AUTOMORPHISM GROUPS OF MINIMAL ALGEBRAS

by

LEX ELLERY RENNER

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Department of Mathematics

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date Oct 11/78
Abstract

Rational homotopy theory is the study of uniquely divisible homotopy invariants. For each nilpotent space $X$ the association

$$X \longrightarrow \text{minimal algebra for } X$$

is a complete determination of these invariants.

If $X$ is a space and $M_X$ its minimal algebra, the algebraic group $\text{Aut } M_X$ and the representation

$$\text{Aut } M_X \longrightarrow \text{GL}(M_X)$$

have considerable influence on the structure of $M_X$. This thesis contains a systematic study of this interaction.

Chapter I contains preliminary results from algebraic group theory and general topology.

In Chapter II I define and study inverse limits of algebraic groups. I prove that many of the known structural properties of algebraic groups remain valid in this more general setting. Emphasis is placed on the conjugacy theorems that are particularly useful for studying minimal algebras.

Chapter III is the main part of the thesis where I develop a structure theory for minimal algebras which relates toroidal symmetry to retracts. Precisely, if $M$ is a minimal algebra then there exists a 1-parameter subgroup

$$\lambda: \mathbb{Q}^* \longrightarrow \text{Aut } M_X$$

such that $\lambda$ extends to
$$\bar{\lambda}: Q \longrightarrow \text{End } M_X$$

with

$$\bar{\lambda}(0) = e = e^2: M_X \longrightarrow M_X.$$ 

Further if $e$ so chosen is minimal then it is uniquely determined up to conjugation by $\text{Aut } M_X$.

In the interesting case where $e = 0_M$, I give a pro-algebraic group theoretic proof of uniqueness of coproduct and product decompositions in the appropriate homotopy category.
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INTRODUCTION

From Sullivan's theory of rational homotopy types [17] one has a well defined correspondence

\[ M : \text{Top} \rightarrow \mathcal{M} \]

which assigns to each nilpotent space \( X \) in \( \text{Top} \) its minimal algebra \( M(X) \) ([8], [10], and [17]).

The usefulness of the method

space \( \rightarrow \) simplicial forms on space \( \rightarrow \) model for space

as exhibited in Sullivan's fundamental paper [17] is three-fold.

1) Categorical: The usual homotopy category of nilpotent spaces localized with respect to 0-equivalence is equivalent to the homotopy category of nilpotent differential graded algebras localized with respect to its family of weak equivalences.

2) Classification: a) The homotopy type of a nilpotent 0-local space \( X \) is uniquely determined by the isomorphism class of \( M(X) \), its minimal algebra.

b) By imposing additional algebraic structures (integral cohomology ring, structure lattice, etc.) Sullivan proves classification results for integral homotopy theory "up to finite ambiguity" using properties of arithmetic groups.

3) Computational: Given a nilpotent space \( X \), one can compute \( \pi_k(X) \otimes \mathbb{Q} \) with its Lie algebra structure and \( H^*(X;\mathbb{Q}) \) with its Massey product structure by examining a presentation of \( M(X) \) a minimal model for \( X \).

One striking consequence of this general algebraic picture is
that Hom sets between 0-local spaces can be computed algebraically using the notion(s) of algebraic homotopy ([8], [10] and [17]).

Under suitable finiteness conditions Sullivan [17] observes that for a minimal algebra $M$, $\text{Aut}(M)$ (resp. $\text{Aut}(M)/\sim$) the group of self-equivalence of $M$ (resp. the group of d.g.a. homotopy classes of self-equivalences) are affine algebraic groups.

We observe here that, more generally, for a minimal algebra $M$ of finite type, $\text{Aut} M$ and $\text{Aut} M / \sim$ are pro-affine $\mathbb{Q}$-groups (2.2.2). Under suitable finiteness conditions much of the important structure theory of $\mathbb{Q}$-algebraic groups remains valid for pro-affine $\mathbb{Q}$-groups.

In chapter I basic notation is set up and well known results from algebraic group theory are collected. Certain types of projective systems are also studied and their usefulness for algebraic groups is introduced.

In chapter II we develop a theory of pro-affine $k$-groups which is general enough for a large class of minimal algebras and still yields "good" rationality and structural properties.

A natural application of this general setup is achieved in chapter III where we focus on the relation of toroidal symmetry to topological closure properties of $\text{Aut}(M) \subseteq \text{End}(M)$ (which is an open imbedding of (pro-) affine varieties). Using the conjugacy theorem of chapter II we interpret the existence of "non-closed tori" $T \subseteq \text{Aut}(M) \subseteq \text{End}(M)$ as structural information about retracts.

The following general result is obtained for a suitable class of minimal algebras $M$.

**Theorem 3.6.2.** Let $M \in M$. Then there exists $M(1) \in M$, unique up to
d.g.a. isomorphism, such that

\[ M(1) \xrightarrow{i} M \xrightarrow{p} M(1) \]

where

\[ p \circ i = 1_{M(1)} \]

ii) there exists \( \lambda : \mathbb{K}^* \rightarrow \text{Aut}(M) \) a 1-parameter subgroup which extends to \( \bar{\lambda} : \mathbb{K} \rightarrow \text{End}(M) \) with \( \bar{\lambda}(0) = i \circ p \).

iii) \( N \in M \) satisfies i) and ii) implies there exists

\[ j : M(1) \rightarrow N \] \text{ and } \[ g : N \rightarrow M(1) \]

such that

\[ q \circ j = 1_{M(1)} \]

Thus we have for all \( M \in M \) an essentially unique "smallest" retract \( M(1) \in M \) such that the idempotent \( i \circ p = e : M \rightarrow M \) that gives it to us is the limit of a 1-parameter subgroup \( \lambda : \mathbb{K}^* \rightarrow \text{Aut}(M) \).

Iterating this process yields for all \( M \in M \) an essentially unique collection \( \{M(k)\}_{k=0}^n \subseteq M \) such that

i) \( M(0) = M \)

ii) \( M(k) \xrightarrow{i_k} M(k-1) \xrightarrow{p_k} M(k) \) satisfies theorem 3.6.2 for all \( k \geq 0 \).

Among other things (3.6.3) this gives us a good insight into the structure of (weak) 0-equivalence in the integral homotopy category.

This can be illustrated by the following interesting special case. Let \( P \) be the full subcategory of 1-connected finite CW complexes satisfying the following condition \( \overline{P} : X \in P \) if whenever \( Y \) is a finite 1-connected CW complex and \( f : X \rightarrow Y \) is a p-equivalence [14] there...
exists \( g : Y \to X \) which is a \( p \)-equivalence.

Thus \( P \) is the subcategory of finite 1-connected CW complexes where \( p \)-equivalence is an equivalence relation. \( P \) is precisely the category of \( p \)-universal spaces of Mimura and Toda which have been studied in [3] and [14].

It is then a consequence of [3] that the following are equivalent.

1) \( X \in P \)
2) \( Y \in P \) and there exists \( Z \) and maps \( f : Z \to X \) and \( q : Z \to Y \) which are \( p \)-equivalences.
3) There exists a prime \( q \neq 0 \) and a map \( f_q : X \to X \) such that
\[
\tilde{f}_{q^#} \otimes 1 : \pi_* (X) \otimes \mathbb{Q} \to \pi_* (X) \otimes \mathbb{Q}
\]
and \( f_{q^#} \otimes 1 = 0 : \pi_* (X) \otimes \mathbb{Z}/p\mathbb{Z} \to \pi_* (X) \otimes \mathbb{Z}/p\mathbb{Z} \)
4) The augmentation map \( \xi_{\tilde{M}(X)} : \tilde{M}(X) \to \tilde{M}(X) \) satisfies
\[
\xi_{\tilde{M}(X)} \in Aut(\tilde{M}(X)) \quad \text{(Zariski closure)} \quad \text{where} \quad \tilde{M}(X) \quad \text{is the minimal model for} \quad X .
\]

One striking observation ((4) above) is that we are dealing with a rational problem. Said differently, the existence of self maps that annihilate \( p \)-torsion functors and induce isomorphisms on \( q \)-torsion functors is independent of \( p \) and \( q \).

Thus Theorem 3.6.2 can be viewed as the rational algebraic picture for the general setup.

Another application of the pro-algebraic conjugacy theorems is given in the latter part of chapter 3 where product and coproduct decompositions
are studied. Generalizing [1] and [2] we prove that for a suitable category $P$ of minimal algebras (those having positive weights and finite-dimensional spherical cohomology) all objects satisfy unique factorization into products and coproducts.

Precisely, if $M \in P$ then $M = \prod_{i=1}^{n} M_i$ where each $M_i$ is $\mathbb{N}$-irreducible (and non-trivial). Furthermore, such a decomposition is unique up to isomorphism and reordering of the factors.

Dually, if $N \in P$ then $N = \prod_{i=1}^{m} N_j$ where each $N_j$ is $\mathbb{N}$-irreducible. Such a decomposition is unique up to isomorphism and reordering.
Chapter 1 - Preliminaries

The purpose of this chapter is to assemble for easy reference well known results from algebraic group theory and general topology. Adequate reference for the rationality theory of algebraic groups may be found in [4] and [9]. The general topology is more or less self contained.

1.1. k-groups

Throughout the chapter (and the entire paper) k denotes a perfect field and K an algebraic closure of k. For the applications we have in mind k will be the rational numbers.

Definition ([4] p. 85):

a) A k-group consists of the following data
   i) a K-affine variety G defined over k:
      ii) k-morphisms \( i : G \rightarrow G \) and \( \mu : G \times G \rightarrow G \) (where \( G \times G \) has the Zariski product topology).
   iii) \( e \in G \) a distinguished element, such that G has the structure of a group with:
       identity = e
       \( i(x) = x^{-1} \)
       \( \mu(x,y) = x \circ y \)

b) A k-morphism \( \phi \) of k-groups G and H is a set map \( \phi : G \rightarrow H \), such that
   i) \( \phi \) is a group morphism
   ii) \( \phi \) is a k-morphism of affine varieties.
c) A $k$-subgroup $H \leq G$ is a subgroup of $G$ which is $k$-closed
(closed in the Zariski $k$-topology).

Remark: Since we assume that $k$ is perfect $H \leq G$ is $k$-closed iff $H$
is defined over $k$. See [4].

The following list of well known results is taken from [4] and [9]. $G$ denotes an arbitrary $k$-group.

1.1.2 ([9] p. 3) (Existence of Quotients)

If $G$ is a $k$-group and $H$ is a $k$-subgroup then there exists
on $G/H$ a unique structure of an algebraic variety (not necessarily
affine) such that,

i) $\pi : G \longrightarrow G/H$ the natural projection is an open $k$-morphism.

ii) $G$ acts $k$-morphically on $G/H$ by left translation.

iii) If $f : G \longrightarrow X$ is a $k$-morphism of varieties then there
exists a unique $k$-morphism $\hat{f} : G/H \longrightarrow X$ such that

$$\pi \downarrow \quad \hat{f}$$

commutes iff $f$ is constant on the fibres.

iv) If $k \leq L \leq K$ where $L$ is a field extension of $k$ and
$
\xi \in (G/H)(L) (= L$-rational points of $G/H)$ then $\pi^{-1}(\xi)$
is defined (and separable) over $L$.

Remark ([4] p. 101): If $H$ is also normal in $G$ then $G/H$ is also an
affine $k$-group with the above mentioned variety structure.

1.1.3 ([9] p. 3) (Existence of rational points)

Let $G$ be a connected $k$-group ($G = G^0$). Then $G(k) \leq G$ is
dense in the Zariski $k$-topology. If $k$ is also infinite then $G(k) \subseteq G$ is also dense in the Zariski $k$-topology.

Note: This can fail if $k$ is not perfect or if $G$ is not connected.

1.1.4 ([4] p. 150) (Jordan-Chevalley decomposition)

Let $G$ be a $k$-group. Then there exists subsets $G_u, G_s \subseteq G$ (uniquely determined) such that

i) $G_s \cap G_u = \{e\}$

ii) $G_s$ and $G_u$ are functorial

iii) for all $x \in G$ there exist unique $x_s \in G_s$ and $x_u \in G_u$ such that $X = x_s \cdot x_u = x_u \cdot x_s$.

iv) if $x \in G(k)$ then $x_s, x_u \in G(k)$. (This can fail if $k$ is not perfect).

v) in any $k$-rational representation $\rho : G \longrightarrow GL(n,K)$ of $G$, $\rho(G_s)$ consists of semi-simple elements and $\rho(G_u)$ consists of unipotent elements.

Definition ([4] p. 200): A torus $T \subseteq G$ is called a torus if $T$ is a closed connected abelian subgroup consisting of semi-simple elements. (Equivalently, if $T \cong K^* \times \ldots \times K^*$ as algebraic groups, where $K^*$ is the multiplicative group of units of $K$).

Definition ([4] p. 204): A torus $T \subseteq G$ is called $k$-split if

i) $T$ is defined over $k$

ii) $\text{Hom}_{k\text{-group}}(T, K^*)$ spans $K[T]$ where $K[T]$ is the affine co-ordinate ring of $T$. 
ii) is equivalent to

\[
\text{ii') } T \cong \mathbb{K}^* \times \ldots \times \mathbb{K}^* \quad \text{as } \text{k-groups (see [4] p. 204).}
\]

1.1.5 ([9] p. 4). If G is a k-group then G has a maximal torus 
T \subseteq G defined over k.

Note: The non-triviality of this statement is that T is defined over k.
Maximal tori exist for dimension reasons.

1.1.6 ([9] p. 11) (Conjugacy of maximal k-split tori). If S, T \subseteq G
are maximal k-split tori then there exists \( g \in G(k) \) such that \( gTg^{-1} = S \).

Remark: 1.1.6 is also true for other classes of subgroups in G. See

It will be useful in limit considerations to keep track of
G(k) (especially for the conjugacy theorems we have in mind). More
specifically we are interested in maps \( \phi: G \rightarrow H \) of k-groups for which
\( \phi(G(k)) = \phi(G)(k) \).

Definition: ([4] p. 357). A connected k-group G is called _k-solvable_
if there is a composition series

\[
G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = \{e\}
\]

such that i) \( G_i \) is a connected k-subgroup

\[
\text{ii) } G_i / G_{i+1} \cong \mathbb{K}^* \text{ or } K
\]

as k-groups, for \( 0 \leq i < n \)
(The terminology in [4] is \textit{k-split}).

1.1.7 ([4] p. 359-62). If $G$ is a connected $k$-group then
i) $G$ is $k$-solvable iff $G$ is trigonizable over $k$.
ii) $G$ unipotent implies $G$ is $k$-solvable.

1.1.8 ([4] p. 363). If $H$ is a $k$-group (not necessarily connected) and $N \subseteq H$ is a connected $k$-solvable subgroup, then in

\[ \pi : H \longrightarrow H/N, \quad \pi(k) : H(k) \longrightarrow (H/N)(k) \]

is surjective.

\textbf{Definition}: A $k$-group morphism $\phi : G \longrightarrow H$ is called \textit{k-proper} if $\phi(N(k)) = \phi(N)(k)$ for all $N \subseteq G$ $k$-closed subgroup.

\textbf{Remark}: Maps which are not k-proper abound in algebraic group theory. As an example let $K$ be an algebraic closure of $Q$, the rational numbers and consider $p_1 : K^* \times K^* \longrightarrow K^*$, the projection onto the first factor.

Let $N \subseteq K^* \times K^*$ be the $k$-closed subgroup defined as $N = \{(x,y)/x \cdot y^2 = 1\}$. Then it is easy to verify that $p_1(N(Q)) \neq p_1(N)(Q)$.

1.1.8. shows us that this fails only because $N \cap \{1\} \times K^*$ is not connected.

\textbf{Lemma 1.1.9}: Let $\phi : G \longrightarrow H$ and $\psi : H \longrightarrow N$ be morphisms of $k$-groups.

a) If $\phi$ is one to one then $\phi$ is k-proper.

b) If $\phi$ and $\psi$ are k-proper $\psi \circ \phi$ is k-proper.
Proof: a) If \( L \leq G \) is a \( k \)-subgroup then \( \phi(L(k)) \leq \phi(L)(k) \).

Conversely, if \( x \in \phi(L)(k) \) then \( \phi^{-1}(x) \in N \) is a \( k \)-closed point.

Thus \( \phi^{-1}(x) \in N(k) \) since \( k \) is perfect.

b) Is obvious.

