=T$ we have (IV). If $\iota T \sqcap T=\emptyset$ then the solid torus V contains an embedded 1-sided torus ιT, a contradiction.

There are four main cases now depending on the number of 1-spheres of $T \sqcap \iota T$ and the compressibility of certain surfaces.

Assume $T \sqcap \iota T$ consists of at least two 1-spheres. Let $A \sqsubset \iota T$ be an innermost annulus: $A \sqcap T=\partial A$. ∂A decomposes T into two annuli A' and A" with $T=A' \sqcup A"$ and

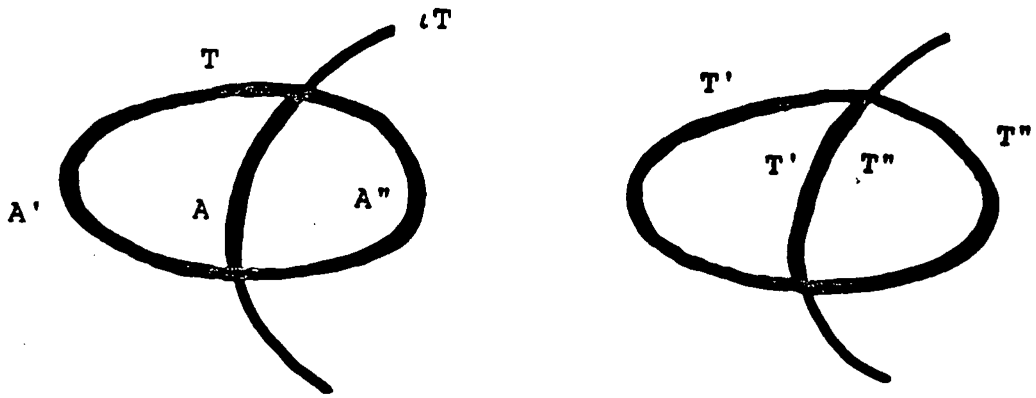


Figure 8.

$\partial A = \partial A' = \partial A'' = A' \sqcap A''$. $T' = A' \sqcup A$ and $T'' = A'' \sqcup A$ are tori or Klein bottles. See Figure 8.

Case 1) T' is incompressible.

Case 1.1) $\iota\partial A = \partial A$ and $\iota A = A'$.

Then $\iota T' = T'$. One sees **Fix** is transversal to T' by considering the standard neighborhoods of ∂A. We arrive at case (I) or (IV).

Case 1.2) Either $\iota\partial A = \partial A$ and $\iota A = A''$ or $\iota\partial A \sqcap \partial A$ is a single 1-sphere S and $\iota A \sqsubset A''$.

In the latter case $\iota S = S$. Let V_1 and V_2 be distinct standard neighborhoods of ∂A and let γ_1 and γ_2 be the two distinct standard annuli that meet both A and A'. Let $T_1 = (A' \sqcup A \sqcup \gamma_1 \sqcup \gamma_2) - \operatorname{int}(V_1 \sqcup V_2)$. Then $T_1 \sqcap \iota T_1 \sqsubset (T \sqcap \iota T) - \partial A$ because $(\gamma_1 \sqcup \gamma_2) \sqcap \iota(\gamma_1 \sqcup \gamma_2) = \emptyset$. T' and T_1 are ambient isotopic so T_1 is incompressible. $\mathbf{Fix} \sqcap (\gamma_1 \sqcup \gamma_2) = \emptyset$ and A is innermost so T_1, ιT_1 and **Fix** are pairwise transversal. Proceed with T_1.

Case 1.3) Either $\iota\partial A \cap \partial A$ is a single 1-sphere S and $\iota A \subset A'$ or $\iota\partial A \cap \partial A = \emptyset$.

Let $\partial A = S \cup S'$. Let V be a standard neighborhood of S and let γ be the standard annulus that meets both A and A'. Let $Ax[0,\epsilon]$ be a sufficiently thin collar of $A=Ax0$ in M such that

$$S'x[0,\epsilon] \subset A',\ Sx[0,\epsilon] \subset T \text{ and}$$
$$(A \cap \partial V)x[0,\epsilon] = (Ax[0,\epsilon]) \cap \partial V.$$

The collar exists since V is a solid torus. In the first case, $\iota S=S$ and $\iota\gamma=\gamma$. By Lemma 5.3 if $(Ax\epsilon) \cap \gamma \neq \emptyset$ we may assume $(Ax\epsilon) \cap V$ and $\iota((Ax\epsilon) \cap V)$ intersect transversally in a 1-sphere S_1 and that both are transversal to **Fix**. In all other cases set $S_1=S$. Define

$$T_1 = (Ax\epsilon) \cup \overline{A'-((S' \cup S)x[0,\epsilon])} \cup (Sx[0,\epsilon])-A'$$

Then $T_1 \cap \iota T_1 \subset ((T \cap \iota T)-\partial A) \cup S_1$. T_1 is incompressible since it is ambient isotopic to T'. T_1, ιT_1 and **Fix** are pairwise transversal. Proceed with T_1.

By case 1) we may now assume T' and T" are compressible.

Case 2) For every annulus $A \subset \iota T$ with $A \cap T = \partial A$, both corresponding surfaces T' and T" are compressible and $T \cap \iota T$ contains more than two 1-spheres.

Then let A_1 and A_2 in ιT be annuli with $A_i \cap T = \partial A_i$ and with $\partial A_i = S_0 \cup S_i$ where S, S_1 and S_2 are 1-spheres with

$S_1 \neq S_2$. Let A, A_1' and A_2' be the three annuli of T that these 1-spheres decompose T into: $\partial A = S_1 \sqcup S_2$ and $\partial A_i' = S_0 \sqcup S_i$, $i=1,2$. See Figure 9.

Define $T_1 = A \sqcup A_1 \sqcup A_2$. T_1 is incompressible. Otherwise T_1 bounds a solid torus or a Klein bottle U. Say $A_1' \subset U$. A_1' is trivial in U by Lemma 5.5. If $A_1' \sqcup A_1$ bounds the trivializing torus then the incompressible $T = A_1' \sqcup A_2' \sqcup A$ is ambient isotopic to $A_1 \sqcup A_2' \sqcup A$ which was compressible by hypothesis. If $A_1' \sqcup A_2 \sqcup A$ bounds the trivializing torus then, since A_1' and A_2 meet on S_0, A_2 must also be trivial in $A_1' \sqcup A_2 \sqcup A$. So T is ambient isotopic to $A_2' \sqcup A_2$ which was assumed compressible.

We have five cases:

Case 2.1) $\iota(S_1 \sqcup S_2) = S_1 \sqcup S_2$ and $\iota S_0 \subset A$. Then $\iota(A_1 \sqcup A_2) = A$.

Case 2.2) $\iota(S_1 \sqcup S_2) = S_1 \sqcup S_2$ and $\iota S_0 \subset A'$.

Then $\iota(A_1 \sqcup A_2) = A_1' \sqcup A_2'$ and $\iota S_0 = S_0$.

Case 2.3) $\iota S_1 = S_1$ and $\iota S_0 = S_0$. Then $\iota A_1 = A_1'$ and $\iota S_2 \subset A_2'$.

Case 2.4) $\iota(S_1 \sqcup S_2) \sqcap (S_1 \sqcup S_2) = \emptyset$.

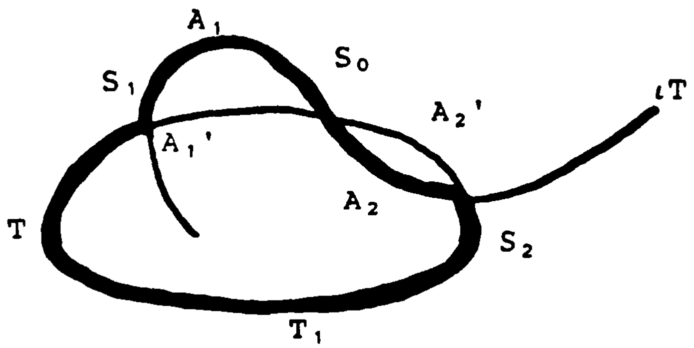


Figure 9.

Case 2.5) $\iota(S_1 \sqcup S_2) \sqcap (S_1 \sqcup S_2)$ is a one 1-sphere.

These cases cover all possibilities. In each case we find a T_i with fewer 1-spheres.

In case 2.5) this follows from the other cases. After relabelling assume S_1 is the 1-sphere in the intersection. Then $\iota S_1 = S_1$. By case 2.3) we assume $\iota S_0 \neq S_0$. Let A_3 be the innermost annulus adjacent to A_1: $A_3 \subset \iota T_1$ with $A_3 \sqcap T_1 = \partial A_3 = S_1 \sqcup S_3$ where $S_3 \neq S_0$. By case 2.3) again we may assume $\iota S_3 \neq S_3$. By case 2.1) and 2.2) we may assume $\iota S_0 \neq S_3$. So we have $\iota(S_0 \sqcup S_3) \sqcap (S_0 \sqcup S_3) = \emptyset$ and case 2.4) gives the reduction.

In case 2.1) use $T_1 = \iota T_1$. **Fix** is transversal to T_1 since the standard annulus meeting A_1 and A is invariant.

In case 2.2) and case 2.3): For $i=1,2$ let V_i be the standard neighborhoods of S_i with γ_i the standard annuli that meet both A and A_i. In all cases $(\gamma_1 \sqcup \gamma_2) \sqcap \iota(\gamma_1 \sqcup \gamma_2) = \emptyset$. Define T_2 to be the incompressible surface ambient isotopic to T_1 given by

$$T_2 = T_1 \sqcup \gamma_1 \sqcup \gamma_2 - \mathrm{int}(V_1 \sqcup V_2).$$

Then T_2, ιT_2 and **Fix** are pairwise transversal and

$$T_2 \sqcap \iota T_2 \subset T_1 \sqcap \iota T_1 - (S_1 \sqcup S_2).$$

In case 2.4): First assume $\iota S_0 \neq S_0$. By symmetry assume $\iota S_0 \neq S_1$. Let $(A_1 \sqcup A_2)x[0,\epsilon]$ be a sufficiently thin collar of $A_1 \sqcup A_2 = (A_1 \sqcup A_1)x0$ in M such that $(S_1 \sqcup S_2)x[0,\epsilon] \subset T$ and $S_0 x[0,\epsilon] \subset A_1$. Define T_2 as

$(A_1 \cup A_2)\times\epsilon \cup \overline{A-((S_1 \cup S_2)\times[0,\epsilon])} \cup ((S_1 \cup S_2)\times[0,\epsilon]-A)$. Then $T_2 \cap \iota T_2 \subset \iota(T \cap \iota T)-S_0$. T_2 is ambient isotopic to incompressible T_1. T_2, ιT_2 and **Fix** are pairwise transversal.

If $\iota S_0=S_0$, proceed as above but replace the condition $S_0\times[0,\epsilon] \subset A_1$ by $S_1\times[0,\epsilon] \subset A$. Use Lemma 5.3 on a standard neighborhood of S_0 to adjust the collar so that $(A_1 \cup A_2)\times\epsilon$ and $\iota(A_1 \cup A_2)\times\epsilon$ intersect transversally in one 1-sphere S_3. Then $T_2 \cap \iota T_2 \subset (\iota(T \cap \iota T)-(S_0 \cup S_1)) \cup S_3$.

Case 3) For each annulus $A \subset \iota T$ with $A \cap T=\partial A$, both corresponding surfaces T' and T" are not incompressible and $T \cap \iota T$ is exactly two 1-spheres.

Set $\iota T=A_{-1} \cup A_1$ with $A_{-1} \cap A_1=\partial A_1=\partial A_{-1}=T \cap \iota T=S_1 \cup S_2$. Then $T=\iota A_{-1} \cup \iota A_1$. There are solid tori or Klein bottles U_i and V_i $(i=\pm1)$ pairwise disjoint on their interiors with $\partial V_i=A_i \cup \iota A_i$ and $\partial U_i=A_i \cup \iota A_{-i}$. None of U_i or V_i are solid Klein bottles. Otherwise, if say V_1 is a solid Klein bottle, then since S_1 decomposes ∂V_1 into two annuli it follows that S_1 bounds a disc in V_1. This contradicts the incompressibility of T. By considering the standard annuli of a standard neighborhood of S_1 we see $\iota V_i=V_i$ and $\iota U_i=U_{-i}$.

Next we show $\iota|V_1$ and $\iota|V_{-1}$ are orientation preserving. If not then by Section 4, $\iota|V_i$ is conjugate to j_A, j_{2D}, j_N, j_M or j_{DP}, the standard involutions on a solid torus. j_{2D} and j_{DP} are not possible since S_1 or S_2 would bound a disc contradicting the incompressibility of T.

If $\iota|V_1$ is conjugate to j_M then say $S_1 \subset$ **Fix** and $S_2 \cap$ **Fix**$=\emptyset$. Then $\iota|V_2$ has a 2-dimensional fixed set component that has only one boundary component. It follows $\iota|V_2$ is also conjugate to j_M. So **Fix** contains a Klein bottle K. There is a regular neighborhood W of K with $\iota\partial W=\partial W$ and $W\cap$ **Fix**$=\emptyset$. Since V_i are solid tori and $K\cap V_i$ is a Möbius band, ∂W is a torus. By Lemma 6.1, ∂W is incompressible. We arrive at (I).

If $\iota|V_1$ is conjugate to j_A, then $[S_1]$ represents a generator of $H_1(V_1)$ and hence there is an ambient isotopy taking ιT to ∂U_{-1} (move A_1 to ιA_1). This contradicts that ιT is incompressible.

Finally suppose $\iota|V_1$ is conjugate to the involution $j_N=\hat{\kappa}x\alpha$ on D^2xS^1. If $\iota S_1=S_1$ then $S_1'=1xS^1$ and $S_2'=-1xS^1$ determine annuli A' and $\iota A'$ of ∂D^2xS^1. It is possible to construct a conjugation $\partial V_1 \longrightarrow \partial D^2xS^1$ taking A_1 to A'. This conjugation extends to a conjugation $V_1 \longrightarrow D^2xS^1$. But $[S_1']$ is a generator of $H_1(D^2xS^1)$ and we get a contradiction as for the j_A case above. If $\iota S_1=S_2$ then use $S_1'=\partial D^2x1$ and $S_2'=\partial D^2x-1$ and proceed as above but this time obtaining a contradiction as for j_{2D} above.

Case 4) $T \sqcap \iota T$ is a single 1-sphere S.

