ALGEBRAIC MONOIDS

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Abstract

Definition: Let k be an algebraically closed field. An algebraic monoid is a triple $(E,m,1)$ such that E is an algebraic variety defined over k, $m : E \times E \longrightarrow E$ is an associative morphism and $1 \in E$ is a two-sided unit for m.

The object of this thesis is to expose several fundamental topics in the theory of algebraic monoids. My results may be divided into three types; general theory of irreducible affine monoids, structure and classification of semi-simple rank one reductive monoids, and theory of general monoid varieties (not necessarily affine).

I General Theory of Affine Monoids

(3.3.6) Existence of Affine Algebraic Monoids

Let G be an irreducible affine algebraic group. Then the following are equivalent.

(i) There exists an irreducible algebraic monoid E such that $G(E) = \{ g \in E \mid g^{-1} \in E \}$ is isomorphic to G and E does not consist entirely of units.

(ii) There exists an irreducible algebraic monoid E such that $G = G(E)$ and $0 \in E$ $(0 \neq 1)$.

(iii) $X(G) = \mathrm{Hom}(G,k^*)$ is a non-trivial abelian group.

(iv) rank $R(G) > 0$, where $R(G)$ is the solvable radical of G.

(4.2.6) Nilpotent Algebraic Monoids

Let E be a nilpotent irreducible algebraic monoid. Then the following are equivalent.

(i) For all $x \in E$ there exists an idempotent $e \in E$ and an element $x^* \in E$, such that $xx^* = e = x^*x$, $ex = xe$ and $ex^* = x^*e$.

(ii) The morphism $m : G(u) x E(s) \longrightarrow E$, $(u,s) \longrightarrow us$, is finite and dominant, where G(u) is the closed subgroup of unipotent elements and E(s) is the closed submonoid of semi-simple elements.

In case E is also a normal variety, m is an isomorphism.

(4.4.14) Reductive and Regular Monoids

(a) Let E be a reductive algebraic monoid. Then E is regular. i.e. For all $x \in E$ there exists $g \in G(E)$ such that $gx = e$ is an idempotent.

(b) Let E be an irreducible algebraic monoid with 0. Then the following are equivalent.

(i) E is regular.

(ii) E is reductive.

(iii) E has no non-trivial nilpotent ideals.

(4.5.2) Connected Algebraic Monoids

Let E be an algebraic monoid with 0. Then the following are equivalent.

(i) E is connected in the Zariski topology.

(ii) There exist idempotents $1, e(1), e(2), \ldots, e(m) = 0$ such that $e(i) > e(i+1)$ for all $i = 0,\ldots,m-1$ and $e(i+1) \in e(i)Ee(i)^0$ (the irreducible component of 1) for all i.

If, in case (ii) we require that each idempotent be minimal, then the number m is uniquely determined and each e(i)Ee(i) is uniquely determined up to isomorphism.

(5.2.1) Structure of Prime Ideals

Let E be irreducible and affine. A prime ideal P of E, is a non-empty subset of of E such that EPE is a subset of P and E − P is multiplicatively closed.

(i) Suppose P is a prime ideal of E. Then there exists a morphism x : E ——> k such that $P = x^{-1}(0)$.

(ii) Let T be a maximal torus of G(E), X its closure in E and W the Weyl group of T. Then there are canonical bijections

primes(E) <——> W-inv.primes(X) <——> W-inv.idempotents(X)

II Reductive Monoids of Semi-simple Rank One

(*) Let E be an irreducible, reductive affine algebraic monoid with 0 such that dim ZG(E) = 1 and rkss G(E) = 1. The restriction on the center is required to avoid the relative arbitrariness of D-monoids.

1. Geometric Structure

(7.2.3) The action GxGxE ——> E, (g,h,x) ——> gxh^{-1} has three orbits, G(E), (E − G) − { 0 } and { 0 }.

(7.4.4) If E is also normal then E is a Cohen-Macaulay algebraic variety.

2. Classification

(7.5.17) Classification I

Let E be as in (*) above. Then G(E) is isomorphic with one of $Gl(k^2)$, $Sl(k^2)xk^*$ or $PGl(k^2)xk^*$. Let G be one of these groups and let Q^+ denote the set of positive rational numbers. Then there is a canonical bijection

$$Q^+ \longleftrightarrow E(G) = \{ E \mid E \text{ as in } (*),\ E \text{ normal},\ G(E) = G \}.$$

For $G = Gl(k^2)$ the correspondence is as follows. Given E, there is a unique bicartesian diagram,

$$\begin{tikzcd} E(m^{-1}) \arrow[r, "\beta"] \arrow[d] & End(k^2) \arrow[d, "\alpha"] \\ E \arrow[r] & E(n) \end{tikzcd}$$

such that all morphisms are finite and dominant. If degree $\alpha = n$ is odd then degree(β) = m is odd and (m,n) = 1. If degree(α) = 2n is even then degree(β) = 2m is even and (m,n) = 1 (and one of m and n is even). In any case, the map $E(G) \longrightarrow Q^+$ given by

$$E \longrightarrow \deg(\alpha)/\deg(\beta)$$

is well defined and one–to–one.

Conversely, if $r \in Q^+$ then r = m/n, where m,n > 0 and (m,n) = 1. It is then possible to construct a bicartesian diagram as above such that deg(α) = n and deg(β) = m if mn is odd, or deg(α) = 2n and deg(β) = 2m if mn is even. Thus we obtain the inverse map $Q^+ \longrightarrow E(G)$,

$$r \longrightarrow E(r).$$

All normal monoids with group $Sl(k^2)xk^*$ are constructed from the monoids with group $Gl(k^2)$ using integral closure and the morphism

$$m : Sl(k^2)xk^* \longrightarrow Gl(k^2),\ m(x,t) = xt.$$

All normal monoids with group $PGl(k^2)xk^*$ are constructed from the monoids with group $Gl(k^2)$ using finite group scheme quotients and the morphism

$$c : Gl(k^2) \longrightarrow PGl(k^2) \times k^*, \quad c(x) = ([x], \det(x)).$$

(7.6.5) Classification II

Let E as in (*) above, be normal. Let T be a maximal torus and let Φ be the roots of the adjoint representation. Let Z denote the Zariski closure of T in E. From general principles (4.1.7 of the text) there exists $F = \{ F(1), F(2) \}$ contained in $X(Z)$ such that $\langle F(1), F(2) \rangle \longrightarrow X(Z)$ is a finite, dominant morphism and each $F(i)$ is non-zero modulo the square of the maximal ideal of functions that vanish at zero. F is called the set of fundamental generators of $X(Z)$.

$(X(T), \Phi(T), F(E))$ is the polyhedral root system of the pair (E,T).

E is uniquely determined up to isomorphism by its polyhedral root system.

The following is a list of all possible polyhedral root systems (X, Φ, F) for the various groups G ((u, v) denotes the free abelian group on u and v, written additively).

(i) $G = Gl(k^2)$

$X = (u, v)$	$\alpha, \beta \in Z$
$\Phi = \{ u-v, v-u \}$	$\alpha > \lvert\beta\rvert \geq 0$
$F = \{ \alpha u+\beta v, \alpha v+\beta u \}$	$(\alpha, \beta) = 1$

(ii) <u>$G = Sl(k^2) \times k^*$</u>

$X = (a, b)$ $\qquad$ $m,n \in N$

$\Phi = \{ 2b, -2b \}$ $\qquad$ $m,n > 0$

$F = \{ ma+nb, ma-nb \}$ $\qquad$ $(m,n) = 1$

(iii) <u>$G = PGl(k^2) \times k^*$</u>

$X = (x, y)$ $\qquad$ $m,n \in N$

$\Phi = \{ y, -y \}$ $\qquad$ $m,n > 0$

$F = \{ mx+ny, mx-ny \}$ $\qquad$ $(m,n) = 1$

III <u>General Monoid Varieties</u>

Let E be an irreducible monoid variety (not necessarily affine).

(8.1.4) If E is quasi-affine then E is affine.

(8.2.3) If E is projective then E is an abelian variety.

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INTRODUCTION

An algebraic monoid E is an algebraic variety which also has the structure of a monoid $m : E\times E \longrightarrow E$, in such a way that m is a morphism of varieties. Well known examples include the monoids associated with finite dimensional associative algebras. The most familiar non-linear example is surely the cusp,

$$\{ (x,y) \in k^2 \mid x^2=y^3 \}.$$

Algebraic monoids also arise in many other contexts. For example, if V is a finite dimensional vector space over the field k and $f:V\bullet V \longrightarrow V$ is a linear map, then $\{ t \in End(k) \mid f(t(v)\bullet t(w))=t(f(v\bullet w))$ for all $v,w \in k \}$ is an algebraic submonoid of End(V) ('$\bullet$' denotes 'tensor product of vector spaces').

At this time there is no comprehensive theory of algebraic monoids nor is there an established paradigm as to what should be the aims of the theory.

Toric monoids have been discussed in [12] and [14] mainly from the point of view of modern algebraic geometry. In [7] affine monoids are briefly encountered as part of the comprehensive introduction to algebraic group theory. Aside from the copious work of M. Putcha [18–26] very little has been done from the point of view of modern semi-group theory. Truly there is no fully developed ideal standard of structure in the theory of monoids.

There are now available several books on algebraic group theory which have properly distilled the necessary prerequisites so as to make the theory available to a wide audience. J. Humphreys' book [13] is a complete introduction to the linear theory over an algebraically closed field and more recently W. Waterhouse [32] has written a coherent introductory text on the three basic approaches to this theory. It is always a source of clarity and depth to keep in mind the interplay (and equivalences) among, Hopf algebras, linear groups and group valued functors.

For example, let $G=k^*$ be the multiplicative group of units of k.

As a linear group we have,

$G=Gl(1)$.

As a Hopf algebra we have,

$k[G]=k[T,T^{-1}]$

$i(T)=T^{-1}$

$d(T)=T\bullet T$

$e(T)=1$.

As a group valued functor we have,

$G(R)=U(R)$,

the units of R, where R is any k–algebra.

Generally speaking the first two viewpoints are equivalent by Hilbert's zeros theorem and the last two are equivalent by Yoneda's lemma.

This correspondence carries over to the case of monoids as well. For example, let $E=End(k)$.

As a linear monoid

$E=End(k)$.

As a bigebra

$k[E]=k[T]$

$d(T)=T\bullet T$

$e(T)=1$.

As a monoid valued functor we have

$E(R)=R$, with the obvious multiplication.

Plainly the only logical difference from linear groups is the absence of inverses.

There is a motif in the theory of algebraic groups which is of fundamental importance for monoids as well. Stated as a problem for monoids this is as follows.

(I) Let E be an irreducible algebraic monoid and let G be its group of units. Assume G is reductive. Let X be the closure in E of a maximal torus T of G and let W be the Weyl group associated with T. To what extent can the structure of E be determined by X and

$$\mathrm{int} : W \longrightarrow \mathrm{Aut}(X)\ ?$$

In particulars,

(i) Can E be determined by (W,X)?

(ii) Are there axioms which characterize the class of D–monoids that are obtained from reductive monoids in this fashion? (One should keep in mind the overwhelming success of <u>root systems</u> in both modern and classical Lie theory, and in algebraic group theory).

The list can be continued. Of course there are other important problems which are not completely encompassed by the above motif. The most important of these, in my opinion, is the following.

(II) Find a complete list of all algebraic monoids E such that E is <u>irreducible, reductive, normal,</u> and <u>rk(G)=rkss(G) + 1.</u>

The condition, rk(G)=rkss(G) + 1, is precisely what is needed to avoid the relative arbitrariness of central D-monoids.

My thesis is guided almost entirely by problems (I) and (II) above. The main results, which are fully exposed in chapters 4 and 7, provide ample evidence that these problems are of fundamental importance in the theory of algebraic monoids. The most significant result of chapter 4 asserts that all irreducible monoids are regular in the sense of von Neumann. As a direct consequence, problem (II) above is solved completely in case rkss(G(E)) = 1.

Chapter 1 contains prelimimary information from algebraic geometry and algebraic group theory. It is offered partly to indicate the level of discourse and also to deduce some preliminary results concerning D-groups.

Chapter 2 introduces the general theory and the basic notation. It is proved that any non-trivial monoid possesses an abundance of idempotents. Thus, a multiplicative Jordan decomposition is possible for many non-units in the monoid E. In particular, the subset of semi-simple elements is well-defined and non-trivial.

In chapter 3 irreducible monoids are discussed. The groups G which occur non-trivially as the group of units of an irreducible monoid are characterized (see Theorem 3.3.6). The remainder of the chapter is more technical and mainly concerned with closure properties of irreducible monoids. Let E be irreducible, and T a maximal torus of G, the group of units of E. Let B be a Borel subgroup of G, B(G) = { B | B is a Borel subgroup of G }, and $e \in E$ an idempotent. The most important background result in this regard (due to M. Putcha) asserts that eEe is in the closure of the centralizer in G of e. From this it follows that E(s) is equal to the union of the gXg^{-1} as g varies over G, where E(s) is the set of semi-simple elements of E and X is the closure of T in E. Analagously, it is proved that E is the union of the gZg^{-1} where Z is the closure of B in E. Thus every element is in the closure of a Borel subgroup. Furthermore, B(x) = { $B \in B(G)$ | x is contained in the closure of B } is closed in B(G). Thus Borel's fixed point theorem can be applied (to B(x)) to prove that if G is reductive then C(T)=X.

Chapter 4 is dedicated to the special properties of the five basic types of monoids. These are D-monoids, nilpotent monoids, solvable monoids regular monoids, and reductive monoids.

Let E be an irreducible D-monoid. Then k[E] = k[X(E)], the monoid algebra of X(E) = Hom(E,k) (this is one standard definition of D-monoids). If E is normal and $0 \in E$ there exists { F(i) | i=1,...,n } contained in X(E) such that <F(1),...,F(n)> ——> X(E) is finite and dominant, n is the number of minimal non-zero idempotents, and { F(i) } is a linearly independent subset modulo m^2, where m is the maximal ideal of functions which vanish at 0. Furthermore { F(i) } is the only such subset. { F(i) } is called the set of <u>fundamental generators</u> of X(E). In the discussion of semi-simple, rank one, monoids it is seen that the fundamental generators are precisely what is needed to synthesise with the root system, in order to classify these monoids in the spirit of classical Lie theory.

Let E be an irreducible nilpotent monoid. It is interesting to know the conditions under which the well-known structure theorem for nilpotent groups can be generalized to monoids. If G is an irreducible nilpotent algebraic group the theorem asserts that $m : G(u)xG(s) \longrightarrow G$ is an isomorphism where G(u) is the subgroup of unipotent elements and G(s) is the subgroup of semi-simple elements. This theorem generalizes to the nilpotent monoid E if and only if E is a Clifford monoid (see Theorem 4.2.6).

Solvable irreducible monoids are important generally because of their relative simplicity combined with the fact that every irreducible monoid is the union of its solvable irreducible submonoids. The main result of this section is a characterization of solvable monoids among irreducible monoids with 0. Let E be an irreducible monoid with 0. Then E is solvable if and only if its subset of nilpotent elements is a two-sided ideal. This result is originally due to M. Putcha. My proof is slightly different, making use of the universal D-monoid associated with a given solvable monoid.

Reductive algebraic monoids are perhaps the most important of all monoids. An irreducible monoid E is reductive if G, its group of units is a reductive group. My preliminary discussion of reductive monoids is mainly concerned with class functions, semi-simple elements and conjugacy classes. Let E be reductive and suppose x is an element of E. Then,

(i) x is semi-simple if and only if the conjugacy class of x is closed in E.

(ii) If T is a maximal torus of G then the centralizer of T in E is equal to the closure of T in E.

(iii) There is a one-to-one correspondence between semi-simple conjugacy classes and orbits of the Weyl group action on the closure of a maximal torus.

(iv) $cl[E] \longrightarrow k[E] \longrightarrow\!\!\!> k[X]$ identifies $cl[E] = \{ f \in k[E] \mid f(xy) = f(yx) \text{ for all } x,y \in E \}$ with the ring of invariant functions of $k[X]$ under the induced action of the Weyl group (X is the closure of a maximal torus).

(v) If E is any irreducible monoid then there exists a morphism $p : E \longrightarrow E'$ such that

(α) p is dominant.

(β) The kernel of p is the unipotent radical of G.

(ρ) If X and X' are the closures of respective maximal tori of E and E' such that $p(X) = X'$ then $p : X \longrightarrow X'$ is an isomorphism.

Thus every irreducible monoid E maps to a reductive monoid E' so as to preserve as much of the original structure as one could possibly hope for.

This construction has two consequences recorded in the thesis.

(1) If E is a von Neumann regular monoid with 0, then E is reductive.

(2) If Q is a prime ideal of E then Q is the inverse image of some prime ideal of E' ($p : E \longrightarrow E'$ as above).

In my proof of the structure theorem for prime ideals

(Theorem 5.2.1), the most important step is a synthesis of this second result and a result on class functions.

The final result of this section is the most significant structure theorem of the thesis. Generalizing a theorem of M. Putcha we find that all reductive monoids are regular in the sense of von–Neumann (see Theorems 4.4.14 and 4.4.15). The structure theory of chapter 7 is a direct consequence.

I have concluded chapter 4 with a short discusion of connected monoids with 0. A monoid is connected if it is connected in the Zariski topology. Using the general theory of chapters 2 and 3, I have obtained the following result (Theorem 4.5.2). A monoid E with 0 is connected if and only if there exists a chain of idempotents $1 = e(0) > e(1) > \ldots > e(k) = 0$ such that $e(i+1)$ is an element of the irreducible component of $e(i)Ee(i)$ containing $e(i)$, for all $i = 1,\ldots,k-1$. This is reinterpreted in the context of rational homotopy theory in chapter 9.

Chapter 5 is exclusively devoted to finding the prime ideals of a given irreducible monoid E in terms of a maximal irreducible D–submonoid. Let E be an irreducible monoid and P be a prime ideal of E. Let T be a maximal torus and let X be its closure in E. Let P(T) be the intersection of P and X. P(T) is a prime ideal of X, invariant under the Weyl group W. Thus, we can construct a W–invariant character c of X such that $c^{-1}(0) = P(T)$. Using the results on class functions there exists a character q <u>on E</u> such that $P = q^{-1}(0)$. Thus, the map $P \longrightarrow P(T)$ induces a one–to–one correspondence between the prime ideals of E and the W–invariant prime ideals of X (see Theorem

5.2.1).

Chapter 6 begins the descent towards the classification and structure theory of semi-simple rank one monoids. The structure theory requires a classification of two-dimensional non-commutative monoids without 0. There are two types. Let k^* be a one-parameter muultiplicative subgroup of G. Then there exists either

$p : k \times E \longrightarrow E$ extending

$k^* \times E \longrightarrow E$, $(t,x) \longrightarrow txt^{-1}$, or

$p : k \times E \longrightarrow E$ extending

$k^* \times E \longrightarrow E$, $(t,x) \longrightarrow t^{-1}xt$,

depending on whether $gxh = gx$ or $gxh = xh$ for all x in E and all g,h in G.

Chapter 7 contains the main computations of my thesis. All reductive, normal semi-simple rank one monoids E, with one-dimensional center are classified in two ways. Using the results of chapter 4 and some representation theory, I construct an essentially unique bicartesian diagram,

$$\begin{array}{ccc} E(2) & \longrightarrow & E(1) \\ \downarrow & & \downarrow \\ E & \longrightarrow & E(3) \end{array}$$

such that all morphisms are finite and dominant, and have linearly reductive kernels (possibly non-reduced). Since all the morphisms can be determined numerically, and E(1) is the monoid of 2-by-2 matrices, the classification follows (see Theorem 7.6.17).

The second classification theorem is established in the proof of the first one. This is accomplished by following the relevant data (roots and fundamental generators) around the diagram from E(1) to E. It is my belief that a classification in the spirit of classical Lie theory should be possible for reductive algebraic monoids with one-dimensional center. The second classification theorem is offered in accordance with this belief. If E is reductive normal and has a 0 and a one-dimensional center then E is uniquely determined by its poyhedral root system $(X(E),\Phi(E),F(E))$ in case the semi-simple rank is one. A statement of this theorem and a list of all the polyhedral root systems is recorded in 7.7.5.

One corollary of the bicartesian diagram above is the following. If E is as above then E is Cohen-Macaulay. This proof also requires the generalization of a theorem of P. Roberts [28] that I have established in the latter part of chapter 1. Hochster has proven that if E is an irreducible normal D-monoid then E is Cohen-Macaulay. The extent to which this result can be generalized to irreducible monoids is not known.

Chapter 8 is a short discussion dedicated to general monoid varieties (not necessarily affine). A well known structure theorem of C. Chevalley asserts that if G is a smooth algebraic group then there exists a unique affine algebraic subgroup N of G such that G/N is an abelian variety. It is not known whether this result extends to algebraic monoid varieties. For example, if G(E) is affine, is E affine?

I have considered two special cases, irreducible quasi-affine monoids and irreducible projective monoids. If E is quasi-affine it is possible to imbed E as an open sub-monoid of some irreducible affine monoid E', $E \longrightarrow E'$. Thus $E'-E$ is a prime ideal of E'. It follows from the results of chapter 5 that E'−E is a principle divisor. Thus, E is actually affine (see Theorem 8.1.4). Using the completeness property of projective varieties we see that all projective irreducible monoids are abelian varieties (see Theorem 8.2.3).

In chapter 9, the final chapter, I have discussed a problem which has its origins in rational homotopy theory. A rational homotopy type may be regarded, by Sullivan's theory [30], as a differential graded algebra M, defined over Q, which is minimal, and free as a graded algebra. The problem I have considered is a special case of, "To what extent is the structure of M influenced by its algebraic monoid of endomorphisms?". I have discussed this problem in a more general context so as to abstract from the peculiarities of rational homotopy theory. Let V(k) be the category of vector spaces over k and let S be an 'algebraic structure' on V(k). An <u>algebraic structure</u> a, is a rule (or functor) which associates with the vector space V, a collection of linear transformations $\{ a(s) \mid s \in S \} = a(S)$ in the union of the Hom(V(m),V(n)) (as m and n vary) satisfying various relations depending on S (V(m) denotes the tensor product of V with itself m times). The collection of pairs (V,a) are the objects of a category $\Omega(k,S)$. If V and W are objects of $\Omega(k,S)$ then $\mathrm{Hom}'(V,W) = \{ f \in \mathrm{Hom}(V,W) \mid a(s)\circ f(m) = f(n)\circ a(s)$ for all s in S $\}$. Assume $\Omega(k,S)$ has a zero object and each V in $\Omega(k,S)$ satisfies suitable finiteness conditions. Then End'(V) is an algebraic monoid and furthermore $0 \in \mathrm{End}'(V)$.

<u>Definition:</u> Let V be an object of $\Omega(k,S)$. Then V has <u>positive weights</u> if 0 is in the Zariski closure of Aut(V) in End(V).

Equivalently, there exists a 1-p.s.g. $t : k^* \longrightarrow Aut(V)$ such that t extends to a morphism $t : k \longrightarrow End(V)$ with $t(0) = 0$.

The importance of this definition was first noticed by R. Body, R. Douglas and D. Sullivan in the context of rational homotopy theory. If X is a finite simply-connected C.W. space then M(X), the minimal model of X, has positive weights if and only if, for every prime p, there exist maps $f(i) : X \longrightarrow X$ such that the homotopy direct limit of $\{ f(i) : X \longrightarrow X \mid i \in N \}$ is homotopy equivalent to the p-localization of X.

R. Body and R. Douglas [2] have proven that if X is a rational homotopy type with positive weights then X satisfies uniqueness of product decompositions in the sense of Krull-Schmidt. This result has since been generalized and dualized by R. Douglas and myself [9]. In chapter 9 I have sketched the main points of this arguement in the more general context the category $\Omega(k,S)$.

In the last section of chapter 9 I have discussed connected algebraic monoids in the context of rational homotopy. The characterization of connected monoids with 0 in chapter 4 fits in neatly with the main theorem of my master's thesis [27;Theorem 3.6.2]. Let M be a minimal algebra. Then End(M) is connected in the Zariski topology if and only if there is a chain $1 = e(0) > e(1) > \ldots > e(k) = 0$ of idempotents in End(M) such that $e(i+1)$ is in the closure of $Aut(e(i)(M))$ for $i = 1,\ldots,k-1$.

I PRELIMINARIES

The theory of algebraic monoids requires much background information from algebraic geometry and algebraic group theory. In this chapter I have assembled many of the prerequisite concepts and results that are needed in subsequent chapters. Occasionaly I have proven a result that is only tacitly available in the literature but more often the results are stated with explicit references and no proofs.