Theorem 1.1.10: Let \( \phi : G \rightarrow H \) be a morphism of \( k \)-groups such that

i) \( \ker \phi \) is unipotent

ii) \( \text{char } k = 0 \).

Then \( \phi \) is \( k \)-proper.

Proof: \( \ker \phi \) is unipotent and \( k \)-closed. Thus it is also connected ([12] p. 101), since \( \text{char } k = 0 \).

Let \( N \leq G \) be a \( k \)-closed subgroup and let \( \phi_N = \phi|_N \). By 1.1.2 we can factor \( \phi_N \) as follows

\[
\begin{array}{c}
N \\
\phi_N
\end{array}
\quad \begin{array}{c}
\rightarrow H \\
p \downarrow i \\
N/(N \cap \ker \phi)
\end{array}
\]

By 1.1.8 \( p(N(k)) = (N/(N \cap \ker \phi))(k) \) and by 1.1.9.a)

\[i((N/(N \cap \ker \phi))(k)) = i((N/(N \cap \ker \phi))(k)) \quad \text{Thus}
\]

\[
\phi(N)(k) = i(p(N))(k)
\]

\[
= i((N/(N \cap \ker \phi))(k))
\]

\[
= i((N/(N \cap \ker \phi))(k))
\]

\[
= i(p(N(k)))
\]

\[
= \phi(N(k)) \quad \text{q.e.d.}
\]
1.2. General Topology

**Limits of Proper Systems**

It was hinted in the introduction that we are interested in a satisfying theory of pro-algebraic $Q$-groups. This involves finding suitable conditions on projective systems and algebraic groups that will allow us to prove delicate structure theorems about a general class of limiting objects. Roughly speaking what we need is a system where inverse limits "behave like images" (we would like to preserve properties like connectedness, compactness, irreducibility and the like.)

The single most important result from general topology is the following.

**Projective Limit Theorem 1.2.1:** Let $\{X_i, \pi_{ij}\}_{i \in I}$ be a projective system of topological spaces $X_i$ such that

i) $X_i \neq \emptyset$ is compact and $T_1$.

ii) $\pi_{ij}$ is a closed continuous map for $i \geq j$.

Then $X = \lim \limits_{\longrightarrow} X_i \neq \emptyset$

**Proof:** [14 p. 57]

Let $S = \left\{ \{A_i\} \mid A_i \subseteq X_i \text{ is closed and non-empty} \right\}$

\[ \pi_{ij}(A_i) \subseteq A_j \]

Then $S \neq \emptyset$ since $\{X_i\} \in S$. Define a partial ordering on $S$ as follows:

$\{A_i\} \leq \{B_i\}$ if $A_i \subseteq B_i$ for all $i$.

Since each $X_i$ is compact, every chain in $(S, \leq)$ has a lower
bound. (This is just the finite intersection property). Thus by Zorn's lemma $S$ has minimal elements.

Suppose $\{A_i\} \in S$ is minimal.

By ii) $B_i = \cap_{j \geq i} \pi_j(A_j)$ is closed non-empty and $\pi_i(B_i) \subseteq B_k$ for $i \geq k$, so $\{B_i\} \in S$. But $\{B_i\} \subseteq \{A_i\}$. Thus $\{B_i\} = \{A_i\}$ by minimality. But then $A_i = \cap_{j \geq i} \pi_j(A_j)$ and thus $\pi_j(A_j) = A_i$ for all $j \geq i$. Fix $i \in I$ and choose $x_i \in A_i$.

Let $C_j = \pi_{-1,j}(x_i) \cap A_j$ if $j \geq i$.

$C_j = A_j$ if $j < i$.

By i) $C_j$ is closed and by construction $\{C_j\} \subseteq \{A_j\}$. Thus $\{C_j\} = \{A_j\}$ again by minimality of $\{A_i\}$. Since $i \in I$ was arbitrary $\{A_i\} = \{x_i\}$.

By definition of $S$ $(x_i) \in \lim_{i \in I} X_i$. q.e.d.

Remark: The conclusion of this theorem is true under more general assumptions. Specifically, if we require that each $\pi_{ij}$ has compact point inverses then the resulting proposition is true for $T_0$ spaces. Also with our assumptions the limiting object is compact. In any case an inverse system satisfying the assumptions of Theorem 1.2.1 will be called a proper system.

Theorem 1.2.2: Let $\{X_i, \pi_{ij}\}$ be as in theorem 1.2.1. Then if $X = \lim_{i \in I} X_i$

i) $\pi_i(X) = \cap_{j \geq i} \pi_j(X_j)$

ii) $X = \lim_{i \in I} \pi_j(X)$
Note: This theorem says that an arbitrary proper system can be replaced by an equivalent one where all the doubly indexed maps are onto. (An inverse system is called proper if it satisfies the assumptions of 1.2.1)

Proof of 1.2.2: i) Clearly \( \pi_i(X) \subseteq \bigcap_{j \geq i} \pi_{ij}(X_j) \). Conversely, let \( x_i \in \bigcap_{j \geq i} \pi_{ij}(X_j) \) and let

\[
F_j = \pi_{ij}^{-1}(x_i) \quad \text{if } j \geq i
\]

\[
F_j = X_j \quad \text{if } j < i
\]

Then \( \{F_j, \pi_{ij}\} \) is a proper system. Thus by theorem 1.2.1 \( \lim F_k \neq \emptyset \).

But \( F_i = \{x_i\} \). Thus there exists \( x \in \lim F_k \subseteq X \) such that \( \pi_i(x) = x_i \).

Thus \( x_i \in \pi_i(X) \). Hence \( \bigcap_{j \geq i} \pi_{ij}(X_j) \subseteq \pi_i(X) \).

ii) Since \( \pi_i(X) \subseteq X_i \) we have \( \lim_\downarrow \pi_i(X) \subseteq X \). But if \( x \in X \) then \( x = (x_i)_{i \in I} \) and \( \pi_i(x) = x_i \). But then each \( x_i \in \bigcap_{j \geq i} \pi_{ij}(X_j) = \pi_i(X) \).

So \( x \in \lim_\downarrow \pi_i(X) \).

Definition: Let \( \{X_i, \pi_{ij}\}_{i \in I} \) and \( \{Y_i, \theta_{ij}\}_{i \in I} \) be inverse systems of topological spaces. A morphism

\[
\phi : \{X_i, \pi_{ij}\}_{i \in I} \longrightarrow \{Y_i, \theta_{ij}\}_{i \in I}
\]

is a collection \( \phi = \{\phi_i\} \) of continuous maps \( \phi_i : X_i \longrightarrow Y_i \) such that
Whenever \( j \geq i \).

Of course \( \{ \phi_i \} \) induces \( \phi = \lim_{\to} \phi_i : \lim_{\to} X_i \to \lim_{\to} Y_i \) as follows: Let \( (x_i) \in \lim_{\to} X_i \) then \( (\phi_i(x_i)) \in \lim_{\to} Y_i \) since

\[
\theta_{ji} \circ \phi_j = \phi_i \circ \pi_{ji}.
\]

Thus define

\[
(\lim_{\to} \phi_i)(x_i) = (\phi_i(x_i))
\]

Remark: i) It is a well known fact about topological spaces that if each \( \phi_i \) is continuous then \( \phi \) is continuous in the inverse limit topologies for \( \lim_{\to} X_i \) and \( \lim_{\to} Y_i \). If the \( X_i \)'s and \( Y_i \)'s also have the structure of a group and the \( \phi_i \)'s respect this structure as well, then \( \lim_{\to} X_i \) and \( \lim_{\to} Y_i \) have natural group structures for which \( \phi = \lim_{\to} \phi_i \) is a group homomorphism.

ii) The above definition of morphism is not the most general one that is useful. It is however convenient and sufficient for the purposes of this paper to have a fixed index set.

**Theorem 1.2.3:** Let \( \phi : \{ X_i, \pi_{ij} \} \to \{ Y_i, \theta_{ij} \} \) be a morphism of proper systems where \( \theta_{ij} \) and \( \pi_{ij} \) are onto when they exist. Then \( \phi = \lim_{\to} \phi_i \) is onto iff \( \phi_i \) is onto for all \( i \).
Proof: Suppose $\phi$ is onto. If $y_i \in Y_i$ there exists $y \in \lim Y_i$ such that $\theta_i(y) = y_i$ (Theorem 1.2.2). But then

\[
\lim X_i \xrightarrow{\phi_i} \lim Y_i
\]

so that $\phi_i$ is onto.

Conversely, suppose $\phi_i$ is onto for all $i$. Then if

\[(y_i) = y \in \lim Y \text{ for each fixed } i \text{ there exists } x_i \in X_i \text{ such that } \phi_i(x_i) = y_i. \quad \text{Thus } \phi_i^{-1}(y_i) = F_i \neq \emptyset, \text{ and since the appropriate maps commute } \pi_j(F_i) \subseteq F_j. \quad \text{Since each } F_i \text{ is also closed, } F = \lim F_i \neq \emptyset \quad \text{(Theorem 1.2.1). But by construction } \phi(F) = (y_i). \quad \text{q.e.d.}
\]

(Actually we proved more than advertised, i.e. If $\{Y_i, \theta_{ij}\}$ is proper and $\phi$ is onto then $\phi_i$ is onto for all $i$, and if $\{X_i, \pi_{ij}\}$ is proper and $\phi_i$ onto for all $i$ then $\phi$ is onto.)

For the remainder of this section we assume that $\pi_{ij}$ is onto whenever $i \geq j$. Theorem 1.2.2 assures us that no generality is lost.

Lemma 1.2.4: Let $\phi : X \longrightarrow Y$ be a map of proper systems. Then

\[\phi(X) = \lim_\leftarrow \phi_i(X_i)\]

Proof: Clearly $\phi(X) \subseteq \lim_\leftarrow \phi_i(X_i)$. Conversely, if $y = (y_i) \in \lim_\leftarrow \phi_i(X_i)$ then $\phi_i^{-1}(y_i) \neq \emptyset$ is closed and $\pi_{ij}(\phi_i^{-1}(y_i)) \subseteq \phi^{-1}_j(y_j)$ so we have a
proper system.

Thus \( \lim_{i \to \infty} \phi_i^{-1}(y_i) \neq \emptyset \). But \( (x) = (x_i) \in \lim_{i \to \infty} \phi_i^{-1}(y_i) \) implies that \( \phi(x) = y \). q.e.d.

Lemma 1.2.5: i) The closed sets for the inverse limit topology on \( X \) are given by

\[
\{ \lim_{i} F_i \mid \pi_{ij}(F_i) \subseteq F_j, F_i \text{ closed} \}.
\]

ii) each \( \pi_i \) is a closed map.

Proof: i) Let \( F_i \subseteq X_i \) be closed and let

\[
F_j = \pi_{ji}^{-1}(F_i) \quad j \geq i
\]

\[
F_j = X_j \quad j \geq i
\]

Then \( \pi_{jk}(F_j) \subseteq F_k \) and we have a proper system. Let \( F = \lim_{i \to \infty} F_i \neq \emptyset \).

Then \( F \subseteq \pi_i^{-1}(F_i) \) by definition. Conversely, if \( x = (x_i) \in \pi_i^{-1}(F_i) \) then \( x_i \in F_i \) and \( j \geq i \) implies \( \pi_{ji}(x) = x_i \). So \( x_j \in \pi_{ij}^{-1}(F_i) = F_j \).

Thus \( \pi_i^{-1}(F_i) = \lim_{j \to \infty} F_j \). Observe also that if \( F^\alpha = \lim_{i \to \infty} F_i^\alpha \) \( \alpha \in \Omega \) some set then

\[
\cap_{\alpha \in \Omega} F^\alpha = \lim_{i \to \infty} \cap_{\alpha \in \Omega} F_i^\alpha.
\]

Thus to complete the proof of i) we only have to show that if

\( F = \lim_{i \to \infty} F_i \) and \( G = \lim_{i \to \infty} G_i \) then \( F \cup G \) is also of this form.

To do this we first observe that if \( F = \lim_{i \to \infty} F_i \) then as in the proof of 1.2.2 \( \pi_i(F) = \cap_{j \geq i} \pi_{ji}(F_i) \) is closed. Thus \( F = \lim_{i \to \infty} \pi_i(F) \) (1.2.2)

and similarly \( G = \lim_{i \to \infty} \pi_i(G) \). But then \( \pi_i(F \cup G) = \pi_i(F) \cup \pi_i(G) \) is closed.

Hence yet another application of 1.2.2 implies that \( F \cup G = \lim_{i \to \infty} \pi_i(F \cup G) \).
ii) This is now easy since if \( F \subseteq X \) is closed, by i),
\[
F = \lim_{\rightarrow} F_i \quad \text{where} \quad F_i \subseteq X_i \quad \text{are closed.} \quad \text{Then by 1.2.2}
\]
\[
\pi_i(F) = \cap_{j \geq i} \pi_i(F_j)
\]
is closed. \( \quad \text{q.e.d.} \)

Theorem 1.2.6: Let \( X = \lim_{\rightarrow} X_i \) be the limit of a proper system. Then if \( X_i \) is connected for all \( i \), \( X \) is connected.

Proof: If \( X = A \cup B \) where \( A \) and \( B \) are closed then \( \pi_i(A) \cup \pi_i(B) = X_i \) and \( \pi_i(A) \) and \( \pi_i(B) \) are closed by 1.2.5. Thus \( \pi_i(A) \cap \pi_i(B) \neq \emptyset \).

Let \( N_i = \pi_i(A) \cap \pi_i(B) \). Then \( \pi_j(N_i) \subseteq N_i \) since \( \pi_j \circ \pi_i = \pi_i \).

Thus we have yet another proper system. So \( \lim N_i = N \neq \emptyset \). Thus
\[
A \cap B = \lim_{\rightarrow} \pi_i(A) \cap \lim_{\rightarrow} \pi_i(B) = \lim_{\rightarrow} (\pi_i(A) \cap \pi_i(B)) = N \neq \emptyset. \quad \text{q.e.d.}
\]

**Coset Topologies for Algebraic Groups**

In order to apply the results of section 1.2 to inverse systems of algebraic groups we must first contrive an appropriate topology so that (unlike the Zariski topology) morphisms are closed maps.

Consequently an arbitrary inverse system of algebraic groups is a proper system of topological spaces (1.2.1).

Since the topology selected (see the remarks preceding 1.2.10) is intrinsic it may be forgotten or resurrected depending upon the whim of the moment.

Recall that a noetherian topological space is one in which any descending chain of closed subsets is stable.
Proposition 1.2.7: Let $X$ be a noetherian topological space and let $F_{\alpha} \subseteq X$ be closed $\alpha \in \Omega$. Then $\bigcap_{\Omega} F_{\alpha} = \bigcap_{F_0} F_{\alpha}$ where $F_0 \subseteq \Omega$ is finite.

Proof: Obvious from the definition.

Proposition 1.2.8: Let $H, N \subseteq G$ be subgroups of the group $G$ and let $x, y \in G$. Then either $xH \cap yN$ is empty or it is a coset of $H \cap N$.

Proof: Note: $xH \cap yN = x(H \cap x^{-1}yN)$ and $H \cap gN \neq \emptyset$ iff $g \in HN$.

Thus $H \cap gN \neq \emptyset$ implies that

$$H \cap gN = H \cap h \cdot gN$$
$$= h \cdot H \cap h \cdot gN$$
$$= h(H \cap N)$$

q.e.d.

We now specialize these simple results to the $k$-group $G$.

Corollary 1.2.9: Let $\{g_H\}_{\alpha}$ be cosets of $k$-closed subgroups $H_{\alpha} \subseteq G$ (a $k$-group). Then either $\bigcap_{\alpha} g_H = \emptyset$ or it is a coset of some $k$-closed subgroup of $G$.

Proof: Since algebraic varieties are noetherian topological spaces use 1.2.7 and 1.2.8 and induction (on 1.2.9).

$\mathcal{W}_k$-Topology

Using propositions 1.2.8 and 1.2.9 we can now define a topology $\mathcal{W}_k(G)$ on a $k$-group $G$ that will be suitable for the formation of proper inverse limits.
Recall that this means we want a topology for which

i) $G$ is compact and $T_1$

ii) $\phi : G \longrightarrow H$ is closed and continuous, where $\phi$ is $k$-group morphism.