Then $\iota S=S$. Let V be a standard neighborhood of S and let α_1, α_2, β_1 and β_2 be the standard annuli with $\alpha_1 \sqcap \alpha_2=\emptyset$, $\beta_1 \sqcap \beta_2=\emptyset$, $\iota\alpha_1=\alpha_1$, $\iota\alpha_2=\alpha_2$ and $\iota\beta_1=\beta_2$.

$$\text{Define } T_1 = (T \sqcup \iota T \sqcup \alpha_1 \sqcup \alpha_2) - \mathrm{int}(V)$$
$$\text{and } T_2 = (T \sqcup \iota T \sqcup \beta_1 \sqcup \beta_2) - \mathrm{int}(V).$$

If T is 2-sided then T_1 is 2-sided. Also $\iota T_1=T_1$. Since T is 2-sided it follows that a sufficiently thin collar $T\mathrm{x}[0,\epsilon]$ of $T=T\mathrm{x}0$ can intersect only one of $\mathrm{int}(\alpha_1)$ and $\mathrm{int}(\alpha_2)$. Hence T_1 cannot separate and therefore T_1 is incompressible. We arrive at (I).

From now on assume T is 1-sided. T_1 and T_2 are tori. This follows since V is a solid torus and either both of the annuli T-$\mathrm{int}(V)$ and ιT-$\mathrm{int}(V)$ are "twisted" relative to V (if T is a Klein bottle) or neither is (if T is a torus).

If either of T_1 or T_2 is incompressible we arrive at (I). Assume then that T_1 and T_2 are compressible. Then T_i bounds a solid torus V_i. If $S \subset V_i$ then V_i contains a 1-sided torus or Klein bottle, a contradiction. So $M=V \sqcup V_1 \sqcup V_2$ with $\mathrm{int}(V)$, $\mathrm{int}(V_1)$ and $\mathrm{int}(V_2)$ pairwise disjoint.

By choice of α_1 and α_2, ι interchanges the components of $\partial\alpha_i$. Therefore $\iota|\alpha_i$ is conjugate to one of $\mathrm{id}\mathrm{x}\tau$, $\kappa\mathrm{x}\tau$ or $\alpha\mathrm{x}\tau$, the standard involutions of $S^1\mathrm{x}I$. Let S_i be a 1-sphere of α_i that is the image of $S^1\mathrm{x}0$ under some conjugation.

Note that S_i does not bound a disc D in V_1, otherwise T would be compressible.

Note also that if there is an annulus A $\subset V_1$ with $\partial A=S_1 \sqcup S_2$ and $\iota A=A$ then we arrive at property (I) and (IV) as follows. Torus V_1 is separated by A. Since ι interchanges the components of ∂a_1, ι interchanges the components of $A-V_1$. A is trivial in V_1 so it follows V_1 can be given a trivial I-bundle structure over A. There is an annulus B $\subset$ V with $\partial B=S_1 \sqcup S_2$ and $\iota B=B$. V is an I-bundle over B. Consider $T_3=A \sqcup B$. It follows $V \sqcup V_1$ is an I-bundle over T_3 with $\partial(V \sqcup V_1)=T_2$ a torus. Moreover T_3 does not separate T so T_3 is 1-sided. If T_3 is a torus we arrive at (IV). If T_3 is a Klein bottle, Lemma 6.1 gives (I).

Since V is a standard neighborhood, $\mathbf{Fix} \sqcap a_1=\emptyset$ if and only if $\mathbf{Fix} \sqcap a_2=\emptyset$. Therefore $\iota|a_1$ and $\iota|a_2$ are conjugate.

Case 4.1) $\iota|a_1$ is conjugate to $\mathrm{id}\mathbf{x}\tau$. Then $\iota|V_1$ has a 2-dimensional fixed set that meets ∂V_1 in two fixed 1-spheres. By Section 4 it follows that S_1 bounds a disc or $S_1 \sqcup S_2$ bound an annulus A fixed by $\iota|V_1$. By the above comments, we arrive at (I) or (IV).

Case 4.2) $\iota|a_1$ is conjugate to $a\mathbf{x}\tau$. Then $\iota|V_1$ is orientation reversing. $\iota|\partial V_1$ is conjugate to $a\mathbf{x}\kappa$ on $S^1 \mathbf{x} S^1$ by a conjugation taking S_i to $S^1\mathbf{x}(-1)^i$. By Section 4 $\iota|V_1$ is conjugate to $\hat{a}\mathbf{x}\kappa$ or $a\mathbf{x}\hat{\kappa}$ by a conjugation extending the one given on the boundaries. In the first case S_1 bounds a disc and in the second case $S_1 \sqcup S_2$ bound an annulus with $\iota A=A$. Again by the above comments we arrive at (I) or (IV).

Case 4.3) $\iota|\alpha_1$ is conjugate to $\kappa x\tau$. Then $\iota|V_1$ and $\iota|V$ are orientation preserving. Now $\iota|V_2$ is orientation reversing if and only if T is a torus. To see this let $S_1=\alpha_1 \sqcap \beta_1$ and without loss say $S_1 \subset T$. Orient S_1. S_1 and ιS_1 bound two annuli A_1 and A_2 of ∂V_2 with $\iota A_1=A_2$. Consider the ways of inducing an orientation on ιS_1. The orientation induced by A_1 and the orientation induced by α_1 are the same if and only if T is a torus. Since $\iota|\alpha_1$ is orientation reversing the orientation induced by α_1 and the orientation induced by ι are opposite. So ι and A_1 induce opposite orientations on ιS_1 if and only if T is a torus. Since $\iota A_1=A_2$ the claim follows.

If T is a torus then $\iota|V_2$ is orientation reversing so $\iota|\partial V_2$ is conjugate to the involution $\alpha x\kappa$ on S^1xS^1 by a conjugation taking S_1 to ixS^1. As in case 4.2) we arrive at (I) or (IV).

If T is a Klein bottle then we arrive at (III). $\iota|\partial V_2$ is fixed point free so $\iota|V_2$ is conjugate to j_S or j_O while $\iota|V_1$ is conjugate to j_{2C}.

QED

IV. INVOLUTIONS ON ORIENTABLE I-BUNDLES OVER TORI AND KLEIN BOTTLES

§7. Involutions on the Trivial I-Bundle Over a Torus

As an application of the annulus theorem we classify the involutions on various I-bundles.

Definition 7.1

Let $W=S^1xS^1xI$ be the trivial I-bundle over the torus $T=S^1xS^1$.

Define the following involutions on W (see Figure 10):

$k_T=idxidx\tau$ having fixed set the torus S^1xS^1x0

$k_{2A}=idx\kappa xid$ having fixed set two annuli $S^1x\pm 1xI$

$k_{2S}=idx\kappa x\tau$ having fixed set two 1-spheres $S^1x\pm 1x0$

$k_A=(\rho\circ(idx\kappa))xid$ having fixed set the annulus S^1x1xI

$k_S=(\rho\circ(idx\kappa))x\tau$ having fixed set the 1-sphere S^1x1x0

$k_{4C}=\kappa x\kappa xid$ having fixed set four 1-cells $\pm 1x\pm 1xI$

$k_{4P}=\kappa x\kappa x\tau$ having fixed set four points $\pm 1x\pm 1x0$

$k_{OF}=\alpha xidxid$

$k_{NI}=\alpha xidx\tau$

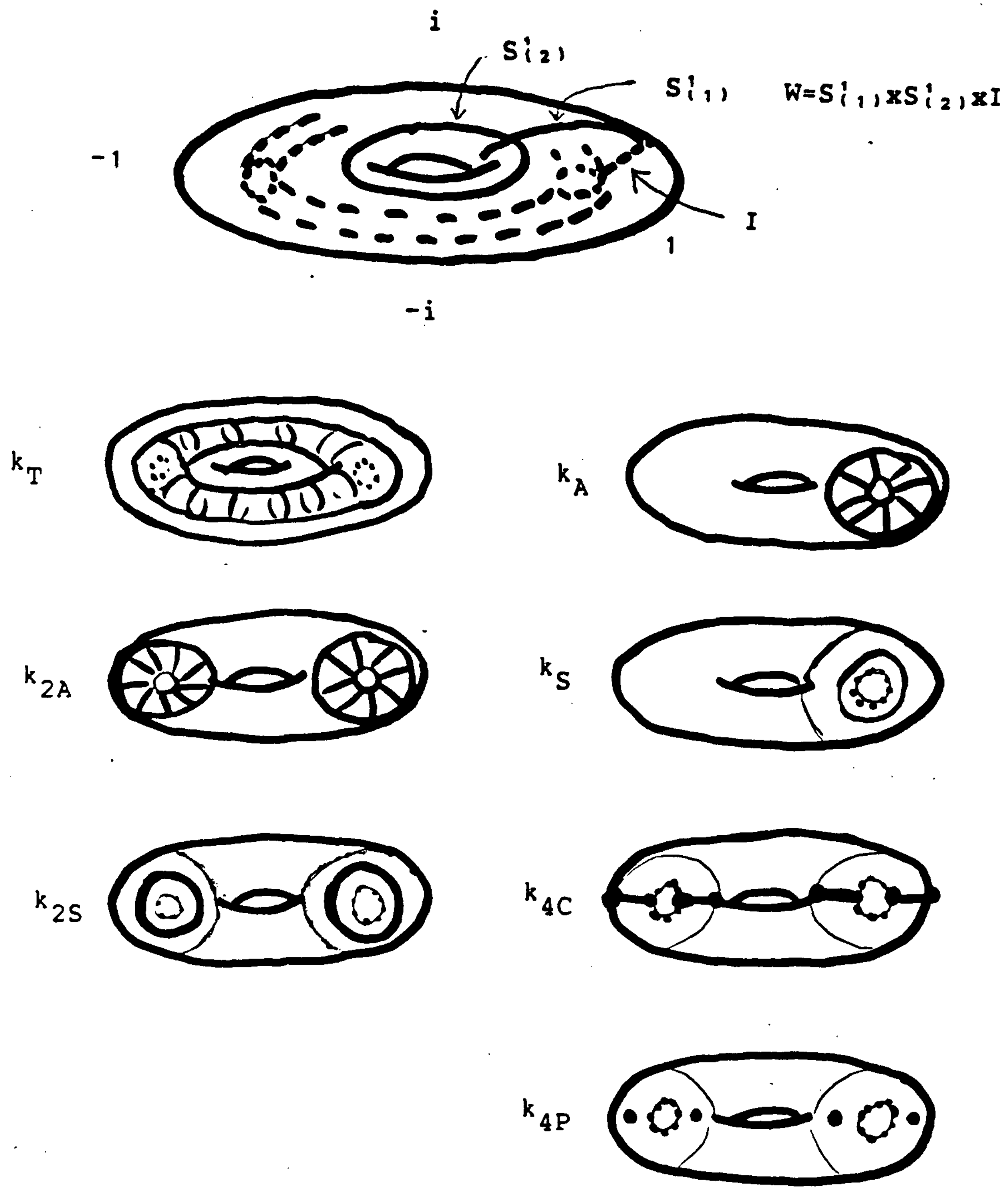
$k_{NF}=\alpha x\kappa xid$

$k_{OI}=\alpha x\kappa x\tau$

Here $\rho\circ(idx\kappa)(z,w)=(z\bar{w},\bar{w})$. The last four involutions are fixed point free. The subscript O means the involution is orientation preserving, N means it is orientation reversing, F means it keeps the boundary components fixed (as sets),

Figure 10.

Fixed point sets for the standard involutions.



and I means it interchanges the two boundary components. The orientation type of the other involutions is determined by the dimensions of their fixed point sets. These eleven involutions are not conjugate, because fixed point sets, orientability type and F/I properties are conjugacy class invariants.

k_A and k_S are conjugate to the following. On S^1xIxI define the identification $d=\alpha x\tau|_{\partial I}xid$. Then $W \cong S^1xIxI/d$. Let $k_A'=(idx\tau xid)/d$ with fix set $(S^1x0xI)/d$ and $k_S'=(idx\tau x\tau)/d$ with fix set $(S^1x0x0)/d$. The other involutions can be given similar alternate conjugate representations using $d=\alpha x\tau|xid$ or $d=idx\tau|xid$.

<u>Lemma 7.2</u>

Let A be an essential annulus in $Tx[-1,1]$. Then $(Tx[-1,1],A) \cong (S^1xS^1x[-1,1],S^1x1x[-1,1])$.

Proof: By an isomorphism take one of the boundary components of A as $S^1x1x[-1,1]$.

First show A meets both boundary components of $Tx[-1,1]$. If not then let A' be the annulus $S^1xJx[-1,1]$ where J is an interval chosen sufficiently close to $1\epsilon S^1$ and small enough so that $A' \cap \partial A=\emptyset$ and A' is in the component of $Tx[-1,1]$ - A that meets $Tx-1$. Then, in the solid torus $Tx[-1,1]/(z,w,-1)\sim(z,w',-1)$, the disc $1xS^1x[-1,1]/\sim$ is a nonseparating 2-cell meeting A' in one nonseparating 1-cell of C. By Lemma 5.5 there is a solid torus V that trivializes A and does not meet A'. Necessarily V does not meet $T\times -1$ so A is also trivial in $T\times[-1,1]$. This is a contradiction.

Next adjust by an isomorphism so that A and $1\times S^1\times[-1,1]$ meet in a single proper 1-cell. Then the isomorphism can be constructed.

QED

<u>Theorem 7.3</u>

Let ι and ι' be involutions on $W=S^1\times S^1\times I$ with isomorphic fixed point sets. If ι and ι' are fixed point free assume, in addition, that both have the same orientation type and that ι interchanges boundary components of ∂W if and only if ι' interchanges boundary components of ∂W.

Then ι and ι' are conjugate. An involution on W is conjugate to one of the eleven involutions listed above.

Proof: Let ι be an involution on $W=S^1\times S^1\times I$. We show it is conjugate to a standard involution. By the Annulus Theorem 5.9 there is an essential annulus A with either $\iota A\cap A=\emptyset$ or $\iota A=A$ and A and Fix transversal. In the latter case, by Lemma 3.5 assume the collar of A is not interchanged. By the previous lemma take A of form $S^1\times 1\times[-1,1]$. With further adjustment take $\iota A=S^1\mathbf{x}-1\mathbf{x}[-1,1]$ if $\iota A\neq A$. Let $W_+=S^1\mathbf{x}\{x+y\cdot i:y\geq 0\}\mathbf{x}[-1,1]$ and $W_-=S^1\mathbf{x}\{x+y\cdot i:y\leq 0\}\mathbf{x}[-1,1]$.

There are three cases:

Case 1) $\iota A=A$, A and **Fix** are transversal and the collar of A is not interchanged.

Case 2) $\iota A\sqcap A=\emptyset$ and $\iota W_+=W_+$.