1.1 Algebraic Geometry And Commutative Algebra

1.1.1 Dimension Theorem [13;p.30]: Suppose $f : X \longrightarrow Y$ is a dominant morphism of irreducible varieties, $r = \dim X - \dim Y$. Suppose W is a closed and irreducible subset of Y and Z is a maximal irreducible component of $f^{-1}(W)$ which dominates W. Then $\dim Z \geq \dim W + r$.

1.1.2 Zariski's main theorem [15;p.414]: Let $f : X \longrightarrow Y$ be a dominant morphism of irreducible varieties. Suppose that every fibre of f is finite. Then there exists a factorization of f, $f = f' \circ j$, where $j : X \longrightarrow Y'$ is an open imbedding and $f' : Y' \longrightarrow Y$ is a finite morphism.

Corollary: If $f : X \longrightarrow Y$, as in 1.1.2, is birational and Y is a normal variety then f is an open imbedding.

1.1.3 Separability [13;p.44]: Let $f : X \longrightarrow Y$ be a dominant morphism of irreducible varieties. For $x \in X$ let $T(x,X)$ denote the tangent space of x to X. If there is a smooth point x of X such that $y = f(x)$ is a smooth point of Y, and $df : T(x,X) \longrightarrow T(y,Y)$ is surjective, then f is separable.

1.1.4 "Nakayama's lemma": Let A be a non-negatively graded ring such that $A(0)$ is a field. Let m be the unique graded maximal ideal and suppose M is a non-negatively graded A-module. Suppose further that $\{ x(i) \mid i \in I \}$ are homogeneous elements of M which generate M/mM. Then $\{ x(i) \}$ generate M.

Proof: Let $x \in M(i)$ be of minimal degree such that x is not an element of $<x(i)>$ (the submodule generated by $\{ x(i) \}$). Now modulo m, $x = \Sigma a(i)x(i)$. So $z = x - \Sigma a(i)x(i) \in mM$. Thus $z = \Sigma m(i)z(i)$, where $m(i) \in m$ and $z(i) \in M$. But $\deg(z(i)) < \deg(z)$ for all i. So $z(j) \in <x(i)>$ for all j. Thus $z = x - \Sigma a(i)x(i) \in <x(i)>$ and so $x = z + \Sigma a(i)x(i)$ is also in $<x(i)>$.

1.1.5 Codimension 2 Lemma [10;p.239]: Suppose X is a normal affine variety and V is a closed subset of codimension larger than or equal to two. Then any morphism from X–V to an affine variety extends uniquely to X.

1.2 Linear Algebraic Groups

Throughout this section, G denotes a linear algebraic group.

1.2.1 Orbits [17;p.66]: Let $GxX \longrightarrow X$ be an action of the algebraic group G on the variety X. For $x \in X$, let $O(x) = \{ gx \mid g \in G \}$ and $G(x) = \{ g \in G \mid gx = x \}$. Then

(i) G(x) is a closed subgroup of G.

(ii) For all $x \in X$, O(x) is open in its closure.

(iii) For all $x \in X$, $\dim O(x) = \dim G - \dim G(x)$.

(iv) For all $n > 0$, $\{ x \in X \mid \dim O(x) > n \}$ is open in X.

1.2.2 Borel subgroups.

Let G be an irreducible algebraic group. A subgroup B, of G is a Borel subgroup if B is solvable and G/B is a complete variety.

(i) Suppose f and g : $G \longrightarrow H$ are morphisms of irreducible algebraic groups such that $f|B = g|B$ for some Borel subgroup B of G. Then $f = g$.

Proof: fg^{-1} is a morphism of varieties from G to H which factors through G/B. Thus, $fg^{-1}(G)$ is complete, irreducible and affine. Hence, $f = g$.

(ii) Normalizer theorem [13;p.143]: If B is a Borel subgroup of G then $N(B) = B$ ($N(B)$ is the normalizer of B in G).

(iii) Borel fixed point theorem [13;p.134]: Suppose X is a complete algebraic variety and G is a solvable irreducible algebraic group. If $GxX \longrightarrow X$ is a group action then $F(X,G)$ (the fixed point set of this action) is non-empty.

(iv) Construction [13;p.145]: Let $B(G) = \{ B \mid B \text{ is a Borel subgroup of } G \}$ and $G/B' = \{ gB' \mid g \in G \}$ where B' is some fixed Borel subgroup. Then by the normalizer theorem

$$s : B(G) \longrightarrow G/B', \; B \longrightarrow gB'$$

(where $B = gB'g^{-1}$) is well-defined and bijective.

Further,

$$\begin{array}{ccc} GxB(G) & \longrightarrow & B(G), \quad (h,B) \longrightarrow hBh^{-1} \\ {\scriptstyle 1xs}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ GxG/B' & \longrightarrow & G/B', \quad (h,gB') \longrightarrow hgB' \end{array}$$

commutes.

Conclusion: The Borel fixed point theorem applies to the action $GxB(G) \longrightarrow B(G)$ which is a priori only set-valued.

1.2.3 Closure:

(i) [29;p.68]. Let $GxX \longrightarrow X$, be a group action, V a closed subset of X and B a Borel subgroup of G such that BV is contained in V. Then GV is closed in X.

(ii) [29;p.70]. Suppose $GxX \longrightarrow X$, X is affine, and $x \in X$ satisfies $tx = x$ for all $t \in T$ a maximal torus of G. Then $O(x)$, the G-orbit of x, is closed in X.

1.2.4 Reductive and geometrically reductive groups:

G is *reductive* if every unipotent normal subgroup of G is trivial.

G is *geometrically reductive* if for every morphism $p : G \longrightarrow Gl(V)$ and every non-zero element v of V left invariant by G, there is a homogeneous polynomial function $f : V \longrightarrow k$, invariant under G such that f(v) is non-zero.

(i) [11]. If G is reductive then G is geometrically reductive.

(ii) [17;p.49]. Suppose G is reductive and X is affine. If $GxX \longrightarrow X$ then F(k[X],G) is a finitely generated k-algebra. Let $p : X \longrightarrow Y$ be the morphism induced from $F(k[X],G) \longrightarrow k[X]$ (so, k[Y] = F(k[X],G)). Then for all y in Y there is a unique closed G-orbit O(x) contained in $p^{-1}(y)$.

(iii) If $GxX \longrightarrow X$ then the union of the set of closed orbits of maximal dimension is open in X.

1.3 Finite D-group Actions

An affine group scheme is a generalized linear algebraic group. Technically, in the study of linear groups and monoids, one is often led quite naturally to consider group schemes which are not necessarily reduced. For example, in characteristic p > 0 the category of affine commutative algebraic groups is not an abelian category, but if the commutative non-reduced group schemes are allowed as well, the resulting category is abelian [32;p.127,ex.12].

In chapter 7 I have been led to consider certain morphisms $f : G \longrightarrow H$ such that kernel(f) is a (not necessarily reduced) finite D-group scheme. This will lead to an important structure theorem about semi-simple, rank one monoids.

<u>Definition:</u> An <u>affine algebraic group scheme</u> G is a representable functor from the category of k-algebras to the category of groups.

By Yoneda's lemma, all the group structures G(R) (as R varies over k-algebras), are the result of morphisms

$e : A \longrightarrow k$ (unit)

$d : A \longrightarrow A \bullet A$ (multiplication (where '$\bullet$' denotes 'tensor product over k'))

$i: A \longrightarrow A$ (inverse)

where A is the representing object for G:

$G(R) = \mathrm{Hom}(A,R)$.

The group axioms imply that

$(d \bullet 1) o d = (1 \bullet d) o d$

$((n o e) \bullet 1) o d = 1 = (1 \bullet (n o e)) o d$ and

$(i \bullet 1) o d = n o e = (1 \bullet i) o d$

where $n : k \longrightarrow A$ is the unit of the k-algebra structure on A. A is thus a Hopf algebra. If G is an affine group scheme we write $A = k[G]$ if A is the Hopf algebra representing G.

<u>Definition:</u> Let G be an affine group scheme over k, an algebraically closed field. Then G is a <u>finite D-group</u> if

(i) dim k[G] is finite.

(ii) $X(G) = \{ a \in k[G] \mid d(a) = a \bullet a \}$ is a k-linear basis of A.

(<u>Note:</u> X(G) is always a group.)

X(G) is the group of <u>characters</u> of G. Thus $k[G] = k[X(G)]$, the <u>group algebra</u> of X(G) over k.

<u>1.3.1 Finite D-group actions:</u> Let X be an affine variety defined over k. Then there is a canonical bijection between actions of

the D-group G on X, $GxX \longrightarrow X$, and direct sum decompositions $k[G] = \Sigma k[G](\alpha)$ such that $\alpha \in X(G)$ and $k[G](\alpha).k[G](\beta)$ is contained in $k[G](\alpha+\beta)$.

Proof: Let $R = k[X]$. Given $R = \Sigma R(\alpha)$ define $f : R \longrightarrow R \otimes k[G]$ as $f(x) = x \otimes \alpha$ for $x \in R(\alpha)$. Clearly this determines an action $GxX \longrightarrow X$.

Conversely, given $f : GxX \longrightarrow X$ we have $f^* : R \longrightarrow R \otimes k[G]$ such that R is $k[G]$-comodule (i.e. f^* is co-associative) and $(1 \otimes e) o f^* = 1$, where e is the augmentation on $k[G]$. One checks, using these two facts, that if $f^*(a) = \Sigma a(\alpha) \otimes \alpha$ then $a = \Sigma a(\alpha)$ and $(a(\alpha))(\beta) = a(\alpha)$ if $\alpha = \beta$ and 0 otherwise. Thus $R = \Sigma R(\alpha)$ where $R(\alpha) = \{ a \in R \mid a = a(\alpha) \}$.

The remainder of the chapter is devoted to the task of sharpening some known results (see [28]) about Cohen-Macaulay rings and finite D-group actions. I have assumed throughout that all rings are Noetherian k-algebras and that k is an algebraically closed field.

Let X be an affine scheme over k and let G be a finite D-group scheme such that $f : GxX \longrightarrow X$ is an action of G on X. For example, if X is an algebraic group and G is a closed finite D-subgroup scheme then $GxX \longrightarrow X$, $(g,x) \longrightarrow gx$, is an action of G on X. Note that such an action may be non-trivial even if $G(k)$ (the set of k-rational points) consists of only one point. Let A be the coordinate ring of X.

1.3.2 Lemma: Let X, A, f be as above and let $A(0) = \{ x \in A \mid f(x) = x \otimes 1 \}$. Then

(i) $A(0)$ is a subalgebra of A.

(ii) The inclusion, $A(0) \longrightarrow A$ is an integral extension of rings.

(iii) $A(\sigma)$ is an $A(0)$–module for all $\sigma \in X(G)$.

(iv) If A is a normal domain then so is $A(0)$.

Proof: (i) – (iii) are straightforward.

(iv). It suffices to prove that $A(0)$ is the intersection M, of K and A since A is normal (here, K is the quotient field of $A(0)$). Let L be the quotient field of A. Now M is an $X(G)$–graded subspace of A. Since $K = K(0) = L(0)$, we must have $A(0) = M$.

1.3.3 Lemma [6;ch.7,4.8]: Let $A \longrightarrow B$ be a finite extension of normal integral domains of the same dimension. Then B is a reflexive A–module.

1.3.4 Lemma[6;ch.7,4.2]: Let A be a normal integral domain and M a finitely generated reflexive A–module, such that $K \bullet M$ is isomorphic to K (where K is the quotient field of A). Then M is isomorphic to a divisorial ideal I of A (i.e. I is the intersection of height one primary ideals).

1.3.5 Lemma: Suppose A is an integral domain such that each local ring of A is a unique factorization domain. Let D be a divisorial ideal of A. Then D is a rank–one projective A–module.

Proof: Well known.

1.3.6 Lemma: Suppose A is a finitely generated k–algebra and $A(0) \longrightarrow A$ is as in lemma 1.3.2. Then $A(0)$ is finitely generated over k and $A(0) \longrightarrow A$ is a finite extension of rings.

Proof: Assume $A = k[x(1),\ldots,x(n)]$, $x(i)$ homogeneous. Then $x(i)^{**}l \in A(0)$ for all $i=1,\ldots,n$ where $l = |X(G)| = \dim(k[G])$ ('**' denotes exponentiation). Let $B = k[x(1)^{**}l,\ldots,x(n)^{**}l]$. Then $B \longrightarrow A(0) \longrightarrow A$. So $B \longrightarrow A(0)$ is finite because $B \longrightarrow A$ is so. Thus $A(0)$ is finitely generated and $A(0) \longrightarrow A$ is finite.

1.3.7 Lemma: Let G be a finite D-group, X a reduced and irreducible affine variety, and u : GxX ——> X an action of G on X. Then for all $\sigma \in X(G)$, $k[X](\sigma) \bullet K(0)$ and $K(0)$ are isomorphic as $A(0)$-modules (where $K(0)$ is the quotient field of $A(0)$).

Proof: Let $A = k[X]$. So $A = \Sigma A(\sigma)$ and without loss of generality $A(\sigma)$ is non-zero for all σ. Consider the $A(0)$-bilinear map $A(\sigma) \bullet A(-\sigma) \longrightarrow A(0)$. Since A is a domain, if $x \in A(-\sigma)$ is non-zero, then $m : A(\sigma) \longrightarrow A(0)$, $m(z) = zx$, is one-to-one and $A(0)$-linear. Thus $A(\sigma)$ is isomorphic with an ideal of $A(0)$ and hence $A(\sigma) \bullet K$ is isomorphic to K.

Definition: Let X be an algebraic variety, $\dim(X) = n$. X is Cohen-Macaulay if for all local rings $O(x)$, $x \in X$, there exists a system of parameters $\{x(1),...,x(n)\}$ of $O(x)$ which forms a regular sequence (see [12]).

1.3.8 D-group coverings: Suppose GxX ——> X is an action of the D-group G on the normal affine variety X. If X/G is smooth then X is Cohen-Macaulay.

Proof: $A = k[X]$ is normal and $A(0) = k[X/G]$ is regular. Consider the inclusion $A(0) \longrightarrow \Sigma A(\sigma)$. $A(\sigma)$ is a reflexive $A(0)$-module by 1.3.3 and $A(\sigma)$ is a rank one $A(0)$-module by 1.3.7. Thus, $A(\sigma)$ is isomorphic to a divisorial ideal by 1.3.4. Hence, $A(\sigma)$ is a rank-one projective $A(0)$-module by 1.3.5. Thus, $A(0) \longrightarrow A$ is a flat morphism. So A is Cohen-Macaulay.

1.3.9 D-group quotients: Suppose X is an irreducible affine Cohen-Macaulay variety and GxX ——> X is an action of the finite D-group on X. Then X/G is Cohen-Macaulay.

Proof: Let $A = k[X]$ and $A(0) = k[X/G]$. A is Cohen-Macaulay as an A-module, and thus as an $A(0)$-module. But A is the direct sum of

$A(0)$ and $A(+)$ as an A–module ($A(+)$ is the direct sum of the $A(\alpha)$ as α varies over all the non–trivial characters). Thus, $A(0)$ is a Cohen–Macaulay $A(0)$–module.

II AFFINE ALGEBRAIC MONOIDS

2.1 Preliminaries

2.1.1 Definition: An affine algebraic monoid E is an triple (E,m,1) such that

(i) E is an affine algebraic variety over k.

(ii) $1 \in E(k)$.

(iii) $m : E \times E \longrightarrow E$ is a morphism of algebraic varieties such that $mo(m \times 1) = mo(1 \times m)$ (associative).

(iv) If $p : E \longrightarrow E$, $p(x) = 1$, then $mo(p,1) = mo(1,p) = 1$ (two-sided unit).

In categorical terminology, an affine algebraic monoid is a representable functor from the category of affine varieties to the category of monoids. An affine variety is completely determined by its affine algebra. So we can reformulate the definition above in terms of commutative algebra.

Let E be an algebraic monoid and let A = k[E] be its coordinate ring. It follows from (iii) and (iv) above that if $d = m^*$, $e : \{1\} \longrightarrow E$ is the inclusion, and $n : k \longrightarrow k[E]$ is the unit of the algebra structure, then

$$(d \bullet 1)od = (1 \bullet d)od \text{ and}$$

$$1 = (noe,1)od = (1,noe)od.$$

A is thus an augmented k-bigebra.

A morphism f of algebraic monoids $f : E \longrightarrow E'$ is a morphism of algebraic varieties such that $fom = m'o(f \times f)$, where m and m' are the multiplications on E and E' respectively, and $f(1) = 1$. Unlike the case for groups the last condition does not follow from the first unless f is dominant.

2.1.2 Translation of functions

Definition: Let E be an algebraic monoid. A rational E-module (V,p) is a morphism of monoids $p : E \longrightarrow End(V)$ such that

(i) For all $v \in V$ there exists a finite dimensional subspace $V(v)$ of V such that $v \in V(v)$ and $p(x)(V(v))$ is contained in $V(v)$ for all x in E.

(ii) If W is a finite-dimensional subspace of V which is E-stable then $p|W : E \longrightarrow End(W)$ is a morphism of algebraic monoids.

Definition: $\rho^* : E \longrightarrow End(k[E])$.

For $x \in E$ let $\rho(x) : E \longrightarrow E$ be defined by $\rho(x)(y) = yx$. Let $\rho^*(x)$ be the induced endomorphism on $k[E]$.

2.1.3 Proposition: $(k[E],\rho^*)$ is a rational E-module.

Proof: See [13;p.62]. The proposition is there stated for groups but the proof is valid for monoids as well.

2.1.4 Proposition: Suppose V is a subspace of $k[E]$ which satisfies

(i) V is finite dimensional.

(ii) $\rho^*(x)(V)$ is contained in V for all x in E.

(iii) V generates $k[E]$ as a k-algebra.

Then $\rho^*|V : E \longrightarrow End(V)$ is a closed imbedding.

Proof: See [13;p.63].

Remark: Putting 2.1.3 and 2.1.4 together we obtain that any affine algebraic monoid E is isomorphic to a closed submonoid of $End(V)$ for some finite dimensional vector space V.

2.2 Elements

2.2.1 Lemma: Suppose G is an algebraic group and X is a Zariski closed subset of G. Then

$N(X) = \{ g \in G \mid Xg \text{ is contained in } X \}$

is a closed subgroup of G.

Proof: One checks that

$N(X) = \{ g \in G \mid \rho^*(I(X)) \text{ is contained in } I(X) \}$ where

$I(X) = \{ f \in k[G] \mid f(x) = 0 \text{ for all } x \text{ in } X \}$.

Thus if $g \in N(X)$ we have

$$\begin{array}{ccc} I(X) & \xrightarrow{\rho^*(g)} & I(X) \\ \downarrow & & \downarrow \\ k[G] & \xrightarrow{\rho^*(g)} & k[G] \\ \downarrow & & \downarrow \\ k[X] & \xrightarrow{\rho^*(g)} & k[X] \end{array}$$

But $\rho^*(g) : k[G] \longrightarrow k[G]$ is an isomorphism and $\rho^*(g)$ acts rationally on $k[G]$. Thus, $\rho^*(g)$ acts rationally on $I(X)$, so $\rho^*(g) : I(X) \longrightarrow I(X)$ is an isomorphism. Hence $(\rho^*(g))^{-1}(I(X)) = I(X)$ and thus $g \in N(X)$, since $\rho^*(g)^{-1} = \rho^*(g^{-1})$.

2.2.2 Corollary: Let G be an algebraic group, S a Zariski closed, multiplicatively closed subset of G. Then S is a closed subgroup of G.

Proof: S is a subset of $N(S)$ by assumption so if $s \in S$ then s^{-1}, $s^{-2} \in N(S)$ by 2.2.1. Thus $1 = ss^{-1} \in S$ and $ss^{-2} = s^{-1} \in S$. Thus S is a subgroup.

2.2.3 Corollary: Let E be an algebraic monoid, $\rho : E \longrightarrow End(V)$ a closed imbedding. Then $G(E)$, the set of elements of E in

Gl(V), is precisely the set of invertible elements of E. Furthermore, G(E) is an algebraic group and there is a morphism $\alpha : E \longrightarrow k = \mathrm{End}(k)$ such that $G(E) = \alpha^{-1}(k^*)$.

<u>Proof:</u> By 2.1.4 there exists a closed imbedding $\rho : E \longrightarrow \mathrm{End}(V)$ for some V. Consider,

$$\begin{tikzcd} S \arrow[r, "\rho"] \arrow[d] & \mathrm{Aut}(V) \arrow[r, "\det"] \arrow[d] & k^* \arrow[d] \\ E \arrow[r] & \mathrm{End}(V) \arrow[r] & k \end{tikzcd}$$

where S is the intersection of E and Aut(V). S is a closed subset of Aut(V) since $\rho(E)$ is closed in End(V). Thus, by 2.2.2 S is an algebraic subgroup of Aut(V). Clearly, S = G(E). Furthermore, if $\alpha = \det\circ\rho$ then $\alpha^{-1}(k^*) = G(E)$.

<u>2.2.4 Lemma:</u> Let $x \in \mathrm{End}(V)$, where V is a finite dimensional vector space over k. Then there is an idempotent $e(x) \in \mathrm{End}(V)$ such that

(i) e(x) is in the Zariski closure of $\{ x, x^2, x^3, \ldots \}$

(ii) For all idempotents f in the Zariski closure of $\{ x, x^2, x^3, \ldots \}$, $fe(x) = e(x)f = f$.

Clearly, e(x) is unique.

<u>Proof:</u> For any endomorphism x, there exists a decomposition x = A + N, where A is invertible when restricted to its image W, and N is nilpotent (N and A commute to zero). Let e(x) be the idempotent with kernel = kernel(A) and image = image(A) and let X be the Zariski closure of $\{ x, x^2, \ldots, \}$ in End(V). By 2.2.2 the intersection S of X and Aut(W) is an algebraic subgroup of Aut(W). Thus $e(x) \in \mathrm{Aut}(W)$. (i) above is satisfied by definition and (ii) is satisfied because e(x) is the identity element of

Aut(W).

Remark: Lemma 2.2.4 may be regarded as a generalization of Fitting's lemma.

2.2.5 Corollary: Let E be an algebraic monoid and let $x \in E$. Then there exists $e(x) \in E$ such that

(i) $e(x)^2 = e(x)$

(ii) $e(x)$ is in the closure of $\{ x, x^2, \ldots, \}$

(iii) $e(x)f = fe(x)$ for all other idempotents satisfying (ii).

Proof: By 2.1.4 there exists a closed imbedding $\rho : E \longrightarrow \mathrm{End}(V)$. If $x \in E$ then the closure of $\{ x, x^2, \ldots, \}$ in End(V) is contained in E. Thus apply 2.2.4.

2.2.6 Corollary: Suppose $g : E \longrightarrow E'$ is a morphism of algebraic monoids. If $e^2 = e \in g(E)$ then there exists $f^2 = f \in E$ such that $g(f) = e$.

Proof: $g(x) = e$ so $g(e(x)) = e$.

Note: Let E be an algebraic monoid. Then there exists $k > 0$ such that for all x in E, $ye(x) = e(x)y = y$, where y is the k-th power of x. This is true for End(V), with $k = \dim(V)$, by the proof of 2.2.4, and in general by 2.1.4.

Notation: Let E be an algebraic monoid.

$$I(E) = \{ e \in E \mid e = e^2 \}.$$

2.2.7 Corollary: Suppose E is an algebraic monoid. Then the following are equivalent.

(i) $G(E) = E$.

(ii) $I(E) = \{ 1 \}$.

Proof: If $x \in E{-}G$ then $e(x) \in E{-}G$.

Notation: Let E be an algebraic monoid, $e \in E$ an idempotent. Then eEe is a closed submonoid of E with identity element e. Let

G(e) be the group of units of eEe.

2.2.8 Proposition:

$$G(e) = \{ x \in eEe \mid e(x) = e \}$$
$$= \{ x \in E \mid xe(x) = x = e(x)x \text{ and } e(x) = e \}.$$

Proof: Clear.

2.2.9 Corollary: Some power of every element of E is in G(e) for some $e \in I(E)$.

Proof: Apply 2.2.8 to the note preceding 2.2.7.

2.2.10 Jordan Decomposition: Suppose $x \in G(e)$. Then there are unique elements x(u) and x(s) in G(e) such that

(i) $x = x(u)x(s) = x(s)x(u)$.

(ii) x(u) is unipotent and x(s) is semi-simple in the group G(e).

(iii) For any morphism $f : E \longrightarrow E'$, $f(x(s)) = f(x)(s)$ and $f(x(u)) = f(x)(u)$.

Proof: (i) and (ii) are clear; for (iii), it suffices to prove that $f(e(x)) = e(f(x))$. But $f(x)f(e(x)) = f(e(x))f(x) = f(x)$. So by 2.2.8, $f(e(x)) = e(f(x))$.

2.3 Examples

2.3.1 Algebras: Let E be a finite dimensional associative algebra. Then E is a linear algebraic variety and the multiplication map is bilinear. Thus E is an algebraic monoid.

