AN ASYMPTOTIC LOOP EXPANSION FOR THE EFFECTIVE POTENTIAL IN THE $P(\phi)_2$ QUANTUM FIELD THEORY

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B.A.Sc., The University of Toronto, 1977

M.Sc., The University of Toronto, 1979

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

Department of Mathematics

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

March 1984

Department of MATHEMATICS

The University of British Columbia
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Date April 22, 1984

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ABSTRACT:

The effective potential $V(\hbar,a)$ for the Euclidean $P(\phi)_2$ quantum field theory is defined to be the Fenchel transform (convex conjugate) of the pressure in an external field, and is shown to be finite. The parameter $\hbar$ is Planck's constant divided by 2π. The classical limit $(\hbar\downarrow 0)$ of the effective potential is shown to be the convex hull of the classical potential $P(a) + \frac{1}{2}m^2a^2$. For values of a for which the classical potential is equal to its convex hull and has a nonvanishing second derivative, the usual one-particle irreducible loop expansion for the effective potential is shown to be asymptotic as $\hbar \downarrow 0$, using a uniformly convergent (as $\hbar \downarrow 0$) high temperature cluster expansion and irreducibility properties of the Legendre transform. For the same values of a, V is shown to be analytic in a for sufficiently small $\hbar$. Finally an example is given for a double well classical potential where the one-particle irreducible loop expansion fails to be asymptotic, and the true asymptotics are obtained.

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Acknowledgement

It is a pleasure to thank Professors Joel Feldman and Lon Rosen for what they have taught me, for helpful suggestions and for their encouragement. I also wish to thank Bruce Sharpe and Peter Sharpe for their company in learning quantum field theory. I am grateful to Joanne Nakonechny for encouragement, and to the National Sciences and Engineering Research Council of Canada for financial support.

Finally I wish to thank Carol Samson for a fast and accurate job of typing the thesis.

Chapter 1: INTRODUCTION AND MAIN RESULTS

§1. The $P(\phi)_2$ Quantum Field Theory

The subjects of axiomatic and constructive quantum field theory arose as an attempt to put quantum field theory on a sound mathematical foundation. The idea, which took shape in the 1950's, was to write down a physically motivated list of properties or axioms that a mathematically well-defined quantum field theory would be expected to satisfy, and then look for examples satisfying the properties. One of the first sets of axioms [WG 64], the Garding-Wightman axioms, involves operator-valued tempered distributions (the fields) acting on a Hilbert space, which transform in an appropriate way under Lorentz transformations. An equivalent set of axioms is formulated in terms of vacuum expectation values of the field [SW 78].

When the axioms were first introduced the only models known to satisfy them were free fields. It was realized that the technical problems involved in the construction of interacting models were less imposing when the dimension of space-time was reduced from four to three or two. Work began on the construction of two-dimensional models, with the hope that a technology could be developed that would be useful in attacking the physically relevant case of four dimensions. A brief account of early progress in the construction of interacting two-dimensional models can be found in the Appendix to [SW 78].

In the early 1970's a new strategy began to emerge, that of analytically continuing the vacuum expectation values from real to imaginary time. This new strategy was called the Euclidean strategy because the replacement of t by it changes the Minkowski metric to the Euclidean metric. In [OS 73] conditions were given on the Schwinger functions (the analytic continuations of the vacuum expectation values) which are equivalent to the Garding-Wightman axioms, in the sense that existence of a family of distributions satisfying the Osterwalder-Schrader axioms guarantees the existence of a unique field theory satisfying the Garding-Wightman axioms, and vice-versa. Motivated by ideas of Feynman [Feyn H 65] the attempt since this time to construct d-dimensional scalar boson field theories has been centred on obtaining Schwinger functions satisfying the Osterwalder-Schrader axioms as the moments of a measure on the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions. In [F 74] a set of axioms (the POS, or probabilistic Osterwalder-Schrader axioms) was introduced for probability measures on $\mathcal{S}'(\mathbb{R}^d)$ which guarantees that the moments of such a measure satisfy the Osterwalder-Schrader axioms. The converse need not be true: the Osterwalder-Schrader axioms do not imply that the Schwinger functions are the moments of a measure. A theory whose Schwinger functions are the moments of a measure is said to be Nelson-Symanzik positive. So far each of the several models constructed in two and three dimensions satisfy the stronger axioms. No interacting models have been constructed yet in four dimensions.

To state the POS axioms we introduce some definitions and notation. Denote the Schwartz space of real-valued test functions on $\mathbb{R}^d$ by $\mathcal{S}(\mathbb{R}^d)$, the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$ by $\mathcal{S}'(\mathbb{R}^d)$, and the action of $\phi \in \mathcal{S}'(\mathbb{R}^d)$ on $f \in \mathcal{S}(\mathbb{R}^d)$ by $\phi(f)$. Although not every $\phi \in \mathcal{S}'(\mathbb{R}^d)$ is a function, we follow common usage and occasionally write $\phi(f) = \int \phi(x) f(x) dx$. Let $\sum$ denote the σ-algebra of subsets of $\mathcal{S}'(\mathbb{R}^d)$ generated by the Borel cylinder sets, i.e., sets of the form

$\{\phi \in \mathcal{S}'(\mathbb{R}^d) : (\phi(f_1),\ldots,\phi(f_n)) \in A\}$ where $f_i \in \mathcal{S}(\mathbb{R}^d)$ $(i = 1,\ldots,n)$, A is a Borel subset of R^n, and n can be any positive integer. Denote by $\sum_+$ the σ-algebra generated by sets of the same form but with the f_i supported in $\{x \in R^d : x_d > 0\}$, and let E denote the Euclidean group of translations, rotations and reflections of R^d. E acts on S' as follows. For $\gamma \in E$ and $f \in S$ let $(\gamma f)(x) = f(\gamma^{-1}x)$ and $(\gamma\phi)(f) = \phi(\gamma^{-1}f)$. Define $\theta \in E$ by $\theta(x_1,\ldots,x_d) = (x_1,\ldots,x_{d-1},-x_d)$. Following [FS 77] the POS Axioms for probability measures ν on $(S'(R^d), \sum)$ are the following:

POS 1: ν is invariant under E.

POS 2: $\int \overline{F\circ\theta}\, F\, d\nu \geq 0$ for all $\sum_+$-measureable functions F on $S'(R^d)$.

POS 3: There is a norm $|||\cdot|||$ continuous on $S'(R^d)$ such that $\int e^{\phi(f)} d\nu$ is uniformly bounded and continuous in the norm on $\{f \in S(R^d) : |||f||| \leq 1\}$

POS 4: The action of the time-translation subgroup τ is ergodic on $(S'(R^d), \sum, d\nu)$. That is, for all $A \in L'(d\nu)$ and $\phi \in S'(R^d)$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(\tau_s\phi)ds = \int A\, d\nu .$$

POS 1 embodies the requirement that the real time field theory be Lorentz covariant. POS 2 gives a positive definite inner product on the Hilbert space of states of the theory. The third axiom is a regularity condition and guarantees that the moments of $d\nu$ exist. Finally, POS 4 is equivalent to uniqueness of the vacuum. The question of whether or not POS 4 is satisfied is closely connected with the study of phase transitions.

The simplest example of a measure satisfying all the POS axioms is the Gaussian measure with mean zero and covariance $C = (-\Delta + m^2)^{-1}$, where Δ

is the Laplacian on R^d and $m > 0$. We shall denote this measure by $d\mu_C$ or $d\mu_{m^2}$ depending on the context. It is the unique measure on $(S'(R^d), \sum)$ whose Fourier transform is $\int e^{i\phi(f)} d\mu_C = e^{-\frac{1}{2}(f,Cf)}$ $L^2(R^d)$; the existence of this measure in guaranteed by Minlos' Theorem [GV IV 64, p. 350].

The Gaussian measure $d\mu_C$ is the starting point for the construction of measures corresponding to two-dimensional interacting scalar boson fields, namely the $P(\phi)_2$ models. These models describe a scalar boson field in two dimensions with mass m and a polynomial self-interaction, and give the quantization of the classical field theory with (Euclidean) Lagrangian density $L(\phi(x), \nabla\phi(x)) = \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2} m^2\phi(x)^2 + P(\phi(x))$. Suppose P is a polynomial on R which is semibounded (bounded below) and let Λ be a square centred at the origin with sides parallel to the coordinate axes. Set

$$d\nu^B_{\Lambda,P} = \frac{e^{-\int_\Lambda :P(\phi(x)):_{C^B} dx} d\mu_{C^B}}{\int e^{-\int_\Lambda :P(\phi(x)):_{C^B} dx} d\mu_{C^B}} \tag{1.1}$$

where $C^B = (-\Delta^B + m^2)^{-1}$ and Δ^B is the Laplacian with B-boundary conditions on $\partial\Lambda$. B may be periodic, free, or Dirichlet for example; a particular choice is often preferred for technical reasons. The Wick dots $: :_{C^B}$ indicate that P has been normal ordered with respect to C_B and are defined in §2.2. One then attempts to show that $d\nu^B_{\Lambda,P}$ approaches a limit as $\Lambda \uparrow R^2$ and that the limiting measure satisfies POS 1-4.

To state the result of [FS 77] concerning the existence of these limits it is necessary to introduce the pressure in an external field μ , defined by

$$\alpha(\mu) = \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda :P(\phi)-\mu\phi:_{C^B}} d\mu_{C^B} \tag{1.2}$$

Here $|\Lambda|$ is the volume of Λ and we have dropped the dummy variable x from the interaction. It is shown in [GRS 76] that the limit in equation (1.2) exists and is independent of the choice of B for a wide class of boundary conditions. It is an easy consequence of Hölder's inequality that α is convex. In [FS 77] it is shown that α is in fact strictly convex. From convexity it follows that the derivative $D\alpha(\mu)$ exists for all but countably many μ, and the right and left derivatives $D^{\pm}\alpha(\mu)$ exist for all μ. Let $P_\mu(x) = P(x) - \mu x$. Fröhlich and Simon prove the following theorem in [FS 77].

<u>Theorem 1.1</u>: If $D^+\alpha(\mu) = D^-\alpha(\mu)$ then the measures $d\nu^B_{\Lambda,P_\mu}$ given by eqn. (1.1) converge to the same limit $d\nu_{P_\mu}$ as $\Lambda \uparrow R^2$ (B = free, periodic, Dirichlet) in the sense of convergence of Fourier transforms, and the measure $d\nu_{P_\mu}$ satisfies POS 1-4. □

More generally, for any μ they construct two measures $d\nu^{\pm}_{P_\mu}$ corresponding to the interaction P_μ which satisfy POS 1-4, and are equal if and only if $D^+\alpha(\mu) = D^-\alpha(\mu)$, in which case $d\nu^{\pm}_{P_\mu} = d\nu_{P_\mu}$. The existence of a μ for which $d\nu^+_{P_\mu} \neq d\nu^-_{P_\mu}$ corresponds to the existence of a phase transition for the theory. Phase transitions are discussed in the following section.

§2. Phase Transitions

In [FS 77] it is shown that

$$D^{\pm}\alpha(\mu) = \int \phi(0)\,d\nu^{\pm}_{P_\mu} \quad \text{for all} \quad \mu \in R \tag{2.1}$$

where the one-point $\int \phi(0)\,d\nu^{\pm}_{P_\mu}$ is the number satisfying

$\int \phi(f)\,d\nu^{\pm}_{P_\mu} = \int \phi(0)\,d\nu^{\pm}_{P_\mu} \int f(x)dx$ for all $f \in \mathcal{S}(R^2)$. Such a number exists by translation invariance of $d\nu^{\pm}_{P_\mu}$. When α is differentiable at μ eqn. (2.1) is what is obtained by formally differentiating eqn (1.1). If α is not differentiable at a point μ_0 then $\int \phi(0)\,d\nu^{+}_{P_{\mu_0}} \neq \int \phi(0)\,d\nu^{-}_{P_{\mu_0}}$ and so $\int \phi(0)\,d\nu_{P_\mu}$ is discontinuous at μ_0. When this happens it is said that there is a phase transition at μ_0, because a continuous change in the parameter μ results in a discontinuous change in the theory.

A number of results have been obtained in the last ten years concerning the existence of phase transitions for various polynomials. In [SG 73] it was shown that for $P(x) = ax^4 - bx^2$ with $a > 0$ and $b \in R$, $\alpha(\mu)$ has an analytic extension to the complement of the imaginary axis and hence a phase transition can only occur for $\mu = 0$. It was shown in [GJS 75] that for some values of a and b a phase transition does occur at $\mu = 0$, and the individual phases were studied in [GJS 76] using a low temperature cluster expansion. Low temperature expansions were used in [I 81] to obtain detailed information about multi-phase theories, where different phases are obtained by varying not only the external field but also the other coefficients of P.

There is a classical intuition at work in [GJS 76] and [I 81] which we now describe. Let $U_0(x) = P(x) + \frac{1}{2} m^2x^2$ be the classical potential, and consider $U_0(\hbar,x) = \hbar^{-1}U_0(\hbar^{\frac{1}{2}}x)$, for $\hbar$ small and positive. Here $\hbar$ is Planck's constant divided by 2π, and $\hbar \downarrow 0$ is the classical limit. If U_0 attains its global minimum more than once, decreasing $\hbar$ has the effect of separating the minima and raising the barrier(s) between them. For deep and widely separated minima of positive curvature (small $\hbar$) the existence of more than one phase is expected, because of the contribution of each minimum to the exponent in the functional integral defining the pressure. If one the other hand U_0 has a uniquely attained global minimum of positive curvature then $U_0(\hbar;\cdot)$ will have that minimum magnified with respect to the others so that for small enough $\hbar$ a unique phase is expected - only the global minimum contributes significantly to the functional integral. Positive curvature is required to ensure a positive mass for each phase. The work of [GJS 76] and [I 81] provides some justification for this intuition in the case where U_0 attains its global minimum more than once. In Lemma 4.4.1 we show that the intuition is justified in the case of a uniquely attained global minimum.

While the classical potential U_0 provides a rule of thumb test for the occurrence of phase transitions, a rigorous test is provided by the effective potential (which as we shall see is a quantum analogue of U_0). The effective potential was first introduced in [GSW 62]. Based on ideas of Jona-Lasinio [J-L 64], it is defined in [CW 73] essentially to be the Legendre transform of the pressure in an external field. Since the Legendre transform does not always exist we find it convenient to use instead the unique convex extension of the Legendre transform, i.e., the convex conjugate or Fenchel transform [Fen 49] of the pressure. We define the effective potential V for the $P(\phi)_2$ model by

$$V(a) = \sup_{\mu \subset R} [\mu a - \alpha(\mu)] , \quad a \in R . \tag{2.2}$$

As we shall see in Theorem 4.1, $V(a)$ is always finite. By definition V is convex and thus cannot have a double well structure. The variable a is known as the classical field, since the supremum in eqn. (2.2) is attained at a value of μ satisfying

$$a = D\alpha(\mu) = \int \phi(0) d\nu_{P_\mu} ,$$

provided such a μ exists.

The importance of V for the study of phase transitions is a consequence of the fact that points of nondifferentiability of a convex function are in a one-one correspondence with linear portions of its convex conjugate. (See §2.1 for a brief review of convex function theory). That is, linear portions of V are in a one-one correspondence with the occurrence of phase transitions. Two examples of the relationship between the one-point function, the pressure, and the effective potential are shown in Figure 1.

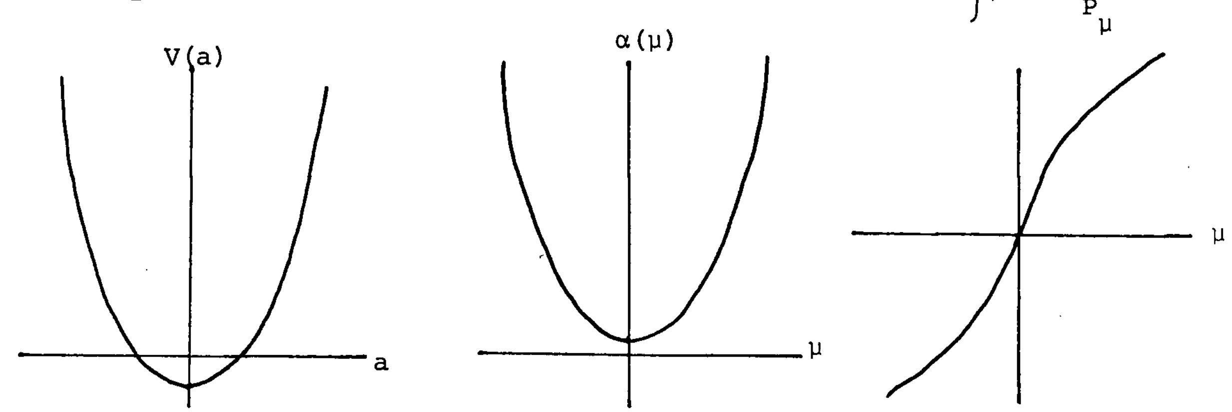
The most common method for the calculation of the effective potential is the loop expansion [CW 73], [Jack 74], which provides a power series expansion in $\hbar$. Until now the dependence of V on $\hbar$ has been suppressed. Putting the $\hbar$'s in explicitly, the pressure is given by

$$\alpha(\hbar,\mu) = \lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{\frac{-1}{\hbar}\int_\Lambda : P(\phi) - \mu\phi :_{\hbar C}} d\mu_{\hbar C} \tag{2.3}$$

and the effective potential by

Figure 1: The effective potential and phase transitions

(a) No phase transition



(b) Phase transition

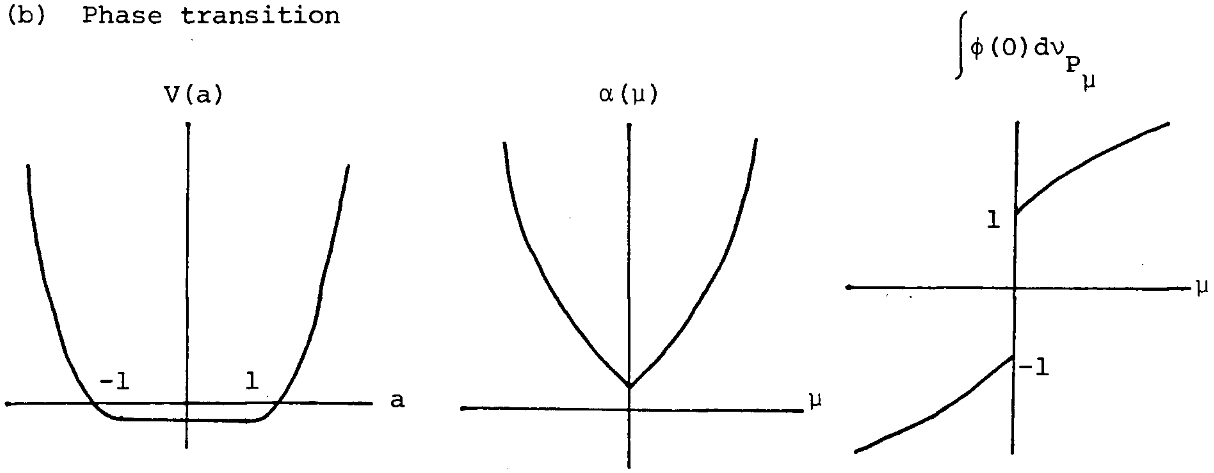


Figure 1 depicts the relationship between the effective potential, the pressure and the one-point function in two examples of an even classical potential.

In (a) the absence of a linear portion in V implies that α is differentiable everywhere and hence the one point function

$$D\alpha(\mu) = \int \phi(0)\, d\nu_{P_\mu} \quad \text{is continuous.}$$

In (b) the interval $[-1, 1]$ on which V is linear with slope zero corresponds to the fact that the pressure α is not differentiable at $\mu = 0$ with left and right derivatives -1 and $+1$ respectively, and hence the one point function jumps from -1 at $\mu = 0^-$ to $+1$ at $\mu = 0^+$.

$$V(\hbar,a) = \sup_{\mu\in R} \; [\mu a - \alpha(\hbar,\mu)] \; . \tag{2.4}$$

V is then approximated by expanding in a power series in $\hbar$ and keeping the first few terms: $V(\hbar,a) \approx \sum_{n=0}^{N} v_n(a)\hbar^n$. Typically N is one or two. In four dimensions, the approximation with $N = 1$ has been used in cosmology [Bran 82]. As we shall see in Theorem 4.2, $v_0(a) = U_o(a)$ for most values of a , so $U_0(a)$ is the classical limit of $V(\hbar,a)$.

In the physics literature it is argued that $v_n(a)$ is given by a certain sum of one-particle irreducible n-loop Feynman graphs. (We describe the graph notation in the next section). The main result of this thesis is a proof that for most values of a the expansion of V as a power series in $\hbar$ is asymptotic in the $P(\phi)_2$ theory, with a proof that the coefficients of the expansion are given by the appropriate sum of graphs. We also give an example (in Theorem 4.5) where the one-particle irreducible loop expansion fails for certain values of a .

§3. Graph Notation

Feyman graphs provide a convenient notation for representing integrals of a form that arises frequently in quantum field theory. In this section the graph notation used in this thesis is explained. To begin with an example and a fixed translation invariant covariance $C(x,y) = C(x-y)$, the graph λ_1  λ_2 λ_3 is by definition equal to

λ_1  λ_2 λ_3

$$= \lambda_1\lambda_2\lambda_3 \int dx_1 \; dx_2 C(0_1x_1)C(0_1x_2)C(x_1x_2)^2 \tag{3.1}$$

The right side of eqn. (3.1) is obtained from the left side by identifying any one vertex as the origin in R^2 and associating with the remaining vertices the variables x_1 and x_2. To every line there corresponds a factor of C evaluated at the endpoints of the line. These factors are multiplied together, integrated over R^2 with respect to x_i, and the result is multiplied by the vertex factors λ_i. This procedure is followed to obtain the value of any graph. Usually the vertex factors depend only on the number of lines emanating from a vertex and are understood to be part of the graph without writing them explicitly.

These graphs arise via Wick's Theorem [GJ 81], which gives the expectation with respect to the Gaussian measure of covariance C (denoted $\langle\cdot\rangle_C$) of a product of Wick-ordered monomials as a sum of graphs. In particular, introducing the semi-colon notation defined by

$$\langle F_1(\phi);F_2(\phi)\rangle_C = \langle F_1(\phi)F_2(\phi)\rangle_C - \langle F_1(\phi)\rangle_C \langle F_2(\phi)\rangle_C$$

$$\langle F_1(\phi);\dots;F_n(\phi)\rangle_C = \sum_{\pi\in P_n} (-1)^{|\pi|+1}(|\Pi|-1)! \prod_{j=1}^{|\pi|} \prod_{i\in\pi_j} \langle F_i(\phi)\rangle_C \tag{3.2}$$

where P_n is the set of partitions of $\{1,2,\dots,n\}$ and $\pi = \{\pi,\dots,\pi_{|\pi|}\}$,

it is a standard fact that for any translation invariant covariance C_Λ that approaches $(-\Delta+m^2)^{-1}$ as $\Lambda \uparrow R_2$, (for example $C_\Lambda = (-\Delta_\Lambda^P+m^2)^{-1}$ with Δ_Λ^P the Laplacian with periodic boundary conditions on $\partial\Lambda$),

$\lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \langle :\phi^{k_1}(\Lambda):;\dots;:\phi^{k_m}(\Lambda):\rangle_{C_\Lambda}$ is given by the sum of all connected graphs with no self-lines, that can be made up of m vertices with k_i legs $(i=1,\dots,m)$ and lines of covariance $(-\Delta+m^2)^{-1}$. Here a self-line is a line that connects a vertex to itself and a connected graph is one for which any two vertices are path-connected by lines in the graph. Graphs are usually taken to include certain combinatorial factors; we explain our convention for combinatorial factors in the next section.

Expectations of the form (3.2) occur most often as derivatives, as follows. Let $S(\lambda,\phi) = \sum_{k=1}^{n} a_k(\lambda):\phi^k(\Lambda):$. Then using induction it can be shown that for certain positive integers C_π

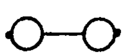
$$\frac{d^k}{d\lambda^k} \ln \int e^{-S(\lambda,\phi)} d\mu_C = \sum_{\pi\in P_k} C_\pi \left\langle -D_1^{|\pi_1|} S(\lambda,\phi); \ldots; -D_1^{|\pi_{|\pi|}|} S(\lambda,\phi)\right\rangle_{S,C} \tag{3.3}$$

where $$\left\langle \cdot \right\rangle_{S,C} = \frac{\int \cdot \, e^{-S(\lambda,\phi)} d\mu_C}{\int e^{-S(\lambda,\phi)} d\mu_C} .$$

If $a_k(\lambda) \to 0$ as $\lambda \to 0$, then

$$\left.\frac{d^k}{d\lambda^k}\right|_0 \ln \int e^{-S(\lambda,\phi)} d\mu_C = \sum_{\pi\in P_k} C_\pi \left\langle -D_1^{|\pi_1|} S(0,\phi); \ldots; -D_1^{|\pi_{|\pi|}|} S(0,\phi)\right\rangle_C$$

Note that a graph with a self-line is infinite because $(-\Delta+m^2)^{-1}(x,x)$ is infinite. In fact a graph with $(-\Delta+m^2)^{-1}$ lines is finite if and only if it is connected and has no self-lines.

A graph is said to be 1-PI (one-particle irreducible) if it is connected and if the removal of any one line leaves a connected graph. A graph is said to be 1-PR (one-particle reducible) if it is not 1-PI .
For example,  and   are 1-PR, while  and  are 1-PI.

A graph G with L lines and V vertices is said to be an n-loop graph, where $n = L - V + 1$. For example,  has $n = 3$,  has $n = 2$.

We close this section with another definition.

<u>Definition 3.1</u>: Given a graph G and $d \in R$, the <u>d-renormalized graph</u> G_d is the graph obtained by removing all self-lines from G , introducing a vertex factor d for each removed self-line, and introducing a factor $c_{k,j}$ for every k-legged vertex of G having j self-lines, where

$c_{k,j} = \dfrac{k!}{2^j j!(k-2j)!}$ is the number of ways of choosing j pairs from k objects. $\square$

For example, $G =$  $\Rightarrow G_d = c_{4,1} d$ 

$G =$  $\Rightarrow G_d = G$

$G =$  $\Rightarrow G_d = c_{6,3} d^3 \bullet$.

As we will see in Theorem 4.3, the $O(\hbar^n)$ contribution to $V(\hbar,a)$ is given by a sum of d-renormalized graphs, for a certain $d = d(a)$. The d-renormalized graphs arise from a Wick re-ordering procedure that introduces the combinational factors $c_{n,j}$ in a natural way (see eqn. (5.5.8)).

§4. <u>Main Results</u>

Before stating the main results, we introduce some notation. For a real function f we denote its convex hull (i.e., the greatest convex function majorized by f) by conv f . Recall the definition of the classical potential: $U_0(x) = P(x) + \frac{1}{2} m^2 x^2$. Let

$B_1 = \overline{\{a \in R : U_0(a) \neq (\text{conv } U_0)(a)\}}$, $B_2 = \{a \in R : U_0''(a) = 0\}$,

and $B = B_1 \cup B_2$.