Case 3) $\iota A\sqcap A=\emptyset$ and $\iota W_+=W_-$.

We show that ι is conjugate to:

in case 1) k_{OF}, k_{NI}, k_T, k_{2A}, k_{2S}, k_A or k_S

in case 2) k_{NF}, k_{OI}, k_{2A}, k_{2S}, k_A, k_S, k_{4C} or k_{4P}.

in case 3) k_{OF}, k_{NI}, k_{NF} or k_{OI}.

Several of the standard involutions are listed in more than one case. Each standard involution (or at least a conjugate of one) can in fact arise in the case it has been listed under. To see this it suffices to display an annulus A' in W, not necessarily of form $S^1\mathbf{x}1\mathbf{x}I$, with properties analogous to those of A. Consider A' as follows: in case 3) take $i\mathbf{x}S^1\mathbf{x}I$; in case 2) take $S^1\mathbf{x}i\mathbf{x}I$; in case 1) take $S^1\mathbf{x}1\mathbf{x}I$ for k_{OF}, k_{NI}, k_T, take $1\mathbf{x}S^1\mathbf{x}I$ for k_{2A}, k_{2S} and take $\{(z,z^2,t):z,t\}$ for k_A, k_S.

Call two involutions on W of **same type** if they have isomorphic fixed sets and, in addition, when they are fixed point free, if they have the same orientation type and simultaneously interchange or do not interchange boundary components. It suffices to show, **first**, that ι has the same type as a standard involution listed under a corresponding case, and **second**, if ι and ι' have the same type and fall into the **same** case 1) - 3) then they are conjugate.

Constant use is made of Section 4. Reserve j to denote standard involutions on the solid torus. All constructions done for ι are to be performed for ι', even if not explicitly stated.

Case 1) $\iota A=A$, A and **Fix** are transversal and the collar of A is not interchanged.

Then $W=S^1\times I\times[-1,1]/d$ where $d=id\times(\tau|\partial I)\times id$. The involution ι induces an involution λ on the solid torus $V=S^1\times I\times[-1,1]$ with the property $\lambda\circ d=d\circ\lambda$ when restricted to $S^1\times\partial I\times[-1,1]$. Fix($\lambda$) is proper since **Fix** is transversal to A. Let A also denote the copy $S^1\times I\times 1$ in V. Since the collar is not interchanged $\lambda(A)=A$. See Remark 3.6. By adjusting ι in a collar of A we may assume $\iota|A$ and $\lambda|A$ are one of the five standard involutions on an annulus (Lemma 3.2).

Let S be a fixed component of ∂A. $[S]\in H_1(V)=\mathbf{Z}$ is a generator. Write $V=D^2\times S^1$. Let $M=S^1\times 1$ and $L=1\times S^1$. Then $[M]$ and $[L]$ generate $H_1(\partial V)=\mathbf{Z}\oplus\mathbf{Z}$ and with a proper choice of orientations $[S]=[L]+a[M]$ where $a\in\mathbf{Z}$.

λ is not conjugate to j_S, j_M, j_{DP} or j_{2P}: If λ were conjugate to j_S then, since j_S is orientation preserving and ∂Fix(j's)$=\emptyset$, $\lambda|A=\alpha\times id$. Therefore S is kept setwise fixed by

λ. So $[S/\lambda]$ represents twice the generator of $H_1(V/\lambda)=\mathbf{Z}$. However $[\mathrm{Fix}(\lambda)]$ also represents a generator of $H_1(V)$ and $[\mathrm{Fix}(\lambda)/\iota]$ is a generator of $H_1(V/\lambda)$. If λ were conjugate to j_M then $\iota_*[S] = \iota_*[L]+a\iota_*[M] = [L]+[M]-a[M] \neq \pm[S]$ contradicting $\iota A=A$. If λ were conjugate to j_{DP} then $\iota_*[S] = -[L]-[M]+a[M] \neq \pm[S]$ contradicting $\iota A=A$. If λ were conjugate to j_{2P} then since j_{2P} is orientation reversing and $\partial\mathrm{Fix}(j_{2P}) = \emptyset$, $\lambda|A=\alpha\times\tau$. Therefore $[S]=\iota_*[S]$. This implies $[L]+a[M] = -[L]+a[M]$, a contradiction.

Hence λ is conjugate to j_A, j_{2D}, j_{2C}, j_O or j_N. Let B be a component of $V-\mathrm{int}(A\sqcup d(A))$ that meets S. We investigate the five possibilities for $\lambda|A$. Since we will see these give rise to involutions of different types, select a conjugation $h:A\longrightarrow A'$ between $\lambda|A$ and $\lambda'|A'$ and choose $S'=h(S)$. This conjugation extends to a conjugation $h:A\sqcup d(A)\longrightarrow A'\sqcup d'(A')$.

Case 1.1) $\lambda|A=\kappa\times\tau$. Let $\mathrm{Fix}(\lambda|A)=\{x,y\}$. Necessarily λ is conjugate to j_{2C}. If x and d(x) are in the same component of $\mathrm{Fix}(\lambda)$ then **Fix** is two 1-spheres. Otherwise **Fix** is one 1-sphere. So ι has the type of k_S or k_{2S}. Now x' and d'(x') are in the same component of $\mathrm{Fix}(\lambda')$ iff x and d(x) are in the same component of $\mathrm{Fix}(\lambda)$. The conjugation extends over **Fix** so extend it to a conjugation $h:V\longrightarrow V'$ between λ and λ'. A conjugation $h:W\longrightarrow W'$ between ι and ι' is induced.

Case 1.2) $\lambda|A=\kappa\times id$. Proceed as in case 1.1) except now $Fix(\lambda|A)$ is two 2-cells, so **Fix** is either two or one annuli. Thus ι has type of k_{2A} or k_A. Now $\iota B=B$. h extends over B since **Fix** separates B into two components. Extend similarly over the annulus $\partial V-(B\sqcup A\sqcup \iota A)$.

Case 1.3) $\lambda|A=id\times\tau$. Then λ is conjugate to j_A so **Fix** is a torus and ι has the type of k_T. Proceed as in 1.1).

Case 1.4) $\lambda|A=\alpha\times\tau$. It follows $\lambda B\sqcap B=\emptyset$. So λ is conjugate to j_N and ι is of type k_{NI}. Proceed as in 1.1).

Case 1.5) $\lambda|A=\alpha\times id$. Then λ is conjugate to j_{2C} or j_O. j_{2C} is not possible since $[S] = \lambda_*[S] = j_{2C*}([L]+a[M]) = -[L]-a[M] = -[S]$. So λ is conjugate to j_O and ι is of type k_{OF}. Let $B_1=B$ and $B_2=\overline{\partial V - (A\sqcup d(A)\sqcup B)}$. Let J be a nonseparating 1-cell of A with $J\sqcap\lambda J=\emptyset$ and let I_i be any path in B_i from ∂J to $d(\partial J)$. B_i/ι is an annulus so by lifting an embedded path that is path homotopic to I_i/ι we may also assume that $I_i\sqcap\iota I_i=\emptyset$. By making proper choices, we arrange that $I_1\sqcup I_2\sqcup J\sqcup d(J)$ bounds a 2-cell in V. A similar property holds for λ' for $J'=h(J)$. Use I_i and I_i' to extend h to a conjugation $\partial V \longrightarrow \partial V'$ and complete the argument as before.

Case 2) $\iota A\sqcap A=\emptyset$ and $\iota W_+=W_+$. Let S be a fixed component of ∂A. Let B be a component of $W_+\sqcap\partial W$ that meets S. Let $\lambda=\iota|W_+$. There are two possibilities:

2a) $\lambda B=B$

2b) $\lambda B \sqcap B = \emptyset$.

In 2a) ι interchanges boundary components of ∂W, while in 2b) it does not. W_+ is a solid torus, so λ is conjugate to one of the standard involution of the solid torus.

This splits the present case into four subcases. In fact

in case 2a) if ι is orientation preserving

then λ is conjugate to j_{2C}.

if ι is orientation reversing

then λ is conjugate to j_A or j_N.

in case 2b) if ι is orientation preserving

then λ is conjugate to j_S or j_O.

if ι is orientation reversing

then λ is conjugate to j_{2P}.

To show this note that in $H_1(W_+)=\mathbf{Z}$, $[S]$ is a generator. Since $S \subset \partial W_+ - \mathbf{Fix}$, λ cannot be conjugate to j_{2D}, j_M or j_{DP}. In case 2a) $\lambda_*[S]=-\mu(\lambda)[S]$, while in case 2b) $\lambda_*[S]=\mu(\lambda)[S]$, where $\mu(\lambda)$ is $+1$ if λ is orientation preserving and -1 if λ is orientation reversing. If λ is conjugate to j_{2C} or j_{2P} then $\lambda[S]=-[S]$. In all other cases $\lambda[S]=[S]$. This establishes the claim.

$\lambda_- = \iota | W_-$ must satisfy the (similar) case 2a) - 2b). Combining λ and λ_- in all the different possible ways gives involutions of types as listed previously. For example, combining a j_A with a j_N gives an involution of type k_A.

Let ι' be of same type as ι. It remains to show they are conjugate. Find an isomorphism $h:A \longrightarrow A'$ and extend by equivariance to $h:A \sqcup \iota A \longrightarrow A' \sqcup \iota' A'$. It suffices to show that h extends to W_+ when $\lambda = \iota | W+$ and $\lambda' = \iota' | W_+$ are conjugate. Take B and B' as above.

Case 2.1) λ is conjugate to j_A. $\mathbf{Fix} \sqcap B$ is a 1-sphere and the components of B-**Fix** are interchanged. Extend h over one of these components and then extend over all of B by equivariance. Similarly for the other annulus of $W_+ \sqcap \partial W$. By the conjugation extendable property of j_A this conjugation extends to all of W_+.

Case 2.2) λ is conjugate to j_N. Then B/ι is a Möbius band. The isomorphism h extends to B/ι. Lift to B and proceed as in 2.1.

Case 2.3) λ is conjugate to j_{2C}. B/ι is a 2-cell with $\mathbf{Fix} \sqcap B/\iota$ being two points. There is an isomorphism which extends the given induced one on $\partial B/\iota$ and takes the two points of $\mathbf{Fix} \sqcap B/\iota$ to $\mathbf{Fix}' \sqcap B'/\iota'$ in either of the two possible ways. This isomorphism lifts and with correct choices h extends as in 2.1.

Case 2.4) λ is conjugate to j_{2P}. Extend h in any way to B and then extend by equivariance to ∂W_+.

Case 2.5) λ is conjugate to j_O or j_S. Then $B \sqcap \lambda B = \emptyset$. Let J be a proper 1-cell of A and let $J' = h(J)$. Select a proper 1-cell I of B with $\partial I = \partial(J \sqcup \lambda J) \sqcap A$ and consider

$C=I\sqcup\lambda I\sqcup J\sqcup\lambda J$. C cannot be used to extend h since even if C bounds a disc in W_+, C' may not bound a disc in W'_+. As before let $W_+\cong D^2xS^1$, $M=S^1x1$ and $L=1xS^1$. Then $[M]$ and $[L]$ generate $H_1(\partial W_+)=Z\oplus Z$ and with correct choices $[S]=[L]+a[M]$ as classes in $H_1(\partial W_+)$, for some $a\epsilon Z$. By changing I assume $[C]=\mu[L]+b[M]$ where μ is 0 or 1 and $b\epsilon Z$. Achieve this by altering the path class of I by concatenating with S and by using the fact that $\lambda_*[S]=[S]$. Now W_+/λ is a torus. Let $p:H_1(\partial W_+)\longrightarrow H_1(\partial W_+/\lambda)$ be the obvious homomorphism. Let $[M_1]$ and $[L_1]$ be generators for $H_1(\partial W_+/\lambda)$ defined as for ∂W_+. Without loss for j_O,

$$p[M]=[M_1] \text{ and } p[L]=2[L_1]$$

and for j_S,

$$p[M]=2[M_1] \text{ and } p[L]=[L_1].$$

For j_S, $p[C]=\mu[L_1]+2b[M_1]$. Since C/λ is double covered by C it follows μ is even. So $\mu=0$. Let $C_1=C$.

For j_O, suppose μ is 0. Then $p[C]=b[M_1]$ so b is even since again C/λ is double covered by C. Then $[(I\sqcup J)/\lambda]=(b/2)[M_1]$ and it follows $(I\sqcup J)/\lambda$ lifts to a 1-sphere. This is a contradiction since $\partial(I\sqcup J)\neq\emptyset$. So $\mu=1$. Then let I_1 be a proper 1-cell in B with $I_1\sqcap I=\partial I_1=\partial I$ such that $C_1=I_1\sqcup\lambda I\sqcup J\sqcup\lambda J$ has class $[C_1]=d[M]$ for some $d\epsilon Z$.

In any event we obtain a curve C_1 with $[C_1]=0\epsilon H_1(W_+)$. Since j_S and j_O determine different μ it follows that for ι' we can define C_1' in the same way as C_1 (i.e.) using I_1' if

C_1 uses I_1. Extend h over I (and I_1 for j_0 case). This h then extends to a conjugation by construction.

Case 3) $\iota A \sqcap A = \emptyset$ and $\iota W_+ = W_-$. Then ι is fixed point free. Suppose ι' is of same type. Let $h:A \longrightarrow A'$ be any isomorphism and extend by equivariance to a conjugation $h:A \sqcup \iota A \longrightarrow A' \sqcup \iota' A'$. Fix a component S of ∂A and let $S'=h(S)$. Let B be the component of $W+ \sqcap \partial W$ that meets S. B is an annulus. Similarly define B'. Since ι and ι' have the same interchange type (F/I property) we have $h(\partial B - S) = \partial B' - S'$. Since they have the same orientation type $h|\partial B$ extends to $h:B \longrightarrow B'$. The isomorphism determined on the annulus $A \sqcup B \sqcup \iota A$ necessarily extends to an isomorphism of the solid tori $W_+ \longrightarrow W_+'$. Extend to $W=W_+ \sqcup W_-$ by equivariance.

