THE FADDEEV-POPOV TECHNIQUE IN GAUGE FIELD THEORIES

by

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ABSTRACT

We adapt the Faddeev-Popov technique to lattice gauge field theories. Our formulation strongly suggests that the Faddeev-Popov formula, which has come into doubt since the discovery of Gribov ambiguities, is in fact correct.

More precisely, we show that Gribov ambiguities can occur in the lattice theory, but that "usually" they do not affect the Faddeev-Popov formula; a method is given for determining when the lattice Faddeev-Popov formula is not valid. We are able to answer in the lattice theory many questions that arise naturally in the continuum theory but which have remained unsettled up to now.

We show that a formal limit of the lattice Faddeev-Popov formula yields the usual continuum formula. We prove some partial results which bear on the problem of proving a rigorous continuum limit.
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INTRODUCTION

Gauge field theories form the core of our theoretical understanding of the interactions of elementary particles. The oldest and most successful such theory is quantum electrodynamics which is an example of an abelian gauge field theory. Nonabelian gauge field theories, first conceived by Yang and Mills in 1954, are currently the leading candidates for the description of the strong interaction and for unified field theories.

The formulation of a gauge field theory presents a number of special features and complications not found in a nongauge model such as a $\phi^4$ theory. At the very outset, when one attempts to quantize a gauge field theory using path integrals, one encounters the difficulty that gauge invariance leads to a kind of overcounting in the integrals which results in infinities. However, unlike some infinities, such as those requiring renormalizations, the infinities due to gauge invariance are somewhat artificial. This is because physical quantities can in principal be derived from a ratio of gauge invariant integrals, so the effects of overcounting can be expected to cancel out. This cancellation can be demonstrated explicitly in an abelian gauge field theory but such a demonstration is not as easy to come by in the nonabelian case.

Another way to view the cause of these infinities in gauge theories is to observe that gauge invariance leads to
the absence of an expected damping term in the integrals, which consequently diverge. In 1967, Faddeev and Popov showed how to restore this damping by what is loosely called "fixing the gauge". The Faddeev-Popov formula for the expectation of a function of the gauge fields became the foundation for much of the later work on nonabelian gauge field theories, most notably in the proof that they are renormalizable. Moreover, the Faddeev-Popov formula seemed to provide a suitable starting point for the rigorous construction of a nonabelian gauge field theory by supplying a formal measure for constructing Schwinger functions.

Unfortunately, an important assumption made in the derivation of the Faddeev-Popov formula was discovered by Gribov in 1977 to be incorrect in the nonabelian case. This defect, manifested by the existence of "Gribov copies", has been generally acknowledged as a serious difficulty even at a nonrigorous level, and many authors have studied its extent and consequences. The full implications of Gribov's discovery are still not known, nor is it clear how much of the work based on the Faddeev-Popov formula should be considered invalid because of it.

A second problem in the Faddeev-Popov derivation was pointed out by Hirschfeld in 1979. This problem, a matter of some missing absolute value signs, seems at first sight to be less serious than Gribov's and has not received nearly as much attention as the latter. However, Hirschfeld argued that it was the key to understanding the effect of Gribov
copies, and that in a sense the two errors cancelled each other out. Much of the work described in this thesis grew out of an attempt to come to a rigorous understanding of Hirschfeld’s observations.

In this thesis we investigate the Faddeev-Popov technique in lattice gauge field theories. By working with the lattice approximation we avoid the difficulty that in the continuum theory many of the fundamental quantities of interest are not well-defined. We show that the Gribov phenomenon can occur on the lattice and give a rigorous proof of a Faddeev-Popov formula which takes it into account.

Our main conclusion is that, barring certain pathologies, the lattice Faddeev-Popov formula is the same regardless of the occurrence or absence of Gribov copies. This suggests that the original Faddeev-Popov formula for the continuum theory, which can be obtained as a formal limit of the lattice formula, is valid in spite of the defects in its original derivation.

We believe that the lattice Faddeev-Popov formula we have developed should be a useful tool in the rigorous construction of a continuum nonabelian gauge field theory obtained by proving the existence of a limit as the lattice spacing goes to zero.

The thesis is organized as follows. In Chapter I we review the Faddeev-Popov technique in its original form and discuss the inadequacies of its derivation. Our formulation
of the Faddeev-Popov technique for lattice gauge theories is given in Chapter II. We use this formulation in Chapter III to clarify several aspects of gauge-fixing in the continuum theory. In Chapter IV we show that the continuum Faddeev-Popov formula can be obtained formally from our lattice version by taking the limit as the lattice spacing goes to zero. We also discuss there some partial results which bear on the problem of taking a rigorous continuum limit for the case of an SU(2) gauge theory in two space-time dimensions.
I. A REVIEW OF THE FADDEEV-POPOV TECHNIQUE

I. THE FADDEEV-POPOV TECHNIQUE IN CONTINUUM GAUGE THEORIES

I.A Review of the Faddeev-Popov Technique

Let \( G \) be a compact Lie group of dimension \( k \) and \( E \) be the Lie algebra of \( G \). Let \( \{ t_a \}_{a=1,2,...,k} \) be a basis for \( E \) which has been normalized to satisfy

\[
\text{tr}(t_a t_b) = -\frac{i}{2} \delta_{ab}
\]  

(1.1)

where \( \text{tr} \) denotes the character in some representation of \( E \).

In the following we use the convention that repeated Roman indices \( a, b, ... \) are summed from 1 to \( k \). We work in \( d = s + 1 \) space-time dimensions and use the conventions that repeated Roman indices \( i, j, ... \) are summed from 1 to \( s \), while repeated Greek indices \( \mu, \nu, ... \) are summed from 0 to \( s \).

A classical gauge field \( A \) is an element of \( \mathfrak{A} \), the set of functions from \( \mathbb{R}^d \rightarrow E \otimes \mathbb{R}^d \). We identify \( A \) with a collection of \( E \)-valued functions \( (A_0, A_1, ..., A_s) \) and write

\[
A_\mu = A_\mu^a t_a.
\]

We shall also need the function space \( \mathfrak{A}_1 \), the set of functions from \( \mathbb{R}^d \rightarrow E \).

The action \( S \) for a Euclidean pure gauge field theory with gauge group \( G \) and coupling constant \( \lambda \) is

\[
S(A) = -\frac{i}{2} \int d^d x \, \text{tr}(F_{\mu\nu} F_{\mu\nu})
\]  

(1.2)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \lambda [A_\mu, A_\nu] \) is the field strength.
A gauge transformation is a function $g: \mathbb{R}^d \rightarrow \mathbb{R}$ which acts on a gauge field by

$$g A_\mu = g A_\mu g^{-1} + \lambda^{-1} g (\partial_\mu g^{-1}).$$  \hfill (1.3)

The function $F_{\mu \nu}$ transforms as

$$F_{\mu \nu}(g A) = g F_{\mu \nu}(A) g^{-1}$$  \hfill (1.4)

so that the action is a gauge invariant quantity:

$$S(g A) = S(A).$$

Physical quantities are obtained in the theory from the expectations of gauge invariant functions $f$, the expectation being given formally by

$$\langle f \rangle = \frac{\int f(A) e^{-S(A)} \, dA}{\int e^{-S(A)} \, dA} \hfill (1.5)$$

where the $dA$ denotes the (nonexistent) product of Lebesgue measures $\prod_{x} \prod_{\alpha, \mu} dA^\alpha_{\mu}(x)$.

The first difficulty encountered in working with eq. (1.5) is that the gauge invariance of the integrands leads to infinities because of a kind of overcounting in the integrals. Imagine partitioning the space $A$ of gauge fields into orbits $(g A)$ and carrying out the integral

$$\int f(A) e^{-S(A)} \, dA$$

of a gauge-invariant function $f$ by first integrating over some surface $Z$ which picks out one representative from each orbit and then over each surface obtained by gauge transforming the fields in $Z$. We obtain
\[ \int f(A) e^{-S(A)} dA = \int \int_{gZ} f(A) e^{-S(A)} dz(A) \ d\gamma \quad (1.6) \]

where \( d\gamma \) and \( dz(A) \) are some as yet unspecified measures on the gauge transformations \( \gamma \) and \( gZ \) respectively.

Since the integrand in eq.(1.6) is constant along each gauge orbit the integral over \( d\gamma \) introduces an infinite constant \( K \):

\[ \int f(A) e^{-S(A)} dA = K \int_{Z} f(A) e^{-S(A)} dz(A). \]

In a ratio of integrals, such as that of eq.(1.5), the constant \( K \) will cancel out, so we should expect that in this case no infinities due to gauge invariance should occur. The restriction of the fields to the surface \( Z \) is known as "fixing the gauge" and \( Z \) itself is known as the "gauge-fixing surface".

\[ \text{Figure 1 Gauge-fixing surface and gauge orbits} \]
To see how to implement this idea, let us approach the problem in a different way by returning to the question of making sense of the formal measure

$$d\mu = \frac{e^{-S(A)}}{\int e^{-S(A)}} \, DA.$$

Write

$$S = S_0 + S_I$$

where

$$S_0 = -\frac{i}{2} \int d^d x \, tr[\{\partial_\mu A_\nu - \partial_\nu A_\mu\}] \tag{1.9}$$

contains the terms in $S$ which are quadratic in $A$. After formally integrating by parts and applying eq. (1.1) we obtain

$$S_0 = \frac{i}{2} \int d^d x \, A_\nu^{a(x)} \frac{\partial_\nu}{\partial_\mu} A_\mu^{b(x)} \tag{1.10}$$

where $D_{\mu\nu}^{ab} = (-\partial_\mu \delta_{\nu}^a + \partial_\nu \delta_{\mu}^b) \delta_{ab}$. We would like to interpret the free measure

$$\frac{e^{-S_0(A)}}{\int e^{-S_0(A)}} \, DA \tag{1.11}$$

as a Gaussian measure with covariance $D^{-1}$ and treat the remainder $S_I$ as interaction terms.

However, we encounter the difficulty that the operator $D$ is not invertible. In momentum space we have

$$\hat{D}_{\mu\nu}^{ab}(k) = (k^2 \delta_{\mu\nu} - k_\mu k_\nu) \delta_{ab}. \tag{1.12}$$

Now $\delta_{\mu\nu} - k_\mu k_\nu/k^2$, thought of as a matrix in the indices $\mu$ and $\nu$, is a projection operator, the projection onto the "transverse subspace". The kernel of $\hat{D}_{\mu\nu}^{ab}(k)$ is the range of the projection $k_\mu k_\nu/k^2$ onto the orthogonal
complement, the "longitudinal subspace".

The fact that $D$ is not invertible is a direct consequence of gauge invariance. For if we define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle A, B \rangle = \frac{i}{2} \int d^d x \, A^a(x) B^a(x),$$

then

$$\langle A, DA \rangle = S(A).$$

Take $A$ to be identically 0 and consider a gauge transformation $g$ of the form $g = e^{Bt}$ where $B \in C^\infty_0(\mathbb{R}^d)$ and $t$ is a fixed element of the Lie algebra of $G$. Then from (1.3)

$$^g A^\mu = - \lambda^{-1} \partial^\mu B_\mu(x) \cdot t$$

so that $[^g A^\mu, g A^\nu] = 0$ and $S_I(\delta A) = 0$. Thus

$$\langle g A, D g A \rangle = S(\delta A)$$

$$= S(A)$$

$$= 0.$$

Since $g A \neq 0$, the equation $\langle g A, D g A \rangle = 0$ implies that the spectrum of $D$ contains 0.

The measure (1.11) is sure to lead to infinities because it does not provide exponential damping in certain "directions", namely for the kernel of $D$. To remedy this situation, we would like to replace $S_0$ by $S_\alpha$ where

$$S_\alpha = S_0 + \frac{1}{i} A \int d^d k \, \hat{A}^a_\mu(k) k_\mu k_\nu \hat{A}^a_\nu(k)$$

$$= S_0 - A \int d^d x \, \text{tr}[(\partial_\mu A^a_\mu(x))^2].$$

(1.13)
The technique for making such a replacement was
originated by Faddeev and Popov [FP] and refined by 't Hooft
[tH1]. We now review their argument in some detail since
much of the rest of the thesis is taken up with analyzing
its shortcomings and improving it. The discussion which
follows is far from rigorous and a number of criticisms of
it could be made. We shall confine our comments to those
which seem appropriate to the level of the argument.

We begin by choosing a gauge-fixing function \( F \) which
is a linear function mapping \( A \to A_1 \). To obtain \( S_\alpha \) as in
eq (1.13) we would choose
\[
F(A) = \partial_\mu A_\mu \quad \text{(Landau gauge)}.
\]

Other common choices include
\[
F(A) = \partial_i A_i \quad \text{(Coulomb gauge)}
\]
\[
F(A) = n_\mu A_\mu \quad \text{(axial gauge)}
\]

where in axial gauge \( n_\mu \) is a fixed vector, most often with
\( n_0 = 1 \) and \( n_1 = 0 \).

Let \( C \) be a function in \( A_1 \) and define \( d(A, C) \) by
\[
1 = d(A, C) \int \delta(F(A) - C) \, dg \quad (1.14)
\]
where the \( \delta \)-function is a product of ordinary functions
\[
\delta(B) = \prod \delta(B^a(x))
\]

\( x, a \)

and \( dg \) is the infinite product of normalized Haar measures
on \( G \)
\[
dg = \prod dg(x).
\]

Notice that because of the right invariance of Haar measure
\( d(A, C) = d(A, C) \).
To be able to define \( d(A,C) \) we of course want the integral in eq. (1.4) to be nonvanishing. We should at least require that the argument of the \( \delta \)-function has a zero. This condition is known as the "attainability of a gauge": given a gauge field \( A \) and a function \( C \) there exists a gauge transformation \( g \) such that \( F(gA) = C \). In the following, we shall ignore the possibility that the integral in eq. (1.4) vanishes.

Suppose \( f \) is a gauge invariant function. We have

\[
\int f(A) e^{-S(A)} \, dA
= \iint d(A,C) \, \delta(F(gA) - C) \, f(A) \, e^{-S(A)} \, dA \, dg
= \iint d(gA,C) \, \delta(F(gA) - C) \, f(gA) \, e^{-S(gA)} \, dA \, dg \quad (1.15)
\]

where in the last line we have used the gauge invariance of \( d(\cdot, C) \), \( f \) and \( S \). Make a change of variables in the \( dA \) integral from \( A \) to \( gA \). The "Lebesgue measure" \( dA \) is invariant under this transformation because it is a translation by \( (\partial g) g^{-1} \) followed by a unitary transformation \( gAg^{-1} \). We obtain

\[
\int f(A) e^{-S(A)} \, dA
= \iint d(A,C) \, \delta(F(A) - C) \, f(A) \, e^{-S(A)} \, dA \, dg
= \int d(A,C) \, \delta(F(A) - C) \, f(A) \, e^{-S(A)} \, dA \quad (1.16)
\]

where we have used the fact that \( \int dg = 1 \). Integrate both sides of (1.16) against a weight function \( E(C) \) and evaluate the \( \delta \)-function to obtain
By choosing \( F(\mathbf{A}) = \partial \mu A_\mu \) and \( E(\mathbf{C}) = \exp(-\alpha \int d^4x \text{tr} (\mathbf{C}^2)) \) we see that we have succeeded in eq.(1.17) in inserting the damping factor needed to modify \( S_0 \) to \( S_\alpha \). The price we have had to pay in doing so is the introduction of the function \( \Delta(\mathbf{A},F(\mathbf{A})). \)

From eq.(1.14)

\[
\Delta(\mathbf{A},F(\mathbf{A}))^{-1} = \int \delta(F(\mathbf{gA}) - F(\mathbf{A})) \, d\mathbf{g}. \quad (1.18)
\]

To evaluate \( \Delta(\mathbf{A},F(\mathbf{A})) \), we assume "uniqueness of gauge fixing": the only gauge transformation \( \mathbf{g} \) for which \( F(\mathbf{gA}) = F(\mathbf{A}) \) is \( \mathbf{g} = 1 \), the identity. Under this assumption, only an infinitesimal neighbourhood \( \mathcal{U} \) of \( 1 \) need be considered in the integral (1.18). For \( \mathbf{g} \in \mathcal{U} \) we can write

\[
\mathbf{g} = e^{\chi} = 1 + \chi + O(\chi^2)
\]

for some \( E \)-valued function \( \chi = \chi_\alpha t_\alpha \). Haar measure in \( \mathcal{U} \) can be written

\[
\,d\mathbf{g} \equiv \,d\mathbf{\chi} \equiv K \prod_{x,a} \,d\chi_\alpha(x)
\quad (1.19)
\]

for some constant \( K \) and

\[
F(\mathbf{gA}) - F(\mathbf{A}) = M(\mathbf{A})\chi + O(\chi^2)
\]

for some linear operator \( M(\mathbf{A}): \mathfrak{g}_1 \to \mathfrak{g}_1 \); \( M(\mathbf{A}) \) is the Jacobian at \( \mathbf{g} = 1 \) of the function \( \mathbf{g} \to F(\mathbf{gA}) \). What we mean by eq.(1.19) is that Haar measure is absolutely continuous with respect to the measure induced on \( \mathcal{U} \) by
I.A REVIEW OF THE FADDEEV-POPOV TECHNIQUE

\[ \Pi_a \, d\gamma_a(x). \] The constant \( K \) represents the Radon-Nikodym derivative at 1. We thus have

\[
d(A,F(A))^{-1} = \int_\mathbb{R} \delta(M(A) \gamma) \, d\gamma = K \lvert \det M(A) \rvert^{-1} \quad (1.20)
\]

where we have used the infinite-dimensional analogue of the formula

\[
\int_{\mathbb{R}^n} \delta(Mx) \, dx = \lvert \det M \rvert^{-1}
\]

\( M \) being a linear map from \( \mathbb{R}^n \to \mathbb{R}^n \).

On putting eq. (1.20) into eq. (1.17) and normalizing we obtain

\[
\langle f \rangle = \frac{\int f(A) \, e^{-S(A)} \, DA}{\int e^{-S(A)} \, DA}
\]

\[
= \frac{\int \lvert \det M(A) \rvert f(A) \, EoF(A) \, e^{-S(A)} \, DA}{\int \lvert \det M(A) \rvert EoF(A) \, e^{-S(A)} \, DA}.
\]

The factor \( \det M(A) \) is known as the Faddeev-Popov determinant. For some reason, perhaps because it was expected that the Faddeev-Popov determinant would always be positive, the absolute value signs in \( \lvert \det M(A) \rvert \) are (almost) invariably omitted without comment in the literature. The result is the Faddeev-Popov formula for the expectation of a gauge invariant function:

\[
\langle f \rangle = \frac{\int \det M(A) \, f(A) \, EoF(A) \, e^{-S(A)} \, DA}{\int \det M(A) \, EoF(A) \, e^{-S(A)} \, DA}.
\]

If we had carried out the preceding argument only for \( C = 0 \)
and had not integrated against \( E(C) \) the result would have been the 
restricted Faddeev-Popov formula

\[
\langle f \rangle = \frac{\int \det M(A) \delta(F(A)) e^{-S(A)} \, dA}{\int \det M(A) \, dA}.
\]

The Faddeev-Popov formula or its restricted version has been 
the starting point for many investigations into nonabelian 
field theories because it allows one to cast a nonabelian 
gauge field theory into a form much like that of non-gauge 
field theories. By choosing \( E \) and \( F \) judiciously one can 
obtain an effective action, such as \( S_a \), which leads to a 
covariance \((-\Delta)^{-1}\). The Faddeev-Popov determinant can be 
written as an integral over fictitious fermion fields 
("ghost fields") in a kind of reversal of the Matthews-Salam 
procedure for integrating out fermions familiar from the 
Yukawa model and quantum electrodynamics. The resulting 
effective action which includes the ghost fields is in a 
form which is particularly suitable for an investigation of 
renormalization questions ([tH1],[tH2]).

To summarize, the derivation of the Faddeev-Popov 
formula lacks proofs of the existence and uniqueness of the 
orbit-surface intersections and neglects to take the 
absolute value of a Jacobian determinant.

Of these omissions, only the uniqueness question has 
received widespread attention in the literature. In the 
abelian case of quantum electrodynamics, this uniqueness can 
be demonstrated explicitly. For example, for Landau gauge
I. A REVIEW OF THE FADDEEV-POPOV TECHNIQUE

\[ \partial_\mu g A_\mu = \partial_\mu (g A_\mu g^{-1} + i^{-1} g (\partial_\mu g^{-1})) \]

\[ = \partial_\mu A_\mu + i^{-1} g g^2 \theta \]

where we have written \( g = e^{i \theta} \). The condition \( \partial_\mu g A_\mu = \partial_\mu A_\mu \)
then reads \( g^{-2} = 0 \). Uniqueness is obtained by requiring
that a solution \( \theta \) go to 0 at infinity. One would
incorporate this requirement into the Faddeev-Popov

*Gribov CG* was the first to investigate whether the

same holds true when \( G \) is nonabelian and discovered that

in general it does not. Specifically, he considered the

case of Coulomb gauge with \( G = SU(2) \), and looked for

solutions \( g \neq 1 \) of the equation

\[ \partial_1 g A_1 = \partial_1 A_1. \quad (1.21) \]

He discovered such solutions for particular classes of \( A \).
The solutions \( g \) could not be eliminated by requiring that

\( g \to 1 \) at infinity.

The fields \( g A \) which satisfy eq. (1.21) have come to be

known as *Gribov copies* and the failure of the traditional
gauge-fixing functions to uniquely specify the fields is

known as the *Gribov ambiguity*.

*Singer [S]* has shown that the Gribov ambiguity is not
peculiar to the Coulomb gauge condition or SU(2). He gave a rigorous formulation of the uniqueness condition in a setting where the space-time manifold, which is normally $\mathbb{R}^4$, is taken to be $S^4$, the unit sphere in $\mathbb{R}^5$. If $\mathcal{G}$ denotes the group of gauge transformations, which can be any compact Lie group, then the orbit space is the quotient $A/\mathcal{G}$ where equivalence is defined to mean "related by a gauge transformation". Singer proved that when $\mathcal{G}$ is nonabelian it is impossible to fix the gauge in this setting, in the sense that there is no continuous function $s:A/\mathcal{G} \to A$ with the property that $\pi s = I$, where $\pi$ is the projection $\pi:A \to A/\mathcal{G}$ and $I$ is the identity transformation.

Taking the gauge fields to be functions on $S^4$ rather than $\mathbb{R}^4$ can be regarded as imposing boundary conditions at $\infty$ on them. When this restriction is removed, it is possible to find a true gauge-fixing condition (axial gauge) as we discuss in Section I.B.
I.B Proposed Remedies for Gribov Ambiguities

The incompleteness of gauge-fixing as described in the previous section has been regarded as a serious problem for the Faddeev-Popov technique. Instead of having to consider the integral in eq.(1.18) only in a neighbourhood of \( \mathcal{A} \), it is necessary to evaluate it in a neighbourhood of each Gribov copy \( g_k \mathcal{A} \). Each copy contributes one determinant so that

\[
\Delta(\mathcal{A},F(\mathcal{A})) = \frac{1}{\sum_k \det M(g_k \mathcal{A})^{-1}}. \tag{1.22}
\]

Equation (1.22) is somewhat symbolic, since in general there is a continuous family of Gribov copies ([G], [BEP]). In any case, such an expression for \( \Delta(\mathcal{A},F(\mathcal{A})) \) would be very difficult to work with, since it appears to require some knowledge of all the solutions to the complicated uniqueness equation \( F(g \mathcal{A}) = F(\mathcal{A}) \). Moreover, the desired representation of \( \Delta(\mathcal{A},F(\mathcal{A})) \) as an integral over ghost fields is lost.

The existence of Gribov copies shows that the Faddeev-Popov technique does not entirely eliminate the overcounting in gauge invariant integrals. Gribov [G] originated the idea that it might be possible to avoid the effect of the copies by restricting the region of integration in some way. That is, it may be possible to
I.B REMEDIES FOR GRIBOV AMBIGUITIES

find a region V \subset A so that for any gauge invariant function f

\[ \left( \int E(C) \, dC \right) \left( \int f(A) \, e^{-S(A)} \, dA \right) \]

= \int_V \text{det} M(A) \cdot f(A) \cdot EoF(A) \cdot e^{-S(A)} \, dA.

This idea has been taking up several other authors ([BEP], [6s], [Z2]) but has not enjoyed much success because of the difficulty of specifying the region V in a way that is both appropriate and concrete. We discuss truncation of functional integrals in more detail in Section III.D.

Singer [S3] suggested that another solution would be to patch together local gauge fixings. He showed that Coulomb gauge can be used to define a function s locally. The idea is to define a partition of unity subordinate to a covering on whose members s is defined, and then apply the Faddeev-Popov technique on each member. This approach has suffers from the difficulty of finding a covering which can be specified in a useful way.

The copies found by Gribov had the property that they were "large" in the sense that a copy would exist if one of the parameters defining A were large enough. Hence one of the common responses to the Gribov ambiguity is to assert that it has no effect on perturbation theory calculations where the behavior of the theory near A_\mu = 0 is investigated.

As mentioned in Section I.A, the axial gauges are examples of true gauges. For example if n_\mu A_\mu = 0, the
condition $n_{\mu}gA_\mu = A_\mu$ requires that $n_{\mu}\partial_\mu(g^{-1}) = 0$ and the condition that $g = 1$ at infinity requires that $g = 1$ everywhere. Many authors have continued to investigate nonabelian gauge fields by using the Faddeev-Popov formula for such a gauge.

Finally, there are those approaches to gauge field theories which do not involve gauge fixing and so avoid the defects of the Faddeev-Popov techniques. One such approach is to study a lattice model (introduced by [Ws]; some reviews are [DI],[Ka],[Kol] and [Ko2]) where the regularization of the integrals provided by the lattice eliminates the infinities due to gauge invariance. By studying the continuum limit of gauge invariant quantities only, there is no need to fix a gauge. One disadvantage of proceeding this way is that the Green's functions of the theory, on which the structure of ultraviolet renormalizations rests, are not gauge invariant.

A second approach is to quantize the classical theory in a different way. Stochastic quantization [PW],[Z1] is a technique which has been advocated for computing physical quantities without the need for gauge fixing terms.

To summarize, the failure to accommodate the Gribov ambiguity has been recognized as a serious shortcoming of the Faddeev-Popov technique and many remedies have been proposed. None of these is entirely satisfactory however, since each requires us to sacrifice generality, concreteness
or familiarity. Moreover, the patch-ups described do not help us to evaluate the soundness of the work based on the Faddeev-Popov formula.

The most desirable resolution of this quandary would be a convincing demonstration that the Faddeev-Popov formula is correct even though its derivation is not. The lone advocate of the point of view that Gribov copies do not invalidate the Faddeev-Popov formula has been Hirschfeld [H]. He was also the only author to call attention to and recognize the importance of the omission of absolute value signs on the Faddeev-Popov determinant.