The remainder of the thesis is devoted to the proof of the following theorems for the $P(\phi)_2$ model.

The first theorem states that the supremum in the definition of $V(\hbar,a)$ is finite.

Theorem 4.1: $V(\hbar,a) < \infty$ for all $\hbar > 0$, $a \in R$.

The second theorem gives the result for the classical limit of the effective potential that was anticipated in §1.2.

Theorem 4.2: $\lim_{\hbar \downarrow 0} V(\hbar,a) = (\text{conv } U_0)(a)$ for every $a \in R$.

For $a \notin B$, the following theorem provides a rigorous proof for the $P(\phi)_2$ model of a standard (non-rigorous) result in quantum field theory [IZ 80] [Ram 81], namely that $V(\hbar,a)$ can be approximated by the first terms of the one-particle irreducible loop expansion.

Theorem 4.3: (a) Let $a \notin B$. Then there exists a $\gamma > 0$ such that $V(\hbar,a)$ is analytic in $\hbar$ for $\hbar \in (0,\gamma)$. Moreover, $V(\hbar,a)$ is C^∞ at $\hbar = 0^+$ (i.e., all right-hand derivates exist at $\hbar = 0$) , and so the expansion $V(\hbar,a) \sim \sum_{n=0}^{\infty} v_n(a)\hbar^n$ is asymptotic, where

$v_n(a) = D_1^n V(0^+,a)/n!$.

(b) Let $a \notin B$. Then $v_0(a) = U_0(a)$ and

$$v_1(a) = -\gamma(a) \equiv -\lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda \frac{P''(a)}{2} : \phi^2 :} d\mu_C .$$

For $n \geq 2$, $-v_n(a)$ is the (finite) sum of all $d(a)$-renormalized 1-PI n-loop diagrams with k-legged vertices taking factors $-P^{(k)}(a)/k!$ $(3 \leq k \leq \deg P)$ and lines corresponding to the free covariance of mass $\sqrt{U_0''(A)}$, where $d(a) = -\frac{1}{4\pi} \log \frac{U_0''(a)}{m^2}$. A combinatorial factor is associated with each graph - see Remark 1 below.

Remark 1: The renormalized graphs in $-v_n(a)$ are to be understood to include combinatorial factors. Given a renormalized graph, let V_{kj} be the number of vertices that originally had k legs and have been renormalized with the removal of j self-lines. The combinatorial factor for the graph is the factor associated with the graph by Wick's theorem divided by $\prod_{j,k} V_{jk}!$. For example, the combinatorial factor of  is $\frac{1}{3!}$ 1728 = 288.

Note that since each vertex has at least 3 lines, the number of graphs contributing to $-v_n(a)$ is finite because the number of loops $n = L - V + 1 \geq \frac{3}{2} V - V + 1 = \frac{1}{2} V + 1$.

As an example of Theorem 4.3 we obtain a renormalized (and rigorous) version of a result of [Jack 74]. Let $U_0(x) = x^4 + \frac{1}{2} x^2$ and $P(x) = x^4$. Since U_0 is convex and $U_0''(a) = 12a^2 + 1 > 0$ for all a, B is the empty set. Then Theorem 4.3 implies that with $d(a) = -\frac{1}{4\pi} \log(1+12a^2)$,

$-v_2(a) = [$  $+$  $]_{d(a)} =$  $+ 3d(a)^2$,

and $-v_3(a) = [$  $+$  $+$  $+$  $+$  $+$  $]_{d(a)}$

$=$  $+ 6d(a)$  $+$  $+$  $+$  $+ 6^2 d(a)^2$ 

Lines are $(-\Delta+1)^{-1}$ lines and 3- and 4-legged vertices take factors 4a and 1 respectively. Amputated legs have been partly drawn to keep clear what the vertex factors should be.

After this research was completed work of Eckmann [E 77] was brought to the author's attention, in which the loop expansion for the Schwinger functions of the $P(\phi)_2$ model is shown to be asymptotic. Theorem 4.3, which gives an asymptotic loop expansion for the effective potential, is a natural follow-up to Eckmann's work. We comment in Chapter 3 where some of our estimates mirror those of Eckmann.

In [F 76], it is shown that for ϕ^4 models V is analytic in certain a, with $\hbar$ fixed. The next theorem gives sufficient conditions for analyticity of V in a for general polynomials.

<u>Theorem 4.4</u>: Let $K \subset B^c$ be compact. There exists a $\gamma > 0$ and an open set $0 \supset K$ such that $V(\hbar,\cdot)$ has an analytic extension to 0 for every $\hbar \in (0,\gamma)$.

There are three main ingredients to the proofs of these theorems. The first step is to reduce analyticity properties of $V(\hbar,a)$ to analyticity properties of $\hbar\alpha(\hbar,\mu)$ by some elementary convex analysis. This is done in §2.1. The proof that the pressure has the required analyticity is via a high temperature cluster expansion [GJS 73], and appears in Chapter 3. The proof of Theorem 4.3(b) uses the third ingredient: an irreducibility analysis in the spirit of [CFR 81], which is the subject of Chapter 5.

Finally, in Chapter 6 we prove the following result which gives an asymptotic expansion for $V(\hbar,a)$ when a is the bad set, for the classical potential $U_0(a) = (a^2 - \frac{1}{8})^2$.

<u>Theorem 4.5</u>: Let $V(\hbar,a)$ denote the effective potential for $m = 1$ and

$P(x) = (x^2 - \frac{1}{8})^2 - \frac{1}{2}x^2$. Then for $|a| < \frac{1}{\sqrt{8}}$, $D_1 V(0,a) = -\gamma(\frac{1}{\sqrt{8}}) = 0$, and for $n \geq 2$, $-\frac{1}{n!} D_1^n V(0,a)$ is given by the sum of all n-loop connected graphs with no self-lines, with three- and four-legged vertices taking factors $\frac{-1}{3!}P^{(3)}(\frac{1}{\sqrt{8}}) = -\sqrt{2}$ and $\frac{-1}{4!}P^{(4)}(\frac{1}{\sqrt{8}}) = -1$ respectively, and lines corresponding to the free covariance of mass 1 . Graphs take combinatorial factors as per Remark 1.

A number of authors [FOR 83], [BC 83], [CF 83] have recently calculated the $O(\hbar)$ contribution to the effective potential corresponding to the classical potential considered in Theorem 4.5. They find that the correct $O(\hbar)$ approximation to the effective potential is the straight line interpolation of the $O(\hbar)$ approximation given for $|a| > \frac{1}{\sqrt{8}}$ by Theorem 4.3. Theorem 4.5 gives a rigorous justification of this fact; the proof is an easy consequence of using the Fenchel transform to define $V(\hbar,a)$ and the known fact that there is a phase transition in this model if $\hbar$ is sufficiently small [GJS 76]. The observation that the n^{th} order contribution for $|a| < \frac{1}{\sqrt{8}}$ is the connected graphs rather than the 1-PI graphs appears to be new.

Chapter 2: PRELIMINARIES

§1. Convex Functions

We begin this section by stating some well-known properties of convex functions. Proofs can be found in [Rock 70] or [RV 73]. Theorems 1.1 and 1.2 below will be used in the proofs of Theorems 1.4.2 and 1.4.3 respectively.

Given a convex function $f : R \to R$, its convex conjugate is defined to be

$$f^*(a) = \sup_{\mu \in R} [\mu a - f(\mu)] \quad , \quad a \in R . \tag{1.1}$$

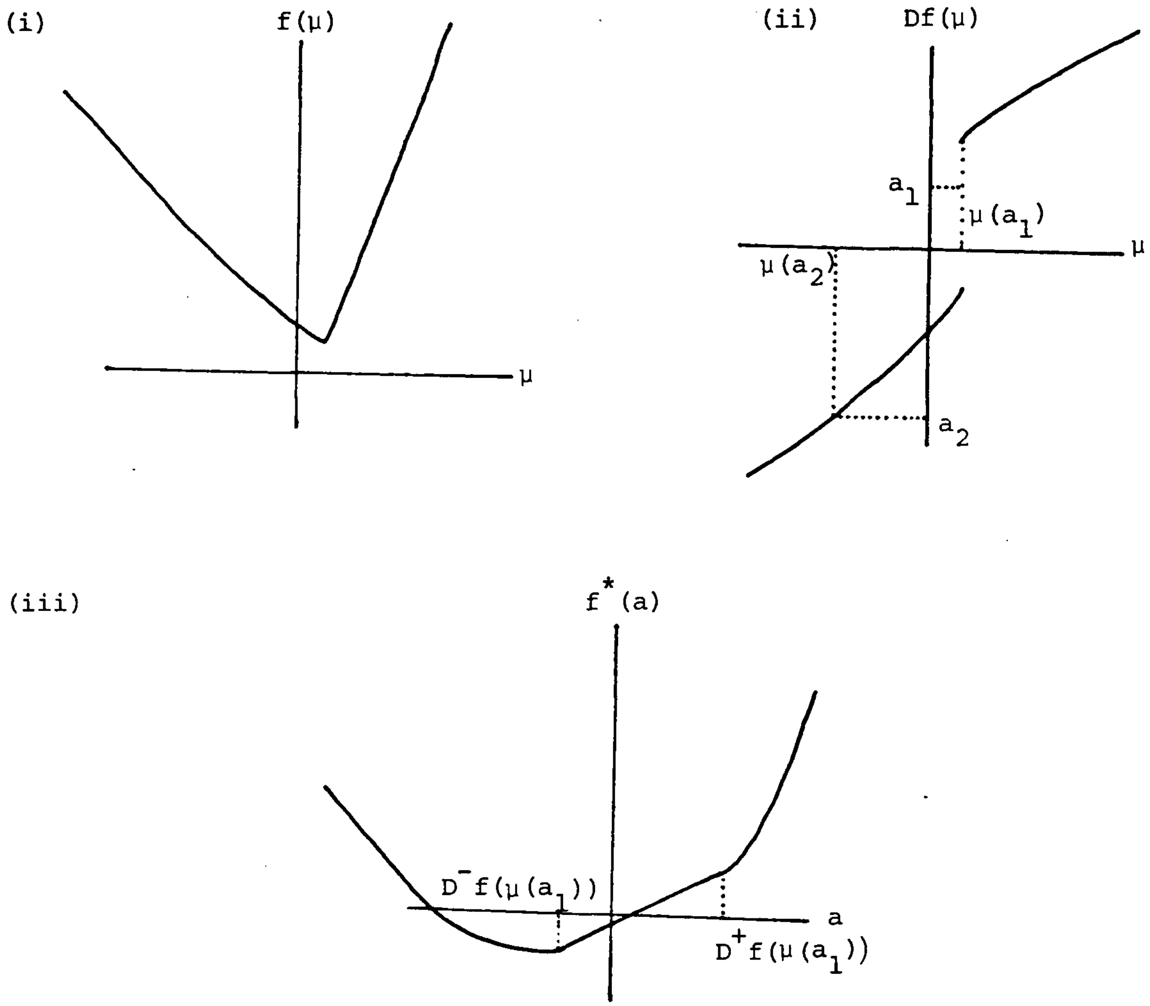
Since f is convex it is continuous and the right and left derivatives $D^{\pm}f$ exist everywhere and are nondecreasing. Let $M = \sup_{\mu} D^+f(\mu)$ and $m = \inf_{\mu} D^-f(\mu)$. Then for $a \in (m,M)$ the supremum in equation (1.1) is finite and is attained at any μ for which $D^-f(\mu) \leq a \leq D^+f(\mu)$. If f is strictly convex then there is one and only one such μ, which we denote $\mu(a)$. If $a < m$ or $a > M$ then the supremum is $+\infty$. In Figure 2 the relationship between Df and $\mu(a)$ is depicted graphically.

We denote by $\mathcal{C}(\mathcal{C}_s)$ the class of convex (strictly convex) functions f on R for which $\lim_{\mu \to \pm\infty} D^{\pm}f(\mu) = \pm\infty$. For $f \in \mathcal{C}$, $f^*(a) < \infty$ for all a.

A property of the convex conjugate we have already mentioned in §1.2 is that points of nondifferentiability of f are in a one-one correspondence with linear portions of f^* (see Figure 2). The precise correspondence is that $D^-f(\mu) \neq D^+f(\mu)$ if and only if f^* is linear with slope μ on the interval $[D^-f(\mu), D^+f(\mu)]$.

For any convex function f, $f^{**} = f$. It is possible to define the conjugate of an arbitrary function f by the formula (1.1) but in general $f^{**} \neq f$. Something can be said however about the relationship of f^{**} to f; to avoid subtleties associated with infinite-valued functions we only mention that for Q a semi-bounded polynomial, $Q^{**} = \text{conv}Q$. Here $\text{conv}Q$ is the convex hull of Q, i.e., the greatest convex function majorized by Q.

Figure 2: The relationship between $Df(\mu)$ and $\mu(a)$ for $f \in C_s$



(i) Given a convex function $f \in C_s$, (ii) the point $\mu(a)$ at which $\sup_\mu[\mu a - f(\mu)]$ is attained is the unique μ for which $a \in [D^- f(\mu), D^+ f(\mu)]$.

(iii) If Df has a jump discontinuity at μ_0, then for $a \in [D^- f(\mu_0), D^+ f(\mu_0)]$ $f^*(a) = \mu_0 a - f(\mu_0)$, so f is linear on $[D^- f(\mu_0), D^+ f(\mu_0)]$ with slope μ_0.

Convex functions are well-behaved with respect to convergence properties. For example, if $f(\hbar,\cdot)$ is convex for all $\hbar > 0$ and $f(\mu) = \lim_{\hbar\downarrow 0} f(\hbar,\mu)$ exists for all μ in a dense subset of R, then $f(\mu) = \lim_{h\downarrow 0} f(\hbar,\mu)$ for all $\mu \in R$, the convergence is uniform on compact subsets of R, and the limit function f is convex. It also follows that

$D^-f(\mu) \leq \underline{\lim}_{\hbar\downarrow 0}\ D_2^-f(\hbar,\mu) \leq \overline{\lim}_{\hbar\downarrow 0}\ D_2^+f(\hbar,\mu) \leq D^+f(\mu)$, for all $\mu \in R$. In particular, if f is differentiable at μ then

$$\lim_{\hbar\downarrow 0} D_2^-f(\hbar,\mu) = \lim_{\hbar\downarrow 0} D_2^+f(\hbar,\mu) = Df(\mu)\ . \tag{1.2}$$

We now give conditions on a family $f(\hbar,\cdot)$ of convex functions which imply smoothness of $f^*(\hbar,a)$, beginning with the following theorem.

<u>Theorem 1.1</u>: Suppose $f(\hbar,\cdot)$ and f are in C_s for all $\hbar > 0$, and suppose $\lim_{\hbar\downarrow 0} f(\hbar,\mu) = f(\mu)$ for all μ in a dense subset of R. Denote by $\mu(\hbar,a)$ and $\mu(a)$ the unique values of μ where $\mu a - f(\hbar,\mu)$ and $\mu a - f(\mu)$ attain their suprema. Then $\lim_{\hbar\downarrow 0} \mu(\hbar,a) = \mu(a)$ and $\lim_{\hbar\downarrow 0} f^*(\hbar,a) = f^*(a)$.

<u>Proof</u>: We first prove that $\lim_{\hbar\downarrow 0} \mu(\hbar,a) = \mu(a)$. Fix $a \in R$ and $\varepsilon > 0$. Choose $\rho \in (0,\varepsilon)$ such that $Df(\mu(a) \pm \rho)$ exist. Let $\alpha = \frac{1}{2}\min\{Df(\mu(a)+\rho) - D^+f(\mu(a)),\ D^-f(\mu(a)) - Df(\mu(a)-\rho)\}$. Since f is strictly convex, $\alpha > 0$. Then $Df(\mu(a)+\rho) > D^+f(\mu(a)) + \alpha \geq a + \alpha$ and similarly $Df(\mu(a)-\rho) < a - \alpha$. By eqn. (1.2) there is a $\delta > 0$ such that

$$\left|D^{\pm}f(\hbar,\mu(a)\pm\rho) - Df(\mu(a)\pm\rho)\right| < \frac{\alpha}{2} \quad \text{for all } \hbar < \delta\ .$$

Therefore $D_2^- f(\hbar,\mu(a) - \rho) < a < D_2^+ f(\hbar,\mu(a) + \rho)$ for all $\hbar < \delta$ and so $\mu(\hbar,a) \in [\mu(a) - \rho,\ \mu(a) + \rho]$ for all $\hbar < \delta$, and hence $\lim_{\hbar \downarrow 0} \mu(\hbar,a) = \mu(a)$.

Now $|f^*(\hbar,a) - f^*(a)| = |\sup_\mu [\mu a - f(\hbar,\mu)] - \sup_\mu [\mu a - f(\mu)]|$.

But if $\sup_x a(x)$ and $\sup_x b(x)$ are attained at x_a and x_b respectively, then

$$|\sup_x a(x) - \sup_x b(x)| \le \max\{|a(x_a) - b(x_a)|\ ,\ |a(x_b) - b(x_b)|\}$$

$$\le \sup_{x \in [x_a, x_b]} |a(x) - b(x)|\ ,\ \text{assuming } x_a \le x_b\ .$$

Therefore for any $\hbar < \delta$, $|f^*(\hbar,a) - f^*(a)| \le \sup_{\mu \in [\mu(a)-\rho,\mu(a)+\rho]} |f(\hbar,\mu) - f(\mu)|$.

Since $f(\hbar,\mu) \to f(\mu)$ uniformly on compact intervals, the right side goes to zero as $\hbar \downarrow 0$. $\square$

Theorem 1.2: Suppose $f(\hbar,\cdot)$ and f belong to the set C_s, with $\lim_{\hbar \downarrow 0} f(\hbar,\mu) = f(\mu)$ for all $\mu \in R$. Let $A = \{a \in R :$ there is no μ with $D^+f(\mu) = D^-f(\mu) = a\}$. Fix $a \notin A$ and suppose that for some $\gamma > 0$ there is an open interval I containing $\mu(a)$, such that f is analytic in $(\hbar,\mu) \in (0,\gamma) \times I \subset C^2$ and

$$|D_2^2 f(\hbar,\mu)| \ge C > 0 \quad \text{for every} \quad (\hbar,\mu) \in (0,\gamma) \times I\ . \tag{1.3}$$

Then for some $\gamma' > 0$, $f^*(\hbar,a)$ is analytic in $\hbar \in (0,\gamma')$.

If in addition the mixed partial derivatives of f are uniformly bounded in $(\hbar,\mu)$, i.e., there are constants $M_{m,n}$ such that

$$|D_1^m D_2^n f(\hbar,\mu)| \le M_{m,n} \quad \text{for every } (\hbar,\mu) \in (0,\gamma) \times I;\ m,n = 0,1,2,\ldots \tag{1.4}$$

then $f^*(\hbar,a)$ is C^∞ at $\hbar = 0^+$ with $D_1^n(0^+,a) = \lim_{\hbar\downarrow 0} D_1^n f^*(\hbar,a)$, $n = 0,1,2,\ldots$.

Remark: If it is assumed that f is C^∞ rather than analytic in $(0,\gamma) \times I$, the same proof gives that $f^*(\cdot,a)$ is C^∞ in $[0,\gamma')$.

Proof: By Theorem 1.1 we can choose $\gamma' < \gamma$ such that $\mu(\hbar,a) \in I$ if $\hbar < \gamma'$. Also, it follows from analyticity of f and the bound (1.3) that there is a neighbourhood $0_\gamma \supset (0,\gamma) \times I$ on which $|D_2^2 f(\hbar,\mu)| > \frac{C}{2}$.

Let $g(\hbar,\mu) = \frac{\partial}{\partial\mu}[\mu a - f(\hbar,\mu)] = a - D_2 f(\hbar,\mu)$ for $(\hbar,\mu) \in V_{\gamma'}$, where we set $V_{\gamma'} = 0_\gamma \cap \{\hbar,\mu) \in C^2 : 0 < \mathrm{Re}\,\hbar < \gamma'\}$. Then $\mu(\hbar,a)$ is uniquely defined by $g(\hbar,\mu(\hbar,a)) = 0$, for $\hbar < \gamma'$. By the fact that $|D_2^2 f(\hbar,\mu)| \geq \frac{C}{2}$ on 0_γ and the implicit function theorem [Hörm 73] it follows that $\mu(\hbar,a)$ is analytic in $\hbar$ in an open neighbourhood $U_{\gamma'} \supset (0,\gamma')$, with $(\hbar,\mu(\hbar,a)) \in V_{\gamma'}$ for all $\hbar \in U_{\gamma'}$. Therefore $f^*(\hbar,a) = \mu(\hbar,a)\, a - f(\hbar,\mu(\hbar,a))$ is analytic in $\hbar \in U_{\gamma'}$.

Suppose now that the bounds (1.4) hold. We show this gives upper bounds on the absolute values of derivatives $D_1^n\mu(\hbar,a)$ uniform in $\hbar \in (0, \gamma')$. In fact $\mu(\hbar,a)$ is defined by the equation

$$g(\hbar,\mu(\hbar,a)) = a - D_2 f(\hbar,\mu(\hbar,a)) = 0 . \tag{1.5}$$

Differentiating eqn. (1.5) with respect to $\hbar$ gives

$$- D_1 D_2 f(\hbar,\mu(\hbar,a)) - D_2^2 f(\hbar,\mu(\hbar,a)) D_1\mu(\hbar,a) = 0 ,$$

i.e.,
$$D_1\mu(\hbar,a) = \frac{- D_1 D_2 f(\hbar,\mu(\hbar,a))}{D_2^2 f(\hbar,\mu(\hbar,a))} . \tag{1.6}$$

It follows from the bounds (1.3) and (1.4) that $|D_1\mu(\hbar,a)|$ is uniformly bounded in $\hbar \in (0,\gamma')$. Repeated differentiation of eqn. (1.6) together with (1.3) and (1.4) gives uniform bounds on the higher order derivatives.

These bounds on $|D_1^n\mu(\hbar,a)|$ and the bound (1.4) imply that $|D_1^n f^*(\hbar,a)| \leq M_n < \infty$ uniformly in $\hbar \in (0,\gamma')$. But this implies that $f^*(\hbar,a)$ is C^∞ at $\hbar = 0^+$. To see this, note that $|D_1^n f^*(x,a) - D_1^n f^*(y,a)| = |\int_x^y D_1^{n+1} f^*(s,a)ds| \leq M_{n+1}(|y| + |x|)$ for all $x, y \in (0,\gamma')$. Therefore $\{D_1^n f^*(\hbar,a)\}_{\hbar>0}$ is Cauchy and so $d_n = \lim_{\hbar\downarrow 0} D_1^n f^*(\hbar,a)$ exists, $n \geq 0$. But for $n \geq 1$,

$$\left|\frac{D_1^{n-1} f^*(\hbar,a) - d_{n-1}}{\hbar} - d_n\right| = \left|\frac{1}{\hbar}\int_0^\hbar (D_1^n f^*(s,a) - d_n)ds\right|$$

$$\leq \sup_{0<s<\hbar} |D_1^n f^*(s,a) - d_n|$$

Since the right side goes to zero as $\hbar \downarrow 0$, $D_1^n f^*(0^+,a)$ exists and equals $\lim_{\hbar\downarrow 0} D_1^n f^*(\hbar,a)$, $n \geq 1$. $\square$

§2. Some Useful Transformations

This section contains some standard facts about Wick ordering and functional integrals that will be needed later. We begin by defining Wick order. Let C be a covariance operator, and let $h \in C_0^\infty(R^2)$ be positive with $\int h(x)dx = 1$. Define the approximate δ function at $x \in R^2$ by $\delta_{r,x}(y) = r^2 h(r(x-y))$, $r \geq 1$. The ultraviolet cutoff field ϕ_r is given by $\phi_r(x) = \phi(\delta_{r,x})$ and the cutoff Wick powers by

$$:\phi_r(x)^n:_C = \sum_{j=0}^{[n/2]} (-1)^j c_{nj}\sigma_r(x)^j \phi_r(x)^{n-2j} \tag{2.1}$$

where $c_{nj} = \dfrac{n!}{(n-2j)!j!2^j}$ and $\sigma_r(x) = \int \delta_{r,x}(y)C(y,z)\delta_{r,x}(z)dydz$.

For $V \subset \Lambda$, $m, m_1 > 0$, and $C = (-\Delta^P + m_1^2\chi_V + m^2\chi_{\Lambda\backslash V})^{-1}$ it is easy to see that there is an M such that $|\sigma_r(x)| \le M \log r$ for large r . (2.2)

If f has compact support and is an element of $L^p(\mathbb{R}^2)$ for some $p > 1$ then $:\phi^n(f):_C = \lim_{r\to\infty} :\phi_r^n(f):$ exists in $L^2(d\mu_C)$, where $:\phi_r^n(f): = \int :\phi_r^n(x):f(x)dx$. This defines the Wick monomials.

The following lemma provides a Wick re-ordering formula.

Lemma 2.1. [GRS 75], [Sp 74].

For V a finite union of lattice squares in Λ and $m, m_1 > 0$, let $C = (-\Delta+m^2)^{-1}$ and $C_1 = (-\Delta+m_1^2\chi_V+m^2\chi_{\Lambda\backslash V})^{-1}$ with periodic boundary conditions on $\partial\Lambda$. Then for any $\hbar > 0$ and $x \in \Lambda$,

$$:\phi^n(x):_{\hbar C} = \sum_{k=0}^{[n/2]} c_{nk}(\hbar d(V,\Lambda,x))^k :\phi(x)^{n-2k}:_{\hbar C_1} ,$$

where $d(\Lambda,\Lambda,x) = \dfrac{-1}{4\pi} \log \dfrac{m_1^2}{m^2} + K(\Lambda)$ with $K(\Lambda) \to 0$ as $\Lambda \uparrow \mathbb{R}^2$, and $|d(V,\Lambda,x)| \le |d(\Lambda,\Lambda,x)|$.

Proof: By a standard result [GJ 81, p. 168],

$$:\phi^n(x):_{\hbar C} = \sum_{k=0}^{[n/2]} c_{nk}[\hbar\delta c_V(x)\}^k :\phi^{n-2k}(x):_{\hbar C_1}$$

where

$$\delta c_V(x) = \lim_{y \to x} [C_1(x,y) - C(x,y)] .$$

Denote by Δ^F the Laplacian with free boundary conditions and by Δ^P the Laplacian with periodic boundary conditions on $\partial\Lambda$. Then for $a > 0$

$$(-\Delta^F + a^2)^{-1}(x,y) = (2\pi)^{-2} \int_{R^2} \frac{e^{ip(x-y)}dp}{p^2+a^2} . \tag{2.3}$$

Writing $nL = (n_1L, n_2L)$ for $n \in Z^2$ and L the side length of Λ,

$$(-\Delta^P + a^2)^{-1}(x,y) = \sum_{n \in Z^2} (-\Delta^F + a^2)^{-1}(x-y+nL) .$$

For $V = \Lambda$,

$$\begin{aligned}\delta c_\Lambda(x) &= \lim_{y\to x} [(-\Delta^P + m_1^2)^{-1}(x,y) - (-\Delta^P + m^2)^{-1}(x,y)] \\ &= \lim_{y \to x} [(-\Delta^F + m_1^2)^{-1}(x-y) - (-\Delta^F + m^2)^{-1}(x-y) + \\ &\qquad \sum_{n \in Z^2 \setminus \{0\}} ((-\Delta^F + m_1^2)^{-1}(x-y+nL) - (-\Delta^F + m^2)^{-1}(x-y+nL))]\end{aligned}$$

By eqn. (2.3), $\lim_{y\to x} [(-\Delta^F + m_1^2)^{-1}(x-y) - (-\Delta^F + m^2)^{-1}(x-y) = \frac{-1}{4\pi} \log \frac{m_1^2}{m^2}$.