QED

Corollary 7.4

Let W and W' be trivial I-bundles over a torus. Involutions conjugate to k_T are conjugation extendable. If ι on W is conjugate to k_A or k_{2A} and ι' on W' is conjugate to ι then a conjugation $h:\partial W \longrightarrow \partial W'$ is conjugation extendable if it satisfies the following condition: Let $\mathbf{Fix}_1$ be a component of $\mathbf{Fix}=\mathrm{Fix}(\iota)$ and let $\mathbf{Fix}_1\mathbf{x}[-1,1]$ be a bicollar of $\mathbf{Fix}_1=\mathbf{Fix}\mathbf{x}0$ such that $\partial\mathbf{Fix}_1\mathbf{x}[-1,1]$ bicollars $\partial\mathbf{Fix}_1$. Similarly for $\mathbf{Fix}_1'$, where $\mathbf{Fix}_1'$ is a component of $\mathbf{Fix}'=\mathrm{Fix}(\iota')$ meeting $h(\partial\mathbf{Fix}_1)$. Then require that h extends to an isomorphism

$$h:\partial W \sqcup \mathbf{Fix}_1 \mathbf{x}[-1,1] \longrightarrow \partial W' \sqcup \mathbf{Fix}_1{}' \mathbf{x}[-1,1].$$

There are conjugations of ∂W that are not extendable!

Proof:

For k_T, the fixed set separates. Let ι on W be conjugate to k_T. Let W_+ be the closure of one of the components. Then W_+ is isomorphic to **T**x[0,1] by an isomorphism taking ∂W_+-**Fix** to Tx0 and **Fix** to Tx1 where $T=S^1xS^1$. Clearly the isomorphism h:Tx0——>T'x0 extends to an isomorphism on W_+ taking Tx1 to T'x1. Extend by equivariance.

For k_A, let ι, ι' and h be as in statement of corollary. It follows h extends to a conjugation $h:\partial W \sqcup$ **Fix**——>$\partial W' \sqcup$ **Fix**'. Cutting W open along **Fix** gives a solid torus V having two copies of **Fix** in its boundary. The involution ι on W is induced by an involution λ on V which interchanges these copies of **Fix**. Similarly for W'. By the condition on the bicollar, $h|(\partial W \sqcup$ **Fix**) is induced from a conjugation $h_1:\partial V$——>$\partial V'$. Now λ is conjugate to j_N so by the conjugation extendable property for j_N, h_1 extends over V and hence induces a conjugation on W extending h.

For k_{2A}, let ι, ι' and h be as in statement of corollary. It follows h extends to a conjugation $h:\partial W \sqcup$ **Fix**$_1$——>$\partial W \sqcup$ **Fix**$_1$'. Since all components of **Fix** and

∂W-**Fix** are annuli h extends to a conjugation $h:\partial W \sqcup \mathbf{Fix} \longrightarrow \partial W \sqcup \mathbf{Fix}'$. Let C and ιC be the two 3-cells that **Fix** decomposes W into. By the bicollar condition $h(\partial C)$ is contained in one of the two 3-cells that **Fix**' decomposes W' into. Say $h(\partial C) \subset \partial C'$. Then extend h to an isomorphism $h:W \sqcup C \longrightarrow W' \sqcup C'$ by coning to a vertex and extend by equivariance to the desired conjugation.

QED

<u>Corollary 7.5</u>

If ι is an orientation preserving involution on $W=S^1 \mathbf{x} S^1 \mathbf{x} I$ then W/ι is isomorphic to one of the following spaces:

$W/k_{2S} \cong D^2 \mathbf{x} S^1$ a solid torus with $\mathbf{Fix}/k_{2S}$ two unknotted 1-spheres $(\pm 1/2)\mathbf{x}S^1$,

$W/k_S \cong D^2 \mathbf{x} S^1$ a solid torus with $\mathbf{Fix}/k_S$ one unknotted 1-sphere $\{(e^{\pi it}/2, e^{2\pi it}):-1 \leq t \leq 1\}$ representing twice a generator of $H_1(D^2 \mathbf{x} S^1)$,

W/k_{OI} an orientable twisted I-bundle over a Klein bottle,

$W/k_{4C} \cong S^2 \mathbf{x} I$ with $\mathbf{Fix}/k_{4C} = \{\text{four points}\}\mathbf{x}I$.

$W/k_{OF} \cong W$.

Proof: ι is conjugate to a standard involution k. Use the representations for the standard involutions. In all cases except for k_{OF}, $W/k \cong S^1 \mathbf{x} \{x+y \cdot i : 0 \leq y\} \mathbf{x} I/(g \sqcup g')$ where g is an identification of S^1x1xI and g' is an identification of $S^1x{-1}xI$ depending on k. For k_S note that $S^1x\{x+y\cdot i:0\leq y\}x0/(g\sqcup g')$ is a Möbius band with boundary Fix/k_S.

QED

§8. Involutions on the Orientable I-Bundle Over a Klein Bottle

Definition 8.1

Let $W=S^1xIxI/d$ be the orientable twisted I-bundle over the Klein bottle S^1xIx0/d, where $d=\kappa x(\tau|\partial I)x\tau$. More explicitly, $W=S^1x[-1,1]x[-1,1]/(z,-1,t)\sim(\bar{z},1,-t)$. $z=\pm i$ is a separating annulus, whereas $z=1$ is a nonseparating Möbius band. See Figure 11. The I-fibers are $zxsxI$. An involution λ on S^1xIxI with $\lambda|\circ d=d\circ\lambda|$ where $\lambda|$ denotes $\lambda|(S^1x\partial IxI)$ induces an involution $k=\lambda/d$ on W. $Fix=Fix(k)$ = $(Fix(\lambda)\sqcup Fix(d^{-1}\circ\lambda|))/d$.

Define the following involutions on W (see Figure 11):

$k_K=idxidx\tau/d$ having fixed set a Klein bottle S^1xIx0/d

$k_{2M}=\kappa xidxid/d$ having fixed set two Möbius bands $\pm1xIxI/d$

$k_{2S}=\kappa\times id\times\tau/d$ having fixed set two 1-spheres $\pm 1\times I\times 0/d$

$k_A=(-\kappa)\times id\times id/d$ having fixed set an annulus $\pm i\times I\times I/d$

$k_S=(-\kappa)\times id\times\tau/d$ having fixed set a 1-sphere $\pm i\times I\times 0/d$

$k_{A2P}=id\times\tau\times id/d$ having fixed set an annulus and two points $(S^1\times 0\times I \sqcup \pm 1\times -1\times 0)/d$

$k_{S2C}=id\times\tau\times\tau/d$ having fixed set a 1-sphere and two 1-cells $(S^1\times 0\times 0 \sqcup \pm 1\times -1\times I)/d$

$k_{2C}=(-\kappa)\times\tau\times id/d$ having fixed set two 1-cells $\pm i\times 0\times I/d$

$k_{2P}=(-\kappa)\times\tau\times\tau/d$ having fixed set two points $\pm i\times 0\times 0/d$

$k_O=\alpha\times id\times id/d$ fixed point free and orientation preserving

$k_N=\alpha\times id\times\tau/d$ fixed point free and orientation reversing

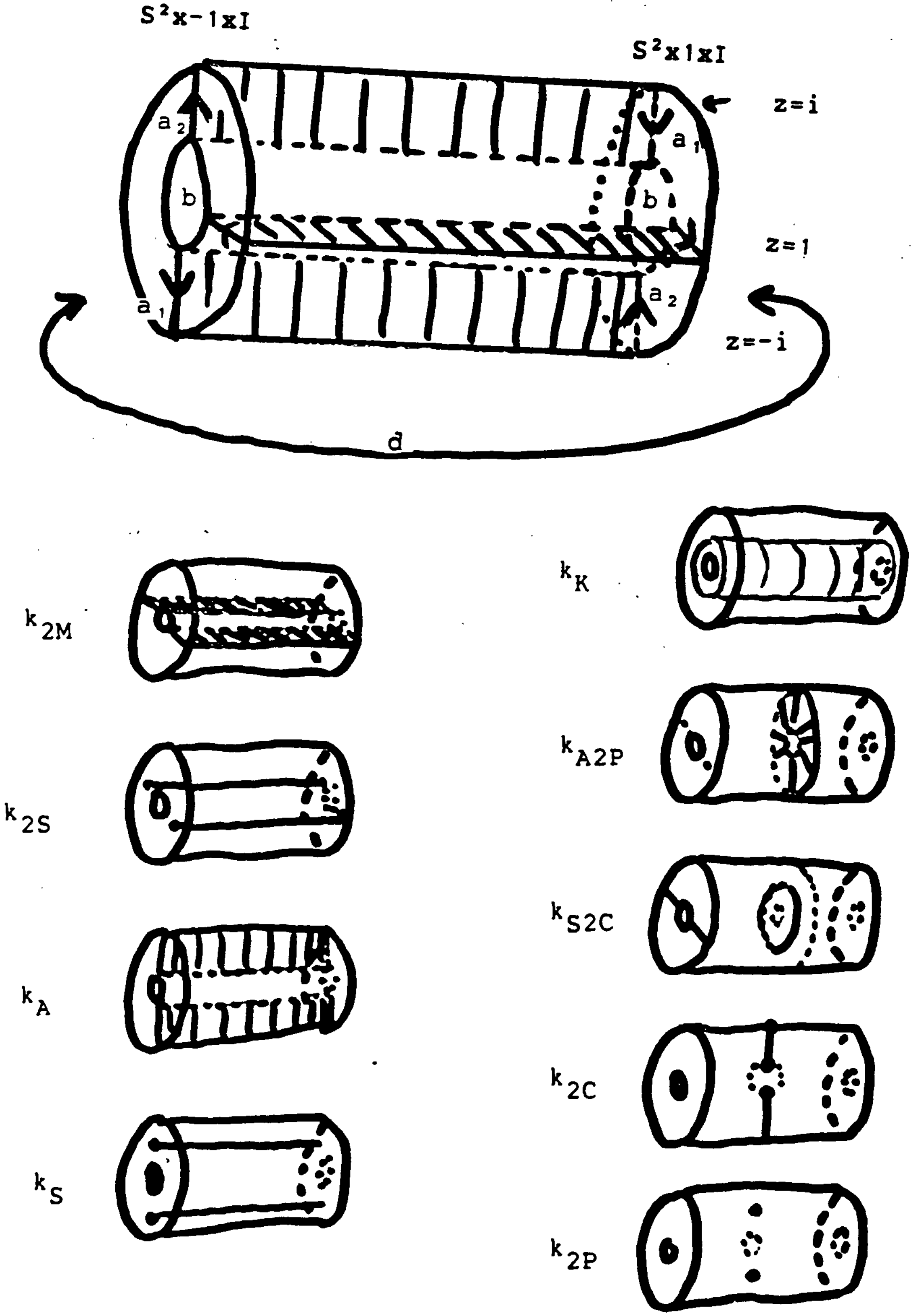
These eleven involutions are not conjugate because their fixed point sets or orientability types are different. Recall that the involutions with even dimensional fixed point sets are orientation reversing and that the ones with odd dimensional fixed sets are orientation preserving since W is orientable.

<u>Lemma 8.2</u>

Let $W=S^1\times I\times I/d$ be the orientable I-bundle over a Klein bottle. Then if A is an essential annulus there is an ambient isotopy moving A so that A is of form $S^1\times -1\times I/d$ if A is nonseparating or of form $\pm i\times I\times I/d$ if A is separating.

Figure 11.

Fixed point sets for the standard involutions.



Proof: Remove components of $A \sqcap (S^1 x i x 0)/d$. For details see [11].

QED

Theorem 8.3

Let ι and ι' be involutions on the orientable I-bundle over a Klein bottle, $W=S^1 xIxI/d$ where $d=\kappa x(\tau|\partial I)x\tau$. Suppose ι and ι' have isomorphic fixed sets and if ι and ι' are fixed point free assume, in addition, that they have the same orientation type.

Then ι and ι' are conjugate. An involution on W is conjugate to one of the eleven involutions listed above.

Proof: The proof is similar to the proof of Theorem 7.3. Let ι be an involution on $W=S^1 xIxI/d$. We show it is conjugate to a standard involution. By the Annulus Theorem 5.9 there is an essential annulus A with either $\iota A \sqcap A=\emptyset$ or $\iota A=A$ and A and **Fix** transversal. In the latter case by Lemma 3.5, assume the collar of A is not interchanged. By the previous lemma take A to be nonseparating of form $S^1 x{-1}xI/d$ or separating of form $\pm ixIxI/d$. In the case where A is nonseparating and $\iota A \sqcap A=\emptyset$ make $\iota A=S^1 x0xI$. Let $W_+=S^1 x[0,1]xI$ and $W_-=S^1 x[-1,0]xI$.

There are five cases:

Case 1) $\iota A=A$, A is nonseparating, A and **Fix** are transversal and the collar of A is not interchanged.

Case 2) $\iota A\cap A=\emptyset$, A is nonseparating and $\iota W_+=W_+$.

Case 3) $\iota A\cap A=\emptyset$, A is nonseparating and $\iota W_+=W_-$.

Case 4) $\iota A=A$, A is separating, A and **Fix** are transversal and the collar of A is not interchanged.

Case 5) $\iota A\cap A=\emptyset$ and A is separating.

We show that ι is conjugate to:

in case 1) k_K, k_{2M}, k_{2S}, k_A, k_S, k_O or k_N

in case 2) k_{A2P}, k_{S2C}, k_{2C} or k_{2P}

in case 4) k_K, k_{2M}, k_{2S}, k_{A2P} or k_{S2C}

in case 5) k_A, k_S, k_{2C}, k_{2P}, k_O or k_N

and that case 3) does not arise.

Several of the standard involutions are listed in more than one case. Each standard involution (or at least a conjugate of one) can in fact arise in the case it has been listed under. To see this it suffices to display an annulus A' in W with properties analogous to those of A. Consider A' as follows: in case 1) take $S^1\mathbf{x}{-1}\mathrm{x}I/d$; in case 2) take $S^1\mathbf{x}(-1/2)\mathrm{x}I/d$; in case 4) take $\pm i\mathrm{x}I\mathrm{x}I/d$; in case 5) take $e^{\pm i\pi/4}\mathrm{x}I\mathrm{x}I/d$.

It suffices to show, **first**, that ι has the same (fixed set) type as a standard involution listed under a corresponding case, and **second**, if ι and ι' have the same type and fall into the **same** case 1) - 5) then they are conjugate.

Constant use is made of Section 4. Reserve j to denote standard involutions on the solid torus. All constructions done for ι are to be performed for ι', even if not explicitly stated.

Case 1) Proceed as for Case 1) of Theorem 7.3. The identification is now $d=\kappa\mathbf{x}(\tau|)\mathbf{x}\tau$ instead of $d=id\mathbf{x}(\tau|)\mathbf{x}id$. Thus two of the five possibilities for $\lambda|A$ give different fixed sets. When $\lambda|A=\kappa\mathbf{x}id$ we have $Fix(\lambda)$ is two 2-cells. Then **Fix** is either two Möbius bands or one annulus. When $\lambda|A=id\mathbf{x}\tau$ we have $Fix(\lambda)$ is an annulus. Then **Fix** is a Klein bottle.