2.3.2 Finite monoids: Let E be a finite (set valued) monoid, $1 \in E$ and $m : E\times E \longrightarrow E$ the multiplication map. Let $k[E] = \mathrm{Hom}(E,k)$ ('Hom' in the category of sets). Then $e : k[E] \longrightarrow k$, $e(f) = f(1)$, and $d : k[E] \longrightarrow k[E]\otimes k[E]$, $d(f) = f \circ m$ induce on E the structure of an algebraic monoid.

Corollary: Let E be a finite monoid, $x \in E$. Then there is an integer n such that the n-th power of x is an idempotent.

<u>Proof:</u> By 2.2.9 the k-th power of x is in G(e(x)) for some k. But G(e(x)) is a finite group.

<u>2.3.3 D-monoids:</u> Let S be a finitely generated submonoid of Z(n), the free abelian group of rank n and let k[S] be the monoid algebra of S over k. Then E(S) = Hom(S,k) (as monoids) is a D-monoid (diagonalizable). D-monoids are characterized by the property of being isomorphic with a closed submonoid of some monoid of diagonal matrices.

<u>2.3.4 Algebraic structures:</u> Let V be a finite dimensional vector space over k and let V(m) denote the m-th tensor product of V over k. Suppose S is a subset of the union of the Hom(V(n),V(m)) as m and n vary. Define End'(V) = { f $\in$ End(V) | f(m)os = sof(n) for all s $\in$ S }. Then End'(V) is an algebraic monoid. This example will be discussed in chapter 9 (see also [8]).

<u>2.3.5</u> Let V be any affine variety defined over k. Let E be the disjoint union of V and a point 1. For x, y $\in$ E define xy = x if x and y are elements of V or y = 1, and xy = y if x = 1. Then E, with this multiplication, is an affine algebraic monoid.

III IRREDUCIBLE ALGEBRAIC MONOIDS

3.1 First Principles

Definition: An algebraic monoid E is *irreducible* if it is so as an algebraic variety.

Unlike the case of algebraic groups, arbitrary algebraic monoids are vastly more general than irreducible algebraic monoids. Example 2.3.5 suggests that algebraic monoids in complete generality are not suitable for axiomatic study.

3.1.1 Proposition: Let E be an algebraic monoid. Then there exists a unique maximal irreducible component E(0) of E such that $1 \in E(0)$.

Proof: Let $\{E(i)\}$ be the set of maximal irreducible components of E. Suppose 1 is an element of both E(0) and E(1). Then E(0)E(1) is irreducible and contains both E(0) and E(1). Thus, by maximality, E(0) = E(1).

Let $E^0 = E(0)$.

3.1.2 Proposition: Let E be an algebraic monoid, and let G be the group of units of E. Then E^0 is the Zariski closure of G^0 in E.

Proof: $1 \in G^0$ and G^0 is irreducible. Thus, G^0 is a subset of E^0. But G^0 is open in E^0. Thus, G^0 is dense in E^0.

3.1.3 Proposition: E^0 is an algebraic submonoid of E.

Proof: E^0E^0 is an irreducible subset of E which contains 1. Thus $E^0E^0 = E^0$ by 3.1.1.

3.1.4 Proposition: Let E be an algebraic monoid and let e be an idempotent of E which is in the Zariski closure of G(E). Then $e \in E^0$.

Proof: Let $f : E \longrightarrow E$ be the morphism of varieties which maps each element to its n-th power. Then for some n, f maps $G(E)$ to $G(E)^0$ (since the later group has finite index in the former). But every idempotent is a fixed point of such a morphism.

3.2 Integral Closure And Normalization

3.2.1 Proposition: Suppose we have the following commutative diagram where A and B are integral domains.

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ A\bullet A & \xrightarrow[j\bullet j]{} & B\bullet B \end{array}$$

If $x \in B$ is integral over A then $d(x) \in B\bullet B$ is integral over $A\bullet A$ ('•' denotes tensor product of vector spaces).

Proof: Clear.

3.2.2 Proposition: Suppose we have morphisms $A \longrightarrow A[1/f] \longrightarrow B$ where B is a normal k-domain and the second morphism is finite and dominant. Then $A'\bullet A' \longrightarrow B\bullet B$ is integrally closed, where A' is the integral closure of A in B.

Proof: $A'[1/f] = B$ since localization commutes with integral closure. Further, $A'\bullet A'$ is normal because A' is. Thus, $A'\bullet A' \longrightarrow B\bullet B$ is integrally closed because $B\bullet B = A'\bullet A'[1/f\bullet f]$.

3.2.3 Theorem: Suppose E is a normal irreducible algebraic monoid and $\rho : G \longrightarrow G(E)$ is a finite dominant morphism of algebraic groups. Then the following diagram can be filled in uniquely

$$\begin{array}{ccc} G & \overset{j'}{\dashrightarrow} & E' \\ \downarrow & & \downarrow \rho' \\ G(E) & \underset{j}{\longrightarrow} & E \end{array}$$

in such a way that

(i) E' is normal and irreducible.

(ii) ρ' is a finite morphism of algebraic monoids.

(iii) j' is an open imbedding.

<u>Proof:</u> Let R be the integral closure of k[E] in k[G]. Then R is normal, ρ'^* is finite and j'^* is an open imbedding ($\rho'^* : k[E] \longrightarrow R$ and $j'^* : R \longrightarrow k[G(E)]$). By 3.2.1 and 3.2.2 the comultiplication d, of k[G] restricted to R is a comultiplication on R. Further, the augmentation of k[G] restricted to R is an augmentation on R. Thus, $(R, d|R, e|R)$ is a normal bigebra, finitely generated over k. Hence, the diagram of algebras dualizes to yield the diagram of monoids advertised in the assertion of the theorem with R = k[E']. This diagram is already uniquely determined by the underlying geometry.

<u>Remarks:</u> Let E be an irreducible algebraic monoid. Then there is a unique irreducible monoid E' and a morphism $n : E' \longrightarrow E$ such that

(i) n is finite, dominant and birational.

(ii) E' is a normal algebraic variety.

The details will be left to the reader.

The construction in Theorem 3.2.3 is an important ingredient in the existence results of the next section.

3.3 Existence Of Algebraic Monoids

The purpose of this section is to characterize the irreducible algebraic groups G for which there is an algebraic monoid E with G(E) = G (non-trivially). It turns out that in case this is possible, E may be chosen with 0.

Observation: Let G be an irreducible algebraic group and suppose

$$X(G) = \mathrm{Hom}(G,k^*) \text{ is trivial.}$$

If G = G(E) for some irreducible monoid E, then G = E.

Proof: There exists ρ : E ——> End(V) a closed imbedding. Further, ρ(G(E)) is contained in Gl(V). By assumsion, the composite G(E) ——> Gl(V) ——> k* is trivial for any character of Gl(V). Thus G(E) is contained in Sl(V). This forces G(E) = E because Sl(V) is closed in End(V).

The remainder of this section is devoted to a proof of the converse.

5 Lemmas

3.3.1 Lemma: Let S be a finitely generated submonoid of some free abelian group and suppose ρ : S ——> N is a monoid map such that $\rho^{-1}(1)$ = { 1 } (where N = { 0, 1, 2,.. }). Then S* = { 0 } where S* = { s ϵ S | −s ϵ S }.

Proof: $\rho^{-1}(0)$ contains S*.

3.3.2 Lemma: Let Z(n) be a free abelian group of rank n and let <a(1),...,a(n)> be the submonoid of Z(n) generated by { a(i) }. Let u ϵ Z(n) be non-zero. Then

$$< mu+a(1),...,mu+a(n) >^* = \{ 1 \}$$

for all sufficiently large m.

Proof: Choose ρ : Z(n) ——> Z such that ρ(u) > 0. Then $m\rho(u)+\rho(a(i)) \geq 0$ for all i if m is sufficiently large. Thus

Lemma 3.3.1 applies.

<u>3.3.3 Lemma:</u> Suppose E is a D–monoid and $j : E \longrightarrow End(V)$ is a morphism such that

(i) $V = \Sigma V(\alpha)$ ($\alpha \in X(E)$)

(ii) $V(0) = \{ v \in V \mid j(t)(v) = v$ for all $t \in E \} = (0)$

(iii) $0 \in E$.

Then $j(0) = 0$ (the zero endomorphism of V).

<u>Proof:</u> One checks that $V(0) = \{ v \in V \mid j(0)(v) = v \}$.

<u>3.3.4 Lemma:</u> Suppose T is a D–group and $\rho : T \longrightarrow Gl(V)$ is a morphism. Let $i : Gl(V) \longrightarrow End(V)$ be the canonical inclusion and suppose $V = \Sigma V(\alpha)$, $\alpha \in X(T)$. Then the image of k[End(V)] under $\rho^* o i^*$ in k[T] is k[α;$V(\alpha)$ is non–zero]. Furthermore, k[α; $V(\alpha)$ is non–zero] is thereby identified with the coordinate ring of the closure of $io\rho(T)$ in End(V).

<u>Proof:</u> Straight forward.

<u>3.3.5 Lemma:</u> Suppose there exist morphisms $u : G \longrightarrow k^* = ZGl(V)$ and $j : G \longrightarrow Sl(V)$ (viewed as morphisms to Gl(V)). Let T be a maximal torus of G and suppose $V = \Sigma V(\alpha)$ (direct sum decomposition) via j. Consider $g(m) : G \longrightarrow Gl(V)$, the morphism obtained by multiplying j and mu. Then, via g(m), $V = \Sigma V'(\alpha+mu)$ where $V'(\alpha+mu) = V(\alpha)$.

<u>Proof:</u> $V(\alpha) = \{ v \in V \mid j(t)(v) = \alpha(t)v$ for all $t \in T \}$. So if $v \in V(\alpha)$ then $g(m)(t)(v) = (\alpha+mu)(t)v$ for all $t \in T$.

<u>Note:</u> Thus, by 3.3.2 and 3.3.3, if m is sufficiently large and u is non–trivial then 0 is an element of the closure of g(m)(T) in End(V).

<u>Conclusion:</u> (putting 3.3.1 – 3.3.5 together)

Assume X(G) is non–trivial. Let $j : G \longrightarrow Sl(V)$ be an imbedding and let $u : G \longrightarrow k^*$ be a non-trivial character. Then for all $l > 0$, $g(m) : G \longrightarrow Gl(V)$ is finite.

Furthermore, by 3.3.2, 3.3.4 and 3.3.5, if l is large enough then the closure of $g(m)$ has a zero of its own for T a maximal torus of G. By 3.3.3 we can choose l large enough so that the zero of $End(V)$ is in the closure of $g(m)$. Hence, letting $\rho = g(m)$, we have

$$G \xrightarrow[\rho]{} G' \longrightarrow Gl(V) \longrightarrow End(V)$$

such that ρ is a finite and dominant and 0 is an element of the closure of G' in $End(V)$.

Let E' be the normalization of the closure of G in $End(V)$. Then we have, $0 \in E'$ and E' is irreducible and normal. Consider the following diagram.

$$\begin{tikzcd} G \arrow[d] & \\ G' \arrow[r] & E' \end{tikzcd}$$

By Theorem 3.2.3 the diagram can be filled in uniquely to yield

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[d] & E \arrow[d, "\rho'"] \\ G' \arrow[r] & E' \end{tikzcd}$$

such that j is an open imbedding and ρ' is finite and dominant. It follows that E also has a zero. In summary, we have established the following result.

<u>3.3.6 Theorem:</u> Let G be an irreducible algebraic group. Then the following are equivalent.

(i) There exists an irreducible algebraic monoid E such that $G(E) = G$ and E is not a group.

(ii) There exists an irreducible monoid E such that $G(E) = G$ and $0 \in E$ (with 0 not equal to 1).

(iii) $X(G)$ is a non-trivial group.

(iv) $\mathrm{rank}(R(G)) > 0$.

3.4 Closure

A useful strategy in the theory of algebraic monoids is to apply the structure theory of algebraic groups and varieties in studying closure properties of various subgroups T of G in E. The purpose of this section is to assemble some of these techniques. I shall assume throughout that E is an irreducible affine algebraic monoid.

3.4.1 Proposition: Let E be an irreducible algebraic monoid with group of units G and let B be a Borel subgroup of $G = G(E)$. Then

(i) E is the union of gZg^{-1} as g varies over G, where Z is the closure of B in E.

(ii) E is the union of gZ as g varies over G.

Proof: G/B is complete and $B \bullet Z$ is contained in Z, where '$\bullet$' denotes either conjugation or left translation. Thus, by 1.2.3 (i), $G \bullet Z$ is closed in E.

Recall from 1.2.2 (iv) that if B(G) is the set of Borel subgroups of G, then we may regard B(G) as a complete algebraic variety in such a way that $G \times B(G) \longrightarrow B(G)$, $(g,B) \longrightarrow gBg^{-1}$, is a morphic group action.

The purpose of the next two lemmas is to prove that if $x \in E$ then $B(x) = \{ B \in B(G) \mid x \text{ is an element of the closure of } B \}$ is a closed non-empty subset of B(G).

<u>3.4.2 Lemma:</u> Suppose $\rho : X \longrightarrow Y$ is an surjective open map of topological spaces and further, that V is a closed and saturated subset of X (i.e. $V = \rho^{-1}(\rho(V))$). Then $\rho(V)$ is a closed subset of Y.

<u>Proof:</u> $\rho(X-V)$ is open in Y and $\rho(X-V) = \rho(X)-\rho(V) = Y-\rho(V)$ since ρ is saturated. Thus $\rho(V)$ is closed in Y.

<u>3.4.3 Lemma:</u> Let $x \in E$ and suppose $x \in Z$, the closure of $B(0) \in B(G)$. Let $V = \{ g \in G \mid g^{-1}xg \in Z \}$. Then $\rho(V)$ is a closed subset of $G/B(0)$, where $\rho : G \longrightarrow G/B(0)$ is defined by $\rho(g) = gB(0)$.

<u>Proof:</u> V is closed since it is a transporter with closed target. So if $g \in V$ then $x \in gZg^{-1}$. Hence $x \in gbZ(gb)^{-1}$ and thus, $gb \in V$ for all $b \in B(0)$. Thus, V is saturated with respect to ρ. By Lemma 3.4.2, $\rho(V)$ is closed in $G/B(0)$.

<u>3.4.4 Proposition:</u> Let $x \in E$. Then $B(x) = \{ B \in B(G) \mid x$ is in the closure of $B \}$ is closed in B(G).

<u>Proof:</u> B(G) may be regarded as a complete variety under the identification given in 1.2.2 (iv). Under this identification $\{ gB(0) \mid g^{-1}xg \in Z(0)$, the closure of $B(0) \}$ corresponds to $\{ B \in B(G) \mid x$ is in the closure of $B \}$.

<u>3.4.5 Proposition:</u> Let E be an irreducible algebraic monoid and let T be a maximal torus of G(E). Suppose $x \in E$ and $xt = tx$ for all $t \in T$. Then there is a Borel subgroup $B \in B(G)$ such that T is contained in B and x is in the closure of B.

<u>Proof:</u> B(x) is closed in B(G) and B(x) is stable under conjugation by T since T centralizes x. By 1.2.2 (iii) T has a fixed point in B(x). Thus there exists $B \in B(x)$ such that T normalizes B. Hence, by 1.2.2 (ii), T is contained in B.

<u>3.4.6 Proposition:</u> Suppose $x \in E$ is semi-simple and T is a maximal torus such that $xt = tx$ for all $t \in T$. Then $x \in X$, the closure of T.

<u>Proof:</u> By 3.4.5 there exists a Borel subgroup B of G such that x is in the closure Z, of B and T is contained in B. Now there exists a representation $Z \longrightarrow T(V)$ (upper triangular). From linear algebra we can assume that both T and x are contained in $D(V)$ the diagonal matrices of $T(V)$. So we have

$\{ x, X \} \longrightarrow Z \longrightarrow T(V) \longrightarrow D(V)$ where the last map is the quotient of $T(V)$ modulo its ideal of nilpotent elements. The composite of all these maps is one-to-one since both X and $\{ x \}$ are contained in $D(V)$. But the image of x is in the image of X because X is the closure of a maximal torus. Thus $x \in X$.

The following fundamental result is due to M. Putcha [21;Theorem 1].

<u>3.4.7 Proposition:</u> Let E be an irreducible algebraic monoid and let $e \in I(E)$ (idempotents). Then

(i) There exists a closed irreducible submonoid E' of E such that Ee is contained in E' and $eE' = eEe$.

(ii) There exists a closed irreducible submonoid E'' of E such that eEe is contained in E'' and E'' is contained in the centralizer of e in E.

<u>3.4.8 Corollary[21]:</u> Let E be an irreducible algebraic monoid and let $e \in I(E)$. Let $C(e) = \{ x \in E \mid xe = ex \}$. Then eEe is contained in $C(e)^{o}$, the identity component of $C(e)$.

<u>Proof:</u> eEe is contained in E'' as in 3.4.7, and E'' is contained in $C(e)$. Thus E'' is contained in $C(e)^{o}$ since E'' is irreducible.

<u>3.4.9 Proposition[21]:</u> Let E be an irreducible algebraic monoid

and let $e \in I(E)$. Let $CG(e) = \{ g \in G(E) \mid ge = eg \}$. Then $CG(e) \longrightarrow G(e)$, $g \longrightarrow ge = eg$, is a surjective morphism of algebraic groups.

Proof: Consider $C(e) \longrightarrow eEe$, $x \longrightarrow ex = xe$. By 3.4.8, this is a dominant morphism when restricted to $C(e)^0$. Thus $CG(e)) \longrightarrow G(e)$ is dominant. But a dominant morphism of algebraic groups is surjective.

3.4.10 Proposition: Let $x \in E$ be semi-simple. Then $x \in X$, the closure of some maximal torus T of $G(E)$.

Proof: Let $e = e(x)$. Then $x \in G(e)$. By 3.4.9 $CG(e) \longrightarrow G(e)$, $g \longrightarrow ge$, is surjective. Thus there exists a maximal torus T in $CG(e)$ such that $x \in eT$. By 3.4.6 $e \in X$, the closure of T. Thus $x \in eT$, which is contained in the closure of T.

3.4.11 Proposition: Let $x \in E$ be semi-simple. Then $Cl(x)$, the conjugacy class of x, is closed in E.

Proof: By 3.4.10 there exists a maximal torus T of $G(E)$ such that $x \in X$, the closure of T. So $txt^{-1} = x$ for all $t \in T$ since X is a commutative monoid. Thus by 1.2.3 (ii) $Cl(x)$ is closed in E.

3.4.12 Proposition: Let E be an irreducible algebraic monoid. Then there are a finite number of conjugacy classes of idempotents in E.

Proof: Let $e \in I(E)$ and let T be a maximal torus of $G(E)$. By Proposition 3.4.10 there exists $g \in G(E)$ such that $geg^{-1} \in X$, the closure of T in E. But the number of idempotents in X is finite since X is isomorphic to a closed submonoid of the diagonal matrices, $D(V)$ for some V.

In subsequent chapters it will be necessary to know that under certain conditions the image of an algebraic monoid is a closed subset of the target monoid. It is easy to construct examples which demonstrate that fairly strict conditions are required.

<u>3.4.13 Proposition:</u> Suppose E and E' are irreducible algebraic monoids with zeros 0 and 0' respectively. If $\rho : E \longrightarrow E'$ is a morphism such that $\rho^{-1}(0') = \{ 0 \}$ then ρ is a finite morphism. In particular, $\rho(E)$ is a closed submonoid of E'.

<u>Proof:</u> Choose a 1–p.s.g. $\alpha : k^* \longrightarrow G(E)$ such that α extends to $\alpha : k \longrightarrow E$ with $\alpha(0) = 0$. This yields an action of k on E by left translation and similarily on E' via ρ.

On the level of coordinate algebras this translates to: $\rho : k[E'] \longrightarrow k[E]$ is a morphism of N–graded algebras (where N $= \{0,1,2,\ldots\}$). Since α converges to zero we obtain $k[E'](0) = k = k[E](0)$. Since $\rho^{-1}(0') = \{ 0 \}$ we have that k[E] is finite dimensional modulo $k[E']^+$. Thus, by 1.1.4, k[E] is a finite k[E']–module.

IV TYPES OF MONOIDS

In this chapter I discuss some of the special properties associated with the five most important types of monoids. These are D–monoids, nilpotent monoids, solvable monoids regular monoids, and reductive monoids.

4.1 D-monoids

Definition: An irreducible D-monoid E is an irreducible algebraic monoid such that G(E) is a torus.

D–monoids are quite varied and have been studied extensively from a geometric point of view in [14]. M. Hochster [12] has proven that normal D–monoids are Cohen–Macaulay. D–monoids have also been studied as algebraic monoids by M. Putcha in [19] and [20]. It is hoped that ultimately the classification of reductive monoids can be reduced to problems concerned with D–monoids.

This section is mostly summary. Pertinent details not mentioned here are well recorded in [14], [19] and [20].

4.1.1 Proposition: There is a categorical equivalence between the category D of irreducible D–monoids and the category M of finitely generated commutative monoids which can be imbedded in a free abelian group. The equivalence is given by functors,

$$E \longrightarrow X(E) = \mathrm{Hom}(E,k) \text{ (D–monoid morphisms)}$$

$$S \longrightarrow E(S) = \mathrm{Hom}(S,k) \text{ (monoid morphisms)}$$

X(E) is the character monoid associated with E.

4.1.2 Proposition: Let E be an irreducible D–monoid and let I(E) $= \{ e \in E \mid e^2 = e \}$. Then

(i) I(E) is finite.

(ii) If $0 \in E$ and if $0 < e(1) < \ldots < e(k) = 1$ is a saturated chain in $I(E)$, then $\dim(E) = k$.

(iii) If $\dim(E) = 2$ and $0 \in E$ then $I(E) = \{ 1, e, f, ef = 0 \}$.

(iv) If $e \in I(E)$ then e is the product all the maximal idempotents which are larger than or equal to e.

<u>4.1.3 Proposition[14;p.12]:</u> There are canonical one-to-one correspondences among $\{ U \mid U$ is an open affine $G(E)$-equivariant subset of $E \}$, $I(E)$ and $\{ X \mid X$ is a $G(E)$-orbit in $E \}$. If U is open affine and $G(E)$-equivariant then there is a unique minimal idempotent $e(U) \in U$.

<u>4.1.4 Proposition[19]:</u> For all maximal idempotents $e \in I(E)-\{1\}$ there is a unique one-dimensional subgroup $G(e)$ of $G(E)$ such that e is an element of the closure of $G(e)$ in E.

<u>4.1.5 Proposition[14]:</u> Let E be a D-monoid. Then E is a normal variety if and only if for all $x \in X(G(E))$, $nx \in X(E)$ implies that $x \in X(E)$.

<u>4.1.6 Proposition:</u> Let E and E' be irreducible D-monoids with zeros 0 and $0'$. Suppose $\rho : E \longrightarrow E'$ is a morphism such that $\rho(0) = 0'$. Then the following are equivalent.

(i) ρ is a finite morphism.

(ii) $\rho|I(E) : I(E) \longrightarrow I(E')$ is one-to-one.

(iii) There exists $n \in N$ such that $nX(E)$ is contained in $\rho^*(X(E'))$.

<u>Proof:</u> (i) => (ii). If $\rho(e) = \rho(f)$ then $\rho(ef) = \rho(e) = \rho(f)$. If e is not f then $ef < e$ and thus efE is a proper subset of eE, whereas $\rho(efE) = \rho(eE)$. Thus $\dim(eE) > \dim(\rho(eE))$. So ρ is not finite.

(ii) => (i). If $\rho : I(E) \longrightarrow I(E')$ is one-to-one then $\rho^{-1}(0')$ is the union of the $G(E)$ orbits of the idempotents that it contains. Thus by 3.4.13 ρ is finite.

(i) => (iii): With an irrelevant loss of generality we may assume that ρ is dominant. Thus we have

$$\begin{array}{ccc} X(E') & \xrightarrow{\rho^*} & X(E) \\ \downarrow & & \downarrow \\ X(G') & \xrightarrow{\rho^*} & X(G) \end{array}$$

If ρ is finite then $mX(G)$ is contained in $\rho^*(X(G'))$ for some $m \in N$. But by assumption, each element of $mX(E)$ is integral over $k[X(E')]$. So, by 4.1.5, there exists $n \in N$ such that $nX(E)$ is contained in $\rho^*(X(E'))$. (iii) => (i) is clear.

<u>4.1.7 Proposition:</u> Let E a normal D–monoid with 0. Then there exists $F(1),\ldots,F(m) \in X(E)$ such that

(i) m is the number of minimal non–zero idempotents of E.

(ii) $\langle F(1),\ldots,F(m)\rangle \longrightarrow X(E)$ is a finite morphism (where $\langle \ldots \rangle$ denotes "the monoid generated by").