Since $(-\Delta^F + a^2)^{-1}(z) \le \text{const}\cdot e^{-a|z|}$ for any $|z| \ge 1$, it follows that $d(\Lambda,\Lambda,x) = \frac{-1}{4\pi} \log \frac{m_1^2}{m^2} + K(\Lambda)$ with $K(\Lambda) \to 0$ as $\Lambda \uparrow R^2$.

To handle the case when $V \neq \Lambda$ we use the following Wiener integral representation for C_1 [GJ 81]:

$$C_1(x,y) = \int_0^\infty dt \int dW^t_{x,y}(w)\, e^{-\int_0^t ds[m_1^2\chi_V(w(s)) + m^2\chi_{\Lambda\setminus V}(w(s))]}$$

where $dW^t_{x,y}$ is Wiener measure on the torus Λ for paths starting at x

and ending at y at time t. Since the exponential factor of the integrand always lies between $e^{-m_1^2 t}$ and $e^{-m^2 t}$, $C_1(x,y)$ always lies between $(-\Delta^P+m_1^2)^{-1}(x,y)$ and $(-\Delta^P+m^2)^{-1}(x,y)$, and hence

$$|C_1(x,y) - C(x,y)| \le |(-\Delta^P+m_1^2)^{-1}(x,y) - (-\Delta^P+m^2)^{-1}(x,y)|$$

Therefore $|\delta c_V(x)| \le |\delta c_\Lambda(x)|$. □

The following four lemmas can all be seen on a formal level by writing

$$d\mu_C(\phi) = \frac{e^{-\frac{1}{2}\int_\Lambda [(\nabla\phi(x))^2 + m^2\phi(x)^2]dx} \prod_{x\in\Lambda} d\phi(x)}{\int e^{-\frac{1}{2}\int_\Lambda [(\nabla\phi(x))^2 + m^2\phi(x)^2]dx} \prod_{x\in\Lambda} d\phi(x)}$$

Lemma 2.2: [Sp 74], [GRS 76].

Let $w_i \in L^2(R^2)$ have compact support and $A(\phi) = \prod_{i=1}^{m} :\phi^{k_i}(w_i):$.

Let $a \in R$, P be a semibounded polynomial and $C = (-\Delta^P+m^2)^{-1}$ with periodic boundary conditions on $\partial\Lambda$. Then for $U_0(x) = P(x) + \frac{1}{2}m^2x^2$,

$$\int A(\phi)\, e^{-\int_\Lambda :P(\phi):} d\mu_C(\phi) = \int A(\psi+a)\, e^{-\int_\Lambda \sum_{k=0}^{n} \frac{U_0^{(k)}(a)}{k!}:\psi^k: - \frac{1}{2}m^2:\psi^2:} d\mu_C(\psi) .$$

Proof: The lemma follows by translation by a. □

Lemma 2.3: For $V \subset \Lambda$, $b + m^2 > 0$, and $C = (-\Delta^P+m^2)^{-1}$ with periodic BC on $\partial\Lambda$,

$$\frac{e^{-b\int_V :\phi^2:}\,d\mu_C}{\int e^{-b\int_V :\phi^2:}\,d\mu_C} = d\mu_{C_1} ,$$

where $C_1 = (-\Delta+(m^2+b)\chi_V + m^2\chi_{\Lambda\setminus V})^{-1}$ with periodic BC on $\partial\Lambda$.

Proof: See [GJ 81, §9.3].

Lemma 2.4: For A, P and C as in Lemma 2.2, and for any $\hbar > 0$,

$$\int A(\phi)\, e^{-\int_V :P(\phi):}\, d\mu_{\hbar C} = \int A(\hbar^{\frac{1}{2}}\phi)\, e^{-\int_V :P(\hbar^{\frac{1}{2}}\phi):}\, d\mu_C .$$

The Wick dots in each integrand match the corresponding measure.

Proof: This lemma follows by scaling the field [GRS 76]. □

Lemma 2.5: For A and P as in Lemma 2.2 and $\sigma > 0$,

$$\int A(\phi)\, e^{-\int_\Lambda :P(\phi):}\, d\mu_{C(\Lambda,m^2)} = \int A_\sigma(\phi)\, e^{-\frac{1}{\sigma^2}\int_{\sigma\Lambda} :P(\phi):}\, d\mu_{C(\sigma\Lambda,\sigma^{-2}m^2)} ,$$

where $C(\Lambda,m^2) = (-\Delta^P+m^2)^{-1}$ with periodic BC on $\partial\Lambda$, and

$$A_\sigma(\phi) = \prod_{i=1}^{m} :\phi^{k_i}(w_i^{(\sigma)}): \ , \text{ with } w_i^{(\sigma)}(x) = w_i(\sigma x) .$$

Proof: See [GJ 81]. □

§3. The Classical Potential

In Chapter 1 we defined the classical potential $U_0(x) = P(x) + \frac{1}{2}m^2x^2$ and the bad set $B = B_1 \cup B_2$, where $B_2 = \{a \in R : U_0''(a) = 0\}$ and $B_1 = \overline{\{a \in R : U_0(a) \neq (\text{conv } U_0)(a)\}}$. It is clear from their definitions that B_2 consists of at most $n-2$ points, where $n = \deg P$, while B_1 consists of a union of at most $\frac{n}{2} - 1$ finite closed intervals.

Let $U_\mu(x) = U_0(x) - \mu x$. Let

$$G_1 = \{\mu \in R : U_\mu \text{ has a uniquely attained global minimum}\} ,$$

and for $\mu \in G_1$ denote the location of the minimum by $\xi(\mu)$. Define $F = \{\mu \in G_1 : U_0''(\xi(\mu)) = 0\}$ and $G = G_1 \backslash F$. It is clear that ξ is strictly increasing on G_1, and hence F is finite. Let

$$m(\mu) = \min_x U_\mu(x) .$$

Then for $\mu \in G_1$, $m(\mu) = U_\mu(\xi(\mu))$.

In this section we prove the following facts about B, G, ξ and μ. The set G^c is finite. The sets B and G are related by ξ : $B^c = \xi(G)$. The functions ξ and m are analytic on G, with $m'(\mu) = -\xi(\mu)$ and $\xi'(\mu) = \frac{1}{U_0''(\xi(\mu))}$ for $\mu \in G$. It is not hard to see from the definition of ξ that ξ is strictly increasing and continuous on G_1, and discontinuous on G_1^c. We show that $\lim_{\mu\to\pm\infty} \xi(\mu) = \pm\infty$. This, together with the facts that $-m'(\mu) = \xi(\mu)$ and ξ is strictly increasing, implies that $-m \in \mathcal{C}_s$.

Finally we prove a technical lemma that will be needed in proving analyticity of the pressure in §3.4.

In preparation for proving that G^c is finite we prove the following lemma.

<u>Lemma 3.1</u>: Suppose $T(x) = \sum_{k=2m}^{n} t_k x^k$ attains its global minimum at $x = 0$ only, where $m, t_n, t_{2m} > 0$. Then there is a $\delta > 0$ such that $T(x) \geq \delta(x^n + x^{2m})$ for all $x \in R$.

<u>Proof</u>: Since T is bounded below, n is even. Therefore

$$T(x) - \frac{1}{2} t_n x^n = \frac{t_n}{2} x^n + \sum_{k=2m}^{n-1} t_k x^k \to \infty \quad \text{as} \quad |x| \to \infty ,$$

and there is a $K > 1$ such that

$$T(x) \geq \frac{1}{2} t_n x^n \geq \frac{1}{2} t_{2m} x^{2m} \quad \text{for} \quad |x| \geq K \tag{3.1}$$

To deal with small $|x|$, observe that

$$T(x) \geq t_{2m} x^{2m} - \sum_{k=2m+1}^{n} |t_k| |x^k| = t_{2m} x^{2m} [1 - \sum_{k=2m+1}^{n} \frac{|t_k|}{t_{2m}} |x|^{k-2m}]$$

Let $\varepsilon = \min\{1, \frac{1}{2}(\sum_{k=2m+1}^{n} \frac{|t_k|}{t_{2m}})^{-1}\}$. Then for $|x| \leq \varepsilon$,

$$T(x) \geq t_{2m} x^{2m} [1 - \varepsilon \sum_{k=2m+1}^{n} \frac{|t_n|}{t_{2m}}] \geq t_{2m} x^{2m} [1 - \frac{1}{2}] = \frac{1}{2} t_{2m} x^{2m} \geq \frac{1}{2} t_{2m} x^n . \tag{3.2}$$

Finally, let $a = \min_{\varepsilon \leq |x| \leq K} T(x) > 0$. For $\varepsilon \leq |x| \leq K$,

$$T(x) \geq a \geq \frac{a}{2K^n} x^n + \frac{a}{2K^{2m}} x^{2m} \geq \frac{a}{2K^n} (x^{2m} + x^n) \tag{3.3}$$

Let $\delta = \min\{\frac{1}{4} t_n, \frac{1}{4} t_{2m}, \frac{a}{2K^n}\}$. By equations (3.1)-(3.3),

$$T(x) \geq \delta(x^n+x^{2m}) \quad \text{for every} \quad x \in R . \quad \square$$

Lemma 3.2: G^c is finite.

Proof: First, since F is finite and $G^c = G_1^c \cup F$, it suffices to show that G_1^c is finite. Note that $U_\mu'(x) = 0$ if and only if $U_0'(x) = \mu$. For $|\mu|$ sufficiently large $U_0'(x) = \mu$ has a unique root and hence U_μ has a uniquely attained global minimum. It follows that there is an $N > 0$ such that $G_1^c \subset [-N, N]$. We claim that for all $\mu \in [-N, N]$, there is a deleted neighbourhood 0_μ of μ such that $0_\mu \cap G_1^c = \phi$. Given the claim, let $0'_\mu = 0_\mu \cup \{\mu\}$. There is a finite subcover $\{0'_{\mu_1},\ldots,0'_{\mu_m}\}$ of $[-N, N]$, and therefore $G_1^c \subset \{\mu_1,\ldots,\mu_\mu\}$. We now prove the claim, considering the cases $\mu \in G_1^c \cap [-N, N]$ and $\mu \in G_1 \cap [-N, N]$ separately.

Suppose $\mu \in G_1$, and let $W(x) = U_\mu(x+\xi(\mu)) - U_\mu(\xi(\mu))$. By Lemma 3.1 there is a $\delta > 0$ such that

$$W(x) \geq \delta x^n \quad \text{for all} \quad x \in R . \tag{3.4}$$

The claim is proved in this case provided it can be shown that there is a $\rho > 0$ such that

$$Z_\varepsilon(x) = U_{\mu+\varepsilon}(x+\xi(\mu)) - U_{\mu+\varepsilon}(\xi(\mu) = W(x) - \varepsilon x \tag{3.5}$$

has a uniquely attained global minimum for all ε with $|\varepsilon| < \rho$. By eqn. (3.4)

$$Z_\varepsilon(x) \geq \delta x^n - \varepsilon x \quad \text{for all} \quad x \in R . \tag{3.6}$$

We consider $\varepsilon > 0$; the case of $\varepsilon < 0$ is similar. Clearly for $\varepsilon > 0$ the global minimum of Z_ε is negative, and occurs in $\{x \in R : x > 0\}$. Now $Z_\varepsilon(x) < 0$ only if $x \in (0,(\varepsilon\delta^{-1})^{1/n-1}) \subset (0,(\rho\delta^{-1})^{1/n-1})$. Thus it suffices to show that there exists a $\rho > 0$ such that $Z'_\varepsilon(x) = 0$ has only one root in $(0, (\rho\delta^{-1})^{1/n-1})$ for all $\varepsilon \in (0,\rho)$. Note that $Z'_\varepsilon(x) = 0$ if and only if $W'(x) = \varepsilon$.

Let $a = \min\{1, \min\{x > 0: W''(x) = 0\}\}$. Then $a > 0$ and W' is one-one on $(0,a)$. Let $\rho = \delta a^{n-1}$. Then $(0,(\rho\delta^{-1})^{1/n-1}) = (0,a)$. Since W' is 1-1 on $(0,a)$, $W'(x) = \varepsilon$ has at most one root in $(0,(\rho\delta^{-1})^{1/n-1})$.

To prove the claim for $\mu \in G_1^c$, again consider the case $\varepsilon > 0$. Let $\xi = \max\{x \in R: U_\mu$ attains its global minimum at $x\}$, and let $W(x) = U_\mu(\xi+x) - U_\mu(\xi)$. The proof here follows the previous case, using the fact that there is a $\delta > 0$ such that $W(x) \geq \delta x^n$, for every $x > 0$, which is clear from the proof of Lemma 3.1. For the $\varepsilon < 0$ case, shift U_μ by $\tilde{\xi} = \min\{x \in R: U_\mu$ attains its global minimum at $x\}$. □

<u>Lemma 3.3</u>: The functions m and ξ are analytic on G , with $m'(\mu) = -\xi(\mu)$ and $\xi'(\mu) = \dfrac{1}{U_0''(\xi(\mu))}$. Furthermore, ξ is strictly increasing on G_1 , continuous on G_1 , and discontinuous on G_1^c ; $\lim_{\mu\to\pm\infty} \xi(\mu) = \pm\infty$; and $-m \in C_s$.

<u>Proof</u>: The derivative U_0' is an entire function, and for $\mu \in G$, $U_0'(\xi(\mu)) = \mu$ and $U_0''(\xi(\mu)) > 0$. By the Inverse Function Theorem [Rudin 74, p. 231], there are open neighbourhoods 0 containing μ and V containing $\xi(\mu)$ such that $U_0'\big|_V$ is invertible and the inverse is analytic on 0 . This inverse is an extension of ξ . Since for $\mu \in G$, $m(\mu) = U_\mu(\xi(\mu)) = U_0(\xi(\mu)) - \mu\xi(\mu)$, m is also analytic on G , with $m'(\mu) = U_0'(\xi(\mu))\xi'(\mu) - \mu\xi'(\mu) - \xi(\mu) = -\xi(\mu)$. To calculate $\xi'(\mu)$, differentiate the equation $U_0'(\xi(\mu)) = \mu$ with respect to μ to obtain

$$\xi'(\mu) = \frac{1}{U_0''(\xi(\mu))} .$$

The fact that ξ is strictly increasing and discontinuous on G_1^c is clear from the definition of ξ . It is also easy to see that ξ is continuous on F , and hence on G_1 . For large μ , $\xi(\mu)$ is the unique root of $U_0'(x) = \mu$. As $\mu \to \pm\infty$ that root diverges to $\pm\infty$, so $\lim_{\mu\to\pm\infty} \xi(\mu) = \pm\infty$. This last fact, together with the strict monotonicity of ξ and the equation $-m'(\mu) = \xi(\mu)$, implies that $-m \in C_s$. $\square$

The following lemma shows how ξ relates B and G .

<u>Lemma 3.4</u>: $B^c = \xi(G)$.

<u>Proof</u>: Suppose $a \in \xi(G)$. Then there is a $\mu_a \in G$ such that $\xi(\mu_a) = a$. Since $U_0''(a) = U_0''(\xi(\mu_a)) > 0$, $a \notin B_2$. We now show $a \notin B_1$. Now $(\text{conv } U_0)(a) = U_0^{**}(a) = \sup_\mu [\mu a - U_0^*(\mu)]$. Since

$$U_0^*(\mu) = \sup_x[\mu x - U_0(x)] = -\min_x U_\mu(x) = -m(\mu) , \tag{3.7}$$

$$(\text{conv } U_0)(a) = \sup_\mu[\mu a + m(\mu)] .$$

But $-m$ is differentiable at μ_a and $D(-m)(\mu_a) = \xi(\mu_a) = a$. Since $-m \in C_s$, this implies that

$$(\text{conv } U_0)(a) = \mu_a a + m(\mu_a) = \mu_a a + U_{\mu_a}(\xi(\mu_a)) = U_0(\xi(\mu_a)) = U_0(a) . \tag{3.8}$$

Since G is a union of open intervals and ξ is strictly increasing and continuous on G , $\xi(G)$ is a union of open intervals. Together with eqn. (3.8), this implies that $a \notin B_1$. Hence $\xi(G) \subset B^c$.

On the other hand, let $a \in B^c$. Suppose, contrary to the statement of the lemma, that $a \notin \xi(G)$. Then $a \in \xi(F)$ or $a \in \xi(G_1)^c$. If

$a \in \xi(F)$ then $U_0''(a) = 0$ so $a \in B_2$. Therefore $a \in \xi(G_1)^c$. By Lemma 3.3 there must be a $\mu_0 \in G_1^c$ for which $a \in [\xi(\mu_0^-), \xi(\mu_0^+)] \subset \xi(G_1)^c$. The interval $[\xi(\mu_0^-), \xi\, \mu_0^+)]$ is nontrivial since $\mu_0 \in G_1^c$ is a point where ξ undergoes a jump discontinuity. Since $\xi(\mu_0^\pm) = D^\pm(-m)(\mu_0)$ by Lemma 3.3, $a \in [D^-(-m)(\mu_0), D^+(-m)(\mu_0)] \subset \xi(G_1)^c$. It follows from the fact that $a \in B^c$, eqn. (3.7), and the correspondence depicted in Figure 2 that

$$U_0(a) = U_0^{**}(a) = (-m)^*(a) = \mu_0 a + m(\mu_0) \quad \text{for all} \quad a \in [D^-(-m)(\mu_0), D^+(-m)(\mu_0)] .$$

But this is impossible because U_0 cannot have a linear segment. □

We close this section with a lemma that will be used in the proof of analyticity of the pressure in §3.4.

<u>Definition 3.5</u>: For $\delta, L > 0$ denote by $\mathcal{T}_{\delta,L}$ the set of all polynomials

$$T(x) = \sum_{k=2}^{n} t_k x^k \quad \text{with} \quad |t_k| \le L \ (k=2,\ldots,n) \quad \text{and} \quad T(x) \ge \delta(x^n + x^2)$$

for all x.

For $T \in \mathcal{T}_{\delta,L}$, small perturbations of the coefficients of T and a small linear perturbation $T(x) \to T(x) - \mu x$ do not change the fact that the polynomial has a unique global minimum, located say at ξ. Translation of the perturbed polynomial so that its global minimum sits at the origin will give a polynomial in $\mathcal{T}_{\delta',L'}$, for some $\delta' > 0$ slightly smaller than δ and L' slightly larger than L. If the perturbations are smooth then ξ will also be smooth. These elementary facts are proved in the following lemma.

Lemma 3.6: Let a_k be analytic in $(0,\gamma_1)$ and C^∞ at 0^+, and let $T(\hbar,x) = \sum_{k=2}^{n} a_k(\hbar)x^k$. Suppose $T(0,\cdot) \in \mathcal{T}_{\delta,L}$. Then there exist

$\delta',L',\gamma,\rho > 0$ such that $T_\mu(\hbar,x) = T(\hbar,x) - \mu x$ has a uniquely attained global minimum at say $\xi(\hbar,\mu)$ for all $(\hbar,\mu) \in [0,\gamma) \times (-\rho,\rho)$, with

$$S(\hbar,\mu;x) \equiv T_\mu(\hbar,\xi(\hbar,\mu)+x) - T_\mu(\hbar,\xi(\hbar,\mu)) \in \mathcal{T}_{\delta',L'} \quad \text{for all}$$

$$(\hbar,\mu) \in [0,\gamma) \times (-\rho,\rho) .$$

Moreover, ξ is analytic on $(0,\gamma) \times (-\rho,\rho)$ and C^∞ on $[0,\gamma) \times (-\rho,\rho)$.

Proof: The proof that $\xi(\hbar,\mu)$ exists is much like part of the proof of Lemma 3.2. First, note that by choosing $\hbar$ sufficiently small we can arrange that $T_0(\hbar,\cdot) \in \mathcal{T}_{\frac{\delta}{2},L+\frac{\delta}{2}}$ for all $\hbar \in [0,\gamma)$, and hence

$$T_\mu(\hbar,x) \geq \frac{\delta}{2}x^2 - \mu x \quad \text{for all} \quad (\hbar,\mu,x) \in [0,\gamma) \times R \times R . \tag{3.9}$$

Consider the case $\mu > 0$, for which the minimum of $T_\mu(\hbar,\cdot)$ is strictly negative and occurs when $x > 0$. It suffices to show that $D_2T_\mu(\hbar,x) = 0$ has at most one root when $T_\mu(\hbar,x) < 0$, i.e., when $x \in (0, \frac{2\mu}{\delta})$ by eqn. (3.9). For $\hbar \in [0,\gamma)$, let $a(\hbar) = \min\{1, \min\{x>0: D_2^2T_0(\hbar,x) = 0\}\}$. Then $D_2T_0(\hbar,\cdot)$ is one-one on $(0,a(\hbar))$. Let $a = \inf_{0\leq\hbar<\gamma} a(\hbar)$. To arrange that $a > 0$ note that $D_2^2T_0(\hbar,x) = D_2^2T_0(0,x) + \sum_{k=2}^{n} k(k-1)(a_k(\hbar) - a_k(0))x^{k-2}$ and that $a(0) > 0$ (since $T(0,\cdot) \in \mathcal{T}_{\delta,L}$) and let $c = \min_{0\leq x\leq \frac{a(0)}{2}} D_2^2T_0(0,x)$, so $c > 0$. Choose γ smaller if necessary to arrange $\left| \sum_{k=2}^{n} k(k-1)(a_k(\hbar) - a_k(0)x^{k-2}\right| \leq \frac{c}{2}$ for all $\hbar \in [0,\gamma)$ and

$0 < x < a(0)$. Then $D_2^2T_0(\hbar,x) = D_2^2T_0(0,x) + \sum_{k=2}^{n} k(k-1)(a_k(\hbar)-a_k(0))x^{k-2}$

$$\geq c - \frac{c}{2} = \frac{c}{2} \quad \text{for all } \hbar \in [0,\gamma),\ 0 < x < \frac{a(0)}{2} ,$$

and hence $a \geq \frac{1}{2}a(0)$.

Since $D_2T_0(\hbar,x)$ is one-one on $(0,a)$ for all $\hbar \in [0,\gamma)$, there is at most one root of $D_2T_0(\hbar,x) = \mu$ in the interval $(0,2\mu\delta^{-1})$ provided $2\mu\delta^{-1} < a$, i.e., $\mu < \frac{1}{2}a\delta$. Thus we take $\rho = \frac{1}{2}a\delta$.

Now the location $\xi(\hbar,\mu)$ of the global minimum of $T_\mu(\hbar,\cdot)$ lies in the interval $(0,a)$, where $a = 2\rho\delta^{-1}$. By taking ρ and γ smaller, we can make $\frac{1}{k!}|D_2^kT_0(\hbar,\xi(\hbar,\mu)) - D_2^kT_0(0,0)|$ as small as desired, uniformly in $\hbar \in [0,\gamma)$, $|\mu| < \rho$ and $k = 2,\ldots,n$. Therefore, since

$$S(\hbar,\mu;x) = \sum_{k=2}^{n} \frac{D_2^kT(\hbar,\xi(\hbar,\mu))}{k!} x^k ,$$

we can arrange that $S(\hbar,\mu;\cdot) \in \mathcal{T}_{\frac{\delta}{4},L+\frac{3\delta}{4}}$ for all $\hbar \in [0,\gamma)$, $|\mu| < \rho$.

It remains to prove that ξ is analytic on $(0,\gamma) \times (-\rho,\rho)$ and C^∞ on $[0,\gamma) \times (-\rho,\rho)$. Let $f(\hbar,\mu;x) = D_2T_0(\hbar,x) - \mu$. Then f is analytic on $0_\gamma \times C \times C$, for some complex open set $0_\gamma \supset (0,\gamma)$. Since $f(\hbar,\mu;\xi(\hbar,\mu)) = 0$, and

$$D_3f(\hbar,\mu;\xi(\hbar,\mu)) = D_2^2T_0(\hbar,\xi(\hbar,\mu)) \geq \frac{\delta}{4} > 0 \quad \text{for} \quad (\hbar,\mu) \in (0,\gamma) \times (-\rho,\rho) ,$$

the Implicit Function Theorem [Hörm 73] implies that there is a neighbourhood $U \supset (0,\gamma) \times (-\rho,\rho)$ on which ξ is analytic. By differentiating the equation $f(\hbar,\mu;\xi(\hbar,\mu)) = 0$ with respect to $\hbar$ or μ and using the uniform lower bound on D_3f it is easy to see that derivatives of ξ are uniformly bounded in $\hbar \in (0,\gamma)$, and hence ξ is smooth at $\hbar = 0^+$. □

CHAPTER 3: THE MAIN ESTIMATES

§1. The Translation

To prove analyticity in $\hbar$ for the effective potential and obtain the desired form for the derivatives at $\hbar = 0$ it is convenient to perform a change of variable, so as to explicitly isolate the leading term. Let $C = (-\Delta+m^2)^{-1}$ with periodic B.C. on $\partial\Lambda$ and recall that $U_\mu(x) = U_0(x) - \mu x$ where $U_0(x) = P(x) + \frac{1}{2} m^2 x^2$. By Lemma 2.2.2, for any fixed $a \in R$

$$\int e^{-\frac{1}{\hbar}\int_\Lambda [:P(\phi):-\mu\phi\ ]} d\mu_{\hbar C} = e^{-\frac{1}{\hbar}|\Lambda|U_\mu(a)} \int e^{-\frac{1}{\hbar}\int_\Lambda [\sum_{k=1}^{n} \frac{U_\mu^{(k)}(a)}{k!}:\phi^k:-\frac{1}{2}m^2:\phi^2:]} d\mu_{\hbar C} \tag{1.1}$$

By definition of the pressure in eqn. (1.2.3), eqn. (1.1) implies

$$\hbar\alpha(\hbar,\mu) = -U_\mu(a) + \hbar\sigma_1(\hbar,\mu-U_0'(a)) \tag{1.2}$$

$$\text{where}\quad \sigma_1(\hbar,j) = \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [\sum_{k=2}^{n} \frac{P^{(k)}(a)}{k!}:\phi^k: - j\phi]} d\mu_{\hbar C} \tag{1.3}$$

Inserting eqn. (1.2) in the definition of V in eqn. (1.2.4) gives

$$\begin{aligned} V(\hbar,a) &= \sup_\mu[\mu a - \hbar\alpha(\hbar,\mu)] \\ &= \sup_{\mu\in R}[\mu a + U_\mu(a) - \hbar\sigma_1(\hbar,\mu-U_0'(a))] \\ &= U_0(a) + \sup_{\mu\in R}[-\hbar\sigma_1(\hbar,\mu)] \end{aligned} \tag{1.4}$$

Next, we perform a mass shift so as to explicitly isolate the $O(\hbar)$ contribution to the effective potential. Let $m_1^2 = U_0''(a) = P''(a) + m^2$. For $a \notin B$, $m_1^2 > 0$. For the remainder of this section we assume $a \notin B$. Let $C_1 = (-\Delta+m_1^2)^{-1}$ with periodic BC on $\partial\Lambda$. By using Lemma 2.2.3 it follows from eqn. (1.3) that

$$\sigma_1(\hbar,\mu) = \lim_\Lambda \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [\sum_{k=3}^n \frac{P^{(k)}(a)}{k!}:\phi^k:_{\hbar C} - \mu\phi]} d\mu_{\hbar C_1}$$

$$+ \lim_\Lambda \frac{1}{|\Lambda|} \int e^{-\frac{1}{\hbar}\int_\Lambda \frac{P''(a)}{2}:\phi^2:} d\mu_{\hbar C} . \tag{1.5}$$

Here and throughout this thesis the Wick dots appearing in an integrand are with respect to the covariance of the measure unless otherwise indicated.