Case 2) $\iota A\sqcap A=\emptyset$, A nonseparating and $\iota W_+=W_+$. Select the component $S=S^1\mathbf{x}-1\mathbf{x}-1$ of ∂A. Let B_+ be the component of $W_+\sqcap\partial W$ that meets S. Let $\lambda_+=\iota|W_+$. There are two possibilities:

$$\text{2a)}\ \lambda_+B_+=B_+$$

$$\text{2b)}\ \lambda_+B_+\sqcap B_+=\emptyset.$$

Similarly for B_- and $\lambda_-=\iota|W_-$. Suppose λ_+ satisfies case 2a). Then $\lambda_+(S)=S^1\mathbf{x}0\mathbf{x}-1$. Since $(\lambda_-|A)\circ d=\lambda_+|A$ evaluating at S gives $\lambda_-(S^1\mathbf{x}1\mathbf{x}1)=S^1\mathbf{x}0\mathbf{x}-1$. Therefore λ_- satisfies case 2b). Similarly if λ_+ satisfies case 2b) then λ_- satisfies case 2a). The conjugacy class of λ is also restricted by the orientation type of ι.

Up to symmetry there are four cases:

Case 2.1) λ_+ is conjugate to j_{2C} and
λ_- is conjugate to j_S.

Case 2.2) λ_+ is conjugate to j_{2C} and
λ_- is conjugate to j_O.

Case 2.3) λ_+ is conjugate to j_A and
λ_- is conjugate to j_{2P}.

Case 2.4) λ_+ is conjugate to j_N and
λ_- is conjugate to j_{2P}.

These give rise to involutions with fixed sets as claimed for this case. The case is completed as case 2) in Theorem 7.3.

Case 3) $\iota A \sqcap A=\emptyset$, A nonseparating and $\iota W_+=W_-$. Take $S=S^1 xIx0$ with some choice of orientation. Then S is a generator of $H_1(W_+)$ and $H_1(W_-)$. Now $\iota|(W_-\sqcap S)=d\circ\iota|(W_+\sqcap S)$ but $d_*[S]=-[S]$. This is a contradiction so this case cannot arise.

Case 5) $\iota A \sqcap A=\emptyset$ and A separates. It follows that ιA also separates and that ιA is contained in one of the two components that A decomposes W into. By a suitable isomorphism, we may assume $A=e^{\pm i\pi/4}xIxI/d$ and $\iota A=e^{\pm i3\pi/4}xIxI/d$. A and ιA decompose W into three solid tori components U_0, U_1, U_2 with $U_1\sqcap U_2=\emptyset$, $U_0\sqcap U_1=A$, $U_0\sqcap U_2=\iota A$ and $\iota U_1=U_2$ and $\iota U_0=U_0$. Moreover, if S is a component of ∂A then

$[S]\epsilon H_1(U_0)$ is a generator and $[S]\epsilon H_1(U_1)$ is twice a generator. $\lambda=\iota|U_0$ is an involution on a solid torus that interchanges the disjoint annuli A and ιA and both annuli have boundaries representing a generator of $H_1(U_0)$. This is the same situation as for λ in case 2) of Theorem 7.3. That argument showed λ is conjugate to j_{2C}, j_A, j_N, j_S, j_O or j_{2P}. Since ι has the same fixed set as $\iota|U_0$ we obtain fixed sets as listed above. Suppose ι' also falls into this case. Then select an isomorphism $h:A \longrightarrow A'$ which we extend by equivariance to a conjugation $h:A \sqcup \iota A \longrightarrow A' \sqcup \iota A'$. The arguments for case 2) in Theorem 7.3 show h extends to a conjugation $h:U_0 \longrightarrow U_0'$. The following claim shows $h|A$ extends to an isomorphism $h:U_1 \longrightarrow U_1'$. Extend to U_2 by equivariance obtaining a conjugation $h:W \longrightarrow W'$ between ι and ι' and concluding this case.

Claim: Let U be a solid torus and A an annulus in ∂U. Suppose a component S of ∂A represents twice the generator of $H_1(U)$. Similarly for A' in U'. Then an isomorphism $h:A \longrightarrow A'$ extends to an isomorphism $h:U \longrightarrow U'$.

To prove this, choose 1-spheres M and L in ∂U so that $[M]$ and $[L]$ generate $H_1(\partial U)$ and $[M]$ is trivial in $H_1(U)$. The choice can be made so that $[S]=a[M]+2[L]$ where "a" is an odd integer. Let I be a proper 1-cell in A meeting both boundary components of A. Similarly for U', using $I'=h(I)$. $B=\overline{\partial V-A}$ is an annulus. There is a proper 1-cell J of B with $\partial J=\partial I$. Let

$S_1 = I \sqcup J$. Then $[S_1] = b[M] + c[L]$ where "c" is an odd integer since $S \sqcap S_1$ is a point. There is an isomorphism of ∂U to itself leaving A fixed which changes $[S_1]$ by a given multiple of $[S]$. So we may choose J so that $[S_1] = d[M] + [L]$ for some $d \in \mathbf{Z}$. Choose J' similarly. Extend $h|A$ to an isomorphism $h: A \sqcup J \longrightarrow A \sqcup J'$. Since int(B-J) is an open 2-cell, h can be extended by coning to an isomorphism $h: \partial U \longrightarrow \partial U'$. Then $(2d-a)h_*[M] = h_*(2[S_1]-[S]) = 2[S_1']-[S'] = (2d'-a')[M']$. Since a and a' are odd and $[M]$ and $[M']$ are generators we get $h_*[M] = \pm[M']$. Hence h extends to $h: U \longrightarrow U'$.

Case 4) $\iota A = A$, A is separating, A and **Fix** are transversal and the collar of A is not interchanged.

$$\text{Let } W_+ = \{z : z = x + y \cdot i, x \geq 0\} \times I \times I/d$$

$$\text{and } W_- = \{z : z = x + y \cdot i, x \leq 0\} \times I \times I/d.$$

Then $\iota W_+ = W_+$ and W_+ and W_- are solid tori. Let S be a component of ∂A. $\lambda_+ = \iota|W_+$ is conjugate to a standard involution j of a torus $D^2 \times S^1$. Let $M = S^1 \times 1$ and $L = 1 \times S^1$. Then on choosing correct orientations $[S] = a[M] + 2[L] \in H_1(\partial(D^2 \times S^1))$ where a is odd. Since $\iota A = A$ it follows $\lambda_{+*}[S] = \mu[S]$ where $\mu = \pm 1$ and μ depends only on $\iota|A$. Checking these conditions for the standard involutions on a solid torus gives:

$\mu = 1$ and λ_+ is conjugate to j_O, j_S or j_M.

$\mu = -1$ and λ_+ is conjugate to j_{2C} or j_{DP}.

Similarly for $\lambda_- = \iota|W_-$. The collar is not interchanged so ι

and $\iota|A$ have the same orientability type. We obtain four cases:

Case 4.1) λ_+ and λ_- are conjugate to j_M and $\iota|A$ is conjugate to $id \times \tau$ or $\alpha \times \tau$.

Case 4.2) λ_+ and λ_- are conjugate to j_{DP} and $\iota|A$ is conjugate to $\kappa \times id$.

Case 4.3) λ_+ and λ_- are conjugate to j_S or j_O and $\iota|A$ is conjugate to $\alpha \times id$.

Case 4.4) λ_+ and λ_- are conjugate to j_{2C} and $\iota|A$ is conjugate to $\kappa \times \tau$.

For case 4.1) we will show the isomorphism class of the fixed set determines $\iota|A$. We will also show the different cases 4.1) - 4.4) have nonisomorphic fixed sets. Therefore given ι' with fixed set isomorphic to that of ι, there is a conjugation $h:A \longrightarrow A'$. We show h extends to a conjugation $h:W_+ \longrightarrow W_+'$. Similarly h extends over W_- and the proof will be complete. Let $B=\overline{\partial W_+ - A}$. B is an annulus.

In case 4.1) $\iota|A$ is conjugate to $id \times \tau$ or $\alpha \times \tau$. In the first case $\mathbf{Fix} \sqcap A=S^1$ so **Fix** is a Klein bottle while in the second case $\mathbf{Fix} \sqcap A=\emptyset$ so **Fix** is two Möbius bands. In the first case B/ι is a Möbius band. It follows $h:A \longrightarrow A'$ extends to a conjugation $h:\partial W_+ \longrightarrow \partial W_+'$. In the second case B/ι is an annulus with $\partial(B/\iota)=(\partial A/\iota) \sqcup (\partial \mathbf{Fix} \sqcap W_+)/\iota$. Extend h/ι and lift to a conjugation $h:\partial W_+ \longrightarrow \partial W_+'$. In both cases the conjugation extendable property of j_M shows this conjugation extends to $h:W_+ \longrightarrow W_+'$.

In case 4.2) let $\mathbf{Fix}_+$ denote the fixed 2-disc component of λ_+. Let $\mathbf{Fix}_+\mathbf{x}[-1,1]$ be a bicollar of $\mathbf{Fix}_+$. Since a component of ∂A has intersection number ± 2 with $\mathbf{Fix}_+$ it follows $\mathbf{Fix}_+\mathbf{x}1$ meets both components of A-$\mathbf{Fix}$. Similarly for $\mathbf{Fix}_-\mathrm{x}1$. Hence $\mathbf{Fix}_+ \sqcup \mathbf{Fix}_-$ is bicollared so it must be an annulus. Thus the fixed set of ι is an annulus and two points. Extend the conjugation to $A \sqcup \partial\mathbf{Fix}_+ \longrightarrow A' \sqcup \partial\mathbf{Fix}_+'$. Since $\partial W_+ - (\partial\mathbf{Fix}_+ \sqcup A)$ is two open 2-cells that are interchanged under ι we can extend to a conjugation $\partial W_+ \longrightarrow \partial W_+'$ and so by the conjugation extendable property of j_{DP} to W_+.

In case 4.3) let S be a fixed component of A. W_+/ι is a solid torus. Let $p:H_1(W_+) \longrightarrow H_1(W_+/\iota)$ be the obvious homomorphism. $[S]=a[M]+2[L]$ where a is an odd integer. First we show j_O is not possible. Compare with case 2.5) of Theorem 7.3. $p[S]=a[M_1]+4[L_1]$. However, S double covers S/ι so it follows a is even, a contradiction.

So only j_S occurs and $\mathbf{Fix}$ is two 1-spheres. Then $p[S]=2a[M_1]+2[L_1]$ and $[S/\iota]=a[M_1]+[L_1]$. Let I be a proper 1-cell in A that meets both boundary components of A such that $I \sqcap \iota I=\emptyset$. Similarly for $I'=h(I)$. There is a proper 1-cell J_1 in W_+/ι with $\partial J_1=\partial(I/\iota)$ and $[J_1 \sqcup I]=\pm[M_1]$. Let $S_0=J \sqcup \iota J \sqcup I \sqcup \iota I$ where J is a lift of J_1 by p^{-1}. Then $[S_0]=\pm[M]$. Similarly for ι'. Extend $h:A \longrightarrow A'$ to

$h:A \sqcup J \longrightarrow A \sqcup J'$ and then by equivariance to $h:A \sqcup J \sqcup \iota J \longrightarrow A' \sqcup J' \sqcup \iota' J'$. Now $\partial W_+ - (A \sqcup J \sqcup \iota J)$ consists of two 2-cells that are interchanged under ι. So h extends to $h:\partial W \longrightarrow \partial W'$. The condition on $[S_0]$ and the conjugation extendable property of j_S show h extends to a conjugation on W.

In case 4.4) λ_+ has fixed set $\mathbf{Fix}_{1+} \sqcup \mathbf{Fix}_{2+}$ where $\mathbf{Fix}_{1+}$ and $\mathbf{Fix}_{2+}$ are proper 1-cells of W_+. Arguments similar to those given already show that $\mathbf{Fix}_{i+} \sqcap A$ cannot be exactly one point, i=1,2. Say then that $\partial \mathbf{Fix}_{2+} \sqsubset A$. Similarly for λ_-. Then the fixed set of ι is two 1-cells $\mathbf{Fix}_{1+}$, $\mathbf{Fix}_{1-}$ and one 1-sphere $\mathbf{Fix}_{2+} \sqcup \mathbf{Fix}_{2-}$. $(\overline{\partial W_+ - A})/\iota$ is a 2-cell and h/ι is given on the boundary. Clearly h/ι can be extended over the 2-cell. On lifting obtain a conjugation $h:\partial W_+ \longrightarrow \partial W_+'$. Since $\mathbf{Fix} \sqcap A$ are two points in the same component of $\mathbf{Fix} \sqcap W_+$ and since also $h(\mathbf{Fix} \sqcap A) = \mathbf{Fix}' \sqcap A'$ are in the same component of $\mathbf{Fix}' \sqcap W_+'$ the conjugation extendable property of j_{2C} gives a conjugation $h:W_+ \longrightarrow W_+'$.

QED

Corollary 8.4

On the orientable I-bundle over a Klein bottle involutions with 2-dimensional fixed sets are conjugation extendable.

Proof: For k_{A2P} the conjugation extends over the fixed set. Then cut open on the fixed set and use the conjugation extendable property of the solid torus involution j_{2P}. The other cases are similar to those for the trivial I-bundle over a torus.

QED

Corollary 8.5

If ι is an orientation preserving involution on an orientable twisted I-bundle $W=S^1xIxI/d$ over a Klein bottle then W/ι is isomorphic to one of the following spaces:

$W/k_{2S} \cong D^2xS^1$ a solid torus with $\mathbf{Fix}/k_{2S}$ two unknotted 1-spheres $\pm 1/2xS^1$,

$W/k_S \cong D^2xS^1$ a solid torus with $\mathbf{Fix}/k_S$ one unknotted 1-sphere $\{(e^{\pi it}/2, e^{2\pi it}): -1 \leq t \leq 1\}$ representing twice a generator of $H_1(D^2xS^1)$,

$W/k_{S2C} \cong D^2xI$ a 3-cell with $\mathbf{Fix}/k_{S2C} = \pm 1/2xI \sqcup ((3/4)(z/|z|))x0$ two 1-cells and one linked 1-sphere,

$W/k_{2C} \cong D^2xI/(\alpha|\partial D^2)x\tau$ an orientable I-bundle over a projective plane with $\mathbf{Fix}/k_{2C}=(\pm 1/2)xI$ two 1-cell fibers,

$W/k_O \cong W$.