This is best accomplished as follows. Let $G$ be a $k$-group and consider the collection $F = \{ \bigcup_{i=1}^{n} g_i H_i \}$ of all finite unions where $H_i \subseteq G$ is a $k$-closed subgroup and $g_i \in G$ are arbitrary. Then let $W_k(G) = F \cup \{ \emptyset \}$. Propositions 1.2.7, 1.2.8 and 1.2.9 assure us that $W_k(G)$ is a topology on $G$.

Notice that allowing the $g_i$ arbitrary is necessary, since if $g, h \in G(k)$ and $gH \cap hN \neq \emptyset$ for some $k$-closed subgroups $H$ and $N$ of $G$ then $gH \cap hN = x(H \cap N)$ where $x \in G$. But we do not necessarily have $x \in G(k)$. This still yields the right topology on $G(k)$ since for a $k$-closed subgroup $H \subseteq G$ we have $gH \cap G(k) \neq \emptyset$ iff there exists $x \in G(k)$ such that $gH = xH$.

**Proposition 1.2.10:** Let $G, H$ be $k$-groups and $\phi : G \longrightarrow H$ be a $k$-group morphism. Then

i) $G$ is $T_1$ and compact in the $W_k$-topology.

ii) $\phi$ is closed and continuous in the $W_k$-topology.

**Proof:**

i) $G$ is $T_1$ by definition and the $W_k$-topology is weaker than the Zariski $K$-topology.

ii) If $N \subseteq G$ is a $k$-closed subgroup, then $\phi(N) \subseteq H$ is $k$-closed ([4] p. 88). Also $\phi$ preserves cosets and unions. $\quad \text{q.e.d.}$
Remark: In general the $\mathcal{W}_k$-topology is neither weaker nor stronger than the Zariski $k$-topology.

Corollary 1.2.11: Suppose also that $\phi : G \rightarrow H$ is $k$-proper. Then if $\phi_k = \phi|_{G(k)}$ $\phi_k : G(k) \rightarrow H(k)$ is a closed map in the induced topology.

Proof: By definition (of $k$-proper) $\phi(N(k)) = \phi(N)(k)$ for $N \leq G$ a $k$-closed subgroup. Now proceed as in 1.2.10. q.e.d.

Homogeneous Spaces

Let $G$ be a $k$-group and $H \leq G$ a $k$-closed subgroup. Then the $\mathcal{W}_k$-topology on $G/H$ is the quotient topology induced from the $\mathcal{W}_k$-topology on $G$ via the natural map $\pi : G \rightarrow G/H$.

A simple argument then shows that if $\phi : G \rightarrow N$ is a morphism of $k$-groups and $\phi(H) \leq M$ where $H \leq G$ and $M \leq N$ are $k$-closed subgroups then the induced map $\bar{\phi} : G/H \rightarrow N/M$ is closed and continuous.

Similarly if $\phi$ is also $k$-proper then $\bar{\phi}_k : G(k)/H(k) \rightarrow N(k)/M(k)$ is closed and continuous.

Warning: $G(k)/H(k)$ is not necessarily the same as $(G/H)(k)$. (It is if $N$ is $k$-solvable and connected. See 1.1.8). $G(k)/H(k)$ is singled out because it is useful in our applications (Especially section 2.3).
Chapter 2 - Pro-Algebraic Geometry

2.1. Pro-Varieties (Inverse limits of algebraic varieties)

A simple but useful property of algebraic varieties (in fact a defining property) is that the diagonal is closed.

i.e. If \( X \) is a \( k \)-variety then \( \Delta X = \{(x, x) \in X \times X\} \) is \( k \)-closed in the Zariski product topology.

An equally useful fact is that the same is true for the projective limit of algebraic varieties. A pro-algebraic variety is, by definition the inverse limit \( X = \lim \limits_{\leftarrow} X_i \) of an inverse system \( \{X_i, \pi_{ij}\} \) of algebraic varieties [12]. Unless the contrary is specifically allowed \( X = \lim \limits_{\leftarrow} X_i \) will be given the inverse limit topology induced from the Zariski topology on the \( X_i \).

Recall that (for sets) if \( \{X_i, \pi_{ij}\}_{i \in I} \) and \( \{Y_i, \theta_{ij}\}_{i \in I} \) are inverse systems then so is \( \{X_i \times Y_i, \pi_{ij} \times \theta_{ij}\}_{i \in I} \) and

\[
\lim_{\leftarrow} X_i \times Y_i = \lim_{\leftarrow} X_i \times \lim_{\leftarrow} Y_i
\]

Proposition 2.1.1: Let \( X = \lim \limits_{\leftarrow} X_i \) be a pro-algebraic variety. Then

\[
\Delta X = \{(x, y) | x = y\} \subseteq X \times X
\]
is closed (in the Zariski topology).

Proof: \( \Delta X_i \subseteq X_i \times X_i \) is closed for all \( i \), and

\[
\Delta X = \{((x_i), (y_i)) \mid x_i = y_i \ \text{for all} \ i\}
= \bigcap_j \{((x_i), (y_i)) \mid x_j = y_j\}
\]
Corollary 2.1.2: Let \( \phi, \psi : X \to Y \) be morphisms of pro-algebraic varieties (i.e.: \( \phi = \lim \phi_i \) where each \( \phi_i \) is a map of varieties and \( \{\phi_i\} \) is a map of inverse systems). Then \( E(\phi, \psi) = \{x \in X | \phi(x) = \psi(x)\} \subseteq X \) is closed.

Proof: If \( \phi \Pi \psi : X \to Y \times Y \) is the unique map so that

\[
\begin{align*}
&\phi &\quad &\phi \Pi \psi \\
X &\quad &\quad &Y \\
&\psi &\quad &\psi \Pi \phi
\end{align*}
\]

then \( E(\phi, \psi) = (\phi \Pi \psi)^{-1}(\Delta Y) \) which is closed by 2.1.1.

2.2. Pro-\( k \)-groups

We are now in a position to develop the elementary properties of pro-affine \( k \)-groups. There will be occasion to distinguish between a pro-\( k \)-group and a \( k \)-pro-group (2.2.2 and 2.2.3). The necessity of this distinction is illustrated in the following example.

Example 2.2.1: Let \( G_n = K^\times \) for \( 0 < n \in \mathbb{Z} \) where \( K \) is an algebraic closure of \( \mathbb{Q} \), the rational numbers. Then \( G_n \) has the structure of a \( \mathbb{Q} \)-group in the usual way.

Define a direction on \( \mathbb{Z}^+ = \{n \in \mathbb{Z} | n > 0\} \) as, \( m \leq n \) if \( m | n \).

Then for \( n \geq m \) define \( \phi_{n,m} : G_n \to G_m \) as \( \phi_{n,m}(x) = x^{n/m} \). One
easily verifies that \( \{ G_n, \phi_{n,m} \}_{n \in \mathbb{Z}} \) is an inverse system of \( \mathbb{Q} \)-groups.

Let \( G = \lim_{\longleftarrow} G_n \) and consider \( G(Q) = \{ (x_n) \in G | x_n \in \mathbb{Q}^n \text{ for all } n \} \). If \( (x_n) \in G(Q) \) then \( \phi_{mn,n}(x_n) = x_m^n = x_n \). Thus \( x_n \) has \( m \)-th roots in \( \mathbb{Q} \) for all \( m > 0 \). Thus \( x_n = 1 \) and hence \( G(Q) = \{ (1) \} \).

Although such systems are interesting, the limit of this one is sadly lacking in the sort of rationality properties that are enjoyed by algebraic groups. We shall see that this cannot happen to inverse systems of \( k \)-groups where all maps are \( k \)-proper.

**Definition 2.2.2:** A **pro-\( k \)-group** is the projective (= inverse) limit of an inverse system \( \{ G_i, \pi_{ij} \} \) of \( k \)-groups.

A morphism of **pro-\( k \)-groups** is the limit of a morphism of inverse systems \( \{ \phi_i \} : \{ G_i, \pi_{ij} \} \longrightarrow \{ H_i, \theta_{ij} \} \) such that each \( \phi_i \) is a \( k \)-group morphism.

Thus if \( \{ \phi_i \} : \{ G_i, \pi_{ij} \} \longrightarrow \{ H_i, \theta_{ij} \} \) then \( \phi = \lim_{\longleftarrow} \phi_i : \lim_{\longleftarrow} G_i \longrightarrow \lim_{\longleftarrow} H_i \) is a morphism of the pro-\( k \)-groups \( G = \lim_{\longleftarrow} G_i \) and \( H = \lim_{\longleftarrow} H_i \).

**Definition 2.2.3:** A **\( k \)-pro group** is a pro-\( k \)-group \( G = \lim_{\longleftarrow} G_i \) where each \( \pi_{ij} : G_i \longrightarrow G_j \) is \( k \)-proper.

Recall that a map \( \phi : G \longrightarrow H \) of \( k \)-groups is called **\( k \)-proper** if \( \phi(N(k)) = \phi(N)(k) \) for each \( k \)-closed subgroup \( N \) of \( G \).

**Remark:** Since the \( k \)-group \( G_i \) has two natural topologies, the Zariski \( k \)-topology and the \( W_k \)-topology, the same can be said of \( \lim_{\longleftarrow} G_i \). Since both topologies are intrinsic, either may be forgotten for strategic
purposes.

In the next proposition we take advantage of the $\mathcal{W}_k$-topology. Recall that for a $k$-group morphism $\phi : G \rightarrow H$, $\phi$ is closed and continuous in the $\mathcal{W}_k$-topology (see 1.2.10). Thus an inverse system $\{G_i, \pi_{ij}\}$ of $k$-groups satisfies the condition of the Projective Limit Theorem (1.2.1) and so we may apply the results of section 1.2 to pro-$k$-groups.

**Notation:** If $G = \varprojlim G_i$ is a pro-$k$-group let $G^0 = \varprojlim G_i^0 \subseteq G$. $G^0$ is called the **connected component of the identity**. (See 1.1.1 for a definition of $G_i^0$).

**Remark:** By 1.2.6 $G^0 \subseteq G$ is connected in the $\mathcal{W}_k$-topology. It is also easy to prove that $G^0$ contains all other connected pro-$k$-subgroups of $G$.

**Proposition 2.2.4:** Let $G = \varprojlim G_i$ and $H = \varprojlim H_i$ be pro-$k$-groups and let $\phi : G \rightarrow H$ be a morphism of pro-$k$-groups. Then

i) $\phi(G)$ is a pro-$k$-subgroup of $H$

ii) $\phi(G^0) = \phi(G)^0$

**Proof:** i) $\phi(G) = \varprojlim \phi_i(G_i)$ by Lemma 1.2.4.

ii) Clearly $\phi(G^0) \subseteq H^0$ so we can assume $H = H^0$, and by Theorem 1.2.2 we can assume all projections are onto.

Thus

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & \phi(G) \\
\downarrow \pi_j & & \downarrow \theta_j \\
G_j & \xrightarrow{\phi_j} & \phi_j(G_j)
\end{array}
$$
and by 7.4.B of [12] \( \phi_j(G_j^0) = \phi_j(G_j)^0 \).

Thus \( \phi(G^0) = \phi\left(\lim G_j^0\right) \)

\[ = \lim \phi_j(G_j^0) \]

\[ = \lim \phi_j(G_j)^0 \]

\[ = \phi(G)^0 \quad \text{q.e.d.} \]

**Theorem 2.2.5**: (Jordan-Chevalley decomposition)

Let \( G \) be a pro-\( k \)-group. Then for all \( x \in G \) there exists \( x_s, x_u \in G \) (uniquely determined) such that

1) \( x \in G(\kron) \) implies \( x_u, x_s \in G(\kron) \). (By definition \( G(\kron) = \lim G_i(\kron) \subseteq \lim' G_i(\kron) \).)

2) \( x = x_s \cdot x_u = x_u \cdot x_s \)

3) \( \pi_i(x_s) \cdot \pi_i(x_u) = \pi_i(x) \) is the classical Jordan-Chevalley decomposition for all \( i \).

4) For any morphism \( \phi : G \rightarrow H \) of pro-\( k \)-groups \( \phi(x_s^i) = \phi(x_s) \) and \( \phi(x_u^i) = \phi(x_u) \).

**Proof**: Do it coordinatewise using the classical Jordan-Chevalley decomposition (1.1.4).

**2.3. Conjugacy Theorems**

**Preliminaries**: If \( G \) is a \( k \)-group and \( X \) is a \( k \)-variety, then a \( k \)-morphically acting on \( X \) is a morphism \( u : G \times X \rightarrow X \).
of $k$-varieties which is also a group action of $G$ on $X$.

Let $\mu : G \times X \to X$ be a $k$-morphic action and let $X(k) \subseteq X$ be the set of $k$-rational points of $X$. It is then well known (see [12] p. 218) that if $Y \subseteq X(k)$ and $Z \subseteq X$ is $k$-closed, then

$$\text{Tran}(Y, Z) = \{x \in G | \mu(x, Y) \subseteq Z\}$$

is $k$-closed in $G$ (in the Zariski $k$-topology). Thus, if $\mu : G \times G \to G$ is defined as $\mu(g, x) = g \cdot x \cdot g^{-1}$, and $T, T'$ are connected $k$-closed conjugate subgroups of $G$, then $\text{Tran}(T(k), T)$ is $k$-closed in $G$. Since $T(k) \subseteq T$ is dense in the Zariski $k$-topology $\text{Tran}(T(k), T') = \text{Tran}(T, T')$.

If also $\text{Tran}(T, T') \cap G(k) \neq \emptyset$ then $\text{Tran}(T, T')$ is a $G(k)$ coset of $N_G(T)$ the normalizer of $T$ in $G$. Hence $\text{Tran}(T, T')$ is closed in the $k$-topology of $G$.

**Theorem 2.3.1**: Let $G = \lim_{i \to \infty} G_i$ be a connected pro-$k$-group (i.e.: $G = G^0$) where $\pi_{ij} : G_i \to G_j$ is onto for $i \geq j$. Let $B_j \neq \emptyset$ be a class of connected $k$-closed subgroups of $G_j$ such that,

- i) If $i \geq j$ and $B_i \in B_i$ then $\pi_{ij}(B_i) \in B_j$.
- ii) $B_i, B'_i \in B_i$ implies there exists $g \in G_i$ such that $gB_i g^{-1} = B'_i$.
- iii) $B_i$ is maximal with respect to condition ii).

Then there exists a non-empty set $B$ such that $B$ consists of pro-algebraic $k$-subgroups of $G$ which satisfy ii) and iii). Furthermore, each $\pi_i : G \to G_i$ induces $\pi_i(B) \to B_i$. 

Proof: By ii) we can define $\overline{\pi}_{ij} : B_i \rightarrow B_j$ as

$$\overline{\pi}_{ij}(B_i) = \pi_{ij}(B_i)$$

Furthermore $\overline{\pi}_{ij}$ is onto (essentially because each $\pi_{ij}$ is onto).

Fix some $i$ and choose $B_i \in B_i$. It is easily verified that

$$B_i \cong G_i/N_{G_i}(B_i).$$

Since $G_i/N_{G_i}(B_i)$ is a homogeneous space for $G_i$ we can put a topology on $B_i$ via this identification. (This is the $\mathcal{W}_k$-topology on $G_i/N_{G_i}(B_i)$ discussed at the end of chapter 1).

If $i \geq j$ and $B_j = \pi_{ij}(B_i)$. Then

$$\begin{array}{ccc}
B_i & \xrightarrow{\overline{\pi}_{ij}} & B_j \\
\downarrow \cong & & \downarrow \cong \\
G_i/N_{G_i}(B_i) & \xrightarrow{\pi_{ij}} & G_j/N_{G_j}(B_j)
\end{array}$$

commutes and thus $\overline{\pi}_{ij}$ is a closed, continuous map with our identification. By homogeneity this topology does not depend on our choice of $B_i$. The net result is that $\{B_i, \overline{\pi}_{ij}\}$ is a proper system (it satisfies the assumptions of Theorem 1.2.1). Thus by Theorem 1.2.1 $\overline{B} = \lim_{+} B_i \neq \emptyset$ and in fact $\overline{\pi}_{i : \overline{B} \rightarrow B_i}$ is onto for each $i$ by Theorem 1.2.2.

By definition of $\overline{B}$, $B \in \overline{B}$ implies that $B = \{B_i | \pi_{ij}(B_i) = B_j\}$. Thus let $\mathfrak{B} = \{B = \lim_{+} B_i | \{B_i\} \in \overline{B}\}$. Then by definition $\mathfrak{B}$ consists of
pro-algebraic $k$-subgroups of $G$.