Hirschfeld reworked the restricted Faddeev-Popov argument in the following way. Instead of beginning with the equation

\[ 1 = \mathcal{D}(A) \int \delta(F(\varphi A)) \, d\varphi \]

and trying to prove that \( \mathcal{D}(A) = \det M(A) \), define \( \eta(A) \) by

\[ \eta(A) = \int \det M(\varphi A) \delta(F(\varphi A)) \, d\varphi. \quad (1.23) \]

Let \( \{g_k A\} \) denote the set of Gribov copies, that is, those fields for which \( F(g_k A) = 0 \). If we restrict the integration in eq.(1.23) to an infinitesimal neighbourhood of \( g_k \), we get

\[
\int_{U_k} \det M(g_k A) \delta(F(g_k A)) \, d\varphi = K \frac{\det M(g_k A)}{|\det M(g_k A)|}
\]

by using the same arguments that lead to eq.(1.20). Thus

\[ \eta(A) = K \sum_k \text{sgn} \det M(g_k A). \quad (1.24) \]
Hirschfeld made the crucial step of identifying $\eta(A)$ with a geometric invariant (the intersection number of the gauge orbit with the gauge fixing surface) to arrive at the rather surprising conclusion that $\eta(A)$ is independent of $A$. Given this fact we can mimic the Faddeev-Popov argument to conclude that

$$\eta \int f(A) \ e^{-S(A)} \ dA$$

$$= \int \det M(A) \ f(A) \ \delta(F(A)) \ e^{-S(A)} \ dA \quad (1.25)$$

for any gauge invariant function $f$. Thus

$$\langle f \rangle = \frac{\int \det M(A) \ f(A) \ \delta(F(A)) \ e^{-S(A)}}{\int \det M(A) \ \delta(F(A)) \ e^{-S(A)}} \quad (1.26)$$

The preceding argument shows how the two major defects in the Faddeev-Popov technique, the neglect of Gribov copies and absolute values, have combined and cancelled each other. The question of the attainability of the gauge appears here as the question of whether or not $\eta = 0$, and hence whether or not we were justified in cancelling it out in passing from eq. (1.25) to eq. (1.26).

In this thesis, we take up Hirschfeld's suggestions and apply them to gauge fixing in lattice gauge theories. In Section III.C we give a rigorous proof in that context of Hirschfeld's main conclusion. However, the main part of our work is based on identifying (the lattice analogue of) $\eta(A)$ with a different geometric invariant (the oriented degree). This identification has several advantages. It allows us to simplify the proof that $\eta$ is independent of $A$ and
dispense with certain assumptions that must be made to carry out Hirschfeld's argument. We are able to obtain the unrestricted form of the Faddeev-Popov formula as easily as the restricted form. Moreover, our approach makes it possible to obtain an explicit expression for $\eta$. This in turn lets us extract from it some information about the orbit-surface intersections and investigate the question of when it vanishes.
II. A Definitions

Let us begin by reviewing the framework of standard lattice gauge theories. Let $\Lambda = \Lambda(\epsilon, (N_\mu))$ be a finite lattice of points $x$ in $\mathbb{R}^d$ of the form $x = (n_0 \epsilon, \ldots, n_d \epsilon)$ where the $n_\mu$ are integers with $|n_\mu| \leq N_\mu$. The basis vector of length $\epsilon$ in the direction $\mu$ is denoted by $e_\mu$. If $x$ and $y$ are adjacent lattice points (i.e., $|x-y| = \epsilon$) we denote the directed bond joining $x$ to $y$ by $\langle x, y \rangle$. The set of bonds joining adjacent points in $\Lambda$ will be denoted by $\Lambda^*$.

As before, the gauge group $G$ is a compact, connected Lie group. A lattice gauge field is a map $a: \Lambda^* \to G$ with the property that

$$a(x,y) = a(y,x)^{-1}. \quad (2.1)$$

The formal correspondence with a continuum gauge field $A$ is given by

$$a(x, x+\epsilon e_\mu) = e^{i \epsilon A_\mu(x+\frac{1}{2} \epsilon e_\mu)}. \quad (2.2)$$

A lattice gauge transformation is a function $g: \Lambda \to G$. Its action is denoted by $a \to ga$ and is defined by

$$(ga)(x,y) = g(x) \ a(x,y) \ g(y)^{-1}. \quad (2.3)$$
II.A  DEFINITIONS

We shall often find it convenient to consider a gauge transformation as a point in the product manifold $\mathfrak{g} = \prod_{x \in \Lambda} G$ and a gauge field as a point in the product manifold $\mathfrak{g}^* = \prod_{x \in \Lambda^*} G$. We denote the Lie algebra of $\mathfrak{g}$ by $\mathfrak{z}$.

$\langle x, y \rangle \in \Lambda^*$

A function $f$ defined on $\mathfrak{g}^*$ is gauge invariant if

$$f(ga) = f(a)$$

for all $g \in \mathfrak{g}$ and $a \in \mathfrak{g}^*$. The action for the lattice theory is a smooth gauge-invariant function $S: \mathfrak{g}^* \to \mathbb{R}$ which is chosen to approximate the continuum action

$$S(A) = -\frac{1}{2} \int \text{Tr} (F_{\mu\nu})^2 \, dx.$$  \hspace{1cm} (2.4)

Often, lattice quantities such as the action are defined in terms of a character for a representation of $G$ and so are arbitrary to the extent that a choice of representation must be made. For our purposes, this choice is not important and it is convenient to simply identify $G$ with one of its representations by a group of finite-dimensional, unitary matrices. In the following, we assume that this identification has been made.

Expectations in the lattice theory are determined by a measure derived from normalized Haar measure $\mu$ on $G$. Since $G$ is compact, $\mu$ is the unique measure for which

$$\mu(G) = 1$$

and
II.A DEFINITIONS

$$\int_G f(g) \, d\mu(g) = \int_G f(gh) \, d\mu(g)$$
\[= \int_G f(hg) \, d\mu(g)\]
\[= \int_G f(g^{-1}) \, d\mu(g)\]

for any \( f \in L^1(G, d\mu) \) and \( h \in G \).

The measure \( d\mu \) can also be obtained from a differential form \([GHV2]\). Let \( R[g]: G \to G \) and \( L[g]: G \to G \) be defined by

\[ R[g](h) = hg \]
\[ L[g](h) = gh. \]

Let \( n \) be the dimension of \( G \) as a manifold. Then there exists a unique \( n \)-form \( \nu \) on \( G \) which is invariant

\[ L[g]^*\nu = R[g]^*\nu = \nu \]

for all \( g \in G \) and such that

\[ \int_G \nu = 1. \]

Then for all smooth functions \( f \) on \( G \) we have

\[ \int_G f \, d\mu = \int_G f\nu. \]

Let \( dg(x) \) and \( da(x,y) \) be \( d\mu \). Then Haar measure on \( \mathbb{S} \) is the product measure \( dg = \prod_{x \in \Lambda} dg(x) \) and on \( \mathbb{S}^* \) it is

\[ da = \prod_{(x,y) \in \Lambda^*} da(x,y). \]

We require that the action \( S(a) \) be such that \( e^{-S(a)} \in L^1(\mathbb{S}^*, da) \). The expectation of a function \( f \in L^2(\mathbb{S}^*, da) \) is then defined to be
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\[ \langle f \rangle = \frac{\int \delta^* f(a) e^{-S(a)} \, da}{\int \delta^* e^{-S(a)} \, da} \quad (2.5) \]

Thanks to the finiteness of Haar measure, the integrals in eq. (2.5) for \( \langle f \rangle \) are well-defined in spite of the gauge invariance of the action. This is one of the ways that the lattice theory provides a regularization of the continuum theory.

We now define the lattice analogue of the continuum damping factor \( E_0F(A) \). As we show below in Theorem 2.14, there are technical difficulties in trying to apply gauge fixing on all of \( \Lambda \). Thus, let \( \Lambda_1 \subset \Lambda \) and define

\[ \delta_1 = G^{\Lambda_1}. \]

Let \( \mathfrak{z}_1 \) denote the Lie algebra of \( \delta_1 \).

A lattice gauge-fixing function is a smooth map

\[ F: \delta^* \to \delta_1. \]

In the examples we discuss below, \( F \) has the form

\[ F(a)(x) = \prod_{(x,y) \in \Lambda^*} a(x,y)^{m(x,y)}. \quad (2.6) \]

where the \( m(x,y) \) are integers. (When \( G \) is nonabelian, the order of the group multiplications in (2.6) must be specified.) We would normally want to choose \( F \) so that the lattice gauge-fixing condition \( F(a) = I \) yields a given continuum condition \( F(A) = 0 \) in some sense as \( \epsilon \to 0 \). In general there are many functions that will satisfy this criterion.

Two examples of gauge fixing functions which we shall carry along with us from now on are those for axial gauge
II.A DEFINITIONS

and Landau gauge. For the former, take

$$\Lambda_1 = \{ x \in \Lambda : x + e_0 \in \Lambda \}$$  \hfill (2.7a)

and

$$F_A(a)(x) = a(x, x + e_0) \quad (x \in \Lambda_1)$$  \hfill (2.7b)

For Landau gauge, take

$$\Lambda_1 = \Lambda^0 \equiv \{ x \in \Lambda : |x_\mu| < \epsilon N_\mu \}.$$  \hfill (2.8a)

and

$$F_L(a)(x) = a(x, x + e_0) a(x, x - e_0) \ldots a(x, x + e_s) a(x, x - e_s).$$  \hfill (2.8b)

The lattice version of

$$E \circ F(A) = e^{d \int \text{Tr}(\partial_\mu A_\mu)^2 \text{d}x}$$

is $E \circ F_L(a)$ with $E: \mathfrak{g}_1 \to \mathbb{R}$ given by

$$E(c) = \exp \left[ 2\epsilon^{-2} \alpha^d - 4 \sum_{x \in \Lambda_1} \text{Re} \text{Tr} (c(x) - 1) \right].$$  \hfill (2.9)

These choices for $F_A$, $F_L$ and $E$ are fairly natural given eq. (2.2). They are based on observations such as

$$a(x, x + e_\mu) a(x, x - e_\mu) - 1 = e^{\epsilon \text{Tr} A_\mu (x + \frac{1}{2} e_\mu)} e^{-\epsilon \text{Tr} A_\mu (x - \frac{1}{2} e_\mu)} - 1$$

$$\sim \epsilon \text{Tr} A_\mu (x + \frac{1}{2} e_\mu) - \epsilon \text{Tr} A_\mu (x - \frac{1}{2} e_\mu)$$

$$\sim \epsilon^2 \partial_\mu A_\mu (x).$$

A more detailed discussion of the $\epsilon \to 0$ limit of these functions is given in IV.C.
II.B The Lattice Faddeev-Popov Formula

We now describe the lattice version of the Faddeev-Popov technique. Given a gauge-fixing function $F$ and a gauge field $a$, define $\xi: \mathfrak{g} \to \mathfrak{g}$ by

$$
\xi(g) = F(a)^{-1}F(ga).
$$

(2.10)

Let $\mathfrak{g}$ denote the Lie algebra of $\mathfrak{g}$. The lattice Faddeev-Popov operator is the map $M(a): \mathfrak{g} \to \mathfrak{g}$ defined by

$$
M(a) = d\xi|_1,
$$

(2.11)

the derivative map of $\xi$ at the identity.

We give the motivation for choosing this definition of $M(a)$ in the next chapter and show in Chapter 7 that formally it yields the correct continuum limit. For now, we can see that $M(a)$ is the right choice because of the following

Theorem 2.1: (Lattice Faddeev-Popov formula) There is a constant, $n$, depending only on $F$ such that for any smooth gauge invariant function $f$ on $\mathcal{X}$ and any smooth function $E: \mathcal{X} \to \mathbb{C}$,

$$
\eta(\int E(c) \, dc)(\int f(a) \, e^{-S(a)} \, da)
$$

$$
= \int \det M(a) \, f(a) \, E(F(a) \, e^{-S(a)} \, da).
$$

(2.12)

Hence if $\eta(\int E(c) \, dc) \neq 0$, 

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\[ \langle \Phi \rangle = \frac{\int f(a) \ e^{-S(a)} \ da}{\int e^{-S(a)} \ da} \]

\[ = \frac{\int \det M(a) \ f(a) \ EoF(a) \ e^{-S(a)} \ da}{\int \det M(a) \ EoF(a) \ e^{-S(a)} \ da} \quad (2.13) \]

It is worth remarking here how Theorem 2.1 "solves the Gribov problem", although the statements which are about to be made are not explained until the next chapter. If there were no Gribov copies, then \( \eta \) would be equal to +1 or -1. The existence of these copies has no effect on eq. (2.12) beyond changing the value of \( \eta \). In a sense, the value of \( \eta \) measures the extent to which Gribov copies lead to overcounting in functional integrals like (2.12). In any case, a change in \( \eta \) has no effect on the value of \( \langle \Phi \rangle \) since \( \eta \) cancels in the normalization, reflecting the fact that the overcounting is identical in the numerator and denominator. A detailed discussion of the implications of Theorem 2.1 for the continuum Faddeev-Popov formula is given in Chapter III.

The theory of the (oriented) degree of a smooth map between compact manifolds is central to the proof of Theorem 2.1 and its consequences, so we pause now to review the main ideas of that theory.

By \( n \)-manifold we mean a smooth, real manifold of dimension \( n \) and without boundary. Let \( M \) and \( N \) denote \( n \)-manifolds which are compact, connected and oriented. (In all our applications, \( M \) and \( N \) will be the same manifold and are a product of compact, connected Lie groups. Because
all Lie groups are orientable [GHV2], the preceding conditions are satisfied.) The tangent space to $M$ at $x \in M$ will be denoted by $T_x(M)$. Suppose $\varphi: M \to N$ is a smooth map and $d\varphi_x$ is its derivative map from $T_x(M) \to T_{\varphi(x)}(N)$.

There are three equivalent definitions which we use for the degree of $\varphi$, each of which is based on a different theorem. We now briefly state these and leave the details to the references.

A point $y \in N$ is a regular value of $\varphi$ if $d\varphi_x$ is surjective for every $x \in \varphi^{-1}(y)$. Suppose $y$ is a regular value of $\varphi$ and $x \in \varphi^{-1}(y)$. The orientation number $\epsilon(\varphi, x)$ of $\varphi$ at $x$ is $+1$ if $\varphi$ preserves orientation at $x$ and $-1$ if $\varphi$ reverses orientation at $x$. Let

$$I(\varphi, (y)) = \sum_{x \in \varphi^{-1}(y)} \epsilon(\varphi, x). \quad (2.14)$$

**Theorem [GP]:** $I(\varphi, (y))$ is independent of $y$. \(\Box\)

**Definition 1:** The oriented degree of $\varphi$ is

$$\deg(\varphi) = I(\varphi, (y))$$

where $y$ is any regular value of $\varphi$.

**Theorem [Sp]:** There is a number, $\eta$, such that if $\omega$ is any $n$-form on $N$

$$\int_M \varphi^*\omega = \eta \int_N \omega. \quad (2.15) \quad \Box$$

**Definition 2:** Define $\deg(\varphi) = \eta$.

Let $H^k(M)$ denote the $k^{th}$ de Rham cohomology vector
II.B THE LATTICE FADDEEV-POPOV FORMULA

space [GHV2] of $M$, and $\psi^\#: H^k(N) \to H^k(M)$ the map induced by the pullback of $\psi$. Since $H^0(M)$ and $H^0(N)$ are one-dimensional vector spaces and $\psi^\#$ is linear, $\psi^\#$ is just multiplication by some constant, $\eta$.

**Definition 3:** Define $\deg(\psi) = \eta$.

**Theorem 2.2 (GHV2):** The oriented degree of a map satisfies

1. If $\psi_1$ and $\psi_2$ are homotopic, then $\deg \psi_1 = \deg \psi_2$.
2. $\deg(\psi_1 \circ \psi_2) = \deg(\psi_1) \cdot \deg(\psi_2)$.
3. If $\psi$ is a diffeomorphism, then $|\deg \psi| = 1$. If $\psi$ is orientation-preserving, then $\deg(\psi) = 1$; if $\psi$ is orientation-reversing, then $\deg(\psi) = -1$. 
4. If $\deg(\psi) \neq 0$, then $\psi$ is surjective. 

The proof of Theorem 2.1 is based on Definition 2 of degree. The next two lemmas are contained in Proposition XIV, Chapter I of [GHV2] but we include the proofs here because they are fundamental for what follows.

**Lemma 2.3:** Let $\psi: S \to S$ and $E: S \to \mathbb{R}$ be smooth maps and let $\nu$ be the invariant form for Haar measure as described above. Then

$$\psi^*(E\nu)(g) = E\circ \psi(g) \cdot J(g) \cdot \nu(g) \quad (2.16)$$

where

$$J(g) = \det d(L[\psi(g)^{-1}] \circ L[g]) a.$$
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Proof: Since $\psi^* (Ev) (g) = \psi^* E (g) \psi^* \nu (g)$ and $\psi^* E (g) = E \circ \psi (g)$, it suffices to show that

$$\psi^* \nu (g) = J (g) \nu (g). \quad (2.17)$$

Let $\xi = L[\psi (g)^{-1}] \circ \psi \circ L [g]$. Then

$$\psi^* \nu (g) = L [g^{-1}] \circ \xi \circ \nu (g)$$

$$= L [g^{-1}] \circ \xi \circ \nu (g) \quad (2.18)$$

where in the last line we have used the left invariance of $\nu$. Now for any $X_1, \ldots, X_n \in T_g (\mathcal{G})$ we have

$$L [g^{-1}] \circ \xi \circ \nu (g) (X_1, \ldots, X_n)$$

$$= \xi \circ \nu (g) (d L [g^{-1}] X_1, \ldots, d L [g^{-1}] X_n)$$

which we write (somewhat loosely) as

$$L [g^{-1}] \circ \xi \circ \nu (g) = \xi \circ \nu (g) \circ d L [g^{-1}] g.$$

Because $\xi (1) = 1,$

$$\xi \circ \nu (g) = \det d \xi_g \nu (1),$$

so from eq. (2.19)

$$L [g^{-1}] \circ \xi \circ \nu (g) = \det d \xi_g \nu (1) \circ d L [g^{-1}] g$$

$$= \det d \xi_g L [g^{-1}] \circ \nu (g)$$

$$= \det d \xi_g \nu (g). \quad (2.20)$$

Combining eq. (2.20) with eq. (2.18) and the fact that $J (g) = \det d \xi_g$ yields eq. (2.17) $\Box$

Lemma 2.4: Let $E: \mathcal{G} \to \mathcal{G}$ and $\psi: \mathcal{G} \to \mathcal{G}$ be smooth maps and $J (g)$ be defined as in Lemma 2.3. Then

$$\int_{\mathcal{G}} E \circ \psi (g) J (g) \, dg = \deg (\psi) \int_{\mathcal{G}} E (h) \, dh. \quad (2.21)$$
Proof: By Lemma 2.3 and definition 2 of degree,
\[ \int_\gamma E \circ \phi(g) \ J(g) \ dg = \int_\gamma \phi^*(E \circ) \]
\[ = \deg(\phi) \int_\gamma E \circ \]
\[ = \deg(\phi) \int_\gamma E(h) \ dh. \]  

If we take \( \phi(g) = F(ga) \) then \( J(g) = \det M(ga) \) (see eq. (2.11)). Thus we have

Corollary 2.5: Let \( F: \mathbb{S}^n \to \mathbb{S} \) be any smooth map and \( a \in \mathbb{S}^n \).

If \( \eta(a) \) is the degree of the map \( \phi: \mathbb{S} \to \mathbb{S} \) defined by \( \phi(g) = F(ga) \), then for all smooth functions \( E: \mathbb{S} \to \mathbb{C} \)
\[ \int_\gamma \det M(ga) \ E \circ F(ga) \ dg = \eta(a) \int_\gamma E(h) \ dh. \]  

Hirschfeld [H] based his work on a quantity closely related to \( \eta(a) \) (cf. Section III.C). As he pointed out, its utility depends on

Lemma 2.6: The quantity \( \eta(a) \) defined in Corollary 2.5 is independent of \( a \).

Proof: This is a consequence of the fact that degree is a homotopy invariant. For let \( a_i \) \((i = 0,1)\) be any two gauge fields. We show below that the maps \( f_i(g) = F(ga_i) \) \((i=0,1)\) are homotopic. Since \( \eta(a_i) = \deg f_i \), it then follows that \( \eta(a_0) = \eta(a_1) \).

The homotopy is obtained as follows. By assumption, \( G \)
is connected and hence so is \( \mathbb{S}^n \). Thus there is a continuous path \( a: [0,1] \to \mathbb{S}^n \) from \( a_0 \) to \( a_1 \). We may take \( a_0 \) to be smooth (see Proposition IX, section 1.11 of
Define \( f_t: [0, 1] \times \mathcal{S} \to \mathcal{S} \) by
\[
f_t(g) = F(ga_t),
\]
Then \( f_t \) interpolates between \( f_0 \) and \( f_1 \). It remains to show that it is smooth. Since \( F: \mathcal{S}^* \to \mathcal{S} \) is smooth it suffices to show that
\[
(t, g) \mapsto ga_t
\]
is smooth. Now a map into a product space is smooth if and only if the projection onto each component is smooth. Thus it is enough to show that for each \( x, y \in \Lambda \), the map
\[
(t, g) \mapsto ga_t(x, y) = g(x) a_t(x, y) g(y)^{-1}
\]
is smooth. By the definition of Lie group, the multiplication map \( \mu(g, h) = gh \) and the inverse map \( \nu(g) = g^{-1} \) are smooth. Also, the projection maps \( g \mapsto g(x) \) and \( a \mapsto a(x, y) \) are smooth. Thus the map (2.23) is a composition of smooth maps and so is itself smooth. This shows that \( f_t \) provides a smooth homotopy between \( f_0 \) and \( f_1 \). \( \Box \)

With these lemmas behind us, we are ready to prove the theorem. The proof is simply an adaptation of the original Faddeev-Popov argument [FP].

**Proof of Theorem 2.1**: The upshot of the lemmas is that for some constant \( n \) depending only on \( F \),
\[
n \int_{\mathcal{S}} E(h) \, dh = \int_{\mathcal{S}} \det M(M^a EoF(M^a a) \, dg. \quad (2.24)
\]
Multiply both sides of eq. (2.24) by
II.B THE LATTICE FADDEEV-POPOV FORMULA

\[ \int_{y^*} f(a) e^{-S(a)} \, da \]

and apply Fubini's theorem to obtain

\[ \eta(\int_{x} E(h) \, dh)(\int_{y^*} f(a) e^{-S(a)} \, da) \]

\[ = (\int_{y} \det M(ga) E \circ F(ga) \, dg)(\int_{y^*} f(a) e^{-S(a)} \, da) \]

\[ = \int_{y} \int_{y^*} \det M(ga) f(a) E \circ F(ga) e^{-S(a)} \, da \, dg. \quad (2.25) \]

Now we wish to make a "change of variables" on the right-hand side of eq.(2.25) from \( a \) to \( ga \). More precisely, we are going to use the fact that if \( \phi \) is any smooth function on \( y^* \)

\[ \int_{y^*} \phi(ga) \, da = \int_{y^*} \phi(a) \, da \quad (2.26) \]

for any \( g \in \mathcal{G} \). To see this, recall that \( da \) is the product of the Haar measures \( da(x,y) \) and that

\[ ga(x,y) = g(x) \, a(x,y) \, g(y)^{-1}. \]

Thus the integral over any of the variables \( a(x,y) \) on the left-hand side of eq.(2.26) differs from that on the right only by a left and a right translation of \( a(x,y) \). Since \( da(x,y) \) is both left and right invariant, eq.(2.26) holds.

Now both \( f \) and \( S \) are gauge invariant, so

\[ \int_{y^*} \det M(ga) f(a) E \circ F(ga) e^{-S(a)} \, da \]

\[ = \int_{y^*} \det M(ga) f(ga) E \circ F(ga) e^{-S(ga)} \, da \]

\[ = \int_{y^*} \det M(a) f(a) E \circ F(a) e^{-S(a)} \, da \]

and eq.(2.25) becomes
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\[ \eta (\int_{\mathcal{V}} E(h) \, dh) (\int_{\mathcal{V}} f(a) \, e^{-S(a)} \, da) \]

\[ = \int_{\mathcal{V}} \int_{\mathcal{V}} \det M(a) f(a) E_0 F(a) e^{-S(a)} \, da \, dg \]

\[ = \int_{\mathcal{V}} \det M(a) f(a) E_0 F(a) e^{-S(a)} \, da \]

where we have used the fact that

\[ \int_{\mathcal{V}} \, dg = 1. \]
II.C The Gauge Degree

Lemma 2.6 allows us to make the following definition.

**Definition:** Let $F$ be a gauge fixing function. The gauge degree associated with $F$ is the number $\eta$ which is the degree of the map $g \rightarrow F(g a)$ for any $a \in \mathcal{G}^*.

As we saw in Theorem 2.1, the lattice Faddeev-Popov formula

$$
\langle f \rangle = \frac{\int \det M(a) f(a) e^{\gamma F(a)} e^{-S(a)} da}{\int \det M(a) e^{\gamma F(a)} e^{-S(a)} da}
$$

is valid provided that the gauge degree is not zero. When the gauge degree is zero, both the numerator and denominator of the right-hand side of the formula are equal to zero.

That fact is reason enough for us to look for a way of determining the gauge degree, but as we shall see in Chapter III, there are other reasons as well. For the gauge degree reflects to some extent the nature and number of orbit-surface intersections. Thus its value gives us some insight into the Gribov copies for that gauge.

In Theorem 2.7 below we obtain a formula for the gauge degree which makes it effectively computable.

Let $F: \mathcal{G}^* \rightarrow \mathcal{G}_1$ be a gauge fixing function of the form

$$
F(a)(x) = \prod_{(x, y) \in \Lambda^*} a(x, y)^{m(x, y)}
$$

(2.6)

where the exponents $m(x, y)$ are integers. For $g \in \mathcal{G}_1$ we
have

\[ F(\mathbf{g}) (x) = \prod_{y \in \Lambda} g(y)^{n(x, y)}. \]  

(2.27)

for some integers \( n(x, y) \). Let \( N : \Lambda \to \Lambda \) be the linear map whose matrix elements are given by

\[ N_{xy} = n(x, y). \]  

(2.28)

The gauge degree is determined by the matrix of exponents \( N \).

**Theorem 2.7:** Let \( G \) be a compact, connected Lie group with rank \( r \). Let \( F \) and \( N \) be as above. Then the gauge degree associated with \( F \) is given by

\[ \eta = (\det N)^r. \]  

(2.29)

**Remarks:**

1. Recall that the rank of a Lie group is the dimension (as a manifold) of a maximal abelian subgroup.

2. As we noted before, there is some ambiguity in the notation used in eqs. (2.6) and (2.27) when \( G \) is nonabelian since the order of the factors in the group multiplications is not indicated. As we shall see in Theorem 2.8 below, changing the order of these factors does not change the gauge degree. Thus to avoid some clumsy notation (cf. eq. (2.27) with eq. (2.30)) we shall use expressions like (2.6) and (2.27).

Theorem 2.7 is an immediate consequence of the following theorem applied to the function \( f(g) = F(\mathbf{g}) \).