Proof: ι is conjugate to a standard involution k. Use the representations for the standard involutions. W/k arises from the following subspaces of W by identifications on their boundaries: $\{x+y\cdot i:0\leq y\}\times I\times I$ for k_{2S} and k_O, $S^1\times I\times[0,1]$ for k_S, and $S^1\times[0,1]\times I$ for k_{S2C} and k_{2C}.

QED

V. INVOLUTIONS ON ORIENTABLE TORUS BUNDLES OVER A 1-SPHERE AND ON UNIONS OF ORIENTABLE TWISTED I-BUNDLES OVER KLEIN BOTTLES

§9. Involutions With 1-Dimensional Fixed Sets

Let $g:T^2 \longrightarrow T^2$ be an isomorphism where $T^2=S^1 x S^1$ and let $d:T^2 x -1 \longrightarrow T^2 x 1$ be defined by $d(x,-1)=(g(x),1)$. Define the **torus bundle** M_g by $M_g=T^2 xI/d$. Then M_g is irreducible and $T^2 x1$ is a nonseparating incompressible 2-sided torus. Up to isotopy $g:T^2 \longrightarrow T^2$ is uniquely determined by $g_*:H_1(T^2) \longrightarrow H_1(T^2)$. Let $S_1=S^1 x1$ and $S_2=1xS^1$. Then with respect to the basis $[S_1]$, $[S_2]$ of $H_1(T^2)$, g_* is given by a matrix $M(g)$ of $GL_2(Z)$. The matrix with respect to a different basis of $H_1(T^2)$ is a conjugate $Q^{-1}\ M(g)\ Q$ of $M(g)$, where $Q \epsilon GL_2(Z)$. M_g is orientable if and only if g is orientation preserving. g is orientation preserving when and only when the $\det(M(g))=1$.

Of interest are the orientable flat space forms $M_1,\cdots,M_5$. See [15]. These are determined by g as follows:

$M_1=S^1xS^1xS^1$: g=id. Then $M(g)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

M_2: $g=\kappa x\kappa$ (i.e.) $g(x,y)=(\bar{x},\bar{y})$. Then $M(g)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$.

M_3: $g=\omega\circ(\kappa x id)\circ\rho$ (i.e.) $g(x,y)=(y,\overline{x}\overline{y})$. Then $M(g)=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$.

M_4: $g=\omega\circ(\kappa x id)$ (i.e.) $g(x,y)=(y,\bar{x})$. Then $M(g)=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$.

M_5: $g=\omega\circ\rho\circ(\kappa x id)$ (i.e.) $g(x,y)=(y,\bar{x}y)$. Then $M(g)=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}$.

Each of these spaces has involutions with 1-dimensional fixed sets.

Let W_1 and W_2 be two orientable twisted I-bundles over a Klein bottle. An isomorphism $d:\partial W_1 \longrightarrow \partial W_2$ determines a **union along the boundaries of two orientable twisted I-bundles over a Klein bottle** $M_d=(W_1 \sqcup W_2)/d$. M_d is double covered by an orientable torus bundle over S^1. M_d is irreducible. Let $T^2=\partial W_1$. The nonseparating annuli of W_1 determine a canonical generator (1,0) of $H_1(T^2)=\mathbb{Z}\oplus\mathbb{Z}$ up to sign. Separating (nontrivial) annuli or Möbius bands of W_1 determine a generator (0,1) up to sign. Note that the involutions k_{2M} and k_{A2P} on W_1 are isomorphisms of W_1 that reverse the signs of these generators. As before, the isomorphism d determines a matrix of $GL_2(\mathbb{Z})$.

An alternate description for these spaces is

$$M = T^2 xI/\ (d_-(x),-1)\sim(x,-1),\ (d_+(x),1)\sim(x,1)$$

where d_- and d_+ are fixed point free orientation reversing isomorphisms. Then T^2x0 decomposes M into two orientable twisted I-bundles over a Klein bottle.

Note that M_2 is a union of orientable twisted I-bundles over a Klein bottle: $S^1x\pm 1xI$ are two Klein bottles.

The orientable flat 3-dimensional space forms have been classified (see Wolf [15]). Up to affine equivalence there are only six such space forms $M_1,\cdots,M_5$ and M_6. Define M_6 by

$$M_6 = S^1xS^1xI/\ (x,y,-1)\sim(-x,-\bar{y},-1),\ (x,y,1)\sim(-\bar{x},-y,1)$$

M_6 is a union of orientable twisted I-bundles over a Klein bottle but is not a torus bundle since $H_1(M_6)=Z_2\oplus Z_2$ is finite. M_6 is also known as the Hantzsche-Wendt manifold (see [4]).

We need two lemmas which describe the position of incompressible tori in M. For details see [11].

<u>Lemma 9.1</u>

Let M be an orientable torus bundle over S^1 and let T be an incompressible torus in M.

If T is nonseparating then $M \cong T^2x[-1,1]/d$ where $d{:}T^2x{-1} \longrightarrow T^2x1$ is an isomorphism.

If T is separating then T decomposes M into W_1 and W_2, two orientable twisted I-bundles over a Klein bottle.

$M=W_1 \sqcup W_2$ and $T=\partial W_1=\partial W_2=W_1 \sqcap W_2$.

Lemma 9.2

Let M be the union of orientable twisted I-bundles over a Klein bottle and let T be an incompressible torus in M.

If T is nonseparating then M is an orientable torus bundle over S^1.

If T is separating then T decomposes M into W_1 and W_2, two orientable twisted I-bundles over a Klein bottle. $M=W_1 \sqcup W_2$ and $T=\partial W_1=\partial W_2=W_1 \sqcap W_2$.

Proof: Two fold cover M by a torus bundle $\tilde{M}$. Argue by cases depending on whether $p^{-1}(T)$ is one or two tori. Note the deck transformation of $\tilde{M}$ is a fixed point free involution. For details see [11].

QED

Lemma 9.3

Let M be an orientable torus bundle over S^1. Suppose M is also a union of orientable twisted I-bundles over a Klein bottle. Then M has a basis as an orientable torus bundle over S^1 so that its matrix is $\begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}$ and a (canonical) basis as a orientable twisted I-bundle over a Klein bottle so that its matrix is $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

Proof: $M=T'\times I/d$ for some identification $d:T'\times -1 \longrightarrow T\times 1$. Let $p:T'\times I \longrightarrow M$ be the induced projection. M contains a Klein bottle K. Isotope K so that $p^{-1}(K)$ consists of annuli A_i meeting both components of $\partial(T'\times I)$. Since K is a Klein bottle one of the A_i must have boundary components representing opposite elements γ and $-\gamma$ of $H_1(T')$. Since M is orientable it follows M has a matrix of the form $\begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}$.

On the other hand, M is a union of orientable twisted I-bundles W_1 and W_2 over a Klein bottle. $W_1 \cap W_2 = T$. T' determines a nonseparating annulus in each of W_1 and W_2. Hence by choosing appropriate generators we may assume the matrix of M, as a union of orientable twisted I-bundles over a Klein bottle, is $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. M has a two fold covering $q:\tilde{M} \longrightarrow M$ by a orientable torus bundle over S^1, $\tilde{M}=U_1 \sqcup U_2$ such that $U_i \cong T^2\times I$ double covers W_i. The deck transformation restricted to U_i is a fixed point free involution k_i on U_i that interchanges boundary components of W_i. Hence k_i is conjugate to $k_{OI}=\alpha\times\kappa\times\tau$. Using this involution one sees that the matrix for the torus bundle $\tilde{M}$ is $\begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$.

The torus T' which determines nonseparating annuli of W_1 and W_2 must lift to two tori and therefore a matrix for $\tilde{M}$ is also

$$\begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & -2a \\ 0 & 1 \end{bmatrix}$$

Abelianizing the fundamental group of a torus bundle M_C

with matrix $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, one sees that $H_1(M_c)=Z\oplus Z\oplus(Z/cZ)$. Also the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ conjugates $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ to $\begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}$ and M_c and M_{-c} are isomorphic.

It follows $b=\pm a$ and so $b=a$ for a suitable choice of generators.

QED

Proposition 9.4

Let M be an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle. Let ι be an involution on M and let T be a separating incompressible torus with $T\sqcap\iota T=\emptyset$. Then there is a nonseparating incompressible torus with $\iota T=T$ and **Fix** and T transversal.

Proof: By Lemmas 9.1 and 9.2, $M=W_1\sqcup W_2'$ with $W_1\sqcap W_2'=T$ where W_1 and W_2' are orientable twisted I-bundles over a Klein bottle. Without loss say ιT is in W_2'. Using the lemmas again we see $M=W_1\sqcup(T'\mathbf{x}[-1,1])\sqcup W_2$ with (for i=1,2)

$$W_i\sqcap(T'\mathbf{x}[-1,1]) = \partial W_i = T'\mathbf{x}(-1)^i$$

$\iota W_1=W_2$, $\iota(T'\mathbf{x}[-1,1])=T'\mathbf{x}[-1,1]$ and with W_i orientable twisted I-bundles over a Klein bottle.

Let A be a nonseparating proper annulus in W_1. Write $\partial A=S_1\sqcup S_2$. There is an essential annulus A_0 in $T'\mathrm{x}[0,1]$ with $\partial A_0=S_1\sqcup\iota S_2$. Then $\partial A_0\sqcap\iota\partial A_0=\emptyset$. By the Partial Annulus

Theorem 5.6 there are disjoint essential annuli A_1 and A_2 transversal to **Fix** with $\partial A_0 \sqcap \iota\partial A_0 = \partial A_1 \sqcup \partial A_2$ and with either $\iota A_i = A_i$ $(i=1,2)$ or $\iota A_1 = A_2$.

Let $T_1 = A \sqcup \iota A \sqcup A_1 \sqcup A_2$. Essential annuli of $T' \mathbf{x}[-1,1]$ must meet both boundary components so T_1 is connected. T_1 is a torus. This follows since A is nonseparating in W_1 so A is "twisted" relative to $T' \mathbf{x}[-1,1]$ as is ιA.

$\iota T_1 = T_1$ and T_1 is transversal to **Fix.** T_1 is nonseparating hence incompressible.

QED

Recall that up to conjugacy there are three orientation preserving involutions j_{2C}, j_S and j_O on a solid torus V, five orientation preserving involutions k_{2S}, k_S, k_{4C}, k_{OF} and k_{OI} on a trivial I-bundle W over a torus and five orientation preserving involutions k_{2S}', k_S', k_{S2C}', k_{2C}' and k_O' on an orientable twisted I-bundle over a Klein bottle.

By applying the Torus Theorem 6.2 to M we obtain the following theorem.

Theorem 9.5

Let M be an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle. Let ι be an involution with a 1-dimensional fixed set component. Then one of the following holds:

1) There is a nonseparating (incompressible) torus T with $\iota T \sqcap T=\emptyset$. T and ιT decompose M into two trivial I-bundles W_1 and W_2 over a torus with $\iota|W_i$ conjugate to k_{2S}, k_S or k_{OI}.

2) There is a nonseparating (incompressible) torus T with $\iota T=T$ and its collar is not interchanged. $M \cong W/d$ where W is a trivial I-bundle over a torus and d is an isomorphism between the boundary components of W. ι is induced from an involution on W that is conjugate to k_{4C}.

3) There is a separating incompressible torus T with $\iota T=T$. M is the union of orientable I-bundles W_1' and W_2' over a Klein bottle with $W_1' \sqcap W_2'=T$. $\iota|W_1'$ and $\iota|W_2'$ are both conjugate to k_{S2C}' or k_{2C}' or are both conjugate to k_{2S}', k_S' or k_O'.

4) M satisfies case (III) of Torus Theorem 6.2 and **Fix**(ι) is exactly one 1-sphere. $\iota|V_1$ is conjugate to j_{2C} and $\iota|V_2$ is conjugate to j_O. M is M_6.

Proof: Apply the Torus Theorem 6.2. Assume for the moment that case (III) does not occur. Then since M is orientable there is a torus T satisfying case (I) or (II) of that theorem. By Proposition 9.4, if $\iota T=T$ then T separates. If $\iota T=T$ and the collar is interchanged then we arrive at case (1) by considering the boundary of a collar of T. We show that case (II) can be eliminated. Then to complete the proof, case (III) is handled.

Suppose case (II) occurs and suppose $\iota S_1=S_1$. Since $A_1 \sqcup A_{-1}$ is a separating incompressible torus it follows by Lemmas 9.1 and 9.2 that $U_{-1} \sqcup V_1$ is an orientable twisted I-bundle over a Klein bottle. ιA_1 is a separating annulus so S^1 bounds a proper Möbius band A of U_{-1}. $K=A \sqcup \iota A$ is a Klein bottle with $\iota K=K$. Then by Lemma 6.1 the boundary of a regular neighborhood of K is an incompressible torus invariant under ι giving case (I).

Suppose $\iota S_1=S_2$. Since ι has a 1-dimensional fixed set and $\mathbf{Fix} \sqcap S_1=\emptyset$, assume $\iota|V_1$ is conjugate to j_S. As above ιA_1 is a separating annulus in the orientable twisted I-bundle over a Klein bottle $U_{-1} \sqcup V_1$. So S_1 represents twice a generator of $H_1(V_1)$. Selecting appropriate generators $[M]$ and $[L]$ of $H_1(\partial V_1)=\mathbf{Z}\oplus\mathbf{Z}$ where $[M]$ is trivial in $H_1(V_1)$, we may assume $[S_1]=a[M]+2[L]$ for some integer a. Let $p:H_1(\partial V_1) \longrightarrow H_1(\partial V_1/\iota)$ be the obvious homomorphism. Then $p[M]=2[M_1]$ and $p[L]=[L_1]$ for suitable generators $[M_1]$ and $[L_1]$ of $H_1(\partial V_1/\iota)$. Then $p[S_1]=2a[M_1]+2[L_1]$ but this contradicts that $S_1 \longrightarrow S_1/\iota$ is an isomorphism.