To complete the proof we must show that $B$ satisfies ii) and iii) above. Let $B = \lim_{+} B_{i}, B' = \lim_{+} B'_{i} \in \mathcal{B}$. Then $T_{i} = \text{Tran}(B_{i}, B'_{i}) \neq \emptyset$ is $W$-closed since it is a coset of $N_{G_{i}}(B_{i})$. Also $\pi_{ij}(T_{i}) \subseteq T_{j}$.

Thus $\{T_{i}, \pi_{ij}\}$ satisfies the assumptions of Theorem 1.2.1, so that $T = \lim_{+} T_{i} \neq \emptyset$. But by definition of $T$, $x = \{x_{i}\} \in T$ means that $x_{i} B_{i} x_{i}^{-1} = B'_{i}$. Hence $x B x^{-1} = B'$.

To prove that $B$ satisfies iii) let $B = \lim_{+} B_{i} \in \mathcal{B}$, $x = \{x_{i}\} \in G$. Then $x B x^{-1} \in \mathcal{B}$ because from Theorem 1.2.2 $x B x^{-1} = \lim_{+} x_{i} B_{i} x_{i}^{-1}$ and by definition $x_{i} B_{i} x_{i}^{-1} \in B'_{i}$.

q.e.d.

Remark: If $\{G_{i}, \pi_{ij}\}$ is an inverse system of $K$-groups (i.e. $k = K$) then the assumptions of Theorem 2.3.1 are satisfied for the following three classes of subgroups.

i) $T_{i} = \{T_{i} \subseteq G_{i} | T_{i}$ is a maximal torus$\}$.

ii) $U_{i} = \{U_{i} \subseteq G_{i} | U_{i}$ is a maximal connected unipotent subgroup$\}$.

iii) $B_{i} = \{B_{i} \subseteq G_{i} | B_{i}$ is a Borel subgroup$\}$.


For the applications we have in mind $k$ is not algebraically closed. Consequently, the results of Theorem 2.3.1 have to be suitably refined to yield conjugacy theorems for $G(k) = \lim_{+} G_{i}(k)$.

Theorem 2.3.2: If $G = \lim_{+} G_{i}$ is a connected $k$-pro-group (i.e. each $\pi_{ij} : G_{i} \longrightarrow G_{j}$ is $k$-proper. See Definition 2.2.3), let $\mathcal{B}_{j} \neq \emptyset$ be a class of connected $k$-subgroups of $G_{i}$ such that
i) If $i > j$ and $B_i \in \mathcal{B}_i$ then $\pi_{ij}(B_i) \in \mathcal{B}_j$.

ii) If $B_i, B'_i \in \mathcal{B}_i$ then there exists $g \in G_i(k)$ such that $gB_ig^{-1} = B'_i$.

iii) $\mathcal{B}_i$ is maximal with respect to ii).

Then there exists a non-empty set $\mathcal{B}$ consisting of pro-$k$-subgroups of $G$ such that conditions ii) and iii) are satisfied for $\mathcal{B}$. Furthermore each $\pi_i : G \longrightarrow G_i$ induces $\overline{\pi_i} : \mathcal{B} \longrightarrow \mathcal{B}_i$.

**Proof:** Let $\mathcal{B}_i(k) = \{B_i(k) | B_i \in \mathcal{B}_i\}$. Then $\mathcal{B}_i(k) \simeq \mathcal{B}_i$.

i.e. Define $r : \mathcal{B}_i \longrightarrow \mathcal{B}_i(k)$

$$r(B_i) = B_i(k)$$

and $c : \mathcal{B}_i(k) \longrightarrow \mathcal{B}_i$

$$c(B_i(k)) = \overline{B}_i(k)$$

(where $\overline{B}_i(k)$ denotes closure in the Zariski $k$-topology).

Then $r$ and $c$ are inverses because $B_i(k) \subseteq B_i$ is dense in the Zariski $k$-topology (1.1.3). But just as in the proof of Theorem 2.3.1

$$\mathcal{B}_i(k) \simeq G_i(k)/N_{G_i(k)}(B_i(k))$$

Hence $\mathcal{B}_i(k)$ and $\mathcal{B}_i$ can be given the topology induced by the $\mathcal{W}_k$-topology on $G_i(k)/N_{G_i(k)}(B_i(k))$. Since by assumption $\pi_{ij} : G_i(k) \longrightarrow G_j(k)$ is closed and continuous $\pi_{ij} : \mathcal{B}_i \longrightarrow \mathcal{B}_j$ is closed and continuous. Thus the proof proceeds exactly as in Theorem 2.3.1. q.e.d.
We specialize these results to maximal $k$-split tori (see the definitions preceding 1.1.5).

**Lemma 2.3.3**: Let $\phi : G \to H$ be an epimorphism of $k$-groups. If $T_d \subseteq G$ is a maximal $k$-split torus then $\phi(T_d) \subseteq H$ is a maximal $k$-split torus.

**Proof**: Let $T_d \subseteq T$ where $T$ is a maximal torus defined over $k$. Then by 34.3 of [12], $T = T_d \cdot T_a$ where $T_a$ is a $k$-anisotropic torus defined over $k$. Furthermore $\phi(T) = \phi(T_d) \cdot \phi(T_a)$ is a maximal torus of $H$ (2.13C of [12]), $\phi(T_d)$ is $k$-split and $\phi(T_a)$ is $k$-anisotropic (p. 219 of [4]). But such a decomposition is unique. q.e.d.

**Remark**: By Lemma 2.3.3 and 1.1.6 the assumptions of Theorem 2.3.2 are satisfied for $T_k(G_i) = \{T_i \subseteq G_i \mid T_i \text{ is a maximal } k\text{-split torus}\}$. Thus we make the following definition.

**Definition**: Let $\{G_i, \pi_{ij} \mid i \in I \}$ be an inverse system of $k$-groups where each $\pi_{ij}$ is $k$-proper (2.2.3). If $T_i \subseteq G_i$ is a $k$-split torus for each $i \in I$ such that $\pi_{ij}(T_i) \subseteq T_j$ whenever $i > j$, then $T = \lim_{i \leftarrow} T_i \subseteq G = \lim_{i \leftarrow} G_i$ is called a $k$-split pro-torus.

If further, each $T_i$ is a maximal $k$-split torus, $T = \lim_{i \leftarrow} T_i$ is called a maximal $k$-split pro-torus.

For further reference we state the following special case of Theorem 2.3.2.

**Theorem 2.3.4**: Let $G = \lim_{i \leftarrow} G_i$ be a $k$-pro-group (2.2.3). Then $G$ has a maximal $k$-split pro-torus $T = \lim_{i \leftarrow} T_i$. If $S = \lim_{i \leftarrow} S_i$ is another
maximal k-split pro-torus then there exists \( g \in G(k) = \lim_{\to} G(k) \) such that \( gSg^{-1} = T \).

**Proof:** This is a simple application of Theorem 2.3.2 to the remark following Lemma 2.3.3. q.e.d.

### 2.4. Imbedded k-pro-groups

In the discussion of pro-algebraic symmetry of minimal algebras (Chapter 3) the topological relation between automorphisms and endomorphisms becomes an important focal point in structural classification. With the help of Theorem 2.3.4 this can yield interesting categorical information (3.6.2).

**Example 2.4.1:** Let \( M(n,K) \) be the set of \( n \times n \) matrices over the algebraically closed field \( K \) and let \( GL(n,K), SL(n,K) \subseteq M(n,K) \) be respectively, the invertible matrices and the matrices of determinant 1.

In the Zariski \( K \)-topology \( \overline{GL(n)} = M(n) \) whereas \( SL(n) \subseteq M(n) \) is closed. In the first case conjugacy properties of \( GL(n,K) \) yield information about \( M(n) \) whereas conjugacy properties of \( SL(n) \) are apriori indifferent to \( M(n) \).

A useful way to get at some of these closure properties is with 1-parameter subgroups.

**1-parameter subgroups**

**Definition:** a) An **imbedded pro-k-group** is the inverse limit \( X = \lim_{\to} X_i \) of affine \( k \)-varieties \( X_i \) such that
i) For each $i$ there exists $G_i \subseteq X_i$ an open affine embedding where $G_i$ is a $k$-group.

ii) $\pi_{ij}|_{G_i}$ is a $k$-group morphism.

b) If in addition each $X_i$ has the structure of an algebraic semi-group consistent with the group structure on $G_i$ and each $\pi_{ij}$ is a semi-group morphism then $X = \lim X_i$ is called a pro-affine $k$-semi-group.

Definition: a) Let $X$ be an imbedded pro-$k$-group. Then a 1-parameter subgroup is a morphism $\lambda : K^* \rightarrow G = \lim G_i \subseteq X$ of pro-$k$-groups.

i.e.: $\lambda = \lim_{i \downarrow} \lambda_i$ where

i) $\lambda_i : K^* \rightarrow G_i \subseteq X_i$ is a $k$-group morphism.

ii)

\[
\begin{array}{ccc}
K^* & \xrightarrow{\lambda_i} & G_i \\
\downarrow{\lambda_j} & & \downarrow{\pi_{ij}} \\
\end{array}
\]

commutes for $i \geq j$

b) A 1-parameter subgroup $\lambda = \lim_{i \downarrow} \lambda_i$ is said to converge if for all $i$

$\lambda_i : K^* \rightarrow G_i$ extends to a morphism

$\overline{\lambda_i} : K \rightarrow X_i$ such that

i)

\[
\begin{array}{ccc}
\lambda_i & \rightarrow & X_i \\
\downarrow{\lambda_i} & & \downarrow{\overline{\lambda_i}} \\
K^* & \rightarrow & G_i \\
\end{array}
\]

commutes.
and ii)

\[ \lambda_i \xrightarrow{\pi_{ij}} X_i \]
\[ \lambda_j \xrightarrow{} X_j \]

commutes.

Notation: i) \( \lambda = \lim \lambda_i \)

ii) \( \lambda(0) = \lim_{t \to 0} \lambda(t) \)

c) If \( \lambda : K \to G \subseteq X \) is a 1-parameter subgroup let

\( \lambda^{-1} = \lambda \circ i : K \to G \subseteq X \) where \( i : K \to K \) is the inverse map.

If neither \( \lambda \) nor \( \lambda^{-1} \) converge \( \lambda \) is said to diverge.

Remark: The notation \( \lambda(0) = \lim_{t \to 0} \lambda(t) \) adopted in part b) makes sense because if \( \lambda_i : K \to G_i \subseteq X \) extends to \( \lambda_i : K \to X_i \) then it extends uniquely. This is an elementary consequence of Corollary 2.1.2.

Lemma 2.4.2: Let \( H \subseteq X \) be a commutative sub-semi-group of \( X \), a pro-affine \( k \)-semi-group. Then \( H \subseteq X \) is commutative.

Proof: Let \( y \in H \) and define

\[ f_y : X \to X \times X \text{ as} \]

\[ f_y(x) = (xy, yx) \]

Then \( H \subseteq f_y^{-1}(\Delta X) = \{ x \in X | xy = yx \} \) which is closed (Proposition 2.1.1).

Thus \( H \subseteq \bigcap_{y \in H} f_y^{-1}(\Delta X) \) for all \( y \in H \) and so

\( \overline{H} \subseteq \bigcap_{y \in H} f_y^{-1}(\Delta X) = \{ x \in X | xy = yx \} \)

for all \( y \in H \).
for \( y \in H \} = Z \\

Repeating the above argument with \( Z \) in place of \( H \) implies that \( \overline{H} \subseteq \{ x \in X | xz = zx \text{ for } z \in Z \} \). But then \( \overline{H} \) is commutative.

q.e.d.

Theorem 2.4.3: a) Let \( X = \lim_{\rightarrow} X_i \) be a pro-affine \( k \)-semi-group. If \( \lambda, \omega : K^* \longrightarrow G \subseteq X \) are 1-parameter subgroups which converge to \( \overline{\lambda}(0) \) and \( \overline{\omega}(0) \) respectively, then there exists \( g \in G \) such that \( g \cdot \overline{\lambda}(0) \cdot g^{-1} \) and \( \overline{\omega}(0) \) commute.

b) If further the induced pro-\( k \)-group \( G = \lim_{\rightarrow} G_i \) is a \( k \)-pro-group (Definition 2.2.3) then there exists \( g \in G(k) = \lim_{\rightarrow} G_i(k) \) such that \( g \cdot \overline{\lambda}(0) \cdot g^{-1} \) and \( \overline{\omega}(0) \) commute.

Proof: a) \( \overline{\lambda}(K) \subseteq \lambda(K^*) \subseteq S \) and \( \overline{\omega}(K) \subseteq \omega(K^*) \subseteq T \) where \( S \) and \( T \) are some maximal \( k \)-split pro-tori. Furthermore \( \overline{S} \) and \( \overline{T} \) are commutative (Lemma 2.4.2) and conjugate (Theorem 2.3.1). Thus a) is proved.

b) If \( G = \lim_{\rightarrow} G_i \) is also a \( k \)-pro-group Theorem 2.3.2 applies as well and b) is proved.

q.e.d.
Chapter 3 - Automorphism Groups of Minimal Algebras

In this chapter a natural application of chapter 2 is achieved by examining the automorphism group of a minimal algebra.

3.1. Preliminaries

Let $A$ be a finitely generated connected graded $k$-algebra, and let $A_K = A \otimes_K K$ where $K$ is an algebraic closure of $k$. Then it is well known that

$$\text{Aut}(A_K) = \{ f : A_K \to A_K | f \text{ is an invertible } K\text{-algebra homomorphism} \}$$

is an imbedded affine $k$-group.

I.e. $G(A_K) = \text{Aut}(A_K) \subseteq \text{End}(A_K) = E(A_K)$ is a principal open subset of $E(A_K)$ where $E(A_K)$ is naturally an affine $k$-variety.

If $d : A \to A$ is a degree +1 linear map and

$$E(A_K, d) = \{ f \in E(A_K) | f \circ d = d \circ f \},$$

$$G(A_K, d) = G(A_K) \cap E(A_K, d)$$

the above conclusions hold for $G(A_K, d) \subseteq E(A_K, d)$ as well.

3.2. Minimal Algebras

Let $M$ be a 1-connected minimal algebra of finite type. Recall
that this means,

i) $M$ is a differential graded algebra over $\mathbb{Q}$.

ii) $M^0 = \mathbb{Q}$, $M^1 = 0$

iii) $M$ is free as a graded commutative algebra.

iv) $d(M) \leq M^+ \cdot M^+$ where $M^+ = \oplus_{n>0} M^n$.

v) $\dim \mathbb{Q} M^n < \infty$ for all $n \geq 0$.

(See [8] or [10] for details).

In many interesting cases such an algebra is not finitely generated so one cannot apply the preliminary remarks of this chapter directly. But in the light of chapter 2 one can do just as well.

For a minimal algebra $M$ define $M_n \subseteq M$ as the subalgebra generated by $\bigoplus_{k<n} M^k$. Clearly $M_n$ is a finitely generated minimal algebra and $M$ can be written as

$$\mathbb{Q} = M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \bigcup_{n>0} M_n = M$$

where each level of the filtration is canonical and functorial (see [8] or [10]).

Thus 3.1 applies to

$$G_n = \text{Aut}(M \otimes \mathbb{K}) \subseteq \text{End}(M \otimes \mathbb{K}) = E_n$$

i.e. $G_n$ is a principal open affine imbedded $\mathbb{Q}$-group in the $\mathbb{Q}$-affine semi-group $E_n$.

By functoriality of the above filtration we have the following commutative diagram of $\mathbb{Q}$-varieties
3.3. \textit{Q}-structures on minimal algebras

For the purpose of studying algebraic groups over \( \mathbb{Q} \), the rational numbers, it is customary (and convenient) to work over \( \overline{\mathbb{Q}} \), an algebraic closure of \( \mathbb{Q} \) and keep track of the rationality properties.

In 3.2 minimal algebras were defined as a special kind of \( \mathbb{Q} \)-algebra. The obvious modification of this definition yields minimal algebras over any field of characteristic 0.

\textbf{Definition:} a) Let \( \overline{M} \) be a minimal algebra over \( \overline{\mathbb{Q}} \) (an algebraic closure of \( \mathbb{Q} \)). A \textit{\( \mathbb{Q} \)-structure on} \( \overline{M} \) consists of a minimal algebra \( M \).
over \( \mathbb{Q} \), such that

1) \( M \leq \overline{M} \)

2) \( K \otimes M \xrightarrow{k} \overline{M} \) is a d.g.a. isomorphism (where \( d(\alpha \otimes x) = \alpha \otimes dx \))

b) A \( \mathbb{Q} \)-morphism \( f : \overline{M} \xrightarrow{} \overline{N} \) of minimal algebras with \( \mathbb{Q} \)-structure is a d.g.a. morphism \( f \) such that \( f(M) \leq N \).