Introducing $$\gamma(a) = \lim_\Lambda \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda \frac{P''(a)}{2}:\phi^2:} d\mu_{\hbar C} \tag{1.6}$$

it is clear from Lemma 2.2.4 that γ is independent of $\hbar > 0$.

Let $$\sigma_2(\hbar,\mu) = \lim_\Lambda \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [\sum_{k=3}^n \frac{P^{(k)}(a)}{k!}:\phi^k:_{\hbar C} - \mu\phi]} d\mu_{\hbar C_1} \tag{1.7}$$

Then by eqns. (1.4) and (1.5),

$$V(h,a) = U_0(a) - \hbar\gamma(a) + \sup_{\mu\in R} [-\hbar\sigma_2(\hbar,\mu)] . \tag{1.8}$$

The next step is to Wick re-order the interaction in σ_2 to match the covariance C_1 . Writing $a_k = P^{(k)}(a)/k!$ $(k=3,\ldots,n)$ and using Lemma 2.2.1, the interaction in $\sigma_2(\hbar,0)$ can be rewritten as

$$\sum_{k=3}^{n} a_k :\phi^k:_{\hbar C} = \sum_{k=0}^{n} q_k(\hbar):\phi^k:_{\hbar C_1} \tag{1.9}$$

where each q_k is a polynomial of degree $[\frac{n-1-k}{2}]$ in $\hbar d = \hbar[(-\frac{1}{4\pi}\log\frac{m_1^2}{m^2})$ plus an $\hbar$-independent term that goes to zero as $\Lambda \uparrow R^2]$. To simplify the notation we drop the Λ-dependent term (which is insignificant for large Λ and disappears in the $\Lambda \uparrow R^2$ limit). In view of Lemma 2.2.1 (see eqn. (5.5.8)),

$$q_0(\hbar) = O(\hbar^2),\ q_1(\hbar) = O(\hbar),\ q_2(\hbar) = O(\hbar),\ q_k(\hbar) = a_k + O(\hbar)\ (3 \leq k \leq n). \tag{1.10}$$

$$\text{Explicitly,}\quad q_0(\hbar) = \sum_{k=2}^{[n/2]} C_{2k,k}\, a_{2k} (\hbar d)^k . \tag{1.11}$$

Inserting eqn. (1.9) in eqn. (1.7) gives

$$\hbar\sigma_2(\hbar,\mu) = -q_0(\hbar) + \hbar \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [\sum_{k=2}^{n} q_k :\phi^k: -(\mu - q_1)\phi]} d\mu_{\hbar C_1} \tag{1.12}$$

$$\text{Let}\quad \sigma(\hbar,j) = \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [\sum_{k=2}^{n} q_k :\phi^k: - j\phi]} d\mu_{\hbar C_1}\ , \tag{1.13}$$

$$\text{so that}\quad \hbar\sigma_2(\hbar,\mu) = -q_0(\hbar) + \hbar\sigma(\hbar,\ \mu - q_1(\hbar)) . \tag{1.14}$$

Inserting eqn. (1.14) into eqn. (1.8) gives

$$V(\hbar,a) = U_0(a) - \hbar\gamma(a) + q_0(\hbar) + \sup_{\mu}[-\hbar\sigma(\hbar,\mu)]\ ,\quad a \notin B\ . \tag{1.15}$$

Observe that $\frac{D^k q_0(0)}{k!} = C_{2k,k} a_{2k} d^k$ gives the value of the d renormalized k loop graph with a single $2k$ legged vertex a_{2k} and legs joined up in pairs. To show that the translated effective potential

$$E(\hbar) = \sup_{\mu}[-\hbar\sigma(\hbar,\mu)] \tag{1.16}$$

is analytic in $\hbar$ with derivatives at $\hbar = 0$ given by the appropriate sum of graphs, we will use Theorem 2.1.2 to reduce the problem to the study of $\hbar\sigma(\hbar,\mu)$. This pressure is studied using a high temperature cluster expansion.

§2. The Cluster Expansion

The main difficulty in proving analyticity of the pressure $\hbar\sigma(\hbar,\mu)$ and the potential $E(\hbar)$ is the infinite volume limit. The high temperature cluster expansion [GJS 73] is often a useful tool in dealing with the infinite volume limit in a weakly coupled theory. (The terminology "high temperature" comes from the fact that weak coupling in quantum field theory is analogous to high temperature in statistical mechanics). By Lemma 2.2.4 we can write σ as

$$\sigma(\hbar,\mu) = \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda [\sum_{k=2}^{n} q_k \hbar^{k/2-1} :\phi^k: - \mu\hbar^{-\frac{1}{2}}\phi]} d\mu_{C_1} . \tag{2.1}$$

By eqn. (1.10), for $k \geq 2$ the coefficient of $:\phi^k:$ is at least $O(\hbar^{\frac{1}{2}})$ so for small $\hbar^{\frac{1}{2}}$ and small $\mu\hbar^{-\frac{1}{2}}$ we are in a weak coupling situation. However the situation is more complicated than the weak coupling case treated in [GJS 73] since $\hbar^{\frac{1}{2}}$ occurs to different powers in the different terms of the interaction.

In this section we write down the cluster expansion and give conditions for and consequences of its convergence. In the next section we prove bounds on the partition function in eqn. (2.1) (for $\mu = 0$) that guarantee convergence uniform in $\hbar$.

To begin, we introduce some notation. Since we are using periodic BC , we identify opposite sides of Λ to obtain a torus. Let $\mathcal{B}_\Lambda$ denote the set of all lattice bonds joining nearest neighbour sites in the periodic lattice $\Lambda \cap Z^2$. For each $b \in \mathcal{B}_\Lambda$ we introduce a parameter $s_b \in [0, 1]$. Let

$$C(s) = \sum_{\Gamma\subset\mathcal{B}_\Lambda} \pi_{b\in\Gamma^c} s_b \pi_{b\in\Gamma^c} (1-s_b) C_{\Gamma^c}$$

where $C_{\Gamma^c} = (-\Delta_\Lambda^{\Gamma^c}+m^2)^{-1}$ with $\Delta_\Lambda^{\Gamma^c}$ the Laplacian with periodic BC on $\partial\Lambda$ and Dirichlet BC on Γ^c , (i.e., $\Delta_\Lambda^{\Gamma^c}$ is the Friedrich's extension [Kato 66] of Δ restricted to $C_0^\infty(\Lambda\backslash\Gamma^c))$. When $s_b = 0$ each nonzero term in $C(s)$ has Dirichlet BC on the bond b , whereas for $s = 1$ each term does not have Dirichlet BC on b but is fully coupled across b . The parameter s_b gives a measure of the amount of coupling in $C(s)$ across the bond b . For $\Gamma \subset \mathcal{B}_\Lambda$, let $\partial^\Gamma = \pi_{b\in\Gamma} \frac{d}{ds_b}$ and

$$s(\Gamma)_b = \begin{cases} s_b & \text{if } b \in \Gamma \\ 0 & \text{if } b \notin \Gamma \end{cases}$$

Let $d\mu(s)$ be Gaussian measure on $\mathcal{S}'(R^2)$ with covariance $C(s)$, and let

$$Z(\Lambda,\lambda,s) = \int e^{-V(\Lambda,\lambda,s)} d\mu(s) \quad \text{where} \quad V(\Lambda,\lambda,s) = \sum_{k=0}^{n} a_k(\lambda) \int_\Lambda :\phi^k:_{C(s)} .$$

and the a_k are functions of $\lambda \in \mathcal{O} \subset \mathbb{C}^m$. We sometimes write $Z_\Gamma(\Lambda,\lambda)$ for $Z(\Lambda,\lambda,\chi_{\Gamma^c})$, the partition function with Dirichlet BC on Γ. Let

$$\langle\cdot\rangle_{\lambda,\Lambda,s} = Z(\lambda,\Lambda,s)^{-1}\int \cdot\; e^{-V(\Lambda,\lambda,s)}d\mu(s) .$$

Then the following expansion holds

$$S_\Lambda^{k_1\ldots k_r}(x_1,\ldots,x_r) \equiv \left\langle :\phi^{k_1}(x_1):\ldots:\phi^{k_r}(x_r):\right\rangle_{\lambda,\Lambda,1}$$

$$= \sum_{X,\Gamma}\int_0^1 \partial^\Gamma \int \prod_{i=1}^{r} :\phi^{k_i}(x_i): e^{-V(X,\lambda,s(\Gamma))}d\mu(s(\Gamma))ds(\Gamma)\,\frac{Z_{\partial X}(\Lambda\backslash X,\lambda)}{Z(\Lambda,\lambda,1)} \tag{2.2}$$

$$\equiv \sum_{X,\Gamma} T^{k_1\ldots k_r}(X,\Gamma,\Lambda,\lambda;x_1,\ldots,x_r) ,$$

where in the summation X ranges over all unions of closed lattice squares in Λ containing $\{x_1,\ldots,x_r\}$ while Γ ranges over subsets of $\mathcal{B}_\Lambda \cap \operatorname{int} X$ such that each component of $X\backslash\Gamma^c$ has nonempty intersection with $\{x_1,\ldots,x_r\}$. The formal derivation of eqn. (2.2) is relatively straightforward involving little more than the fundamental theorem of calculus. The hard work goes into bounding the right side of eqn. (2.2) with bounds independent of Λ and λ. The proof of the following theorem is implicit in [GJS 73].

<u>Theorem 2.1</u>: Suppose $|a_k(\lambda)| \le L$ for all $\lambda \in \mathcal{O}$, $k \in \{0,1,2,\ldots,n\}$, and that the following bounds hold: For some $p > 1$,

$$\left|\int e^{-pV(\Lambda,\lambda,s)}d\mu(s)\right| \le e^{K|\Lambda|} , \text{ for all } \lambda \in \mathcal{O} ,\ \Lambda ,\ s \tag{2.3}$$

and

$$|Z_{\partial\Delta}(\Delta,\lambda)| \geq \frac{1}{2} \text{ , for all } \lambda \in 0 \text{ , and for any unit lattice square } \Delta \text{ .} \quad (2.4)$$

Then for any $C > 0$ there is an $M > 0$ depending only on K and L (and C) such that for $m > M(K,L)$ and $w \in L^2(\Lambda^r)$, and for all Λ and $\lambda \in 0$,

$$\sum_{\{X,\Gamma:|X|\geq D\}} \left|\int T^{k_1\ldots k_r}(X,\Gamma,\Lambda,\lambda;x_1,\ldots,x_r)w(x_1,\ldots,x_r)dx\right| \leq |w|e^{-C(D-r)} \quad (2.5)$$

where $|\cdot|$ is a translation invariant norm on $L^2(\Lambda^r)$. □

There are three main ideas in the proof of Theorem 2.1. The first is that the number of terms in the sum over X and Γ in eqn. (2.2) having a fixed value of $|X|$ can be bounded above by $e^{C_1|X|}$ where C_1 is a constant depending only on the geometry. The second key result is that for any constant $C > 0$ there is a constant $C_2 > 0$, depending on the K of eqn. (2.3), such that for m sufficiently large (depending on C) ,

$$\left|\int_0^1 \partial^\Gamma \int \prod_{i=1}^{r} :\phi^{k_i}(x_i):e^{-V(X,\lambda,\ s(\Gamma))}d\mu(s(\Gamma))ds(\Gamma)w(x_1,\ldots,x_r)dx\right| \leq e^{-C|\Gamma|+C_2|X|}|w| .$$

The decay factor $e^{-C|\Gamma|}$ comes from estimating the derivatives of $C(s)$ produced by applying the derivatives ∂^Γ to the Wick dots or the measure $d\mu(s(\Gamma))$ on the left side. See [GJS 73] for details. The bound (2.3) is used with Hölder's inequality to control the exponential of the interaction. Finally, using the lower bound (2.4) it can be shown using ideas from statistical mechanics and the two estimates just stated that for m sufficiently large (again depending on K) there is a constant C_3 such that

$$\left|\frac{Z_{\partial X}(\Lambda\backslash X,\lambda)}{Z(\Lambda,\lambda,1)}\right| \le e^{C_3|X|} .$$

Combining the above three estimates with the fact that $|\Gamma| \ge \frac{1}{2}(|X|-r)$ because of the constraints on Γ, the factor $e^{-C|\Gamma|}$ gives convergence for C (hence m) sufficiently large.

Now let $\psi_i = :\phi^{k_i}(w_i):$ where $w_i \in L^2(\Lambda)$, and let $\psi_{i,x}$ be the translation of ψ_i by $x \in R^2$. In [GJS 73] it is shown how to use Theorem 2.1 to prove that there are positive constants K_1 and m' such that

$$|\langle \psi_{1,x};\psi_{2,y}\rangle_{\lambda,\Lambda,1}| \le K_1 e^{-m'|x-y|} ,$$

with K_1 depending on ψ_1,ψ_2. The most important consequence of Theorem 2.1 for our purposes is the following generalization of the above bound to higher order truncated expectation values, which is due to Dimock [Dim 74]. (See [EMS 75] for related estimates).

Theorem 2.2: Suppose the hypotheses of Theorem 2.1 hold. Then there are positive constants K_2 and m_* such that

$$\sup_{\Lambda}\ \sup_{\lambda\in O} |\langle \psi_{1,x_1};\dots;\psi_{N,x_N}\rangle_{\lambda,\Lambda,1}| \le K_2\, e^{-m_*\delta(x_1,\dots,x_N)/N} ,$$

where $\delta(x_1,\dots,x_N) = \sup_{1\le i\le j\le N} |x_i - x_j|$, K_2 depends on the ψ_i, and $\sup_{\Lambda}$ ranges over squares Λ. $\square$

An immediate Corollary of Theorem 2.2 is the following result, which is the main result we need from this section.

Corollary 2.3: Suppose the hypothesis of Theorem 2.1 hold. Then there is a positive constant M_r such that

$$\sup_{\Lambda} \sup_{\lambda\in 0} \frac{1}{|\Lambda|} \left| \left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda): \right\rangle_{\lambda,\Lambda,1} \right| \leq M_r \tag{2.6}$$

where $\sup_{\Lambda}$ ranges over squares Λ of integer side-length.

<u>Proof</u>: Denoting the unit lattice square centred at $j \in Z^2$ by Δ_j, we have

$$\frac{1}{|\Lambda|} \left| \left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda): \right\rangle_{\lambda,\Lambda,1} \right|$$

$$\leq \frac{1}{|\Lambda|} \sum_{j_1,\ldots,j_r \in Z^2\cap\Lambda} \left| \left\langle :\phi^{k_1}(\Delta_{j_1}):;\ldots;:\phi^{k_r}(\Delta_{j_r}) \right\rangle_{\lambda,\Lambda,1} \right|$$

$$= \frac{1}{|\Lambda|} \sum_{j_1,\ldots,j_r \in Z^2\cap\Lambda} \left| \left\langle :\phi^{k_1}(\Delta_0):;:\phi^{k_2}(\Delta_{j_2-j_1}):;\ldots;:\phi^{k_r}(\Delta_{j_r-j_1}): \right\rangle_{\lambda,\Lambda,1} \right| \tag{2.7}$$

$$\leq \frac{1}{|\Lambda|} \sum_{j_1,\ldots,j_r \in Z^2\cap\Lambda} K_2 \, e^{-m_*[|j_2-j_1|+\ldots+|j_r-j_1|]/r^2} \tag{2.8}$$

$$\leq \frac{K_2}{|\Lambda|} \sum_{j_1\in Z^2\cap\Lambda} \sum_{\ell_2,\ldots,\ell_r \in Z^2} e^{-m_*[|\ell_2|+\ldots+|\ell_r|]/r^2}$$

$$\leq K_2 \cdot \text{const.}$$

where (2.7) follows from translation invariance of the periodic theory and (2.8) follows by Theorem 2.2. □

§3. <u>Convergence of the Cluster Expansion</u>

To prove analyticity of $\hbar\sigma(\hbar,\mu)$ in $\hbar$ and μ, the first step is to perform a translation in the functional integral defining σ to remove the linear term from the interaction. For $a = \xi(\mu_a) \notin B$, the classical potential occurring in $\sigma(\hbar,\mu)$ in eqn. (1.13) satisfies (for $\hbar = \mu = 0$)

$$\sum_{k=2}^{n} q_k(0)x^k + \frac{1}{2} m_1^2 x^2 = \sum_{k=2}^{n} \frac{P^{(k)}(a)}{k!} x^k + \frac{1}{2} m^2 x^2 = U_{\mu_a}(\xi(\mu_a)+x) - U_{\mu_a}(\xi(\mu_a) \in T_{\delta,L}$$

for some $\delta,L > 0$, by Lemmas 2.3.4 and 2.3.1. For small $\hbar$ and μ the translation will replace the $q_k(0)$ by $\hbar$ and μ dependent coefficients that are close to the $q_k(0)$, so by Lemma 2.3.6 the classical potential will remain in some $T_{\delta',L'}$. We prove convergence of the cluster expansion in just this context.

The idea for the proof of Lemma 3.1 below originated in work of Spencer [Sp 74]. After this research was completed the author learned of a paper by Eckmann [E 77] where an estimate basically the same as eqn. (3.1) is proved by essentially the same method and used to prove convergence of a cluster expansion uniform in small real $\hbar$.

Let $S_{\theta,\gamma} = \{z \in C : 0 < \mathrm{Re}\, z < \gamma ,\ 0 < |\mathrm{Arg}\, z| < \theta\}$.

<u>Lemma 3.1</u>: Let $T(\hbar,x) = \sum_{k=2}^{n} a_k(\hbar)x^k$ and $a_1(\hbar) = O(\hbar^{\frac{1}{2}})$ where the a_k are continuous in some $\overline{S}_{\theta',\gamma'}$. Suppose $\mathrm{Re}T(0,\cdot) \in T_{\delta,L}$ for some $\delta,L > 0$. Then there exist $\theta,\gamma > 0$ such that

$$\left| \int e^{-\int_V [\hbar^{-1} :T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi - \frac{1}{2} m^2 :\phi^2:]} d\mu_{m^2}(s) \right| \le e^{K|V|} \tag{3.1}$$

for every $\hbar \in S_{\theta,\gamma}$ and for every s, and for every finite union V of lattice squares in Λ. The constant K depends on δ and L.

Before proving Lemma 3.1 we outline the proof. First we reduce the problem to real $\hbar$ using an elementary argument and use conditioning

[GRS 76] to reduce to the case $s = 1$. We then shift mass from the measure to the interaction, leaving a mass term $\frac{1}{2}\delta\phi^2$ in the volume V in the measure. The resulting new interaction (Wick re-ordered to match the measure) evaluated at the ultraviolet cutoff field ϕ_r is bounded below by $-\text{const}(\log r)^{n/2}$, uniformly in $\hbar$, using the fact that for $\hbar$ sufficiently small $T(\hbar,x) \geq \frac{1}{2}\delta(x^n+x^2)$. An appeal to a result of [DG 74] completes the proof.

Proof of Lemma 3.1: The first step is to reduce the problem to the case of real $\hbar$ and a. Note that

$$\left|\int e^{-\int_V [\frac{1}{\hbar}:T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:]} d\mu_{m^2}(s)\right|$$

$$\leq \int e^{-\int_V (:\mathrm{Re}[\frac{1}{\hbar}T(\hbar,\hbar^{\frac{1}{2}}\phi)]:+\mathrm{Re}a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:)} d\mu_{m^2}(s)$$

For small θ and γ, $\mathrm{Re}a_1(\hbar) = O(\mathrm{Re}\hbar^{\frac{1}{2}})$. Let $t = \hbar^{\frac{1}{2}} = p+iq$, and for $2 \leq k \leq n$, let $b_k(t) = p^{2-k}\mathrm{Re}(a_k(t^2)t^{k-2})$ for $t \neq 0$ and $b_k(0) = \mathrm{Re}a_k(0)$. Then

$$\mathrm{Re}[\hbar^{-1}T(\hbar,\hbar^{\frac{1}{2}}\phi)] = \sum_{k=2}^{n} \mathrm{Re}[a_k(t^2)t^{k-2}]\phi^k = \sum_{k=2}^{n} b_k(t)p^{k-2}\phi^k \tag{3.2}$$

But

$$|b_k(t) - \mathrm{Re}a_k(0)| \leq \left|\frac{\mathrm{Re}(a_k(t^2)t^{k-2})}{p^{k-2}} - \frac{\mathrm{Re}(a_k(0)t^{k-2})}{p^{k-2}}\right| + \left|\frac{\mathrm{Re}(a_k(0)t^{k-2})}{p^{k-2}} - \mathrm{Re}a_k(0)\right|$$

Since $t^m = \sum_{\ell=0}^{m} \binom{m}{\ell} p^{m-\ell}(iq)^\ell = p^m[1 + \sum_{\ell=1}^{m} \binom{m}{\ell}(\frac{iq}{p})^\ell]$, it follows that

$|\frac{t^m}{p^m} - 1|$ can be made arbitrarily small by taking θ (and hence $|\frac{q}{p}|$) small. Therefore

$$|b_k(t) - \mathrm{Re}\,a_k(0)| \le |a_k(t^2)-a_k(0)||\frac{t^{k-2}}{p^{k-2}}| + |a_k(0)||\frac{t^{k-2}}{p^{k-2}} - 1|$$

can be made as small as desired by taking θ & γ sufficiently small, so there exist θ,γ such that $\sum_{k=2}^{p} b_k(t)\phi^k \in T_{\frac{\delta}{2},L+\frac{\delta}{2}}$ for all $t^2 \in S_{\theta,\gamma}$.

In view of eqn. (3.2), it suffices to consider $\hbar$ real and replace a_k by $b_k \in R$.

The second step is to use conditioning to reduce the estimate (3.1) to the case $s \equiv 1$. To see this, note that as forms $-\Delta_\Lambda^\Gamma + m^2 \subset -\Delta_\Lambda + m^2$ [Kato 66, Thm. 2.10, p. 326] and hence $(-\Delta_\Lambda^\Gamma+m^2)^{-1} \le (-\Delta_\Lambda+m^2)^{-1}$ [Kato 66, Thm. 2.2.1, p. 330]. Since $C(s)$ is a convex combination of covariances of the form $(-\Delta_\Lambda^\Gamma+m^2)^{-1}$, $C(s) \le (-\Delta_\Lambda+m^2)^{-1}$ and hence by the Conditioning Comparison Theorem [GRS 76, Thm. III.1, p. 256] it follows that

$$\int e^{-\int_V [\frac{1}{\hbar}:T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi - \frac{m^2}{2}:\phi^2:]} d\mu_{m^2}(s)$$

$$\le \int e^{-\int_V [\frac{1}{\hbar}:T(\hbar,\hbar^{\frac{1}{2}})\phi):+a_1(\hbar)\phi - \frac{1}{2}m^2:\phi^2:]} d\mu_{m^2} ,$$

where $d\mu_{m^2} = d\mu_{m^2}(1)$.

The third step is to perform a mass shift in the covariance. By

Lemma 2.2.3

$$\int e^{-\int_V [\frac{1}{\hbar}:T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi - \frac{1}{2}m^2:\phi^2:]} d\mu_{m^2}$$

$$= \frac{\int e^{-\int_V [\frac{1}{\hbar}:T(\hbar,\hbar^{\frac{1}{2}}\phi):_{m^2}+a_1(\hbar)\phi - \frac{1}{2}m^2:\phi^2:_{m^2}]} e^{-\int_V (\frac{1}{2}m^2-\frac{\delta}{2}):\phi^2:^\delta} d\mu^\delta}{\int e^{-\int_V (\frac{1}{2}m^2 - \frac{\delta}{2}):\phi^2:^\delta} d\mu^\delta} \qquad (3.3)$$

where $d\mu^\delta$ is Gaussian measure with periodic covariance $(-\Delta+\delta\chi_V+m^2\chi_{\Lambda\setminus V})^{-1}$.

The two quadratic terms $\frac{1}{2}m^2:\phi^2:$ essentially cancel; the term $-\frac{\delta}{2}\phi^2$ will be cancelled using $\hbar^{-1}T(\hbar,\hbar^{\frac{1}{2}}\phi) \geq \frac{\delta}{2}\hbar^{\frac{n}{2}-1}\phi^n + \frac{\delta}{2}\phi^2$. Wick order with respect to $d\mu^\delta$ is denoted $:\;:^\delta$. Applying Jensen's inequality to the denominator of the right side of eqn. (3.3) we obtain

$$\int e^{-\int_V [\frac{1}{\hbar}:T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:]} d\mu_{m^2} \leq \int e^{-\int_V :A(\phi):^\delta} d\mu^\delta$$

where $:A(\phi):^\delta = \hbar^{-1}:T(\hbar,\hbar^{\frac{1}{2}}\phi):_{m^2} +a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:_{m^2} + (\frac{1}{2}m^2 - \frac{\delta}{2}):\phi^2:^\delta$.

By Lemma 2.2.1 A has the form

$$A(x) = \hbar^{-1}T(\hbar,\hbar^{\frac{1}{2}}x) - \frac{\delta}{2}x^2 + \sum_{k=0}^{n-2} \tilde{a}_k(\hbar)\hbar^{\frac{k}{2}}x^k \qquad (3.4)$$

where the $\tilde{a}_k$ are bounded in absolute value by a constant depending only on δ and L .