Now consider case (III). Let V_1'' be the closure of the component of $M-(T \sqcup \iota T)$ that contains V_1. Let V_1' be the solid torus obtained from V_1'' by replacing S by two copies S_1 and S_2. Then $V_1' \cong V_1''/(S_1 \sim S_2)$. Similarly for V_2' and V_2''. Let W_1 be a regular neighborhood of T. ∂W_1 is an incompressible torus. By Lemmas 9.1 and 9.2, $W_2=\overline{M-W_1}$ is an orientable twisted I-bundle over a Klein bottle. ιT-int(W_1) is a separating annulus. Let $\partial(\iota T-W_1)=C_1 \sqcup C_2$. See Figure 12. It follows C_1, C_2, S_1 and S_2 represent twice a generator in both $H_1(V_1')$ and $H_1(V_2')$. Let ι_1 be the involution on V_1' induced by ι. Similarly for ι_2. From the Torus Theorem 6.2, ι_1 is conjugate to j_{2C} and ι_2 is conjugate to j_S or j_O. The argument given above for case (II) shows ι_2 is not conjugate to j_S since ι_2 interchanges components of $\partial V_2-(S_1 \sqcup S_2)$. Also the 1-cell S_1/ι_1 in V_1'/ι_1 cannot meet both components of Fix(ι_1) because otherwise its lift S_1 would be a 1-sphere representing an odd multiple of a generator of $H_1(V_1')$. Hence S_1 meets only one component of Fix(ι_1). Therefore

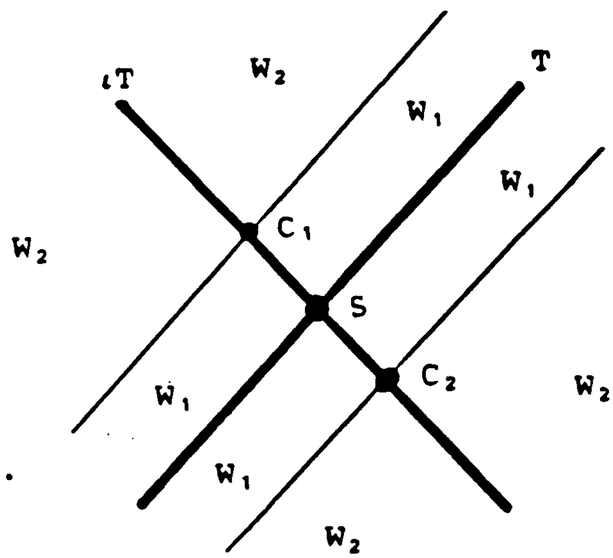


Figure 12.

$Fix(\iota)$ is one 1-sphere.

Since C_1 is a boundary component of the nonseparating annulus $\iota T \sqcap W_1$ of W_1 and of the separating annulus $\iota T \sqcap W_2$ of W_2 we see that a (1,0)-generator of W_1 and a (0,1)-generator of W_2 correspond.

Let C_2 be a 1-sphere in T meeting S transversally in one point. Let A be the annulus of V_2' determined by T. Since ι_2 is conjugate to j_O and ∂A represents twice a generator of $H_1(V_2')$, there is a proper 2-cell D of V_2' with $\iota_2 D \sqcap D = \emptyset$. Also arrange that D meets A in two proper nonseparating 1-cells of A, one of which is C_1, and that D meets $\iota_2 A$ in two proper nonseparating 1-cells. Then $\iota_2(\partial D \sqcap \iota_2 A)$ and $\partial D \sqcap A$ are four disjoint nonseparating proper 1-cells of A. Now consider A as an annulus in V_2'. Since ι_1 is conjugate to j_{2C} and $(j_{2C})_*$ on $H_1(V)$ is multiplication by -1, it follows that $\iota_1 \iota_2(\partial D \sqcap \iota_2 A) \sqcup (\partial D \sqcup A)$ bounds a disc in V_2. But $(\iota_1 \iota_2)|\iota T = id$ so ∂D bounds a disc D_1 in V_1'. By choice of C_1, $(D \sqcup D_1) \sqcap W_1$ is two Möbius bands so its

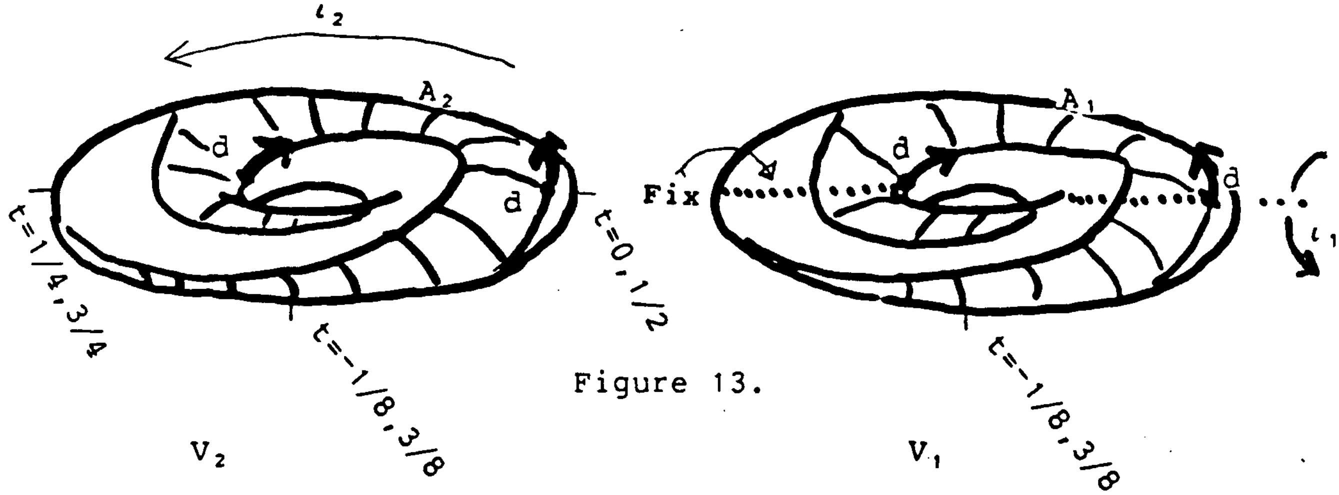


Figure 13.

boundary is a (0,1)-generator of W_1. On the other hand $(D \sqcup D_1) \sqcap W_2$ is a nonseparating annulus of W_2 since it has two boundary components and does not separate V_1'. This gives M_6.

To show how M_6 arises in this manner we give the following construction. See Figure 13. Let $V_j'=D^2 x S^1$ and

$$A_j=\{(e^{2\pi i(t+v)},e^{4\pi it}) : t\epsilon \mathbf{R},\ 0\leq v\leq 1/4\}$$

Let $\iota_1=j_O=id\mathbf{x}a$ and $\iota_2=j_{2C}=\hat{\kappa}\mathbf{x}\kappa$. Let d be the identification

$$(e^{2\pi it},e^{4\pi it}) \sim (e^{2\pi i(1-t)},e^{4\pi i((3/4)-t)})$$

on ∂A_j. Let $V_j''=V_j'/d$. Define $h:A_2 \longrightarrow A_1$ to be induced from the identity and define

$$h|(\iota_2 A_2) = \iota_1 \circ h \circ \iota_2 | A_2 = \hat{\kappa}\mathbf{x}(-\kappa)$$

(ie) $h:\partial V_2' \longrightarrow \partial V_1'$ is $h(x) = x$ if $x\epsilon A_1$ and $h(x) = (\hat{\kappa}\mathbf{x}-\kappa)(x)$ otherwise. Then $M_6 = V_1' \sqcup V_2' \ / \ h \sqcup d$ and involution $\iota_1 \sqcup \iota_2$ has fixed set one 1-sphere. (In the previous construction one can take $D \sqcup D'=\{t=-1/8\})$

QED

<u>Corollary 9.6</u>

Let ι be an involution on an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle with a 1-dimensional fixed set. Then the fixed set is one, two, three or four 1-spheres.

Corollary 9.7

Let M an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle. Let ι be an involution on M with a 1-dimensional fixed set.

Then M/ι is a lens space, $\mathbf{P}^3$, $\mathbf{P}^3\#\mathbf{P}^3$, or the boundary union of a solid torus with an orientable twisted I-bundle over a Klein bottle.

Proof: Use Corollary 7.5 and Corollary 8.5. Consider the cases of Theorem 9.5.

In 1) when each of $\iota|W_i$ is conjugate to k_{2S} or k_S, M/ι is a union along the boundaries of two solid tori. When one of $\iota|W_i$ is conjugate to k_{OI} then M/ι is the boundary union of a solid torus with an orientable twisted I-bundle over a Klein bottle.

In 2) W/k_{2C} is $S^2 \mathbf{x} I$. The identification of $S^2 \mathbf{x} -1$ with $S^2 \mathbf{x} 1$ gives the lens space $S^2 \mathbf{x} S^1$.

In 3) for k_{2S}', k_S' and k_O' we get the same spaces as in 1). For k_{S2C}' and k_{2C}', capping W/k_{S2C}' gives a 3-sphere and capping W/k_{2C}' gives $\mathbf{P}^3$.

In 4) V/ι is a 3-cell and $V_{\sqcap}V_1/\iota$ is two 2-cells. V_1/ι is a 3-cell so $V_{\sqcup}V_1/\iota$ is a solid torus. Since V_2 is also a solid torus M/ι is a lens space.

QED

Note that torus bundles may also be unions of twisted I-bundles over a Klein bottle. (See Lemma 9.3)

In the following let $Q \in GL_2(\mathbb{Z})$. Let $E \in GL_2(\mathbb{Z})$ be in the subgroup K of $GL_2(\mathbb{Z})$ generated by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

Note $\mathbb{Z} \longrightarrow \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ induces an exact sequence:

$$1 \longrightarrow K \longrightarrow GL_2(\mathbb{Z}) \longrightarrow GL_2(\mathbb{Z}_2) \longrightarrow 1$$

Theorem 9.8

Let M be an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle. Then M has an involution ι with fixed set exactly n 1-spheres (n>0) if and only if one of a), b) or c) below holds.

a) M is a torus bundle with matrix conjugate to one of the following for some Q or E:

(1) $Q^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Q \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ n = 2 or 4

(2) $Q^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Q \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ n = 1 or 3

(3) $Q^{-1} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} Q \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ n = 2

(4) EP where $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ n = 4

(5) EP where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ n = 1 or 3

(6) EP where $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ n = 2

b) M is a union of orientable twisted I-bundles over a Klein bottle with matrix (with respect to some set of canonical generators) one of the following for some Q or E: In (7) - (9) the inverses of matrices are to be taken over the field of rationals but the product matrix is required to be integral.

(7) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} Q \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ n = 2, 3 or 4

(8) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} Q \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ n = 1 or 2

(9) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} Q \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ n = 1 or 2

(10) EP where P = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ n = 2 or 4

(11) EP where P = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$
n = 1 or 3

(12) EP where P = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ n = 3

(13) EP where P = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$
n = 2

c) M is M_6 and n = 1

<u>Remark 9.9</u>

The proof shows how in each case (1) - (13) the involution on M arises from involutions on orientable torus bundles over S^1 or orientable twisted I-bundles over a Klein bottle.

Proof: Apply Theorem 9.5.

Suppose case (1) of Theorem 9.5 occurs: There is a nonseparating torus with $\iota T \sqcap T = \emptyset$.

Let W_1 and W_2 be the two components of M determined by T and ιT. Let $h_1: W_1 \longrightarrow T^2 xI$ and $h_2: W_2 \longrightarrow T^2 xI$ be conjugations between $\iota|W_i$ and the standard involutions k_{2S}, k_S or k_{OI} with $h_i(T) = T^2 x1$. Let $d: T^2 x1 \longrightarrow T^2 x1$ be the identification induced by T. Then the involutions induce an identification $\iota_2 \circ d \circ \iota_1 : T^2 x{-1} \longrightarrow T^2 x{-1}$. Consider the effect of these isomorphisms in $H_1(T^2) = H_1(S^1 x S^1) = Z \oplus Z$ with respect to basis $[S^1 x1]$ and $[1xS^1]$ where orientations are induced from $S^1 \subset C$. Then $M \cong T^2 xI/q$ where the matrix of q is $M(d)^{-1} M(\iota_2) M(d) M(\iota_1)$. The matrix of k_{2S} and k_{OI} is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and of k_S is $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$. We obtain (1) - (3) above.

Suppose case (2) of Theorem 9.5 occurs: There is a nonseparating torus with $\iota T = T$ and the collar of T is not interchanged.

Cut M open on T. Then $M \cong TxI/d'$ where $d': Tx{-1} \longrightarrow Tx1$ is an isomorphism. ι is induced from an involution λ on TxI that satisfies $d' \circ (\lambda | Tx{-1}) = (\lambda | Tx1) \circ d'$. Let $h: TxI \longrightarrow T^2 xI$ be a conjugation between λ and k_{4C} (with $h(Tx{-1}) = T^2 x{-1}$). Define $d = h| \circ d' \circ (h|T^2 x{-1})^{-1}$. Then $M \cong T^2 xI/d$ and ι is conjugate to the involution on $T^2 xI/d$ induced from the involution $k_{4C} = \kappa x \kappa x id$ on $T^2 xI$. The matrix of M is conjugate to the matrix of $d: T^2 = T^2 x{-1} \longrightarrow T^2 x1 = T^2$. Note d satisfies

$d\circ(\kappa x\kappa)=(\kappa x\kappa)\circ d$.

Now $T^2/\kappa x\kappa$ is a 2-sphere. The class of all isomorphisms of $T^2/\kappa x\kappa$, up to isotopy, that keep the four points $\pm 1x\pm 1$ fixed is generated by two Dehn twists: a twist on $S^1xi/\kappa x\kappa$ and a twist on $ixS^1/\kappa x\kappa$, see [2]. A twist on $S^1xi/\kappa x\kappa$ lifts to a twist on S^1xi together with a twist on $S^1x{-i}$ in the "same" direction. With respect to the basis $[S^1x1]$, $[1xS^1]$ of $H_1(T^2)$ the lifted twist has matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$). Hence if d keeps all four points fixed (and therefore **Fix** is four 1-spheres) then the matrix of d is in the subgroup K of $GL_2(\mathbb{Z})$.

In general d induces a permutation on the four points $\pm 1x\pm 1$. Label these points as $1=1x1$, $2=-1x1$, $3=1x-1$ and $4=-1x-1$. The number of 1-sphere fixed components of ι is the number of orbits of the permutation induced by d on $\{1,2,3,4\}$. Note that $\rho_1(z,w)=(zw,w)$ commutes with $\kappa x\kappa$, induces the permutation (34) and has matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. ρ_1 is in fact a Dehn twist. $\rho_2(z,w)=(-zw,w)$ has the same matrix and induces the permutation (12). $\rho_3(z,w)=(z,zw)$ commutes with $\kappa x\kappa$, induces the permutation (24) and has matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $\rho_4(z,w)=(z,-zw)$ has the same matrix and induces the permutation (13). Also note that the following isomorphisms commute with $\kappa x\kappa$ and have matrix the identity: αxid inducing permutation (12)(34), idxα inducing (13)(24) and αxα inducing (14)(23).

There is a composition r of these isomorphisms such that $d=d'\circ r$ and such that d' induces the identity permutation: Use $a\times id$, $id\times a$ and $a\times a$ to reduce the possibilities to permutations (34), (24), (14), (12)(34), (243), (234), (1423), (1234) and (1243). Then use the ρ's to generate these permutations. So the matrix of d is of form EP where E is the matrix of d' and hence is in K and P is of form listed in (4) - (6).