(iii) $\{ F(i) \}$ is a linearly independent set modulo m^2, the square of the ideal of functions that vanish at 0.

Furthermore, $\{ F(i) \}$ is the only such subset of characters satisfying all these properties. $F = \{ F(i) \}$ is the set of <u>fundamental generators</u> of $X(E)$.

<u>Proof:</u> Let $e \in I(E)$ be a minimal non–zero idempotent. Then by 4.1.2 (ii), dim $eE = 1$. Since E is normal, eE is normal, because it is a retract of E. Thus eE is isomorphic to k. So $k[eE] = k[F(e)]$, where $F(e)$ is the unique character which generates $k[eE]$. $E \longrightarrow eE$, $x \longrightarrow ex$ is a morphism of D–monoids so $X(eE) \longrightarrow X(E)$ is a morphism of monoids. Consider, $\rho : E \longrightarrow Z$, $\rho(x) = (ex)$, e minimal, where Z is the direct product of all the eE

as e ranges over the minimal idempotents of E. Then ρ is a morphism of D-monoids such that $\rho(e)$ is non-zero for every minimal idempotent e in E. Thus, $\rho^{-1}(0) = \{ 0 \}$, because otherwise it contains a minimal idempotent. Hence ρ is finite by 3.4.13. On the level of characters we have $\langle F(1),\ldots,F(m)\rangle \longrightarrow X(E)$ (where the F(e) have been relabled). This proves (i) and (ii). Each F(i) is non-zero modulo m^2 because eE is a retract of E. Thus { F(i) } is a linearly independent subset of m/m^2. This proves (iii).

If { x(i) } is a subset of X(E) such that $\langle x(1),\ldots,x(m)\rangle \longrightarrow X(E)$ is finite then it follows that each x(i) is some power of one of the F(i). Thus { x(i) } satisfies (iii) if and only if { x(i) } = { F(i) }.

4.2 <u>Nilpotent Monoids</u>

<u>Definition:</u> Let E be an irreducible algebraic monoid. E is <u>nilpotent</u> if G(E) is a nilpotent algebraic group.

A well known structure theorem asserts that if G is a nilpotent algebraic group then $G(u) = \{ u \in G \mid u \text{ is unipotent} \}$ and $G(s) = \{ s \in G \mid s \text{ is semi-simple} \}$ are closed subgroups of G and G is isomorphic to the direct product of G(u) and G(s). The purpose of this section is to characterize the class of nilpotent monoids for which a generalization of this theorem is possible. M. Putcha has obtained similar results for commutative monoids in [19].

<u>4.2.1 Lemma:</u> Suppose E is an irreducible nilpotent monoid. Let T be the maximal torus of G(E). Then every semi-simple element of E is in the closure of T.

<u>Proof:</u> By 3.4.10 every semi-simple element of E is in the

closure of a torus.

Definition: Let E be an algebraic monoid. If E is the union of the $G(e)$ as e varies over all idempotents, then E is a Clifford monoid (see 2.2.7 – 2.2.10 for a discussion of $G(e)$).

4.2.2 Lemma: Suppose E is a nilpotent Clifford monoid. Let $e \in I(E)$ be a maximal idempotent. Then $E(e) = \{ x \in E \mid xe = ex = e \}^0$ is a one-dimensional D-submonoid of E.

Proof: $I(E(e)) = \{ 1, e \}$ and e is the zero element of $E(e)$. Suppose $x \in E(e)$ and x is not e. Then $e(x)$ is not equal to e either, because if $e = e(x)$ then $x \in G(e)$ since E is Clifford. But then e is the only element common to both $G(e)$ and $E(e)$. Thus, $e(x) = 1$, since $I(E) = \{ 1, e \}$. Hence, $x \in G(E(e))$. So $E(e)$ is the union of $G(E(e))$ and $\{ e \}$. Thus, $\dim E(e) = 1$, since by 2.2.3, $\dim(E - G(E)) = \dim(G) - 1$. Furthermore, $G(E(e))$ is a D-group, because irreducible unipotent monoids are groups.

4.2.3 Lemma: Let E and e be as in 4.2.2. Then $G(u) \longrightarrow eG(u)$, $u \longrightarrow eu$, is a finite morphism.

Proof: Let $K = \{ v \in G(u) \mid ev = e \}$. By 4.2.2, K^0 is irreducible, and semi-simple. Thus, $K^0 = \{ 1 \}$ and so K is finite.

4.2.4 Lemma: Let E be as in 4.2.2 and suppose $e \in I(E)$ is any idempotent. Then $G(u) \longrightarrow eG(u)$ is finite.

Proof: Let $1 > e(1) > \ldots > e(k) = e$ be a saturated chain of idempotents. Then $e(i)G(u) \longrightarrow e(i+1)G(u)$ is finite for each i by 4.2.3 applied to the Clifford monoid $e(i)Ee(i)$. Since $ex = e(k)\ldots e(1)x$, $G(u) \longrightarrow eG(u)$ is the composite of finite morphisms. Thus $G(u) \longrightarrow eG(u)$ is finite.

4.2.5 Lemma: Let E be as in 4.2.2. Then $m : E(s) \times G(u) \longrightarrow E$,

$(x,u) \longrightarrow xu$ is a finite birational morphism of algebraic monoids.

Proof: m is birational by the well known result for algebraic groups. Suppose (x,u) satisfies $xu = e$ (so $e(x) = e$). Then $eu = x^*xu = x^*e = x^*$ (for some $x^* \in G(e)$). But then x^* is a semi-simple element of $eG(u)$. Hence, $x^* = e$ and thus $x = e$. It follows that m is one-to-one and birational. m is onto because image(m) contains G(E) and all idempotents. The same is true for the normalization of E. Thus, by 1.1.2, m is finite.

4.2.6 Theorem: Suppose E is irreducible and nilpotent. Then the following are equivalent.

(i) E is Clifford.

(ii) The morphism $m : E(s)xG(u) \longrightarrow E$, $(x,u) \longrightarrow xu$ is finite and dominant.

If, in addition, E is normal, then m is an isomorphism.

Proof: (i) => (ii). Lemma 4.2.5.

(ii) => (i). Both groups and D-monoids are Clifford monoids. It follows that E(s)xG(u) is Clifford. Thus E is Clifford as it is the image of a Clifford monoid.

If E is normal then it follows from 4.2.5 and 1.1.2 that m is an isomorphism.

4.3 Solvable Monoids

Definition: Let E be an irreducible algebraic monoid. Then E is solvable if G(E) is a solvable algebraic group.

In this section I prove two general results about solvable algebraic monoids.

4.3.1 Theorem: Let E be a solvable irreducible algebraic monoid. Then there exists an irreducible D-monoid X and a morphism

$$\rho : E \longrightarrow X$$

such that for any morphism $f : E \longrightarrow Y$ where Y is a D-monoid there is a unique morphism $f^* : X \longrightarrow Y$, such that $f^* \circ \rho = f$. Furthermore, for all maximal tori T of $G(E)$ the composite $Z \longrightarrow E \longrightarrow X$ is an isomorphism, where Z is the closure of T in E.

<u>4.3.2 Theorem:</u> Let E be an irreducible algebraic monoid with zero. Then the following are equivalent.

(i) E is solvable.

(ii) $N = \{ x \in E \mid x \text{ is nilpotent} \}$ is a two sided ideal of E.

<u>Proof of 4.3.1:</u> Let $X(E)$ be the characters of E. Then $X(E)$ is a linearly independent multiplicative subset of $k[E]$. Let $R = k[X(E)]$, the monoid algebra of $X(E)$ over k, and let T be a maximal torus of $G(E)$. Let Z be the closure of T in E. Thus, the composite $R \longrightarrow k[E] \longrightarrow k[Z]$ is one-to-one, since the same is true when restricted to $G(E)$, and $G(E)$ is dense in E.

<u>Claim:</u> $R \longrightarrow k[Z]$ is an isomorphism.

<u>Proof of claim:</u> There exists a closed imbedding $E \longrightarrow T(V)$ (upper triangular matrices) for some V. We may assume also that Z is contained in $D(V)$, the diagonal matrices. Suppose we have a character $\rho : Z \longrightarrow k$. Then ρ can be lifted to a character $\alpha : D(V) \longrightarrow k$ since $Z \longrightarrow D(V)$ is a closed imbedding. But α lifts to $T(V)$ because the inclusion $D(V) \longrightarrow T(V)$ splits. Restricting this lifting to E yields: Every character on Z lifts to E. This proves the claim.

Now suppose $f : E \longrightarrow Y$ is a morphism, where Y is a D-monoid. On the level of characters this yields $f^* : X(Y) \longrightarrow X(E)$. But $X(E)$ is contained in R. So f factors through $X = \operatorname{Spec}(R)$.

Proof of 4.3.2: (i) => (ii). By 4.3.1 there is a morphism $\rho : E \longrightarrow X$ such that for every maximal torus T in $G(E)$ the composite $Z \longrightarrow E \longrightarrow X$ is an isomorphism. Thus, since every semi-simple element is in the closure of a torus, 0 is the only semi-simple element of $\rho^{-1}(0)$. Now $\rho^{-1}(0)$ is a closed ideal of E; so if $x \in \rho^{-1}(0)$, then $e(x) \in \rho^{-1}(0)$. Thus $\rho^{-1}(0)$ is the set of nilpotent elements of E.

(ii) => (i). Suppose E is not solvable. Then C(T) is a proper subset of N(T) (i.e. the Weyl group is non-trivial). Let $\sigma \in W = N(T)/C(T)$ be a non-trivial element. So $\mathrm{int}(\sigma) : I(Z) \longrightarrow I(Z)$ is non-trivial, since by 4.1.7, the automorphisms of a D-monoid with 0 are faithfully represented on the idempotents. It follows that $\mathrm{int}(\sigma)$ acts non-trivially on the minimal idempotents of Z. So let $e, f \in I(Z)$ be minimal non-zero idempotents such that $\sigma e = f\sigma$. Thus $(\sigma e)^2 = \sigma e f \sigma = \sigma 0 \sigma = 0$ since e and f are distinct minimal idempotents of Z. But then σe is nilpotent, yet $\sigma e \sigma^{-1}$ is not. Hence the nilpotent elements do not form an ideal.

Much more can be said about the structure of solvable algebraic monoids. In [24;Theorem 23] M. Putcha gives numerical and semigroup characterizations of solvable algebraic monoids.

4.3.3 Proposition: Let E be a solvable irreducible algebraic monoid and let $x, y \in E(s)$. Suppose $xy = yx$ and $\rho(x) = \rho(y)$ where ρ is the universal morphism to a D-monoid. Then $x = y$.

Proof: By 4.3.1, it suffices to prove this for $E = T(V)$ (upper triangular matrices). Let C(x) be the centralizer of x in E. Then C(x) and ZC(x) are both irreducible, since they are both linear subspaces of T(V). Let T be a maximal torus of C(x) such that $y \in X$, the closure of T in C(x). But we also have $x \in X$

since x is a central semi-simple element. Thus $x = y$ since $\rho|X$ is one-to-one.

4.4 Reductive And Regular Algebraic Monoids

Definition: Let E be an irreducible algebraic monoid. E is reductive if $G(E)$ is a reductive algebraic group.

Definition: (a) A monoid E is regular if for all $x \in E$ there exists $g \in G(E)$ and $e \in I(E)$ such that $gx = e$.

(b) A monoid E is von Neumann regular if for all $x \in E$ there exists $a \in E$ such that $xax = x$.

Let E be an irreducible algebraic monoid. It is then a consequence of [21;Theorem 13] that E is regular if and only if E is von Neumann regular. Thus, I have often taken the liberty of using the definition that is most convenient.

The main result of this section (see Theorems 4.4.14 and 4.4.15) asserts that all reductive monoids are regular.

Let us recall some properties concerning conjugacy classes in semi-simple algebraic groups.

4.4.1 Proposition[29;p.87-92]: Let G be a semi-simple algebraic group.

(i) If $x \in G$ then $Cl(x)$ is closed in G if and only if x is semi-simple.

(ii) Let $cl[G] = \{ f \in k[G] \mid f(xy) = f(yx) \text{ for all } x, y \in G \}$. Then $cl[G]$ is the ring of invariant functions under the induced action of conjugation.

(iii) Let T be a maximal torus of G, W the Weyl group. Then

$$\begin{array}{ccc} cl[G] & \longrightarrow & k[G] \\ \Big\| & & \Big\downarrow \\ k[T]' & \longrightarrow & k[T] \end{array}$$

commutes, where $k[T]'$ is the the ring of invariant functions in $k[T]$ under the induced action of W.

Remark: If G is an irreducible reductive group then $(G,G) = G'$ is semi-simple and G is commensurable with the product of G' and $Z(G)^0$. It follows from this and 4.4.1 above, that 4.4.1 is also valid for reductive groups.

Let $cl[E] = \{ f \in k[E] \mid f(xy) = f(yx) \text{ for all } x,y \in E \}$.

4.4.2 Proposition: Let E be an irreducible algebraic monoid, T a maximal torus. Suppose for $x, y \in X$, the closure of T, there exists $g \in G$ such that $gxg^{-1} = y$. Then there exists $w \in N(T)$ (normalizer) such that $wxw^{-1} = y$.

Proof: $g^{-1}Tg$ and T are contained in $CG(x)^0$, the identity component of the centralizer of x in $G(E)$. Thus there exists $z \in CG(x)^0$ such that $zg^{-1}Tgz^{-1} = T$. But then $zg^{-1} \in N(T)$, yet $(zg^{-1})^{-1}x(zg^{-1}) = gxg^{-1} = y$.

If $x \in E$ is semi-simple then by 3.4.10, x is in the closure of some maximal torus. Thus, by 4.4.2 the semi-simple conjugacy classes are canonically parametrized by X/W.

4.4.3 Theorem: Let E be a reductive algebraic monoid and let T be a maximal torus of G. Let X be the closure of T in E. Then the inclusion $cl[E] \longrightarrow k[E]$ followed by the projection $k[E] \longrightarrow\!\!\!\!\!\to k[X]$ induces an isomorphism of $cl[E]$ onto $k[X]'$, the ring of invariant functions under the induced action of W on $k[X]$.

Proof: $cl[E]$ is the intersection in $k[G]$ of $k[E]$ and $cl[G]$. It follows from this that (when restricted to T) cl[E] is the intersection in k[T] of k[X]' and cl[G]. Thus, by the remark following 4.4.1, cl[E] is identified with k[X]'.

<u>Remark:</u> From general principles (1.2.4) we have,

(i) cl[E] is a finitely generated k–algebra since G is reductive and cl[E] is a ring of invariants of G.

(ii) The morphism cl : E ——> E(cl) = Spec(cl[E]), induced from the inclusion cl[E] ——> k[E] satisfies: each fibre of the morphism cl contains precisely one <u>closed</u> conjugacy class.

<u>4.4.4 Theorem:</u> Let E be a reductive algebraic monoid and let $x \in E$. Then

(i) x is semi–simple if and only if Cl(x) (the conjugacy class of x in E) is closed in E.

(ii) If T is a maximal torus of G then the centralizer of T in E is equal to the closure of T in E.

<u>Proof:</u> Let X be the closure of T in E. From 4.4.3 we have X/W is canonically isomorphic with E(cl).

If $x \in E$ is semi–simple then by 3.4.11 Cl(x) is closed in E. Conversely, if Cl(x) is closed in E then by the remark above Cl(x) is the only closed conjugacy class in $cl^{-1}(cl(x))$. But from our identification of X/W with E(cl) we obtain that X intersects every closed conjugacy class. This proves (i).

If $x \in C(T)$ then Cl(x) is closed by 1.2.3 (ii). Thus by (i) above x is semi–simple. Hence, x is in the closure of T by 3.4.6.

<u>4.4.5 Proposition:</u> let E be reductive and let $U(E) = \{ x \in E \mid \dim Cl(x) = \dim G - \mathrm{rk}\, G \}$. Then U(E) is a non–empty open subset of E.

Proof: By 1.2.1 (iv) $U'(E) = \{ x \in E \mid \dim Cl(x) \geq \dim G - rkG \}$ is open in E. Since E is irreducible, and $U(G) = \{ x \in G \mid \dim Cl(x) = \dim G - rkG \}$ is non-empty and open (see [29]), it follows that $U(E) = U'(E)$.

4.4.6 Proposition: $\{ x \in U(E) \mid x \text{ is semi-simple} \}$ is open in E.

Proof: This follows from 1.2.4 (iii) and 4.4.4 (i).

Remark: By 4.4.5 we have

$$\dim Cl(x) \leq \dim G - rk\ G$$

for all $x \in E$.

4.4.7 Lemma: Suppose G is an algebraic group and $\rho : G \longrightarrow Gl(V)$ is a rational representation such that V is completely reducible. Then the unipotent radical of G is contained in the kernel of ρ.

Proof: Without loss of generality assume V is a simple G-module. Let UR(G) be the unipotent radical of G and let W be the invariants of UR(G) in V. Since UR(G) is unipotent, W is non-zero and since UR(G) is normal, W is a G-submodule of V. Thus W = V.

4.4.8 Theorem: Suppose E is an irreducible algebraic monoid. Then there exists an irreducible reductive algebraic monoid E' and a morphism $\rho : E \longrightarrow E'$ such that

(i) ρ is dominant

(ii) kernel(ρ) = UR(G), the unipotent radical of G.

(iii) If T is a maximal torus of G(E) such that $\rho(T) = T'$ then $\rho : X \longrightarrow X'$ is an isomorphism, where X and X' are the closures of T and T' respectively.

Proof: There exists a representation $\sigma : E \longrightarrow End(V)$ such that σ is a closed imbedding. Let F be a composition series of the

E–module V. Thus, F is a linearly ordered collection of subspaces { V(i) } of V such that V(i+1)/V(i) is a simple E–module for all i. Let End(V,F) = { f $\in$ End(V) | f(V(i)) is contained in V(i) for all i }. Thus, by definition of F, α factors through the inclusion End(V,F) ——> End(V). There is a canonical morphism q : End(V,F) ——> End(Gr(V)), where Gr(V) is the graded object associated with the filtration F of V. By 4.4.7 UR(G) is in the kernel of $q \circ \alpha$. Thus UR(G) = ker($q \circ \alpha$) since ker($q \circ \alpha$) is unipotent. Since q is a morphism of algebras with nilpotent kernel, q restricts to an isomorphism on the level of maximal D–submonoids. Thus, it follows that the closure of $q \circ \alpha$(E) in End(Gr(V),F) satisfies conclusions (i)–(iii) of the theorem.

<u>Remark:</u> Theorem 4.4.8. is useful in the discussion of prime ideals in Chapter 5 and in the proof that reductive monoids are regular.

The following application is inspired by close analogy with a well–known result from classical ring theory. Let R be a finite dimensional associative algebra over an algebraically closed field k. If R is von Neumann regular then R is a semi–simple ring.

<u>4.4.9 Corollary:</u> Suppose E is a regular algebraic monoid with 0. Then E is reductive.

<u>Proof:</u> If E is not reductive, let ρ: E ——> E' be as in 4.4.8. Then by 1.1.1 dim $\rho^{-1}(0') > 0$. So let x $\in$ $\rho^{-1}(0')$ be non–zero. If E is regular then there exists a $\in$ E such that xax = x. But then xa = $(xa)^2$ is non–zero. So $\rho^{-1}(0')$ contains non–zero idempotents. This is impossible by 4.4.8 (iii) since by 3.4.10

every idempotent is in the closure of a maximal torus. This contradiction implies that E is not regular.

The remainder of this section is devoted to the proof that all reductive irreducible monoids are regular. This result has been proved by M. Putcha in characteristic zero using Weyl's theorem on the complete reducibility of rational representations. The main ideas of Putcha's proof have survived in my treatment, even though Weyl's theorem is not true in general. It is curious that Haboush's theorem (1.2.4 (i)) is not required in the proof. The proof requires the following result of M. Putcha.

<u>4.4.10 Proposition[23;Theorem 1.4]:</u> Let E be an irreducible algebraic monoid with group of units G. Let $e \in I(E)$ and let $E(e) = \{ x \in E \mid xe = ex = e \}^{0}$. Then $GE(e)G = \{ a \in E \mid e \in EaE \}$.

<u>4.4.11 Lemma:</u> Let E be a reductive monoid and let E(e) be as above. Then E(e) is reductive.

<u>Proof:</u> $CG(e) = \{ g \in G \mid ge = eg \}$ is reductive since the conjugacy class of e is closed. By 3.4.9 $CG(e) \longrightarrow G(e)$, $g \longrightarrow eg$, is a morphism of algebraic groups. Thus G(E(e)) is reductive since it is the identity component of the kernel of this map.

<u>4.4.12 Lemma:</u> Suppose E is a regular irreducible algebraic monoid with 0. Let $N = \{ x \in E \mid x \text{ is nilpotent} \}$. Then N is a closed subset of codimension larger than or equal to two.

<u>Proof:</u> Clearly, N is closed. Since E is regular it has no ideals consisting entirely of nilpotent elements. But every closed irreducible subset of E—G of codimension one in E is a maximal irreducible component of E—G. Furthermore, each maximal

irreducible component of E–G is an ideal because E is irreducible.

4.4.13 Lemma: Suppose ρ : E ——> E' is a finite dominant morphism of irreducible monoids with 0. If E' has no non–zero nilpotent ideals then E has no non–zero nilpotent ideals.

Proof: If V is a nilpotent ideal of E then X, the closure of $\rho(V)$ in E', is a nilpotent ideal (since ρ is dominant). Thus, by assumption, V is contained in $\rho^{-1}(0)$, which is finite. It follows that V = { 0 }.

4.4.14 Theorem: Suppose that E is an irreducible reductive algebraic monoid. Then E is regular.

Proof: We may assume that E is a normal variety since the image of a regular monoid is regular. Assume also, for the moment, that E has a zero, and inductively that all reductive monoids of dimension less than dim(E) are regular. Now, as in the proof of 4.4.8, there exists a morphism ρ : E ——> E" such that ρ is generically finite–to–one and dominant, and E" has a faithful completely reducible representation. Further, if X and X" are the closures of respective maximal tori, then ρ : X ——> X" is an isomorphism. Let E' be the monoid associated with the integral closure of k[E"] in k[E] (see 3.2.3). Thus we have α : E ——> E' birational and β : E' ——> E" finite and dominant with $\beta \circ \alpha = \rho$.

Let f : E" ——> End(V) be a faithful completely reducible representation. Assume that there exists a nilpotent ideal N of E" and let t > 0 be its index of nilpotency. Let W be the subspace of V spanned by NV. Clearly the index of nilpotency of N restricted to W is t – 1. Thus W is a proper subspace of V.

Further, W is E"-invariant since EN is contained in N. Since V is completely reducible, there exists a subspace U of V such that V is the direct sum of U and W, and U is E"-stable. But by definition of W, NU is contained in W. Thus NU = { 0 } since U and W are complementary. Thus N has index of nilpotency t − 1. This contradiction proves that E" has no non-zero nilpotent ideals. It follows easily that E" can have no ideals consisting entirely of nilpotent elements. Let $z \in E"$ be an arbitrary element. Then E"zE" contains non-nilpotent elements. Thus there exists a non-zero idempotent $e \in E"zE"$. By 4.4.10 there exists $g,h \in G$ such that $gzh \in E"(e)$. But E"(e) is reductive (by 4.4.11) and of dimension strictly less than dim E". Hence inductively E"(e) is regular, so there exists $u,v \in G(E(e))$ such that $ugzhv = f = f^2$. Thus E" is regular. Now $\beta : E' \longrightarrow E"$ is finite and dominant so by 4.4.13 E' has no non-zero nilpotent ideals. Thus E' is regular as well by the same arguement.

Consider the morphism $\alpha : E \longrightarrow E'$. Recall that α is birational and induces an isomorphism $\alpha : X \longrightarrow X'$ on maximal irreducible D-submonoids. Clearly, if N and N' are the respective sets of nilpotent elements of E and E', then $\alpha^{-1}(N') = N$. Thus, $\alpha : E - N \longrightarrow E' - N'$. Let $e \in E - N$ be an idempotent. Then e is in the closure of $C(e)^0 = C(e)$. Now α restricts to a morphism $\alpha : CG(e) \longrightarrow CG'(\alpha(e))$. Restricting α to the closure of respective maximal tori yields an isomorphism. Thus, α also induces an isomorphism on Weyl groups (of CG(e) and $CG'(\alpha(e))$). It follows that $\alpha : CG(e) \longrightarrow CG'(\alpha(e))$ is bijective and hence, $\alpha : Cl(e) \longrightarrow Cl(\alpha(e))$ is bijective as well (since $Cl(e) = G/CG(e)$). Thus, α is one-to-one when restricted to idempotents, since by 4.4.2, α preserves the conjugacy classes.