The fourth step is to provide a lower bound on $:A(\phi_r):^{\delta}$, where ϕ_r is the ultraviolet cutoff field defined in §2.2. By eqns. (3.4), (2.2.1) and (2.2.2)

$$:A(\phi_r):^{\delta} = \hbar^{-1}:T(\hbar,\hbar^{\frac{1}{2}}\phi_r):^{\delta} - \frac{\delta}{2}:\phi_r^2:^{\delta} + \sum_{k=0}^{n-2} \tilde{a}_k(\hbar)\hbar^{\frac{k}{2}}:\phi_r^k:^{\delta}$$

$$= \hbar^{-1}T(\hbar,\hbar^{\frac{1}{2}}\phi_r) - \frac{\delta}{2}\phi_r^2 + \sum_{k=0}^{n-2} c_k(\hbar,r)\hbar^{\frac{k}{2}}\sigma_r^{\frac{n-k}{2}}\phi_r^k$$

with $|c_k(\hbar,r)|$ uniformly bounded in small $\hbar$ and large r . The key to obtaining the lower bound is that since $T(\hbar,x) = \sum_{k=2}^{n} b_k x^k \in \mathcal{T}_{\frac{\delta}{2},L+\frac{\delta}{2}}$, it follows that

$$\hbar^{-1}T(\hbar,\hbar^{\frac{1}{2}}\phi_r) \geq \frac{\delta}{2}\hbar^{\frac{n}{2}-1}\phi_r^n + \frac{\delta}{2}\phi_r^2$$

and so

$$:A(\phi_r):^{\delta} \geq \frac{\delta}{2}\hbar^{\frac{n}{2}-1}\phi_r^n + \sum_{k=0}^{n-2} c_k\hbar^{\frac{k}{2}}\sigma_r^{\frac{n-k}{2}}\phi_r^k$$

$$= \sigma_r^{\frac{n}{2}}[\frac{\delta}{2}\hbar^{-1}x^n + \sum_{k=0}^{n-2} c_k x^k]$$

where $x = \hbar^{\frac{1}{2}}\sigma_r^{-\frac{1}{2}}\phi_r$. For $0 < \hbar \leq 1$, it follows that

$$:A(\phi_r):^{\delta} \geq \sigma_r^{\frac{n}{2}}[\frac{\delta}{2}x^n + \sum_{k=0}^{n-2} c_k x^k] .$$

Since the c_k are bounded uniformly in $\hbar$ and r, this implies

$$
\begin{aligned}
:A(\phi_r):^{\delta} &\geq -\mathrm{const}\cdot\sigma_r^{\frac{n}{2}} \\
&\geq -\mathrm{const}\cdot(\log r)^{\frac{n}{2}} ,
\end{aligned}
\tag{3.5}
$$

by eqn. (2.2.2), where the constant is independent of $\hbar$ and r.

The final step is to appeal to Proposition 2.9 of [DG 74], which uses a decomposition of $\mathcal{S}'(R^2)$ to show that a bound of the form (3.5), together with standard estimates for Gaussian expectations of Wick ordered products, imply the upper bound

$$\int e^{-\int_V :A(\phi):^{\delta}} d\mu^{\delta} \leq e^{K|V|}$$ and hence (3.1). □

Lemma 3.1 corresponds to the bound (2.3) needed for convergence of the cluster expansion. However it must be improved for the following reason. In general the mass m_1 will not be large, and a scaling transformation must be performed to increase it. However this scaling affects K. Since the size of the required mass depends on K, there is a problem. The bound (3.7) below is an adequate modification of (3.1); see Corollary 3.3.

Theorem 3.2: Let $T(\hbar,x) = \sum_{k=2}^{n} a_k(\hbar)x^k$ and $a_1(\hbar) = O(\hbar^{\frac{1}{2}})$ where the a_k are continuous in some $\overline{S}_{\theta,\gamma}$. Suppose $\mathrm{Re}\, T(0,x) \in \mathcal{T}_{\delta,L}$ for some $\delta, L > 0$, and fix $m,\varepsilon > 0$. Then there exist $\theta,\gamma,b > 0$ such that if $|a_2(0) - \frac{1}{2}m^2| < b$ then

$$\left| \int e^{-\int_V [\frac{1}{\hbar} : T(\hbar, \hbar^{\frac{1}{2}}\phi) : + a_1(\hbar)\phi - \frac{1}{2}m^2 : \phi^2 :]} d\mu_{m^2}(s) \right| \leq e^{\varepsilon |V|} \tag{3.7}$$

for every $\hbar \in S_{\theta,\gamma}$, for every s, and for every finite union V of unit lattice squares in Λ. Moreover,

$$\left| \int e^{-\int_\Delta [\frac{1}{\hbar} : T(\hbar, \hbar^{\frac{1}{2}}\phi) : + a_1(\hbar)\phi - \frac{1}{2}m^2 : \phi^2 :]} d\mu_{m^2}(s) \right| \geq \frac{1}{2} , \tag{3.8}$$

for every $\hbar \in S_{\theta,\gamma}$, for every s, and for every unit lattice square Δ.

<u>Proof</u>: The proof follows [Sp 74].

For $\Delta \subset V$ we define

$$\psi_\Delta = e^{-\int_\Delta [:\frac{1}{\hbar} T(\hbar, \hbar^{\frac{1}{2}}\phi) : + a_1(\hbar)\phi - \frac{1}{2}m^2 : \phi^2 :]} - 1 . \tag{3.9}$$

Then

$$\begin{aligned} \int e^{-\int_V [:\frac{1}{\hbar} T(\hbar, \hbar^{\frac{1}{2}}\phi) : + a_1(\hbar)\phi - \frac{1}{2}m^2 : \phi^2 :]} d\mu_{m^2}(s) &= \int \prod_{\Delta \subset V} (\psi_\Delta + 1) d\mu_{m^2}(s) \\ &= \sum_{X \subset V} \int \prod_{\Delta \subset X} \psi_\Delta \, d\mu_{m^2}(s) . \end{aligned} \tag{3.10}$$

We claim that there is a $\gamma = \gamma(\varepsilon, \delta, L)$ such that for $h < \gamma$,

$$\left| \int \prod_{\Delta \subset X} \psi_\Delta \, d\mu_{m^2}(s) \right| \leq \varepsilon^{|X|} . \tag{3.11}$$

We will show how (3.11) implies the theorem and then prove (3.11).

Given (3.11), it follows from eqn. (3.10) that

$$\left|\int e^{-\int_V [:\frac{1}{\hbar} T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:]} d\mu_{m^2}(s)\right| \leq \sum_{X\subset V} \varepsilon^{|X|}$$

$$= \sum_{m=0}^{|V|} \binom{|V|}{m} \varepsilon^m \leq (1+\varepsilon)^{|V|} \leq e^{\varepsilon|V|}$$

which proves (3.7). The bound (3.8) follows from (3.11) with $X = \Delta$.

It remains only to prove the inequality (3.11). To simplify the notation, let $:S(V): = \int_V [\frac{1}{\hbar}:T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:]$. By the Fundamental Theorem of Calculus,

$$\psi_{\Delta_i} = -\int_0^1 d\lambda_i :S(\Delta_i):e^{-\lambda_i:S(\Delta_i):} . \tag{3.12}$$

By eqn. (3.12) and Hölder's inequality

$$\left|\int \prod_{\Delta\subset X} \psi_\Delta \, d\mu_{m^2}(s)\right| \leq \left\|\prod_i :S(\Delta_i):\right\|_{p'} \sup_{0\leq\lambda_i\leq 1} \left\| e^{-\sum_i \lambda_i:S(\Delta_i):} \right\|_p \tag{3.13}$$

where $p > 1$ will be chosen below to be near one. The norm $\|\cdot\|_p$ is the norm in $L^p(d\mu_{m^2}(s))$.

By assumption the coefficients of S are $O(\hbar^{\frac{1}{2}})$ or $O(b)$. For $\hbar < \gamma$, it follows from standard estimates on Gaussian integrals [GJ 81] that for given fixed p' ,

$$\left\|\prod_{\Delta_i\subset X} :S(\Delta_i):\right\|_{p'} \leq (\max\{\gamma^{\frac{1}{2}},b\}\cdot M)^{|X|} \tag{3.14}$$

for some constant M independent of $\hbar$ and s.

To bound the other factor on the right side of eqn. (3.13), we cannot use Lemma 3.1 directly, because when $\lambda_i = 0$ the classical potential will not be in any $\mathcal{T}_{\delta,L}$. However the proof of Lemma 3.1 can be modified to overcome this difficulty, as we will now show. As in the proof of Lemma 3.1 we assume that $\hbar$ and T are real and that $s = 1$. Note that for $p > 1$ and $\gamma \in (0,m^2)$,

$$\int e^{-p\lambda_i :S(V):} d\mu_{m^2} = \frac{\int e^{-p\lambda_i :S(V):_{m^2}} e^{-\frac{1}{2}(m^2-\gamma)\int_V :\phi^2:} d\mu^\gamma}{\int e^{-\frac{1}{2}(m^2-\gamma)\int_V :\phi^2:} d\mu^\gamma}$$

$$\leq \int e^{-p\lambda_i :S(V):_{m^2} - \frac{1}{2}(m^2-\gamma)\int_V :\phi^2:} d\mu^\gamma$$

by Lemma 2.2.3 and Jensen's inequality. But by Lemma 2.2.1

$$p\lambda_i[:\hbar^{-1}T(\hbar,\hbar^{\frac{1}{2}}\phi_r):_{m^2}+a_1(\hbar)\phi_r - \tfrac{1}{2}m^2:\phi_r^2:_{m^2}] + \tfrac{1}{2}(m^2-\gamma):\phi_r^2:^\gamma$$

$$\geq p\lambda_i[\hbar^{-1}:T(\hbar,\hbar^{\frac{1}{2}}\phi_r):_{m^2} + a_1(\hbar)\phi_r - \tfrac{1}{2}m^2:\phi_r^2:_{m^2}] + \tfrac{1}{2}(m^2-\gamma):\phi_r^2:_{m^2} - C$$

$$= \lambda_i[p\hbar^{-1}:T(\hbar,\hbar^{\frac{1}{2}}\phi_r):_{m^2} + pa_1(\hbar)\phi_r - \tfrac{\delta}{2}:\phi_r^2:_{m^2}]$$

$$+ [\lambda_i(\tfrac{\delta}{2} - \tfrac{p}{2}m^2) + \tfrac{1}{2}(m^2-\gamma)]:\phi_r^2:_{m^2} - C \tag{3.15}$$

If $T(0,\cdot) \in \mathcal{T}_{\delta,L}$ then for $p \in (1,2)$, $pT(0,\cdot) \in \mathcal{T}_{\delta,2L}$, so the estimates of the fourth step of the proof of Lemma 3.1 show that

$$p\hbar^{-1}{:}T(\hbar,\hbar^{\frac{1}{2}}\phi_r){:}_{m^2} + pa_1(\hbar)\phi_r - \frac{\delta}{2}{:}\phi_r^2{:}_{m^2} \geq -M_1(\log r)^{\frac{n}{2}} \tag{3.16}$$

(Compare the left side of the above inequality with the expression for ${:}A(\phi){:}^{\delta}$ given above eqn. (3.4)). As for the second term on the right side of eqn. (3.15), we choose p and γ such that

$$0 \leq \min_{0\leq\lambda_i\leq 1} [\lambda_i(\delta-\tfrac{p}{2}m^2) + \tfrac{1}{2}(m^2-\gamma)] = \min_{\lambda_1\in\{0,1\}} [\lambda_i(\delta-\tfrac{p}{2}m^2) + \tfrac{1}{2}(m^2-\gamma)]$$

$$= \min\{\tfrac{1}{2}(m^2-\gamma),\ \delta - (\tfrac{p-1}{2})m^2 - \tfrac{1}{2}\gamma\} \tag{3.17}$$

Clearly $p = 1 + \frac{\delta}{m^2}$ and $\gamma = \min\{\frac{m^2}{2},\frac{\delta}{2}\}$ satisfy (3.17). By eqns. (3.15), (3.16) and (3.17), we have

$$p\lambda_i[\hbar^{-1}{:}T(\hbar,\hbar^{\frac{1}{2}}\phi_r{:}_{m^2} + a_1(\hbar)\phi_r - \tfrac{1}{2}m^2{:}\phi_r^2{:}_{m^2}] + \tfrac{1}{2}(m^2-\gamma){:}\phi_r^2{:}^{\gamma}$$

$$\geq -M_1(\log r)^{\frac{n}{2}} - c_1\log r - c \geq M_2(\log r)^{\frac{n}{2}} .$$

It now follows as in the last step of the proof of Lemma 3.1 that

$$\sup_{0\leq\lambda_i\leq 1} \left\| e^{-\sum_i \lambda_i{:}S(\Delta_i){:}} \right\|_p \leq e^{K|X|} . \tag{3.18}$$

Using the bounds (3.18) and (3.14), eqn. (3.11) follows from eqn. (3.13) by taking b and γ sufficiently small. □

The main consequence of Theorem 3.2 is the following.

Corollary 3.3: For an interaction T and a function $a_1(\hbar) = O(\hbar^{\frac{1}{2}})$ as in Theorem 3.2 there exist $\theta,\gamma,b > 0$ such that the cluster expansion for the interaction $\frac{1}{\hbar} T(\hbar,\hbar^{\frac{1}{2}}\phi)+a_1(\hbar)\phi - \frac{1}{2}m^2\phi^2$ and mass m converges with bounds depending only on m,δ and L, independent of Λ and of $\hbar \in S_{\theta,\gamma}$. In particular, eqn. (2.6) holds for this interaction, with M_r independent of $\hbar \in S_{\theta,\gamma}$.

Proof: Theorem 2.1 and Corollary 2.3 cannot be immediately applied because the mass may not be large. To overcome this problem we consider the theory obtained by replacing the given theory, abbreviated (T,a_1,m), by a theory $(\sigma^{-2}T,\sigma^{-2}a_1,\sigma^{-1}m)$ where $\sigma > 0$ is chosen sufficiently small that $\sigma^{-1}m > M(1,1)$, where $M(K,L)$ is the lower bound on the mass for convergence of the cluster expansion given in Theorem 2.1. For $\mathrm{Re}T(0,\cdot) \in \mathcal{T}_{\delta,L}$ with $|a_2(0) - \frac{1}{2}m^2| < b$ we have $\sigma^{-2}\mathrm{Re}T(0,\cdot) \in \mathcal{T}_{\sigma^{-2}\delta,\sigma^{-2}L}$ and $|\sigma^{-2}a_2(0)-\sigma^{-2}m^2| < \sigma^{-2}b$. By Theorem 3.2 applied to the theory $(\sigma^{-2}T,\sigma^{-2}a_1,\sigma^{-1}m)$ there are $b,\theta,\gamma > 0$ such that

$$\left|\int e^{-\frac{1}{\sigma^2}\int_\Lambda [\hbar^{-1}:T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:]} d\mu_{(\frac{m}{\sigma})^2}(s)\right| \le e^{|\Lambda|} \tag{3.19}$$

and

$$\left|\int e^{-\frac{1}{\sigma^2}\int_\Delta [\hbar^{-1}:T(\hbar,\hbar^{\frac{1}{2}}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:]} d\mu_{(\frac{m}{\sigma})^2}(s)\right| \ge \frac{1}{2} \tag{3.20}$$

for $\hbar \in S_{\theta,\gamma}$ and $|\sigma^{-2}a_2(0) - \frac{1}{2}\sigma^{-2}m^2| < \sigma^{-2}b$. By taking γ and b to be smaller the coefficients of $\hbar^{-1}T(\hbar,\hbar^{\frac{1}{2}}\phi) + a_1(\hbar)\phi - \frac{1}{2}m^2\phi^2$ can be made less than one in absolute value.

The bounds (3.19) and (3.20) correspond to the bounds (2.3) and (2.4) of Theorem 2.1. (Clearly the $\frac{1}{\sigma^2}$ multiplying the integral $\int_\Lambda$ on the left side of eqn. (3.19) can be replaced by $\frac{p}{\sigma^2}$ for p sufficiently close to one by further decreasing γ and b). By Theorem 2.1 we obtain a uniformly convergent cluster expansion for the theory $(\sigma^{-2}T,\sigma^{-2}a_1, {}^{-1}m)$. But by Lemma 2.2.5, a generalized Schwinger function for the theory $(\sigma^{-2}T,\sigma^{-2}a_1,\sigma^{-1}m)$ is equal to the corresponding generalized Schwinger function for the theory (T,a_1,m) provided we also replace Λ by $\sigma^{-1}\Lambda$ and w by $w^{(\sigma^{-1})}$. Since σ^{-1} is just a constant the Corollary is proved. □

§4. Analyticity of the Pressure

We are now in a position to prove that for some $\varepsilon,\gamma > 0$ and open set $0_\gamma \supset (0,\gamma)$ the pressure $\hbar\sigma(\hbar,\mu)$ of eqn. (1.13) is jointly analytic in $(\hbar,\mu) \in 0_\gamma \times D_\varepsilon$, where $D_\varepsilon = \{z \in C : |z| < \varepsilon\}$, and is C^∞ at $\hbar = 0^+$. The strategy of the proof is the following. Given $T(\hbar,x) = \sum_{k=2}^{n} a_k(\hbar)x^k$ with a_k analytic in $(0,\gamma)$ and $T(0,\cdot) \in T_{\delta,L}$, let

$$\hbar\tau(\hbar,\mu) = \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [:T(\hbar,\phi):-\frac{1}{2}m^2:\phi^2:-\mu\phi]} d\mu_{\hbar C} . \tag{4.1}$$

By Lemma 2.3.6, if $\hbar$ and $|\mu|$ are sufficiently small then $T(\hbar,x) - \mu x$ has a uniquely attained global minimum, at say $\xi(\hbar,\mu)$, with ξ analytic in $(\hbar,\mu) \in (0,\gamma') \times (-\varepsilon',\varepsilon')$ and C^∞ at $\hbar = 0^+$. By Lemmas 2.3.6 and 2.2.4, translating the field in eqn. (4.1) by $\xi(\hbar,\mu)$ gives rise to a new pressure whose classical potential lies in some $T_{\delta',L'}$ uniformly in small $\hbar$ and $|\mu|$. For $|a_2(0) - \frac{m^2}{2}|$ sufficiently small we can appeal to Corollary 3.3 to conclude that $\hbar\tau(\hbar,\mu)$ is analytic in $(\hbar^{\frac{1}{2}},\mu) \in (0,\gamma'') \times (-\varepsilon'',\varepsilon'')$ and C^∞ at $\hbar^{\frac{1}{2}} = 0^+$. To improve this to C^∞ at $\hbar = 0^+$ we will show that odd order derivatives with respect to $\hbar^{\frac{1}{2}}$ vanish at $\hbar^{\frac{1}{2}} = 0$. In Lemma 4.2 below it is shown that the vanishing of the odd order derivatives implies the required smoothness at $\hbar = 0$.

The following Lemma will be used to show that the above-mentioned odd order derivatives at $\hbar^{\frac{1}{2}} = 0$ vanish. For the analyticity of the pressure we only need $f = 0$ in Lemma 4.1. However we prove the more general result because it will be needed in Theorem 5.1.3. The parameter t corresponds to $\hbar^{\frac{1}{2}}$.

<u>Lemma 4.1</u>: Let $B_t(x) = t^{-2}T(t^2,tx) - \frac{1}{2}m^2x^2 - f(t)x$,

where $T(t^2,x) = \sum_{k=2}^{n} a_k(t^2)x^k$ with a_k analytic in $(0,\gamma)$ and C^∞ in $[0,\gamma)$, $\mathrm{Re}\,T(0,\cdot) \in \mathcal{T}_{\delta,L}$, $|a_2(0) - \frac{1}{2}m^2| < b$ where b is specified by Corollary 3.3, and where $f(t) = tg(t^2)$ with g analytic is $(0,\gamma)$ and C^∞ in $[0,\gamma)$. Let

$$\langle\cdot\rangle_{P,\Lambda} = \frac{\int \cdot\, e^{-\int_\Lambda :P(\phi):} d\mu_{m^2}}{\int e^{-\int_\Lambda :P(\phi):} d\mu_{m^2}} .$$

Then

$$\lim_{t\downarrow 0} \frac{1}{|\Lambda|} \left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):\right\rangle_{B_t,\Lambda} = \frac{1}{|\Lambda|} \left\langle :\phi^{k_1}(\phi):;\ldots;:\phi^{k_r}(\Lambda):\right\rangle_{B_0,\Lambda} \tag{4.2}$$

uniformly in Λ. In particular, if $k_1+\ldots+k_r$ is odd the limit in (4.2) is zero, since B_0 is quadratic.

<u>Proof</u>: It follows from Corollary 3.3 and the Fundamental Theory of Calculus that

$$\left|\frac{1}{|\Lambda|} \left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):\right\rangle_{B_t,\Lambda} - \frac{1}{|\Lambda|}\left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):\right\rangle_{B_0,\Lambda}\right|$$

$$= \frac{1}{|\Lambda|} \left| \int_0^t \frac{d}{ds} \left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):\right\rangle_{B_s,\Lambda} ds\right|$$

$$= \frac{1}{|\Lambda|} \left| \int_0^t \left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):;\int_\Lambda \frac{d}{ds}:B_s(\phi):\right\rangle_{B_{s,\Lambda}} ds \right|$$

$\leq Mt$ uniformly in Λ. □

Lemma 4.2: Let 0 be an open neighbourhood of $(0,\sqrt{\gamma})$ on which a function f is analytic. Suppose f is C^∞ at 0^+ with $f^{(2k+1)}(0^+) = 0$, $k = 0,1,2,\ldots$. Let $f_1(x) = f(x^{\frac{1}{2}})$, $x \in U = 0^2 \cap (C\setminus\{x \in R : x \leq 0\})$. Then f_1 is analytic on U and C^∞ at 0^+.

Proof: The only thing to check is that f_1 is C^∞ at 0^+. But this is obvious since

$$f_1(x) = f(\sqrt{x}) \sim f(0) + \frac{1}{2!} f^{(2)}(0)x + \frac{1}{4!} f^{(4)}(0)x^2 + \frac{1}{6!} f^{(6)}(0)x^3 + \ldots .$$

□

Theorem 4.3: Let $T(\hbar,x) = \sum_{k=2}^{n} a_k(\hbar)x^k$ where the a_k are analytic in an interval $(0,\rho)$ and C^∞ at 0^+, and $T(0,\cdot) \in T_{\delta,L}$. Then for $|a_2(0) - \frac{1}{2}m^2|$ sufficiently small there exist $\gamma > 0$ and complex open neighbourhoods $0_\gamma \supset (0,\gamma)$ and D containing 0 such that

$$\hbar\tau(\hbar,\mu) = \hbar \lim_\Lambda \tau_\Lambda(\hbar,\mu) \equiv \hbar \lim_\Lambda \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [:T(\hbar,\phi):-\frac{1}{2}m^2:\phi^2:-\mu\phi]} d\mu_{\hbar C}$$

is jointly analytic in $(\hbar,\mu) \in 0_\gamma \times D$ and C^∞ at $\hbar = 0^+$, with uniformly bounded derivatives. Moreover, there is a $c > 0$ such that

$$|D_2^2 \hbar\tau(\hbar,\mu)| \geq c \text{ for all } (\hbar,\mu) \in 0_\gamma \times D. \tag{4.3}$$

Proof: By Lemma 2.3.6 there exist $\gamma',\varepsilon' > 0$ such that for $\mu \in (-\varepsilon',\varepsilon')$ and $\hbar \in [0;\gamma')$ $T_\mu(\hbar,x) = T(\hbar,x) - \mu x$ has a uniquely attained global minimum, at say $\xi(\hbar,\mu)$, with

$$S(\hbar,\mu;x) = T_\mu(\hbar,x+\xi(\hbar,\mu)) - T_\mu(\hbar,\xi(\hbar,\mu)) \in T_{\delta',L'} \quad \text{for all}$$

$$(\hbar,\mu) \in [0,\gamma') \times (-\varepsilon',\varepsilon') \ . \tag{4.4}$$

Moreover ξ is analytic in $V_{\gamma'} \times D_{\varepsilon'}$, where $D_{\varepsilon'} = \{z \in C : |z| < \varepsilon'\}$ and $V_{\gamma'}$ is an open neighbourhood of $(0,\gamma')$, and C^∞ at $\hbar = 0^+$.

Using Lemma 2.2.2, translating in τ_Λ by ξ and using Lemma 2.2.3 gives

$$\begin{aligned}\hbar\tau_\Lambda(\hbar,\mu) &= -T_\mu(\hbar,\xi(\hbar,\mu)) + \frac{\hbar}{|\Lambda|}\ln\int e^{-\frac{1}{\hbar}\int_\Lambda [:S(\hbar,\mu;\phi):-\frac{1}{2}m^2:\phi^2:]} d\mu_{\hbar C} \\ &= -T_\mu(\hbar,\xi(\hbar,\mu)) + \frac{\hbar}{|\Lambda|}\ln\int e^{-\int_\Lambda [\frac{1}{\hbar}:S(\hbar,\mu;\hbar^{\frac{1}{2}}\phi):-\frac{1}{2}m^2:\phi^2]} d\mu_C \ ,\end{aligned} \tag{4.5}$$

for all $(\hbar,\mu) \in V_\gamma \times D_{\varepsilon'}$. Since $\frac{1}{2}D_3^2S(0,\mu;0) = \frac{1}{2}D_2^2T(0,\xi(0,\mu))$ we can make $\frac{1}{2}D_2^2S(0,\mu;0)$ as close as desired to $\frac{1}{2}m^2$ by taking ε' and $|a_2(0) - \frac{1}{2}m^2|$ sufficiently small. Then by Corollary 3.3 and eqn. (4.4) expectations of the form $\frac{1}{|\Lambda|}\left\langle :\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):\right\rangle_{\tilde{S},\Lambda}$ are bounded in absolute value independent of Λ, $\hbar$, μ, where

$$\tilde{S}(\hbar^{\frac{1}{2}},\mu;x) = \hbar^{-1}S(\hbar,\mu;\hbar^{\frac{1}{2}}x) - \frac{1}{2}m^2x^2.$$

The first term on the right side of eqn. (4.5) is analytic in $(\hbar,\mu) \in V_{\gamma'} \times D_{\varepsilon'}$ and C^∞ at $\hbar = 0^+$, and does not depend on Λ. Its derivatives are uniformly bounded. To see that $\hbar\tau(\hbar,\mu)$ is analytic, and C^{∞} at $\hbar = 0^{+}$, we show that the infinite volume limit of the second term on the right side of eqn. (4.5) is analytic in $(\hbar^{\frac{1}{2}},\mu)$ using Vitali's theorem, then use Lemma 4.1 to show that odd derivatives with respect to $t = \hbar^{\frac{1}{2}}$ vanish at $t = 0$ and appeal to Lemma 4.2.