The set of \( \mathbb{Q} \)-endomorphisms of \( \overline{M} \) is denoted \( \text{End}(\overline{M})(\mathbb{Q}) \) and the invertible ones \( \text{Aut}(\overline{M})(\mathbb{Q}) \). Clearly, the diagram

\[
\begin{array}{ccc}
\text{End}(M) & \xrightarrow{1 \otimes 0} & \text{End}(\overline{M}) \\
\downarrow & & \downarrow \\
\text{Aut}(M) & \xrightarrow{1 \otimes 0} & \text{Aut}(\overline{M})
\end{array}
\]

identifies \( \text{Aut}(M) \) and \( \text{End}(M) \) with \( \text{Aut}(\overline{M})(\mathbb{Q}) \) and \( \text{End}(\overline{M})(\mathbb{Q}) \) respectively. For notational convenience we let

\[
\begin{align*}
E_n &= \text{End}(\overline{M}_n), \quad E_n(\mathbb{Q}) = \text{End}(\overline{M}_n)(\mathbb{Q}) \\
G_n &= \text{Aut}(\overline{M}_n), \quad G_n(\mathbb{Q}) = \text{Aut}(\overline{M}_n)(\mathbb{Q})
\end{align*}
\]

Then \( G_n \) is a \( \mathbb{Q} \)-group and \( G_n(\mathbb{Q}) \) its subset of \( \mathbb{Q} \)-rational points.

In what follows we shall always adhere to the following notation.

\( \overline{M} \) is a minimal algebra over \( K \) (the algebraic closure of \( \mathbb{Q} \)) with \( \mathbb{Q} \)-structure \( M \leq \overline{M} \)

\[
\overline{M}^n = \{ x \in \overline{M} \mid \text{degree } x = n \}
\]

\( \overline{M}_n = \) the subalgebra of \( \overline{M} \) generated by \( \otimes \overline{M}^k \) for \( k \leq n \).
$M^+ = \bigoplus_{K \geq 1} \overline{M^K}$

$D(\overline{M}) = \mu(\overline{M^+} \otimes \overline{M^+})$ where $\mu: \overline{M} \otimes \overline{M} \rightarrow \overline{M}$ is the algebra structure.

$Q(\overline{M}) = \overline{M^+} / D(\overline{M})$

$Z(\overline{M}) = \{ x \in \overline{M} \mid dx = 0 \}$

$B(\overline{M}) = \{ x \in \overline{M} \mid x = dy \text{ for some } y \in \overline{M} \}$

$H(\overline{M}) = Z(\overline{M}) / B(\overline{M})$

$S(\overline{M}) = Z(\overline{M}) / (Z(\overline{M}) \cap D(\overline{M}))$

Each of the above is naturally graded, $Z(\overline{M}) \subseteq \overline{M}$ is a subalgebra and $B(\overline{M}) \subseteq Z(\overline{M})$ is an ideal.

If $\overline{V}$ is any vector space over $K$ a $Q$-structure on $\overline{V}$ is a $Q$-vector space $\overline{V} \subseteq \overline{V}$ such that $K \otimes V \rightarrow \overline{V}$ is an isomorphism.

**Proposition 3.3.1:** Let $\overline{M} \subseteq \overline{M}$ be minimal algebra with $Q$-structure. Then $\overline{M} \subseteq \overline{M}$ induces $Q$-structures on $\overline{M^n}$, $\overline{M^n}$, $D(\overline{M})$, $Q(\overline{M})$, $Z(\overline{M})$, $B(\overline{M})$, $H(\overline{M})$ and $S(\overline{M})$.

Thus for example $\overline{M^n} \subseteq \overline{M^n}$ is a $Q$-structure on $\overline{M^n}$ in the above sense.

**Proof:** The proof is entirely elementary and does not depend on the properties of minimal algebras. As a sample let $\overline{M} = \oplus M^n$. Then

$K \otimes (\oplus M^n) \rightarrow \overline{M} = \oplus \overline{M^n}$ and $K \otimes (\oplus M^n) \rightarrow \oplus (K \otimes M^n)$.

Thus $K \otimes M^n \rightarrow \overline{M^n}$.

q.e.d.
We now prove a simple but critical lemma that allows us to apply the results of chapter 2 to a sufficiently large class of minimal algebras. Recall that for a minimal algebra $M$, $G_n = \text{Aut}(M_n)$, and $r_{n,n-1} : G_n \rightarrow G_{n-1}$ is the restriction of maps:

**Lemma 3.3.2**: Let $\overline{M}$ be a minimal algebra with $Q$-structure $M \subseteq \overline{M}$. Suppose that $S^n(\overline{M}) = Z^n(\overline{M})/(Z^n(\overline{M}) \cap D(M)) = 0$. Then $r_{n,n-1} : G_n \rightarrow G_{n-1}$ is $Q$-proper (see definition 2.2.3).

**Proof**: Let $K_n = \ker r_{n,n-1}$. Then $K_n = \{ f \in \text{Aut}(M_n) | f|_{M_{n-1}} = 1_{M_{n-1}} \}$. Since $S^n(\overline{M}) = 0$, $f \in K_n$ implies that

$$f|_{\overline{M}_n} = 1|_{\overline{M}_n} + \eta$$

where

$$\eta(\overline{M}_n) \subseteq D^n(\overline{M})$$

Thus $f$ is unipotent and consequently $K_n$ is a unipotent subgroup of $G_n$. By Theorem 1.1.10 $r_{n,n-1}$ is $Q$-proper. q.e.d.

Thus we can now properly examine the inverse limit of $\{G_n, r_{n,m}\}$ introduced at the beginning of this chapter. Recalling notation we have

$$G_n = \text{Aut}(\overline{M}_n)$$

and $r_{n,m} : G_n \rightarrow G_m$ is the restriction of maps. Let

$$G = \lim_{\leftarrow} G_n \quad \text{and} \quad G(Q) = \lim_{\leftarrow} G(Q)_n.$$

**Theorem 3.3.3**: Let $\overline{M}$ be a minimal algebra with $Q$-structure $M \subseteq \overline{M}$. If $\dim_K S(\overline{M}) < \infty$ then $G = \lim_{\leftarrow} G_n$ is a $Q$-pro-group (Definition 2.2.3). Thus any two maximal $Q$-split pro-tori $S$ and $T$ are conjugate under $G(Q)$.

**Proof**: By lemma 3.3.2 all maps $r_{n,m} : G_n \rightarrow G_m$ are $Q$-proper for
n,m > N, where N is chosen so that \( S^k(M) = 0 \) for \( k > N \). Thus Theorem 2.3.2 applies and the theorem is proved. q.e.d.

Remark: There are other classes of pro-algebraic subgroups of \( G \) which satisfy the conclusion of Theorem 3.3.3. Our interest in toroidal symmetry is prompted by its usefulness, where applicable, in structural analysis.

3.4. Weight Splittings of Minimal Algebras

Definition: Let \( \overline{M} \) be a minimal algebra with \( Q \)-structure \( M \subseteq \overline{M} \). An \( n \)-weight splitting of \( \overline{M} \) is a direct sum decomposition \( M = \bigoplus_{\alpha \in \mathbb{Z}^n} a M \) (over \( Q \)) such that

1. \( M^m = \bigoplus_{\alpha \in \mathbb{Z}^n} (\alpha M \cap M^m) \)
2. \( d(\alpha M) \subseteq \alpha M \)
3. \( \alpha M \cap M \subseteq \alpha + \beta M \)

(where \( \alpha M \cap M = \mu(\alpha M \otimes M) \))

Clearly such a decomposition \( M = \bigoplus_{\alpha \in \mathbb{Z}^n} a M \) induces \( \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}^n} a \overline{M} \)

where \( \overline{M} = M \otimes K \).

In general a minimal algebra \( \overline{M} \) with \( Q \)-structure \( M \subseteq \overline{M} \) may possess many \( n \)-weight splittings (for various \( n \)). The following theorem implies that weight splittings and \( (Q\text{-split}) \) toroidal symmetry are essentially the same. Recall that \( K^* \) is the set of non-zero algebraic
numbers and \( G = \text{Aut}(\overline{M}) \) where \( \overline{M} \) is a minimal algebra with \( Q \)-structure \( M \subseteq \overline{M} \).

**Theorem 3.4.1**: Each \( n \)-weight splitting \( \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}^n} \alpha \overline{M} \) determines a \( Q \)-group morphism \( \phi : K^* \times \ldots \times K^* \longrightarrow \text{Aut}(\overline{M}) \) and conversely.

**Proof**: If \( \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}^n} \alpha \overline{M} \) is an \( n \)-weight splitting define

\[ \phi : K^* \times \ldots \times K^* \longrightarrow \text{Aut}(\overline{M}) \]

as follows. Let \( t = (t_1, \ldots, t_n) \in K^* \times \ldots \times K^* \) and \( x \in \alpha \overline{M} \) where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \). Then \( \phi(t)(x) = t_1^\alpha_1 \cdots t_n^\alpha_n \cdot x \).

\( \phi(t) \) is uniquely determined by these conditions since

\[ \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}^n} \alpha \overline{M} \quad \text{and} \quad \phi(t) \circ d = d \circ \phi(t) \quad \text{since} \quad d(\alpha \overline{M}) \subseteq \alpha \overline{M} \]

and

\[ \phi(t)(x \cdot y) = \phi(t)(x) \cdot \phi(t)(y) \quad \text{since} \quad \alpha \overline{M} \cdot \beta \overline{M} \subseteq \alpha + \beta \overline{M} \].

Conversely, if \( \phi : K^* \times \ldots \times K^* \longrightarrow \text{Aut}(\overline{M}) \) is a \( Q \)-group morphism let \( T = \phi(K^* \times \ldots \times K^*) \) and consider \( X(T) = \text{Hom}(T, K^*) \). It is well known that \( X(T) \cong \mathbb{Z}^n \) for some \( n \) (see [4] p. 205). For \( \alpha \in X(T) \) let \( \alpha \overline{M} = \{ x \in \overline{M} \mid \phi(t)(x) = \alpha(\phi(t)) \cdot x \quad \text{for all} \quad \phi(t) \in T \} \).

Clearly \( \overline{M} = \bigoplus_{\alpha \in X(T)} \alpha \overline{M} \) since the \( \alpha \overline{M} \) are just the various eigenspaces of \( T \) which is diagonalizable.

**Proposition 3.4.2**: Let \( \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}^n} \alpha \overline{M} \) be an \( n \)-weight splitting and let \( \phi : K^* \times \ldots \times K^* \longrightarrow \text{Aut}(\overline{M}) \) be the corresponding \( Q \)-group morphism (as in the proof of 3.4.1). Then this induces the following direct sum
decompositions. \( \overline{M}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \alpha \overline{M} \), \( B(\overline{M}) = \bigoplus_{\alpha} B(\overline{M}) \), \( Z(\overline{M}) = \bigoplus_{\alpha} Z(\overline{M}) \), \( H(\overline{M}) = \bigoplus_{\alpha} H(\overline{M}) \), \( Q(\overline{M}) = \bigoplus_{\alpha} Q(\overline{M}) \), \( S(\overline{M}) = \bigoplus_{\alpha} S(\overline{M}) \).

Furthermore, all eigenvalues of \( \phi(K^* \times \ldots \times K^*) \) are generated multiplicatively on \( S(\overline{M}) \).

Proof: Let \( T = \phi(K^* \times \ldots \times K^*) \). Each of the above spaces (\( \overline{M}_n \), \( B(\overline{M}) \), etc.) are functorial and the induced maps on automorphism groups are maps of \( \mathbb{Q} \)-groups. For example \( B : \text{Aut}(\overline{M}) \longrightarrow \text{Aut}(B(\overline{M})) \) is a \( \mathbb{Q} \)-group morphism. But then \( B(T) \) is a \( \mathbb{Q} \)-split torus (see [4] p.219) and thus diagonalizable over \( \mathbb{Q} \). Hence we can write \( B(\overline{M}) = \bigoplus_{\alpha} B(\overline{M}) \).

To prove the second part we observe that \( Q(\overline{M}) = S(\overline{M}) \oplus h(\overline{M}) \) (unnaturally). Inductively \( \overline{x} \in h^n(\overline{M}) \) implies that \( 0 \neq \alpha \overline{x} \in M_{n-1} \) (where \( \overline{x} \) represents \( \overline{x} \in h^n(\overline{M}) \)) so that the condition \( d(\alpha \overline{M}) \subseteq \alpha \overline{M} \) implies that the eigenvalues corresponding to \( h^n(\overline{M}) \) are determined by \( M_{n-1} \).

q.e.d.

3.5. Types of Weight Splittings

We now single out some of the most important types of weight splittings.

Definition: Let \( \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}^n} \alpha \overline{M} \) be an \( n \)-weight splitting of \( \overline{M} \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

i) If \( \alpha^m(\overline{M}) = 0 \) whenever \( \sum_{i=1}^{n} \alpha_i \neq m \) then \( \bigoplus_{\alpha \in \mathbb{Z}^n} \alpha \overline{M} \) is called a
homology diagonal weight splitting.

ii) If \( \delta \bar{M} = 0 \) for \( \alpha_i < 0 \) and \( \delta \bar{M} = \bar{M}^0 \) then \( \Phi \sum_{\alpha \in \mathbb{Z}} \alpha \bar{M} \) is called a positive weight splitting.

iii) If \( \delta \bar{M} = 0 \) for \( \alpha_i < 0 \) then \( \Phi \sum_{\alpha \in \mathbb{Z}} \alpha \bar{M} \) is called a non-negative weight splitting.

iv) If \( \delta \bar{M} = 0 \) for \( \alpha i \neq 0 \) then \( \Phi \sum_{\alpha \in \mathbb{Z}} \alpha \bar{M} \) is called the trivial weight splitting.

Remark: i) In [17] it is shown that a minimal algebra \( \bar{M} \) has a homology diagonal weight splitting if and only if there exists a d.g.a. map \( \phi : \bar{M} \to H(\bar{M}) \) such that

\[
\begin{array}{ccc}
\bar{M} & \xrightarrow{\phi} & H(\bar{M}) \\
\downarrow{\iota} & & \downarrow{\pi} \\
\mathbb{Z}(\bar{M}) & \xrightarrow{\pi} & H(\bar{M})
\end{array}
\]

commutes.

A topological space \( X \) is called formal if its minimal algebra \( M(X) \) satisfies the above property (since in such cases \( M(X) \) is a formal consequence of \( H^*(X;\mathbb{Q}) \)).

Easily verified examples of formal spaces include spheres and Eilenberg-Maclane spaces. Using the classical Hodge theory and other techniques it is shown in [7] that Kähler manifolds are formal.

ii) Morgan [15] proves that open smooth complex varieties have positive weight rational homotopy type (i.e., The associated minimal
algebra has a positive weight decomposition). Actually he proves more than this so a brief summary is in order.

By the work of Hironaka [11] any non-singular open complex variety $X$ can be written $X = V - \cup D_i$ where $V$ is smooth and projective and $\{D_i\}$ are smooth divisors of $V$ with normal crossings. Generalizing Hodge theory Deligne [5], [6] obtains mixed Hodge structures for the de Rham cohomology of such varieties. By carrying this construction through, first to the differential forms and then to the minimal algebra Morgan obtains his result. He also observes that the rational homotopy (minimal algebra) of an open smooth variety is a "formal consequence" of the cohomological properties of its Hironaka resolution.

Proposition 3.5.1: Let $\overline{M} = \sum_{\alpha \in \mathbb{Z}} \alpha^M$ be a non-negative $n$-weight splitting. Then $O^\overline{M}$ is a minimal algebra with $Q$-structure $O^M \leq O^\overline{M}$.

Proof: The conditions $O^M \wedge O^M \leq O^M$ and $d(O^M) \leq O^M$ imply that $O^M$ is a d.g.a. and clearly $K \otimes O^\overline{M} \rightarrow O^\overline{M}$ is an isomorphism.

Since the weight splitting is non-negative $\overline{M} = \sum_{\alpha \neq 0} \alpha^\overline{M}$ is a differential graded ideal and clearly $O^\overline{M} = M/\overline{M}$.