Now suppose that $\alpha(x) = \alpha(e)$ for some $x \in E-N$. Thus, $\alpha(e(x)) = e(\alpha(x)) = \alpha(e)$. So, $e(x) = e$, since α is one-to-one when restricted to idempotents. By 4.4.10, and the induction hypothesis, $x \in E$ is a regular element. So, there exists $g,h \in G(E)$ such that $gxh = f$ is an idempotent. But then, $\alpha(g)\alpha(e)\alpha(h) = \alpha(g)\alpha(x)\alpha(h) = \alpha(f)$. Hence, by [21] $\alpha(e)$ and $\alpha(f)$ are conjugate. Thus, e and f are conjugate, since α preserves conjugacy. Thus, we may assume that $gxh = e = e(x)$. It follows that $x \in G(e)$ because in any representation of E, $\mathrm{rank}(x) = \mathrm{rank}(e)$.

By the proof of 4.4.11, eEe is reductive. Thus we have $\alpha : eEe \longrightarrow \alpha(e)E'\alpha(e)$ such that

(i) α is dominant.

(ii) α is one-to-one when restricted to the closure of a maximal torus.

(iii) eEe is reductive.

It follows from the induction hypothesis that $\alpha|eEe$ is finite-to-one. Thus, since $\alpha^{-1}(\alpha(e))$ is contained in $G(e)$, $\alpha^{-1}(\alpha(e))$ is finite. Since every element of $E'-N'$ is a unit times an idempotent, it follows that $\alpha : E-N \longrightarrow E'-N'$ is finite-to-one. Hence, $\alpha : E - N \longrightarrow E' - N'$ is onto and finite-to-one because E' is regular. By 1.1.2, α induces an isomorphism of $E - N$ onto $E' - N'$. Let $U' = E' - N'$. Identifying U' with $E - N$ via α we have a morphism $U' \longrightarrow E$. Thus by 1.1.5 there is a unique morphism $f : E' \longrightarrow E$ extending $U' \longrightarrow E$ (recall that the codimension of N' in E' is larger than one).

Thus α is an isomorphism because $\alpha \circ f = 1$. So E is regular.

Now assume that E is reductive but does not necessarily have a zero. Let $e \in I(E)$ be a minimal idempotent and let $x \in E$. Then without loss of generality $xe = ex = ke$ for some $k \in G(E)$. But then by 4.4.10 there exists $g, h \in G(E)$ such that $gxh \in E(e)$. By definition, $e \in E(e)$ is the zero of $E(e)$. Further, by 4.4.11 $E(e)$ is reductive. Thus, by the above arguement, $E(e)$ is regular. Hence, there exist $u,v \in G(E(e))$ such that ugxhv is an idempotent of $E(e)$. But then E is regular.

We thus have a significant generalization of a fundamental theorem of modern algebra.

<u>4.4.15 Theorem:</u> Let E be an irreducible algebraic monoid with zero. Then the following are equivalent.

(i) E is regular.

(ii) E is reductive.

(iii) E has no non-trivial nilpotent ideals.

<u>Proof:</u> 4.4.9 and 4.4.14.

4.5 <u>Connected Monoids With Zero</u>

<u>Definition:</u> Let E be an algebraic monoid with 0. Then E is <u>connected</u> if E is connected in the Zariski topology (equivalently, if k[E] has no non-trivial idempotents, since any non-trivial idempotent yields a direct product decomposition).

Let E be a connected monoid with 0 and let E^0 be the irreducible component of 1. Let T be a maximal torus of G^0 and let $e \in X$ be a minimal idempotent of X, the closure of T in E^0. e is not the identity element of E since this would imply that G^0 is closed in E, thereby contradicting the connectedness of E. Let $e(1) = e$, and $E(1) = eEe$. Then E(1) is an algebraic monoid with 0 and identity element e. If e' is another minimal idempotent then $E'(1)$ is isomorphic with $E(1)$ since, in any irreducible monoid all minimal idempotents are conjugate.

Assuming $e(1)$ is non-zero the procedure can be applied to $E(1)$. Thus, we obtain a sequence of idempotents
$1 = e(0) > e(1) > \ldots > e(k) = 0$ and monoids
$E(l) = e(l)E(l-1)e(l)$, $l = 1,\ldots,k$, such that, for all l, $e(l)$ is a minimal idempotent of $E(l-1)^{o}$.
If another such sequence
$1 = f(0) > f(1) > \ldots > f(m) = 0$ is chosen, then $k = m$ and $E(l)$ is isomorphic with $E'(l)$ for all l.

There is a converse to this result. For this we need a lemma.

<u>4.5.1 Lemma:</u> Suppose k^* acts on the affine variety X in such a way that the action extends to k. Suppose that $F(X,k)$, the fixed point set of this action, is connected. Then X is connected.

<u>Proof:</u> The action of k on X induces a direct sum decomposition $k[E] = \Sigma k[E](\alpha)$ where α ranges over $N = X(k)$. Thus, the composite, $k[E](0) \longrightarrow k[X] \longrightarrow k[F(X,k)]$ is an isomorphism ($F(X,k)$ is the fixed point set of the action). Now I, the set of idempotents of $k[X]$, is finite. So I is contained in $k[X](0)$. Thus, the co-ordinate ring of X has no non-trivial idempotents and X is thus connected.

<u>4.5.2 Theorem:</u> Let E be an algebraic monoid with zero. Then the following are equivalent.

(i) E is connected in the Zariski topology.

(ii) There is a chain of idempotents $1 = e(0) > e(1) > \ldots > e(k) = 0$ such that $e(i+1) \in e(i)Ee(i)^{o}$ for $i = 0,\ldots,k-1$.

(iii) For all non-zero idempotents $e \in I(E)$, $G(eEe)$, the group of units of eEe, is not closed in eEe.

Proof: (i) => (ii). Already given. (ii) => (i). Inductively we may assume that $e(1)Ee(1)$ is connected. Since $e(1) \in E^0$ there exists a 1-p.s.g. $\rho : k^* \longrightarrow G^0$ such that ρ extends to $\rho : k \longrightarrow E^0$ with $\rho(0) = e(1)$. Thus, $e(1)E$ is connected since the fixed point set of the action $f(t,x) = x\rho(t)$ on $e(1)E$ is $e(1)Ee(1)$ (so 4.5.1 applies). But then E is connected (again by 4.5.1) since $e(1)E$ is the fixed point set of the action $g(t,x) = \rho(t)x$ on E. (i) => (iii). This follows from the fact if E is connected then eEe is connected. (iii) => (ii). If G is not closed in E then by 2.2.7 there exists a non-trivial idempotent $e \in E^0$. Thus inductively we can construct a chain of idempotents as in (ii).

V IDEALS

The purpose of this chapter is to record some of the general properties of ideals. The main result is a structure theorem for prime ideals (5.2.1). I have assumed throughout that E is an irreducible algebraic monoid. An ideal of E is a subset J, of E such that EJE is contained in J.

5.1 Preliminary Results

5.1.1 Proposition: Let E be solvable and let I be an ideal of E. Then the following are equivalent.

(i) If some power of x is in I then x is in I (I is radical).

(ii) $I = \rho^{-1}(\rho(I))$ where $\rho : E \longrightarrow X$ is the universal morphism to a D–monoid (4.3.1).

Proof: (ii) => (i). Any ideal in a D–monoid is radical as is any pullback of a radical ideal.

(i) => (ii). ρ is onto, so $\rho(I)$ is an ideal of X. Thus $\rho(I)$ is the union of a finite number of orbits of idempotents (under the action of right translation). Let $eG(X)$ be an orbit of $\rho(I)$. Now $\rho^{-1}(eG(X)) = \{ x \in E \mid$ some power of x is in $G(e(x))$, $\rho(e(x)) = e \}$. So $\rho^{-1}(\rho(I)) = \{ x \in E \mid$ some power of x is in $I \}$, since, if $x \in I$ then $e(x) \in I$.

5.1.2 Corollary: Let E be irreducible and solvable. Then there is a canonical one–to–one correspondence between radical ideals of E and radical ideals of X, where $\rho : E \longrightarrow X$ is the universal D–monoid associated with E.

5.1.3 Proposition: Let E be irreducible and suppose that I is an ideal of E. Let Z be the closure in E of some Borel subgroup B of $G(E)$. If the intersection $I(B)$ of I with Z is a radical ideal then I is closed in E.

Proof: If $I(B)$ is radical it is closed by 5.1.2. But I is the union of all the conjugates of $I(B)$. Thus I is closed by 1.2.3 (i).

5.1.4 Corollary: Suppose P is a prime ideal of E (i.e. P is an ideal such that $E - P$ is multiplicatively closed). Then P is a closed subset of E.

Proof: Any prime ideal is radical.

5.1.5 Corollary: Suppose P and Q are prime ideals of E such that $P(s) = Q(s)$ (they contain the same semi-simple elements). Then $P = Q$.

Proof: If $P(s) = Q(s)$ then the same is true of $P(B)$ and $Q(B)$ (the intersections of the closure of B with P and Q respectively), where B is a Borel subgroup. Thus $\rho(P(B)) = \rho(Q(B))$ where ρ is the universal morphism from the closure of B to a D-monoid. Thus, by 5.1.1 $P(B) = Q(B)$. So $P = Q$.

5.1.6 Proposition: Let E be irreducible, T a maximal torus of $G(E)$, and W its Weyl group. Suppose I is a W-invariant prime ideal of Z, the closure of T in E. Then there exists a W-invariant character $z \in X(Z)$ such that $I = z^{-1}(0)$.

Proof: Assume Z is normal. Then $Z - I$ is a normal algebraic monoid variety. Let $e \in Z - I$ be the minimal idempotent. By Sumihiro's theorem [31;Corollary 2] there exists a T-invariant affine open subset U of $Z - I$ with $e \in U$. Thus $U = Z - I$ since any open subset of $Z - I$ with $e \in U$ intersects every other T-orbit. So $Z - I$ is affine and $Z - I \longrightarrow Z$ is an open imbedding. Consider $k[Z] \longrightarrow k[Z - I]$. If $A = \{ f \in k[Z] \mid f(I) = 0 \}$ then $Ak[Z - I] = k[Z - I]$. Since $A = (x(i))$, for some $x(i) \in X(Z)$, we have $\Sigma a(i)x(i) = 1$ for some $\{ a(i) \}$ in

$k[Z - I]$. Let $a(i) = \Sigma b(i,j)y(i,j)$, $b(i,j) \in k[Z - I]$, $y(i,j) \in X(Z - I)$. Then $\Sigma b(i,j)x(i)y(i,j) = 1$. After collecting terms we have $1 = \Sigma n(i,j)x(i)y(i,j)$ where all the $x(i)y(i,j)$ are distinct. Thus $x(i)y(i,j) = 1$ for some i and j because characters are linearly independent. Let $x = x(i)$ and $y = y(i,j)$. Then $x \in k[Z - I]$ is a unit. Thus, $k[Z - I] = k[Z][1/x]$ and consequently $A = r((x))$ ($r((x))$ denotes the radical of (x)). Let $w \in W$. Then $A = r(w^*(x))$ since A is W-invariant. Thus $A = r((z))$ where z is the product of all $w^*(x)$ as w ranges over W. Further, z is W-invariant and $I = z^{-1}(0)$. Hence, it remains to find z in case Z is not necessarily normal. Let $n : Z' \longrightarrow Z$ be the normalization of Z. The Weyl group acts on Z' as well; thus, choose $z \in k[Z]$ as above, so that $z^{-1}(0) = I' = n^{-1}(I)$ and z is W-invariant. By 4.5.1 some power, say v, of z is an element of X(Z). But $z^{-1}(0) = v^{-1}(0)$ in Z'. Hence the inverse image in Z' of $v^{-1}(0)$ is equal to $n^{-1}(I)$. Thus $v^{-1}(0) = I$ since n is onto.

5.2 The Structure Of Prime Ideals

5.2.1 Theorem: Let E be an irreducible algebraic monoid, T a maximal torus of G(E), W the Weyl group of T. Then

(i) If P is a prime ideal of E there exists a character $x \in X(E)$ such that $P = x^{-1}(0)$.

(ii) There are canonical bijections among the set of primes of E, the set of W-invariant primes of X (the closure of T in E) and the set of W-invariant idempotents of X.

Proof: Assume E is reductive. Let P be a prime ideal of E and let X be the closure of a maximal torus T of G(E). Consider P(T), the intersection of P and X. P(T) is a W-invariant prime ideal of X. Thus, by 5.1.6 there exists a W-invariant character

x on X such that $P(T) = x^{-1}(0)$. By 4.4.3, x lifts uniquely from $k[X]$ to $x \in cl[E]$. If S is another maximal torus then $S = gTg^{-1}$ so it follows that x is a character on E and that $x^{-1}(0)$ has the same semi-simple elements as P. Thus, by 5.1.5 $P = x^{-1}(0)$.

Now assume that E is not necessarily reductive. By 4.4.8 there exists a morphism $\rho : E \longrightarrow E'$ such that E' is reductive, $\rho : X \longrightarrow X'$ is an isomorphism, and $\rho^{-1}(1) = UR(G)$, the unipotent radical of G. Then $\rho(P(T)) = \rho(P)(T)$, since if $x \in \rho(P)(T)$ then there exists $s \in P$ semi-simple such that $\rho(s) = x$. But then there exists $u \in UR(G)$ such that $usu^{-1} \in P(T)$. Thus, $\rho(usu^{-1}) = \rho(s) = x$. Hence, by the remark following 4.4.1, there exists $v \in cl[E]$ such that $v^{-1}(0)$ intersected with X is equal to $P(T)$ (since $cl[E']$ is contained in $k[E]$). Hence, by 5.1.5, $P = v^{-1}(0)$. This proves (i).

Proof of (ii). If P is a prime ideal of E then $P(T)$ is W-invariant. If $P(T) = Q(T)$ then $P = Q$ by 5.1.5. Conversely, if I is a W-invariant prime then by 5.1.6 there exists a W-invariant character $v \in X(T)$ such that $I = v^{-1}(0)$. If $E \longrightarrow E'$ is the morphism of 4.4.8 to the reductive monoid E', then v can be lifted to a class function on E' which is apriori a class function on E. It follows that v is actually a character on E' and thus, on E. Since $v^{-1}(0)(T) = I(T)$ we see that every W-invariant prime of X occurs in this fashion.

If I is a W-invariant prime then $e(I)$, the minimal idempotent of $X - I$, is a W-invariant idempotent. Conversely, if $e \in X$ is a W-invariant idempotent then it follows from 4.1.2 (iv), that $I(e)$, the union of all fX as f ranges over all maximal idempotents not strictly larger than e, is a W-invariant prime ideal of X such that $e \in X - I$ is the minimal idempotent.

Remark: Theorem 5.2.1 demonstrates an important motif in the theory of algebraic monoids. One hopes that ultimately much of the theory of reductive monoids can be reduced to problems concerning D-monoids and their symmetries.

VI TWO-DIMENSIONAL REGULAR MONOIDS

The classification and structure theory of semi-simple rank one monoids, according to the next chapter, requires a deeper understanding of lower-dimensional monoids. The purpose of this chapter is to expose the properties of two-dimensional monoids which are relevant to these developments. For completeness I have also included the case (case 2 below) which is not needed in subsequent chapters.

6.1 Structural Properties

Definition: A monoid E is regular if for all $x \in E$ there is an idempotent e and a unit g such that $gx = e$.

Note that any two-dimensional irreducible monoid is solvable.

6.1.1 Proposition: Let E be a two-dimensional irreducible (non-trivial) algebraic monoid. Then the following are equivalent.

(i) E is regular.

(ii) Either E is a D-monoid or it does not have a zero.

(iii) E is Clifford (see 4.2).

Proof: Let $\rho : E \longrightarrow X$ be the universal D-monoid associated with E (as discussed in 4.3).

(i) => (ii). Plainly, a D-monoid is regular. So assume ρ is not an isomorphism. Since E is non-trivial, X is not 0-dimensional. Thus $\dim X = 1$. If $0 \in E$ then $I(E) = \{ 0, 1 \}$ since $I(X) = \{ 0, 1 \}$. Thus $\rho^{-1}(0) = \{ x \in E \mid x \text{ is nilpotent } \}$. Further, by 1.1.1, $\dim(\rho^{-1}(0)) \geq \dim(\rho)$. Thus E cannot be regular since the set of nilpotent elements is a two-sided ideal of E. Hence, if E is regular E cannot have a zero.

(i) => (iii). D-monoids are Clifford so again we may assume that E does not have a zero and that $\dim X = 1$.

(a) Assume E is commutative. Then $I(E) = \{ 1, e \}$ and e is non-zero. Since by 4.2.1 $es = se = e$ for all semi-simple elements s, we must have that $eG(u) = G(u)e$ is one dimensional. Hence, the morphism $G(u)xE(s) \longrightarrow E$ is finite-to-one. The image is open and multiplicatively closed and contains all semi-simple elements. Thus, by 5.1.5, it is onto. Hence, E is Clifford.

(b) Assume E is non-commutative. Then if $e \in E$ is an idempotent $\dim Cl(e) = 1$, since otherwise, $Cl(e) = \{ e \}$. So, e would be an element of the closure of every 1-p.s.g. of G. This is absurd since the union of the 1-p.s.g.'s is dense in G (this would force e to be the zero of E). Let V be a component of $E - G$. Then V is a two-sided ideal of E since its codimension is one. Thus, V contains an idempotent by 2.2.5. Thus, $Cl(e)$ is a subset of V. But the dimensions are the same so $Cl(e) = V$. Hence $E - G = V$ is irreducible of dimension one and $E - G = Cl(e)$. Clearly E is thus Clifford.

(iii) => (i). Any Clifford monoid is regular.

<u>Remark:</u> From the proof of 6.1.1 we have the following result. Suppose E is non-commutative, $\dim E = 2$ and $e \in E - G$ is an idempotent. Then E is the union of $G(E)$ and $Cl(e)$.

<u>Case 1:</u> E non-commutative, $\dim E = 2$.

Then either

(a) $eE = Cl(e)$ and

$Ee = \{ e \}$ or

(b) $Ee = Cl(e)$ and

$eE = \{ e \}$.

Proof: If $e, f \in Cl(e)$, then $ef = fe$ implies $e = f$ by 4.3.3. Thus, $eEe = \{ e \}$. Since e is not the zero of E, either $eE = Cl(e)$ or $Ee = Cl(e)$. Thus the conclusion follows.

For the remainder of case 1 I shall assume that $eE = Cl(e)$ and $Ee = \{ e \}$ (the other case is similar). The example to keep in mind is the set of two-by-two upper-triangular matrices $(a(i,j))$ such that $a(1,1) = 1$.

Note that for all $x, y \in E$, $xey = ey$.

6.1.3 Theorem: Let E be as above and let $e \in E$ be a non-trivial idempotent. Let $\rho : k^* \longrightarrow G$ be a 1-p.s.g. such that e is in the closure of $\rho(k^*)$. Then the action $(g,x) \longrightarrow g^{-1}xg$ of k^* on E extends to an action of k on E.

Proof: $g^{-1}xg = xg$. So the action clearly extends.

Case 2: E is commutative.

As in the proof of 6.1.1 the morphism $m : G(u)xE(s) \longrightarrow E$, $m(u,s) = us$, is finite and birational.

The remainder of this section is pre-occupied with several results concerning low-dimensional monoids. They are all ingredients in the structure theory of the next chapter.

6.1.5 Proposition: Let E be a reductive monoid with 0, one-dimensional center and semi-simple rank one. Let e be an idempotent not equal to 0 or 1. Let $R(e) = \{ g \in G \mid eg = ege \}$ and $L(e) = \{ g \in G \mid ge = ege \}$. Then R(e) and L(e) are opposite Borel subgroups.

Proof: The intersection of R(e) and L(e) is the centralizer of e in G which is a maximal torus. Thus R(e) and L(e) are 'opposite'. We prove R(e) is Borel.

There exists $g \in G$ unipotent, such that eg is not equal to

e. This follows from the formula immediately preceding 6.1.3 applied to either the closure of $k^*B(u)$ or $k^*B(u)^-$, where B and B^- are the Borel subgroups containing the centralizer of e in G, and k^* is the 1—p.s.g. whose closure contains e. Similarily, there exists $m \in G$ unipotent such that me is not equal to e.

By 3.4.7 Ee is contained in the closure of R(e). Because of m above dim Ee $\geq$ 2, and because of g, dim R(e) $\leq$ 3. Thus dim R(e) = 3 and dim Ee = 2 is the only possibility. Thus R(e) is Borel.

<u>6.1.6 Corollary:</u> Let E be as in 6.1.5. Then dim Ee = dim eE = 2.

VII SEMI-SIMPLE RANK ONE, REDUCTIVE MONOIDS

This chapter is an exposition of the main computations of the thesis. These results include a classification of all normal reductive monoids with zero and one–dimensional center, in case the semi–simple rank is one.

7.1 is a discussion of the possible groups and the possible monoid types which arise in this way.

7.2 is a record of some of the immediate corollaries that result from the von Neumann regularity of the underlying monoid.

In 7.3 a procedure is devised whereby finite morphisms between certain monoids can be replaced by morphisms with D–group kernels.

7.4 contains more technical preliminaries and a proof that normal, reductive monoids with zero and one–dimensional center are Cohen–Macaulay as algebraic varieties in case the semi–simple rank is one.

In 7.5 and 7.6 two classification theorems are established. The first makes use of certain bicartesian squares associated with the monoids in question and the second is based on a computation of the characters of a maximal irreducible D–submonoid.

7.1 Rank Two, Semi-simple Rank One, Reductive Groups

7.1.1 Proposition: Suppose G is a non–abelian reductive group, rk G = 2, and rkss G = 1. Then G is isomorphic to one of $Gl(k^2)$, $Sl(k^2)xk^*$, or $PGl(k^2)xk^*$.

Proof: Case 1. $(G,G) = Sl(k^2)$. Consider the morphism $m : Sl(k^2)xk^* \longrightarrow G$, $m(x,t) = xt$ (here k^* is the identity component of the center of G). If the kernel of m is non–trivial (scheme theoretically) it follows that $\ker(m) = \{ ([a,a],a) \in Sl(k^2)xk^* \mid a^2 = 1 \}$, where $[x,y]$ denotes the diagonal matrix with given entries. Hence, $G = Sl(k^2)xk^*/\ker(m) = Gl(k^2)$.

Case 2. $(G,G) = PGl(k^2)$. In this case the kernel of m has to be trivial, since $PGl(k^2)$ has no non–trivial finite normal D–subgroups. Thus G is isomorphic with $PGl(k^2)xk^*$.

<u>7.1.2 Proposition:</u> Suppose E is an irreducible algebraic monoid such that $G = G(E)$ is as in 7.1.1 and let T be a maximal torus of $G(E)$. Then there are three possibilities for the closure X, of T in E.

(i) $I(X) = \{ 1 \}$, in which case $G(E) = E$.

(ii) $I(X) = \{ 1, e \}$, in which case there exists a morphism $G'xk \longrightarrow E$ which is finite and dominant $(G' = (G,G))$.

(iii) $I(X) = \{ 1, e, f, 0 \}$, in which case 0 is the zero of E as well.

<u>Proof:</u> (i) follows from 2.2.7.

(ii) Suppose $I(X) = \{ 1, e \}$. Then if w is an element of the normalizer of T, $wew^{-1} = e$. Hence, e is contained in the closure of a W–invariant irreducible torus, S. But then S is central, because G is reductive. Thus, e is central and hence, e is in the closure of every maximal torus. Therefore, $I(E) = \{ 1, e \}$. Consider $E \longrightarrow eE$, $x \longrightarrow ex$. Since e is not the zero, eg is not equal to e for some $g \in G'$. Thus $G' \longrightarrow eE$ is finite to one and dominant since G has no non–trivial normal subgroups of positive dimension. Hence $m : G'xk \longrightarrow E$, $(x,y) \longrightarrow xy$, is finite–to–one and dominant. Thus it is also onto because the complement of the image is an ideal with no semi–simple elements. Thus m is finite by 1.1.2.

(iii) $I(T) = \{ 1, e, f, 0 \}$. If w is a non-trivial element of the normalizer of T, then the fixed idempotents of w are 0 and 1. Thus 0 is a central idempotent of E and by the conjugacy of maximal tori, 0 is the 0 of every maximal torus. Thus 0 is the 0 of E since the semi-simple elements of G are dense in E.

7.2 Properties Of Semi-simple Rank One Monoids

The purpose of this section is to record some of the geometric properties of semi-simple rank one monoids. Throughout I have assumed without further mention that E is irreducible, $0 \in E$, $\dim Z(G(E)) = 1$, and rkss $G(E) = 1$. Let $G = G(E)$.

<u>7.2.1 Proposition:</u> N, the set of nilpotent elements of E, is irreducible of dimension two.