Let $\zeta_{\Lambda}(t,\mu) = \frac{1}{|\Lambda|} \ln \int e^{-\int_{\Lambda} :\tilde{S}(t,\mu;\phi):} d\mu_C$, $(t,\mu) \in (V_{\gamma'} \cap S_{\frac{\pi}{8},\gamma}) \times D_{\varepsilon}$,

where $0 < \gamma \leq \gamma'$ and $0 < \varepsilon \leq \varepsilon'$ are such that the logarithm is well defined. Since

$$|D_1\zeta_{\Lambda}(t,\mu)| = \frac{1}{|\Lambda|} \left|\left\langle \int_{\Lambda} :D_1\tilde{S}(t,\mu;\phi): \right\rangle_{\tilde{S},\Lambda}\right| \quad \text{and}$$

$$|D_2\zeta_{\Lambda}(t,\mu)| = \frac{1}{|\Lambda|} \left|\left\langle \int_{\Lambda} :D_2\tilde{S}(t,\mu;\phi): \right\rangle_{\tilde{S},\Lambda}\right| \quad \text{are uniformly bounded in } \Lambda,t,\mu \text{ ,}$$

the same is true of $|\zeta_{\Lambda}(t,\mu)|$ so by Vitali's theorem $\zeta(t,\mu) = \lim\limits_{\Lambda} \zeta_{\Lambda}(t,\mu)$ is analytic in $(t,\mu) \in (V_{\gamma'} \cap S_{\frac{\pi}{8},\gamma}) \times D_{\varepsilon}$. Moreover all derivatives of ζ are bounded uniformly in (t,μ). We now show that $D_1^{2k+1}D_2^{\ell}\zeta_{\Lambda}(t,\mu) \to 0$ as $t \downarrow 0$, uniformly in Λ, for $k,\ell = 0,1,2,\ldots$. In fact, by eqn. (1.3.3),

$$D_1^{2k+1}\zeta_{\Lambda}(t,\mu) = \frac{1}{|\Lambda|} \sum_{\pi\in P_{2k+1}} c_{\pi} \left\langle \int_{\Lambda} :D_1^{|\pi_1|}\tilde{S}(t,\mu;\phi):;\ldots;\int_{\Lambda} :D^{|\pi_{|\pi|}|}\tilde{S}(t,\mu;\phi): \right\rangle_{\tilde{S},\Lambda} \tag{4.6}$$

where P_n is the set of partitions of $\{1,\ldots,n\}$, π_i are the elements of a partition π, and c_π are positive integers. Differentiation of eqn. (4.6) with respect to μ gives

$$D_1^{2k+1} D_2^{\ell} T_{\Lambda}(t,\mu) = \frac{1}{|\Lambda|} \sum_{\pi \in P_{2k+1}} \sum_{\sigma \in P_\ell} \Big\langle \int_\Lambda :D_1^{|\pi_1|} D_2^{|\sigma_1|} \tilde{S}:;\ldots;$$

$$\int_\Lambda :D_1^{|\pi_{|\pi|}|} D_2^{|\sigma_{|\pi|}|} \tilde{S}:;\int_\Lambda :D_2^{|\sigma_{|\pi|+1}|} \tilde{S}:;\ldots;\int_\Lambda :D_2^{|\sigma_{|\sigma|}|} \tilde{S}:\Big\rangle_{\tilde{S},\Lambda} \tag{4.7}$$

where we allow $|\sigma_1|,\ldots,|\sigma_{|\pi|}|$ to be zero.

Since $\tilde{S}(t,\mu;\phi) = \sum_{k=2}^{n} \frac{D_2^k T(t^2,\xi(t^2,\mu))}{k!} t^{k-2}\phi^k - \frac{1}{2}m^2\phi^2$, the $t = 0$ contribution to $D_1^j D_2^i \tilde{S}$ is a linear combination of terms of the form $c(\mu)\phi^r(\Lambda)$ where r is odd if j is odd and r is even if j is even. By eqn. (4.4) and Lemma 4.1 (with $f = 0$), as $t \to 0$ the right side of eqn. (4.7) approaches uniformly in Λ a sum of terms of the form

$$c(\mu) \Big\langle :\phi^{r_1}(\Lambda):;\ldots;:\phi^{r_{|\sigma|}}(\Lambda):\Big\rangle_{\tilde{S}(0,\mu;\cdot),\Lambda} \tag{4.8}$$

where $r_1,\ldots,r_{|\pi|}$ have the same parity as $|\pi_1|,\ldots,|\pi_{|\pi|}|$, and $r_{|\pi|+1},\ldots,r_{|\sigma|}$ are all even (in fact equal 2). Since $|\pi_1|+\ldots+|\pi_{|\pi|}| = 2k+1$ is odd, $r_1+\ldots+r_{|\sigma|}$ is also odd. The expectation in eqn. (4.8) is invariant under $\phi \to -\phi$ since $\tilde{S}(0,\mu;\cdot)$ is quadratic, and hence equals zero. It now follows from Lemma 4.2 that $\hbar D_2^{\ell}\tau(\hbar,\mu)$ is C^∞ at $\hbar = 0^+$.

It remains to prove the lower bound (4.3). Using the notation

$\int (F_1(\phi);F_2(\phi))d\nu = \int F_1(\phi)F_2(\phi)d\nu - \int F_1(\phi)d\nu \int F_2(\phi)d\nu$, we have

$$\hbar D_2^2\tau_\Lambda(\hbar,\mu) = \frac{\frac{1}{\hbar|\Lambda|}\int(\phi(\Lambda);\phi(\Lambda))\, e^{-\frac{1}{\hbar}\int_\Lambda[:T(\hbar,\phi): - \frac{1}{2}m^2:\phi^2: - \mu\phi\,]} d\mu_{\hbar C}}{\int e^{-\frac{1}{\hbar}\int_\Lambda[:T(\hbar,\phi): - \frac{1}{2}m^2:\phi^2: - \mu\phi\,]} d\mu_{\hbar C}}$$

$$= \frac{\frac{1}{\hbar|\Lambda|}\int((\phi(\Lambda)+\xi(\hbar,\mu));(\phi(\Lambda)+\xi(\hbar,\mu)))\, e^{-\frac{1}{\hbar}\int_\Lambda[:S(\hbar,\mu;\phi):-\frac{1}{2}m^2:\phi^2:]} d\mu_{\hbar C}}{\int e^{-\frac{1}{\hbar}\int_\Lambda[:S(\hbar,\mu;\phi):-\frac{1}{2}m^2:\phi^2:]} d\mu_{\hbar C}}$$

$$= \frac{\frac{1}{\hbar|\Lambda|}\int(\phi(\Lambda);\phi(\Lambda))\, e^{-\frac{1}{\hbar}\int_\Lambda[:S(\hbar,\mu;\phi): - \frac{1}{2}m^2:\phi^2:]} d\mu_{\hbar C}}{\int e^{-\frac{1}{\hbar}\int_\Lambda[:S(\hbar,\mu;\phi): - \frac{1}{2}m^2:\phi^2:]} d\mu_{\hbar C}}$$

$$= \frac{1}{|\Lambda|}\left\langle \phi(\Lambda);\phi(\Lambda)\right\rangle_{\tilde{S},\Lambda}$$

where we used Lemmas 2.2.3 and 2.2.4. By Lemma 4.1,

$$\lim_{\hbar\downarrow 0} \hbar D_2^2\tau(\hbar,\mu) = \lim_{\Lambda} \frac{1}{|\Lambda|}\left\langle \phi(\Lambda);\phi(\Lambda)\right\rangle_{\tilde{S}(0,\mu;\cdot),\Lambda} \tag{4.9}$$

The right side of eqn. (4.9) is continuous in μ and equals

$\int_{R^2} (-\Delta+2a_2(0))^{-1}(x)dx > 0$ for $\mu = 0$. Therefore, taking ε and γ smaller if necessary, the lower bound (4.3) holds. □

CHAPTER 4: SMOOTHNESS OF THE EFFECTIVE POTENTIAL

§1. Proof of Theorem 1.4.1

Theorem 1.4.1: $V(\hbar,a) < \infty$ for all $\hbar > 0$, $a \in R$.

Proof: To show that $V(\hbar,a) = \sup_\mu [\mu a - \hbar\alpha(\hbar,\mu)]$ is finite, it suffices to show that $\lim_{\mu\to\pm\infty} \hbar D_2^{\pm}\alpha(\hbar,\mu) = \pm\infty$ (recall Figure 2(ii)). Since $\hbar$ plays no important role in this discussion we drop it from the notation. We write

$$\langle\cdot\rangle_{P,\Lambda,m^2} = \frac{\int \cdot\, e^{-\int_\Lambda :P(\phi):} d\mu_C}{\int e^{-\int_\Lambda :P(\phi):} d\mu_C}$$

and $\langle\ \rangle^{\pm}_{P,m^2}$ for the $\begin{smallmatrix}\text{right}\\ \text{left}\end{smallmatrix}$ continuous infinite volume expectation [FS 77]. Then by eqn. (1.2.1) $D^{\pm}\alpha(\mu) = \langle\phi(0)\rangle^{\pm}_{P(x)-\mu x,m^2}$. Note that it suffices to prove that $\lim_{\mu\to\infty} D^{+}\alpha(\mu) = \infty$ for all semibounded P because this implies that

$$\lim_{\mu\to-\infty} D^{-}\alpha(\mu) = \lim_{\mu\to-\infty} \langle\phi(0)\rangle^{-}_{P(x)-\mu x,m^2} = \lim_{\mu\to-\infty} \langle-\phi(0)\rangle^{+}_{P(-x)+\mu x,m^2}$$

$$= -\lim_{\mu\to\infty} \langle\phi(0)\rangle^{+}_{P(-x)-\mu x,m^2}$$

$$= \infty .$$

Also since α is differentiable except on a countable set it suffices to show that $D^{+}\alpha(\mu)$ is unbounded on the set of positive μ's for which $D\alpha(\mu)$ exists. For such μ,

$$\lim_{\Lambda\uparrow R^2} \left\langle \phi(0) \right\rangle_{P(x)-\mu x,\Lambda,m^2} = \left\langle \phi(0) \right\rangle^{+}_{P(x)-\mu x,\mu^2} = \left\langle \phi(0) \right\rangle^{-}_{P(x)-\mu x,m^2}$$

For large μ, let $\xi(\mu)$ denote the unique point at which $U_\mu(x) = P(x) + \frac{1}{2}m^2x^2 - \mu x$ attains its global minimum. By Lemma 2.3.3 ξ is increasing and $\lim_{\mu\to\infty} \xi(\mu) = \infty$. Therefore it suffices to show that $\left| \left\langle \phi(0) \right\rangle_{P(x)-\mu x,\Lambda,m^2} - \xi(\mu) \right|$ is bounded above uniformly in Λ and large μ. This upper bound is a consequence of the cluster expansion of [Sp 74]. We spell out the details.

By Lemma 2.2.2, writing $P_\mu(x) = P(x) - \mu x$,

$$\left\langle \phi(0) \right\rangle_{P_\mu,\Lambda,m^2} = \xi(\mu) + \left\langle \phi(0) \right\rangle_{S(\mu,\cdot),\Lambda,m^2}$$

where $S(\mu,x) = U_\mu(x+\xi(\mu)) - U_\mu(\xi(\mu)) - \frac{1}{2}m^2x^2 = \sum_{k=2}^{n} a_k(\mu)x^k - \frac{1}{2}m^2x^2$, with $a_k(\mu) = U_\mu^{(k)}(\xi(\mu))/k!$. Therefore it suffices to show that $\left| \left\langle \phi(0) \right\rangle_{S(\mu,\cdot),\Lambda,m^2} \right|$ is bounded uniformly in Λ and large μ.

Applying scaling (Lemma 2.2.5) and mass-shift (Lemma 2.2.3) transformations, we obtain

$$\left\langle \phi(0) \right\rangle_{S(\mu,\cdot),\Lambda,m^2} = \left\langle \phi(0) \right\rangle_{\xi^{2-n}S(\mu,\cdot),\xi^{\frac{n}{2}-1}\Lambda,\xi^{2-n}m^2} = \left\langle \phi(0) \right\rangle_{Q,\xi^{\frac{n}{2}-1}\Lambda,\xi^{2-n}a_2} \, . \tag{1.1}$$

Here Q is defined by

$$:Q(\phi):_{\xi^{2-n}a_2} = \xi^{2-n}\sum_{k=3}^{n} a_k :\phi^k:_{\xi^{2-n}m^2} = \sum_{k=3}^{n} b_k \xi^{2-k} :\phi^k:_{\xi^{2-n}m^2}$$

where we have introduced $b_k(\mu) = \xi^{k-n} a_k(\mu)$. Note that $\lim_{\mu\to\infty} b_k(\mu)$ exists and is finite.

By Lemma 2.2.1 and the fact that $\left|\log \frac{\xi^{2-n}a_2}{\xi^{2-n}m^2}\right| = 0(\log \xi^{n-2})$,

$$Q(x) = \sum_{k=3}^{n} b_k \xi^{2-k} x^k + \sum_{k=0}^{n-2} 0(\xi^{-k}(\log\xi^{n-2})^{\frac{n-k}{2}})x^k .$$

The constant term in Q can be cancelled in eqn. (1.1), so the classical potential occurring on the right side of eqn. (1.1) is

$$W(x) = \sum_{k=3}^{n} b_k \xi^{2-k} x^k + \sum_{k=1}^{n-2} 0(\xi^{-k}(\log\xi^{n-2})^{\frac{n-k}{2}})x^k + \frac{1}{2}\xi^{2-n}a_2 x^2$$

$$= \sum_{k=2}^{n} (b_k+\lambda_k)\xi^{2-k}x^k + \lambda_1 x$$

where $\lambda_k = 0(\xi^{-2}(\log\xi^{n-2})^{\frac{n-k}{2}})$, $k \geq 2$; $\lambda_1 = 0(\xi^{-1}(\log\xi^{n-2})^{\frac{n-1}{2}})$.

Since $\sum_{k=2}^{n} b_k x^k = \xi^{-n}\sum_{k=2}^{n} a_k(\xi x)^k = \xi^{-n}[U_\mu(\xi x+\xi) - U_\mu(\xi)]$ has a uniquely attained global minimum at zero, for large μ W will have a uniquely attained global minimum, at say $\eta(\mu)$, with $\eta(\mu) \to 0$ as $\mu \to \infty$, Translation by η (Lemma 2.2.2) gives

$$\left\langle\phi(0)\right\rangle_{Q,\xi^{\frac{n}{2}-1}\Lambda,\xi^{2-n}a_2} = \eta(\mu) + \left\langle\phi(0)\right\rangle_{Q_1,\xi^{\frac{n}{2}-1}\Lambda,\xi^{2-n}a_2} \qquad (1.2)$$

where $Q_1(x) = W(x+\eta) - W(\eta) - \frac{1}{2}\xi^{2-n}a_2x^2$.

The expectation on the right side of eqn. (1.2) has the classical potential

$$W_1(x) = Q_1(x) + \frac{1}{2}\xi^{2-n}a_2x^2 = \sum_{k=2}^{n} \frac{W^{(k)}(\eta)}{k!} x^k .$$

Let $$T(x) = \xi^{-2}W_1(\xi x) . \tag{1.3}$$

Then $W_1(x) = \xi^2 T(\xi^{-1}x)$. Here ξ^{-1} plays the role of $h^{\frac{1}{2}}$.

Now $T(x) = \sum_{k=2}^{n} \frac{W^{(k)}(\eta)}{k!} \xi^{k-2}x^k$, and

$$\frac{1}{k!} W^{(k)}(\eta)\xi^{k-2} = \frac{1}{k!} \xi^{k-2} \sum_{j=k}^{n} (b_j+\lambda_j)\xi^{2-j}j(j-1)\ldots(j-k+1)\eta^{j-k}$$

$$= \sum_{j=k}^{n} (b_j+\lambda_j)\xi^{k-j}\binom{j}{k}\eta^{j-k}$$

$$\sim b_k \quad \text{for large } \mu , k \geq 2 . \tag{1.4}$$

Recall Lemma 4.1 from [Sp 74]:

<u>Lemma 1.1</u>: Let $U_0(x) = \sum_{k=2}^{n} t_k x^k$ be bounded below, $U_\mu(x) = U_0(x) - \mu x$, and for large μ let $\xi(\mu)$ be the location of the uniquely attained global minimum of U_μ . Then there exist $M, c > 0$ such that for all $\mu > M$

$$U_\mu(x+\xi(\mu)) - U_\mu(\xi(\mu)) \geq c \sum_{k=2}^{n} \xi(\mu)^{n-k}|x|^k \quad \text{for all } x \in R . \quad \square$$

By this lemma, $\sum_{k=2}^{n} b_k x^k = \xi^{-n}[U_\mu(\xi x+\xi) - U_\mu(\xi)]$

$$\geq \xi^{-n} c \sum_{k=2}^{n} \xi^{n-k}|\xi x|^k = c \sum_{k=2}^{n} |x|^k . \tag{1.5}$$

By eqns. (1.4) and (1.5) it follows that for μ sufficiently large

$$T(x) \geq \frac{c}{2} \sum_{k=2}^{n} |x|^k .$$

Since the coefficients of T are uniformly bounded for large μ by eqn. (1.4), $T \in \mathcal{T}_{\frac{c}{2},L}$ for some $L > 0$. It follows from eqn. (1.3) and Corollary 3.3.3 that $\left|\left\langle \phi(0) \right\rangle_{Q_1, \xi^{\frac{n}{2}-1}\Lambda, \xi^{2-n} a_2}\right|$ is bounded uniformly in Λ and large μ. Since $\eta(\mu) \to 0$ as $\mu \to \infty$ it follows from eqns. (1.2) and (1.1) that $\left|\left\langle \phi(0) \right\rangle_{S(\mu,\cdot),\Lambda,m^2}\right|$ is bounded uniformly in Λ and large μ. $\square$

§2. Proof of Theorem 1.4.2.

Theorem 1.4.2: For every $a \in R$, $\lim_{\hbar \downarrow 0} V(\hbar,a) = (\mathrm{conv} U_0)(a)$.

Proof: Because of the fact that $\hbar\alpha(\hbar,\cdot)$ is strictly convex [FS 77] and $\hbar D_2^{\pm}\alpha(\hbar,\mu) \to \pm\infty$ as $\mu \to \pm\infty$ (as shown in §4.1), it follows that $\hbar\alpha(\hbar,\cdot) \in \mathcal{C}_s$. In Theorem 2.1 below we will show that $\lim_{h\downarrow 0} \hbar\alpha(\hbar,\mu) = -m(\mu)$ for all μ. Using this, and the fact that $-m \in \mathcal{C}_s$ by Lemma 2.3.3, it follows from Theorem 2.1.1 and eqn. (2.3.7) that

$$\lim_{\hbar\downarrow 0} V(\hbar,a) = -m^*(a) = U_0^{**}(a) = (\mathrm{conv} U_0)(a) \text{ , for all } a \in R .$$

We now prove the promised Theorem, which is a Laplace's method type result for functional integrals on $\mathcal{S}'(R^2)$. For related results in the context of Gaussian integrals on $C[0,1]$, see [Sim 79], [ER 82].

Theorem 2.1: $\lim_{\hbar\downarrow 0} \hbar\alpha(\hbar,\mu) = -m(\mu)$, for all $\mu \in R$.

Proof: Let $\alpha_\Lambda(\hbar,\mu) = \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [:P(\phi):-\mu\phi]} d\mu_{\hbar C}$, and fix $\mu \in G$.

Let $T(x) = U_\mu(x+\xi(\mu)) - U_\mu(\xi(\mu))$. By Lemma 2.3.1, $T \in T_{\delta,L}$ for some $\delta, L > 0$. By Lemma 2.2.2,

$$\hbar\alpha_\Lambda(\hbar,\mu) = -U_\mu(\xi(\mu)) + \hbar \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [:T(\phi): - \frac{1}{2}m^2:\phi^2:]} d\mu_{\hbar C} \tag{2.1}$$

By Jensen's inequality the argument of the logarithm on the right side of eqn. (2.1) is bounded below by one, and by Lemma 2.2.4 and Lemma 3.3.1 it is bounded above by $e^{K|\Lambda|}$ if $\hbar$ is sufficiently small. These bounds and eqn. (2.1) show that $|\hbar\alpha_\Lambda(\hbar,\mu) + m(\mu)| \to 0$ uniformly in Λ , as $\hbar \downarrow 0$, for $\mu \in G$. But by Lemma 2.3.2 G is dense in R and hence $\lim_{\hbar\downarrow 0} \hbar\alpha(\hbar,\mu) = -m(\mu)$ for all $\mu \in R$ by convexity. □

§3. Proof of Theorem 1.4.3(a):

Theorem 1.4.3(a): Let $a \notin B$. There exists a $\gamma > 0$ such that $V(\hbar,a)$ is analytic in $\hbar$ for $\hbar \in (0,\gamma)$. Moreover $V(\hbar,a)$ is C^∞ at $\hbar = 0^+$, and so the expansion $V(\hbar,a) \sim \sum_{n=0}^{\infty} v_n(a)\hbar^n$ is asymptotic, where $v_n(a) = \frac{D_1^n V(0^+,a)}{n!}$.

Proof: Recall eqn. (3.1.15):

$$V(\hbar,a) = U_0(a) - \hbar\gamma(a) + q_0(\hbar) + \sup_\mu[-\hbar\sigma(\hbar,\mu)] \ , \ a \notin B \ ,$$

where q_0 and σ are functions of a . Fix $a \notin B$. Since q_0 is a polynomial we need only show that $E(\hbar) = \sup_{\mu} [-\hbar\sigma(\hbar,\mu)]$ is analytic on $(0,\gamma)$ and C^{∞} at $\hbar = 0^{+}$. We show this using Theorem 2.1.2.

Note that it suffices to show that

$$\lim_{\hbar \downarrow 0} \hbar\sigma(\hbar,\mu) = -m_0(\mu) \text{ , for all } \mu \in R \text{ ,} \tag{3.1}$$

where $m_0(\mu) = \min_x [\sum_{k=3}^{n} q_k(0)x^k + \frac{1}{2} m_1^2 x^2 - \mu x]$. In fact, writing (as in Theorem 2.1.1) $\mu(0)$ for the location of the supremum in $\sup_{\mu} [+m_0(\mu)]$, it follows from Lemma 2.3.3 that $\mu(0)$ is the unique root of $-m_0'(x) = 0$. By Lemma 2.3.3 this root is the unique μ for which $\sum_{k=3}^{n} q_k(0)x^k + \frac{1}{2}m^2x^2 - \mu x$ attains its global minimum at zero. Since $a \notin B$, there are $\delta, L > 0$ such that

$$\sum_{k=3}^{n} q_k(0)x^k + \frac{1}{2}m_1^2x^2 \in T_{\delta,L} \text{ ,} \tag{3.2}$$

and so $\mu(0) = 0$. Now given eqn. (3.1), it follows from (3.2) and Theorem 3.4.3 that $\hbar\sigma(\hbar,\mu)$ satisfies the analyticity requirements of Theorem 2.1.2, as well as the necessary bounds on the derivatives, and hence E is analytic in $(0,\gamma)$ and C^{∞} at $\hbar = 0^{+}$. It remains to prove eqn. (3.1).

We show that (3.1) holds for $\mu \in G(0)$, where for $\lambda \geq 0$ $G(\lambda) = \{\mu \in R: \sum_{k=3}^{n} q_k(\lambda)x^k + \frac{1}{2}m_1^2x^2 - \mu x$ has a uniquely attained global minimum and has positive curvature at that minimum$\}$. The set $G(0)$ is dense in R by Lemma 2.3.2, so (3.1) holds for all μ if it holds for $\mu \in G(0)$, by convexity.

Let $\sigma_\Lambda(\hbar,\lambda,\mu) = \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [\sum_{k=2}^{n} q_k(\lambda):\phi^k: - \mu\phi]} d\mu_{\hbar C}$,

and let $\sigma_\Lambda(\hbar,\mu) = \sigma_\Lambda(\hbar,\hbar,\mu)$, so $\sigma(\hbar,\mu) = \lim_\Lambda \sigma_\Lambda(\hbar,\mu)$. By the Fundamental Theorem of Calculus,

$$|\hbar\sigma_\Lambda(\hbar,\mu) + m_0(\mu)| \leq |\hbar\sigma_\Lambda(\hbar,0,\mu) + m_0(\mu)| + \hbar \int_0^\hbar |D_2\sigma_\Lambda(\hbar,\lambda,\mu)| d\lambda \, . \tag{3.3}$$

By Theorem 2.1, the infinite volume limit of the first term on the right side of (3.2) goes to zero as $\hbar \downarrow 0$. As for the second term, fix $\mu \in G(0)$ and $\gamma > 0$ sufficiently small that $\mu \in G(\lambda)$ for $\lambda \in (0,\gamma)$. In the expectation $\hbar D_2\sigma_\Lambda(\hbar,\lambda,\mu)$, translate the field by the location $\xi(\lambda,\mu)$ of the global minimum of $\sum_{k=3}^{n} q_k(\lambda)\phi^k + \frac{1}{2}m_1^2\phi^2 - \mu\phi$, scale the field $\phi \to \hbar^{\frac{1}{2}}\phi$, shift the quadratic term of the interaction over to the measure, and Wick re-order the interaction to match the new measure. Then by Corollary 3.3.3, $\hbar|D_2\sigma_\Lambda(\hbar,\lambda,\mu)|$ is bounded uniformly in Λ and in small $\hbar$ and λ , and therefore the second term on the right side of (3.3) is $O(\hbar)$ uniformly in Λ . □

Note that it was also proven in Theorem 2.1.2 that the point $\mu(\hbar)$ at which $\sup_\mu [-\hbar\sigma(\hbar,\mu)]$ is attained is analytic and bounded on $(0,\gamma)$ and hence C^∞ at $h = 0^+$. In particular

$$\lim_{\hbar\downarrow 0} \mu(\hbar) = \mu(0) = 0 \tag{3.4}$$

§4. <u>Proof of Theorem 1.4.4.</u>

<u>Theorem 1.4.4:</u> Let $K \subset B^c$ be compact. Then there is a $\gamma > 0$ and an

open set $0 \supset K$ such that $V(\hbar,\cdot)$ has an analytic extension to 0 for every $\hbar < \gamma$.

Proof: Fix $a \notin B$. Since $\hbar\alpha(\hbar,\cdot)$ is strictly convex, it follows from §4.1 that $\hbar\alpha(\hbar,\cdot) \in C_s$, and hence there is a unique $\mu(\hbar,a)$ for which

$$V(\hbar,a) = \mu(\hbar,a)a - \hbar\alpha(\hbar,\mu(\hbar,a)) .$$

Similarly by Lemma 2.3.3 $-m \in C_s$ so there is a unique $\mu(a)$ at which $\sup_\mu [\mu a + m(\mu)]$ is attained. It follows from Theorems 2.1 and 2.1.1 that $|\mu(\hbar,a) - \mu(a)|$ can be made arbitrarily small by taking $\hbar$ sufficiently close to zero.

Let $K \subset B^c$ be compact. We show $V(\hbar,a) = \mu(\hbar,a)a - \hbar\alpha(\hbar,\mu(\hbar,a))$ is analytic in a neighbourhood of K by showing the following.

Lemma 4.1: For $a \in K$ there is a neighbourhood 0_a containing $\mu(a)$ and $\gamma_a > 0$ such that $\hbar\alpha(\hbar,\cdot)$ has an analytic extension to 0_a for every $\hbar < \gamma_a$. That is, for all $\hbar < \gamma_a$ there is no phase transition in the neighbourhood 0_a of $\mu(a)$.

Lemma 4.2: There is an open disk V_a containing a to which $\mu(\hbar,\cdot)$ has an analytic extension for every $\hbar < \gamma_a$, with $\mu(\hbar,V_a) \subset 0_a$.