Thus $O^\overline{M} = M/\overline{M}$, and so $O^\overline{M}$ is a retract of $\overline{M}$. From this it is easy to deduce that

$$Q(O^\overline{M}) = O^\overline{M} \quad \text{and} \quad D(O^\overline{M}) = D(\overline{M}) \cap O^\overline{M}.$$ 

Thus $O^\overline{M}$ is minimal. q.e.d.
3.6. 1-parameter subgroups and Weight Splittings

Notation: Let $\bar{M}$ be a minimal algebra with $Q$-structure $M \subseteq \bar{M}$ such that $\dim_K S(\bar{M}) < \infty$. If $\bar{M}$ has a positive 1-weight splitting $\bar{M} = \bigoplus_{\alpha \in \mathbb{Z}} a^\alpha \bar{M}$, we say $\bar{M}$ has positive weights. The category of all such algebras will be denoted $P$. By standard abuse of notation we write $\bar{M} \in P$ if $\bar{M}$ has positive weights.

At the other extreme we let $N$ be the subcategory of all minimal algebras $\bar{M}$ (with the above finiteness condition) which have no non-trivial non-negative weight decomposition. Again we write $\bar{M} \in N$ if $\bar{M}$ satisfies the above property.

Proposition 3.6.1: i) $\bar{M} \in P$ if and only if there exists a one parameter subgroup $\lambda : K^* \rightarrow \text{Aut}(\bar{M})$ such that $\lim_{t \rightarrow 0} \lambda(t) = e_{\bar{M}}$. (Recall $e_{\bar{M}}$ is the composite $\bar{M} \rightarrow K \rightarrow \bar{M}$).

ii) $\bar{M} \in N$ if and only if for any one parameter subgroup $\lambda : K^* \rightarrow \text{Aut}(\bar{M})$, $\lambda(K^*)$ is closed in $\text{End}(\bar{M})$.

iii) $\bar{M}$ possesses a non-trivial non-negative 1-weight splitting if and only if there exists a one-parameter subgroup $\lambda : K^* \rightarrow \text{Aut}(\bar{M})$ such that $\lim_{t \rightarrow 0} \lambda(t) = e$ some idempotent $e \neq 1_{\bar{M}}$ and $\lambda(t)\big|_{e(\bar{M})} = 1_{e(\bar{M})}$. 

Proof: For the "only if" parts of i) and iii) let $\bar{M} = \bigoplus_{\alpha \in \mathbb{Z}} a^\alpha \bar{M}$ be a non-negative 1-weight splitting. Define $\bar{\lambda} : K \rightarrow \text{End}(\bar{M})$ as follows.

$$\bar{\lambda}(t)(x) = \begin{cases} t^\alpha \cdot x & \text{if } x \in a^\alpha \bar{M}, \alpha \neq 0 \\ 0 & \text{if } x \in a^\alpha \bar{M}, t = 0, \alpha \neq 0 \\ x & \text{if } x \in a^0 \bar{M} \end{cases}$$
Then \( \lambda = \overline{\lambda} \mid K^* \) is a one-parameter subgroup such that

\[
\lim_{t \to 0} \lambda(t) = e \text{ is an idempotent } e \neq 1_M.
\]

In case the given weight splitting is positive \( (\mathcal{O}_M = M^O = \mathbb{Q}) \)
\( \lambda(0) = e \). The "if" parts of a) and c) follow from Theorem 3.4.1 and the
fact that if a one-parameter subgroup converges as \( t \to 0 \) then all
non-zero eigenspaces have non-negative weights.

ii) follows from iii) since a convergent 1-parameter subgroup
always converges to an idempotent. \( \quad \) q.e.d.

Remark: In the proof of Proposition 3.6.1 we made no reference to the
pro-algebraic character of the results. We can get away with this here
because our assumptions include the existence of a certain limiting object.
Our results on inverse limits are only needed for coherence and existence
information.

Theorem 3.6.2: Let \( \overline{M} \) be a minimal algebra with \( \mathbb{Q} \)-structure \( M \leq \overline{M} \)
such that \( \dim_K \overline{S(M)} < \infty \). Then there exists a minimal algebra \( \overline{M}(1) \)
unique up to d.g.a. isomorphism such that

i) There exists d.g.a. maps \( j_1 : \overline{M}(1) \to \overline{M} \) and
\( p_1 : \overline{M} \to \overline{M}(1) \) with \( p_1 \circ j_1 = 1_{\overline{M}(1)} \).

ii) There exists a one-parameter subgroup \( \lambda : K^* \to \text{Aut}(\overline{M}) \)
such that \( \lim_{t \to 0} \lambda(t) = j_1 \circ p_1 \).

iii) If \( \overline{N} \to \overline{M} \to \overline{N} \) satisfies i) and ii) above then there exists
\( r : \overline{N} \to \overline{M}(1) \) and \( \lambda : \overline{M}(1) \to \overline{N} \).
such that \( r \circ l = \left\lfloor \frac{1}{M(1)} \right\rfloor \)

Remark: For \( \overline{M}(1) \) and \( \overline{N} \) as above it is also easy to deduce from the following proof that there exists a one parameter subgroup \( \omega : K^* \longrightarrow \text{Aut}(\overline{N}) \) such that \( \lim_{t \to 0} \omega(t) = l \circ r \).

Proof: Clearly any two d.g.a.'s satisfying i), ii) and iii) are isomorphic.

Let \( T \subseteq \text{Aut}(\overline{M}) \) be a maximal \( \mathbb{Q} \)-split pro-torus (2.3.4). Since \( \dim_k S(\overline{M}) < \infty \), \( \dim T < \infty \) (see 3.4.2).

Let \( \Gamma \subseteq \overline{T} \) (closure in the projective limit Zariski topology) be defined as follows.

\[ e_i \in \Gamma \text{ if there exists a one parameter subgroup } \lambda_i : K^* \longrightarrow T \text{ such that } \lim_{t \to 0} \lambda_i(t) = e_i. \]

Then \( \Gamma \) is a finite commutative semi-group because \( \overline{T} \) is finite dimensional and commutative (2.4.2), and the product of one-parameter subgroups is yet another. Thus let

\[ e = \prod_{e_i \in \Gamma} e_i. \]

Of course, \( e = \lim_{t \to 0} \lambda(t) \) where

\[ \lambda : K^* \xrightarrow{\Delta} K^* \times \ldots \times K^* \xrightarrow{\prod \lambda_i} T, \]

\( \Delta(t) = (t, \ldots, t) \),

and \( (\prod \lambda_i(t_1, \ldots, t_n)) = \prod \lambda_i(t_i) \).
Thus let $\bar{M}(1) = e(M) \subseteq \bar{M}$. Clearly $\bar{M}(1)$ satisfies conditions i) and ii).

Suppose $(\bar{N}, j, q)$ satisfies i) and ii) above. Then $q \circ j = 1_{\bar{N}}$ and there exists $\omega : K^* \longrightarrow \text{Aut}(\bar{M})$ such that $\lim_{t \to 0} \omega(t) = j \circ q$.

But then $j \circ q \in \omega(K)$ and $\omega(K) \leq S$ where $S$ is some maximal $Q$-split pro-torus of $\text{Aut}(\bar{M})$. By Theorem 2.3.4 there exists $u \in \text{Aut}(\bar{M})(Q) = \text{Aut}(M)$ such that $u S u^{-1} = \bar{T}$. Thus $e' = u(j \circ q) u^{-1} \epsilon \bar{T}$ is an idempotent, so that $e' \epsilon \Gamma$. By definition of $e'$ above $e' \circ e = e \circ e' = e$.

Clearly this implies that $e(\bar{M}) = \bar{M}(1)$ is a retract of $e' (\bar{M}) = \bar{N}$.

q.e.d.

3.6.3: Notice that this process can be iterated as follows. For each $\bar{M}$ satisfying the conditions of Theorem 3.6.2 we can form

$$\bar{M}(1) \leq \bar{M} \longrightarrow \bar{M}(1) , \ p_1 \circ j_1 = 1_{\bar{M}(1)}$$

and

$$\lambda_1 : K^* \longrightarrow \text{Aut}(\bar{M}) \text{ such that } \lim_{t \to 0} \lambda_1(t) = j_1 \circ p_1 .$$

But then Theorem 3.6.2 applies to $\bar{M}(1)$. Thus we can find $(\bar{M}(2), j_2, p_2)$ such that

$$\bar{M}(2) \leq \bar{M}(1) \longrightarrow \bar{M}(2) , \ p_2 \circ j_2 = 1_{\bar{M}(2)}$$

and

$$\lambda_2 : K^* \longrightarrow \text{Aut}(\bar{M}(1)) \text{ such that } \lim_{t \to 0} \lambda_2(t) = j_2 \circ p_2 , \ (\text{where } j_2 \circ p_2 \text{ is "smallest" in the sense of }$$
condition iii) of Theorem 3.6.2).

Continuing in this way we obtain a finite collection

\[ \{(M(\ell), j_\ell, p_\ell, \lambda_\ell)\}_{\ell=1}^m \]

such that

i) \( M(\ell) \subseteq M(\ell-1) \xrightarrow{j_\ell} M(\ell) \) and \( p_\ell \circ j_\ell = 1_{M(\ell)} \)

ii) \( \lambda_\ell : K \longrightarrow Aut(M(\ell-1)) \) is a one parameter subgroup such that

\[ \lim_{t \to 0} \lambda_\ell(t) = j_\ell \circ p_\ell \]

iii) If \( N \subseteq M(\ell-1) \longrightarrow N \cdot \) satisfies i) and ii) then \( M(\ell) \) is a retract of \( N \).

By proposition 3.6.1 iii) the one-parameter subgroup \( \lambda_\ell : K \longrightarrow Aut(M(\ell-1)) \) induces a non-negative \( 1 \)-weight splitting

\[ \overline{M}(\ell-1) = \bigoplus_{\alpha \in \mathbb{Z}} \overline{M}(\ell-1) \] where \( \overline{M}(\ell-1) = M(\ell) \)

Let \( I(\overline{M}(\ell))^+ \) be the ideal in \( \overline{M}(\ell-1) \) generated by \( \overline{M}(\ell)^+ = \overline{M}(\ell-1)^+ \).

Then \( I(\overline{M}(\ell))^+ \) is stable under the differential so \( \overline{M}(\ell-1)/I(\overline{M}(\ell))^+ \) makes sense as a d.g.a. For simplicity we write \( \overline{M}(\ell-1)/M(\ell) \) for \( \overline{M}(\ell-1)/I(\overline{M}(\ell))^+ \). Then \( M(\ell-1)/\overline{M}(\ell) \) is a minimal algebra. The differential is decomposable because the differential of \( M(\ell-1) \) is decomposable.

\( \overline{M}(\ell-1)/\overline{M}(\ell) \) is free as a graded algebra because \( M(\ell) \) is a retract of \( \overline{M}(\ell-1) \).

Along with Sullivan [17] we call

\[ \overline{M}(\ell) \longrightarrow \overline{M}(\ell-1) \longrightarrow \overline{M}(\ell-1)/M(\ell) \]
an algebraic fibration with section.

3.6.3. We can rephrase our results as follows. For any minimal algebra $\overline{M}$ satisfying the conditions of Theorem 3.6.1 there exists a tower of algebraic fibrations with sections.

\[
\begin{array}{c}
\overline{M}/\overline{M}(1) \leftarrow \overline{M} \\
\downarrow j_1 \quad \downarrow p_1 \\
\overline{M}(1)/\overline{M}(2) \leftarrow \overline{M}(1) = \overline{M}(1) \\
\downarrow j_2 \quad \downarrow p_2 \\
\overline{M}(2)/\overline{M}(3) \leftarrow \overline{M}(2) = \overline{M}(2) \\
\downarrow j_3 \quad \downarrow p_3 \\
\vdots \quad \vdots \\
\overline{M}(m) = \overline{M}(m) \\
\downarrow j_m \quad \downarrow p_m \\
\end{array}
\]

and one parameter subgroups

\[
\lambda_\ell : K^* \longrightarrow \text{Aut}(\overline{M}(\ell-1))
\]

which induce maps of algebraic fibrations.
Note: i) The above data is essentially unique if we require each 
\( (\overline{M}(\ell-1), \overline{M}(\ell)) \) to be chosen in accordance with Theorem 3.6.2.

ii) Each \( \overline{M}(\ell-1)/\overline{M}(\ell) \) has positive weights induced from \( \lambda_\ell : K^* \to \text{Aut}(\overline{M}(\ell-1)/\overline{M}(\ell)) \) because \( \overline{M}(\ell) \) is the eigenspace of \( \lambda_\ell \) with eigenvalue 1.

iii) Since the process has to terminate the minimal algebra \( \overline{M}(m) \) has the property that if \( \lambda : K^* \to \text{Aut}(\overline{M}(m)) \) is a one parameter subgroup then \( \lambda(K^*) \) is closed in \( \text{End}(\overline{M}(m)) \). Thus (*) can be viewed as a composition series for the minimal algebra \( \overline{M} \). The successive "quotients" have positive weights and the series terminates with a minimal algebra \( \overline{M}(m) \) that has no non-trivial non-negative weights. Equivalently, \( \text{Aut}(\overline{M}(m)) \) is closed in \( \text{End}(\overline{M}(m)) \).

3.7. Products, Coproducts and Toroidal Symmetry

The coproduct in the category of minimal algebras is the graded tensor product ([8] p. 3). It is unlikely that this category has products. However, the homotopy category of minimal algebras and d.g.a.
homotopy classes of maps ([8] and [17]) has products. We can construct the product as follows.

Let $\overline{M}$, $\overline{N}$ be minimal algebras (with $Q$-structure). Define $\overline{M} \times \overline{N} = \overline{M}^+ \oplus \overline{N}^+ \oplus K$. Then $\overline{M} \times \overline{N}$ is a 1-connected d.g.a. with coordinate wise differential, addition and multiplication. Here, of course $(\overline{M} \times \overline{N})^n = \overline{M}^n \oplus \overline{N}^n$ if $n > 0$. It is easy to verify that that $\overline{M} \times \overline{N}$ is the product of $\overline{M}$ and $\overline{N}$ in the category of connected differential graded algebras. But $\overline{M} \times \overline{N}$ is not a minimal algebra in general. To remedy this situation we proceed as follows.

By Theorem 2.5 of [8] there exists a minimal algebra $\overline{M} \otimes \overline{N}$ and a morphism $\rho : \overline{M} \otimes \overline{N} \longrightarrow \overline{M} \times \overline{N}$ of differential graded algebras, such that $\rho^*$ is an isomorphism on $H^*$.

Suppose $f : \overline{A} \longrightarrow \overline{M}$, $g : \overline{A} \longrightarrow \overline{N}$ are morphisms of d.g.a.'s (preserving the tacitly assumed $Q$-structures). Then there exists a map $\phi : \overline{A} \longrightarrow \overline{M} \times \overline{N}$ such that

\[
g \quad \overline{N} \quad \overline{M} \times \overline{N} \quad \overline{M} \otimes \overline{N} \quad \overline{A} \quad \phi \quad \overline{M} \]

commutes.

By Theorem 5.19 of [10] there exists $f \circ g : \overline{A} \longrightarrow \overline{M} \otimes \overline{N}$ unique up to d.g.a. homotopy such that

\[
f \circ g \quad \overline{M} \otimes \overline{N} \quad \overline{A} \quad \overline{M} \times \overline{N} \quad \rho \quad \overline{A} \quad \phi \quad \overline{M} \times \overline{N} \]

commutes up to homotopy.
Thus

\[
\begin{array}{c}
\text{A} \\
\downarrow \quad f \circ g \\
\text{M} \circ \text{N}
\end{array}
\quad \begin{array}{c}
P_{M} \circ \rho \\
\downarrow \\
P_{N} \circ \rho
\end{array}
\quad \begin{array}{c}
\text{A} \\
\downarrow g
\end{array}
\quad \begin{array}{c}
\text{N}
\end{array}
\quad \text{homotopy commutes.}
\]

To prove uniqueness suppose \( \tilde{f} \sim f' : \text{A} \rightarrow \text{M} \) and \( \tilde{g} \sim g' : \text{A} \rightarrow \text{N} \).