<u>Proof:</u> It follows easily, since E is regular, that the set of nilpotent elements of the closure of a Borel subgroup, is irreducible of dimension one. Thus, since Borel subgroups are all conjugate and of codimension one, N is irreducible of dimension two.

<u>7.2.2 Proposition:</u> $\dim(E - G) = 3$ and $E - G$ is irreducible.

<u>Proof:</u> $E - G = x^{-1}(0)$ for some character $x : E \longrightarrow k$ by 5.2.1. So $\dim(E - G) = 3$ by Krull's principal ideal theorem. Let $z \in E - G$. Since E is regular, there exists $g \in G$ such that $gz = e$, a non-zero idempotent. Since all idempotents not equal to 0 or 1 are conjugate, $GzG = (E - G) - \{ 0 \}$. Thus $E - G$ is equal to the closure of GzG, which is irreducible.

<u>7.2.3 Corollary:</u> The action $G \times G \times E \longrightarrow E$, given by $(g,h,x) \longrightarrow gxh^{-1}$ has three orbits, $\{ 0 \}$, $(E - G) - \{ 0 \}$ and G.

<u>7.2.4 Corollary:</u> $E - G$ is the only non-trivial two-sided ideal.

<u>7.2.5 Proposition:</u> Let T be a maximal torus and let X be its

closure in E. Then $E - \{0\}$ and $X - \{0\}$ are smooth algebraic varieties, assuming E is normal.

Proof: Let Sing(E) be the singular locus of E. By Krull's characterization of normality, codim(Sing(E)) $\geq$ 2. Thus, by 7.2.4, Sing(E) is contained in $\{0\}$ because Sing(E) is a two-sided ideal of E.

By 4.4.4 (ii) $X - \{0\}$ is the fixed point set of the action of T on $E - \{0\}$ by inner automorphisms. Thus, $X - \{0\}$ is smooth since $E - \{0\}$ is smooth and T is linearly reductive.

In the next section we shall see that if E is normal then both E and T are Cohen-Macaulay.

7.2.6 Construction: Big Cell.

Let B and B^- be opposite Borel subgroups and let $e^2 = e \in X$, the closure of the maximal torus associated with B and B^-. Now $T(e) = X - fX$ is the unique open submonoid of X such that $I(T(e)) = \{1, e\}$ (where f is the other non-trivial idempotent of X). Further, T(e) is affine and $T(e) - T = ZG^0e$. Let Z and Z^- be the closures of B and B^- in E and let k^* be the 1-p.s.g of T which converges to e. Notice that e is in the closure of $k^*B(u)$ which is a two-dimensional regular monoid. Thus, the results of chapter 6 may be applied.

Assume that

$$Ze = Ee \text{ and}$$

$$eZ^- = eE$$

(as in 3.4.7 and 6.1.6).

Consider the morphism of varieties $m : B(u) \times T(e) \times B^-(u) \longrightarrow E$, $m(x,y,z) = xyz$. m is birational by the well known construction from group theory. To show that m is finite-to-one it suffices to show that $m^{-1}(e)$ is a finite set since $B(u)x(Te)xB^{-}(u)$ is an orbit under the action

$$B(u)x(ZG)^{o}xB^{-}(u)x(B(u)x(T(e))xB^{-}(u)) \longrightarrow (B(u)x(T(e))xB^{-}(u))$$

$$(u,t,v)*(x,y,z) = (ux,ty,zv^{-1}).$$

Suppose that $xyz = e$, $x \in B(u)$, $y = te$, $t \in Z(G)^{0}$ and $z \in B^{-}(u)$. So, $etz = x^{-1}e$ and thus, $x^{-1}e$ commutes with e. By the remark following 6.1.1 (applied to the closure of $B(u)k^{*}$), $x^{-1}e = e$. Similarily, $ez = e$. Thus, $te = e$ as well. But, $\{ (x,t,z) \in B(u)x(ZG)^{o}xB^{-}(u) \mid xe = te = ez = e \}$ is finite since, by assumption, $\dim(B(u)e) = \dim(ZG^{o}e) = \dim(eB^{-}(u)) = 1$.

<u>7.2.7 Proposition:</u> Assume E is normal. Then $m : BxT(e)xB^{-} \longrightarrow E$ is an open imbedding.

<u>Proof:</u> m is finite–to–one and birational. Since E is normal, m is an open imbedding by 1.1.2.

<u>7.2.8 Corollary:</u> Suppose E is normal and E' is another algebraic monoid. Let T be a maximal torus of $G(E)$. Suppose we have morphisms $\rho : G(E) \longrightarrow E'$ and $\alpha : T(e) \longrightarrow E'$ such that $\rho|T = \alpha|T$. Then there exists a unique morphism $\beta : E \longrightarrow E'$ such that $\beta|G(E) = \rho$ and $\beta|T(e) = \alpha$.

<u>Proof:</u> Let $U = B(u)T(e)B^{-}(u)$ be as in 7.2.7. Define $\beta' : U \longrightarrow E'$, $\beta'(x,y,z) = \rho(x)\alpha(y)\rho(z)$. Thus β' agrees with ρ on $G(E)$ and with α on $T(e)$. Thus there exists $\beta'' : V \longrightarrow E'$ extending both β and ρ (where V is the union of U and $G(E)$). But the codimension of $E - V$ is greater than or equal to two since V intersects $E - G$ and $E - G$ is irreducible. Thus, by 1.1.5 there exists a unique morphism $\beta : E \longrightarrow E'$ extending β''.

7.3 Constructing Morphisms And Applications

Let k be an algebraically closed field of characteristic $p > 0$. If E is an algebraic monoid defined over k, with group of units G, then there exist non-trivial purely inseparable morphisms $\rho : E \longrightarrow E$. The purpose of this section is to classify these morphism in case the group G is isomorphic to $Sl(k^2)xk^*$. Since the results recorded here are elementary in nature, proofs will often be omitted or sketched.

7.3.1 Proposition: Let B be the subgroup of upper-triangular matrices of $Sl(k^2)$ and let $\rho : B \longrightarrow B$ be a bijective morphism. Then there exists $n \in N$ and $g \in B$ such that $g\rho(a(i,j))g^{-1} = (F(n)(a(i,j)))$ for all $(a(i,j)) \in B$, where $F(n) : k \longrightarrow k$ is the Frobenius morphism composed with itself n times.

7.3.2 Proposition: Let $\rho : Sl(k^2) \longrightarrow Sl(k^2)$ be a bijective algebraic group homomorphism. Then there exists $g \in Sl(k^2)$ and $n \in N$ such that $g\rho((a(i,j)))g^{-1} = (F(n)(a(i,j)))$ for all $(a(i,j)) \in Sl(k^2)$.

Proof: There exists $g \in Sl(k^2)$ such that $\rho' = g\rho g^{-1}$ satisfies $\rho'(B) = B$ (B as above). Thus, by 7.3.1, $\rho'|B = F(n)|B$ for some $n \in N$. Thus, by 1.2.2 (i), $\rho' = F(n)$.

7.3.3 Proposition: Let $\rho : G(1) \longrightarrow G(2)$ be a morphism of algebraic groups, where G(1) and G(2) are each isomorphic to one of $Gl(k^2)$, $Sl(k^2)$ or $PGl(k^2)$. Let $\alpha(i) : Sl(k^2)xk^* \longrightarrow G(i)$, i = 1,2, be given by $\alpha(g,t) = (\beta(i)(g))t$, where $\beta(i) : Sl(k^2) \longrightarrow (G(i),G(i))$ is the universal covering map and $k^* \longrightarrow G(i)$ is the identity component of the center of G(i). Then there exists a unique morphism $\rho' : Sl(k^2) \longrightarrow Sl(k^2)$ such that $\rho o \alpha(1) = \alpha(2) o \rho'$.

Proof: ρ lifts to ρ' on the level of Borel subgroups. The morphism ρ' is as in Proposition 7.3.1. This morphism extends to all of $Sl(k^2)xk^*$. Thus, by 1.2.2 (i), $\rho o \alpha(1) = \alpha(2) o \rho'$.

7.3.4 Proposition: Suppose we have the following solid arrow diagram in the category of algebraic monoids, where G(i), i=1,2, is the group of units of E(i), i=1,2 and E(i)*, i=1,2, is constructed in accordance with 3.2.3 applied to the finite morphisms α and β. Assume that all horizontal morphisms are finite and dominant. Assume further, that E(1) is normal. Then the dotted arrow can be filled in uniquely.

$$\begin{tikzcd}
E(1) \arrow[r, dashed] & E(2) \\
G(1) \arrow[u] \arrow[r, "n"] & G(2) \arrow[u] \\
Sl(k^2)xk^* \arrow[u, "\alpha"] \arrow[d] \arrow[r] & Sl(k^2)xk^* \arrow[u, "\beta"'] \arrow[d] \\
E(1)^* \arrow[r] \arrow[uuu, bend left] & E(2)^* \arrow[uuu, bend right]
\end{tikzcd}$$

Proof: $n^*(k[E(2)])$ is contained in the intersection (in $k[Sl(k^2)xk^*]$) of $k[G(1)]$ and $k[E(1)^*]$. But $k[E(1)]$ is equal to the intersection of $k[G(1)]$ and $k[E(1)^*]$, since E(1) is normal. Thus the arrow exists.

7.3.5 Proposition: Suppose $\rho : E(1) \longrightarrow E(2)$ is a finite dominant morphism of normal algebraic monoids (G(i) = G(E(i)) as in 7.3.3). Then either

(i) There exists a finite dominant morphism $\alpha : E(1) \longrightarrow E(2)$ such that kernel(α) is a finite D-group; or

(ii) There exists a commutative diagram

$$\begin{tikzcd}
E(3) \arrow[r] \arrow[d] & E(1) \arrow[d] \\
E(2) \arrow[r] & E'
\end{tikzcd}$$

such that every morphism is finite and dominant and every kernel is a finite D-group.

<u>Proof:</u> By 7.3.3 and 7.3.4 it suffices to prove this if $G(1) = G(2) = Sl(k^2)$. So we have $\rho : Sl(k^2)xk^* \longrightarrow Sl(k^2)xk^*$. Since ρ is bijective, we may assume, by 7.3.2, that $\rho = (F(n),sF(m))$, where $F(n)$ is as in 7.3.1 and $(s,p) = 1$.

<u>Case 1:</u> $n < m$.

Consider the diagram,

$$\begin{tikzcd}
Sl(k^2)xk^* \arrow[r, "{(F(n),F(n))}"'] \arrow[rr, bend left, "{(F(n),sF(m))}"] \arrow[d] & Sl(k^2)xk^* \arrow[r, "{(1,sF(m-n))}"'] \arrow[d] & Sl(k^2)xk^* \arrow[d] \\
E(1) \arrow[r, "\beta"] \arrow[rr, bend right, "\rho"'] & E(1) \arrow[r, dashed] & E(2)
\end{tikzcd}$$

β exists by 7.2.8 since on the level of characters

$$F(n) : X \longrightarrow X, \ F(n) \text{ as in } 7.3.1,$$

is the desired extension.

Thus, on the level of characters, we have

$$\begin{tikzcd}
X(1) & X(1) \arrow[l] & X(2) \arrow[l] \arrow[ll, bend right] \\
X(1)' \arrow[u] \arrow[ur, phantom, "A"] & X(1)' \arrow[l] \arrow[u] & X(2)' \arrow[l, dashed] \arrow[u] \arrow[ll, bend left]
\end{tikzcd}$$

The dotted arrow exists because the diagram A is a pullback. Hence, again, by 7.2.8, we can fill in the dotted arrow (of case 1) to a morphism $E(1) \longrightarrow E(2)$.

<u>Case 2:</u> n > m.

Let $E' = E(2)/K$, where $K = \{ x \in G(2) \mid (F(n-m))(x) = 1 \}$. Then we have

$$\begin{tikzcd} E(1) \arrow[r, "{(F(n),sF(m))}"] & E(2) \arrow[r, "{(1,F(n-m))}"] & E' \end{tikzcd}$$

$f = (1,F(n-m))$ is the desired morphism $E(2) \longrightarrow E'$. Composing $(1,F(n-m))$ and $(F(n),sF(m))$ we obtain $g = (F(n),sF(n))$. Noting that $E' = E(1)/\ker(g)$, we also obtain the following diagram,

$$\begin{tikzcd} Sl(k^2)xk^* \arrow[r, "{(1,s)}"'] \arrow[rr, bend left, "{(F(n),sF(n))}"] \arrow[d] & Sl(k^2)xk^* \arrow[r, "{(F(n),F(n))}"'] \arrow[d] & Sl(k^2)xk^* \arrow[d] \\ E(2) \arrow[r, dashed] \arrow[rr, bend right, "g"'] & E' \arrow[r, "\beta"] & E' \end{tikzcd}$$

β exists (by 7.2.8) just as in case 1 above.

On the level of characters we have

$$\begin{tikzcd} X(1) & X' \arrow[l] & X' \arrow[l] \\ X(1)^* \arrow[u] & X'^* \arrow[l, dashed] \arrow[u] & X'^* \arrow[l] \arrow[u] \arrow[ll, bend left] \end{tikzcd}$$

where * denotes the characters of the closure of the torus in question. The dotted arrow exists because the image of X'^* in $X(1)$ is finite over the image of X'^* in $X(1)^*$. Applying 7.2.8 again we obtain, in this case

$$E(1) \longrightarrow E'.$$

From above, we also have a morphism $f : E(2) \longrightarrow E'$. Thus, taking the pullback of these two morphisms and restricting the resulting diagram to (normalized) identity components, we obtain the diagram advertised in (ii) above.

7.4 Cohen-Macaulay Monoids

Let E be a reductive algebraic monoid such that dim $Z(G(E)) = 1$, rkss $G(E) = 1$ and $0 \in E$.

7.4.1 Lemma: There exists a representation $\rho : E \longrightarrow \mathrm{End}(V)$ such that

(i) ρ is an irreducible representation.

(ii) ρ is a finite morphism.

Proof: By the proof of 4.4.8 there exists an irreducible representation $\rho : E \longrightarrow \mathrm{End}(W)$ of E such that $\rho(0) = 0$ and no idempotent of E is sent to 0. Let V be an E–simple summand of W such that some idempotent e of E is non–zero on V. By 7.2.3, $\rho|V : E \longrightarrow \mathrm{End}(V)$ has a trivial kernel. It follows from 3.4.13 that $\rho|V$ is a finite morphism.

Let $\rho : E \longrightarrow \mathrm{End}(V)$ be finite and irreducible as in 7.4.1. Then we have the following commutative diagram:

$$\begin{array}{ccc} Sl(k^2) & \xrightarrow{F(n)} & \mathrm{End}(k^2) \\ \downarrow & & \downarrow m \\ E & \xrightarrow[\rho]{} & \mathrm{End}(V) = \mathrm{End}(m(K^2)) \end{array}$$

where m denotes the m–th symmetric power and F(n) is as in 7.3.1. This follows from the fact that every irreducible representation of (G(E),G(E)) is isogenous to a symmetric power of the canonical two–dimensional representation of $Sl(k^2)$. Clearly,

(i) $m : \mathrm{End}(k^2) \longrightarrow \mathrm{End}(m(k^2))$ is finite.

(ii) $\rho(Z(E)^0) = Z(\mathrm{End}(m(k^2)) = m(Z(\mathrm{End}(k^2)))$.

Thus, $\rho(E) = m(\mathrm{End}(k^2))$.

Hence, if we let $E(1) = \mathrm{image}(m)$, then we obtain, $m : \mathrm{End}(k^2) \longrightarrow E(1)$ and $\rho : E \longrightarrow E(1)$ such that

(i) both m and ρ are finite morphisms.

(ii) kernel(m) is a finite D-group ($X(\mathrm{kernel}(m)) = Z/mZ$).

By 7.3.5 applied to ρ above, we have;

7.4.2 Proposition: Let E be a reductive, normal algebraic monoid with 0, such that dim $ZG(E) = 1$ and rkss $G(E) = 1$. Then there exists either

(i) a morphism $\rho : E \longrightarrow m(\mathrm{End}(k^2))$ such that ρ is finite and dominant ('m' denotes m-th symmetric power) and kernel(ρ) is a finite D-group, or

(ii) morphisms $\beta : E \longrightarrow E'$ and $\alpha : m(\mathrm{End}(k^2)) \longrightarrow E'$ such that both β and α are finite and dominant and have finite D-group kernels.

7.4.3 Note: In case (ii) we may assume that E' is normal. Then there exists an isomorphism $E' \longrightarrow ml(\mathrm{End}(k^2)$ such that with this identification, $\alpha \circ m = ml$ (the ml-th symmetric power), where $l = \mathrm{degree}(\alpha)$. Thus in either case (7.4.2 (i) or (ii)) we have morphisms $m : \mathrm{End}(k^2) \longrightarrow m(\mathrm{End}(k^2))$ and $\rho : E \longrightarrow m(\mathrm{End}(k^2))$ such that both m and ρ are finite and dominant and have finite D-group kernels.

7.4.4 Theorem: Let E be as in 7.4.2. Then E is Cohen-Macaulay.

Proof: We have from 7.4.3, the following diagram, where $R = \mathrm{End}(k^2)$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & R \arrow[d, "m"] \\ E \arrow[r, "\rho"'] & m(R) \end{tikzcd}$$

Here X is the normalization of the identity component of the pull-back of m and ρ. All morphisms have finite D-group kernels and R is a smooth variety. By 1.3.8, X is Cohen-Macaulay and thus, by 1.3.9, E is Cohen-Macaulay.

7.4.5 Theorem: Let E be as in 7.4.2 and let T be a maximal torus of G(E). Then the closure of T in E is a normal algebraic variety.

Proof: Again from 7.4.3 we have

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & R \arrow[d, "m"] \\ E \arrow[r, "\rho"'] & m(R) \end{tikzcd}$$

where all morphisms are finite and dominant and R is a regular variety. It suffices to prove that if T is a maximal torus of X then Z, its closure in X, is normal. This follows from the fact that $\beta(Z)$ is isomorphic to $Z/\beta^{-1}(1)$, so, by 1.3.2 (iv), $\beta(Z)$ is normal if Z is.

Now let W be a maximal irreducible D-submonoid of R and let $Z = \alpha^{-1}(W)$ (a priori non-reduced). Let $n = \deg\alpha = \dim k[\beta^{-1}(1)]$. Since R is regular and X is Cohen-Macaulay, α is a flat morphism. Thus, $\alpha|Z : Z \longrightarrow W$ is flat of degree n. We have inclusions $Z^*(red) \longrightarrow Z(red)$ and $j : Z(red) \longrightarrow Z$, where Z(red) is the reduced variety associated with Z and $Z^*(red) = \rho^{-1}(T)$ (T is the group of units of W). Now Z*(red) is a commutative subgroup of G(X) consisting entirely of semi-simple elements. Thus Z*(red) is actually a maximal torus. Since $\beta^{-1}(1)$ is contained on Z*(red), $\rho|T$ is flat of degree = n = dim $k[\beta^{-1}(1)]$. Thus j is an isomorphism because otherwise deg $\rho|T$ = deg ρ < deg α = n. Plainly, Z is then equal to the closure of Z*. Hence, $\alpha|Z : Z \longrightarrow W$ is flat and thus Z is Cohen-Macaulay. But then Z is normal because by 7.2.5 the singular locus has codimension larger than or equal to two (by 7.2.5).

7.5 Classification I

By 7.4.3 and the proof of 7.4.4 we have, for E normal, reductive with $0 \in E$, rk G(E) = 2 and rkss G(E) = 1, the following commutative diagram in the category of algebraic monoids;

7.5.1 Diagram

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "g"'] & \mathrm{End}(k^2) \arrow[d, "\alpha"] \\ E \arrow[r, "\beta"'] & E'' \end{tikzcd}$$

such that all morphisms are finite and dominant and all kernels are finite D-groups.

We would like to have as rigid a diagram as is possible, so as to maximize its technical efficiency in further developments. To this end, we may assume that K, the intersection of ker(f) and ker(g), is scheme theoretically trivial because both f and g factor uniquely through $E' \longrightarrow E'/K$. Thus, the composite, $\ker(g) \longrightarrow E' \longrightarrow \mathrm{End}(k^2)$, is a closed imbedding. Further, f(ker(g)) is contained in $\ker(\alpha)$ since diagram 7.5.1 commutes. Letting H = f(ker(g)) we see that β factors through

$End(k^2)/H \longrightarrow E''$ since g is the universal morphism vanishing on ker(g).

Thus, summing up, we may assume that, diagram 7.5.1 satisfies the following properties.

(i) Every kernel is central.

(ii) ker(f) and ker(g) have trivial scheme theoretic intersection.

(iii) $f : ker(g) \longrightarrow ker(\alpha)$ is an isomorphism.

(iv) $g : ker(f) \longrightarrow ker(\beta)$ is an isomorphism.

(v) The diagram is bicartesian.

If Z' is the closure of some maximal torus in E' and $Z = g(Z')$, $Z^* = f(Z')$ and $Z'' = \alpha(Z^*) = \beta(Z)$ then we have the following commutative diagram in the category of algebraic monoids.

<u>7.5.2 Diagram:</u>

$$\begin{array}{ccc} Z' & \xrightarrow{f} & Z^* \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Z & \xrightarrow[\beta]{} & Z'' \end{array}$$

such that every morphism is finite and dominant and the diagram is bicartesian on the group level (i.e. It is both a pull–back and a push–out).

<u>$G(E) = Gl(k^2)$</u>

Restricting the diagram 7.5.2 to the centers of each group we have the following commutative diagram in the category of algebraic groups.

7.5.3 Diagram:

$$\begin{array}{ccc} ZG' & \xrightarrow{f} & k^* \\ g\downarrow & & \downarrow\alpha \\ k^* & \xrightarrow[\beta]{} & k^* \end{array}$$

Further, 7.5.3 is bicartesian because all the kernels in diagram 7.5.1 are central. Since G' = $Gl(k^2)$ or $Sl(k^2)xk^*$, ZG' = k^* or $(Z/2Z)xk^*$.

If G' = $Gl(k^2)$ then 7.5.3 becomes,

$$\begin{array}{ccc} k^* & \xrightarrow{f} & k^* \\ g\downarrow & & \downarrow\alpha \\ k^* & \xrightarrow[\beta]{} & k^* \end{array}$$

Thus, degreeα and degreeβ are both odd, since if degreeα is even then G(E") (in 7.5.1) is isomorphic to $PGl(k^2)xk^*$ and thus, degreeβ is even as well. But then the pull-back (k^* at upper left) could not be irreducible (the number of irreducible components of the pull-back is equal to the greatest common divisor of deg(α) and deg(β)). Similarily, degreeβ is odd. Furthermore, degreeα and degreeβ are relatively prime for the same reason.

If G' = $Sl(k^2)xk^*$ then 7.5.3 becomes

$$\begin{array}{ccc} (Z/2)xk^* & \xrightarrow{f} & k^* \\ g\downarrow & & \downarrow\alpha \\ k^* & \xrightarrow[\beta]{} & k^* \end{array}$$

Since f and g are the restrictions of morphisms $Sl(k^2)xk^* \longrightarrow$

$Gl(k^2)$ with finite D-group kernels, $f(i,x) = i+mx$ and $g(i,x) = i+nx$ for some m and n (where $i \in Z/2$ is the non-trivial element viewed as an element of k^*. Here, I have written k^* additively). Thus, $(m,n) = 1$ because by assumption the intersection of ker(f) and ker(g) is trivial. But m and n cannot both be odd because $i+mi = i+ni = 0$ for m and n odd (and by assumption, ker(f) and ker(g) have trivial intersection). Thus, $(m,n) = 1$ and one of m and n is even.

Conversely, if $(m,n) = 1$ and one of m and n is odd then $i+mx = i+nx$ has no solution for x (because this implies that x has order 2 and no element of order 2 satisfies the equation).

Let us summarize these results as follows:

7.5.4 Proposition: Let

$$\begin{tikzcd} ZG' \arrow[r, "f"] \arrow[d, "g"'] & k^* \arrow[d, "\alpha"] \\ k^* \arrow[r, "\beta"'] & k^* \end{tikzcd}$$

be the diagram of 7.5.3.

(a) Then there are two possibilities.

(i) $G' = Gl(k^2)$

Then $ZG' = k^*$, $\alpha(x) = nx$, $\beta(x) = mx$, $(m,n) = 1$ and mn is odd.

(ii) $G' = Sl(k^2)xk^*$

Then $ZG' = (Z/2Z)xk^*$, $\alpha(x) = 2nx$, $\beta(x) = 2mx$, $(m,n) = 1$ and mn is even.

(b) Furthermore, all diagrams defined in (a) occur as the restriction of the appropriate diagram 7.5.1 to the centers of the various groups.

<u>Proof:</u> It remains to verify (b).