**Theorem 2.8:** Let \( G \) be a compact, connected Lie group...
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with rank r. Let \( D \geq 1 \) and \( \pi_1: G^D \to G \) be the canonical projection onto the \( i \)th component. Suppose \( f : G^D \to G^D \) has the form

\[
(\pi_1 \circ f)(g) = g_{j_1}^m g_{j_2}^m \cdots g_{j_p}^m
\]

for \( i = 1,2,\ldots,D \) where \( g \) denotes \( (g_1, g_2, \ldots, g_D) \in G^D \) and each \( m_{ij} \) is an integer. Each \( j_k \in \{1,2,\ldots,D\} \) and different \( j_k \)'s can have the same value. Define

\[
\eta_{ij} = \sum_{(j_k : j_k = j)} m_{ij}. \tag{2.30}
\]

Then

\[
\text{deg } f = (\det N)^r
\]

where \( N = (\eta_{ij}) \).

Remarks: 1. In the ambiguous notation of eq. (2.27) we would write

\[
(\pi_1 \circ f)(g) = \prod_{1 \leq j \leq D} g_{j}^{\eta_{ij}}
\]

for eq. (2.30).

2. Theorem 2.8 generalizes the well known result that the degree of the power map \( g \to g^n \) on \( G \) is \( n^r \).

The proof of Theorem 2.8 requires some results from the theory of the cohomology of compact Lie groups. We give a quick review here leading to the main theorem in the subject, Theorem 2.9 below. The interested reader is referred to [GHV2] for proofs.

The de Rham cohomology algebra of \( G \) is the graded
algebra given by the direct sum

\[ H(G) = \sum_{k=0}^{n} H^k(G) \]

where \( n \) is the dimension of \( G \). We also need the graded algebra

\[ H^+(G) = \sum_{k=1}^{n} H^k(G). \]

The Kunneth theorem gives an isomorphism

\[ H(G^D) \cong H(G) \otimes H(G) \otimes \cdots \otimes H(G) \] (\( D \) copies)

by the map

\[ \lambda(a_1 \otimes a_2 \otimes \cdots \otimes a_D) = \pi_1 a_1 \cdot \pi_2 a_2 \cdot \cdots \cdot \pi_D a_D. \]

In view of this theorem, we normally identify \( H(G^D) \) with \( \otimes^D H(G) \).

Let \( \mu: G \times G \to G \) be the multiplication map \( \mu(g, h) = gh \). An element \( a \in H^+(G) \) is \textbf{primitive} if

\[ \mu^# a = a \otimes 1 + 1 \otimes a. \]

Let \( P_G \) denote the graded subspace of \( H(G) \) which consists of the primitive elements.

\textbf{Theorem 2.9:} Let \( G \) be a compact, connected Lie group with rank \( r \).

(a) Any primitive element of \( H(G) \) has odd degree.

(b) \( \dim P_G = r \).

(c) The inclusion map \( P_G \to H(G) \) extends to an isomorphism

\[ \lambda_G: \Lambda P_G \to H(G). \]

where \( \Lambda P_G \) denotes the exterior algebra over the vector
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The proof of Theorem 2.8 is based on the following three lemmas.

Lemma 2.10: Let $\mu_N: G^N \to G$ be the multiplication map

$$\mu_N(g_1, \ldots, g_N) = g_1 g_2 \cdots g_N.$$ 

Suppose $a \in H^*(G)$ is primitive. Then

$$\mu_N^a = \sum_{i=1}^{D} \pi_i^a.$$

Proof: We shall need the following fact. Suppose $M_i$, $N_i$ ($i=1,2$) are any manifolds and $f_i: M_i \to N_i$ are smooth maps.

Let $f_1 \otimes f_2: M_1 \times M_2 \to N_1 \times N_2$ be the map

$$(f_1 \otimes f_2)(x_1, x_2) = (f_1(x_1), f(x_2)).$$

I claim that

$$(f_1 \otimes f_2)^\#(a_1 \otimes a_2) = f_1^\# a_1 \otimes f_2^\# a_2.$$ 

For let $\pi_1$ and $\bar{\pi}_1$ denote the projection maps on $N_1 \times N_2$ and $M_1 \times M_2$ respectively. Then, if $a_i \in H(N_i)$,

$$(f_1 \otimes f_2)^\#(a_1 \otimes a_2) = (f_1 \otimes f_2)^\#(\pi_1^# a_1 \cdot \bar{\pi}_1^# a_2)$$

$$= (f_1 \otimes f_2)^\# a_1 \cdot (f_1 \otimes f_2)^\# a_2$$

$$= \pi_1^# f_1^# a_1 \cdot f_2^# a_2$$

$$= f_1^# a_1 \otimes f_2^# a_2.$$ 

(2.31)

We now use induction on $D$ to prove the lemma. The lemma is true when $D = 2$ by the definition of primitive element. Note that

$$\mu_D = \mu_2 \circ (I \otimes \mu_{D-1})$$ 

(2.32)
II.C THE GAUGE DEGREE

where $1: G \to G$ is the identity map. Thus

$$\mu_D#a = (1 \otimes \mu_D^{-1})#\mu_D#a$$

$$= (1 \otimes \mu_D^{-1})#(1 \otimes a + a \otimes 1)$$

$$= 1 \otimes \mu_D^{-1}a + a \otimes 1$$

by eq. (2.31). By the induction hypothesis, we are finished. □

Lemma 2.11: Let $P_m: G \to G$ (m an integer) be the m-power map

$$P_m(g) = g^m.$$ 

Then if $a \in H^+(G)$ is primitive,

$$P_m#a = ma.$$ 

Proof: It suffices to prove this for $m \geq 0$ and $m = -1$, since $P_m# = P_{-1}# \circ P_m#$. 

If $m \geq 0$, let $\Delta_m: G \to G^m$ be the diagonal map

$$\Delta_m(g) = (g, g, \ldots, g).$$

Then $P_m = \mu_m \circ \Delta$. By Lemma 2.10, and the fact that

$$\pi_1 \circ \Delta_m = I,$$

$$P_m#a = \Delta_m^#(\sum_{i=1}^m \pi_i#a)$$

$$= m \sum_{i=1}^m (\pi_i \circ \Delta_m)#a$$

$$= mx.$$ 

If $m = -1$, let $\psi: G \to G \times G$ be $g \to (g, g^{-1})$. Then $\mu_2 \circ \psi$ is the constant map 1 so
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$$0 = (\mu_2 \circ \psi) \# \alpha$$

$$= \psi \# \mu_2 \alpha$$

$$= \psi \# (\pi_1 \# \alpha + \pi_2 \# \alpha)$$

$$= P_1 \# \alpha + P_{-1} \# \alpha$$

$$= \alpha + P_{-1} \# \alpha.$$ 

Hence $P_{-1} \# \alpha = -\alpha$. \(\square\)

Lemma 2.12: Let $f$ be as in Theorem 2.8 and $\alpha \in H^*(G)$ be primitive. Then

$$(\pi_1 \circ f) \# \alpha = \sum_{j=1}^D n_{ij} \pi_j \# \alpha.$$ 

Proof: Define $Q: G^D \rightarrow G^{P_1}$ by

$$Q(g) = (g_{j_1} a_{j_2}, \ldots, g_{j_{p_1}} a).$$

Then

$$\pi_1 \circ f = \mu_{p_1} \circ Q.$$ 

If we let $\overline{\pi}_k: G^{P_1} \rightarrow G$ be the canonical projection onto the $k$th component, then $\overline{\pi}_k \circ Q = P_{\pi_{j_k}} \circ \pi_{j_k}$. Thus

$$(\mu_{p_1} \circ Q) \# \alpha = Q\# (\sum_{k=1}^{p_{j_k}} \overline{\pi}_k \# \alpha)$$

$$= \sum_{k=1}^{p_{j_k}} (\pi_k \circ Q) \# \alpha.$$
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\[ \frac{p_i}{k=1} \prod_{m_{ij} \neq 0} \alpha_{k-1} = \prod_{j=1}^{D} n_{ij} \# \alpha_k \]

Proof of Theorem 2.8: By definition 3 of degree (see Section II.B), it suffices to show that

\[ f^\# \alpha = (\det N)^r \alpha \]

for any nonzero \( \alpha \) in the top cohomology vector space of \( G^D \) (i.e., \( \alpha \in H^D_k(G^D) \) where \( k = \dim G \)). We shall construct a suitable \( \alpha \) from the primitive generators of \( H(G) \).

First, we claim that there are primitives \( \alpha_1, \alpha_2, \ldots, \alpha_r \in H^+(G) \) so that \( \alpha_1 \alpha_2 \cdots \alpha_r \) is a nonzero element of the top cohomology vector space of \( G \). For, by Theorem 2.9, the subspace \( P_\alpha \) of primitive elements of \( H(G) \) has dimension \( r \). Let \( (\alpha_1, \ldots, \alpha_r) \) be a basis of \( P_\alpha \). The linear independence of the \( \alpha_i \) implies that (see Lemma VIII section 4.17 of [GHV2])

\[ \alpha_1 \alpha_2 \cdots \alpha_r \neq 0. \]

Since the map \( \lambda_\alpha : \Lambda P_\alpha \to H(G) \) is an isomorphism (Theorem 2.9), \( \alpha_1 \cdots \alpha_r \) must have top degree since everything in \( H(G) \) can be constructed from the \( \alpha_k \). This
proves the claim.

If $\beta \in H^d(G)$, define $\beta^D \in H^d(G^D)$ by

$$\beta^D = \beta \otimes \beta \otimes \cdots \otimes \beta \quad \text{(D copies).}$$

Set

$$\alpha = a^D_1 a^D_2 \cdots a^D_r.$$

Note that up to a possible minus sign, $\alpha$ equals $(a_1 a_2 \cdots a_r)^D$ and so is not zero. Also, $\alpha \in H^k(G^D)$.

We now show that

$$f^# a^D_k = (\det N) a^D_k$$

from which it follows that

$$f^# \alpha = (f^# a^D_1) \cdot (f^# a^D_2) \cdot \cdots \cdot (f^# a^D_r)$$

$$= (\det N) f^# \alpha. \quad (2.33)$$

Let $\beta = a_k$ for any $k$ and recall (Theorem 2.9(a)) that $\beta$ has odd degree. Then

$$f^# \beta^D = f^# (\pi^#_1 \beta \cdot \pi^#_2 \cdots \cdot \pi^#_D \beta)$$

$$= (f^# \pi^#_1 \beta) \cdot (f^# \pi^#_2 \beta) \cdots \cdot (f^# \pi^#_D \beta).$$

The fact that $\pi^#_1 \beta \cdot \pi^#_2 \beta = -\pi^#_2 \beta \cdot \pi^#_1 \beta$ and Lemma 2.12 now give

$$f^# \beta^D = (\det N) \beta^D. \quad \Box$$
II.D Validity of the lattice Faddeev-Popov formula

Now that we know how to calculate the gauge degree (Theorem 2.7) we can check in particular examples whether or not it is zero, and hence whether or not the lattice Faddeev-Popov formula is valid. Some insight into the "bad" case ($n = 0$) can be obtained from Theorem 2.13 below.

First, note that the Faddeev-Popov determinant is most easily calculated using the fact that if $\psi: \mathfrak{g} \to \mathfrak{g}$ maps $\mathfrak{g}$ into $\mathfrak{g}$, then

$$\psi(e^x) = 1 + d\psi \xi y + O(\xi^2). \quad (2.34)$$

It is also convenient to introduce the standard ordered basis of $\mathbb{R}^{\Lambda_1}$. The basis elements $(b_x)_{x \in \Lambda_1}$ (which we regard as functions from $\Lambda_1 \to \mathbb{R}$) are defined by $b_x(y) = \delta_{xy}$ and they are ordered lexicographically. That is,

$b_x > b_y$ if for some $p \in (0,1,\ldots,s)$

$$x_1 = y_1 \text{ for } i < p$$

and

$$x_p > y_p.$$

Theorem 2.13: The gauge degree is zero if and only if the Faddeev-Popov determinant at $\mathfrak{g}$ is 0:

$$n = 0 \iff \det M(\mathfrak{g}) = 0. \quad (2.35)$$

Proof: We shall show that

$$\det M(\mathfrak{g}) = (\det N)^k \quad (2.36)$$

where $k$ is the dimension of $G$ and $N$ is the matrix
described in eq. (2.22). The equivalence (2.35) is then obvious since
\[ \eta = (\det N)^T. \]

In the following, we shall freely identify the isomorphic vector spaces
\[ \mathcal{E}_1 \cong \otimes_{x \in \Lambda_1} E \cong \mathbb{R}^{\Lambda_1} \otimes E. \]

Recall that
\[ M(\mathbb{1}) = d\varphi_{\mathbb{1}} \]
where
\[ \varphi(g) = F(\Theta \mathbb{1}). \]

Let \( \gamma \in \mathcal{E} \). Then (see eq. (2.27))
\[ \varphi(e^\gamma(x)) = \prod_{y \in \Lambda} (e^\gamma(y)) \cdot n(x, y) \]
\[ = \prod_{y \in \Lambda} e^{n(x, y) \gamma(y)} \]
\[ = 1 + \sum_{y \in \Lambda} n(x, y) \gamma(y) + o(\gamma^2). \]

Applying eq. (2.34), we see that
\[ d\varphi_{\mathbb{1}} = N \otimes I \quad (2.37) \]
where \( I : E \to E \) is the identity map. Since the dimension of \( E \) is \( k \),
\[ \det M(\mathbb{1}) = \det (N \otimes I) = (\det N)^k. \]

Recall that a gauge fixing function \( F \) is defined with respect to a sublattice \( \Lambda_1 \subset \Lambda \) (see Section II.A). The next theorem shows that we run into trouble if we try to fix the gauge on all of \( \Lambda \).
Theorem 2.14: If $\Lambda_1 = \Lambda$ then the gauge degree associated with $F$ is zero:

$$\Lambda_1 = \Lambda \implies n = 0.$$ 

Proof: We shall show that

$$\sum_{y \in \Lambda} n(x, y) = 0 \quad (2.38)$$

where the $n(x, y)$ are defined in eq.(2.27). Thus the kernel of $N$ (see eq.(2.22)) is nontrivial, since it contains the functions on $\Lambda$ which are constant. Thus $n = (\det N)^r = 0$.

We prove (2.38) assuming that $G$ is abelian. The proof for the nonabelian case is the same except that the notation is considerably more complicated.

We have

$$F(a)(x) = \prod_{y \in \Lambda} a(x, y) m(x, y)$$

and

$$F(Ga)(x) = \prod_{y \in \Lambda} [g(x)a(x, y)g(y)^{-1}] m(x, y)$$

so that

$$F(G^2)(x) = \prod_{y \neq x} g(x)^m(x, y) g(y)^{-m(x, y)}.$$ 

From eq.(2.27) we see that

$$n(x, x) = \sum_{y \neq x} m(x, y)$$

and

$$n(x, y) = -m(x, y) \quad (y \neq x).$$

Thus
II.D VALIDITY OF THE LATTICE FADDEEV-POPOV FORMULA

\[ \sum_{y \in \Lambda} n(x, y) = n(x, x) + \sum_{y \in \Lambda} n(x, y) \]

\[ = \sum_{y \in \Lambda} m(x, y) - \sum_{y \not= x} m(x, y) \]

\[ = 0. \quad Q.E.D. \]

The reader might well be wondering at this point whether there are any cases for which the gauge degree is not zero. One way to get around Theorem 2.14 is to take \( \Lambda_1 \) to be a proper sublattice of \( \Lambda \). We now illustrate how this works for axial gauge and Landau gauge.

Axial gauge is the easiest to analyze by our methods. In this case the gauge fixing function is (cf. eq.(2.7))

\[ F_{A}(a)(x) = a(x, x + e_0). \]

We use eq.(2.34) to calculate \( M(a) \). Let

\[ \chi = \{ \chi(x) \}_{x \in \Lambda_1} \in \mathcal{E}_1, \quad a \in \mathfrak{g}^* \text{ and } \hat{\chi}(g) = F_{A}(a)^{-1} F_{A}(g a). \]

Then

\[ \hat{\chi}(e^{\chi})(x) \]

\[ = F_{A}(e^{\chi} a)(x) F_{A}(a)(x)^{-1} \]

\[ = e^{\chi(x)} a(x, x + e_0) e^{-\chi(x + e_0)} a(x, x + e_0)^{-1} \]

\[ = (1 + \chi(x)) a(x, x + e_0) (1 - \chi(x + e_0)) a(x, x + e_0)^{-1} + O(\chi^2) \]

\[ = 1 + \chi(x) - a(x, x + e_0) \chi(x + e_0) a(x, x + e_0)^{-1} + O(\chi^2) \]

\[ = 1 + \chi(x) - ad(a)(x, x + e_0) \chi(x + e_0) + O(\chi^2) \]

where \( ad(a): E \to E \) for \( a \in G \) is the adjoint map \( ad(a) = e^{a \chi} a^{-1} \).

Applying eq.(2.34), we deduce that
II.D VALIDITY OF THE LATTICE FADDEEV-POPOV FORMULA

\[ M(a)_{xx} = I \]

\[ M(a)_{x,x+e_0} = -\text{ad}[a(x,x+e_0)] \]

\[ M(a)_{xy} = 0 \text{ if } y \not= x \text{ or } x + e_0. \]

In the standard ordered basis of \( \mathbb{R}^1 \)

\[ b_{x+e_0} > b_x \]

so the matrix \( M(a) \) is upper triangular with diagonal elements equal to one. Thus \( \det M(a) = 1 \) for all \( a \in \mathfrak{g}^* \).

Theorem 2.13 tells us that in particular, \( \eta \neq 0 \) for axial gauge.

A rigorous version of the lattice Faddeev-Popov formula for axial gauge has also been given by [OS]. They showed that for any gauge invariant function \( f \)

\[ \langle f \rangle = \frac{\int f^A(a) e^{-S^A(a)} \, da}{\int e^{-S^A(a)} \, da} \]

where \( f^A \) and \( S^A \) are obtained from \( f \) and \( S \) by setting all arguments \( a(x,x+e_0) \) to \( \mathbb{1} \). We can obtain this formula from eq. (2.13) by choosing \( E(c) = \delta(c) \), where by \( \delta(c) \) we mean \( \prod_x \delta(c_x) \) the latter \( \delta \)-function being the one appropriate to Haar measure [OS].

We now turn to the more complicated Landau gauge.

Given \( x \in \Lambda \), define

\[ \text{ad}(\pm \mu) = \text{ad}[a(x,x+e_0)a(x,x-e_0)a(x,x+e_1)\cdots a(x,x \pm e_\mu)] \]

so, for example,

\[ \text{ad}(1) = \text{ad}[a(x,x+e_0)a(x,x-e_0)a(x,x+e_1)] \]

\[ \text{ad}(-1) = \text{ad}[a(x,x+e_0)a(x,x-e_0)a(x,x+e_1)a(x,x-e_1)]. \]
For $F_L$ as given by eq. (2.8b) and $g = e^{\xi} \epsilon \xi$ we have

$$F_L(a)(x) = (I + \chi(x)) a(x, x + e_\xi) (I - \chi(x + e_\xi))$$
$$\times (I + \chi(x)) a(x, x - e_\xi) (I - \chi(x - e_\xi)) \times \ldots$$
$$\times (I + \chi(x)) a(x, x - e_{s\xi}) (I - \chi(x - e_{s\xi}))$$
$$\times \left[a(x, x + e_\xi) \ldots a(x, x - e_{s\xi})\right]^{-1} + O(\chi^2)$$
$$= 1 + (1 - \text{ad}(-s) + \sum_\mu [\text{ad}(+\mu) + \text{ad}(-\mu)]) \chi(x)$$
$$- \sum_\mu \text{ad}(+\mu) \chi(x + e_\mu) - \sum_\mu \text{ad}(-\mu) \chi(x - e_\mu)$$
$$+ O(\chi^2). \quad (2.39)$$

The matrix element $M(a)_{xy}$ is the coefficient of $\chi(y)$ in the preceding expression. Thus

$$M(a)_{xx} = 1 - \text{ad}(-s) + \sum_\mu [\text{ad}(+\mu) + \text{ad}(-\mu)]$$
$$M(a)_{x, x + e_\mu} = -\text{ad}(e_\mu) \quad (2.40)$$
$$M(a)_{xy} = 0 \text{ if } |x-y| > \epsilon.$$  

For the gauge degree we need only consider $M(1)$. From eq. (2.40) we have

$$M(1)_{xx} = 2d_\Lambda$$
$$M(1)_{x, x + e_\mu} = -1 \quad (2.41)$$
$$M(1)_{xy} = 0 \text{ if } |x-y| > \epsilon.$$  

Thus

$$M(1) = \epsilon^2 1 \quad (2.42)$$

where $d_\Lambda$ is the finite-difference Laplacian with Dirichlet
boundary conditions on $\Lambda$ [GRS2],[GJ]. The eigenvalues of $\Delta^E_\Lambda$ are [GJ]

$$\lambda_k = \sum_{\mu=0}^{N-1} 4\epsilon^2 \sin^2(k_{\mu} \pi/4N_{\mu})$$  \hspace{1cm} (2.43)

with $k_{\mu} \epsilon \{1,2,\ldots,2N_{\mu}-1\}$. No eigenvalue is zero, so $
\det M(\Omega) \neq 0$ and the gauge degree $\eta \neq 0$ for this choice of $\Lambda_1$.

We have just seen that by choosing boundary conditions carefully it is not too hard to arrange that $\eta \neq 0$. However, it would be better to have more freedom to make this choice. For example, one natural way to define a gauge fixing function on all of $\Lambda$ is to impose periodic boundary conditions. If we had done this for Landau gauge in eq. (2.8b), the difference would be that we would have $\Delta^E_p$, the Laplacian with periodic boundary conditions, instead of $\Delta^E_\Lambda$ in eq. (2.42). In this case, $\det M(\Omega) = 0$ because the kernel of $\Delta^E_p$ contains the constant functions and so is nontrivial.

This example provides an illustration of Theorem 2.14 and at the same time suggests a way to circumvent it. If the constant functions are the source of the trouble, why not factor them out at the beginning? The following lemma and theorem are a partial implementation of this idea.

Suppose $\mathcal{H}$ is a normal subgroup of $\mathcal{G}$ and let $[g]$ denote the image of $g \in \mathcal{G}$ under the canonical projection $\mathcal{G} \to \mathcal{G}/\mathcal{H}$.

**Lemma 2.15:** Suppose $F: \mathcal{G}^* \to \mathcal{G}$ is such that for any $a \in \mathcal{G}^*$
the map \( \tilde{\psi}_a: \mathcal{S}/\mathcal{H} \rightarrow \mathcal{S}/\mathcal{H} \)

\[
\tilde{\psi}_a([g]) = [F(\theta_a)]
\]  

(2.44)

is well-defined. Then \( \eta = \deg(\tilde{\psi}_a) \) is the same for all \( a \). 
Furthermore, for any smooth function \( E \) on \( \mathcal{S}/\mathcal{H} \) and gauge invariant function \( f \) on \( \mathcal{S}^* \),

\[
\eta\left( \int_{\mathcal{S}/\mathcal{H}} E([c]) \, d[c] \right) \int_{\mathcal{S}^*} f(a) \, e^{-S(a)} \, da
\]

\[
= \int_{\mathcal{S}^*} \det \tilde{M}(a) \, f(a) \, E([F(a)]) \, e^{-S(a)} \, da
\]

where \( \tilde{M}(a) = d\theta_a \) for

\[
\theta([g]) = \tilde{\psi}_a([g]) \tilde{\psi}_a([\mathbb{1}])^{-1}.
\]

Proof: First note that the map \( [g] \rightarrow \tilde{M}(\theta_a) \) is well-defined. For suppose that \( [g_1] = [g_2] \). \( \tilde{M}(\theta_a) \) is the derivative at \( \mathbb{1} \) of the map

\[
\theta_1([h]) = [F(hg_1)] [F(\theta_a)]^{-1}.
\]

Since \( [hg_1] = [hg_2] \) and \( \tilde{\psi}_a \) is well-defined,

\[
[F(\theta_a)] = [F(\theta_2)]
\]

and

\[
[F(hg_1)] = [F(hg_2)].
\]

Thus \( \theta_1 = \theta_2 \) and \( \tilde{M}(\theta_1) = \tilde{M}(\theta_2) \).

Apply Lemma 2.4 with \( \mathcal{S} \) replaced by \( \mathcal{S}/\mathcal{H} \) and \( \psi \) by \( \tilde{\psi}_a \) to obtain

\[
\int_{\mathcal{S}/\mathcal{H}} \det \tilde{M}(\theta_a) \, E([F(a)]) \, d[g]
\]

\[
= \deg(\tilde{\psi}_a) \int_{\mathcal{S}/\mathcal{H}} E([h]) \, d[h]. \quad (2.45)
\]

The proof that \( \deg(\tilde{\psi}_a) \) is independent of \( a \) is the same.
as that of Lemma 2.6. The remainder of the proof of the present theorem runs parallel to the proof of Theorem 2.1:

\[ \eta(\oint_{\mathcal{C}^{cl}} g(c) \, dc) \left( \oint_{\mathcal{G}^c} f(a) \, e^{-S(a)} \, da \right) \]

\[ = \oint_{\mathcal{G}^{c}} \oint_{\mathcal{G}^c} \det \tilde{M}(g) \, f(a) \, E(\{F(g)\}) \, e^{-S(g)} \, da \, dg \]

\[ = \oint_{\mathcal{G}^{c}} \oint_{\mathcal{G}^c} \det \tilde{M}(a) \, f(a) \, E(\{F(a)\}) \, e^{-S(a)} \, da \, dg \]

\[ = \oint_{\mathcal{G}^{c}} \oint_{\mathcal{G}^c} \det \tilde{M}(a) \, f(a) \, E(\{F(a)\}) \, e^{-S(a)} \, da \, dg \]

\[ = \oint_{\mathcal{G}^c} \oint_{\mathcal{G}^c} \det \tilde{M}(a) \, f(a) \, E(\{F(a)\}) \, e^{-S(a)} \, da. \]

\[ \text{Theorem 2.16: Let } \mathcal{G} \text{ be abelian. Using the notation of Lemma 2.13, let } M(a) \text{ be the Faddeev-Popov operator associated with } F. \text{ Suppose the kernel of } M(\mathfrak{g}) \text{ coincides with the Lie algebra of } \mathcal{H}. \text{ Then } \eta \neq 0 \text{ and} \]

\[ \langle f \rangle = \frac{\oint \det \tilde{M}(a) \, f(a) \, E(\{F(a)\}) \, e^{-S(a)} \, da}{\oint \det \tilde{M}(a) \, E(\{F(a)\}) \, e^{-S(a)} \, da}. \quad (2.46) \]

\[ \text{Proof: Let } \tilde{\psi} = \tilde{\psi}_{\mathfrak{g}}. \text{ Because } \mathcal{G} \text{ is abelian we have} \]

\[ \theta([h]) = \tilde{\psi}([hg]) \tilde{\psi}(g)^{-1} = [F(hg)F(g)^{-1}] = [F(h\mathfrak{g})] \]

\[ \text{so that } \tilde{M}(a) \text{ is independent of } a; \text{ we shall denote it simply by } \tilde{M}. \text{ By taking } E = 1 \text{ in eq.}(2.22) \text{ we see that} \]

\[ \det \tilde{M} = \text{deg } \tilde{\psi}. \]

We also see from eq.\((2.47)\) that \( \theta([h]) = \tilde{\psi}(h) \) so

\[ \tilde{M} = d\theta_{\mathfrak{g}} = d\tilde{\psi}_{\mathfrak{g}} \]

and consequently
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\[ \text{deg } \tilde{\Psi} = \det \tilde{d} \Psi. \]

Let \( \mathcal{F} \) denote the Lie algebra of \( \mathcal{H} \). The Lie algebra of \( \mathcal{H}/\mathcal{H} \) is isomorphic to \( \mathcal{E}/\mathcal{F} \) [GHV2]. Let \( \Psi(g) = F(g) \).