Then there exists \( F : \text{A} \rightarrow \text{M} \) and \( G : \text{A} \rightarrow \text{N} \) such that

\[
F \circ \lambda = f, \quad F \circ \lambda' = f'
\]

\[
G \circ \lambda = g, \quad G \circ \lambda' = g'
\]

Thus define

\[
H : \text{A} \rightarrow \text{M} \times \text{N}
\]

(\( H = (F \times G) \circ \Delta \))

Retracing our steps we have proved that whenever \( f : \text{A} \rightarrow \text{M} \) and \( g : \text{A} \rightarrow \text{N} \) are d.g.a. maps there exists a map \( f \circ g : \text{A} \rightarrow \text{M} \circ \text{N} \) unique up to homotopy such that

\[
\begin{array}{c}
f \\
\text{A} \\
\downarrow \quad f \circ g \\
\text{M} \circ \text{N}
\end{array}
\quad \begin{array}{c}
P_{M} \circ \rho \\
\downarrow \\
P_{N} \circ \rho
\end{array}
\quad \begin{array}{c}
\text{A} \\
\downarrow g
\end{array}
\quad \begin{array}{c}
\text{N}
\end{array}
\]
homotopy commutes.

Before relating these ideas to toroidal symmetry we observe that

\[
\begin{array}{c}
H^+(\bar{M} \odot \bar{N}) \xrightarrow{\sim} H^+(\bar{M} \times \bar{N}) \xrightarrow{\sim} H^+(\bar{M}) \oplus H^+(\bar{N}) \\
\end{array}
\]

and that \(\bar{M}, \bar{N}\) are retracts of \(\bar{M} \odot \bar{N}\).

The subcategory of minimal algebras which exhibits the most accessible relationship between toroidal symmetry and coproducts and products is \(\mathcal{P}\). Recall that \(\mathcal{P}\) is the category of minimal algebras which have positive weights.

Let \(Z^n = Z \times \ldots \times Z\) and \(Z \rightarrow Z^n\) be the inclusion on the \(i\)-th factor. Define \(VZ_i = \bigcup_{i=1}^{n} Z_i \leq Z^n\).

**Theorem 3.7.1:** Let \(\bar{M} = \bigoplus_{\alpha \in Z^n} \bar{M}\) be a positive \(n\)-weight splitting and let \(\bar{M} = \bigoplus_{\alpha \in Z^n} \bar{M}\). Then

\[
\begin{array}{c}
\bar{M} = \bigoplus_{\alpha \in Z^n} \bar{M} \\
\end{array}
\]

i) Each \(\bar{M}_i\) is a minimal algebra.

ii) There are maps \(\bigoplus_{\alpha \in Z^n} \phi \bar{M} \xrightarrow{\Pi j_i} \bar{M} \xrightarrow{\Pi P_i} \bigoplus_{\alpha \in Z^n} \phi \bar{M}\).

iii) \(\Pi j_i\) is an isomorphism if and only if \(Q(\bar{M}) = \bigoplus_{\alpha \in VZ_i} Q(\bar{M})\).

iv) \(\Pi P_i\) is an isomorphism if and only if \(H(\bar{M}) = \bigoplus_{\alpha \in VZ_i} H(\bar{M})\).

**Proof:** i) follows from proposition 3.5.1.

ii) For each \(\bar{M}_i\) we have maps \(\bar{M}_i \xrightarrow{j_i} \bar{M} \xrightarrow{P_i} \bar{M}_i\) such that
Thus there exist unique (up to homotopy) maps.

\[ p_i \circ j_i = l_{\bar{M}_i} \]

\[ \oplus \bar{M}_i \xrightarrow{j_i} \bar{M} \quad \text{and} \quad M \xrightarrow{\pi p_i} \oplus \bar{M}_i \]

such that

\[ \oplus \bar{M}_i \xrightarrow{\mu j_i} \bar{M} \quad \text{and} \quad M \xrightarrow{\pi p_j} \oplus \bar{M}_i \]

commute (up to homotopy).

iii) If \( \mu j_i : \oplus \bar{M}_i \xrightarrow{\epsilon} \bar{M} \) then \( Q(\mu j_i) : Q(\oplus \bar{M}_i) \xrightarrow{\epsilon} Q(M) \).

But \( \oplus Q(j_i) : \oplus Q(\bar{M}_i) \xrightarrow{\epsilon} Q(\oplus \bar{M}_i) \).

Thus \( \oplus Q(j) : \oplus Q(\bar{M}_i) \xrightarrow{\epsilon} Q(\bar{M}) \) and

\[ \oplus Q(\bar{M}_i) \xrightarrow{\epsilon} Q(\bar{M}) \]

\[ \quad \xrightarrow{\epsilon} \quad \oplus Q(\bar{M}) \]

\[ \alpha \in \nu Z_i \]

Thus if \( \mu j_i \) is an isomorphism \( \oplus Q(\bar{M}) = Q(\bar{M}) \). Conversely, if

\[ \alpha \in \nu Z_i \]

\[ \oplus Q(\bar{M}) = Q(M) \] then
Thus $Q(\sqcup j_i)$ is an isomorphism. Hence $\sqcup j_i$ is an isomorphism because it induces an isomorphism on (free) generating sets.

iv) The proof of iv) is exactly dual to iii) using the fact that any map between minimal algebras which induces an isomorphism on cohomology is an isomorphism. q.e.d.

Remark: In the above theorem if we let $e_i = j_i \circ p_i : \overline{M} \rightarrow \overline{M}$. Then $e_i \circ e_i = e_i$ and $e_i \circ e_j = e_j \circ e_i$. Condition iii) is then equivalent to

$$iii') \quad e_i \circ e_j = \epsilon \quad \text{if} \quad i \neq j, \quad \text{and} \quad \sum_{j=1}^{n} Q(e_j) = 1 \quad \frac{Q(M)}{Q(M)}.$$

Condition iv) is equivalent to

$$iv') \quad e_i \circ e_j = \epsilon \quad \text{if} \quad i \neq j, \quad \text{and} \quad \sum_{j=1}^{n} H(e_j) = 1 \quad \frac{H(M)}{H(M)}.$$

Here of course $\epsilon \bigm|_{\overline{M}}$ is the composite $\overline{M} \rightarrow K \rightarrow \overline{M}$. 
Before using this theorem to prove uniqueness of decomposition results for \( P \) we must digress slightly and prove a stronger version of Proposition 3.6.1. i). In order to do this we need a few elementary lemmas about imbedded algebraic groups.

**Lemma 3.7.2:** Let \( G \) be an algebraic group and let \( \rho : G \to \text{Gl}(n,K) \subseteq M(n,K) \) be a representation of \( G \), where \( M(n,K) \) is the set of all \( n \times n \) matrices over \( K \).

If \( 0 \in \overline{\rho(G)} \) (Zariski closure) then \( 0 \in \overline{\rho(T)} \) where \( T \) is any maximal torus of \( G \) (Here \( V = K^n \)).

**Proof:** (R. Body). Let \( D(n) \subseteq M(n,K) \) be the (Zariski closed) subset of all matrices with zeros off the diagonal. Let \( \chi : D(n) \to K^n \) be the composite \( D(n) \subseteq M(n,K) \twoheadrightarrow K[t]/(t^n) \cong K^n \) where \( \phi(M) = \det(tI - M) \).

Then \( \chi \) is a finite dominant map ([12] p. 31). Thus it is also closed.

Without loss of generality there exists a maximal torus \( T \subseteq G \) such that \( \rho(T) \subseteq D(n) \). Notice that \( \overline{\rho(T)} \subseteq D(n) \). Thus

\[
0 \in \chi(\overline{\rho(G)}) \subseteq \chi(\overline{\rho(G)}) = \chi(\overline{\rho(T)}) = \chi(\overline{\rho(T)})
\]

The first inclusion is true by continuity. The first equality is true since

i) \( \chi(\alpha) = \chi(\alpha \alpha^{-1}) \quad \alpha \in \text{Gl}(n,K) \)

and

ii) \( \chi(\alpha) = \chi(\alpha_s) \quad (\alpha_s \text{ is the semi-simple part of } \alpha \text{ (1.1.4)}) \).

The second equality follows from the fact that \( \chi \) is a closed map. But since \( \chi^{-1}(0) = \{0_V\}, 0_V \in \overline{\rho(T)} \).

q.e.d.
Lemma 3.7.3: Suppose \( \rho : G \rightarrow GL(n,K) \subseteq M(n,K) \) is a representation of the algebraic group \( G \) such that \( 0 \not\in \rho(G) \). Then there exists a one-parameter subgroup \( \lambda : K^* \rightarrow \rho(G) \) such that \( \lambda \) extends to \( \bar{\lambda} : K \rightarrow \bar{\rho}(G) \) with \( \bar{\lambda}(0) = 0 \).

Proof: By lemma 3.7.2 \( 0 \not\in \bar{\rho}(T) \) where \( T \subseteq G \) is some torus. But \( \rho(T) \subseteq \bar{\rho}(T) \) is a toroidal imbedding in the sense of [13]. Furthermore \( 0 \) is a fixed point under the action

\[
\rho(T) \times \bar{\rho}(T) \rightarrow \bar{\rho}(T)
\]

Thus by Theorem 2 of [13] there exists a one-parameter subgroup \( \lambda : K^* \rightarrow \rho(T) \) such that \( \lambda \) extends to \( \bar{\lambda} : K \rightarrow \bar{\rho}(T) \) with \( \bar{\lambda}(0) = 0 \). q.e.d.

Rationality properties are also nicely preserved under this construction in the following sense.

Lemma 3.7.4: Let \( G \) be a \( k \)-group and \( \rho : G \rightarrow GL(n,K) \subseteq M(n,K) \) be a \( k \)-rational representation. If \( 0 \not\in \bar{\rho}(G) \) then there exists a one parameter subgroup \( \lambda : K^* \rightarrow \rho(G) \) defined over \( k \) such that \( \lambda \) extends to \( \bar{\lambda} : K \rightarrow \bar{\rho}(G) \) with \( \bar{\lambda}(0) = 0 \).

Proof: By lemma 3.7.3 there exists a one parameter subgroup \( \lambda : K^* \rightarrow \rho(G) \) extending to \( \bar{\lambda} : K \rightarrow \bar{\rho}(G) \) with \( \bar{\lambda}(0) = 0 \). Now \( \lambda(K^*) \subseteq S \) some maximal torus \( S \) defined over \( k \) (see p. 4 of [9]). By p. 219 of [4] we can write \( S = S_d \times S_a \) where \( S_d \) is \( k \)-split and \( S_a \) is \( k \)-anisotropic.

Thus \( \text{Hom}_{K\text{-group}}(K^*,S) = \text{Hom}_{K\text{-group}}(K^*,S_d) \oplus \text{Hom}_{K\text{-group}}(K^*,S_a) \). Thus we
can write \( \lambda = (\lambda_d, \lambda_a) \). But \( \text{Hom}_{K\text{-group}}(K^*, S_d) = \text{Hom}_{K\text{-group}}(K^*, S_d) \)

because \( S_d \) and \( K^* \) are \( k \)-split. Thus \( \lambda_d : K^* \to \rho(G) \) is a \( k \)-group morphism. Furthermore by p. 218 of [4] \( S_a \subseteq \bigcap_{a \in \text{Hom}_{K\text{-group}}(S, K)} \ker \psi^a \).

Thus \( \det^{-1}(1) \supseteq S_a \). Hence \( \lambda_d \) extends to \( 0 \) and

\[
\lambda_d(0) = 0.
\]

q.e.d.

Lemma 3.7.4 applies at once to yield the following version of Proposition 3.6.1 i).

Proposition 3.7.5: Let \( \overline{M} \) be a finitely generated minimal algebra with \( Q \)-structure \( M \subseteq \overline{M} \). Then the following are equivalent

i) \( \overline{M} \) has positive weights.

ii) \( \varepsilon \in \text{Aut}(\overline{M}) \subseteq \text{End}(\overline{M}) \).

iii) There exists a one parameter subgroup \( \lambda : K^* \to \text{Aut}(\overline{M}) \) extending to \( \overline{\lambda} : K \to \text{End}(\overline{M}) \) with \( \overline{\lambda}(0) = \varepsilon \).

Proof: By proposition 3.6.1 we need only prove that ii) implies iii). Since \( \overline{M} \) is finitely generated there exists a finite dimensional subspace \( V \subseteq M \) such that \( \rho : \text{Aut}(\overline{M}) \to \text{GL}(V) \) is a representation satisfying the assumptions of Lemma 3.7.4.

q.e.d.

Proposition 3.7.6: Let \( \overline{M} \) be a minimal algebra with \( Q \)-structure \( M \subseteq \overline{M} \) such that \( \dim_K S(\overline{M}) < \infty \). Then \( \overline{M} \) has positive weights if and only \( \overline{M}_n \) has positive weights for all \( n \geq 0 \).

Proof: If \( \overline{M} \) has positive weights then so does each \( \overline{M}_n \), because each
weight space \( \overline{M}_n \) is an eigenspace of some diagonalizable map under which \( \overline{M}_n \) is invariant.

Conversely, if \( \dim_K S(\overline{M}) < \infty \) then there exists \( N > 0 \) such that

the restriction \( r_n : \text{Aut}(\overline{M}_n) \rightarrow \text{Aut}(\overline{M}_{n-1}) \) has a unipotent kernel

(Lemma 3.3.2), for \( n > N \). Since each \( \overline{M}_n \) has positive weights there exists an integer \( L > 0 \) such that \( \dim T_n = \dim T_{n+1} \) for \( n > L \),

where \( T_n \leq \text{Aut}(\overline{M}_n) \) is a maximal \( \mathbb{Q} \)-split torus (The \( \mathbb{Q} \)-split rank cannot keep decreasing).

Hence if \( T_n \leq \text{Aut}(\overline{M}_n) \) and \( T_{n-1} \leq \text{Aut}(\overline{M}_{n-1}) \) are respective maximal \( \mathbb{Q} \)-split tori with \( r_n(T_n) \leq T_{n-1} \) then \( r_n : T_n \rightarrow T_{n-1} \)

for \( n > \max\{N,L\} \).

By Lemma 3.3.2 and Theorem 2.3.2 there exists \( \{T_n \mid T_n \leq \text{Aut}(\overline{M}_n) \} \) is a maximal \( \mathbb{Q} \)-split torus and \( r_n(T_n) \leq T_{n-1} \).

Now let \( \overline{M}_n = \bigoplus_{\alpha \in \mathbb{Z}} a_n \)

\( (n > \max\{N,L\}) \) and let \( \lambda_n : K^* \rightarrow \text{Aut}(\overline{M}_n) \) be the associated one-parameter subgroup (Theorem 3.4.1). By the above remarks we have the following diagram
Thus we can fill in the dotted arrows with \( Q \)-group morphisms
\[
\lambda_{n+1} : K^* \longrightarrow T_{n+1}
\]
such that the whole diagram commutes. This determines a morphism of \( Q \)-pro-groups
\[
\lambda = \lim_{\longrightarrow} \lambda : K^* \longrightarrow \lim_{\longrightarrow} T_n = T
\]
By Theorem 3.4.1 this determines a 1-weight splitting \( \tilde{M} = \bigoplus_{\alpha \in \mathbb{Z}} \tilde{M}_\alpha \).

Since \( \{ \alpha \in \mathbb{Z} \mid \alpha \tilde{M} \neq 0 \} \) is additively generated by \( \{ \alpha \in \mathbb{Z} \mid \alpha S(\tilde{M}) \neq 0 \} \)
(Proposition 3.4.2) the above weight splitting is positive. \( \text{q.e.d.} \)

**Theorem 3.7.7**: Let \( \tilde{M} \) be a minimal algebra with \( Q \)-structure \( M \subseteq \tilde{M} \) such that \( \dim_K S(\tilde{M}) < \infty \). Then the following conditions are equivalent

i) \( e_\tilde{M} \in \text{Aut}(\tilde{M}) \)

ii) \( \tilde{M} \) has positive weights

iii) There exists a one parameter subgroup \( \lambda : K^* \longrightarrow \text{Aut}(\tilde{M}) \)

extending to \( \lambda : K \longrightarrow \text{End}(\tilde{M}) \) with \( \lambda(0) = e_{\tilde{M}} \).

**Proof**: ii) and iii) are equivalent by Proposition 3.6.1 and clearly iii) implies i). Thus it suffices to show that i) implies ii).