Since K is compact, $\bigcup_{a\in K} V_a$ has an open subcover $\{V_{a_1},\dots,V_{a_N}\}$, and $V(h,\cdot)$ is analytic on $\bigcup_{i=1}^N V_{a_i}$ for all $\hbar < \min_{1\le i\le N} \gamma_{a_i}$. It remains to prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1: Fix $\mu \in G$. Making the a dependence of σ_1 explicit by writing $\sigma_1(\hbar,\mu;a)$ for $\sigma_1(\hbar,\mu)$, it follows from eqn. (3.1.2) that

$$\hbar\alpha(\hbar,\mu) = -U_\mu(\xi(\mu)) + \hbar\sigma_1(\hbar,0;\xi(\mu)) . \tag{4.1}$$

By Lemma 2.3.3 and the above equation, it suffices to show that for fixed $a_0 \notin B$ there is a fixed neighbourhood 0 of a_0 and a $\gamma_0 > 0$ such that $\hbar\sigma_1(\hbar,0;a)$ is analytic in $a \in 0$ for all $\hbar < \gamma_0$.

Let $\sigma_{1,\Lambda}(\hbar,0;a) = \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda \sum_{k=2}^{n} \frac{P^{(k)}(a)}{k!} \hbar^{\frac{k}{2}-1} :\phi^k:} d\mu_C$, so that

$\sigma_1(\hbar,0;a) = \lim_\Lambda \sigma_{1,\Lambda}(\hbar,0;a)$. We will give bounds on $|\sigma_{1,\Lambda}(\hbar,0;a)|$ uniform in a, $\hbar$ and Λ for $\hbar < \gamma_0$ and $|a-a_0| < \varepsilon$, with $\varepsilon,\gamma_0 > 0$. The Lemma will then follow by Vitali's Theorem.

Now $P''(a_0) + m^2 > 0$ since $a_0 \notin B$, so by Lemma 2.2.3

$$\sigma_{1,\Lambda}(\hbar,0;a) = \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda \sum_{k=2}^{n} [\frac{P^{(k)}(a)}{k!} \hbar^{\frac{k}{2}-1} :\phi^k:_C - \frac{P''(a_0)}{2} :\phi^2:_C]} d\mu_{C_0}$$

$$+ \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda \frac{P''(a_0)}{2} :\phi^2:} d\mu_C \,, \tag{4.2}$$

where C_0 is the periodic covariance of mass $m^2 + P''(a_0)$. The second term on the right side of eqn. (4.2) is bounded uniformly in Λ. The first term will be bounded by using Corollary 3.3.3 to give a uniform bound on its derivative with respect to a. To apply Corollary 3.3.3 the Wick order of the interaction must match the measure. Using Lemma 2.2.1 to Wick re-reorder we obtain a new interaction

$:S(\hbar,a;\phi):_{C_0} = \sum_{k=0}^{n} s_k(\hbar,a):\phi^k:_{C_0}$, where $s_k = [\frac{P^{(k)}(a)}{k!} + 0(h)]\hbar^{\frac{k}{2}-1}$, $3 \le k \le n$,

$s_2 = \frac{1}{2}(P''(a) - P''(a_0)) + 0(\hbar)$, $s_1 = 0(\hbar^{\frac{1}{2}})$ and $s_0 = 0(1)$. Now

$\sum_{k=2}^{n} \frac{P^{(k)}(a_0)}{k!} x^k + \frac{1}{2} m^2 x^2 \in T_{\delta,L}$ (for some δ, L) since $a_0 \in B$, so for $|a-a_0|$ and $\hbar$ sufficiently small, $\mathrm{Re} \sum_{k=2}^{n} s_k \hbar^{1-\frac{k}{2}} x^k + \frac{1}{2}(m^2+P''(a_0)x^2 \in T_{\frac{\delta}{2},L+\frac{\delta}{2}}$,

and Corollary 3.3.3 can be applied. □

Note that the convergence of the cluster expansion obtained in the proof of Lemma 4.1 shows that for some $\gamma > 0$ and open disk U containing $\mu(a)$,

$$|D_2^2 \hbar\alpha(\hbar,\mu)| = \left| -\frac{d^2}{d\mu^2} U_\mu(\xi(\mu)) + O(\hbar) \right|$$

$$= \left| \frac{1}{U_0''(\xi(\mu))} + O(\hbar) \right| \geq c > 0 \quad \text{for } \hbar < \gamma,\ \mu \in O \tag{4.3}$$

by eqn. (4.1) and Lemma 2.3.3.

Lemma 4.2 is a consequence of the following generalization of the inverse function theorem [Rudin 74], [Rudin 76].

Lemma 4.3: Suppose $f(\hbar,\cdot)$, $\hbar > 0$ are analytic functions in a neighbourhood O of a point μ_0, and that

$$|D_2 f(\hbar,\mu)| \geq c > 0 \quad \text{for all } \hbar < \gamma,\ \mu \in O \tag{4.4}$$

and $$|D_2^2 f(\hbar,\mu)| \leq M \quad \text{for all } \hbar < \gamma,\ \mu \in O\ . \tag{4.5}$$

Suppose also that there exist $\mu_\hbar$ with $\lim_{\hbar \downarrow 0} \mu_\hbar = \mu_0$, and $f(\hbar,\mu_\hbar) = a_0$ independent of $\hbar$. Then there is a $\gamma > 0$ and an open neighbourhood

V of a_0 such that for all $0 < \hbar < \gamma$ $f(\hbar,\cdot)$ has an analytic inverse function on V, with $f^{-1}(\hbar,V) \subset 0$.

<u>Proof</u>: Denote the open disk of radius r centred at μ by $D(\mu,r)$, and choose $r \leq \frac{c}{2M}$ such that $D(\mu_0 r) \subset 0$. Choose $\gamma > 0$ such that $\mu_\hbar \in D(\mu_0, \frac{r}{2})$ for all $\hbar < \gamma$. Then $B_\hbar = \overline{D(\mu_\hbar , \frac{r}{2})} \subset D(\mu_0,r)$ if $\hbar < \gamma$. Fix $a \in D(a_0, \frac{c^2}{8M})$ and let

$$g(\hbar,\mu) = \mu + [D_2 f(\hbar,\mu_0)]^{-1}(a-f(\hbar,\mu)) .$$

Note that for any given $\hbar$, $g(\hbar,\cdot)$ has a fixed point in $B_\hbar$ if and only if $a = f(\hbar,\mu)$ has a solution $\mu \in B_\hbar$. We will show that for $\hbar$ sufficiently small $g(h,\cdot)$ has a unique fixed point in $B_\hbar$, and therefore $f(\hbar,\cdot)$ has an inverse on $V = D(a_0, \frac{c^2}{8M})$. The inverse function must be analytic in view of (4.4) and the analytic Inverse Function Theorem.

Note that

$$|D_2 g(\hbar,\mu)| = |1 - [D_2 f(\hbar,\mu_0)]^{-1} D_2 f(\hbar,\mu)| \leq \frac{1}{c} |D_2 f(\hbar,\mu) - D_2 f(\hbar,\mu_0)| \leq \frac{1}{c} Mr \leq \frac{1}{2}$$

for $\mu \in D(\mu_0,r)$, by eqn. (4.5) and the fact that $r \leq \frac{c}{2M}$. Therefore

$$|g(\hbar,\mu_1) - g(\hbar,\mu_2)| \leq \frac{1}{2} |\mu_1 - \mu_2| \quad \text{if } \hbar < \gamma , \mu_1, \mu_2 \in B_\hbar .$$

Moreover if $\mu \in B_\hbar$ then

$$\begin{aligned} |g(\hbar,\mu) - \mu_\hbar| &\leq |g(\hbar,\mu) - g(\hbar,\mu_\hbar)| + |g(\hbar,\mu_\hbar) - \mu_\hbar| \\ &\leq \frac{1}{2}|\mu - \mu_\hbar| + |D_2 f(\hbar,\mu_0)|^{-1} |a - f(\hbar,\mu_\hbar)| \\ &\leq \frac{1}{2}\frac{r}{2} + \frac{1}{c}\frac{c^2}{8M} \leq \frac{c}{4M} \quad \text{if } \hbar < \gamma , \end{aligned}$$

so $g(\hbar,\mu) \in B_\hbar$ if $\hbar < \gamma$. But since $g(\hbar,\cdot)$ is a contraction from $B_\hbar$ to itself, it has a unique fixed point, for all $\hbar < \gamma$ [Rudin 76]. □

Proof of Lemma 4.2: Let $f(\hbar,\mu) = \hbar D_2\alpha(\hbar,\mu)$. Then $f(\hbar,\cdot)$ is analytic in 0_a by Lemma 4.1. By eqn. (4.3), $|D_2 f(\hbar,\mu)| \geq c > 0$ for $\hbar < \gamma$, $\mu \in 0_a$. By the uniform convergence of the cluster expansion obtained in the proof of Lemma 4.1 it also follows that

$$|D_2^2 f(\hbar,\mu)| \leq M \quad \text{if} \quad \hbar < \gamma \ , \ \mu \in 0_a \ .$$

By Theorems 2.1 and 2.1.1 $\lim_{\hbar \downarrow 0} \mu(\hbar,a) = \mu(a)$. Since $f(\hbar,\mu(\hbar,a)) = a$, all hypotheses of Lemma 4.1 are satisfied, and the result follows. □

CHAPTER 5: THE LOOP EXPANSION

§1. $-D_1^N V(0,a)$ is a sum of Graphs

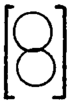
In this section we fix $a \in B$ and prove that for $N \geq 2$, $-D_1^N V(0,a)$ is equal to a finite sum of graphs with lines corresponding to the free covariance of mass $m_1(a)$, where $m_1^2(a) = P''(a) + m^2 = U_0''(a)$. The proof of Theorem 1.4.3(b) will then be completed by identifying the graphs topologically.

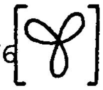
Recall eqn. (3.1.15):

$$V(\hbar,a) = U_0(a) - \hbar\alpha(a) + q_0(\hbar) + \sup_\mu[-\hbar\ (\hbar,\mu)] \ , \quad a \notin B \tag{1.1}$$

where γ, q_0 and σ are given by eqns. (3.1.6), (3.1.11) and (3.1.13) respectively. By eqn. (3.1.11),

$$\frac{1}{k!} D^k q_0(0) = \begin{cases} c_{2k,k}\, a_{2k}\, d^k & k = 2,\ldots,\frac{n}{2} \ , \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

i.e., in the notation of Definition 1.3.1 $\frac{1}{2}D^2 q_0(0) = a_4 \Big[$  $\Big]_d$,

$\frac{1}{3!}D^3 q_0(0) = a_6 \Big[$  $\Big]_d$ and so on, where $d = \frac{-1}{4\pi} \log \frac{m_1^2}{m^2}$.

Let

$$E(\hbar) = \sup_{\mu}[-\hbar\sigma(\hbar,\mu)] = -\hbar\sigma(\hbar,\mu(\hbar)) \ .$$

In section 4.3 it was shown that E is analytic in $(0,\gamma)$ and C^∞ at 0^+ . By Leibnitz' Rule

$$D^N E(0) = \lim_{\hbar \downarrow 0} D^N E(\hbar) = -\lim_{\hbar \downarrow 0}[\hbar \frac{d^N}{dh^N}\sigma(h,\mu(h)) + N\frac{d^{N-1}}{dh^{N-1}}\sigma(\hbar,\mu(h))] \ . \qquad (1.3)$$

We will show that $E(0) = DE(0) = 0$ and that $\left.\frac{d^N}{d\hbar^N}\right|\sigma(\hbar,\sigma(\hbar))$ is a finite sum of graphs with lines of mass m_1 . Recalling the notation $v_N(a) = \frac{1}{N!}D_1^N V(0,a)$ from Theorem 1.4.3, this shows that $v_0(a) = U_0(a)$, $v_1(a) = -\gamma(a)$, and that $-D_1^N V(0,a)$ is a sum of graphs. The topological structure of the graphs contributing to $-v_N(a)$ for $N \geq 2$ will be shown in the remainder of Chapter 5 to be as stated in Theorem 1.4.3(b).

The first step is the following lemma.

Lemma 1.1: For some $\gamma > 0$, $\sigma(h,\mu(h))$ is C^∞ in $\hbar \in [0,\gamma)$, with $\sigma(0,\mu(0)) = 0$.

Proof: As was just mentioned, E is C^∞ in $[0,\gamma)$, and therefore the same is true of $\hbar\sigma(\hbar,\mu(\hbar)) = -E(\hbar)$. We now show that $\lim_{\hbar\downarrow 0} \sigma(\hbar,\mu(\hbar)) = 0$, which will prove the lemma.

By eqn. (3.1.13),

$$\sigma(\hbar,\mu) = \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_{\Lambda}[\sum_{k=2}^{n} q_k(\hbar):\phi^k: - \mu\phi]} d\mu_{\hbar C_1} .$$

Let $Q_\mu(\hbar,x) = \sum_{k=2}^{n} q_k(\hbar)x^k + \frac{1}{2} m_1^2 x^2 - \mu x$. By Lemma 2.3.6, $Q_\mu(\hbar,\cdot)$ has a uniquely attained global minimum, at say $\xi(h,\mu)$ with ξ smooth, provided $\hbar$ and μ are sufficiently small. By Lemma 2.2.2,

$$\sigma(\hbar,\mu) = -\frac{1}{\hbar}Q_\mu(\hbar,\xi(\hbar,\mu)) + \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_{\Lambda}[:T(\phi):-\frac{1}{2}m_1^2:\phi^2:]} d\mu_{\hbar C_1} \tag{1.4}$$

where $T(x) = Q_\mu(\hbar,\xi(\hbar,\mu)+x) - Q_\mu(\hbar,\xi(\hbar,\mu)) \in \mathcal{T}_{\delta,L}$ uniformly in small $\hbar$ and μ . Evaluating eqn. (1.4) at $\mu = \mu(\hbar)$ and using Lemma 2.2.4 gives

$$\sigma(\hbar,\mu(\hbar)) = -\frac{1}{\hbar}Q_{\mu(\hbar)}(\hbar,\xi(\hbar,\mu(\hbar)) + \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int e^{-\int_{\Lambda}[\frac{1}{\hbar}:T(\hbar^{\frac{1}{2}}\phi):-\frac{1}{2}m_1^2:\phi^2:]} d\mu_{C_1} . \tag{1.5}$$

But since $\mu(0) = \xi(0,\mu(0)) = 0$, regularity of μ and ξ imply that $\mu(\hbar) = O(\hbar)$ and $\xi(\hbar,\mu(\hbar)) = O(\hbar)$. It follows by substituting into $Q_{\mu(\hbar)}(\hbar,\xi(\hbar,\mu(\hbar)))$ that $Q_{\mu(\hbar)}(\hbar,\xi(\hbar,\mu(\hbar))) = O(\hbar^2)$, and therefore

$$-\frac{1}{\hbar}Q_{\mu(\hbar)}(\hbar,\xi(\hbar,\mu(\hbar))) \to 0 \quad \text{as} \quad \hbar \downarrow 0 .$$

To show that the second term on the right side of eqn. (1.5) goes to zero as $\hbar \downarrow 0$, we call it $\beta(\hbar) \equiv \lim_{\Lambda} \beta_\Lambda(\hbar)$ and note that the cluster expansion converges for $\beta_\Lambda(\hbar)$ uniformly in small $\hbar$, by Corollary 3.3.3.

In particular, there is a constant M such that

$\left|\frac{\partial}{\partial \hbar^{\frac{1}{2}}} \beta(\hbar)\right| \le M$ for all small $\hbar$; and therefore

$$|\beta_\Lambda(\hbar) - \beta_\Lambda(0)| = |\beta_\Lambda(\hbar) - 0| \le \int_0^{\hbar^{\frac{1}{2}}} |\frac{\partial}{\partial x} \beta_\Lambda(x^2)|dx \le M\hbar^{\frac{1}{2}} . \qquad \Box$$

Corollary 1.2: $E(0) = DE(0) = 0$.

Proof: Since $E(\hbar) = -\hbar\sigma(\hbar,\mu(\hbar))$, $E(0) = 0$ is an immediate consequence of Lemma 1.1. Also, $DE(\hbar) = -\sigma(\hbar,\mu(\hbar)) + \hbar \frac{d}{d\hbar}\sigma(\hbar,\mu(\hbar))$ goes to zero as $\hbar \downarrow 0$ by Lemma 1.1. $\Box$

Before stating the main result of this section, we introduce some notation. Let

$$q_{kj} = \frac{1}{j!} D^j q_k(0) \qquad (j = 0,.,\ldots,\tfrac{n}{2}) \tag{1.6}$$

so that $q_k(\hbar) = \sum_{j=0}^{\frac{n}{2}} q_{kj}\hbar^j$.

Recall from Theorem 1.4.3 the notation $v_N(a) = D_1^N V(0,a)/N!$.

Theorem 1.3: For $a \notin B$, $v_0(a) = U_0(a)$ and $v_1(a) = -\gamma(a)$. For $N \ge 2$, $-D_1^N V(0,a)$ is equal to $-D^N q_0(0)$ plus $- D^N E(0)$. The derivative $-D^N E(0)$ is given by a linear combination of graphs with no self lines, with positive or negative coefficients, made up of lines of mass m_1 and vertices $-q_{kj}$, $k = 2, 3,\ldots,n$; $j = 0, 1,\ldots,\frac{n}{2}$. The graph corresponding to $-D^N q_0(0)$ is given under eqn. (1.2).

Proof: By eqns. (1.1) and (1.3) and Corollary 1.2 the only thing to check is that for $N \geq 2$, $-D^N E(0) = N \left.\frac{d^{N-1}}{d\hbar^{N-1}}\right|_0 \sigma(\hbar,\mu(\hbar))$ is a sum of graphs with vertices $-q_{kj}$, $k \neq 0$.

By definition of σ and Lemma 2.2.4, $\sigma(\hbar,\mu) = \lim_{\Lambda} \sigma_\Lambda(\hbar,\mu)$, where

$$\sigma_\Lambda(\hbar,\mu) = \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda \sum_{k=2}^{n} [q_k(\hbar)\hbar^{\frac{k}{2}-1} :\phi^k: - \mu\hbar^{-\frac{1}{2}}\phi]} d\mu_{C_1} .$$

Let $f(\hbar^{\frac{1}{2}}) = \mu(\hbar)\hbar^{-\frac{1}{2}}$. By eqn. (4.3.4) there is a function g , C^∞ on $[0, \gamma)$, such that $f(\hbar^{\frac{1}{2}}) = \hbar^{\frac{1}{2}} g(\hbar)$. (1.7)

Also $\sigma_\Lambda(\hbar,\mu(\hbar)) = \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda \sum_{k=2}^{n} [q_k(\hbar)\hbar^{\frac{k}{2}-1} :\phi^k: - f(\hbar^{\frac{1}{2}})\phi]} d\mu_{C_1}$.

For $x \in R$, let $\zeta(t,x) = \lim_{\Lambda} \zeta_\Lambda(t,x)$, where

$$\zeta_\Lambda(t,x) = \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda \sum_{k=2}^{n} [q_k(t^2)t^{k-2} :\phi^k: - x\phi]} d\mu_{C_1} .$$

Then $\zeta(t,f(t)) = \sigma(t^2,\mu(t^2))$.

We see by substituting t^2 for $\hbar$ in the asymptotic expansion for $\sigma(\hbar,\mu(\hbar))$ that $\left.\frac{d^n}{d\hbar^n}\right|_0 \sigma(\hbar,\mu(\hbar)) = \frac{n!}{(2n)!} \left.\frac{d^{2n}}{dt^{2n}}\right|_0 \zeta(t,f(t))$, so it suffices to show that $\left.\frac{d^{2n}}{dt^{2n}}\right|_0 \zeta(t,f(t))$ is given by a linear combination of appropriate graphs. To show this, we begin by introducing some notation. Let

$S(t,\phi,\Lambda) = \sum_{k=2}^{n} q_k(t^2)t^{k-2}:\phi^k(\Lambda): - f(t)\phi(\Lambda)$. Then by eqn. (1.3.3),

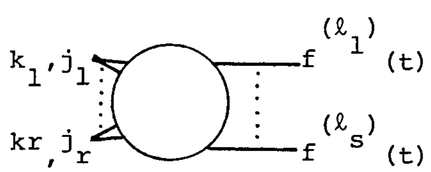
$$\frac{d^k}{dt^k}\zeta_\Lambda(t,f(t)) = \sum_{\pi\in P_k} c_\pi \frac{1}{|\Lambda|}\left\langle -D_1^{|\pi_1|}S(t,\phi,\Lambda);\ldots;-D_1^{|\pi_{|\pi|}|}S(t,\phi,\Lambda)\right\rangle_{t,\Lambda} \tag{1.8}$$

where $\langle\cdot\rangle_{t,\Lambda}$ is the expectation with respect to the interaction $S(t,\phi,\Lambda)$ and the notation P_k, π, c_π is as in eqn. (1.3.3). The expectations on the right side of eqn. (1.8) are finite sums of positive integers times positive powers of t times expressions of the form (recall eqn. (1.6))

$$\frac{1}{|\Lambda|}\left\langle -q_{k_1j_1}:\phi^{k_1}(\Lambda):;\ldots;-q_{k_rj_r}:\phi^{k_r}(\Lambda):;f^{(\ell_1)}(t)\phi(\Lambda);\ldots;f^{(\ell_s)}(t)\phi(\Lambda)\right\rangle_{t,\Lambda}, \tag{1.9}$$

with $k_i \in \{2,\ldots,n\}$, $j_i \in \{0,1,\ldots,\frac{n}{2}\}$, $r \geq 0$, $s \geq 0$, $\ell_i \geq 1$.

We denote the infinite volume limit of the expression (1.9) graphically by



(1.10)

We now show that the vertex factors $f^{(\ell_i)}(0)$ are actually graphs which hood onto the corresponding legs. To simplify the notation we use  to denote a linear combination of terms of the form (1.10) with vertex factors 1 instead of $f^{(\ell_i)}(t)$; which linear combination will be apparent from the context. The coefficients of the linear combination will include combinatorial factors and powers of t .

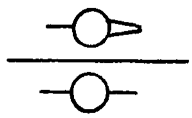
Derivatives of f are calculated as follows. Since $D_2\sigma(\hbar,\mu(\hbar)) = 0$ by definition of $\mu(\hbar)$, it follows from the fact that $D_2\zeta(t,f(t)) = tD_2\sigma(t^2,\mu(t^2))$ that

$$D_2\zeta(t,f(t)) = 0 \ . \tag{1.11}$$

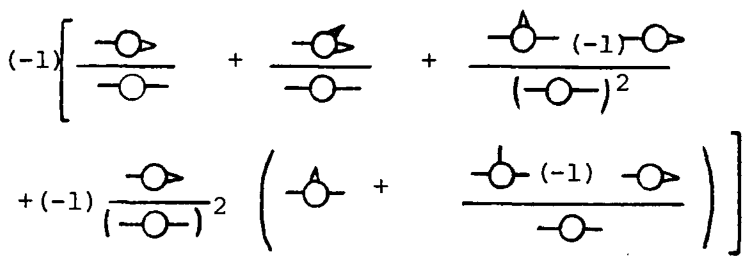
By eqn. (1.7), f is C^∞ in t. Differentiating eqn. (1.11) with respect to t gives

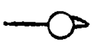
$$Df(t) = \frac{-D_1D_2\zeta(t,f(t))}{D_2^2\zeta(t,f(t))} \ . \tag{1.12}$$

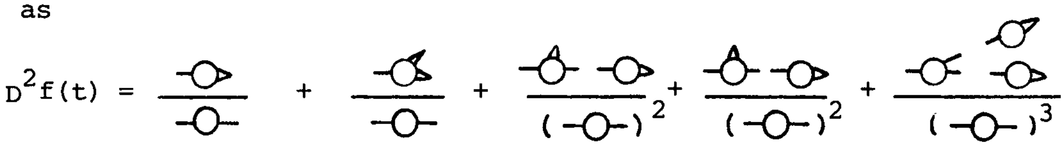
Using the graph notation described in the last paragraph, eqn. (1.12) can be written

$Df(t) = \quad (-1)$  (1.13)

As explained below, differentiation of eqn. (1.13) gives

$D^2f(t) =$  (1.14)

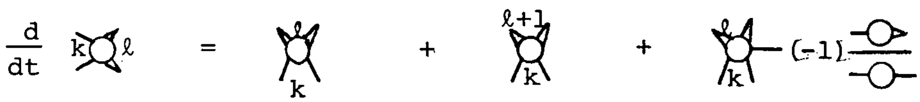
The terms on the right side of eqn. (1.14) arise as follows. The first three terms come from differentiating the numerator  of eqn. (1.13): the first term comes from differentiating t's appearing as coefficients of  ; the second term from differentiating the $\sum_{k=2}^{n} q_k(t^2)t^{k-2}:\phi^k:$ part of the interaction; the third term from differentiating the $f(t)\phi$ part of the interaction and using eqn. (1.13). The last term on the right side of eqn. (1.14) comes from differentiating the factor $\frac{1}{\ }$. Since there is no t dependent coefficient as a factor in , there are only two terms in the derivative of . Dropping minus signs we can rewrite eqn. (1.14) as



(1.15)

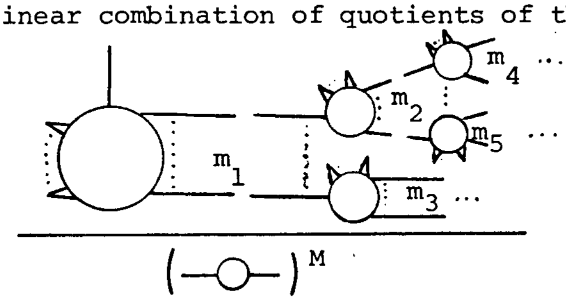
In the last three numerators of (1.15) note how all but one of the single legged vertices can be matched in pairs, and that the power of in the denominator exceeds the number of matched pairs by one.

We will now show how eqn. (1.15) generalizes to higher order derivatives. By the same reasoning used to differentiate above,



(1.16)

Using the formula (1.16) it follows from eqn. (1.13) and induction that $D^k f(t)$ is a linear combination of quotients of the form

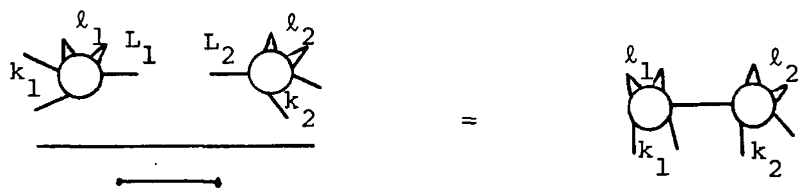


(1.17)

where the diagram eventually terminates; $M - 1$ is the total number of matched pairs of legs, i.e., $M = m_1+m_2+m_3+\ldots+1$; and there is only one unmatched leg. To see this, suppose $D^{k-1}f(t)$ is of the form (1.17) and note that differentiation of any factor of the numerator (using (1.16)) produces a sum of terms of the form (1.17). Also, using the quotient rule to differentiate the denominator gives terms of the form (1.17) by matching one leg of $\frac{d}{dt}$ to the unmatched leg of the numerator. (There will still be one unmatched leg left over).