Case (i). Define $\alpha : Gl(k^2) \longrightarrow Gl(k^2)$, $\alpha(x) = [\det(x)^{**}m, \det(x)^{**}m]x$ and $\beta : Gl(k^2) \longrightarrow Gl(k^2)$, $\beta(x) = [\det(x)^{**}n, \det(x)^{**}n]x$. On the level of the center, $\alpha(x) = x^{**}(2m+1)$ and $\beta(x) = x^{**}(2n+1)$. So choose m and n such that $(2m+1, 2n+1) = 1$ (here '**' denotes exponentiation and $[u,v]$ denotes the diagonal matrix with given entries).

Case (ii). Define $\alpha : Gl(k^2) \longrightarrow PGl(k^2)xk^*$, $\alpha(x) = ([x], \det(x)^{**}m)$ and $\beta : Gl(k^2) \longrightarrow PGl(k^2)xk^*$, $\beta(x) = ([x], \det(x)^{**}n)$. Then on the level of the center $\alpha(x) = x^{**}2m$ and $\beta(x) = x^{**}2n$. So choose m and n so that $(m,n) = 1$ and mn is even.

The procedure I have adopted in the classification is to follow the diagram 7.5.2 from Z* to Z" to Z, keeping track of the induced map on the level of characters.

<u>7.5.5 X(Z"). Z" as in 7.5.2;</u> degrèe α odd.

<u>Notation:</u> The diagonal two–by–two matrix $(a(i,j))$, will be written as $[a(1,1), a(2,2)]$ and the characters of a D–monoid will always be written additively.

$Z^* = \{ [a,b] \mid a,b \in k \}$ and $\ker(\alpha) = \{ [x,x] \mid x^{**}n = 0 \}$ for some odd value of n ('**' denotes exponentiation). Thus, if $u : [a,b] \longrightarrow a$, and $v : [a,b] \longrightarrow b$ are the generators of $X(Z^*)$, we have a short exact sequence

$$X(Z") \xrightarrow[\alpha^*]{} X(Z^*) \xrightarrow[j]{} Z/nZ$$

where $j(u)=j(v)$ is the generator of Z/nZ. Thus, by observation, $X(Z") = ((n-1)u/2+(n+1)v/2, (n+1)u/2+(n-1)v/2) = (z,w)$.

Since Z" is normal (see 7.4.5 and 4.1.5), X(Z") is equal to the intersection in X(T*) of X(Z*) and X(T"). Thus, it follows that

$X(Z") = \{ x \in (z,w) \mid lx \in \langle (n+1)z/2+(1-n)w/2, (n+1)w/2+(1-n)z/2 \rangle$ for some l $\}$.

$(n+1)z/2+(1-n)w/2$ and $(n+1)w/2+(1-n)z/2 \in X(Z")$ are called the <u>fundamental generators</u> of $X(Z")$ (see 4.1.7).

<u>7.5.6</u> Summing up, we have $\alpha^* : X(Z") \longrightarrow X(Z^*)$ with

$$\alpha^*(z) = (n-1)u/2+(n+1)v/2$$

$$\alpha^*(w) = (n+1)u/2+(n-1)v/2$$

The fundamental generators of $X(Z")$ are $(n+1)z/2+(1-n)w/2$ and $(n+1)w/2+(1-n)z/2$.

<u>7.5.7 X(Z"); Z" as in 7.5.2;</u> degree α even.

$Z^* = \{ [a,b] \mid a,b \in k \}$ and $\ker(\alpha) = \{ [a,a] \mid a^{**}k = 1 \}$.

Here, $k = 2n$. From the proof of 7.5.4 (or directly), we have

$\alpha : T^* \longrightarrow T"$, $\alpha([a,b]) = (ab^{-1}, (ab)^{**}n)$.

Hence, if $X(T") = (z,w)$, where z and w are the projections onto the first and second factors, we have,

$$\alpha^* : X(Z") \longrightarrow X(Z^*)$$

$$\alpha^*(z) = u-v$$

$$\alpha^*(w) = n(u+v).$$

In this case the fundamental generators are $w+nz$ and $w-nz$. Hence,

$X(Z") = \{ x \in (z,w) \mid lx \in \langle w+nz, w-nz \rangle$ for some l $\}$

Note that we could compute a presentation of $X(Z")$ directly from this.

<u>7.5.8</u> Summing up, we have $\alpha^* : X(Z") \longrightarrow X(Z^*)$, with

$$\alpha^*(z) = u-v$$

$$\alpha^*(w) = n(u+v).$$

The fundamental generators of $X(Z")$ are $w+nz$ and $w-nz$.

<u>7.5.9 X(Z);</u> degree β odd.

From the proof of 7.5.4, $\beta : Gl(k^2) \longrightarrow Gl(k^2)$ is given by $[a,b] \longrightarrow [a**((m+1)/2), b**((m-1)/2)]$ when restricted to T, the set of diagonal matrices (here,'**' denotes exponentiation). Thus, if $T = \{ [a,b] \mid a,b \in k^* \}$, and u and v denote the characters $u : [a,b] \longrightarrow a$, $v : [a,b] \longrightarrow b$, then

$$\beta^*(z) = (m+1)u/2+(m-1)v/2 \text{ and}$$

$$\beta^*(w) = (m-1)u/2+(m+1)/2.$$

So if $X(T") = (z,w)$ and $X(Z")$ is as in 7.5.5 we have

<u>7.5.10</u>

$$\beta^*((n+1)z/2+(1-n)w/2) = (m+n)u/2+(m-n)v/2$$

$$\beta^*((n+1)w/2+(1-n)z/2) = (m+n)v/2+(m-n)u/2$$

Since β^* is finite and Z is normal, we obtain

$X(Z) = \{ x \in (u,v) \mid lx \in \langle(m+n)u/2+(m-n)v/2, (m+n)v/2+(m-n)u/2\rangle$ for some l }, and $F = \{ (m+n)u/2+(m-n)v/2, \ (m+n)v/2+(m-n)u/2 \}$ is the set of fundamental generators of X(Z).

<u>7.5.11 X(Z);</u> degree β even.

From the proof of 7.5.4 $\beta : Gl(k^2) \longrightarrow PGl(k^2)xk^*$ is given by $\beta([a,b]) = (ab^{-1}, (ab)**m)$ for some m, when restricted to the diagonal group T of $Gl(k^2)$. Thus if $u([a,b]) = a$ and $v([a,b]) = b$ then we have

$$\beta^*(z) = u-v$$

$$\beta^*(w) = m(u+v).$$

So if $X(T) = (z,w)$ and $X(Z")$ is as in 7.5.7 we have

<u>7.5.12</u>

$$\beta^*(w+nz) = (m+n)u/2+(m-n)v/2$$

$$\beta^*(w-nz) = (m+n)v/2+(m-n)u/2$$

Since β^* is finite and Z is normal, we obtain

$X(Z) = \{ x \in (u,v) \mid lx \in \langle(m+n)u+(m-n)v, (m+n)v+(m-n)u\rangle$ for some

l } and F = { $(m+n)u+(m-n)v$, $(m+n)v+(m-n)u$ } is the set of fundamental generators of $X(Z)$ because $(m+n,m-n)=1$ whenever $(m,n)=1$ and mn is even.

7.5.13 Construction of $X(Z)$, $G(E) = Gl(k^2)$. Summary.

Given E, there is a bicartesian diagram in the category of algebraic monoids.

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "g"'] & End(k^2) \arrow[d, "\alpha"] \\ E \arrow[r, "\beta"'] & E'' \end{tikzcd}$$

Let Z be the closure in E of some maximal torus T of $G(E)$ and let $X(T) = (u,v)$.

Case (i): degreeα = n is odd.

Then degreeβ = m is odd and $(n,m) = 1$. Further,

$X(Z) = \{ x \in (u,v) \mid lx \in \langle (m+n)u/2+(m-n)v/2,(m-n)u/2+(m+n)/2\rangle$ for some l }

Case (ii): degreeα = $2n$ is even.

Then degreeα = $2m$ is even, $(m,n) = 1$ and mn is even. Further,

$X(Z) = \{ x \in (u,v) \mid lx \in \langle (m+n)u+(m-n)v,(m+n)v+(m-n)u\rangle$ some l }

In both cases,

$$w(u) = v$$

$$w(v) = u$$

for the non-trivial element $w \in W$, the Weyl group of T. Thus,

$$w = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

relative to the basis { u, v } of $X(T)$.

To construct all possible character monoids (4.1.1), that occur in this fashion, let

$$\alpha, \beta \in Z$$

$$\alpha > |\beta|$$

$$(\alpha, \beta) = 1$$

If $\alpha+\beta$ is odd then

$X(\alpha,\beta) = \{ x \in (u,v) \mid lx \in \langle \alpha u+\beta v, \alpha v+\beta u \rangle \text{ some } l \}$.

These are the character monoids of case (i) where $m = \alpha+\beta$ and $n = \alpha-\beta$.

If $\alpha+\beta$ is even then

$X(\alpha,\beta) = \{ x \in (u,v) \mid lx \in \langle \alpha u+\beta v, \alpha v+\beta u \rangle \text{ some } l \}$.

These are the character monoids of case (ii), where $m = (\alpha+\beta)/2$ and $n = (\alpha-\beta)/2$.

<u>$G(E) = Sl(k^2)xk^*$</u>

To classify the monoids with group $Sl(k^2)xk^*$ I have used the results concerning $Gl(k^2)$ and some general results about D–group actions.

Let E be as in 7.5.1 and suppose $G(E) = Sl(k^2)xk^*$. There is a canonical morphism,

$m : Sl(k^2)xk^* \longrightarrow Gl(k^2)$, $m(x,\beta) = x[\beta,\beta]$.

On the toric level, $m([\alpha,\alpha^{-1}],\beta) = [\alpha\beta,\alpha^{-1}\beta]$.

m is isomorphic to the quotient morphism of $Sl(k^2)xk^*$ by the subgroup $K = \{ ([\alpha,\alpha],\alpha) \mid \alpha^2 = 1 \}$.

Thus we have

$$\begin{array}{ccc} Sl(k^2)xk^* & \xrightarrow{m} & Gl(k^2) \\ \Big\downarrow & & \Big\downarrow j \\ E & \longrightarrow & E/K \end{array}$$

Here, $k[E/K] = k[E](0)$ (where 0 is the trivial character on K), as in 1.3.1 applied to the action $KxE \longrightarrow E$, $(x,y) \longrightarrow xy$. By

1.3.2 (iv), E/K is a normal algebraic monoid, and further, $j : Gl(k^2) \longrightarrow G(E/K)$ is an isomorphism. Hence, $E/K = E'$ is a monoid of the type just classified. It follows from the definition of m, that (on the level of maximal tori)

$m|T : T \longrightarrow T'$ induces

$$m^* : X(T') \longrightarrow X(T),$$
$$m^*(u) = a+b$$
$$m^*(v) = a-b$$

where $X(T) = (a,b)$ and $X(T') = (u,v)$.

So if Z' is the closure of T' in E', then

$X(Z') = \{ x \in (u,v) \mid kx \in \langle \alpha u+\beta v, \alpha v+\beta u \rangle \text{ for some } k \}$

as summarized in 7.5.13. Thus, since $m^*(u) = a+b$ and $m^*(v) = a-b$, we have

$$m^*(\alpha u+\beta v) = (\alpha+\beta)a+(\alpha-\beta)b$$
$$m^*(\alpha v+\beta u) = (\alpha+\beta)a+(\beta-\alpha)b.$$

<u>Case (i):</u> $\alpha+\beta$ is odd.

Then as in 7.5.13 (i), $(\alpha+\beta, \alpha-\beta) = 1$ and hence $F = \{ (\alpha+\beta)a+(\alpha-\beta)b, (\alpha+\beta)a+(\beta-\alpha)b \}$ is the set of fundamental generators.

Thus, to construct the possible character monoids (4.1.1) that occur in this fashion, let $m,n > 0$, $(m,n) = 1$, mn odd (here, $m=\alpha+\beta$ and $n=\alpha-\beta$). Then

$X(T(m,n)) = \{ x \in (a,b) \mid kx \in \langle ma+nb, ma-nb \rangle \text{ for some } k \}$.

If $w \in W$ is the non-trivial element of the Weyl group of T, then $w(a) = a$ and $w(b) = -b$, so

$$w = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

relative to the basis $\{ a, b \}$ of $X(T)$.

<u>Case (ii):</u> $\alpha+\beta$ is even.

Then as in 7.5.13 (i), $((\alpha+\beta)/2,(\alpha-\beta)/2) = 1$ and hence, $F = \{ (\alpha+\beta)a/2+(\alpha-\beta)b/2, (\alpha+\beta)a/2+(\beta-\alpha)b/2 \}$ is the set of fundamental generators.

Thus to construct the possible character monoids that occur in this fashion, let $m,n > 0$, $(m,n) = 1$, mn even (here $m=(\alpha+\beta)/2$, $n=(\alpha-\beta)/2$). Then

$X(Z(m,n)) = \{ x \in (a,b) \mid kx \in \langle ma+nb, ma-nb \rangle$ for some $k \}$.

If $w \in W$ is the non-trivial element of the Weyl group of T, then $w(a) = a$ and $w(b) = -b$, so

$$w = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

relative to the basis $\{ a, b \}$ of $X(T)$.

<u>7.5.14 $X(Z)$; $G(E) = Sl(k^2)xk^*$. Summary.</u>

Given E, there exists

$$m : E \longrightarrow E'$$

such that $G(E') = Gl(k^2)$ and $m : Sl(k^2)xk^* \longrightarrow Gl(k^2)$ is given by $m(x,\beta) = x[\beta,\beta]$.

Let T be a maximal torus of $G(E)$ and let Z be the closure in E. Using the morphism m, and our classification of the monoids E', we obtain: $Z = Z(m,n)$ for some $m,n > 0$, $(m,n) = 1$, where

$X(Z(m,n)) = \{ x \in (a,b) \mid kx \in \langle ma+nb, ma-nb \rangle$ for some $k \}$.

If $w \in Aut(Z)$ is the non-trivial element, then

$$w = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

relative to the basis $\{ a, b \}$ of $X(T) = X(T(m,n))$.

<u>$G(E) = PGl(k^2)xk^*$</u>

Let E be as in 7.5.1 and suppose $G(E) = PGl(k^2)xk^*$. There is a canonical morphism,

$c : Gl(k^2) \longrightarrow PGl(k^2)xk^*$, $c(x) = ([x], \det(x))$.

If $T' = \{ [a,b] \mid ab \in k^* \}$ and $T = c(T')$, then the sequence $u(2) \longrightarrow T' \longrightarrow T$ is exact, where $u(2) = \{ [a,a] \mid a^2 = 1 \}$. Thus on the level of characters, we have

$$c^* : X(T) \longrightarrow X(T'),$$

$$c^*(x) = u+v$$

$$c^*(x) = u-v$$

where $X(T) = (x,y)$ and $X(T') = (u,v)$.

By Theorem 3.2.3, the diagram $Gl(k^2) \longrightarrow PGl(k^2)xk^* \longrightarrow E$ can be completed uniquely to a diagram,

$$\begin{tikzcd}
Gl(k^2) \arrow[r, "c"] \arrow[d, "j"'] & PGl(k^2)xk^* \arrow[d] \\
E' \arrow[r, "f"'] & E
\end{tikzcd}$$

such that f is finite and dominant, $j : Gl(k^2) \longrightarrow G(E')$ is an isomorphism, and E' is normal. Thus, we have $c^* : X(Z) \longrightarrow X(Z')$, where Z' and Z are the closures of respective maximal tori.

By the results of 7.5.13,

$X(Z') = \{ x \in (u,v) \mid kx \in \langle \alpha u+\beta v, \alpha v+\beta u \rangle$ for some $k \}$. Thus, since $c^*(x) = u+v$ and $c^*(y) = u-v$, we have

$$c^*(\alpha x+\beta y) = (\alpha+\beta)u+(\alpha-\beta)v$$

$$c^*(\alpha y+\beta x) = (\alpha+\beta)u+(\beta-\alpha)v.$$

<u>Note:</u> The image of $X(T)$ in $X(T')$ is the subset of all elements $ru+sv$ such that $r+s$ is even.

<u>Case (i):</u> $\alpha+\beta$ is odd.

By the note, $\alpha u+\beta v$ and $\alpha v+\beta u$ are not elements of $X(Z)$. By 4.1.7 the fundamental generators of $X(Z)$ are multiples of $\alpha u+\beta v$ and $\alpha v+\beta u$ since $c^* : X(Z) \longrightarrow X(Z')$ is a finite morphism. Thus, again by the note, $F = \{ 2(\alpha u+\beta v), 2(\alpha v+\beta u) \}$ is the set of fundamental generators.

Since $\alpha+\beta$ is odd we may write (uniquely)

$$\alpha = (m+n)/2,$$

$$\beta = (m-n)/2$$

where $m,n > 0$, $(m,n) = 1$ and mn is odd.

Thus, to construct the possible character monoids that occur in this fashion, let $m,n > 0$, $(m,n) = 1$, mn odd. Then
$X(T(m,n)) = \{ v \in (x,y) \mid kv \in \langle mx+ny, mx-ny\rangle \text{ for some } k \}$.
If w is the non-trivial element of W, the Weyl group of T, then $w(x) = x$ and $w(y) = -y$. So

$$w = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

relative to the basis $\{ x, y \}$ of $X(T)$.

<u>Case (ii):</u> $\alpha+\beta$ is even.

In this case (by the note preceding case (i)) we have $\alpha u+\beta v$, $\alpha v+\beta u \in X(Z)$. Since these are the fundamental generators for $X(Z')$, $F = \{ \alpha u+\beta v, \alpha v+\beta u \}$ is the set of fundamental generators of $X(Z)$. Since $\alpha+\beta$ is even, we can write $\alpha=m+n$ and $\beta=m-n$, where $m,n > 0$ $(m,n)=1$ and mn is even. Thus,

$$F(1) = (m+n)u+(m-n)v = mx+ny \text{ and}$$

$$F(2) = (m+n)v+(m-n)u = mx-ny$$

Thus to construct the possible character monoids that occur in this fashion, let $m,n > 0$, $(m,n) = 1$, mn even (here $\alpha=m+n$, $\beta=m-n$). Then

$X(Z(m,n)) = \{ v \in (a,b) \mid kv \in \langle mx+ny, mx-ny \rangle \text{ for some } k \}$.

If w is the non-trivial element of W, the Weyl group of T, then $w(x) = x$ and $w(y) = -y$. So

$$w = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

relative to the basis $\{ x, y \}$ of $X(T)$.

<u>7.5.15 $X(Z)$; $G(E) = PGl(k^2)xk^*$. Summary.</u>

Given E, there exists

$$c : E' \longrightarrow E$$

such that $G(E') = Gl(k^2)$ and $c : Gl(k^2) \longrightarrow PGl(k^2)xk^*$ is given by $c(x) = ([x], \det(x))$.

Let T be a maximal torus of $G(E)$ and let Z be the closure of T in E. Using the morphism c, and our classification of the monoids E', we obtain; $Z = Z(m,n)$ for some $m,n > 0$, $(m,n) = 1$, where

$X(Z(m,n)) = \{ v \in (x,y) \mid kv \in \langle mx+ny, mx-ny \rangle \text{ for some } k \}$.

If $w \in Aut(Z)$ is the non-trivial element, then

$$w = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

relative to the basis $\{ x, y \}$ of $X(T) = X(T(m,n))$.

<u>7.5.16 Remark:</u> A comparison of 7.5.14 and 7.5.15 demonstrates that the data collected from the monoids with group $Sl(k^2)xk^*$ is identical to the data collected from the monoids with group $PGl(k^2)xk^*$. Thus our description is only characteristic if we know the unit groups. However, when we embellish this description by fitting in the root systems, the resulting numerical data will completely distinguish the monoids from one another.

The following theorem is a summary of the results obtained from 7.5.1 to 7.5.15.

<u>7.5.17 Classification I:</u> Let G be one of the groups $Sl(k^2)xk^*$, $Gl(k^2)$ or $PGl(k^2)xk^*$ and let Q^+ denote the set of positive rational numbers. Then there is a canonical one–to–one correspondence.

$$Q^+ \longleftrightarrow E(G) = \{ E \mid E \text{ as in } (*),\ E \text{ normal},\ G(E) = G \}.$$

For $G = Gl(k^2)$ the correspondence is as follows. Given E there is a unique bicartesian diagram,

$$\begin{tikzcd} E(m^{-1}) \arrow[r] \arrow[d] & End(k^2) \arrow[d, "\alpha"] \\ E \arrow[r, "\beta"'] & E(n) \end{tikzcd}$$

such that all morphisms are finite and dominant and each kernel is a finite D–group. If degree $\alpha = n$ is odd then degree $\beta = m$ is odd and $(m,n) = 1$. If degree $\alpha = 2n$ is even then degree $\beta = 2m$ is even, $(m,n) = 1$ and one of m and n is even. In any case, the map $E(G) \longrightarrow Q^+$ given by

$$E \longrightarrow \deg(\alpha)/\deg(\beta)$$

is well defined and one–to–one.

Conversely, given $r \in Q^+$, $r = m/n$, where $m,n > 0$ and $(m,n) = 1$. It is then possible to construct a bicartesian diagram as above such that $\deg \alpha = n$ and $\deg \beta = m$ if mn is odd, or $\deg \alpha = 2n$ and $\deg \beta = 2m$ if mn is even. Thus we obtain the inverse map $Q^+ \longrightarrow E(G)$,

$$r \longrightarrow E(r).$$

All normal monoids with group $Sl(k^2)xk^*$ are constructed from the monoids with group $Gl(k^2)$ using integral closure and the morphism

$$m : Sl(k^2)xk^* \longrightarrow Gl(k^2),\ m(x,t) = xt.$$

All normal monoids with group $PGl(k^2)xk^*$ are constructed from the monoids with group $Gl(k^2)$ using finite D–group scheme quotients and the morphism

$$c : Gl(k^2) \longrightarrow PGl(k^2)xk^*,\ c(x) = ([x],\det(x)).$$

7.6 Polyhedral Root Systems And Classification II

In 7.5.16 we observed that the correspondence $E \longrightarrow X(Z)$ is not a complete invariant unless $G(E) = Gl(k^2)$. The purpose of this section is to find the root system Φ, amidst the characters $X(T)$, and to see how it relates to the set of fundamental generators F of $X(Z)$. This will lead to a complete numerical invariant, $E \longrightarrow (X(T),\Phi(T),F)$, the polyhedral root system.

7.6.1 Lemma: Suppose $f : G \longrightarrow G'$ is an epimorphism of reductive algebraic groups such that $\ker(f)$ is contained in the center of G. Let T and T' be maximal tori of G and G', respectively, such that $f(T) = T'$. Let Φ and Φ' be the roots (weights of the adjoint representation). Then $(f|T)^*(\Phi') = \Phi$.

Proof: Let B^+ and B^- be opposite Borel subgroups containing T, so that T is the intersection of B^+ and B^-. Let $g = T(G)$, the tangent space of G at 1. Then $g = g^+ + t + g^-$ (direct sum) where $g^+ = T(B^+(u))$, $g^- = T(B^-(u))$ and $t = T(T)$ (the tangent spaces at the identity). Further, $df : g \longrightarrow g'$ preserves these direct summands since $f(B^+(u)) = B'^+(u)$ and $f(B^-(u)) = B'^-(u)$. Now, $df : g^+ \longrightarrow g'^+$ and $df : g^- \longrightarrow g'^-$ are isomorphisms since $\ker(f)$ is central. Thus, it follows that $f^*(\Phi') = \Phi$.

7.6.2 Polyhedral Root System for E; $G(E) = Gl(k^2)$.

From 7.5.1 we have

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & End(k^2) \arrow[d, "\alpha"] \\ E \arrow[r, "\beta"'] & E'' \end{tikzcd}$$

Since $\ker(\alpha)$ and $\ker(\beta)$ are central, we apply 7.6.1 to follow the roots around the diagram from $End(k^2)$ to E.

Let Z^* and Z'' be the closures of maximal tori in $End(k^2)$ and E'', respectively, such that $\alpha(Z^*) = Z''$. α induces $\alpha^* : X(Z'') \longrightarrow X(Z^*)$.

Case (i): degreeα odd.

From 7.5.5, we have

$$\alpha^*(z) = (n-1)a/2+(n+1)a/2$$

$$\alpha^*(w) = (n+1)a/2+(n-1)b/2$$

(where $X(Z'') = (z,w)$ and $X(Z^*) = (a,b)$).

By 7.6.1, $\alpha^*(\Phi'') = \Phi^* = \{ a-b,\ b-a \}$. Thus $\Phi'' = \{ a-b,\ b-a \} = \{ w-z,\ z-w \}$.