Then \( \tilde{\Psi} \) is the map induced by \( \Psi \). By hypothesis \( \mathcal{F} \) is the kernel of the linear map \( \tilde{d} \Psi \), so that there is an induced map \( (\tilde{d} \Psi) : \mathcal{E}/\mathcal{F} \to \mathcal{E}/\mathcal{F} \) which is nonsingular. But under the isomorphism of Lie algebras just mentioned, \( (\tilde{d} \Psi) \) is mapped into \( d \tilde{\Psi} \), so that the latter is nonsingular. Thus \( \text{deg}(\tilde{\Psi}) = \det (d \tilde{\Psi}) \neq 0. \]  

I conjecture that Theorem 2.16 is also true when \( G \) is nonabelian. However, as we shall see below, it is difficult to satisfy the hypotheses of Lemma 2.15 when \( G \) is nonabelian.

Let us now apply Theorem 2.16 to the example that introduced it. Suppose we wish to use Landau gauge when \( G = U(1) \) and \( \Lambda \) is the periodic lattice

\[ \Lambda = \{(n_0, \ldots , n_s) : -N \leq n_\mu \leq N \}, \]

with boundary points identified. As we discussed before Lemma 2.15, \( \mathcal{M}(1) = -\epsilon^2 d \mathcal{E} \) in this case. The kernel \( \mathcal{F} \) of \( d \mathcal{E} \) consists of the constant functions, i.e.,

\[ \mathcal{F} = \{ \gamma : \Lambda \to \mathbb{E} \mid \gamma(x) = c \text{ for some } c \in \mathbb{E}, \text{ all } x \in \Lambda \}. \]

This can easily be seen by taking the Fourier transform. Let

\[ \tilde{\Lambda} = \{(n_0 b_0, \ldots , n_s b_s) : b_\mu = \frac{\pi}{N \mu}, -N \mu \leq n_\mu \leq N \mu - 1 \}. \]

The functions \( \{ e_k \}_{k \in \tilde{\Lambda}} \) given by \( e_k(x) = e^{ik \cdot x} \) form a
basis for $C^\Lambda$ and we have the relations

$$\hat{f}(k) = \sum_{x \in \Lambda} f(x) e_k(x)$$

$$f(x) = \frac{1}{\mu(2N)} \sum_{k \in \Lambda} e_k(x) \hat{f}(k).$$

A short calculation shows that

$$-\delta_P e_k = \lambda_k e_k$$

where $\lambda_k = 4\varepsilon^{-2} \sum_{\mu=0}^S \sin^2(\frac{\pi k\varepsilon}{\mu}).$

If $f \in \text{Ker}(\delta_P)$ then

$$\delta_P f(x) = 0 \implies \frac{1}{\mu(2N)} \sum_{k} \lambda_k e_k \hat{f}(k) = 0$$

$$\implies \hat{f}(k) = 0 \text{ for all } k \neq 0$$

$$\implies f(x) = \frac{1}{\mu(2N)} \hat{f}(0)$$

i.e., $f$ is a constant function.

The subgroup $\mathcal{H} \subset \mathcal{Y}$ which has $\mathbb{F}$ for its Lie algebra

is the group of constant functions $\Lambda \to \mathbb{G}$. Since $\mathbb{G}$ is

abelian, $\mathcal{H}$ is a normal subgroup.

The map $\psi_a$ of Lemma 2.15 is well-defined for any

$F: \mathbb{S}^* \to \mathcal{Y}$. To see this, note first that for any $h \in \mathcal{H}$ and

$a \in \mathbb{S}^*$

$h_a = a$.

If $[g_1] = [g_2]$ then there is an $h \in \mathcal{H}$ such that

$g_1 = g_2 h$ and

$$\psi_a([g_1]) = [F(g_1 a)]$$
II.D VALIDITY OF THE LATTICE FADDEEV-POPOV FORMULA

\[ F(g^h_2 a) \]
\[ = [F(g^h_2 a)] \]
\[ = \tilde{\psi}_a([g_2]). \]

Thus Theorem 2.16 applies and we have an Faddeev-Popov formula for the abelian periodic lattice model. The price we have had to pay to overcome the \( \eta = 0 \) problem is that now the damping factor \( E \) is a function of \([F(a)]\) instead of \( F(a) \). The function \( E \) suggested in eq. (2.9) is not of this form.

It is more difficult to apply Lemma 2.15 to a nonabelian model. The subgroup \( H \) consisting of the constant functions of \( Y \) is the natural choice to make, but it is not a normal subgroup so \( Y/H \) is a homogeneous space rather than a Lie group. More seriously, the map \( \tilde{\psi}_a \) is not well-defined in general when \( G \) is nonabelian.
CHAPTER III
CONTINUUM FADDEEV-POPOV TECHNIQUE REVISITED

III.A Relation Between the Lattice and Continuum Arguments

Much of the discussion in the physics literature of the Gribov phenomenon in continuum theories is based on implicit analogies between finite- and infinite-dimensional field theories. One of the main purposes of the work described in this thesis is to clarify these discussions, first of all by putting the finite-dimensional (lattice) theory on a firm foundation. We have done so in Chapter II.

In this section we take the next step, which is to make more explicit the parallels between the continuum Faddeev-Popov technique described in Chapter I and the lattice Faddeev-Popov technique in Chapter II.

Specifically, we rederive Corollary 2.5 and Lemma 2.6 in a heuristic fashion and in a way which mimics the continuum argument.

Of central interest is the Faddeev-Popov determinant itself. In the continuum theory, it arises as a Jacobian for a change of variables $\chi \to F(e^{\chi}a)$ on the Lie algebra of the gauge group. We show below (Corollary 3.2) that the lattice Faddeev-Popov determinant is the Jacobian for the change of variables $g \to F(ga)$ on the gauge group.

To begin, let us derive a change of variables formula
III.A RELATION BETWEEN LATTICE AND CONTINUUM

for integrals taken with respect to Haar measure.

Theorem 3.1 below generalizes the familiar formula for Lebesgue measure on $\mathbb{R}^n$

$$\int_{U} E\varphi(x) |J(x)| \, dx = \int_{\varphi(U)} E(y) \, dy \quad (3.1)$$

where $J$ is the Jacobian determinant

$$J(x) = \det d\varphi_x. \quad (3.2)$$

**Theorem 3.1:** Let $G$ be a Lie group (not necessarily compact) and $U$ be an open subset of $G$. Suppose that $E: G \to \mathbb{R}$ is a smooth function with compact support and $\varphi: U \to G$ is a diffeomorphism onto its image. Then

$$\int_{\varphi(U)} E(h) \, dh = \int_{U} E\varphi(g) |J(g)| \, dg \quad (3.3)$$

in which $dg$ and $dh$ denote left Haar measure on $G$ and $J$ is the Jacobian determinant

$$J(g) = \det d\bar{\xi}_g \quad (3.4)$$

where

$$\bar{\xi} = L[\varphi(g)^{-1}] \circ \varphi \circ L[g]. \quad (3.5)$$

Moreover, $J(g) \neq 0$ for any $g \in U$ and

$$\int_{U} E\varphi(g) \, dg = \int_{\varphi(U)} \frac{E(h)}{|J(\varphi^{-1}(h))|} \, dh. \quad (3.6)$$

**Proof:** We first claim that $J(g) > 0$ if and only if $\varphi$ preserves orientation at $g$. We orient $G$ by choosing a positive orientation class $O^+$ of $T_g(G)$ and declaring that a basis $B$ of $T_g(G)$ is positively oriented if and only if $dL[\varphi^{-1}]_g B \in O^+$. In other words, left translation is an orientation preserving map.
Consequently, \( \varphi \) preserves orientation at \( g \) if and only if \( \varphi \) preserves orientation at \( f \). Since \( \varphi(f) = f \), \( \varphi \) preserves orientation at \( f \) if and only if \( \det d\varphi > 0 \), that is, if and only if \( J(g) > 0 \). This proves the claim.

Write \( U \) as the disjoint union of \( U_+ \) and \( U_- \) where \( U_+ = \{ g \in U \mid J(g) > 0 \} \), \( U_- = \{ g \in U \mid J(g) < 0 \} \). If \( \omega \) is any compactly supported \( n \)-form (\( n = \dim B \)) then

\[
\int_{U_+} \varphi^* \omega = \int_{\varphi(U_+)} \omega \\
\int_{U_-} \varphi^* \omega = -\int_{\varphi(U_-)} \omega
\]

since \( \varphi \) preserves orientation on \( U_+ \) and reverses orientation on \( U_- \) [GP].

Take \( \omega = Ev \) and apply Lemma 2.3 to obtain,

\[
\int_{U_-} E \circ \varphi(g) |J(g)| \, dg = - \int_{U_-} E \circ \varphi(g) J(g) \, dg \\
= - \int_{U_-} \varphi^*(Ev) \\
= \int_{U_-} E(h) \, dh
\]

and similarly for the integral over \( U_+ \). Finally, since \( \varphi \) is a diffeomorphism \( \varphi(U) \) is the disjoint union of \( \varphi(U_+) \) and \( \varphi(U_-) \). Equation (3.3) is obtained by adding the integrals over \( U_+ \) and \( U_- \).

The "moreover" is a consequence of the fact that \( J(g) = 0 \) only if \( d\varphi \) is singular, but it never is because \( \varphi \) and \( L(h) \) for any \( h \) are diffeomorphisms.

Equation (3.6) is obtained by replacing the function \( E \) with \( E_h \) for any diffeomorphism \( h \).
Corollary 3.2: The FP determinant, \( \det M(a) \), is the Jacobian determinant \( J(\Omega) \) for the change of variables \( \varphi(g) = F(ga) \).

Proof: By eqs. (3.4) and (3.5), \( J(\Omega) = \det d\xi_\Omega \) where
\[
\xi(g) = L(\varphi(\Omega)^{-1}) \circ \varphi \circ L[\Omega](g)
= F(a)^{-1} F(ga),
\]
This function \( \xi \) is the same as the function \( \xi \) defined in eq. (2.10). Thus by eq. (2.11),
\[
\det M(a) = \det d\xi_\Omega = J(\Omega).
\]

We now apply these results to the FP technique. The original FP argument, expressed in the notation of the lattice theory, is based on the false equality (cf. eq. (1.14))
\[
\det M(a) \int_\Omega \delta(F(ga)c^{-1}) \, dg = 1. \tag{3.7}
\]
Let us begin instead by defining the quantity
\[
\eta(a, c) = \int_\Omega \det M(ga) \delta(F(ga)c^{-1}) \, dg. \tag{3.8}
\]
The \( \delta \)-function in eqs. (3.7) and (3.8) is the one appropriate to Haar measure, as described by [OS]. However, we are arguing somewhat informally here and do not require anything from \( \delta \) other than that it behave as a \( \delta \)-function should:
\[
\int_\Omega E(h) \delta(hc^{-1}) \, dh = E(c).
\]
The connection between \( \eta(a, c) \) and the gauge degree \( \eta \)
given by eq. (2.13) can be made by taking $E(h) = \delta(hc^{-1})$ in the latter equation.

Let us suppose that $c$ is a regular value of $\varphi(g) = F(ga)$. Then by the inverse function theorem [GP], $\varphi$ is a local diffeomorphism at each $g \in \psi^{-1}(c)$. Since $\varphi$ is compact, we conclude that $\psi^{-1}$ is a finite set. Let

$$\psi^{-1}(c) = \{g_1, g_2, \ldots, g_n\}$$

and let $U_k$ be a neighbourhood of $g_k$ on which $\varphi$ is a diffeomorphism.

We can then use Theorem 3.1 and rewrite eq. (3.8) as

$$\eta(a, c) = \sum_{k=1}^{n} \left( \int_{\varphi(U_k)} \det M(ga) \delta(F(ga)c^{-1}) \, dg \right)$$

$$= \sum_{k=1}^{n} \left( \int_{\varphi(U_k)} \frac{\det M(\varphi^{-1}(h)a)}{\det M(\varphi^{-1}(h)a)} \delta(hc^{-1}) \, dh \right)$$

$$= \sum_{k=1}^{n} \frac{\det M(g_k a)}{|\det M(g_k a)|}$$

$$= \sum_{k=1}^{n} \text{sgn} \det M(g_k a)$$

(3.9)

where $\text{sgn}(x) = x/|x|$ for $x \neq 0$.

In order that eq. (3.8) be a fruitful replacement for eq. (3.7), we need to show that $\eta(a, c)$ is independent of $a$ and $c$. Of course, we have already seen this since $\eta(a, c)$ is the degree of $\varphi(g) = F(ga)$. However, there is some intuitive insight to be gained by pursuing the matter again in the present context.

As we saw in the proof of Theorem 3.1,
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\[ \text{sgn det } M(q_k a) = \varepsilon(\varphi, g_k) \]  

(3.10)

where \( \varepsilon(\varphi, g_k) \) is the orientation number of \( \varphi \) at \( g_k \).

Thus from eq. (3.9)

\[
\eta(a, c) = \sum_{g_k \in \varphi^{-1}(c)} \varepsilon(\varphi, g_k)
\]

\[
= I(\varphi, (c))
\]

\[
= \deg \varphi
\]  

(3.11)

where we are using definition 1 of degree (see Section II.B).

Consider how we would obtain \( \eta(a, c) \) in a simple example where \( \varphi: U(1) \rightarrow U(1) \). Since \( U(1) = S^1 \) (the unit circle) we can plot the graph of \( \varphi \) in a unit square with opposite edges identified. (See Fig. 2)

The orientation number \( \varepsilon(\varphi, g_k) \) is +1 or -1 depending on whether the slope of \( \varphi \) is positive or negative when it intersects the line determined by \( c \). The homotopy invariance of \( I(\varphi, (c)) \) and its independence of \( c \) are intuitively obvious from this picture. If \( \varphi \) (and hence its graph) or \( c \) changes in a continuous manner, the points of intersection are created or disappear in pairs with opposite orientation number. Hence the sum over \( g_k \) in (3.11) remains the same.

We are now in a position to discuss in more detail the remarks made after the statement of Theorem 2.1. If there were no Gribov copies, then \( \varphi^{-1}(c) \) would contain only one point \( g \). There would be only one term in the sum (3.9) and consequently \( |\eta(a, c)| = 1 \). Equation (3.7) is correct
Figure 2 Orientation numbers for the map $g \to F^{(g)a}$
in this case (up to a sign) and is equivalent to eq. (3.8).

When there are Gribov copies, there are several terms
in the sum (3.9), one for each copy. The quantity $\eta$ then
measures not the total number of copies, but the net number
or algebraic sum of the number of copies, with the sign of a
copy's contribution determined by the sign of the
Faddeev-Popov determinant at that point.
Some Questions from the Continuum Theory

The lattice Faddeev-Popov technique we have described in Chapter II allows us to analyze the behavior of gauge field orbits and gauge-fixing surfaces in more detail than in the continuum model. In this section we address the lattice analogues of some of the issues raised in Chapter I for the continuum theory. The topics discussed are:

(i) the attainability of gauge conditions
(ii) the existence of Gribov copies
(iii) the gauge invariance of the Faddeev-Popov determinant.

Not surprisingly, the discussion relies heavily on our knowledge of the gauge degree.

Throughout this section we use the usual notation of $F: \mathcal{G}^* \to \mathcal{G}$ for the gauge-fixing function and $\eta$ for the gauge degree associated with $F$.

(i) Attainability of gauge conditions: Fix a gauge field $a \in \mathcal{G}^*$ and let $c \in \mathcal{G}$. The question to be investigated is, does there exist a gauge transformation $g \in \mathcal{G}$ such that

$$ F(ga) = c? \tag{3.12} $$

As we discussed in Chapter I, it is important to have an affirmative answer in order to carry out the Faddeev-Popov argument.
We do not have to look far for an example where eq. (3.12) does not hold. For example, let $\Lambda$ be a periodic lattice

$$\Lambda = \{ x = (n_0 \epsilon, \ldots, n_s \epsilon) | -N_\mu \leq n_\mu \leq N_\mu \}$$

with opposite boundary points identified,

$$F(a)(x) = a(x, x + e_0)$$

and take $c = 1$. Fix $a \in \mathcal{G}$ and let us seek $g \in \mathcal{G}$ such that

$$F(\mathcal{G}a)(x) = 1$$

that is, such that

$$g(x) a(x, x + e_0) g(x - e_0)^{-1} = 1$$

(3.13)

for all $x \in \Lambda$. Suppose we had such a $g$. Multiply the equations (3.13) together for all $x$ in $\Lambda$ with a fixed value of $x_0$. We obtain

$$g(x) a(x, x + e_0) g(x + e_0)^{-1} g(x + 2e_0) a(x + e_0, x + 2e_0) \cdots g(x - e_0) a(x - e_0, x) g(x)^{-1} = 1.$$  

(3.14)

On simplifying and rearranging (3.14) we find that

$$a(x, x + e_0) a(x + e_0, x + 2e_0) \cdots a(x - e_0, x) = 1.$$  

(3.15)

Of course, eq. (3.15) is not true in general so that even axial gauge can be unattainable.

However, we already know that axial gauge on a periodic lattice is pathological because its gauge degree is zero (see Theorem 2.14). The next theorem shows that these facts are closely connected.

**Theorem 3.3:** If the gauge degree associated with $F$ is not
III.B SOME QUESTIONS FROM THE CONTINUUM THEORY

zero, then

\[ F(g_a) = c \]

has a solution in \( g \in S \) for any \( a \in S^\times, c \in S \).

**Proof:** The gauge degree is the degree of the map \( \varphi(g) = F(g_a) \). Any map with nonzero degree is surjective (see Theorem 2.2(4)). \( \square \)

In particular, for axial and Landau gauge as given by eqs. (2.7) and (2.8) the gauge can always be attained.

(ii) Existence of Gribov copies: Gauge-fixing in the lattice model could not be considered a good guide to the continuum theory if Gribov copies never occurred on the lattice. Lovelace [L] has shown that there can be gauges without copies in the lattice theory. However, we can see that there are gauges where copies occur from Theorem 3.4: Let \( \eta \) be the gauge degree associated with the gauge fixing function \( F \). Fix a gauge field \( a \in S^\times \).

Suppose \( c = F(a) \) is a regular value of \( \varphi(g) = F(g_a) \). Then the number of solutions in \( g \in S \) of

\[ F(g_a) = c \]

is at least \( |\eta| \).

In particular, if \( |\eta| > 1 \), then for almost all \( c \in S \) the equation \( F(g_a) = c \) has multiple solutions.

**Proof:** Suppose first that \( c \) is a regular value of \( \varphi \).

Use definition 1 of degree (see Section II.B) to write
III.B SOME QUESTIONS FROM THE CONTINUUM THEORY

$$\eta = I(\varphi, (c))$$

$$= \Sigma_{g \in \varphi^{-1}(c)} \xi(\varphi, g)$$

for $$\varphi(g) = F(g^a)$$. Then

$$|\eta| \leq \Sigma_{g \in \varphi^{-1}(c)} |\xi(\varphi, g)| \leq \Sigma_{g \in \varphi^{-1}(c)} 1$$

which proves that there are at least $$|\eta|$$ Gribov copies.

Suppose $$|\eta| > 1$$. By Sard's theorem [GP], almost all $$c \in \mathcal{S}$$ are regular values of $$\varphi$$ and as above, any such $$c$$ has multiple solutions. $$\square$$

Corollary 3.5: Gribov copies occur in Landau gauge (eq. (2.8)).

Proof: We have calculated the gauge degree in Section II.D to be

$$\eta = \det(\xi^2 A^\mu_\lambda).$$

From eq. (2.6), the eigenvalues of $$\xi^2 A^\mu_\lambda$$ are

$$a_k = \Sigma_{\mu=0}^S \beta_{k\mu}^2$$

where $$\beta_{k\mu} = 2\sin(k\mu \pi/4N_\mu)$$ for $$k\mu \in \{1, 2, \ldots, 2N_\mu - 1\}$$. By the arithmetic-geometric mean inequality,

$$a_k \geq (s+1) \left( \pi \prod_{0 \leq \mu \leq S} \beta_{k\mu}^2 \right)^{1/(s+1)}$$

so

$$|\eta| \leq \Pi_k a_k$$

$$\leq \Pi_k (s+1) \Pi_k \left( \Pi_\mu \beta_{k\mu}^2 \right)^{1/(s+1)}.$$
\[ \prod_{k \mu} (\beta^2_{k \mu})^{1/(s+1)} = \prod_{k_0} \prod_{k_s} (\beta^2_{k_0} \cdots \beta^2_{k_s})^{1/(s+1)} \]

\[ = (\prod_{k_0} \beta^2_{k_0})^{s+1} \cdots (\prod_{k_s} \beta^2_{k_s})^{s+1} \]

\[ = \prod_{\mu} (\prod_{k_\mu} \beta^2_{k_\mu}). \]

Consequently,

\[ |\eta| = \prod_{k} (s+1) \prod_{\mu} (\prod_{k_\mu} \beta^2_{k_\mu}). \]

Thus in order to prove that \( |\eta| > 1 \), it certainly suffices to show that \( \prod_{\mu} \beta^2_{k_\mu} > 1 \). In fact we now show that

\[ \prod_{\mu} \beta^2_{k_\mu} = N_\mu. \quad (3.16) \]

Consider

\[ \prod_{1 \leq k \leq 2N-1} 4 \sin^2 \left( \frac{k\pi}{4N} \right) = \prod_{1 \leq k \leq 2N-1} 4 \sin \left( \frac{k\pi}{4N} \right) \sin \left( \frac{(2N-k)\pi}{4N} \right) \]

\[ = \prod_{1 \leq k \leq 2N-1} 4 \sin \left( \frac{k\pi}{4N} \right) \cos \left( \frac{k\pi}{4N} \right) \]

\[ = \prod_{1 \leq k \leq 2N-1} 2 \sin \left( \frac{k\pi}{2N} \right). \quad (3.17) \]

Write the polynomial \( P(z) = (z^{2N} - 1)/(z - 1) \) as product of its linear factors to obtain

\[ P(z) = \prod_{1 \leq k \leq 2N-1} (z - e^{2\pi i k/2N}). \]

Thus

\[ \lim_{z \to 1} P(z) = 2N = \prod_{1 \leq k \leq 2N-1} (1 - e^{2\pi i k/2N}) \]

\[ = \prod_{1 \leq k \leq 2N-1} e^{\pi i k/2N} (e^{-\pi i k/2N} - e^{\pi i k/2N}) \]
Taking the modulus of both sides of eq. (3.18) yields

\[ 2N = \prod_{1 \leq k \leq 2N-1} 2\sin\left(\frac{\pi k}{2N}\right). \]  

Equation (3.18) yields the desired eq. (3.16). \( \square \)

(iii) Gauge invariance of the Faddeev-Popov determinant: In the original derivation of the Faddeev-Popov determinant it appeared that it was gauge invariant. The existence of Gribov copies makes this doubtful. However, one still sees arguments in the literature which make use of this supposed invariance. (See for example the discussion of the Slavnov-Taylor identities in [12].)

We now show that in the lattice theory, gauge invariance holds only in very special cases. Of course, one such case is that where \( G \) is abelian, for then \( M(a) \) is independent of \( a \) (cf. eqs. (2.10) and (2.11)).

Theorem 3.6: Suppose the gauge group \( G \) is nonabelian. Then the Faddeev-Popov determinant \( \det M(a) \) is gauge invariant if and only if it is a constant and that constant is 0, 1 or -1.

Proof: Take \( E \) to be the constant function 1 in eq. (2.22) to obtain

\[ \eta = \oint \det M(0) \, dg. \]

If \( \det M(a) \) were gauge invariant, then we would have
for every field $a$, and the Faddeev-Popov determinant would be a constant.

Moreover, suppose that $k$ is the dimension of $G$. From eqs. (2.29) and (2.36) we have

$$(\det N)^r = (\det N)^k$$

But the rank $r$ of $G$ is the same as the dimension of $G$ if and only if $G$ is abelian. Hence it must be that

$\det N = 0$ or $|\det N| = 1$. \(\square\)

In particular the Faddeev-Popov determinant for Landau gauge is not gauge invariant.
III.C Alternative Interpretation of the Gauge Degree

Hirschfeld's study [H] of Gribov copies is similar to ours in many respects, but he offers a different intuition based on the geometry of orbit-surface intersections. In this section we review that geometric intuition and give rigorous proofs of Hirschfeld's main conclusions within the framework we have developed.

Let us consider again eq. (3.9)

\[ \eta(a,c) = \sum_{g} \det M(ga) \delta(F(gac^{-1})) \, dg \]

\[ = \sum_{k=1}^{n} \text{sgn} \det M(g_{k}a) \]  

(3.20)

where the sum is over those \( g_{k} \in \mathcal{G} \) such that \( F(g_{k}a) = c \).

One can picture the situation when there are Gribov copies as in Fig. 3. The gauge orbit \( 0 = \{ ga \mid g \in \mathcal{G} \} \) has several points of intersection with the gauge-fixing surface \( \mathcal{I} = \{ b \in \mathcal{G}^{*} \mid F(b) = c \} \). Hirschfeld argues that \( \text{sgn} \det M(g_{k}a) \) measures the sense of travel of the gauge orbit as it passes through the surface at \( g_{k}a \). Thus in Fig. 3 \( \text{sgn} \det M(g_{1}a) = +1, \) \( \text{sgn} \det M(g_{2}a) = -1, \) and so on.

As in Section III.A, the picture makes it clear why \( \eta(a,c) \) is independent of \( a \) and \( c \). If \( a \) or \( c \) changes its value in a continuous manner, the gauge orbit \( 0 \) and the gauge fixing surface will be deformed in a continuous...
manner. Any points of intersection which are created or disappear during this transformation do so in pairs with opposite values of \( \text{sgn det } M^{g_k a} \) so that the sum in eq. (3.20) does not change.

To make this argument more precise, we again make use of some notions of oriented intersection theory [GP]. Our goal is to show that in the absence of pathologies, \( \eta(a,c) \) is the oriented intersection number of the map \( f(g) = g_a \) with \( Z \). The conclusion that \( \eta(a,c) \) is independent of \( a \) is then a consequence of the fact that the oriented intersection number is a homotopy invariant. However, it would take more work using this approach to show that \( \eta(a,c) \) is independent of \( c \).