Suppose case (3) of Theorem 9.5 occurs: There is a separating torus with $\iota T=T$.

Then M is a union of two orientable twisted I-bundles W_1 and W_2 over Klein bottles. Let $d:\partial W_1 \longrightarrow \partial W_2$ be the identification. As above, by changing $\iota|W_i$ by a conjugation on W_i, we may assume that $\iota_1=\iota|W_1$ and $\iota_2=\iota|W_2$ are standard involutions. Then $\iota_2 d=d\iota_1$. Select representatives of $H_1(\partial W_1)=Z\oplus Z$ in the canonical way: (1,0) arising from a nonseparating annulus of W_1 and (0,1) from a separating annulus.

Suppose $\iota|\partial W_1$ is fixed point free. $T_1=\partial W_1/\iota_1$ is a torus. We can select a basis for T_1 such that $p_1:\partial W_1 \longrightarrow T_1$ has matrix $M(p_1)=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ if ι_1 is conjugate to k_{2S}' or k_S' and matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ if ι_1 is conjugate to k_O'. Similarly for T_2. $d:\partial W_1 \longrightarrow \partial W_2$ induces an isomorphism $q:T_1 \longrightarrow T_2$ with $p_2\circ d=q\circ p_1$. Then the matrix of d is (as a product in $GL_2(Q)$) the matrix $M(p_2)^{-1}QM(p_1)$ where Q is the matrix of q.

Conversely, up to isotopy a matrix $Q \in GL_2(Z)$ determines an isomorphism $q: T_1 \longrightarrow T_2$ and this isomorphism lifts if $M(p_2)^{-1} Q M(p_1) \in GL_2(Z)$. This gives (7) - (9).

Suppose $\iota|\partial W_1$ is not fixed point free. Then $\iota|\partial W_1$ is conjugate to $\kappa x \kappa$. ι_1 and ι_2 are conjugate to $k_{2C}' = (-\kappa) x \tau x \tau / d$ or to $k_{S2C}' = id x \tau x \tau / d$. For convenience use the representation $\kappa x \tau x id/d$ for k_{S2C}' instead.

For both k_{2C}' and k_{S2C}': $S_1 = S^1 x 0 x 1$ is invariant, meets both fixed 1-cells and represents $(1,0) \in H_1(\partial W_1)$, that is, is a boundary component of a nonseparating annulus. For k_{2C}': $S_2 = 1 x 1 x I \sqcup 1 x -1 x I$ is invariant, meets both fixed 1-cells and represents (0,1). For k_{S2C}': $S_2 = i x 1 x I \sqcup -i x -1 x I$ is invariant, meets only one fixed 1-cell and represents (0,1). The curves S_1 and S_2 give a way to assign labels 1, 2, 3 and 4 to the four points of $\mathbf{Fix} \sqcap \partial W_1$ (e.g. 1 to $S_1 \sqcap S_2$). The fixed sets of k_{2C}' and k_{S2C}' match these labels in a different way. The matching can be arranged to occur in the same way if a twist on S_2 is performed in the k_{S2C}' case. The matching and d determine the number of components of **Fix**.

Proceed as in the case above where T was a nonseparating torus with $\iota T = T$. In the above listing (12) and (13) arise from combining a k_{2C}' and a k_{S2C}'. (11) and (12) arise from combining a k_{2C}' and a k_{2C}' or from combining a k_{S2C}' and a k_{S2C}'.

QED

Let M be an orientable torus bundle over S^1 with involution ι. Call ι **fiber preserving** if there is a fibration $p:M \longrightarrow S^1$ such that $\iota(p^{-1}(x))=p^{-1}(\iota(x))$ for all $x \in S^1$ and if **Fix**=Fix(ι) is transversal to each fiber $p^{-1}(x)$.

Note the involution $\iota(x,y,z)=(y,x,-z)$ on $S^1 x S^1 x S^1$ is not fiber preserving, in this sense, with respect to fibration obtained by projection to the last coordinate.

Corollary 9.10

Let M be an orientable torus bundle over S^1. Then M has a fiber preserving involution with fixed set exactly n 1-spheres $(n>0)$ if and only if the matrix of M is conjugate to one of (4), (5) or (6) of Theorem 9.8.

Proof: Let $x \in S^1$ be such that the fiber $T=p^{-1}(x)$ meets the fixed set. Then $\iota T=T$. Since T and **Fix** are transversal the collar of T is not interchanged. So T satisfies case (2) of Theorem 9.5. Now follow the proof of Theorem 9.8.

QED

§10. Involutions With 2-Dimensional Fixed Sets

In the previous section the space forms M_1, M_2 and M_6 were defined. Recall also Definition 3.1. Define the following involutions on these spaces.

On M_1:

$\alpha_{2T}=id\times\kappa\times id/d$ having fixed set two tori $S^1\times\pm1\times I/d$

$\alpha_T=\omega\times id/d$ having fixed set the nonseparating torus $\{(z,z)\mid z\in C\}\times I/d$

On M_2:

$\beta_{2K}=id\times\kappa\times id/d$ having fixed set two Klein bottles $S^1\times\pm1\times I/d$

$\beta_K=\omega\times id/d$ having fixed set the Klein bottle $\{(z,z)\mid z\in C\}\times I/d$

$\beta_T=id\times(-\kappa)\times id/d$ having fixed set separating torus $S^1\times\pm i\times I/d$

$\beta_{T4P}=\kappa\times\kappa\times\tau/d$ having fixed set a torus and four points $(S^1\times S^1\times -1\sqcup\pm1\times\pm1\times0)/d$

On M_6:

$\gamma_{K2P}=(-\kappa)\times\alpha\times id/d$ having fixed set a Klein bottle and two points $(S^1\times S^1\times1\sqcup1\times\pm1\times-1)/d$

Here d denotes the identification for the corresponding space form M_i.

Lemma 10.1

Let M be an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle. Let ι be an involution with fixed set containing a Klein bottle. Then there is a separating incompressible torus T with $\iota T=T$ and $T \sqcap \mathbf{Fix}=\emptyset$.

Proof: By Lemma 6.1 the boundary of a regular neighborhood of the Klein bottle K is an incompressible torus. Since K is fixed under ι, arrange that the regular neighborhood is invariant under ι and that is meets **Fix** only at K. Then let T be the boundary of this regular neighborhood.

QED

Theorem 10.2

Let M be an orientable torus bundle over S^1 or a union of orientable twisted I-bundles over a Klein bottle. Let ι be an involution on M with a 2-dimensional fixed set component. Then M is isomorphic to M_1, M_2 or M_6 and ι is conjugate to one of the seven involutions defined above.

Proof: We apply the Torus Theorem 6.2.

Some economy could be acheived by showing that the fixed set contains an incompressible torus or Klein bottle F. When F is a torus, F could then be used to construct a torus T with $\iota T \sqcap T=\emptyset$. The present approach has the advantage that it generalized to the case of orientation reversing involutions. It also parallels the proof of Theorem 7.3.

M is orientable so case (IV) of the Torus Theorem 6.2 does not arise and 2-sided Klein bottles do not occur. Cases (II) and (III) do not arise since ι is orientation reversing. So there is an incompressible torus T with either $T \sqcap \iota T=\emptyset$ or $\iota T=T$ and T and **Fix** transversal.

By Proposition 9.4 there are three cases:

Case 1) $\iota T \sqcap T=\emptyset$ and T does not separate.

Case 2) $\iota T=T$, T does not separate and the bicollar of T is not interchanged.

Case 3) $\iota T=T$ and T separates.

By the last lemma we may assume T does not meet any Klein bottle components of **Fix**.

We show ι is conjugate to:

in case 1) α_{2T}, α_T, or β_{T4P}.

in case 2) α_T if $T \sqcap$**Fix** is one 1-sphere,

α_{2T}, α_T, or β_T otherwise.

in case 3) β_{2K}, β_K, β_T, β_{T4P} or γ_{K2P}.

Several of the standard involutions are listed in more than one case. Each standard involution can arise in the case it is listed under. Namely consider T=: $S^1 x i x I$, $\{(x,ix,t):x,t\}$, $S^1 x S^1 x(1/2)$, $S^1 x S^1 x 0$, $S^1 x S^1 x 0$,

$\{(x,\bar{x},t):x,t\}$, S^1xS^1x0, $S^1x\pm ixI$, $\{(x,\pm ix,t);x,t\}$, $\pm ixS^1xI$, $S^1x\pm ixI$ and S^1xS^1x0, respectively.

It suffices to show ι has the same fixed set as a standard involution listed under the corresponding case, and that if ι and ι' have isomorphic fixed sets and fall into the same case 1)- 3) then they are conjugate.

We make use of Lemmas 9.1 and 9.2, Corollary 7.4 and Corollary 8.4.

Case 1) $\iota T \sqcap T=\emptyset$ and T does not separate.

$\iota T \sqcup T$ decomposes M into two components $W_i \cong T^2xI$ with $\partial W_i=T \sqcup \iota T$. Since there is a fixed set, $\iota W_i=W_i$. The boundary components of W_i are interchanged so $\iota|(W_i)$ is conjugate to k_T, k_{4P} or k_{NI}. Since ι has a 2-dimensional fixed set assume $\iota|(W_1)$ is conjugate to k_T. Fixed sets of the same type as for the standard involutions α_{2T}, α_T, or β_{T4P} are obtained. Given ι' with isomorphic fixed set, let $h:W_2 \longrightarrow W_2'$ be a conjugation between $\iota|(W_2)$ and $\iota|(W_2')$. Then $(h|):\partial W_1=\partial W_2 \longrightarrow \partial W_2'=\partial W_1'$ is a conjugation which by the conjugation extendable property for k_T extends to a conjugation $h:W_1 \longrightarrow W_1'$. Hence ι and ι' are conjugate.

Case 2) $\iota T=T$, T does not separate and ι does not interchange the collar.

Then M is isomorphic to W/d where $W=T^2xI$ and $d:T^2x1 \longrightarrow T^2x-1$ is an isomorphism. ι is induced by an involution λ on W that does not interchange the components of ∂W. Since λ has a two dimensional fixed component, λ is conjugate to k_A or k_{2A}.

Case 2.1) λ is conjugate to k_A. Then $\partial Fix(\lambda)=S_1 \sqcup S_2$ has two components so $T \sqcap \mathbf{Fix}$ has one component. Orient S_1. Then annulus $Fix(\lambda)$ induces an orientation on S_2. If $d_*[S_1]=-[S_2]$ then **Fix** is a Klein bottle meeting T. So $d_*[S_1]=[S_2]$ and **Fix** is a torus. Let ι' be conjugate to ι and assume ι' falls into this case also. Construct $W'/d' \cong M'$ as above. Construct a conjugation $T^2\mathbf{x}1 \longrightarrow (T^2\mathbf{x}1)'$ between $\lambda|(T^2\mathbf{x}1)$ and $\lambda'|(T^2\mathbf{x}1)'$. Extend to a conjugation $h:\partial W \longrightarrow \partial W'$ by defining $h(T^2\mathbf{x}-1)=d'\circ h\circ(d^{-1})$. Since **Fix** is a torus in an orientable manifold, h extends over a bicollar of **Fix**. Then the conjugation extendable property of k_A shows h extends to a conjugation $h:W \longrightarrow W'$ between λ and λ'. h induces a conjugation between ι and ι'

Case 2.2) λ is conjugate to k_{2A}. Then $Fix(\lambda)$ has two annular components A_1 and A_2 and **Fix** meets T in two 1-spheres. Let $S_{ij}=T^2\mathbf{x}(-1)^i \sqcap A_j$. Pick orientations so that each represents the same element of $H_1(W)$. $d:T^2\mathbf{x}1 \longrightarrow T^2\mathbf{x}-1$ is orientation preserving and must take S_{11} to S_{21} or S_{22}. There are four subcases:

1) $d(S_{11})=S_{21}$ and $d_*[S_{11}]=[S_{21}]$. Then **Fix** is two tori.

2) $d(S_{11})=S_{21}$ and $d_*[S_{11}]=-[S_{21}]$. Then **Fix** is two Klein bottles. These meet T so this case does not occur.

3) $d(S_{11})=S_{22}$ and $d_*[S_{11}]=[S_{22}]$. Then **Fix** is a

nonseparating torus since d must interchange the components of W-**Fix**.

4) $d(S_{11})=S_{22}$ and $d_*[S_{11}]=-[S_{22}]$. Then **Fix** is one separating torus.

If ι' is also given then it will fall into the same one of these subcases. So a conjugation can be constructed between λ and λ' as in Case 2.1.

Case 3) $\iota T=T$, T separates. Then $M=W_1 \sqcup W_2$ with $T=W_1 \sqcap W_2$ where W_i are orientable twisted I-bundles over a Klein bottle. We have $\iota W_1=W_1$ and $\iota|(W_i)$ is orientation reversing. So $\lambda_i=\iota|W_i$ is conjugate to k_K, k_{2M}, k_A, k_{A2P}, k_{2P} or k_N. Since $\iota|\partial W_1=\iota|\partial W_2$ it follows:

Case 3.1) Both λ_1 and λ_2 are conjugate to k_K, k_{2P} or k_N

Case 3.2) Both λ_1 and λ_2 are conjugate to k_{2M}, k_A or k_{A2P}.

In case 3.1) by symmetry assume λ_1 is conjugate to k_K. **Fix** is two Klein bottles, a Klein bottle and two points, or just one Klein bottle. If ι' is also given construct a conjugation by taking any conjugation between λ_2 and λ_2' and extending to W_1 using the conjugation extendable property of k_K.

In case 3.2) the fixed set always intersects T. By construction we avoided Klein bottles meeting T. Therefore λ_i is not conjugate to k_{2M}.

Next, it is possible that λ_1 is conjugate to k_A and λ_2 is conjugate to k_{A2P}. Let $\mathbf{Fix} \sqcap T = S_1 \sqcup S_2$. Torus T induces an orientation on S_2 once an orientation on S_1 is fixed. In k_A the annular fixed set induces the same orientation on S_2 as T does. In k_{A2P} the annular fixed set induces an orientation on S_2 opposite to the one T induces. Therefore T would be a Klein bottle.

Hence we have λ_1 and λ_2 are conjugate to k_A and the fixed set is a torus or λ_1 and λ_2 are conjugate to k_{A2P} and the fixed set is a torus and four points. If ι' is given it is easy to construct a conjugation between ι' and ι since k_A and k_{A2P} have the conjugation extendable property.

QED

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