If \( e \in \text{Aut}(\tilde{M}) \) then \( e_{\tilde{M}_n}^{-1} \epsilon e_{\tilde{M}_n} \in \text{Aut}(\tilde{M}_n) \). Thus by proposition 3.7.5 \( \tilde{M}_n \) has positive weights (\( \tilde{M}_n \) is finitely generated). Since this holds for all \( n \geq 0 \) \( \tilde{M} \) has positive weights by Theorem 3.7.6. \( \text{q.e.d.} \)

**Remark**: Condition i) of Theorem 3.7.7 is a purely topological condition on the imbedding \( \text{Aut}(\tilde{M}) \subseteq \text{End}(\tilde{M}) \). The proof of the following proposition exhibits the usefulness of such a theorem.
Proposition 3.7.8: Suppose $\bar{M}$ is a finitely generated minimal algebra and $\bar{M}$ has positive weights. If $\bar{N}$ is a retract of $\bar{M}$ then $\bar{N}$ has positive weights.

Proof: [2] Suppose $\bar{N} \xrightarrow{j} \bar{M} \xrightarrow{r} \bar{N}$ where $r \circ j = 1$. Clearly $\bar{N}$ is finitely generated. Define $\psi : \text{End}(\bar{M}) \longrightarrow \text{End}(\bar{N})$ as $\psi(f) = r \circ f \circ j$. Then $\psi$ is a morphism of $Q$-varieties.

Since $\bar{M}$ has positive weights there exists a one-parameter subgroup $\lambda : K^{*} \longrightarrow \text{Aut}(\bar{M})$ which extends to $\bar{\lambda} : K \longrightarrow \text{End}(\bar{M})$ with $\bar{\lambda}(0) = e$. Thus $e \in \bar{\lambda}(K) \subseteq L \subseteq \bar{L}$ (in fact equal) (Zariski closure in $\text{End}(\bar{M})$). $L$ is irreducible (not the union of two proper closed subsets) since $K$ is irreducible and $\bar{\lambda}$ is continuous. Thus $\bar{L}$ is irreducible.

But then $\psi(L)$ is irreducible because $\psi$ is continuous. Also $\psi(L) \subseteq \psi(L)$ is open because $\psi(L)$ is constructible.

Let $D = \text{Aut}(\bar{N}) \cap \psi(L)$. Then $1 \notin D \neq \emptyset$. Thus $D \subseteq \psi(L)$ is open and non-empty. Hence $D \subseteq \psi(L)$ is open. Since $\psi(L)$ is irreducible $D = \psi(L)$. Thus $\psi(L) \subseteq D \subseteq \text{Aut}(\bar{N})$. Finally, $e = \psi(e) \in \psi(L) \subseteq \text{Aut}(\bar{N})$. Therefore, by Theorem 3.7.7 $\bar{N}$ has positive weights. q.e.d.

Corollary 3.7.9: Let $\bar{M}$ be a minimal algebra with positive weights such that $\dim_{K} S(\bar{M}) < \infty$. If $\bar{N}$ is a retract of $\bar{M}$ then $\bar{N}$ has positive weights.

Proof: If $\bar{N} \xrightarrow{j} \bar{M} \xrightarrow{r} \bar{N}$ satisfies $r \circ j = 1$ then $\bar{N} \xrightarrow{j_{n}} \bar{M} \xrightarrow{r_{n}} \bar{N}_{n}$
satisfies \( r_n \circ j_n = 1_{N_n} \) for all \( n \geq 0 \). Thus Proposition 3.7.8 applies and \( N_n \) has positive weights for all \( n \geq 0 \). By Proposition 3.7.6 \( N \) has positive weights.

q.e.d.

**Proposition 3.7.10:** Let \( \overline{M}, \overline{N} \) be minimal algebras such that \( \dim_K S(\overline{M}) < \infty \) and \( \dim_K S(\overline{N}) < \infty \). Then the following are equivalent.

1. \( \overline{M} \) and \( \overline{N} \) have positive weights.
2. \( \overline{M} \otimes \overline{N} \) has positive weights.
3. \( \overline{M} \circ \overline{N} \) has positive weights.

**Proof:** Clearly iii) or ii) implies i) because \( \overline{M} \) and \( \overline{N} \) are retracts of \( \overline{M} \otimes \overline{N} \) and \( \overline{M} \circ \overline{N} \). Thus we need only show that i) implies ii) and that i) implies iii).

i) implies ii).

Let \( \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}} \overline{M}^{\alpha} \) and \( \overline{N} = \bigoplus_{\alpha \in \mathbb{Z}} \overline{N}^{\alpha} \) be positive weight splittings of \( \overline{M} \) and \( \overline{N} \) respectively. Then \( \overline{M} \otimes \overline{N} = \bigoplus_{\alpha \in \mathbb{Z}} (\bigoplus_{\beta + \gamma = \alpha} \overline{M}^{\beta} \otimes \overline{N}^{\gamma}) = \bigoplus_{\alpha \in \mathbb{Z}} \alpha (\overline{M} \otimes \overline{N}) \) is a positive weight splitting of \( \overline{M} \otimes \overline{N} \) where \( (\overline{M} \otimes \overline{N}) = \bigoplus_{\beta + \gamma = \alpha} \overline{M}^{\beta} \otimes \overline{N}^{\gamma} \).

i) implies iii).

Recall that \( \overline{M} \otimes \overline{N} \) is a minimal algebra for \( \overline{M} \times \overline{N} \) so there exists a weak equivalence \( \rho : \overline{M} \otimes \overline{N} \longrightarrow \overline{M} \times \overline{N} \).

Suppose \( \overline{M} = \bigoplus_{\alpha \in \mathbb{Z}} \overline{M}^{\alpha} \) and \( \overline{N} = \bigoplus_{\beta \in \mathbb{Z}} \overline{N}^{\beta} \) are positive weight splittings
for \( M \) and \( N \) respectively and define \( \lambda_p : M \times N \to M \times N \) as
\[
\lambda_p(x, y) = (p^a x, p^b y)
\]
for \( x \in M \) and \( y \in N \) and \( p \neq 1 \in Q^* \).

Clearly, \( \lambda_p \) is a d.g.a. automorphism. Thus by Theorem 2.13 of [8] there exists \( \omega : M \otimes N \to M \otimes N \) such that
\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\omega} & M \times N \\
\downarrow & & \downarrow \lambda_p \\
M \otimes N & \xrightarrow{\rho} & M \times N \\
\end{array}
\]
commutes.

\( \omega \) is an automorphism because the above diagram forces it to be a weak equivalence.

\[ \omega^* : H(M \otimes N) \to H(M \otimes N) \]
is diagonalizable because \( \lambda_p \) is. Thus \( \omega_s = \omega^* \) where \( \omega_s \) is the semisimple part of \( \omega \) (see Theorem 2.2.5).

A straightforward induction argument on \( \{(M \otimes N)_n\}_{n \geq 0} \) shows that if \( \omega_s \in \text{Aut}(M \otimes N) \) is semisimple and \( \omega_s^* \) is diagonalizable then \( \omega_s \) is in fact diagonalizable.

Thus \( \omega_s \) is diagonalizable and induces a positive weight splitting on \( H(M \otimes N) \) (because \( \lambda_p \) does). By Proposition 3.4.2 \( \omega_s \) induces a positive weight splitting on \( M \otimes N \) because the eigenvalues of \( \omega_s \) are already multiplicatively generated on \( S(M \otimes N) \).

q.e.d.

In the language of [1] Proposition 3.7.10 says that the category \( P \) of minimal algebras \( M \) with positive weights satisfying
\[ \dim_K S(M) < \infty \]
is productive with respect to \( \otimes \) and \( \oplus \).

\textbf{Definition:} Let \( M \in P \), \( M \not\in Q = K \).
i) If $\overline{M} \cong \overline{N} \otimes \overline{L}$ implies that $\overline{M} \cong \overline{N}$ or $\overline{M} \cong \overline{L}$ then $\overline{M}$ is called $\otimes$-irreducible.

ii) If $\overline{M} \cong \overline{N} \oplus \overline{L}$ implies that $\overline{M} \cong \overline{N}$ or $\overline{M} \cong \overline{L}$ then $\overline{M}$ is called $\oplus$-irreducible.

Clearly any $\overline{M} \in P$ can be written $\overline{M} \cong \oplus_{i=1}^{n} \overline{M}_i$ where each $\overline{M}_i \in P$ is $\otimes$-irreducible. One starts with any non-trivial such splitting and reiterates the process on each factor. The process terminates because $\dim_K S(M) < \infty$. Each $\overline{M}_i \in P$ because of Proposition 3.7.10.

Similarly any $\overline{N} \in P$ can be written $\overline{N} \cong \oplus_{i=1}^{m} \overline{N}_i$ where each $\overline{N}_i \in P$ is $\oplus$-irreducible.

As in the proof of Proposition 3.7.10 there are weight splittings

$$
\overline{M} = \bigoplus_{\alpha \in Z} \alpha \overline{M} \quad \text{and} \quad \overline{N} = \bigoplus_{\beta \in Z} \beta \overline{N}
$$

such that

$$
\overline{M}_i \cong \bigoplus_{\alpha \in Z_i} \alpha \overline{M} \quad \text{and} \quad \overline{N}_i \cong \bigoplus_{\beta \in Z_j} \beta \overline{N}
$$

where $Z = Z_i \longrightarrow Z^n$ is the inclusion on the $i$-th factor. By Theorem 3.7.1

$$
Q(\overline{M}) = \bigoplus_{\alpha \in \nu Z_i} \alpha Q(\overline{M}) \cong \bigoplus_{i=1}^{n} Q(\overline{M}_i)
$$

and

$$
H(\overline{N}) = \bigoplus_{\beta \in \nu Z_j} \beta H(\overline{N}) \cong \bigoplus_{j=1}^{m} H(\overline{N}_j)
$$

The above weight splittings determine $Q$-group morphisms

$$
\phi : K^* \times \ldots \times K^* \longrightarrow Aut(\overline{M})
$$

$n$-factors.
and

\[ \psi : K^* \times \ldots \times K^* \to \text{Aut}(N) \]

\[ m \text{-factors} \]

where

\[ \phi(t_1, \ldots, t_n)(x) = t_1^{\alpha_1} \ldots t_n^{\alpha_n} \cdot x, \quad x \in \bar{M}, \alpha = (\alpha_1, \ldots, \alpha_n) \]

and

\[ \psi(s_1, \ldots, s_m)(x) = s_1^{\beta_1} \ldots s_m^{\beta_m} \cdot x, \quad x \in \bar{N}, \beta = (\beta_1, \ldots, \beta_m) \].

Let \( \lambda_i : K^* \to K^* \times \ldots \times K^* \) satisfy \( \lambda_i(t) = (t, t, \ldots, t, 1, t, \ldots, t) \)

\[ n \text{-factors} \]

(i-th factor).

and \( \omega_j : K^* \to K^* \times \ldots \times K^* \) satisfy \( \omega_j(s) = (s, s, \ldots, s, 1, s, \ldots, s) \)

\[ m \text{-factors} \]

(j-th factor).

Then

\[ \phi_i = \phi \circ \lambda_i : K^* \to \text{Aut}(M) \]

and

\[ \psi_j = \psi \circ \omega_j : K^* \to \text{Aut}(N) \]

are one-parameter subgroups such that

\[ \lim_{t \to 0} \phi_i(t) = e_i \quad \text{where} \quad e_i(M) = \bar{m}_i \]

and

\[ \lim_{t \to 0} \psi_j(s) = f_j \quad \text{where} \quad f_j(N) = \bar{n}_j \]

Thus

\[ \{e_1, \ldots, e_n\} \subseteq \phi(K^* \times \ldots \times K^*) \subseteq \text{Aut}(M) \]

and

\[ \{f_1, \ldots, f_m\} \subseteq \psi(K^* \times \ldots \times K^*) \subseteq \text{Aut}(N) \]

By Theorem 3.7.1

(1) \( e_i^2 = e_i \)

(2) \( e_i \circ e_j = e_{ij} \) if \( i \neq j \).
C3) \[ \sum_{i=1}^{n} Q(e_i) = 1 \]

and dually

P1) \[ f_j^2 = f_j \]

P2) \[ f_j \circ f_K = e_N \text{ if } j \neq K \]

P3) \[ \sum_{j=1}^{m} H(f_j) = 1 \]

Conversely, (again by Theorem 3.7.1) any collection \( \{e_1, \ldots, e_n\} \subseteq \text{End}(M) \) such that C1) - C3) hold determines a Q-splitting \( \overline{M} \equiv \bigoplus_{i=1}^{n} e_i(M) \).

Dually, any collection \( \{f_1, \ldots, f_m\} \subseteq \text{End}(N) \) such that P1) - P3) hold determines a Q-splitting \( \overline{N} \equiv \bigoplus_{j=1}^{m} f_j(N) \).

A brief summary is in order here. If \( \overline{M}, \overline{N} \in P \) then a Q-splitting \( \overline{M} \equiv \bigoplus_{i=1}^{n} \overline{M_i} \) determines

i) \( \{e_1, \ldots, e_n\} \subseteq \text{End}(\overline{M}) \) satisfying C1) - C3) such that

\( e_i(\overline{M}) \equiv \overline{M_i} \).

ii) A maximal Q-split torus \( T \subseteq \text{Aut}(\overline{M}) \) such that \( \{e_1, \ldots, e_n\} \subseteq \overline{T} \) the Zariski closure of \( T \). A Q-splitting

\( \overline{N} \equiv \bigoplus_{j=1}^{m} \overline{N_i} \) determines

i') \( \{f_1, \ldots, f_m\} \subseteq \text{End}(\overline{N}) \) satisfying P1) - P3) such that
\[ f_j(\overline{N}) \cong \overline{N}_j. \]

ii) A maximal \( \Theta \)-split torus \( S \leq \text{Aut}(\overline{N}) \) such that \( \{f_1, \ldots, f_m\} \subseteq \overline{S} \) the Zariski closure of \( S \).

Notation: If \( \{e_1, \ldots, e_n\} \subseteq \text{End}(M) \) satisfy Cl) - C3) and \( e_i(M) \) is \( \Theta \)-irreducible for all \( i \), \( \{e_1, \ldots, e_n\} \) is called a \( \Theta \)-splitting.

Dually, if \( \{f_1, \ldots, f_n\} \subseteq \text{End}(\overline{N}) \) satisfies Pl) - P3) and \( f_j(\overline{N}) \) is \( \Theta \)-irreducible for each \( j \), \( \{f_1, \ldots, f_n\} \) is called a \( \Theta \)-splitting.

If \( \{e_1, \ldots, e_n\} \subseteq \overline{T} \) and \( \{e'_1, \ldots, e'_{n'}\} \subseteq \overline{T'} \) are \( \Theta \)-splittings then by Theorem 2.3.4 there exists \( g \in \text{Aut}(\overline{M})(\Theta) \) such that \( \{ge_1g^{-1}, \ldots, ge_ng^{-1}\} \subseteq \overline{T} \). Since \( \overline{T} \) is commutative (Lemma 2.4.2) \( ge_i^{'}g^{-1} \) commutes with \( e_j \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, n \).

Dually, if \( \{f_1, \ldots, f_n\} \subseteq \overline{S} \) are \( \Theta \)-splittings then by the same reasoning there exists \( h \in \text{Aut}(\overline{N})(\Theta) \) such that \( \{hf_1, \ldots, hf_n, h^{-1}\} \subseteq \overline{S} \).

In the language of \([1]\) \( P \) is a flexible, productive category of minimal algebras with respect to both \( \Theta \)-splittings and \( \Theta \)-splittings. Thus Theorem 2 of \([1]\) implies that each non-trivial \( \overline{M} \in P \) satisfies

i) \( \overline{M} \cong \bigoplus_{i=1}^{n} \overline{M}_i \)

ii) Each \( \overline{M}_i \) is \( \Theta \)-irreducible

iii) If \( \overline{M} \cong \bigoplus_{i=1}^{n'} \overline{M} \) satisfies i) and ii) then \( n = n' \) and \( \overline{M}_i \cong \overline{M}_{\sigma(i)} \), where \( \sigma \) is a permutation on \( n \)-letters.
The dual argument yields the following.

Each non-trivial $\overline{N} \in P$ satisfies

i) $\overline{N} \equiv \prod_{j=1}^{m} \overline{N}_j$

ii) Each $\overline{N}_j$ is $\varnothing$-irreducible.

iii) If $\overline{N} \equiv \prod_{j=1}^{m'} \overline{N}_j'$ satisfies i) and ii) then $m = m'$ and

$\overline{N}_j \equiv \overline{N}'_{\tau(j)}$ where $\tau$ is a permutation of $M$. 
References