Suppose \( X, Y \) and \( Z \) are oriented manifolds without boundary, \( X \) is compact, \( Z \) is a closed submanifold of \( Y \) and \( \dim X + \dim Z = \dim Y \). A smooth map \( f:X \rightarrow Y \) is transversal to \( Z \) if for every \( x \in f^{-1}(Z) \)

\[
df_x T_X(x) + T_{f(x)}(Z) = T_{f(x)}(Y). \tag{3.21}
\]

A compactness argument shows that \( f^{-1}(Z) \) is a finite set in this case. Moreover, because \( \dim X + \dim Z = \dim Y \), the sum in eq. (3.21) is a direct sum

\[
df_x T_X(x) \oplus T_{f(x)}(Z) = T_{f(x)}(Y). \tag{3.22}
\]

and \( df_x \) is an isomorphism onto its image. Thus \( df_x \) and the given orientation of \( T_X(x) \) induce an orientation on \( df_x T_X(x) \) which along with the given orientation of \( T_{f(x)}(Z) \) determine an orientation of \( T_{f(x)}(Y) \). If this
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Figure 3 Sign of det $M^{G_k\alpha}$

orientation of $T_f(x)(Y)$ agrees with the given orientation of $T_f(x)$ then the orientation number $\epsilon(f,x)$ of $f$ at $x$ is $+1$; otherwise $\epsilon(f,x) = -1$.

**Definition:** Let $X$, $Y$, $Z$ and $f$ be as above. The intersection number of $f$ with $Z$ is the integer

$$I(f,Z) = \sum_{x \in f^{-1}(Z)} \epsilon(f,x). \quad (3.23)$$

**Theorem 3.7:** ([GP]) Homotopic maps have the same intersection number.

We now resume our proof of the claim that $\eta(a,c)$ is
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an oriented intersection number. We use the standard
criterion [GP] that $Z = F^{-1}(w)$ is a submanifold if $w$ is
a regular value of $F$.

The next theorem shows the relation between the $+$ and
- signs of Figures 2 and 3.

**Lemma 3.8:** Let $X$, $Y$ and $W$ be oriented manifolds
without boundary, $X$ compact and $\dim X = \dim W$. Let
$f : X \to Y$ and $F : Y \to W$ be smooth maps (Fig. 4).

Suppose that $w \in W$ is a regular value of $F$ and let
$Z = F^{-1}(w)$. Suppose that $f$ is transversal to $Z$.

Then $Z$ can be oriented in such a way that for all
$x \in f^{-1}(Z)$, the orientation number $\epsilon(f, x)$ of $f$ at $x$
is $+1$ if and only if $F \circ f$ preserves orientation at $x$.

**Proof:** We orient $Z$ by the preimage orientation. A
discussion of preimage and direct sum orientations can be
found in [GP], but we review the essential steps for the
case at hand.

Let us use the notation $O^+(V)$ to denote the class of
positively oriented bases of an oriented vector space $V$.
For $z \in Z$, let $H$ be a subspace of $T_z(Y)$ such that

$$H \oplus T_z(Z) = T_z(Y).$$

We first orient $H$ and then use the direct sum orientation
to obtain an orientation of $T_z(Z)$.

Since $w$ is a regular value of $F$, we have

$$dF_z T_z(Y) = T_w(W).$$

But $dF_z T_z(Z) = \{0\}$ so in fact
The orientation of $T(W)$ provides an orientation of $dF_zH$, which in turn provides an orientation for $H$ determined by declaring $B \in 0^+(H)$ if and only if $dF_zB \in 0^+(T(W))$.

By transversality and complementarity of dimensions,

$$df_xT_x(X) \oplus T_z(Z) = T_z(Y)$$

so that it is legitimate to take $H = df_xT_x(X)$.

To summarize, we have oriented $Z$ in such a way that $B_2 \in 0^+(T_z(Z))$ if and only if there exists $B_1 \in 0^+(df_xT_x(X))$ such that $B_1 \oplus B_2 \in 0^+(T_z(Y))$.

The proof is now just a matter of unravelling this tangle of definitions. Let $x \in f^{-1}(Z)$, $B_1 \in 0^+(T_x(X))$ and $B_2 \in 0^+(T_y(Z))$. Then

$F \circ f$ preserves orientation at $x$

$$\iff d(F \circ f)_x B_1 \in 0^+(T_w(W))$$

$$\iff df_xB_1 \in 0^+(df_xT_x(X)) \text{ (definition of } 0^+(df_xT_x(X)))$$

$$\iff df_xB_1 \oplus B_2 \in 0^+(T_z(Y)) \text{ (definition of } 0^+(T_z(Y)))$$

$$\implies \epsilon(f,x) = +1 \text{ (definition of orientation number).} \Box$$

Theorem 3.9: Fix $a \in \mathfrak{g}^*$ and let $f: \mathfrak{g} \to \mathfrak{g}$ be the map which gauge transforms $a$:

$$f(g) = g_a.$$ 

Suppose the gauge-fixing function $F: \mathfrak{g} \to \mathfrak{g}$ has $c$ as a regular value so that $Z \equiv F^{-1}(c)$ is a submanifold of $\mathfrak{g}^*$. Further suppose that $f$ is transversal to $Z$. 

$$dF_zH = T(W). \quad (3.26)$$
Then $\eta(a,c)$, as given by eq. (3.20), is the oriented intersection number of $f$ with $Z$.

**Proof:** We shall show that $F_{\Phi}$ preserves orientation at a point $g \in Z$ if and only if $\text{sgn det } M(g_{a}) = +1$. Then by Lemma 3.8

$$\eta(a,c) = \sum \text{sgn det } M(g_{ka})$$

$$= \sum e(f,g)$$

$$= I(f,Z).$$

We first orient $Z$ in the following standard way. Choose an orientation class $O^+(\mathcal{E})$ for the Lie algebra $\mathcal{E}$ of $Z$. For any point $g \in Z$ define

$$O^+(T_{g}(Z)) = L[g]_{\mathbb{A}}(O^+(\mathcal{E}))$$

where $L[g](h) = gh$.

Fix $g \in Z$ and suppose $B \in O^+(\mathcal{E})$. Let $\psi \equiv F_{\Phi}$.

Then $F_{\Phi}$ preserves orientation at $g$

$$\Leftrightarrow (d\psi \circ L[g]_{\mathbb{A}})(B) \in O^+(T_{\psi}(g)(Z))$$

$$\Leftrightarrow (dL[\psi(g)]^{-1}) d\psi \circ L[g]_{\mathbb{A}}(B) \in O^+(\mathcal{E})$$

$$\Leftrightarrow d\xi_{g}(B) \in O^+(\mathcal{E})$$

where

$$\xi = L[\psi(g)]^{-1} \circ \psi \circ L[g].$$

Since $d\xi_{g} : \mathcal{E} \to \mathcal{E}$, $d\xi_{g}(B) \in O^+(\mathcal{E})$ if and only if $\text{sgn det } d\xi_{g} = +1$. But (see eq. (2.11))

$$d\xi_{g} = M(g_{a})$$
so the claim is proven. □

Theorems 3.9 and 3.7 allow us to conclude that \( \eta(a,c) \) is independent of \( a \) and hence allow another approach to proving the lattice FP formula. However, the assumptions of Theorem 3.9, that \( c \) is regular value of \( F \) and that \( f \) is transversal to \( Z \), can be difficult to check in realistic examples.

Calculations in some especially simple cases suggest that \( F^{-1}(c) \) is "usually" a submanifold and that the more risky assumption is that \( f \) is transversal to \( Z \). A hint that there is some connection between nontransversal intersections and gauge degrees that are zero may be seen in the following theorem.

\textbf{Theorem 3.10:} Suppose \( a \in S^* \), \( F: S^* \to S \) and suppose \( c \) is a regular value of \( F \) (Fig 5). Then the gauge orbit \( f(g) = g_\alpha \) has a nontransversal intersection with \( Z = F^{-1}(c) \) at \( g \) if and only if the FP determinant \( \det M(g_\alpha) = 0. \)

\textbf{Proof:} Let \( k = \dim S \), \( m = \dim S^* \) and \( g \in F^{-1}(Z) \). Note that \( \det M(g_\alpha) = 0 \) if and only if \( d(F \circ f)_g \) is singular, because

\[ M(g_\alpha) = dL[F \circ f(g)^{-1}]_g \circ d(F \circ f)_g \circ dL(g)_g \]

and \( dL(h) \) is a bijection for any \( h \in S \).

Suppose first that \( \det M(g_\alpha) \neq 0 \). We need to show that

\[ df_g T_g(S) \otimes T_z(Z) = T_z(S^*) \] (3.27)
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Figure 5. See Theorem 3.9.
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(the direct sum reflects the fact that \( \dim Z = m - k \)). We shall show

(i) \( \dim df_Tg(S) = k \)

(ii) \( df_Tg(S) \cap T_z(Z) = \{0\} \).

If \( \dim df_Tg(S) < k \) then \( \dim d(F \circ f)_Tg(S) < k \). But this is impossible because \( \det M(S) \neq 0 \) so \( d(f \circ f)_g \) is an isomorphism. Thus (i) is proven.

As for (ii) note that \( dF_TZ(Z) = \{0\} \) because \( F \mid Z \) is constant. In particular,

\[
dF_T(df_Tg(S) \cap T_z(Z)) = \{0\}.
\]

If \( df_Tg(S) \cap T_z(Z) \neq \{0\} \), then a nontrivial subspace of \( df_Tg(S) \) would be mapped to \( \{0\} \) by \( dF_T \). But this is impossible because \( d(G \circ f)_g \) is an isomorphism. Thus we have shown that eq.(3.27) holds and \( f \) is transversal at \( g \).

On the other hand, suppose that eq.(3.27) holds. Because \( c \) is a regular value of \( F \), \( dF_T \) surjective. Apply \( dF_T \) to both sides of eq.(3.27). Because \( dF_T(Z) = \{0\} \) we obtain

\[
dF_Tdf_Tg(S) = T_g(S)
\]

Hence \( d(F \circ f)_g \) is onto and nonsingular, so \( \det M(S) \neq 0 \). □

To summarize, let us compare the differences between Hirschfeld's identification of \( \eta \) as an oriented intersection number and our identification of \( \eta \) as an oriented degree. Theorem 3.9 shows that, given certain
assumptions, these points of view are equivalent. However, while $\eta$ as an oriented intersection number has a more direct geometric interpretation (Fig. 3), $\eta$ as a degree is technically simpler and more general: $\eta$ is the degree even without the assumptions of regularity and transversality made in Theorem 3.9; the fact that $\eta(a,c)$ is independent of $c$ is easily demonstrated only with this identification.
III.D Truncation of the Functional Integral

Several authors, beginning with Gribov [6], have considered the possibility that the equation

\[(\int_{\mathcal{E}^*} E(c) \, Dc) \left( \int f(a) \, e^{-S(a)} \, da \right) \]

\[= \int_V \det M(a) f(a) \, E \cdot F(a) \, e^{-S(a)} \, da, \quad (3.28)\]

which is false when \( V = \mathcal{E}^* \) unless \( n = 1 \), might nevertheless be true if \( V \) is taken to be an appropriate subset of \( \mathcal{E}^* \). Gribov has argued that such a truncation could provide a mechanism for quark confinement. This idea has received some support from Bender, Eguchi and Pagels [BEP] but has been argued against by Greensite [Gs]. In any case, the suggestion that truncation of the functional integral could be a way to improve the Faddeev-Popov argument has been pursued by some authors [Z2], [Gs]. From our point of view such an improvement is not needed, but truncation could conceivably provide another way to deal with the problem of a gauge degree that is zero. Another goal of truncation is to provide a density in the Faddeev-Popov formula which is manifestly positive so as to facilitate numerical studies of the lattice equations.

In this section we examine in the context of lattice gauge theories some of the suggestions made for accomplishing this truncation. The idea is to choose \( V \) in
such a way that if \( a \in V \) then any Gribov copy of \( a \) lies outside \( V \). Recall that the gauge degree \( \eta \) measures the net number of times that the function \( \psi(g) = F(ga) \) wraps \( S \) around itself. The hope is that restricting the gauge fields to lie in \( V \) will restrict \( \psi \) in such a way that its image covers \( S \) exactly once. The effect would be to replace \( \eta \) by \( 1 \) so that eq. (3.28) would hold.

To begin, let \( V \) be a measurable subset of \( S^* \) and for any \( a \in S^* \) define

\[
U(a) = \{ g \in S | ga \in V \}. \tag{3.29}
\]

It is easy to see that

(i) \( 1 \in U(a) \iff a \in V \)

(ii) \( U(a) = U(ga) \).

Define

\[
\theta(a) = \int_{U(a)} \det M(ga) E \circ F(ga) \, dg
\]

\[
= \int_S 1_{U(a)}(g) \det M(ga) E \circ F(ga) \, dg \tag{3.30}
\]

where \( 1_Q \) denotes the characteristic function of the set \( Q \). Now

\[
g \in U(a) \iff g \in U(ga) \quad (by \ (ii))
\]

\[
\iff 1 \in U(ga)
\]

\[
\iff ga \in V \quad (by \ (i)).
\]

Hence \( 1_{U(a)}(g) = 1_V(ga) \) and so

\[
\theta(a) = \int_S 1_V(ga) \det M(ga) E \circ F(ga) \, dg. \tag{3.31}
\]

From (3.31) and the right invariance of Haar measure we see
that $\theta$ is gauge invariant. Let us apply the Faddeev-Popov technique. We suppose that $\theta(a) \neq 0$ for any $a \in S^*$. Then

$$\int f(a) e^{-S(a)} da = \int \frac{1}{\theta(a)} f(a) \det M(g_a) \det M(a) e^{S(a)} da \, dg$$

$$= \int \frac{1}{\theta(a)} f(a) \det M(a) e^{S(a)} da$$

$$= \int \frac{1}{\theta(a)} f(a) \det M(a) e^{S(a)} da. \quad (3.32)$$

The difficulty, of course, is finding the appropriate subset $V$ so that $\theta(a)$ is equal to $\int E(c) dc$ or at least is independent of $a$.

Gribov [GJ makes the following choice:

$$V = \{ a \in S^* \mid M(a) \text{ has no negative eigenvalues} \}.$$ 

In the case of Landau gauge, where $M(1)$ is the Dirichlet Laplacian, $V$ contains a neighbourhood of $1$.

It is doubtful that Gribov's essentially analytic condition captures the geometric idea that is needed to define $V$. To see what can go wrong, consider the graphs of the two functions $F(g_{a_1})$ and $F(g_{a_2})$ shown in Fig. 6. By applying the change of variables formula discussed in Section II.A, we obtain

$$\theta(a_1) = \int \det M(g_{a_1}) E(g_{a_1}) \, dg$$

$$= \int E(c) dc$$

while
III.D TRUNCATION OF THE FUNCTIONAL INTEGRAL

\[ \theta(a_2) = \int_{U(a_2)} \det M(g a) E \circ F(g a) \, dg \]

\[ = \int_{W_1} E(c) \, dc + \int_{W_2} E(c) \, dc \]

\[ = \int_{W_1} E(c) \, dc + \int_{W_1 \cap W_2} E(c) \, dc \]

so that in general \( \theta(a_1) \neq \theta(a_2) \). As Gribov recognized, it is quite possible that an extra condition on \( V \) is needed to ensure that \( F(g a) \) restricted to \( U(a) \) is one-to-one.

![Diagram showing \( F(g a_1) \) and \( F(g a_2) \) with \( W_1 \) and \( W_2 \)]

\[ u(a_2) = u_1 U u_2 \]

Figure 6 Example for Gribov's truncation scheme

Greensite [Gs] works with the restricted Faddeev-Popov formula where \( E \) is a \( \delta \)-function. He arrives at the expression

\[ \int f(a) e^{-S(a)} \, da \]

\[ = \int \frac{1}{N(a)} \det M(a) \delta(F(a)) f(a) e^{-S(a)} \, da \quad (3.33) \]
where $N(a)$ is the number of Gribov copies of the field $a$. Equation (3.33) is quite close to our eq.(2.12). The reason that $N(a)$ appears instead of $\eta$ is that in deriving eq.(3.33), Greensite neglected to take the absolute value of the Faddeev-Popov determinant when making his change of variables.

Greensite argues for truncation from eq.(3.33), but says no more than that $V$ is a region which contains one and only one intersection point of each orbit with the gauge fixing surface $F(a) = \Omega$. If we add the condition that $\det M(a) > 0$ at each such intersection point then $\Theta(a) = 1$ as desired. However, this description of $V$ is too imprecise.

A more concrete description of a choice of $V$ is given by Zwanziger [ZI]. His choice of $V$ is based on a particular gauge, the "background gauge". A field $a$ is in $V$ if it is on the gauge-fixing surface and $M(a)$ is positive definite. The objections that were raised above for Gribov's $V$ apply here as well and Zwanziger is not able to show that his choice of $V$ results in $\Theta(a)$ being independent of $a$.

To summarize, the attempts to avoid the Gribov problem by truncating the functional integral have had only limited success. Either the description of the region $V$ is too imprecise to be useful or the choice of $V$ does not seem likely to work.
Despite the superficial appeal of truncation, a consideration of diagrams such as Fig. 6 leads one to the conclusion that it is rather unnatural and consequently difficult to formulate. We do not believe that it holds much promise for modifying either the lattice or continuum version of the Faddeev-Popov argument.
IV.A INTRODUCTION

CHAPTER IV CONTINUUM LIMIT

IV.A Introduction

One of the major motivations for developing the lattice theory as described in the preceding chapters is to provide a tool for the rigorous construction of a continuum nonabelian gauge field theory. One could hope to carry out this construction by obtaining the Schwinger functions of the continuum theory as a limit as $\epsilon \to 0$ of appropriate lattice functions. The philosophy behind the approach we have taken is to try to bring a gauge field theory into a form which resembles a nongauge field theory such as the $P(\varphi)$ or Yukawa models, so as not to have to depart too drastically from the techniques used in their construction. In particular, we wish to avoid the necessity of working at all times only with gauge invariant functions. By breaking gauge invariance, we can use noninvariant objects like Schwinger functions as intermediate constructs and in the end restrict our attention to the physically more relevant gauge invariant quantities.

In this section, we survey some of the issues involved in taking the continuum limit of the lattice model developed in the preceding chapters, concentrating on the example of quantum chromodynamics in two space-time dimensions ($\text{QCD}_2$) with gauge group $\text{SU}(2)$. We then introduce the partial
results we have been able to obtain and which are described
in the remaining sections of this chapter. We also describe
the problems we have not been able to solve and indicate
what work remains to be done.

(a) Continuum theory

Let us begin by writing down the formalism of continuum
QCD. The action for this theory is

\[ S(A, \gamma, \bar{\gamma}) = S_g(A) + S_m(\gamma, \bar{\gamma}) \]

\[ = \int \mathcal{Z}_g \, d^2x + \int \mathcal{Z}_m \, d^2x \]  \hspace{1cm} (4.1)

where

\[ \mathcal{Z}_g = -\frac{1}{2} \text{tr} \left( F_{\mu \nu} F_{\mu \nu} \right) \]

\[ F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + \lambda [A_{\mu}, A_{\nu}] \]

\[ \mathcal{Z}_m = \bar{\gamma} \left( \mathcal{E} + m + \Lambda \right) \gamma \]

in which \( \mathcal{E} \) stands for \( \Sigma \gamma_{\mu} \gamma_{\mu} \), the \( \gamma_{\mu} \) being a
representation of the Clifford algebra

\[ \gamma_{\mu}^* = \gamma_{\mu} \]

\[ (\gamma_{\mu}, \gamma_{\nu}) = 2\delta_{\mu \nu} \].

The action (4.1) is invariant under the combined local gauge
transformations

\[ \psi(x) \rightarrow g(x) \psi(x) \]

\[ \bar{\psi}(x) \rightarrow g(x)^* \bar{\psi}(x) \]

\[ A_{\mu}(x) \rightarrow g(x) [A_{\mu}(x) + \lambda^{-1} \partial_{\mu} \log(x)] g(x)^{-1}. \]

Before gauge fixing, the expectation of a function \( f(A, \gamma, \bar{\gamma}) \)
of the gauge field \( A \) and the matter fields \( \gamma, \bar{\gamma} \) is given
by the formal expression

$$\langle f \rangle = \frac{\int f(A, Y, \bar{Y}) e^{-S(A, Y, \bar{Y})} DA \, dY \, d\bar{Y}}{\int e^{-S(A, Y, \bar{Y})} DA \, dY \, d\bar{Y}}.$$  

We did not explicitly include matter fields such as $Y, \bar{Y}$ in our discussion of gauge fixing, but we could have without changing any of the results. For if $f(A, Y, \bar{Y})$ is a gauge invariant function, we can imagine carrying out the fermion integration $dY \, d\bar{Y}$ to obtain

$$\langle f \rangle = \frac{\int \check{f}(A) e^{-S_g(A)} DA}{\int e^{-S_g(A)} DA} \quad \text{(4.2)}$$

where $\check{f}(A)$ is a gauge invariant function of $A$. The Faddeev-Popov technique can then be applied as before.

If we employ gauge fixing using Landau gauge to make the replacement

$$S_g(A) \rightarrow S_{g, \alpha}(A) = S_g(A) - \alpha \int d^2x \, \text{tr}[(\partial_\mu A_\mu(x))^2]$$

the quadratic part of the gauge field action becomes

$$\frac{1}{2} \int A_\mu^a(x) [-\delta_{\mu\nu} + (1-\alpha) \delta_\mu \delta_\nu] \delta_{ab} A_\nu^b(x) \, d^2x. \quad \text{(4.3)}$$

The simplest version of (4.3) is obtained by making the choice $\alpha = 1$, which yields what is known as Feynman gauge in which the covariance is $(-d)^{-1} \delta_{\mu\nu} \delta_{ab}$. The Faddeev-Popov operator $M(A)$ operates on $L^2(\mathbb{R}^2) \otimes \mathbb{R}^2 \otimes \mathbb{E}$ and is given by

$$M(A)B = -dB - \lambda \partial_\mu [A_\mu, B].$$

Thus the formal continuum expression for the expectation of a gauge invariant function in Feynman gauge
IV.A INTRODUCTION

is

\[
\langle f \rangle = \frac{\int \det M(A) f(A, Y, \bar{Y}) e^{-S_{g,1}(A) - S_m(Y, \bar{Y})} DA \; dY \; d\bar{Y}}{\int \det M(A) e^{-S_{g,1}(A) - S_m(Y, \bar{Y})} DA \; dY \; d\bar{Y}}.
\]

(4.4)

We want to modify (4.4) a little before discussing how to obtain it as a continuum limit. The matter fields \(Y, \bar{Y}\) appear in QCD very much as they do in QED ([WC], [W]) and we shall drop them so as to be able to concentrate on those things that are special to the nonabelian case. The model that remains is sometimes referred to as a "pure gauge theory".

There is little hope of making sense of the Faddeev-Popov determinant unless we normalize it by writing

\[
M(A) = (-\Delta)(I + L(A))
\]

and cancelling the common factor \(\det (-\Delta)\) top and bottom in (4.4).

Let us write

\[
S_{g,1} = S_0 + S_1
\]

where

\[
S_0(A) = \frac{1}{2} \int A^a_\mu(x) (-\Delta) \delta^\mu_\nu \delta_{ab} A^b_\nu(x) \; d^2x.
\]

\[
S_1(A) = \lambda \int \text{tr} \, \delta \mu A_\nu [A_\mu, A_\nu] \; d^2x + \lambda^2 \int \text{tr} \, [A_\mu, A_\nu]^2 \; d^2x.
\]

Although we have not reflected it in our notation, we should keep in mind that we are working in Feynman gauge.

Thus the formal expression we wish to make rigorous sense of in pure QCD\(_2\) with Feynman gauge is
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\[ \langle f \rangle = \frac{\int f(A) \det(1+L(A)) \, e^{-S_I(A)} \, d\mu}{\int \det(1+L(A)) \, e^{-S_I(A)} \, d\mu} \quad (4.5) \]

in which we are writing \( d\mu(A) \) for the formal measure

\[ \frac{e^{-S_0(A)} \, dA}{\int e^{-S_0(A)} \, dA} \]

which can be interpreted as Gaussian measure with covariance \((-\partial)^{-1} \delta_{\mu \nu} \delta_{ab}\). At this point we do not require that \( f \) be gauge invariant, but rather we have in mind that \( f(A) \) is, for example, a polynomial in \( A \).

To follow the traditions of constructive field theory, we should first attempt to construct the finite volume version of (4.5). We put the theory in the finite volume \( \Lambda \subset \mathbb{R}^2 \) by taking the measure \( d\mu \) to be Gaussian measure with covariance \((-\partial_\Lambda)^{-1} \delta_{\mu \nu} \delta_{ab} \) where \(-\partial_\Lambda \) is the Laplacian operator with Dirichlet boundary conditions on \( \Lambda \) [GRS2]. Also, the integrals defining \( S_I \) will now extend only over \( \Lambda \) instead of \( \mathbb{R}^2 \).

It is still too much to expect to be able to work with eq. (4.5) as it stands because of ultraviolet divergences that must be removed by renormalization. That is, we must expect to have to make the replacements

\[ \det (I + L) \rightarrow \det_{\text{ren}} (I + L) \]

\[ S_I \rightarrow S_{I, \text{ren}} \quad (4.6) \]

by subtracting counterterms.
(b) Lattice theory

We now turn to the lattice version of eq. (4.5). For the gauge invariant action we choose the action due to Wilson [W]

\[ S^W(a) = -\frac{1}{2} \lambda^{-2} \epsilon^{-2} \sum_P \text{Re} \text{tr} [U_P(a) - 1] \]

where the sum ranges over all plaquettes (i.e., elementary lattice squares) \( P \) and \( U_P(a) \) is the plaquette variable

\[ U_P(a) = a(x, x+e_\mu) a(x+e_\mu, x+e_\mu+e_\nu) a(x+e_\mu+e_\nu, x+e_\nu) a(x+e_\nu, x). \]

The Faddeev-Popov determinant \( \det M_\epsilon(a) \) that arises from the gauge fixing operation is given by eq. (2.40). We normalize \( M_\epsilon \) and define \( L_\epsilon \) by

\[ (-\Delta_\lambda) (I + L_\epsilon(a)) = M_\epsilon(a). \]

Thus the lattice counterpart to eq. (4.5) is

\[ \langle \phi_\epsilon \rangle = \frac{\int f_\epsilon(a) \det(1+L_\epsilon(a)) e^{-S_\epsilon(a)} da}{\int \det(1+L_\epsilon(a)) e^{-S_\epsilon(a)} da}, \quad (4.7) \]

(c) Continuum limit: a suggested approach

We are now in a position to pose the two fundamental problems of the continuum limit.

**Problem 1**: What are the appropriate renormalizations to make in eq. (4.6)?

**Problem 2**: How can we obtain the renormalized eq. (4.5) as the limit of the renormalized eq. (4.7)?

With regard to Problem 1, it is natural to begin with those renormalizations suggested by perturbation theory and power counting. We discuss these matters in Section IV.B.
IV.A INTRODUCTION

We now give an outline of a possible approach to solving Problem 2. It is doubtful that the scheme we are about to describe can be carried out in as simple a form as we are suggesting. Nevertheless, it gives us a place to start and any progress made here will certainly be helpful in eventually obtaining the solution to Problem 2.

To begin, we should first break up the lattice action somehow into terms corresponding to free and interaction parts

$$S_\varepsilon = S_{0,\varepsilon} + S_{1,\varepsilon}$$

We discuss further below how one might want to make this split-up.