If Z is the closure of the maximal torus T such that $\beta(T) = T''$, we have $\beta : X(Z'') \longrightarrow X(Z) = (u,v)$. From 7.5.9 we obtain,

$$\beta^*(z) = (m+1)u/2+(m-1)v/2 \text{ and}$$

$$\beta^*(w) = (m+1)v/2+(m-1)u/2.$$

Thus

$$\beta^*(z-w) = u-v \text{ and}$$

$$\beta^*(w-z) = v-u.$$

Hence, $\Phi = \{ u-v,\ v-u \}$, so, gathering all the relevant data,

$$X = (u,\ v) \qquad \alpha,\beta \in Z$$
$$\Phi = \{\ u-v,\ v-u\ \} \qquad \alpha > |\beta| \geq 0$$
$$F = \{\ \alpha u+\beta v,\ \alpha v+\beta u\ \} \qquad (\alpha,\beta) = 1$$

Here, $\alpha=(m+n)/2$ and $\beta=(m-n)/2$.

Case (ii): degreeσ even.

From 7.5.7 we have $\sigma^* : X(Z'') \longrightarrow X(Z^*)$ and thus,

$$\sigma^*(z) = a-b$$
$$\sigma^*(w) = n(a+b).$$

Thus $\Phi'' = \{\ z,\ -z\ \}$.

If Z is the closure of the maximal torus T such that $\beta(T) = T''$, we have $\beta^* : X(Z'') \longrightarrow X(Z) = (u,v)$.

From 7.5.11 we obtain,

$$\beta^*(z) = u-v \text{ and}$$
$$\beta^*(w) = m(u+v).$$

Hence, $\Phi = \{\ u-v,\ v-u\ \}$. So in this case again we obtain,

$$X = (u,\ v) \qquad \alpha,\beta \in Z$$
$$\Phi = \{\ u-v,\ v-u\ \} \qquad \alpha > |\beta| \geq 0$$
$$F = \{\ \alpha u+\beta v,\ \alpha v+\beta u\ \} \qquad (\alpha,\beta) = 1$$

Here $\alpha=m+n$ and $\beta=m-n$.

7.6.3 Polyhedral Root System for E, $G(E) = Sl(k^2)xk^*$.

The morphism $m : Sl(k^2)xk^* \longrightarrow Gl(k^2)$, $m(x,\beta) = x[\beta,\beta]$, induces $m^* : X(T') \longrightarrow X(T)$, $m^*(u) = a+b$, $m^*(v) = a-b$, where T' and T are maximal tori of $Gl(k^2)$ and $Sl(k^2)$, respectively, and $X(Z'') = (u,v)$, $X(Z) = (a,b)$. Thus, since $m^*(\Phi') = \Phi$, we have

$$\Phi = \{\ 2b,\ -2b\ \}.$$

So, gathering all the relevant data (see 7.5.14) we have,

$X = (a,\ b)$	$m,n \in N$
$\Phi = \{\ 2b,\ -2b\ \}$	$m,n > 0$
$F = \{\ ma+nb,\ ma-nb\ \}$	$(m,n) = 1$

7.6.4 Polyhedral Root System for E; $G(E) = PGl(k^2)xk^*$.

The morphism $c : Gl(k^2) \longrightarrow PGl(k^2)xk^*$, $x \longrightarrow ([x],\det(x))$, induces $c^* : X(T) \longrightarrow X(T')$, $c^*(x) = u+v$ and $c^*(y) = u-v$, where T' and T are maximal tori of $Gl(k^2)$ and $PGl(k^2)xk^*$, respectively, $X(T') = (u,v)$, and $X(T) = (x,y)$. Thus, since $c^*(\Phi) = \Phi'$, we have

$$\Phi = \{\ y,\ -y\ \}.$$

So, gathering all the relevant data (see 7.5.15) we have,

$X = (x,\ y)$	$m,n \in N$
$\Phi = \{\ y,\ -y\ \}$	$m,n > 0$
$F = \{\ mx+ny,\ mx-ny\ \}$	$(m,n) = 1$

Definition: Let E be an irreducible algebraic monoid with 0 and let T be a maximal torus of G(E). Let X denote the characters of T, Φ the roots, and F the fundamental generators (see 4.1.7). Then (X,Φ,F) is the polyhedral root system of (E,T).

7.6.5 Classification II: Let G be one of the groups $Gl(k^2)$, $Sl(k^2)xk^*$ or $PGl(k^2)xk^*$. Then any normal algebraic monoid E with 0 and group of units G is uniquely determined by its polyhedral root system $(X(T),\Phi(E),F(E))$. The following is a list of all possible polyhedral root systems for each group G ((u,v) denotes the free abelian group on the generators u and v).

(i) <u>$G = Gl(k^2)$</u>

$X = (u,v)$ $\qquad$ $\alpha,\beta \in Z$

$\Phi = \{\, u-v,\ v-u \,\}$ $\qquad$ $\alpha > |\beta| \geq 0$

$F = \{\, \alpha u+\beta v,\ \alpha v+\beta u \,\}$ $\qquad$ $(\alpha,\beta) = 1$

(ii) <u>$G = Sl(k^2)xk^*$</u>

$X = (a,b)$ $\qquad$ $m,n \in N$

$\Phi = \{\, 2b,\ -2b \,\}$ $\qquad$ $m,n > 0$

$F = \{\, ma+nb,\ ma-nb \,\}$ $\qquad$ $(m,n) = 1$

(iii) <u>$G = PGl(k^2)xk^*$</u>

$X = (x,y)$ $\qquad$ $m,n \in N$

$\Phi = \{\, y,\ -y \,\}$ $\qquad$ $m,n > 0$

$F = \{\, mx+ny,\ mx-ny \,\}$ $\qquad$ $(m,n) = 1$

<u>7.6.6 Remark:</u> It is interesting to note that (i), (ii) and (iii) above exhaust all the reasonable possibilities among two-dimensional normal D-monoids. Precisely, let X be a two-dimensional irreducible normal D-monoid with 0. Then X is isomorphic to the maximal irreducible D-submonoid of some E as in 7.6.5 if and only if X has a non-trivial automorphism.

VIII IRREDUCIBLE MONOID VARIETIES

The purpose of this chapter is to initiate the study of more general monoid varieties.

Definition: An algebraic monoid variety is an irreducible (not necessarily affine) algebraic variety E, defined over the algebraically closed field k, such that

(i) $1 \in E$

(ii) $m : E \times E \longrightarrow E$ is an associative morphism of algebraic varieties with 1 as two-sided unit.

On the two extremes we have the quasi-affine monoids and the projective monoids. This chapter is devoted to the proofs of the following results.

(1) If E is irreducible and quasi-affine then E is affine.

(2) If E is irreducible and projective then E is an abelian variety.

8.1 Quasi-affine Monoids

Let U be a quasi-affine variety defined over k and let k[U] denote the set of global sections of the structure sheaf O(U). U is not determined by k[U]. For example, if $U = k^2 - \{0\}$, then U is quasi-affine and $k[U] = k[k^2]$ (see also 1.1.5).

Let $j : U \longrightarrow X$ be an open imbedding, where X is affine and let $J = (f(1),\ldots,f(n))$ be the ideal of regular functions on X which vanish on $X - j(U)$. The induced morphism satisfies

(i) $k[X][1/f(i)] \longrightarrow k[U][1/f(i)]$ is an isomorphism for all $i = 1,\ldots,n$.

(ii) k[U] is the intersection of the $k[U][1/f(i)]$ as i varies from 1 to n.

(iii) U is isomorphic to the union of the X(i) in X, where

$X(i) = \{ x \in X \mid f(i)(x) \text{ is non-zero } \}$.

The affine variety X is somewhat arbitrary. Any k-algebra R, contained in k[U] such that

(i) R is finitely generated over k.

(ii) { f(i) } is contained in R.

(iii) $R[1/f(i)] = k[U][1/f(i)]$, $i = 1,\ldots,n$.

induces an isomorphism of U onto the union of the Y(i), i = 1,...,n, where Y is the affine variety associated with R.

<u>8.1.1:</u> U is uniquely determined up to isomorphism by (k[U],{ f(i) }).

<u>Remarks:</u> U is affine if and only if { f(i) } generates the unit ideal of k[U] (see [1], chapter 3, exercise 24).

Any finite subset { g(i) } of k[U] such that r((f(i))) = r((g(i))) (radical) works equally well.

Suppose now that E is a quasi-affine irreducible algebraic monoid and let (k[E],{ f(i) }) be as above.

The morphism $m : E\times E \longrightarrow E$ induces $d : k[E] \longrightarrow k[E]\otimes k[E] = k[E\times E]$ in such a way that

$n : k \longrightarrow k[E]$ (k-algebra structure)

$e : k[E] \longrightarrow k$ (unit)

$d : k[E] \longrightarrow k[E]\otimes k[E]$ (multiplication)

induces on k[E] the structure of a bigebra.

By a well-known result,

<u>8.1.2:</u> $k[E] = \text{colimit}(k[E](\alpha))$, where each $k[E](\alpha)$ is a finitely generated sub-bigebra of k[E] (see [32] p.24. The proof there is stated for Hopf algebras, but works equally well for bigebras).

<u>8.1.3 Proposition:</u> Suppose E is an irreducible quasi-affine

algebraic monoid. Then there exists an irreducible affine algebraic monoid E' and a morphism $g : E \longrightarrow E'$ such that g is an open imbedding.

Proof: Let (k[E],{ f(i) }) be as in 8.1.1. By 8.1.2 there exists a bigebra R, contained in k[E] such that { f(i) } is a subset of R and $R[1/f(i)] = k[E][1/f(i)]$ for all i. But this is the same as being given an open imbedding $g : E \longrightarrow E'$ of algebraic monoids where E' is the affine monoid associated with R.

8.1.4 Theorem: Suppose E is an irreducible quasi-affine algebraic monoid. Then E is affine.

Proof: By 8.1.3 there exists $g : E \longrightarrow E'$ an open imbedding, where E' is affine. Thus, g induces an isomorphism on unit groups. Hence, $E' - g(E)$ is a prime ideal of E', because g(E) is multiplicatively closed. Thus by 5.2.1 (i) there exists a character $x \in k[E']$ such that $E' - E = x^{-1}(0)$. Thus, $E = E' - x^{-1}(0)$ is affine.

8.2 Projective Monoids

Let E be an irreducible projective monoid variety, and let $m : ExE \longrightarrow E$ be the multiplication morphism. Consider $m^{-1}(1) = \{ (x,y) \in ExE \mid xy = 1 \}$. Suppose $(x,y) \in m^{-1}(1)$ and let $x : E \longrightarrow E$, $x(z) = xz$. Thus $x \circ y = 1$. Hence yE is dense in E, because they have the same dimension. But $yE = \{z \in E \mid yxz = z\}$. So yE is closed in E and thus yE = E. But then there exists $z \in E$ such that $yz = 1$. So, $x = x(yz) = (xy)z = z$. Thus $xy = 1$ if and only if $yx = 1$. Therefore, the morphism $g : m^{-1}(1) \longrightarrow E$, $g(x,y) = x$, is one-to-one. Since the dimension of every component of $m^{-1}(1)$ is larger than or equal to dim E (see 1.1.1), g is dominant. It follows that $m^{-1}(1)$ is irreducible.

Furthermore, g is bijective since $g(m^{-1}(1))$ is closed in E ($m^{-1}(1)$ is a complete variety). Thus,

8.2.1: If E is projective then every element $x \in E$ is invertible.

Remark: It does not follow automatically from 8.2.1 that E is a group scheme. Even though $i : E \longrightarrow E$, $i(x) = x^{-1}$, is well defined as a set map, we have no a priori guarantee that i is a morphism of varieties.

Now, it follows easily that $m : ExE \longrightarrow E$ is separable and since E is homogeneous, m is smooth. Thus $m^{-1}(1)$ is irreducible and smooth. Define

$$i : m^{-1}(1) \longrightarrow m^{-1}(1),\ i(x,y) = (y,x) \text{ and}$$

$$u : m^{-1}(1)xm^{-1}(1) \longrightarrow m^{-1}(1),\ u((x,y),(u,v)) = (xu,vy).$$

Plainly, $(m^{-1}(1),u,i,(1,1))$ defines on $m^{-1}(1)$ the structure of an algebraic group, with multiplication u, inverse i, and unit (1,1). Furthermore, $g : m^{-1}(1) \longrightarrow E$, $g(x,y) = x$, is a bijective morphism of algebraic monoids. Thus, to complete the discussion, it suffices to demonstrate that g is separable.

8.2.2 Lemma: Let $Z = m^{-1}(1)$ and let g be as above. Then g is separable.

Proof: Let $j : Z \longrightarrow ExE$ be the inclusion and let $p(i) : ExE \longrightarrow E$, $i = 1,2$, be the projection morphisms. Let $m : ExE \longrightarrow E$ be the multiplication morphism. Using the projections p(1) and p(2), the tangent space of ExE at (1,1) is identified with the direct sum, TE+TE of the tangent space at 1 of E with itself. Further, if TZ is the tangent space of Z at (1,1),

$$TZ \xrightarrow[dj]{} TE+TE \xrightarrow[dm]{} TE$$

is exact, because m is a smooth morphism. Using the bigebra structure of the local ring $O(1,E)$, it follows that $dm(x,y) = x + y$. Thus $dj(TZ) = \{ (x,-x) \mid x \in TX \}$. Hence, $dg : TZ \longrightarrow TE$ is an isomorphism since $g = p(1) \circ j$. Thus, by 1.1.3, g is separable.

<u>Definition:</u> An <u>abelian variety</u> is an irreducible projective algebraic group.

<u>8.2.3 Theorem:</u>[1] Suppose E is an irreducible projective algebraic monoid. Then $E = G(E)$ is an abelian variety.

<u>Proof:</u> $m^{-1}(1)$ is a projective algebraic group and thus an abelian variety. $g : m^{-1}(1) \longrightarrow E$, $g(x,y) = x$, is a bijective separable morphism of smooth algebraic monoids and thus an isomorphism.

<u>Remark:</u> Abelian varieties are much studied in algebraic geometry. It is a remarkable fact that every projective algebraic group is commutative.

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[1] D. Mumford has obtained similar, more general results (see <u>Abelian Varieties</u>, Tata, Bombay, page 44); accordingly, our associativity assumption on E is superfluous.

IX APPLICATIONS TO RATIONAL HOMOTOPY THEORY

There have been some recent applications involving algebraic monoids to problems in topology and algebra. Although such matters may be considered digressive from the themes of this thesis, they have been, at least for myself, ingressive to many of the problems in the theory of algebraic monoids. The purpose of this chapter is to describe, in general terms how algebraic monoids are related to several non-trivial problems in rational homotopy theory.

9.1 Algebraic Categories And Positive Weights Spaces

In [30] Sullivan establishes a complete and algebraic description of uniquely divisible homotopy invariants. He then observes that if X is a simply-connected C.W.-space then $Aut(X(0))$ is an algebraic group defined over Q, where $X(0)$ is the 0-localization of X. Furthermore, there is a differential graded algebra, $M(X)$ such that $End(X(0))$ is isomorphic to $End(M(X))$ modulo d.g.a. homotopy.

In [3] R. Body and D. Sullivan consider the following class of 'sufficiently divisible' simply-connected C.W.-spaces. Let $Z(p) = \{ r \in Q \mid r=m/n,\ (n,p)=1 \}$.

> X is sufficiently p-divisible if for any map $f : X \longrightarrow Y$ such that $f^* : H^*(Y;Z(p)) \longrightarrow H^*(X;Z(p))$ is an isomorphism there exists a map $g : Y \longrightarrow X$ such that $g^* : H^*(X;Z(p)) \longrightarrow H^*(Y;Z(p))$ is an isomorphism.

They establish the following fundamental results about this class of spaces.

9.1.1 Theorem[3]: Let X be a simply-connected C.W.-space. Then the following are equivalent.

(i) X is sufficiently p-divisible for some p.

(ii) X is sufficiently p-divisible for all p.

(iii) X is sufficiently 0-divisible ($Z(0) = Q$).

(iv) 0, the basepoint morphism of $M(X)$, is in the Zariski closure of $Aut(M(X))$ in $End(m(X))$.

(v) $M(X) = M$ has positive weights. i.e. There exists a direct sum decomposition (compatible with the usual grading on M), $M = \Sigma M(\alpha)$ such that $d(M(\alpha))$ is contained in $M(\alpha)$ for all α, $M(\alpha)M(\beta)$ is contained in $M(\alpha+\beta)$ for all α and β and, and $M(\alpha) = 0$ for all $\alpha < 0$.

The important observation here is that we now have a completely algebraic definition of 'sufficiently divisible'.

In [2], [8] and [9] the following question is considered for X a simply-connected C.W.-space (with some mild finiteness conditions).

> Does $X(0)$ satisfy unique factorization in the homotopy category, with respect to the formation of products?

Because of Sullivan's rational homotopy theory, this is now a question of pure algebra.

> Does $M(X)$ satisfy unique factorization with respect to the formation of graded tensor product?

In [9] this question is answered affirmatively in case X is sufficiently divisible in the above sense (the dual question regarding coproducts is also considered). Since we are here concerned with how algebraic monoids are involved, I will generalize and modify the context accordingly. So let us consider the following categories, called <u>algebraic categories</u> (see [8]).

Let $V(k)$ be the category of vector spaces over the field k and let S be a category whose objects are in one-to-one correspondence with the non-negative integers, $N = \{ 0, 1, 2, \ldots \}$. Associated with S is the category $\Omega(S)$.

If C is a category let $|C|$ denote the class of objects of C.

The objects of $\Omega(S)$ are pairs $(V,\alpha(V))$ (or just (V,α)) where $V \in |V(k)|$ and $\alpha(V)$ is a functor from S to $V(k)$ such that $\alpha(0) = k$, $\alpha(1) = V$ and $\alpha(m) = V(m)$ for all m, where $V(m)$ is the the tensor product of V with itself m times. So, in particular, if $x \in \hom(m,n)$ and $y \in \hom(n,p)$ then $\alpha(y)\circ\alpha(x) = \alpha(y\circ x)$.

The morphisms of $\Omega(S)$ are the linear maps in $V(k)$ which preserve the S–structure. $\mathrm{Hom}'(V,W) = \{ f \in \mathrm{Hom}(V,W) \mid f(n)\circ\alpha(V)(x) = \alpha(W)(x)\circ f(m)$, for all $x \in \hom(m,n)$, and all $m,n \in |S| \}$. Here $f(n)$ denotes the tensor product of f with itself n times. Assume further, that $0 \in S$ is the zero object. For each $m \in |S|$, there are unique morphisms $n : 0 \longrightarrow m$ and $e : m \longrightarrow 0$. We shall also, assume that the morphisms of $\Omega(S)$ preserve this structure. It follows that k is the zero object of $\Omega(S)$.

Remark: The definition above is easiest to apply in practice, if S can be realized as a subcategory of SETS. The purpose of the definition is to provide a context for abstracting from the idiosyncracies of various algebraic categories in order to display the essence of how an object may be influenced by its algebraic monoid of endomorphisms.

Let $\Omega(S)$ be an algebraic category and let (V,α) be an object of $\Omega(S)$. Assume that V satisfies some sufficiently strict finiteness conditions (still very general in practice). Then

$G = \mathrm{Aut}(V,\alpha)$ is the algebraic group of units of the algebraic monoid $E = \mathrm{End}(V,\alpha)$.

<u>Proof:</u> $f \in \mathrm{End}(V)$ is in $\mathrm{End}(V,\alpha)$ if and only if $\alpha(x) o f(k) = f(l) o \alpha(x)$ for all $k,l \in |S|$ and all $x \in \hom(k,l)$. Thus, by our finiteness assumption, $\mathrm{End}(V,\alpha)$ is an algebraic subset of $\mathrm{End}(V)$. Clearly, $\mathrm{Aut}(V,\alpha)$ is the associated algebraic group of units.

<u>Definition:</u> Let (V,α) be an object of $\Omega(S)$. Then (V,α) has <u>positive weights</u> if $0(V)$ is an element of the closure of $\mathrm{Aut}(V,\alpha)$ in $\mathrm{End}(V,\alpha)$.

Thus, condition (iv) of 9.1.1 can be formulated in this very general setting.

<u>9.1.2 Theorem:</u> Suppose (V,α) has positive weights. Then $V = \bullet V(i)$, $i = 1,\ldots,m$, in such a way that each $V(i)$ is $\bullet$–irreducible in $\Omega(S)$. Furthermore, if $V = \bullet W(j)$ $j = 1,\ldots,n$, where each $W(j)$ is $\bullet$–irreducible in $\Omega(S)$, then $m = n$ and there exists $p : \{1,\ldots,m\} \longrightarrow \{1,\ldots,n\}$, bijective, such that $V(i)$ and $W(p(i))$ are isomorphic.

<u>Sketch of proof:</u> If $V = \bullet V(j)$ let $e(i) : V \longrightarrow V$ be given by the composite of

$$e\bullet\ldots\bullet 1\bullet\ldots\bullet m : V(1)\bullet\ldots\bullet V(m) \longrightarrow k\bullet\ldots\bullet V(i)\bullet\ldots\bullet k \text{ and}$$

$$n\bullet\ldots\bullet 1\bullet\ldots\bullet n : k\bullet\ldots\bullet V(i)\bullet\ldots\bullet k \longrightarrow V(1)\bullet\ldots\bullet V(m) = V$$

where $e : W \longrightarrow k$ and $n : k \longrightarrow W$ are the unique morphisms to and from the zero object, k.

Then $e(i)^2 = e(i)$, $i = 1,\ldots,m$ and $e(i) o e(j) = e(j) o e(i) = 0(V)$ if i is not equal to j.

Because V has positive weights, we can construct a maximal k–split torus T of $\mathrm{Aut}'(V)$ such that $\{ e(i) \mid i = 1,\ldots,m \}$ is contained in the closure of T in End'(V). If { f(j) | j = 1,...,n } is another set of splitting idempotents, then we can assume that { f(j) } is in the closure of T as well because maximal k-split tori are all conjugate [5]. But then { e(i) } = { f(j) }, because by the results of [9], { e(i)of(j) | j = 1,...,n } determines a •-product decomposition of V(i) for each i. Since V(i) is •-irreducible by assumption, e(i)of(j) = e(i) for some j. Similarily, f(j)oe(k) = f(j) for some k. But then e(i) = e(i)of(j) = e(i)o(f(j)oe(k)) = e(i)oe(k). Thus, e(i)oe(k) is non-zero. So, e(i) = e(k). Thus, it follows that any two irreducible •-product decompositions are equivalent in the sense advertised.

<u>9.1.3 Examples:</u> Theorem 9.1.2 (applied to the relevant •-product) applies to any of the following categories.

(i) simply-connected minimal differential algebras and morphisms.

(ii) simply-connected minimal differential graded coalgebras and morphisms.

(iii) connected minimal differential graded Lie algebras and morphisms.

(iv) connected minimal Lie coalgebras and morphisms.

(v) any of (i) – (iv) without the differential and minimality restrictions.

Categories (i) – (iv) all give rise to the same homotopy theory if the characteristic of the ground field is 0 [16]. Each of the categories defined in (v) may be considered a subcategory of one of the categories defined in (i) – (iv). Furthermore, all objects defined in (v) have positive weights. One question, however, is left open by Theorem 9.1.2. How does one generalize the result to the situation of objects without positive weights? No counterexamples are known.

9.2 Homotopy Types With Connected Endomorphism Monoid

The main result of [27] is a structure theorem of Sullivan's minimal algebras based on a synthesis of algebraic fibrations and idempotents that adhere to Q-split tori. Assuming, for simplicity, that M is a finitely generated minimal algebra defined over an algebraically closed field k, of characteristic 0, the result is as follows.

9.2.1 Theorem: There exists a sequence $1 = e(0) > e(1) > \ldots > e(m)$ of idempotents in End(M) such that $e(i+1)$ is an element of the closure of Aut(e(M)) in End(e(M)) and $e(i+1)$ is a minimal such idempotent. Futhermore, if F(i) is the quotient d.g.a. of $e(i-1)(M)$ by the ideal generated $e(i)(M)^{+}$, then F(i) is a minimal algebra with positive weights. The series terminates at e(m)(M) because Aut(e(m)(M)) is closed in End(e(m)(M)).

The integer m is uniquely determined and each e(i)(M) is uniquely determined up to isomorphism.

Theorem 9.2.1 fits in neatly with the characterization of connected monoids given in Theorem 4.5.2.

9.2.2 Corollary: Let M be as in 9.2.1. Then the following are equivalent.

(i) End(M) is connected in the Zariski topology.

(ii) The sequence $1 > e(1) > \ldots > e(m)$ terminates with $e(m) = 0(M)$.

(iii) For all non-zero idempotents $e \in \mathrm{End}(M)$, Aut(e(M)) is not closed in End(e(M)).

The same result may be applied to any of the categories $\Omega(S)$ considered in 9.1.

It is not known, however, whether the unique factorization results for positive weight spaces can be extended to rational homotopy types with connected endomorphism monoid.

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