As in previous lattice models in quantum field theory ([GRS1], [BFS1 - 3], [WC]) we consider the lattice theory to be embedded in the continuum theory: we write

$$a(x, x+\frac{1}{2}e_\mu) = e^{\pm\varepsilon A_\mu,\varepsilon(x+\frac{1}{2}e_\mu)}$$

(4.8)

as in eq.(2.2) and regard $A_\mu,\varepsilon(y)$ as a continuum field $A_\mu$ smeared with some smooth approximation $f_\varepsilon,\gamma$ to the $\delta$-function with concentrated at $y$. For example, in [GRS1] $f_\varepsilon,\gamma$ is chosen in such a way that

$$\int A_\mu^a(f_\varepsilon, x) A_\nu^b(f_\varepsilon, y) \, d\mu = \delta_{\mu\nu} \delta_{ab} (-\delta_\Lambda)(x, y).$$

The choice of lattice embedding is to a large degree merely a matter of technical convenience and for the purposes of this discussion, we shall assume that the [GRS1] embedding is the one we are using.
Another aspect of the lattice embedding to consider is the relation between the free lattice measure

\[
\frac{e^{-S_0(a)}}{\int e^{-S_0(a)} \, da} \quad \text{to the continuum Gaussian measure } \, du. \text{ Haar measure } \, da \text{ on } S^* \text{ is equivalent to the measure obtained from Lebesgue measure on a subset of the Lie algebra } \mathcal{E}^*. \text{ That is, for some function } R_{\epsilon} \text{ (the Radon-Nikodym derivative) we have}
\]

\[
\frac{e^{-S_0(a)}}{\int e^{-S_0(a)} \, da} = \frac{e^{-S_0,\epsilon(e^{\lambda A_\epsilon})} R_{\epsilon}(A_\epsilon) \, dA_\epsilon}{\int e^{-S_0,\epsilon(e^{\lambda A_\epsilon})} R_{\epsilon}(A_\epsilon) \, dA_\epsilon}
\]

in which \(dA_\epsilon\) is Lebesgue measure on \(\mathcal{E}^*\).

If the factor \(R_{\epsilon}\) were not present, the natural choice for \(S_{0,\epsilon}\) would be

\[
S_{0,\epsilon}(e^{\lambda A_\epsilon}) = \frac{1}{2} \sum_x \epsilon^2 a^{\mu,\epsilon}_x (x) (-\delta^\lambda_{\mu}) \delta_{\mu \nu} \delta_{ab} A^{b,\epsilon}_x (x)
\]

which is the finite-difference approximation to

\[
S_0(A) = \frac{1}{2} \sum_x a^{\mu}_x (x) (-\delta^\lambda) \delta_{\mu \nu} \delta_{ab} A^{b}_x (x) \, d^2 x.
\]

With the [GRS1] embedding and this choice of \(S_{0,\epsilon}\) we have for any integrable function \(F(A_\epsilon)\)

\[
\int F(A_\epsilon) \, d\mu = \int F(A_\epsilon) e^{-S_0,\epsilon(A_\epsilon)} \, dA_\epsilon.
\]

Thus

\[
\langle f_\epsilon \rangle = \frac{\int f_\epsilon(e^{\lambda A_\epsilon}) \det(I + L_\epsilon(e^{\lambda A_\epsilon})) e^{S_{1,\epsilon}(e^{\lambda A_\epsilon})} \, d\mu}{\int \det(I + L_\epsilon(e^{\lambda A_\epsilon})) e^{S_{1,\epsilon}(e^{\lambda A_\epsilon})} \, d\mu}.
\]
The next task is to renormalize the lattice quantities. Assuming this has been done, we would have a complete solution to Problem 2 if we could show that the following converge in all $L^p(d\mu)$ ($p \geq 1$):

1. $f_\epsilon(e^{\lambda A\epsilon}) \to f(A)$
2. $\det_{\text{ren}}(I + L_\epsilon(e^{\lambda A\epsilon})) \to \det_{\text{ren}}(I + L(A))$
3. $e^{SI_{\text{ren}}(e^{\lambda A\epsilon})} \to e^{-SI_{\text{ren}}(A)}$
4. $R_\epsilon(A_\epsilon) \to \text{constant}$.

(d) Continuum limit: discussion and partial results

So as not to mislead the reader, let us point out right away that there are serious difficulties in the preceding scheme.

First of all, the limit (iv) is probably not true in general. For example, for the case of $SU(2)$ in two dimensions we show in Section IV.D the formal limit

$$R_\epsilon(A_\epsilon) \to \exp\left(\frac{1}{24} \lambda^2 \sum_{a,\mu, \Lambda} |A^a_\mu(x)|^2 \, d^2 x\right). \quad (4.9)$$

This a surprising and unwelcome result for several reasons. However we argue in IV.D that the formal limit (4.9) is misleading and that in fact $R_\epsilon$ makes no contribution in the continuum limit.

A second difficulty with the scheme outlined in part (c) above is relating the action $S_\epsilon(a)$ which is based on gauge fields in the gauge group to $S_0(A)$ where the gauge field is in the Lie algebra. The intuition is that
the lattice action ought to provide Gaussian damping for the lattice gauge fields which are far from $1$ so that it is the fields near $1$ which determine the continuum limit. However, the periodic nature of the group integrals can spoil this intuition. The problem is comparable to trying to obtain

$$\int_{\mathbb{R}} f(x) \, e^{-x^2/2} \, dx$$

(4.10)

as a limit as $\epsilon \to 0$ of

$$\int_{-\pi/\epsilon}^{\pi/\epsilon} f(x) \, e^{-\left(1-\cos(k\epsilon x)\right)/\epsilon^2 k^2} \, dx.$$  (4.11)

For any $k$, we have the pointwise convergence of $e^{-\left(1-\cos(k\epsilon x)\right)/\epsilon^2 k^2}$ to $e^{-x^2/2}$. Moreover, when $k = 1$ the desired convergence (4.10) $\to$ (4.11) can be shown using the dominated convergence theorem together with the estimate

$$1 - \cos \epsilon x \geq \frac{2}{\pi^2} \epsilon^2 \quad \text{for} \quad x \in [-\pi/\epsilon, \pi/\epsilon].$$

However when $k = 2$ this argument fails. The difference is that when $k = 1$, the exponent has a unique maximum at $x = 0$, whereas when $k = 2$ there are additional maxima at $x = \pm \pi/2$ and the exponential provides no damping near the boundaries of the region of integration. In general, (4.11) will not converge (depending on what $f$ is).

What all this has to do with integrals in lattice gauge theories is spelled out in Section IV.C. In brutally simplified terms the analogy is this. Equations (4.10) and
IV.A INTRODUCTION

(4.11) are the integrals for continuum and lattice expectations respectively. The term \( \cos(kx) \) appears in the latter instead of \( x^2 \) because the lattice gauge field is in the gauge group rather than the Lie algebra. The region of integration \([-\pi/\varepsilon, \pi/\varepsilon]\) arises from expressing Haar measure on the group as Lebesgue measure on a subset of the Lie algebra. Values of \( k \) greater than 1 occur because, for example, the plaquette variables contain a product of four gauge fields, corresponding to \( k = 4 \), and the gauge invariant action fails to have a unique minimum. One could hope that fixing the gauge would result in an action which does have a unique minimum, although there is no reason to expect this \textit{a priori}. What does happen in general is that gauge fixing reduces \( k \), but not all the way to 1. This is an important point because it is difficult to see how to get convergence of the free measure without this unique minimum property.

Let us now discuss the results we have obtained in carrying out the program described in (c) above.

With regard to (ii), we show in Section IV.E the formal convergence of \( L_\varepsilon(e^{\lambda A_\varepsilon}) \rightarrow L(A) \). In Section IV.F we show that the continuum Faddeev-Popov operator \( L(A) \) is in the Carleman classes \( C_{2+\delta} \) for all \( \delta > 0 \) and for almost all \( A \). This means in particular that the cutoff determinant \( \det_3(I+L(A)) \) is well-defined a.e. and constitutes the first step in renormalizing the Faddeev-Popov determinant.
In two other gauge field models which have been constructed to date, QED$_2$ [WC,W] and Higgs$_2$ [BFS1-3], a lattice limit was employed which allowed the proving of diamagnetic inequalities. In our case, a diamagnetic inequality would be

$$|\text{det}(I + L_\epsilon(a))| \leq 1$$  \hspace{1cm} (4.12)

for all $a$ and $\epsilon$. Inequalities of this sort were crucial in establishing convergence and LP properties of determinants in QED$_2$ and Higgs$_2$ and will no doubt be important in QCD$_2$ as well. We prove eq. (4.12) in Section IV.G, but for a one-dimensional model only.
IV.B Power Counting in QCD

We now give a sketch of what renormalizations we should expect to have to make in pure QCD to eliminate ultraviolet divergences. Our reasoning is based on a general power counting formula which we now develop following [11].

Let \( G \) be a Feynman graph without any internal loops.

Suppose \( G \) has:

- \( L \) independent momentum integrations
- \( I_B \) internal boson lines
- \( E_B \) external boson lines
- \( I_F \) internal fermion lines
- \( E_F \) external fermion lines
- \( I_D \) derivatives on internal lines
- \( E_D \) derivatives on external lines
- \( V \) vertices

Then the superficial degree of divergence in \( d \) dimensions is

\[
\omega(G) = dL - 2I_B - I_F + I_D
\]

and the integral corresponding to the graph \( G \) will diverge if \( \omega(G) \geq 0 \). Momentum conservation at the vertices implies

\[
L = I_B + I_F - V + 1
\]

so

\[
\omega(G) = d - dV + (d-2)I_B + (d-1)I_F + D.
\]

Each vertex \( v \) arises from an interaction term \( T_v \) in the Lagrangian. Let

- \( B_v \) = number of bosons in \( T_v \)
- \( F_v \) = number of fermions in \( T_v \)
IV.B POWER COUNTING IN QCD

\[ D_v = \text{number of derivatives in } T_v. \]

Because we have assumed there are no loops in \( G \), each internal line is incident on two vertices and each external line is incident on one vertex. Thus

\[ I_B = \frac{1}{2} \Sigma B_v - \frac{1}{2} E_B \]
\[ I_F = \frac{1}{2} \Sigma F_v - \frac{1}{2} E_F \]
\[ I_D = \Sigma D_v - E_D. \]

Putting these relations into the previous expression for \( \omega(G) \) yields

\[ \omega(G) = d - dV + \frac{1}{2}(d-2)(\Sigma B_v - E_B) + \frac{1}{2}(d-1)(\Sigma F_v - E_F) \]
\[ + \Sigma D_v - E_D \]

where

\[ \omega_v = \frac{1}{2}(d-2)B_v + \frac{1}{2}(d-1)F_v + D_v - d. \]

In the \( d = 2 \) case we are going to consider, these expressions reduce to

\[ \omega(G) = 2 + \Sigma \omega_v - \frac{1}{2}E_F - E_D \quad (4.13a) \]
\[ \omega_v = \frac{1}{2}F_v + D_v - 2. \quad (4.13b) \]

To apply the power counting formulas (4.13) to QCD, we take the conventional step [R,12] of writing the Faddeev-Popov determinant as an integral over a pair of anticommuting "ghost" fields \( \eta, \bar{\eta} \) as follows

\[ \det M(A) = \int \frac{d^2 \eta}{d \eta} \quad (4.13c) \]
For purposes of power counting the ghost fields behave as bosons because of the $\bar{\eta}(-\partial)\eta$ term. The interaction term $\bar{\eta}\partial_{\mu}[A_{\mu},\eta]$ is a vertex with three bosons and one derivative. When the gauge-ghost interaction is added to those in (4.1) we have a total of four interactions. Let us use the following notation to denote the different fields:

\[ \begin{align*}
\text{---} & = \text{gauge field} \\
\text{----} & = \text{ghost field} \\
\text{-----} & = \text{derivative of ghost field} \\
\text{~~~~} & = \text{derivative of gauge field}
\end{align*} \]

The vertices are

\[
\begin{align*}
(3A)A^2 & \quad A^4 & \quad \bar{\eta}\partial(A\eta) \\
\text{\includegraphics[width=0.3\textwidth]{vertex1}} & \text{\includegraphics[width=0.3\textwidth]{vertex2}} & \text{\includegraphics[width=0.3\textwidth]{vertex3}} \\
\omega_{v_3} = -1 & \omega_{v_4} = -2 & \omega_{v_m} = -1
\end{align*}
\]

If we let $N_v$ denote the number of vertices of type $v$, then from (4.13) the power counting formula for pure QCD is

\[
\omega(G) = 2 - N_3 - 2N_4 - N_g - E_D.
\]

Table 1 shows the possible connected graphs with nonnegative degree of divergence. Those graphs with no external lines (vacuum graphs) can
### Table 1  Superficially divergent graphs in QCD$_2$

<table>
<thead>
<tr>
<th>$N_3$</th>
<th>$N_4$</th>
<th>$N_5$</th>
<th>$E_0$</th>
<th>$\omega(G)$</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td><img src="image" alt="Graphs" /></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td><img src="image" alt="Graphs" /></td>
</tr>
</tbody>
</table>

---
ghost field  
derivative of ghost field
---
gauge field  
derivative of gauge field
be renormalized "trivially" by constant counterterms.

Graphs with loops, which we have been ignoring from the beginning, are eliminated by Wick ordering. The remaining nontrivial counterterms are those required to renormalize the graphs

\[
\begin{align*}
\text{Graph 1} & \\
\text{Graph 2} & 
\end{align*}
\]

each of which has \( u(G) = 0 \). These graphs appear to require a gauge field mass counterterm (as in the Yukawa model), but such a term would violate gauge invariance. However, gauge invariance in the form of the Slavnov-Taylor identities implies that these graphs, when added together, are less divergent than power counting suggests (see [12]). Their effective degree of divergence is negative.

Note that this improved degree of divergence is a result of cancellations between terms that arise from \( \det M(A) \) and \( e^{-S_f} \). It is for this reason that we said in Section IV.A that it might be too optimistic to expect those two factors to be in \( L^P(du) \) separately.

Thus we have seen that there are no nontrivial renormalizations required in QCD. However, the cancellations due to gauge invariance can be expected to be subtle and quite nontrivial to exhibit in a rigorous construction. It is clearly important to have a gauge invariant regularization of the theory, a fact which argues for using a lattice model.
IV.C The Action

In this section we show that the formal limit of the lattice action in Feynman gauge (as described in IV.A) yields its expected continuum counterpart. To convert that formal limit into a rigorous limit, it is first necessary to ensure that it is the gauge fields near the group identity which make the most important contribution to the continuum limit. In the latter part of this section we discuss some technical problems we have encountered in trying to accomplish that goal.

(a) Formal limit

We have seen that the action for lattice Feynman gauge is

\[ S(a) = -\frac{1}{2} \lambda^{-2} \varepsilon^{d-4} \Sigma_p \text{Re} \text{tr}[U_p(a) - 1] \]

\[ -\lambda^{-2} \varepsilon^{d-4} \Sigma_x \text{Re} \text{tr}[V_x(a) - 1] \]

where \( U_p(a) \) is the plaquette variable

\[ U_p(a) = a(x, x+e_\mu) a(x+e_\mu, x+e_\mu+e_\nu) a(x+e_\mu+e_\nu, x+e_\nu) a(x+e_\nu, x) \]

and \( V_x(a) \) is the "cross" variable

\[ V_x(a) = a(x, x+e_0) a(x, x-e_0) \cdots a(x, x+e_\mu) a(x, x-e_\mu). \]

We take the formal limit by using the relation

\[ a(x, x+\varepsilon e_\mu) = e^{\mp \varepsilon A_\mu(x+\frac{\varepsilon}{2} e_\mu)}. \quad (4.14) \]

The proofs of the convergence of the Wilson action and the
damping term to their continuum counterparts are very similar. Since the former is well-known (see for example [Ko2]), we give the proof for the latter only.

Let \( A_\mu^a: \mathbb{R}^d \to \mathbb{R} \) be smooth with compact support in \( \Lambda \) and let \( a \) be the lattice gauge field obtained from \( A \) by eq. (4.14). Consider the damping term

\[
\exp(-\lambda^{-2} \epsilon d^{-4} \sum_x \text{Re tr}[V_x(a) - I]).
\]

We claim that

\[
\text{tr}[V_x(a) - I] = \frac{1}{2} \lambda^2 \epsilon d^{-4} \text{tr}[\partial_\mu A_\mu(x)]^2 + O(\epsilon^5)
\]

where

\[
\partial_\mu f(x) = \epsilon^{-1} [f(x + \frac{1}{2} \epsilon e_\mu) - f(x - \frac{1}{2} \epsilon e_\mu)].
\]

From the Baker-Campbell-Hausdorff formula [Sp]

\[
e^\epsilon A e^\epsilon B = e^{\epsilon (A+B)} + \frac{\epsilon}{2} \epsilon^2 [A,B] + O(\epsilon^3)
\]

we have

\[
a(x,x+\epsilon e_\mu) a(x,x-\epsilon e_\mu)
\]

\[
= e^{\epsilon \lambda A_\mu(x+\frac{1}{2} \epsilon e_\mu)} e^{-\epsilon \lambda A_\mu(x-\frac{1}{2} \epsilon e_\mu)}
\]

\[
= e^{\lambda \epsilon^2 \partial_\mu A_\mu(x)} - \frac{\epsilon}{2} \epsilon^2 \lambda^2 [A_\mu(x+\frac{1}{2} \epsilon e_\mu), A_\mu(x-\frac{1}{2} \epsilon e_\mu)] + O(\epsilon^3).
\]

Since by assumption \( \partial_\mu A_\mu \) is continuous

\[
[A_\mu(x+\frac{1}{2} \epsilon e_\mu), A_\mu(x-\frac{1}{2} \epsilon e_\mu)] = [A_\mu(x), A_\mu(x)] + O(\epsilon)
\]

\[
= O(\epsilon)
\]

so that

\[
a(x,x+\epsilon e_\mu) a(x,x-\epsilon e_\mu) = e^{\lambda \epsilon^2 \partial_\mu A_\mu(x)} + O(\epsilon^3).
\]

Consequently
IV.C THE ACTION

\[ V_X(a) = e^{\lambda \epsilon^2 \sum \mu A_\mu(x) + O(\epsilon^3)} \]  

in which the function represented by \( O(\epsilon^3) \) is in the Lie algebra.

Now \( G \) is a unitary group (that is, we have assumed we are working in a unitary representation of \( G \)) so that any element \( A \) of the Lie algebra \( E \) of \( G \) is antihermitian. It follows that \( \text{Re} \ tr(A) = 0. \) Hence for any \( A, B \in E \)

\[ \text{Re} \ tr[\epsilon^2 A + \epsilon^3 B] = \text{Re} \ tr[1 + \epsilon^2 A + \epsilon^3 B + \frac{1}{2} \epsilon^4 A^2 + O(\epsilon^5)] \]

\[ = \text{Re} \ tr[1 + \frac{1}{2} \epsilon^4 tr(A^2) + O(\epsilon^5)]. \]

Consequently,

\[ -\lambda^{-2} \epsilon d^{-4} \sum_k \text{Re} tr[V_X(a) - 1] \]

\[ = -\frac{1}{2} \sum_{k, \mu} \epsilon d \text{tr}[\partial_\mu A_\mu(x)]^2 + O(\epsilon) \]

\[ \to -\frac{1}{2} \Lambda \text{tr}[\partial_\mu A_\mu(x)]^2 d^d x \]

as \( \epsilon \to 0. \)

(b) Unique minimum of lattice action

We discussed in Section IV.A the need to have an effective action \( S(a) \) which has a unique minimum at \( a = \Lambda \) in order that it be the fields near \( \Lambda \) which determine the continuum limit, the fields away from \( \Lambda \) being exponentially damped. This requirement can be met by making the appropriate choices of the gauge-fixing function \( F \), the damping term \( E \) and the sublattice \( \Lambda_1 \). However, there are other considerations as well. Specifically we want to
choose $F$, $E$ and $\Lambda_1$ so that the following conditions hold:

(i) the effective action $E_0F(a) e^{-S(a)}$ has a unique minimum at $a = 1$.

(ii) the gauge degree $\eta$ is nonzero.

(iii) the quadratic part of the effective action leads to a tractable Gaussian measure in the continuum limit.

The rather vague condition (iii) is meant to express our desire to obtain from the lattice action a well-defined Gaussian measure for the noninteracting part of the gauge field measure. To illustrate what we have in mind, let us consider an example which is ideal from the point of view of condition (iii).

If we expand the Wilson action to leading order in $\epsilon$ and second order in $A$, we get

$$S(a) = -\frac{1}{2} \sum_{x \in \Lambda} \epsilon^d \text{tr} \left( \partial_{\mu} A_\nu - \partial_{\nu} A_\mu \right) (x + \frac{1}{2} e_\mu + \frac{1}{2} e_\nu I)^2 + O(A^3). \quad (4.18)$$

The damping term for Feynman gauge

$$E_0F(a) = \exp (-\lambda^{-2} \epsilon^d \sum_{x} \text{Re} \text{tr} \left[ V_x(a) - I \right]) \quad (4.19)$$

which to the same approximation is

$$E_0F(a) = \exp (-\sum_{x,\mu} \epsilon^d \text{tr} \left[ \partial_{\mu} A_\mu(x) \right]^2 + O(A^3)). \quad (4.20)$$

We now impose Dirichlet boundary conditions by taking the
IV.C THE ACTION

range of the sums in eqs. (4.18) and (4.19) to be all plaquettes and points in the infinite lattice \( \mathbb{Z}^d \), but with the gauge fields taking the value 1 on all bonds not in \( \Lambda \), so that there is only a finite number of nonzero terms in each sum. By combining eqs. (4.18) and (4.20) and performing a summation by parts we obtain the quadratic part of the effective action:

\[
-\frac{1}{J} \Sigma \epsilon^d \text{tr} \left[ (\delta^\epsilon_A y - \delta^\epsilon A^\mu x)(x + \frac{1}{2} e_\mu \frac{1}{2} e_\nu) \right]^2 - \Sigma \epsilon^d \text{tr} \left[ \delta^\epsilon A^\mu x \right]^2
\]

\[
= -\Sigma \epsilon^d \text{tr} \left[ A_y (x + \frac{1}{2} e_\nu) \cdot (\delta^\epsilon A^\mu x + \frac{1}{2} e_\mu) \right]
\]

\[
= \frac{1}{J} \Sigma \epsilon^d A^\mu_y (x + \frac{1}{2} e_\mu) \left( -\delta^\epsilon \delta^\epsilon \mu \nu A_y (x + \frac{1}{2} e_\nu) \right)
\]

where \( d^\epsilon \) is the lattice Laplacian with DBC.

The quadratic form is that given by the operator

\[-d^\epsilon \delta^\epsilon \mu \nu,\]

which suggests that the noninteracting lattice measure converges to Gaussian measure with covariance

\[ (-d^\epsilon \delta^\epsilon) \delta^\epsilon \mu \nu.\]

We now return to the question of choosing the components of the lattice model. It has turned out to be unexpectedly difficult to do this and we have not yet been able to find choices which satisfy more than two of the three conditions (i) - (iii).

To illustrate the problem, let us consider again the effective action described above which satisfies condition (iii). We claim that it also satisfies condition (i). For \( G \) consists of unitary matrices so \( |\text{tr} a| \leq \text{tr} 1 \) for any \( b \in G \). Thus \( S_{\text{eff}} \) has a minimum when \( U_p(a) = 1 \) and
IV.C THE ACTION

\[ V_x(a) = 1 \] for all \( P \) and \( x \). It is not hard to convince oneself, by drawing diagrams for small lattices, that this can happen only when \( a = 1 \).

However, to obtain the damping term (4.19), we would need to fix the gauge on all of \( \Lambda \) (that is, take \( \Lambda_1 = \Lambda \)) which, by Theorem 2.14, would result in the gauge degree being zero, violating (ii).

Suppose we modify the preceding example by choosing \( \Lambda_1 \) as in (2.8a). We then satisfy (ii) but violate (i). For, again by considering diagrams of small lattices, it is not difficult to find a gauge field \( a \) such that

\[ a(x,y) = \pm 1 \] and such that each \( U_P(a) \) and \( V_x(a) \) depends on an even number of bonds where \( a(x,y) \) has the value \( -1 \). An example is shown in the diagram below. The heavy dots indicate the lattice sites in \( \Lambda_1 \); the gauge fields have the value \( 1 \) on the bonds labelled by "+" and \( -1 \) on the bonds labelled "-". For such a field we have \( U_P(a) = U_P(1) \) and \( V_x(a) = V_x(1) \), so the minimum is not unique.

Finally, just to illustrate that conditions (i) and (ii) are not mutually exclusive, we present an example of a lattice model which satisfies both (i) and (ii). Consider
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the lattice $\Lambda$ shown below. We take $S(a)$ to be the Wilson action for the plaquettes shown, with the understanding that the gauge field has the value $\mathbb{1}$ on those bonds with a stroke drawn through them (on the boundary of $\Lambda$). This action is invariant under gauge transformations $g$ for which $g(x) = \mathbb{1}$ for $x \in \partial \Lambda$. Take $\Lambda_1 = \Lambda^0$ as in (2.8) but let the gauge-fixing function $F$ be a map into $G^{\Lambda_1}$ where $\Lambda_1$ is the sublattice consisting of the points which are circled. Define $F(a)(x) = V_x(a)$ for $x \in \Lambda_1$.

To see that (ii) is satisfied, consider the condition for the existence of gauge copies

$$V_x(g\mathbb{1}) = \mathbb{1} \text{ for all } x \in \Lambda_1.$$  \hspace{1cm} (4.21)

By considering (4.21) first for all the points $x$ at the lower boundary of $\Lambda_1$ and then for the points on the next row up and so on, we see that (4.21) can hold only when $g = \mathbb{1}$. Thus there are no Gribov copies in this gauge and $|\eta| = 1$.

Similar considerations show that if $V_x(a) = \mathbb{1}$ and $U_P(a) = \mathbb{1}$ for all $x$ and $P$ then $a = \mathbb{1}$ so (i) holds. The defect of this model is that the effective action has an
asymmetrical and awkward form and it is not obvious what the corresponding Gaussian measure should be.

We do not believe that the conditions (i) - (iii) are inherently incompatible, but neither does it seem likely that they will be satisfied very easily. Probably the resolution of this difficulty lies in obtaining (ii) and (iii) at the expense of (i) and then somehow eliminating the invariance which the existence of multiple minima indicates still remains after breaking gauge invariance with the damping term.

To carry out such an elimination, it would be necessary that gauge invariant functions also share the residual invariance. That this might be so is suggested by the following heuristic considerations. When the gauge group is abelian, an argument [WC] based on the Stone-Weierstrass theorem shows that a continuous gauge invariant function $f(a)$ is determined by the Wilson loop variables, which are products of the gauge fields on lattice bonds forming a closed loop. By a lattice version of Stokes theorem, these loop variables can be written in terms of the plaquette variables $U_p(a)$. Thus for any gauge field $a$ for which $U_p(a) = U_p(1)$ for all plaquettes $P$ we have $f(a) = f(1)$, which is the desired invariance.
IV.D Haar Measure and Lebesgue Measure

In this section we investigate the relation between lattice Haar measure on the gauge group and continuum "Lebesgue" measure on its Lie algebra by examining the continuum limit of the Radon-Nikodym derivative $R_\varepsilon(A_\varepsilon)$ (see Section IV.A). In Theorem 4.1 below we find that, contrary to what one might expect, $R_\varepsilon$ does not become negligible in the formal continuum limit.

Let $\sigma$ be the 2x2 Pauli spin matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and put $\mathbf{t} = -i\mathbf{\sigma}/2$. Given $\mathbf{A} \in \mathbb{R}^3$ write $A = A \cdot \mathbf{t}$. Then for any $a \in SU(2)$ there is an $\mathbf{A} \in \mathbb{R}^3$ such that $a = e^{iA} = \cos(|A|/2) \mathbb{1} + 2\mathbf{A} \cdot \text{sin}(|A|/2)$. If $da$ denotes Haar measure on $SU(2)$, then for any $f \in L^1(SU(2), da)$ we have the following expression for the Haar integral of $f$ ([Wi], p.152)

$$\int_{SU(2)} f(a) da = \pi^{-2} \int_{|A| \leq 2\pi} f(e^{iA}) s(|A|/2)^2 dA$$

where $s(x) = \frac{\sin x}{x}$. Thus if we use the relation

$$a(x, x+e_\mu) = e^{ieA_\mu} e^{i(x+x+e_\mu)}$$ (4.22)

the lattice measure

$$da = \prod_{x,\mu} da(x, x+e_\mu)$$
can be written (up to a normalization constant)
\[
da = R_\varepsilon(A_\varepsilon) \prod_{x, \mu} \delta_{\mu, \varepsilon}(x + \frac{1}{2} e_\mu)
\]
where
\[
R_\varepsilon(A_\varepsilon) = \prod_{x \in \Lambda_\varepsilon} \delta(\varepsilon \lambda \| A_{\mu, \varepsilon}(x) / 2\|)^2 \cdot 12 \pi \varepsilon \lambda (\| A_{\mu, \varepsilon}(x) \|)
\]
(4.23)
and
\[
1_K = 1(x : |x| \leq K).
\]

**Theorem 4.1:** Let $\Lambda \subset \mathbb{R}^2$. Fix $A_\mu$ such that each $A_\mu \in C_0^\infty(\Lambda)$. Define $A_{\mu, \varepsilon}(x) = A_\mu([x])$ where $[x]$ denotes the point in the lattice $\Lambda_\varepsilon$ nearest to $x$. Then
\[
R_\varepsilon(A) \to \exp(-\frac{1}{12} \sum_{\mu} \int \lambda \| A_{\mu, \varepsilon}(x) \|^2 \ dx)
\]
as $\varepsilon \to 0$. $\Box$

**Proof:** Choose $\varepsilon$ so small that $\|A_{\mu, \varepsilon}\|_{\infty} < 2\pi / \varepsilon \lambda$. Then
\[
\log R_\varepsilon(A_\varepsilon) = 2 \sum_{x, \mu} \log \delta(\varepsilon \lambda \| A_{\mu, \varepsilon}(x) / 2\|)
\]
Now
\[
s(x) = 1 - \frac{1}{6} x^2 + O(x^4)
\]
and
\[
\log(1 + x) = x + O(x^2)
\]
so
\[
\log R_\varepsilon(A_\varepsilon) = 2 \sum_{x, \mu} \log(1 - \frac{\varepsilon^2 \lambda^2 \| A_{\mu, \varepsilon}(x) \|^2}{24}) + O(\varepsilon^4)
\]
Thus

$$= \sum_{x, \mu} \left[-\frac{\epsilon^2 \lambda^2 \mathbf{A}_\mu(x)^2}{24} + O(\epsilon^4)\right]$$

$$= -\frac{\lambda^2}{12} \sum_{x, \mu} \epsilon^2 \mathbf{A}_\mu(x)^2 + O(\epsilon^2)$$

Thus

$$\log R\epsilon(A_\epsilon) \to -\frac{\lambda^2}{12} \sum \mathbf{A}_\mu(x)^2 \ dx$$

as $\epsilon \to 0$.

The limit obtained in Theorem 4.1 is puzzling, and especially so is the appearance of what seems to be a mass term for the gauge field in two dimensions. Such a term is gauge variant and does not appear in the usual formulation of the continuum theory. It arises not from any physical input, but simply from the fact that we have used Haar measure as an approximation to Lebesgue measure. If this term could be shown to persist in a more rigorous form of the continuum limit, it would indicate a serious deficiency in that approximation.

However we now argue that Theorem 4.1 is misleading because, in gauge invariant integrals at least, the Radon-Nikodym derivative does not actually occur in quite the same form as (4.23). That conclusion is based on the following theorem.

**Theorem 4.2:** Let $f$ be a gauge invariant function in a two dimensional lattice theory. Then there is a function $f_1$ such that
\[ \int f(a) \, da = \int f_1(U) \, dU \]

where \( U = \{ U_p(a) \}_{P \in \Lambda} \) and \( dU \) is the product of Haar
measures \( \prod_{P \in \Lambda} dU_p \).

Before proving Theorem 4.2 let us explain how it
rescues us from Theorem 4.1. With our standard
approximation (eq. (4.14)) \( a = e \epsilon \lambda A \) we have

\[ U_p(a) = e^{\epsilon^2 \lambda \mu \nu, \epsilon (P)} \]

where \( F_{\mu \nu, \epsilon} \) is a finite dimensional approximation to the
continuum field strength tensor \( F_{\mu \nu} \). When we use the
variables \( F_{\mu \nu, \epsilon} \) as the coordinates of the group, the
Radon-Nikodym derivative for Haar measure is

\[ R_\epsilon^{(1)}(F_\epsilon) = \prod_{P \in \Lambda} s(\epsilon^2 \lambda \mu \nu, \epsilon (P) \| /2)^2 \, 12 \pi \epsilon \lambda (\| F_{\mu \nu, \epsilon} (P) \|) \].

In this case the formal limit is

\[ \log R_\epsilon^{(1)}(F_\epsilon) = 2 \sum_P \log \left( 1 - \frac{\epsilon^4 \lambda^2}{24} \| F_{\mu \nu, \epsilon} (P) \|^2 + O(\epsilon^6) \right) \]

\[ = - \frac{\lambda^2}{12} \epsilon^2 \sum_P \epsilon^2 \| F_{\mu \nu, \epsilon} (P) \|^2 + O(\epsilon^4) \]

\[ \to 0 \]

as \( \epsilon \to 0 \) since \( \epsilon \epsilon^2 \| F_{\mu \nu, \epsilon} \|^2 \to \frac{1}{2} \sum F_{\mu \nu} \| d^2 x \). Hence
\( R_\epsilon^{(1)} \to 1 \). We discuss below the situation for gauge variant
integrals.

We now turn to the proof of Theorem 4.2. Given a
finite lattice \( \Lambda \) define a tree to be a set of bonds of \( \Lambda \)
which contains no cycles (closed loops). The following
lemma is valid in any number of dimensions.
Lemma 4.3: Let $T$ be a tree. Then for any gauge field $a$ there is a gauge transformation $g$ such that $g a(x,y) g^{-1} = a(x,y)$ for all $<x,y> \in T$. Moreover, $g(x) = 1$ when the vertex $x$ lies outside $T$.

Proof: The proof is by induction on $|T|$ = the number of bonds in $T$. The case $|T| = 1$ is trivial. Suppose the lemma holds whenever $|T| < N$ and let $T$ be a tree with $|T| = N$.

Let $<x,y> \in T$ be a bond with the property that one of its endpoints, $y$ say, is the endpoint of no other bond in $T$. Such a bond $<x,y>$ must exist for otherwise $T$ would contain a cycle. Apply the induction hypothesis to find the gauge transformation $g$ for the tree $T - <x,y>$. Set $g(y) = a(x,y) g(x)^{-1}$. Then

$$g a(x,y) = g(x) a(x,y) g(y)^{-1} = 1.$$

Now suppose as usual that $\Lambda$ is a rectangular lattice and let $T$ be the tree consisting of the bonds shown in the diagram below, i.e., $T$ is the set of all horizontal bonds and all bonds on the right-hand side of $\Lambda$. Let $a|T$ denote $(a(x,y) | <x,y> \in T)$, the set of bond variables for bonds in $T$. We wish to make a change of variables from the bond variables $(a(x,y) | <x,y> \in \Lambda)$ to $a|T \cup \{U_p\}_{p \in \Lambda}$, a subset of the bond variables together with the plaquette variables. (Since we are discussing the plaquette variables $U_p$ themselves and not just $\text{Tr}(U_p)$ we need to make some
The change of variables just described is an invertible transformation. For, given the values of $a(x,y)$ on $T$ and the plaquette variables $U_p$ we can uniquely determine $a(x,y)$ on all of $\Lambda$ as follows. For each plaquette in the column on the right-hand side of $\Lambda$ the values of $a$ on three out of four of the plaquette's bonds are known, since these bonds are in $T$. We also know $U_p$ and hence can determine the value of the fourth bond. But that means that for the next column of plaquettes to the left, we have determined three out of four of its bonds, and so on.

**Proof of Theorem 4.2:** Let $T$ be as above. From the preceding discussion we know that for some function $f_2$ we have

$$f(a) = f_2(a|T, U).$$
Given \( a \), let \( g \) be the gauge transformation given by Lemma 4.3 so that
\[
f(a) = f(ga) = f_2(1, gU).
\] (4.24)

For each bond \( \langle x, y \rangle \) which is not in \( T \) use the translation invariance of Haar measure to write
\[
da(x, y) = dU_P
\]
where \( P \) is the plaquette which has \( \langle x, y \rangle \) as its left boundary. Define \( d_{aT} = \prod_{\langle x, y \rangle \in T} da(x, y) \). Then using (4.24) we have
\[
\int f(a) da = \int f(a) dU da_T
= \int f_2(1, gU) dU da_T
\]
in which \( g \) depends on \( a_T \). Now \( gU_P = g(x)U_P g(x)^{-1} \), so by the invariance of Haar measure,
\[
\int f_2(1, gU) dU = \int f_2(1, U) dU.
\] (4.25)

Define \( f_1(\cdot) = f_2(1, \cdot) \). Then by (4.25) we have
\[
\int f(a) da = \int f_1(U) dU da_T = \int f_1(U) dU.
\]

Let us now consider how the preceding argument would be different in the gauge variant lattice theory. In the general case of an integral where the integrand depends on \( \langle a(x, y) \rangle \langle x, y \rangle \in T \) we could expect Theorem 4.1 to have an effect in the continuum limit. However, in our lattice model the action is a function of \( U_P \) and \( V_x \), the plaquette variables and the cross variables. If we could show that the change of variables \( \langle a(x, y) \rangle \Rightarrow \)
(U_\nu)_{\mu \nu} \text{ is well-defined we could write}
\int f(a) \, da = \int f(U,V) \, dU \, dV.

Once again Theorem 4.1 would be an inappropriate guide to the continuum limit because

U_\nu = e^{\epsilon_2 F_{\mu \nu}, \epsilon(P)}
V_\nu = e^{\epsilon_2 A_{\mu}, \epsilon(x)}.

Both sets of variables \( F_{\mu \nu}, \epsilon \), \( A_{\mu}, \epsilon \) have an \( \epsilon^2 \) dependence instead of the \( \epsilon \) dependence of the variables \( A_{\mu}, \epsilon \). By the same argument used in the gauge invariant case, the Radon-Nikodym derivative makes no contribution in the formal continuum limit.

It is plausible that the change of variables 
\( (a) \rightarrow (U,V) \) is well-defined since, heuristically, \( U \sim F_{\mu \nu} \)
for which \( U_{\nu}(a) \) and \( V_{\nu}(a) \) are everywhere \( \epsilon \). Hence the change of variables \( (a) \rightarrow (U,V) \) is not invertible.

This appears to be mainly a technical problem and does not contradict the intuition just described. We conclude that, at least at a formal level of argument, the Radon-Nikodym derivative does not make a contribution to the continuum limit.
IV.E Faddeev-Popov Determinant - Formal Limit

The lattice Faddeev-Popov determinant for Landau gauge $M_e(a)$ was shown in Section II.D to be

$$M(a)_{xx} = I - \text{ad}(-s) + \sum_{\mu=0}^{5} \left[ \text{ad}(+\mu) + \text{ad}(-\mu) \right]$$

$$M(a)_{x+x_{\pm e_{\mu}}} = -\text{ad}(+\mu) \quad (2.40)$$

$$M(a)_{xy} = 0 \text{ if } |x-y| > \epsilon$$

where

$$\text{ad}(+\mu) = \text{ad}(a(x,x+e_{0}) a(x,x-e_{0}) a(x,x+e_{1}) ... a(x,x_{\pm e_{\mu}})).$$

The continuum Faddeev-Popov operator $M(A)$ operates on $L^2(\mathbb{R}^2) \otimes \mathbb{R}^2 \otimes E$ and is given by

$$M(A)B = -\partial B - \lambda \mu [A_{\mu}, B].$$

We now show that if $B^a \in C^\infty_0(A)$ then

$$\epsilon^{-2} M_e(\epsilon^A)B \to M(A)B.$$

We have from eq. (2.40) that

$$\epsilon^{-2}(M_e(a)B)(x) = \epsilon^{-2} \Sigma_y M(a)_{xy} B(y)$$

$$= \epsilon^{-2} [I - \text{ad}(-s)] B(x)$$

$$+ \epsilon^{-2} \Sigma_{\mu} \text{ad}(+\mu) \left[ B(x) - B(x_{-e_{\mu}}) \right]$$

$$+ \epsilon^{-2} \Sigma_{\mu} \text{ad}(-\mu) \left[ B(x) - B(x_{e_{\mu}}) \right]$$

$$= \epsilon^{-2} [I - \text{ad}(-s)] B(x)$$

$$- \epsilon^{-1} \Sigma_{\mu} \text{ad}(+\mu) \partial_{\mu} B(x+\frac{1}{2}e_{\mu})$$

$$+ \epsilon^{-1} \Sigma_{\mu} \text{ad}(-\mu) \partial_{\mu} B(x-\frac{1}{2}e_{\mu}). \quad (4.26)$$

By eq. (4.14)
IV.E FADDEEV-POPOV DETERMINANT - FORMAL LIMIT

\[ a(x, x+e_\mu) a(x, x-e_\mu) = 1 + \lambda \epsilon^2 \partial_\mu A_\mu(x) + O(\epsilon^3) \]

with the result for terms in (4.26) being

(i) \[ \epsilon^{-2} [\epsilon - \text{ad}(-s)] B(x) \]

\[ = -\lambda \sum \partial_\mu [\partial_\mu A_\mu(x), B(x)] + O(\epsilon) \]

(ii) \[ -\epsilon^{-1} \text{ad}(\mu) \partial_\mu B(x + \frac{1}{2} e_\mu) \]

\[ = -\epsilon^{-1} \partial_\mu B(x + \frac{1}{2} e_\mu) - \lambda [A_\mu(x + \frac{1}{2} e_\mu), \partial_\mu B(x + \frac{1}{2} e_\mu)] \]

\[ + O(\epsilon) \]

(iii) \[ \epsilon^{-1} \text{ad}(-\mu) \partial_\mu B(x - \frac{1}{2} e_\mu) \]

\[ = \epsilon^{-1} \partial_\mu B(x - \frac{1}{2} e_\mu) + O(\epsilon). \]

Putting these equations into eq. (4.26) we obtain

\[ \epsilon^{-2} M_\epsilon (a) B(x) = \sum \partial_\mu (-\lambda [\partial_\mu A_\mu(x), B(x)] - \partial_\mu \partial_\mu B(x)) \]

\[ -\lambda [A_\mu(x + \frac{1}{2} e_\mu), \partial_\mu B(x + \frac{1}{2} e_\mu)] + O(\epsilon) \]

\[ + -\partial B(x) - \lambda \sum \partial_\mu [A_\mu(x), B(x)] \]

as \( \epsilon \to 0. \)
IV.F Faddeev-Popov Operator - Trace Class Properties

Let \( A \subset \mathbb{R}^2 \) be the rectangle

\[
A = \{ x \in \mathbb{R}^2 \mid |x_\mu| \leq \frac{1}{2}, \mu = 1,2 \}
\]

and \( \Delta_A \) be the Laplacian operator with Dirichlet boundary conditions on \( A \). The unnormalized Faddeev-Popov operator \( M(A) \) is given by

\[
M(A)B = -\Delta_A B - \lambda \partial_\mu [A_\mu, B].
\]

Define the normalized operator \( L(A) \) by

\[
L(A) = (-\Delta_A) [I + L(A)].
\]

In this section we show that there is a Hilbert space \( \mathcal{H} \) on which \( L(A) \) is compact and in the Carleman class \( C_{2+\varepsilon} \) for all \( \varepsilon > 0 \) for almost all \( A \). This means in particular that \( \det_3(I+L(A)) \) is well-defined and constitutes the first step in obtaining a well-defined renormalized Faddeev-Popov determinant. For a definition of the Carleman norms \( \| \cdot \|_p \) and \( \det_p \) see [Si1]. Our basic approach is taken from the techniques used in the Yukawa model ([Se],[BR]).

The gauge group \( G \) can be any compact Lie group. If we take a basis \( \{ t_a \} \) for its Lie algebra \( E \) we have

\[
[t_a, t_b] = f_{abc} t_c
\]

which defines the structure constants \( f_{abc} \). If we write

\[
A_\mu = A_\mu^a t_a
\]
and

\[ L(A)B = L(A)_{ab}B_b a \]

we have

\[ L_{ab} = -\lambda \epsilon^{cba} (-\partial_\Lambda)^{-1} \partial_\mu A_\mu^c. \]  \hspace{1cm} (4.27)

The notation \( \partial_\mu A_\mu^c \) is used to emphasize that the operator composition is intended.

We now define the Hilbert space \( \mathcal{H} \) on which the Faddeev-Popov operator acts. Let \((\cdot,\cdot)\) denote the inner product on \( L^2(\Lambda, dx) \) and let \( \mathcal{H}^{1/2} \) be the completion of \( D((-\partial_\Lambda)^{1/2}) \) with the inner product

\[ (f, g)^{1/2} = (f, (-\partial_\Lambda)^{1/2} g). \]

We define

\[ \mathcal{H} \equiv \mathcal{H}^{1/2} \otimes \mathcal{E}. \]

The field \( A_\mu^a \) is the Gaussian random process with covariance

\[ \langle A_\mu^a(f_\mu) A_\nu^b(g_\nu) \rangle = \delta_{\mu \nu} \delta_{ab} (f_\mu, (-\partial_\Lambda)^{-1} g_\nu). \]

We let \( d\mu \) denote the corresponding measure, which is the free measure on the gauge fields in Feynman gauge.

The main point of this section is to prove

**Theorem 4.4:** The operator \( L \) defined in eq.(4.27) is in \( C^{2+\epsilon} \) a.e. \( (d\mu) \) for all \( \epsilon > 0 \).

The approach we take is to show that

\[ \int \| L(A) \|^{2+\epsilon} d\mu < \infty. \]
To begin, we discuss the operator \((-d_A)^{-1} \partial_\mu\) appearing in eq. (4.27). Let $D = (-d_A)^{1/2}$, $m_D^2 = \inf \text{spec}(-d_A) > 0$,

$$D_0 = (-d + m_D^2)^{1/2}$$

where $\Delta$ is the Laplacian with free boundary conditions.

**Lemma 4.5:** (a) If the operator $A > 0$ and $0 < \beta < 1$, then

$$A^{-\beta} = \frac{\sin \beta \pi}{\pi} \int_0^\infty t^{-\beta} (A + t)^{-1} \, dt.$$  \hspace{1cm} (4.28)

(b) For all $\alpha \in [0,2]$ there is a constant $c = c(\alpha)$ so that

$$D^{-\alpha} \leq cD_0^{-\alpha}.$$  \hspace{1cm} (4.29)

**Proof:** (a) This follows from the spectral theorem and the fact that for $\lambda > 0$

$$\int_0^\infty \frac{t^{-\beta}}{(\lambda + t)^{1+\beta}} \, dt = \pi \frac{\sin \pi \beta}{\lambda^\beta}.$$  \hspace{1cm}

Thus for any $f \in L^2$

$$(f, A^{-\beta} f) = \int \lambda^{-\beta} d(f, \mathcal{E}_\lambda f)$$

$$= \frac{\sin \beta \pi}{\pi} \int \int \frac{t^{-\beta}}{(\lambda + t)^{1+\beta}} d(f, \mathcal{E}_\lambda f)$$

$$= \frac{\sin \beta \pi}{\pi} \int_0^\infty t^{-\beta} f((A+t)^{-1}f) \, dt$$

by Fubini's theorem. The result for $(f, A^{-\beta} g)$ then follows from the polarization identity.

(b) We have the form inequalities [GRS2]
\[-d_A \leq m_0^2\]
\[-d_A \geq -d\]

which imply that for all \( t \geq 0,\)
\[(-d_A + t)^{-1} \leq \frac{1}{2} (-d + m_0^2 + 2t).\]

So by a theorem in [K] (p. 330)
\[(-d_A + t)^{-1} \leq 2 (-d + m_0^2 + 2t)^{-1}. \quad (4.30)\]

In particular, \( D^{-2} \leq 2 D_0^{-2}.\)

From (a), for \( \alpha < 2\)
\[(f, D^{-\alpha} f) = \frac{\sin \alpha \pi/2}{\pi} \int_0^\infty t^{-\alpha/2} (\mu, (-d_A + t)^{-1} f) \, dt \]
\[\leq K \int_0^\infty t^{-\alpha/2} (\mu, (-d + m_0^2 + t)^{-1} f) \, dt \]
\[\leq K' (f, D_0^{-\alpha} f). \quad \square\]

Lemma 4.6: If \( \alpha \geq 1, \) \( D^{-\alpha} \partial_\mu \) extends from \( C_0^\infty (\Lambda) \) to a bounded operator on \( \mathcal{H}_2.\)

Proof: \( D^{-\beta} \) is bounded on \( \mathcal{H}_2 \) for any \( \beta > 0 \) so it suffices to prove the theorem for the case \( \alpha = 1.\)

By Lemma 4.5(b)
\[ D^{-1} \leq K D_0^{-1}. \]

Thus if \( f \in C_0^\infty (\Lambda) \) we have
\[\|D^{-1} \partial_\mu f\|_2^2 = (\partial_\mu f, D_0^{-1} \partial_\mu f) \leq K (\partial_\mu f, D_0^{-1} \partial_\mu f) = K (f, f)\]
We now define the operator \((-d_{\lambda})^{-1}\) to be the unique bounded extension of \(D^{-2}\partial_{\mu}L_{0}^{\omega}(\Lambda)\).

We now return to the \(C_{p}\) properties of \(L\). These are based on the \(x\)-space behavior of the integral kernels of \((-d_{\lambda})^{-\alpha}(x,y)\) and \(\frac{\partial}{\partial x_{\mu}}(-d_{\lambda})^{-\alpha}(x,y)\) which are given by the next two lemmas. The letters \(K\), \(K_{j}\) and so on used below denote different constants on different occasions. However, at all times, these constants depend at most on \(\Lambda\). In particular, they are independent of \(x,y \in \Lambda\) and of \(m\).

**Lemma 4.7:** The integral kernel of \(C^{\alpha}\) obeys

\[
K|x-y|^{-2(1-\alpha)} \leq C^{\alpha}(x,y) \leq K \ln|x-y| \quad (\alpha = 1)
\]

for all \(x,y \in \Lambda\).

**Lemma 4.8:** The first derivative of \(C^{\alpha}\) obeys

\[
|\partial_{\mu}C^{\alpha}(x,y)| \leq K|x-y|^{-2(1-\alpha)-1}, \quad 0 < \alpha < 1
\]

for all \(x,y \in \Lambda\).

Lemmas 4.7 and 4.8, which are proven below, allow us to prove the main result.

**Theorem 4.9:** For each \(\epsilon > 0\), \(L \in C_{2+\epsilon}\) a.e.
Proof: Let $d\mu$ be the Gaussian measure associated with $A^a_y$ and define the $\|B\|_{p,q}$ norms in the usual way: If $B$ is a map from the gauge fields $A^a_y$ into the space of bounded operators on $\mathcal{H}$,

$$\|B\|_{p,q} = \left( \int |B|^p \, d\mu \right)^{1/q}.$$ 

By complex interpolation [Si1] and Holder's inequality, for all $\delta > 0$

$$\|L\|_{2+\epsilon, 2+\epsilon} \leq \|D^{-2\epsilon}\delta L\|_{2,2} \|D^{(2-\epsilon)\delta L}\|_{4,4}$$

so it suffices to show

(i) $\|D^{-\alpha}L\|_{2,2} < \infty \quad (0 < \alpha < \frac{1}{2})$

(ii) $\|D^\beta L\|_{4,4} < \infty \quad (\beta < \frac{1}{2})$

to conclude that $\|L\|_{2+\epsilon}$ is finite a.e.

Let $*$ denote the adjoint on $\mathcal{H}$ and $\dagger$ the adjoint on $L^2(\Lambda)\otimes \mathcal{E}$. Then in general $B^* = D^{-1}B^\dagger D$ so

$$(D^{-\delta L})^*(D^{-\delta L}) = D^{-1}L^\dagger + D^1 - 2\delta L.$$ 

A direct computation then shows

$$\|D^{-\alpha}L\|_2^2 = K\|L\|_2^2 \sum_\Lambda D^{-1}(x,y) D^{-2}(x,y) \frac{\partial}{\partial y_\mu} D^{-3/2-\alpha}(y,z)$$

$$\times \frac{\partial}{\partial x_\nu} D^{-3/2-\alpha}(z,x) \frac{\partial}{\partial y_\mu} D^{-3/2-\alpha}(y,z)$$

so that

$$\|D^{-\alpha}L\|_{2,2}^2 \leq K \sum_\Lambda D^{-1}(x,y) D^{-2}(x,y) \left| \frac{\partial}{\partial y_\mu} D^{-3/2-\alpha}(y,z) \right|$$

$$\times \left| \frac{\partial}{\partial x_\nu} D^{-3/2-\alpha}(z,x) \right| \, dx \, dy \, dz \quad (4.31)$$

Now apply Lemmas 4.7 and 4.8 to eq.(4.31). We obtain
IV.F FADDEEV-POPOV OPERATOR - TRACE CLASS PROPERTIES

\[ ||D^{-\alpha}L||^2_{2,2} \leq K \int_{\Lambda} |x-y|^{-1} \ln \frac{1}{|x-y|} \]
\[ \times |y-z|^{-3/2+\alpha} |z-x|^{-3/2+\alpha} \, dx \, dy \, dz. \]  
(4.32)

Recall Lemma C6 of [BR], which says that
\[ \int_{\Lambda} d^2z \, |z-x|^{-\beta} |z-y|^{-\gamma} \leq K |x-y|^{2-\beta-\gamma} \]  
(4.33)
if \( 0 < \beta, \gamma \leq 2 \) and \( \beta + \gamma > 2 \). This estimate, combined with
\[ \ln \frac{1}{|x-y|} \leq K_6 |x-y|^{-\delta} \]
for all \( \delta > 0 \) and some \( K_6 \), yields, from eq. (4.32)
\[ ||D^{-\alpha}L||^2_{2,2} \leq K \int_{\Lambda} |x-y|^{-2+2\alpha-\delta} \, dx \, dy < \infty \]
if we chose \( \delta < 2\alpha \). This proves (i) above.

The proof of (ii) is similar, being based on the expression
\[ ||D\partial L||^2_{4,4} \leq K \int_{\Lambda} D^{-1}(x_1,x_2) \, D^{-1}(x_4,x_5) \]
\[ \times \frac{\partial}{\partial x_2^\mu} \, D^{-3/2+\beta}(x_2,x_3) \, \frac{\partial}{\partial x_4^\nu} \, D^{-3/2+\beta}(x_3,x_4) \]
\[ \times \frac{\partial}{\partial x_5^\rho} \, D^{-3/2+\beta}(x_5,x_6) \, \frac{\partial}{\partial x_1^\lambda} \, D^{-3/2+\beta}(x_6,x_1) \]
\[ \times [\delta_{\mu\nu} \delta_{\rho\lambda} D^{-2}(x_2,x_4) \, D^{-2}(x_5,x_1) \]
\[ + \delta_{\mu\lambda} \delta_{\nu\rho} D^{-2}(x_2,x_5) \, D^{-2}(x_4,x_1) \]
\[ + \delta_{\mu\rho} \delta_{\nu\lambda} D^{-2}(x_2,x_1) \, D^{-2}(x_4,x_5) ] \prod_{1 \leq i < j \leq 6} dx^i. \]

We now turn to the proofs of the estimates used above.
Proof of Lemma 4.7: The estimate for $a = 1$ follows the well-known behaviour of the Dirichlet Green's function in two dimensions (see e.g., [CH]). For $0 < a < 1$, we make use of a Wiener integral representation of $C^a(x,y)$.

By the spectral theorem and the fact that for $\lambda > 0$

$$\lambda^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-\lambda t} \, dt \quad (4.34)$$

we have that the distributional kernel of $(-d_\lambda)^{-a}$ is given by

$$(-d_\lambda)^{-a}(x,y) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{d_\lambda t} \, d\mu_t(x,y) \, dt. \quad (4.35)$$

Let $\mu_{0,x,y;t}$ denote Wiener measure conditioned to start at $x$ at time 0 and be at $y$ at time $t$, with normalization

$$\int d\mu_{0,x,y;t} = (2\pi t)^{-1/2} e^{-\|x-y\|^2/2t}.$$

Then (see [Si2])

$$e^{d_\lambda t}(x,y) = \int_Q d\mu_{0,x,y;2t}$$

where $Q = \{\omega \mid \omega(s) \in \Lambda \text{ for all } s \in [0,t]\}$. This expression shows clearly that

$$0 \leq e^{d_\lambda t}(x,y) \leq (4\pi t)^{-1} e^{-\|x-y\|^2/4t} \quad (4.36)$$

Putting eq. (4.36) into eq. (4.35), we obtain

$$0 \leq C^a(x,y)$$

$$\leq \frac{1}{4\pi \Gamma(a)} \int_0^\infty t^{a-1-d/2} e^{\|x-y\|^2/4t} \, dt$$

$$\leq K \|x-y\|^{2a-2}$$

by scaling $t \to \|x-y\|^2 t$. □
The proof of Lemma 4.8 is considerably more complicated. The strategy is to use Lemma 4.5(a) to express $C^\beta$ in terms of the massive Green's functions $C_m^\Lambda(x,y)$, and then differentiate both sides of eq. (4.28). Thus the proof is reduced to establishing estimates for $\partial_\mu C_m^\Lambda(x,y)$. These are obtained from the method of images expression for $C_m^\Lambda$.

We shall prove Lemma 4.8 in a series of further lemmas.

**Lemma 4.10:** Let $C_m$ be the Green's function for $-\Delta + m^2$ with free boundary conditions. Then for some constants $K_1$, $K_2$ and all $x \in \mathbb{R}^2$

$$|\partial_\mu C_m(x)| \leq m e^{-m|x|} [K_1 (m|x|)^{-\frac{1}{2}} + K_2 (m|x|)^{-1}]. \quad (4.37)$$

**Proof:** By performing one $p$ integration in the $x$-direction, we obtain

$$C_m(x) = \frac{1}{(2\pi)^2} \int e^{i p \cdot x} \frac{d^2 p}{p^2 + m^2}$$

$$= K \int_0^\infty \frac{e^{-\mu(k)m|x|}}{\mu(k)} dk \quad (4.38)$$

where

$$\mu(k) = (k^2 + 1)^{\frac{1}{2}}. \quad (4.39)$$

Note that for some constants $a, b > 0$

$$\mu(k) \geq 1 + ak^2, \quad k \leq 1$$

$$\mu(k) \geq 1 + bk, \quad k \geq 1. \quad (4.40)$$

Now from (4.38)

$$\partial_\mu C_m(x) = K m \frac{x}{|x|} \int_0^\infty \frac{e^{-\mu(k)m|x|}}{\mu(k)} dk. \quad (4.41)$$
A scaling argument combined with eq. (4.40) shows
\[ \int_{-\infty}^{\infty} e^{-\mu(k) m|x|} \, dk \leq e^{-m|x|} \int_{-\infty}^{\infty} e^{-\epsilon k^2 m|x|} \, dk \]
\[ \leq K (m|x|)^{-\frac{3}{2}} e^{-m|x|}. \quad (4.42) \]

Similarly,
\[ \int_{-\infty}^{\infty} e^{-\mu(k) m|x|} \, dk \leq K (m|x|)^{-1} e^{-m|x|}. \quad (4.43) \]

Equations (4.41) through (4.43) give the desired result. \( \square \)

By the method of images
\[ C_m(x, y) = \sum_{n \in \mathbb{Z}^2} (-1)^{|n|} C_m(x - p_n y) \]
where \(|n| = |n_1| + |n_2|\) and
\[ (p_n y)_\mu = (-1)^{|\mu|} (y_\mu - n_\mu^1 \mu). \]

**Lemma 4.11.** (a) \(|x - p_n y| \geq |x - y|\) for all \(n \in \mathbb{Z}^2\).

(b) If \(|n_\mu| \geq 2\) for some \(\nu\), then
\[ |x - p_n y| \geq K |n| \]
for all \(x, y \in \Lambda\).

**Proof:** (a) I claim that in fact
\[ |(x - p_n y)_\mu| \geq |(x - y)_\mu| \]
for each \(\mu\). If \(|n_\mu| = 0\) this is obvious; if \(|n_\mu| \geq 2\) then
\[ |(x - p_n y)_\mu| \geq |n_\mu|_\mu - |x_\mu| - |y_\mu| \]
\[ \geq (|n_\mu| - 1)_\mu \]
\[ \geq |(x - y)_\mu| \]
since \(|x_\mu|, |y_\mu| \leq 1_\mu/2\) for \(x, y \in \Lambda\).
Suppose $|n_\mu| = 1$. Then we must show
\[ \epsilon(x-y)_\mu \neq l_\mu + \nu(x+y)_\mu \] (4.44)
for all choices of $\epsilon, \nu \in (-1,1)$, i.e.,
\[(\epsilon-\nu)x_\mu - (\epsilon+\nu)y_\mu \neq l_\mu. \] (4.45)

But only one term on the left-hand side of eq.(4.45) is nonzero, and its absolute value is bounded by $2(1/2) = 1/2$.

(b) Suppose $|n_\nu| \geq 2$. Then
\[ |(x-n_\nu y)_\nu| \leq |n_\nu||y| - |x_\nu| - |y_\nu| \]
\[ \leq (|n_\nu| - 1)|y| \]
\[ \leq K(|n_\nu| + 1). \] (4.46)

If the other component of $n$, $n_\nu$, say, satisfies $|n_\nu| = 2$, then the theorem is proved by summing eq.(4.46).

If $|n_\nu| \leq 1$ then
\[ |x-n_\nu y| \leq |(x-n_\nu y)_\nu| \leq K(|n_\nu| + 1) \leq K|n|. \]

Lemma 4.12: The series
\[ \sum \frac{(-1)^{|n|}}{n!} \frac{\partial}{\partial x^\mu} C_m(x-n_\nu y) \] (4.47)
converges absolutely and uniformly for $x, y$ such that $|x-y|$ is bounded below uniformly. Thus
\[ \frac{\partial}{\partial x^\mu} C_m(x,y) = \sum \frac{(-1)^{|n|}}{n!} \frac{\partial}{\partial x^\mu} C_m(x-n_\nu y) \] (4.48)
for all $x, y \in \Lambda$.

Proof: Combining Lemmas 4.10 and 4.11 we have
\[ |\frac{\partial}{\partial x^\mu} C_m(x-n_\nu y)| \leq K_1 e^{-K_2|n|}. \]
Although the series (4.47) is absolutely convergent, to obtain the desired estimates from eq. (4.48), we need to take into account the cancellations arising from the alternating signs of the terms.

For this purpose, let us introduce some notation. If \( f : \mathbb{Z}^d \to \mathbb{C} \), let

\[
\Delta f(n) = f(n + e_\mu) - f(n)
\]

where \( e_\mu \) is the vector in \( \mathbb{Z}^d \) with 1 in the \( \mu \)-th component and 0 in the others. Let

\[
\Delta^{(d)} = \Delta_1 \cdots \Delta_d.
\]

**Lemma 4.13:** If \( f : \mathbb{Z}^d \to \mathbb{C} \) is such that \( \sum_{n \in \mathbb{Z}^d} |f(n)| < \infty \), then

\[
\sum_{n \in \mathbb{Z}^d} (-1)^{|n|} f(n) = \sum_{n \in \mathbb{Z}^d} \Delta^{(d)} f(n), \quad (4.49)
\]

**Proof:** A straightforward induction on \( d ). \( \Box \)

**Lemma 4.14:** If \( n \in 2\mathbb{Z}^d + 1 \),

\[
|\Delta^{(d)} \frac{\partial}{\partial x_\mu} C_m(x - p_\mu y)| \leq K \sup_{|s_\mu| \leq b_\mu} |D^{d_1} \cdots D^{d_d} C_m(a_n + s)|
\]

where

\[
a_{n, y} = -y_y + (n_y + \frac{1}{2})_y
\]

\[
b = x + \frac{1}{2} l
\]

\[
D_y = \frac{\partial}{\partial x_\mu}
\]

**Proof:** I claim that
\[ d^{(d)} C_m(x-p_n y) = \Pi_{1 \leq j \leq d} \int_{-b_j}^{b_j} ds_j D_d \cdots D_1 C_m(a_n + s). \] (4.51)

For example,

\[ d_1 C_m(x-p_n y) \]

\[ = C_m((-1)^{n+1} y - (n+1) \lambda) - C_m((-1)^n (y-n\lambda)) \]

\[ = C_m((a_n^{(1)} + b_1 \lambda) - C_m(a_n^{(1)} - b_1 \lambda) \]

where

\[ a_n^{(1)} = -y + (n+\frac{1}{2}) \lambda \]

\[ a_n^{(1)} = -x + y + n\lambda - y, \quad y \neq 1. \]

Thus

\[ d_1 C_m(x-p_n y) = \int_{-b_1}^{b_1} ds_1 D_1 C_m(a_n^{(1)} + s \lambda). \]

The general formula eq.(4.52) is proved by induction similarly.

A short calculation using the rules for differentiating integrals shows that

\[ d^{(d)} \frac{\partial}{\partial x^\mu} C_m(x-p_n y) = \frac{\partial}{\partial x^\mu} d^{(d)} C_m(x-p_n y) \]

\[ = \Pi_{1 \leq j \leq d} \int_{-b_j}^{b_j} ds_j D_d \cdots D_1 C_m(a_n + s). \]

The theorem follows from this equation, by taking

\[ K = \Pi_{1 \leq j \leq d} \frac{1}{2} b_j. \]  

Lemma 4.15: For all \( x \in \mathbb{R}^2 \)
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\[ |D_\mu D_2 D_1 C_m(x)| \leq (K_1 |x|^{-3} + K_2 |x|^{-2} + K_3^\frac{1}{2} |x|^{-5/2} + K_4^\frac{1}{2} |x|^{-3/2} + K_5^3/2 |x|^{-3/2} + K_6^5/2 |x|^{-1/2}) e^{-m|x|}. \]

The \( K_j \) are independent of \(|x|\) and \( m \).

**Proof:** Start from eq. (4.38)

\[ C_m(x) = K \int_0^\infty e^{-\mu(k)m|x|/\mu(k)} \, dk \]  

and calculate

\[ D_1 C_m(x) = -K m \frac{x_1}{|x|} \int_0^\infty e^{-\mu(k)m|x|} \, dk \]

\[ D_2 D_1 C_m(x) = K m \frac{x_1 x_2}{|x|^2} \int_0^\infty e^{-\mu(k)m|x|} (\frac{1}{|x|} + \mu(k)) \, dk. \]

Without loss of generality we may assume the \( \mu = 2 \). Now

\[ D_2^2 D_1 C_m(x) = K m \int_0^\infty \left[ \frac{x_1 x_2}{|x|^2} - \frac{2 x_1}{|x|^4} - \frac{x_1^2}{|x|^2} \mu(k)m \right] \]

\[ x \left[ \frac{1}{|x|} + \mu(k)m \right] - \frac{x_1 x_2}{|x|^4} e^{-\mu(k)m|x|} \, dk. \]

Using \( |x_1| \leq |x| \) we obtain

\[ |D_2^2 D_1 C_m(x)| \leq m \int_0^\infty \left[ \frac{K_1}{|x|^2} + \frac{K_2}{|x|} \mu(k)m + \frac{K_3}{|x|} \right] \]

\[ + K_4^2 \mu(k)^2 \, e^{-\mu(k)m|x|} \, dk. \]  

Using eq. (4.40) we find

\[ \int_0^1 \mu(k) e^{-\mu(k)m|x|} \, dk \leq e^{-m|x|} \int_0^1 e^{-ak^2m|x|} \, dk \]

\[ = K(m|x|)^{-\frac{1}{2}} e^{-m|x|} \]

and
\[ F_{\text{Faddeev-Popov Operator}} - \text{Trace Class Properties} \]

\[ \int_{-\infty}^{\infty} \mu(k)e^{-\mu(k)|x|} \, dk \leq K \int_{-\infty}^{\infty} e^{-|x|} \, dk \]

That is,

\[ \int_{0}^{\infty} \mu(k)e^{-\mu(k)|x|} \, dk \]

\[ \leq K (|x|)^{-n-1} e^{-|x|} \quad \text{(4.53)} \]

Equation (4.53) applied to eq.(4.52) gives the desired result. \( \square \)

\textbf{Lemma 4.16:} The bound

\[ \frac{1}{\partial \mu^2} C^A(x,y) \leq (K_1 m_1^2 |x-y| + K_2 |x-y|^{-1}) e^{-|x|} + f(m) \]

holds uniformly for \( x, y \in \Lambda \) where \( f \) is a function such that

\[ \int_{0}^{\infty} m^{-2\alpha} f(m) \, dm < \infty. \]

\textbf{Proof:} The idea is to split up the method of images sum (4.48) into two pieces. One piece involves terms with small \( n \), and is estimated by the known behaviour of \( \partial \mu C^A(x,y) \). The sum of the remaining terms is shown to be bounded by the function \( f(m) \).

From Lemmas 4.12 and 4.13, we have

\[ \partial \mu C^A(x,y) = \sum_{n \in \mathbb{Z}^2+1} d^{(2)} \partial \mu C^A(x-p_n y) \quad \text{(4.54)} \]

Let \( Q = (n \in \mathbb{Z}^2+1 : |n_y| < 2 \text{ for all } y) \). Then by combining eqs. (4.14) and Lemma 4.11(a), we have
To estimate \( \Sigma_{\mathcal{Q}} \), recall Lemmas 4.13 and 4.14. Also note that if \( n \in \mathcal{Q}^c \), then \( |n_y| \geq 3 \) for some \( y \) and so

\[
|a_n + s| \leq K|n| \quad (4.56)
\]

which can be seen from an argument similar to that of Lemma 4.11(b). Combining these facts yields

\[
|d^{(2)} \partial_{\mu} C_m(x-p_n y)|
\]

\[
\leq \left( \sum_{j=1}^{6} K_j m^{a_j |n| - \beta_j} \right) e^{-K m |n|} \quad (n \in \mathcal{Q}^c) \quad (4.57)
\]

where \( a_j, \beta_j \leq 0 \) and \( \beta_j + a_j = 2 \). We now "steal" a little bit from the exponential term in (4.57) via

\[
e^{-K m |n|} = e^{-K m |n|/2} e^{-K m |n|/2} \leq K_{\epsilon_j} (m |n|)^{-\epsilon_j} e^{-K m |n|/2}. \quad (4.58)
\]

We choose \( \epsilon_j \) so that

\[
\alpha_j + 2(1 - a) - \epsilon_j > 0
\]

\[
\beta_j + \epsilon_j > 2.
\]

This is possible because \( 1 - a > 0 \) and \( \alpha_j + \beta_j > 2 \).

From eq. (4.57), eq. (4.58) and the fact that

\[
\sum_{n \in \mathcal{Q}^c} |n|^{-\delta} < \infty
\]

if \( \gamma > 2 \) we obtain

\[
\Sigma_{\mathcal{Q}^c} |d^{(2)} \partial_{\mu} C_m(x-p_n y)| \leq \left( \sum_{j=1}^{6} K_j m^{a_j - \epsilon_j} \right) e^{-K m}. \quad (4.59)
\]

Take \( f(m) \) to be the right-hand side of eq. (4.59).
Equations (4.55) and (4.59) give the desired bounds. □

Proof of Lemma 4.8: By Lemma 4.5(a)

\[ C^a(x,y) = \frac{\sin \beta \pi}{\pi} \int_0^\infty t^{-\alpha} (-\Delta + t)^{-1}(x,y) \, dt \]

\[ = K \int_0^\infty m^{1-2\alpha} C^\Lambda_m(x,y) \, dm \]

where we have made a change of variables \( t \to m^2 \) to obtain the last line. A dominated convergence argument using the bounds of Lemma 4.16 allows us to differentiate under the integral to obtain

\[ \partial_\mu C^a(x,y) = K \int_0^\infty m^{1-2\alpha} \partial_\mu C^\Lambda_m(x,y) \, dm. \]

Using Lemma 4.16 again, we obtain

\[ |\partial_\mu C^a(x,y)| \leq K_1 |x-y|^{-\frac{3}{2}} \int_0^\infty m^{3/2-2\alpha} e^{-m|x-y|} \, dm \]

\[ + K_2 |x-y|^{-1} \int_0^\infty m^{1-2\alpha} e^{-m|x-y|} \, dm \]

\[ + \int_0^\infty m^{1-2\alpha} f(m) \, dm. \]

By scaling \( m \to m/|x-y| \), the first two terms are bounded by

\[ K |x-y|^{-3+2\alpha}. \]

The third term is bounded by a constant, which can be bounded by

\[ K' |x-y|^{-3+2\alpha}. \]
In this section we prove

**Theorem 4.17:** Let $M(a)$ be the Faddeev-Popov operator for Landau gauge (eq. (2.40)) in $d = 1$ dimension. Then

$$|\det M(a)| \leq \det M(\emptyset). \quad (4.60)$$

The inequality (4.60) is of a class known as diamagnetic inequalities in which a determinant or partition function in an external field is bounded by its value in zero external field. The term "diamagnetic" is used because one of the first inequalities of this type expressed the diamagnetic response of a system of spinless particles to an external magnetic field [Si2].

Diamagnetic inequalities have been proven for Euclidean lattice fermions with antiperiodic boundary conditions [BFS1], [W] and for lattice bosons for a wide variety of boundary conditions [BFS1]. The proof we give for QCD$_1$ is rather clumsy and does not readily extend to the two (or higher) dimensional case. The importance of Theorem 4.17 is to suggest that a diamagnetic inequality for the Faddeev-Popov determinant does in fact exist.

**Proof of Theorem 4.17:** Label the lattice points as shown

```
  0  1  2  ...  n  n+1
```

and write $U(a)_{ij}$ for $\text{ad}(a_{ij})$. We shall usually omit the
argument \(a\) and just write \(U_{ij}\). For interior points \(i\), the Faddeev-Popov operator \(M(a)\) is given by

\[
M(a)_{ii} = 1 + U_{ii}
\]

\[
M(a)_{i,i+1} = -U_{i,i+1}
\]

\[
M(a)_{i,i-1} = -U_{i,i-1} + \frac{1}{U_{i,i}}
\]

In matrix form, \(M(a)\) is

\[
\begin{bmatrix}
1+U_{12} & -U_{12} & & & \\
-U_{21}U_{23} & 1+U_{23} & -U_{23} & & \\
& -U_{32}U_{32} & 1+U_{32} & -U_{32} & \\
& & & -U_{n-1,n}U_{n-1,n} & 1+U_{n-1,n} \\
& & & & -U_{n,n+1}U_{n,n+1} & 1+U_{n,n+1}
\end{bmatrix}
\]

Because \(G\) is compact and \(a \to \text{ad}(a)\) is a finite-dimensional representation of \(G\), each \(U_{ij}\) is a unitary matrix. Note also that \(U_{ij} = (U_{ji})^{-1}\) and \(U_{ij}(I) = I\).

We shall perform a series of manipulations (a reduction to triangular form) on the matrix \(M(a)\) to show that

\[
\det M(a) = \det(S_k V_k(a))
\]

where each \(V_k(a)\) is a product of \(U_{ij}(a)\)'s. In particular, \(V_k(I) = I\). Thus by Lieb's inequality [Si1] we have

\[
|\det(S_k V_k(a))|^2 \leq \det(S_k |V_k(a)|) \det(S_k |V_k(a)|^*) \det(S_k I)^2
\]

\[
= \det(S_k I)^2
\]
= \text{det}(\Sigma_k V_k(\mathbb{I}))^2.$

and equation (4.60) follows.

To begin, multiply $M(a)$ on the left by the block diagonal matrix

$$
\begin{bmatrix}
U_{21} & 0 \\
& U_{32} \\
& & \ddots \\
& & & U_{n+1,n}
\end{bmatrix}
$$

The result is $M_1$

$$
\begin{bmatrix}
U_{21} + 1 & -1 \\
-U_{21} & U_{32} + 1 & -1 \\
& \ddots & \ddots & \ddots \\
& & -U_{k,k-1} & U_{k+1,k+1} & -1 \\
& & & \ddots & \ddots & \ddots \\
& & & & -U_{n,n-1} & U_{n+1,n+1}
\end{bmatrix}
$$

and

$$
\text{det } M = \text{det}(\prod_{i} U_{1,i,i+1}) \text{ det } M_1. \tag{4.61}
$$

Next, replace row$(k)$ in $M_1$ by row$(k) + \text{row}(k-1)$, starting with row$(2)$. The result is $M_2$
We now start the reduction process proper. Multiply $M_2$ on the right by

$$
\begin{bmatrix}
1 & \\
1 + U_{12} & 1 \\
0 & \\
0 & \\
0 & \\
0 & \\
0 & \\
0 & 1
\end{bmatrix}
$$

which has the effect of multiplying the second column on the right by $1 + U_{12}$. The result is $M_3$. 

and

$$\text{det } M_2 = \text{det } M_1. \quad (4.62)$$
To complete the first stage of the reduction, replace column(2) by column(2) + column(1). The result is $M_4$

\[
\begin{bmatrix}
U_{12} + 1 & 0 \\
1 & U_{32}(1+U_{12}) + 1 & -1 \\
1 & 1 & 0 & \ldots & 0 & U_{k+1,k} + 1 & -1 \\
1 & 1 & \ldots & 0 & U_{n+1,n+1}
\end{bmatrix}
\]

and

\[
\det M_4 = \det M_3.
\] (4.64)

Let $M_2'$ denote the matrix consisting of the lower right block of $M_4$, as shown. Then

\[
\det M_4 = \det(1+U_{12}) \det M_2'
\] (4.65)

which combined with (4.63) and (4.64) yields
We claim that it is permissible to cancel the factor $\det (1+U_{12})$ in eq. (4.66). For if $\det (1+U_{12}) = 0$, replace $U_{12}$ by a sequence $U_{12}^{(i)}$ such that $U_{12}^{(i)} \to U_{12}$ and $\det U_{12}^{(i)} \neq 0$. We can cancel the factor $\det U_{12}^{(i)}$ for each $n$ and take the limit after to obtain

$$\det M_2 = \det M'_2.$$  

(4.67)

Note that $M'_2$ has the same form as $M_2$ but has one less row and column. We repeat the reduction $(n-1)$ times to obtain a matrix $M_2^{(n)}$ with only one entry which is a sum of products of the $U_{ij}$ and $\det M_2^{(n)} = \det M_2$. By eq. (4.61) we thus have

$$\det M(a) = \det (\prod_{i} U_{i,i+1}) \det M_2^{(n)}$$

$$= \det (\prod_{i} U_{i,i+1}) \det (\Sigma \prod_{i} U_{ij})$$

$$= \det (\Sigma V_k(a)).$$

\[\square\]
V. Open Problems

We now give a summary of some of the more important questions left unresolved in this thesis and of the possible ways in which the results we have given could be extended. Several of these problems and extensions are related and all are directed toward the achievement of the main goal, which is to give a rigorous construction of QCD in two (or more) dimensions by proving the Osterwalder-Schrader axioms hold for the continuum limit of a gauge variant lattice approximation.

Problem 1 - Unique minimum for the lattice action:

Find a choice of sublattice $\Lambda_i$ and gauge-fixing function $F$ for which the gauge variant lattice action is reasonable and has a unique minimum at $a = 1$. (Cf. Section IV.C) Failing this, find a way to take the continuum limit which does not rely on the intuition that fields far from 1 are being damped out by the action.

It is possible that the failure to solve this problem indicates there is some physics going on that has not been taken into account. Find out what it is.

Problem 2 - Diamagnetic inequality:

Prove the diamagnetic inequality (Section IV.G) for the Faddeev-Popov determinant in higher dimensions and for a
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wider class of boundary conditions.

Problem 3 - Zero gauge degree:

Find a way to avoid the problem of zero gauge degree when $A_1 = A$. For example, extend Theorem 2.16 (Section II.D) to the nonabelian case.

Problem 4 - Truncation of the functional integral:

We were not optimistic about the prospects for truncating the functional integral (Section III.D), but in any case the framework we have developed should be adequate to prove or disprove the following theorem (cf. eq. (3.28)):

There exists a subset $V$ of the lattice gauge fields such that the Faddeev-Popov determinant is positive on $V$ and for any gauge invariant function $f$

$$\left( \int e^{c \cdot \Phi} \right) \left( \int e^{-S(a)} da \right) = \int_V \det M(a) f(a) e^{\Phi(a)} e^{-S(a)} da.$$ 

Problem 5 - Renormalization and gauge invariance:

By using a lattice version of the Slavnov-Taylor identities or some other means, demonstrate the cancellations between the lattice versions of the superficially divergent graphs alluded to in Section IV.B.

Problem 6 - Gauge invariance and plaquette variables:

Prove or disprove that the equation

$$\int f(a) da = \int f_1(U) dU$$
expressing the integral of a gauge invariant function in terms of an integral over plaquette variables holds in more than two dimensions. (Cf. Theorem 4.5, Section IV.D.)
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