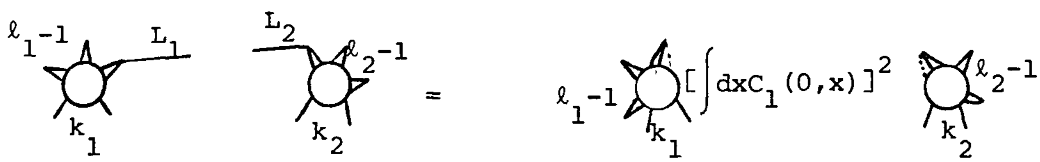
In the limit $t \to 0$ the measure in (1.9) becomes $d\mu_{C_1}$ by Lemma 3.4.1. Hence by Wick's Theorem $D^k f(0)$ is a linear combination of products of connected graphs without self-lines, with vertices and lines as in the statement of the Theorem as well as one-legged vertices which match up in $M - 1$ pairs as depicted in (1.17), divided by $(\bullet\!\!-\!\!\!-\!\!\bullet)^M$. Thus there is one power of $\bullet\!\!-\!\!\!-\!\!\bullet$ for each matched pair of legs, with one power left over. The unmatched leg in (1.17) should be thought of as being matched to the corresponding leg of (1.10), and the extra power of $\bullet\!\!-\!\!\!-\!\!\bullet$ in the denominator as corresponding to these legs. As we will now show, at $t = 0$ each factor of $\bullet\!\!-\!\!\!-\!\!\bullet$ in the denominator serves to link together one matched pair of legs to create a connected graph.

We will now show that at $t = 0$



(1.18)

where each circle denotes a connected graph with no vertices other than those explicitly drawn. In fact, each of the lines L_1 and L_2 must be connected to a multi-legged vertex; choose these to be the vertices fixed at zero when evaluating the graphs. Then the numerator can be written



(1.19)

where the dashed lines indicate the absence of L_1 and L_2. One of the factors $\int dx C_1(0,x)$ on the right side of eqn. (1.19) cancels the denominator on the left side of eqn. (1.18). The remaining factor serves to link up the two graphs on the right side of eqn. (1.19). To see this, take one of the graphs under the integral $\int dx C_1(0,x)$ and use translation invariance to fix the fixed vertex of that graph at x instead of at the origin. Since the remaining graph has one vertex fixed at zero, $C_1(0,x)$ links the two graphs together. This proves eqn. (1.18).

Theorem 1.3 now follows by repeated application of eqn. (1.18) to see that at $t = 0$ the $M - 1$ matched pairs of legs in (1.17) can be joined by cancelling $M - 1$ factors of •——• , in the denominator, and that the single unmatched leg of (1.17) can be joined to the appropriate unmatched leg of (1.10) by cancelling the remaining factor of •——• in the denominator, resulting in a connected graph. □

§2. The Test for Irreducibility

Irreducibility properties of a graph depend only on the topological structure of the graph and not on the rules for evaluating the graph. In this section we define the notion of a topological graph and show how a function can be assigned to a topological graph in such a way as to provide a test for whether or not the graph is 1-PI .

Definition 2.1: A topological graph is a collection of finitely many vertices, each having a finite number of legs (half-lines joined at one end to the vertex), such that every leg of every vertex is paired with some other leg to form a line.

Examples:  .

As in §1.3, a topological graph is said to be connected if its vertices are path connected by its lines. A topological graph is 1-PI if the removal of any one line from the graph leaves a connected graph.

There are many ways to assign a function to a topological graph in such a way as to be able to use the function to test the graph for irreducibility properties. The choice we make is guided by our strategy for identifying the topological structure of the graphs contributing to $-D_1^N V(0,0)$. That strategy is to introduce a lattice theory and an "effective potential" for the lattice theory which generates exactly the same topological graphs as the continuum effective potential, but with different rules of evaluation. These rules of evaluation make the irreducibility test straightforward.

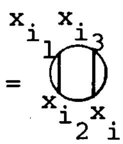
We now explain the method of [Sp 75] for testing a graph for one-particle irreducibility in the context we need. For a fixed positive integer m , we consider the lattice L_{2m} of $2m$ points $\{x_1,\ldots,x_{2m}\}$, thought of as consisting of the two sublattices $\{x_1,\ldots,x_m\}$ and $\{x_{m+1},\ldots,x_{2m}\}$. Write $m_1^2 = U_0''(a)$ as usual and let

$$C(\lambda) = m_1^{-4} \begin{bmatrix} R_1 & \lambda R \\ \lambda R & R_2 \end{bmatrix} , \lambda \in [0, 1] , \tag{2.1}$$

where $(R_1)_{ij} = \begin{cases} m_1^2 , & i = j \\ r_{ij} , & i \neq j \end{cases}$, $(R_2)_{ij} = \begin{cases} m_1^2 , & i = j \\ r_{m+i,m+j} & i \neq j \end{cases}$, and

$R_{ij} = r$ for all i, j . The matrices R, R_1, and R_2 are all $m \times m$, the r_{ij} are strictly positive with $r_{ij} \leq r$, $r_{ij} = r_{ji}$ for all i and j , and $r > 0$ is chosen sufficiently small that $C(\lambda)$ is positive definite for all $\lambda \in [0, 1]$ and all $r_{ij} \in (0,r)$. (It is possible to so choose r since for $r = 0$, $C(\lambda) = m_1^{-2} I$. See Lemma 4.1). The fact that $C(\lambda)$ is positive definite is not relevant for the irreducibility test; positive definiteness is required because $C(\lambda)$ will be the covariance of a Gaussian measure on R^{2m} in the next section. The variable λ measures the coupling between the sets $\{x_1, x_2, \ldots, x_m\}$ and $\{x_{m+1}, x_{m+2}, \ldots, x_{2m}\}$. Observe that $DC(0) = m_1^{-4} \begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix}$ has nonnegative entries.

<u>Definition 2.1</u>: Let L_{2m} (the lattice of $2m$ points) consist of the $2m$ points labeled $\{x_1, \ldots, x_{2m}\}$. A topological graph G is <u>imposed on L_{2m}</u> by assigning each vertex of G to a different point in L_{2m}. Such an assignment is called an <u>imposition of G on L_{2m}</u>. An <u>admissible imposition (AI)</u> is an imposition for which at least one vertex is assigned to each of the sublattices $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_{2m}\}$. □

Now consider a graph with $2m$ vertices or less that has been imposed on L_{2m}. For example, $G =$ 

, where the i_j are different elements of $\{1, \ldots, 2m\}$. The rule for evaluating such a graph is to form the product with one factor of $C(\lambda)_{i_j i_k}$ for each line joining x_{i_j} to x_{i_j}. The graph G depicted above has the value $G(\lambda) = C(\lambda)^2_{i_1 i_2} C(\lambda)_{i_1 i_3} C(\lambda)_{i_2 i_4} C(\lambda)^2_{i_3 i_4}$.

The test for irreducibility is the following [Sp 75].

<u>Lemma 2.2</u>: A topological graph G with V vertices is 1-PI if and only if $DG(0) = G(0) = 0$ for every AI of G on L_{2m}, for some $m \geq V$.

<u>Proof</u>: Note that $C(0)_{ij} \geq 0$ for all i and j with $C(0)_{ij} = 0$ if and only if $i \in \{1, 2, \ldots, m\}$ and $j \in \{m+1, \ldots, 2m\}$ or vice-versa. Also, $DC(0)_{ij} \geq 0$ for all i and j.

We first consider connectedness. Since $G(\lambda)$ is a product of $C(\lambda)_{i_j i_k}$, $G(0) = 0$ for every AI if and only if at least one $C(0)_{i_j i_k} = 0$

for every AI . This happens if and only if at least one line joins $\{1,\ldots,m\}$ to $\{m+1,\ldots,2m\}$ for every AI , i.e., if and only if G is connected.

Now we consider one-particle irreducibility. Note that $DG(0)$ is a sum of products, each of which consists of one $DC(0)_{i_j i_k}$ multiplied by the remaining $C(0)_{i_\ell i_m}$. Each such product is greater than or equal to zero, so $DG(0) = 0$ for every AI if and only if each such product is zero for every AI . This happens if and only if at least two factors $C(\lambda)_{ij}$ occur in $G(\lambda)$ with $i \in \{1,\ldots,m\}$ and $j \in \{m+1,\ldots,2m\}$ or vice-versa for every AI , i.e. if and only if G is 1-PI . $\square$

Note that the above proof goes through if we take $m = 1$ and do not require that different vertices be assigned to different lattice points. The requirement that different vertices be assigned to different lattice points will be needed in Theorem 5.6.

§3. The Lattice Theory

In this section we introduce a lattice analogue to the effective potential $E(h)$ which generates exactly the same topological graphs as E but which assigns values to the graphs in such a way that the irreducibility test of §5.2 can be applied. In the lattice theory we include space-time dependent coupling constants g_{kji} , which will be used to reduce the analysis of graphs with vertices summed over the lattice to the analysis of graphs with fixed vertices. Because of these space-time dependent coupling constants, it is necessary (although it is not obvious at first glance) to make the external field space-time dependent to preserve the irreducibility of the effective potential. It is because of the space-time dependent coupling constants and external field that it is more convenient to work on a lattice theory than a continuum theory. (See the remark after Theorem 4.3).

The lattice interaction in an external field $\mu \in R^m$ is given by

$$I_\mu(\hbar,g,x) = \sum_{i=1}^{2m} [\sum_{k=2}^{n} \sum_{j=0}^{\frac{n}{2}} q_{kj}\hbar^j g_{kji} x_i^k - \mu_i x_i] \tag{3.1}$$

where $\mu=(\mu_1,\ldots,\mu_{2m})\in R^{2m}$, $x = (x_1,\ldots,x_{2m})\in R^{2m}$, and the q_{kj} are defined in eqn. (1.6). The variable g_{kji} serves to label the quantity $\hbar^j x_i^k$ in I_μ . The vector g has components g_{kji} $(k=2,\ldots,n;j=0,\ldots,\frac{n}{2};i=1,\ldots,2m)$ and is restricted to lie in the subset $\overline{C}_\varepsilon \subset R^{N_m}$, $N_m = 2m(\frac{n}{2}+1)(n-1)$, defined as follows. The positive constant ε will be fixed below.

<u>Definition 3.1</u>: For $\varepsilon > 0$, $C_\varepsilon \subset R^{N_m}$ is the open cone with vertex at the origin, axis the line segment $\{(t,t,t,\ldots,t) \in R^{N_m} : 0 < t < 1\}$, and radius ε at its wide end. □

By taking ε small and any coordinate g_{kji} near 1 , we can ensure that the coefficients of $\sum_{k=2}^{n} \sum_{j=0}^{\frac{n}{2}} q_{kj}\hbar^j g_{kji} x_i^k$ are close to those of the polynomial $\sum_{k=2}^{n} q_k(\hbar)x_i^k$, for all $g \in \overline{C}_\varepsilon$.

We now prove a useful fact about C_ε. Let $\underline{P} : \overline{C}_\varepsilon \to [0, 1]$ denote the mapping which takes a vector in $\overline{C}_\varepsilon$ to the first component of its orthogonal projection on the axis of $\overline{C}_\varepsilon$.

<u>Lemma 3.2</u>: For any $g \in \overline{C}_\varepsilon$ and any component g_{kji} of g ,

$$|g_{kji} - \underline{P}g| \leq \varepsilon\, \underline{P}g$$

<u>Proof</u>: Let $\underline{P}_1 g$ denote the projection of $g \in \overline{C}_\varepsilon$ on the axis of $\overline{C}_\varepsilon$.

By the triangle inequality $|g_{kji} - \underset{\sim}{P}g| \le |g - \underset{\sim}{P}_1 g|$. But by the cone geometry, $|g - \underset{\sim}{P}_1 g| \le \varepsilon \dfrac{|\underset{\sim}{P}_1 g|}{\sqrt{N_m}} = \varepsilon \underset{\sim}{P}g$. $\square$

The import of this lemma is that by choosing ε small, we can make the coefficients of $\sum_{k=2}^{n} \sum_{j=0}^{\frac{n}{2}} q_{kj} \hbar^j g_{kji} x_i^k$ near to those of $\underset{\sim}{P}g \sum_{k=2}^{n} q_k(\hbar) x_i^k$.

The analogue of the pressure in the lattice theory is given by

$$T_{2m}(\hbar,g,\mu,\lambda) = \ln \int e^{-\frac{1}{\hbar}:I_\mu(\hbar,g,x):} d\gamma_{\hbar C(\lambda)} \ , \ (\hbar,g,\mu,\lambda) \in (0,\infty) \times \overline{C}_\varepsilon \times R^m \times [0,1], \tag{3.2}$$

where $d\gamma_D$ is Gaussian measure on R^{2m} with covariance D , i.e.,

$$d\gamma_D = \frac{e^{-\frac{1}{2}xD^{-1}x} dx}{\int e^{-\frac{1}{2}xD^{-1}x} dx}$$

and the Wick dots are with respect to the covariance $\hbar C(\lambda)$, i.e.,

$$:x_i^k: = \sum_{j=0}^{[\frac{k}{2}]} c_{kj} (-1)^j [\hbar C(\lambda)_{ii}]^j x_i^{k-2j} \ .$$

Because T_{2m} has not been normalized by dividing by the volume it generates lattice graphs that have all vertices summed over rather than having one vertex fixed as with the continuum pressure.

The lattice analogue of $E(\hbar)$ is the Legendre transform Γ_{2m} (evaluated with the classical field equal to zero) given by

$$\Gamma_{2m}(\hbar,g,\lambda) = \sup_{\mu \in R^{2m}} [-\hbar T_{2m}(\hbar,\mu,g,\lambda)] \ , \ (\hbar,g,\lambda) \in (0,\infty) \times C_\varepsilon \times [0,1]. \tag{3.3}$$

The following lemma will be used in the proof that Γ_{2m} is finite.

Lemma 3.3: Let $d\nu = g(x)dx$ be a finite positive measure on R', with $g > 0$ and $e^{\pm jx} \in L'(d\nu)$. Let $d\nu_j = \dfrac{e^{jx}d\nu}{\int e^{jx}d\nu}$. Then $\lim_{j\to\pm\infty} \int x d\nu_j = \pm\infty$.

Proof: It suffices to prove that $\lim_{j\to\infty} \int x d\nu_j = +\infty$, since

$\int x d\nu_{-j} = -\int x d\nu_j^-$ where $d\nu_j^- = \dfrac{e^{jx}g(-x)dx}{\int e^{jx}g(-x)dx}$, and $d\nu^- = g(-x)dx$ satisfies the hypotheses of the Lemma.

To prove the $j \to +\infty$ case, we begin by showing that given any $a < 1$ and $y > 0$ there is a $J(y)$ such that

$$\int_y^\infty d\nu_j \geq a \quad \text{for every} \quad j \geq J(y) .$$

In fact, let $\varepsilon > 0$ and choose $x_0 < y$ such that $\int_{x_0}^{y} d\nu \leq \varepsilon$. Choose J_0 such that $e^{j(x_0-y)} < \varepsilon$ for $j \geq J_0$. Then

$$\int_{-\infty}^\infty e^{jx}d\nu = \int_{-\infty}^{x_0} e^{jx}d\nu + \int_{x_0}^{y} e^{jx}d\nu + \int_y^\infty e^{jx}d\nu \leq e^{jx_0}\int_{-\infty}^{x_0} d\nu + e^{jy}\varepsilon + \int_y^\infty e^{jx}d\nu$$

$$= e^{jy}[\varepsilon \int_{-\infty}^{x_0} d\nu + \varepsilon + e^{-jy}\int_y^\infty e^{jx}d\nu]$$

so $\int_y^\infty d\nu_j \geq [e^{-jy}\int_y^\infty e^{jx}d\nu][\varepsilon \int_{-\infty}^{x_0} d\nu + \varepsilon + e^{-jy}\int_y^\infty e^{jx}d\nu]^{-1} \geq a$

for ε sufficiently small.

But for $y > 0$ and $j \geq J(y)$,

$$\int_{-\infty}^{\infty} x d\nu_j = \int_{-\infty}^{-y} x d\nu_j + \int_{-y}^{y} x d\nu_j + \int_{y}^{\infty} x d\nu_j \geq \int_{-\infty}^{-y} x d\nu_j - y(1-a) + ya$$

$$= \int_{-\infty}^{-y} x d\nu_j + (2a-1)y \ . \tag{3.4}$$

And if $y > 0$ and $j > 0$ then

$$\left|\int_{-\infty}^{-y} x d\nu_j\right| = \int_{y}^{\infty} x d\nu_{-j}^{-} \leq \int_{0}^{\infty} x d\nu_{-j}^{-} = \frac{\int_{0}^{\infty} x\, e^{-jx} g(-x)\, dx}{\int_{-\infty}^{\infty} e^{-jx} g(-x)\, dx}$$

$$\leq \frac{\int_{0}^{\infty} x\, g(-x)\, dx}{\int_{-\infty}^{0} g(-x)\, dx} \equiv c \tag{3.5}$$

By eqns. (3.4) and (3.5), $\int_{-\infty}^{\infty} x d\nu_j \geq -c + (2a-1)y$ if $j \geq J(y)$. The Lemma then follows by taking $a = \frac{3}{4}$, since y can be taken to be arbitrarily large. $\square$

Theorem 3.4: The lattice Legendre transform $\Gamma_{2m}(\hbar,g,\lambda)$ is finite for $(\hbar,g,\lambda) \in (0,\infty) \times C_\varepsilon \times [0, 1]$, and the supremum in its definition is attained at a unique point $\mu(\hbar,g,\lambda)$.

Proof: The variables $\hbar,g,\lambda,m$ play no role in the proof so we drop them from the notation and simply write

$$\Gamma = \sup_{\mu \in R^{2m}} [-T(\mu)] = -\inf_{\mu \in R^{2m}} T(\mu) \quad . \tag{3.4}$$

Now $T(\mu) = \ln \int e^{-:I_0(x):} e^{\mu x} \, d\gamma_C(x)$, so for $\theta \in [0, 1]$ and $\mu \neq \nu$,

$$T(\theta\mu + (1-\theta)\nu) = \ln \int e^{\theta(-:I_0(x):+\mu x)} e^{(1-\theta)(-:I_0(x):+\nu x)} d\gamma_C(x)$$

$$\leq \ln \left[\left(\int e^{-:I_0(x):+\mu x} d\gamma_C\right)^{\theta} \left(\int e^{-:I_0(x):+\nu x} d\gamma_C\right)^{1-\theta} \right]$$

$$= \theta T(\mu) + (1-\theta)T(\nu) \quad , \tag{3.5}$$

by Hölder's inequality. By the conditions for strict inequality in Hölder's inequality given in [Rudin 74, p. 66], Hölder's inequality is strict here provided there are no constants α, β (not both zero) such that $\alpha \, e^{-:I_0(x):+\mu x} = \beta \, e^{-:I_0(x):+\nu x}$ a.e.$(d\gamma)$. Since there are no such α, β if $\mu \neq \nu$ the inequality in (3.5) is strict:

$$T(\theta\mu + (1-\theta)\nu) < \theta T(\mu) + (1-\theta)T(\nu) \quad .$$

That is, T is strictly convex. It follows that if T is bounded below then the supremum in eqn. (3.4) is finite and is attained at a unique point.

By a standard theorem [Rock 70, Thm. 27.2], T is bounded below if $\lim_{t\to\infty} \frac{\partial}{\partial t} T(t\mu) > 0$ for every $\mu \neq 0$. We use Lemma 3.3 to show more, that in fact $\lim_{t\to\infty} \frac{\partial}{\partial t} T(t\mu) = +\infty$. By definition of T ,

$$\frac{\partial}{\partial t} T(t\mu) = \frac{\int \mu x \, e^{-:I_0(x):+t\mu x} \, d\gamma_C}{\int e^{-:I_0(x):+t\mu x} \, d\gamma_C}$$

Expand the Wick dots, write $d\gamma_C = \text{const } e^{-\frac{1}{2}xC^{-1}x} dx$, and choose an i for which $\mu_i \neq 0$. Let $z = \mu \cdot x$ and $y = (x_1, \ldots, \hat{x}_i, \ldots, x_{2m}) \in R^{2m-1}$ Then for some polynomial P in $2m$ variables,

$$\frac{\partial}{\partial t} T(t\mu) = \frac{\int z \, e^{tz} \left(\int e^{-P(z,y)} dy \right) dz}{\int e^{tz} \left(\int e^{-P(z,y)} dy \right) dz} ,$$

which goes to $+\infty$ as $t \to \infty$ by Lemma 3.3. □

§4. Regularity of the Lattice Legendre Transform

In this section it is shown that Γ_{2m} is C^∞ as a function of $(\hbar, g, \lambda) \in [0, \gamma) \times \overline{C}_\varepsilon \times [0, 1]$. Because the lattice interaction involves the basic polynomial $\sum_{k=2}^{n} q_k x^k + \frac{1}{2} m_1^2 x^2$, the lattice and continuum theories have similar structures. The proofs in this section are based on the same ideas as the proofs of smoothness of the continuum pressure and effective potential, and can be omitted in a first reading.

We begin with a lemma about $C(\lambda)^{-1}$. Recall from the definition of $C(\lambda)$ (eqn. (2.1)) that the nondiagonal elements of $C(\lambda)$ are required to be less than or equal to r .

Lemma 4.1: For any $\varepsilon > 0$ there is an $r_0 > 0$ such that $C(\lambda)^{-1}$ exists for all $r < r_0$, and $xC(\lambda)^{-1}x \geq (m_1^2 - \varepsilon)|x|^2$ for all $x \in R^{2m}$, $r < r_0$, $\lambda \in [0, 1]$.

Proof: For $r = 0$, $C(\lambda) = m_1^{-2}I$, so $C(\lambda)^{-1} = m_1^2 I$. By choosing r sufficiently small the spectrum of $C(\lambda)$ can be confined to a neighbourhood of m_1^{-2} small enough to guarantee that the spectrum of $C(\lambda)^{-1}$ is within ε of m_1^2 . □

The following Lemma provides a multi-dimensional analogue to Lemma 2.3.6, and is the key to proving smoothness of T_{2m} . After T_{2m} has been shown to be smooth Lemma 2.1.2 will be applied to obtain smoothness of Γ_{2m} .

Lemma 4.2: There exist $r_0, \varepsilon, \rho, \gamma > 0$ such that the polynomial in $x \in R^{2m}$

$$J_\mu(\hbar,g,\lambda;x) = :I_0(\hbar,g,x):_{\hbar C(\lambda)} - \mu x + \tfrac{1}{2}xC(\lambda)^{-1}x \qquad (4.1)$$

(after undoing the Wick ordering) has a uniquely attained global minimum, at say $\xi(\hbar,g,\mu,\lambda)$, for all $(\hbar,g,\mu,\lambda,r) \in [0, \gamma) \times \overline{C}_\varepsilon \times D_\rho \times [0, 1] \times [0, r_0]$, where $D_\rho = \{x \in R^{2m}: |x| < \rho\}$. Also, $\xi \in C^\infty([0, \gamma) \times \overline{C}_\varepsilon \times D_\rho \times [0, 1])$. Moreover, there exists a $c > 0$ such that

$$K(\hbar,g,\mu,\lambda;x) = J_\mu(\hbar,g,\lambda;x+\xi(\hbar,g,\mu,\lambda)) - J_\mu(\hbar,g,\lambda;\xi(\hbar,g,\mu,\lambda)) \geq c|x|^2 \qquad (4.2)$$

for every $(\hbar,g,\mu,\lambda,x) \in [0, \gamma) \times \overline{C}_\varepsilon \times D_\rho \times [0, 1] \times R^{2m}$.

Proof: Recall from eqn. (4.3.2) that for $a \notin B$ there is a $\delta > 0$ such that

$$\sum_{k=2}^{n} q_{k0}y^k + \tfrac{1}{2}m_1^2y^2 \geq \delta \sum_{k=2}^{n} |y|^k \text{ for every } y \in R^1 . \qquad (4.3)$$

By Lemma 4.1 $r_0 > 0$ can be chosen sufficiently small that

$$\tfrac{1}{2}xC(\lambda)^{-1}x \geq (\tfrac{1}{2}m_1^2 - \tfrac{\delta}{4})|x|^2 \text{ for all } (\lambda,r,x) \in [0, 1] \times [0, r_0] \times R^{2m} . \qquad (4.4)$$

By Lemma 3.2 it is possible to choose γ and ε small enough that the coefficient of x_i^k in $:I_0(\hbar,g,x):_{\hbar C(\lambda)}$ (after undoing the Wick ordering) is within $\frac{\delta}{4}\underline{P}g$ of $(\underline{P}g)q_{k0}$ for all $(\hbar,g,\lambda) \in [0,\gamma) \times \overline{C}_\varepsilon \times [0,1]$ and $i \in \{1,2,\ldots,2m\}$, $k \in \{2,\ldots,n\}$. This can be done because the coefficient is a sum of terms each of which is linear in one component of g and which are at least $O(\hbar)$ for all terms except the $g_{k0i}q_{k0}x_i^k$ term. Using this, together with eqn. (4.4), we have

$$J_0(\hbar,g,\lambda;x) \geq \sum_{i=1}^{2m} \sum_{k=2}^{n} [(\underline{P}g)q_{k0}x_i^k - (\underline{P}g)\tfrac{\delta}{4}|x_i|^k] + (\tfrac{1}{2}m_1^2 - \tfrac{\delta}{4})|x|^2$$

$$+ C_1(\hbar,g,\lambda)x + C_0(\hbar,g,\lambda) .$$

where C_1 and C_0 come from the Wick ordering, are $O(\hbar)$, and are a sum of monomials in components of g. Therefore, using eqn. (4.3),

$$J_0(\hbar,g,\lambda;x) \geq (\underline{P}g) \sum_{i=1}^{2m} \Big[(\sum_{k=2}^{n} q_{k0}x_i^k + \tfrac{1}{2}m_1^2x_i^2) - \tfrac{\delta}{4}|x_i|^k - \tfrac{\delta}{4}|x_i|^2 \Big]$$

$$+ (1-\underline{P}g)(\tfrac{1}{2}m_1^2 - \tfrac{\delta}{4})|x|^2 + C_1x + C_0$$

$$\geq (\underline{P}g)\tfrac{\delta}{2}|x|^2 + (1-\underline{P}g)(\tfrac{1}{2}m_1^2 - \tfrac{\delta}{4})|x|^2 + C_1x + C_0$$

But $(\underline{P}g)\frac{\delta}{2} + (1-\underline{P}g)(\frac{1}{2}m_1^2 - \frac{\delta}{4}) = \frac{1}{2}m_1^2 - \frac{\delta}{4} + \underline{P}g(\frac{3\delta}{4} - \frac{1}{2}m_1^2)$

$$\geq \tfrac{1}{2}m_1^2 - \tfrac{\delta}{4} + \tfrac{3\delta}{4} - \tfrac{1}{2}m_1^2 = \tfrac{\delta}{2} ,$$

since $\frac{1}{2}m_1^2 \geq \delta$ by eqn. (4.3) (and the fact that $q_{20} = 0$). Therefore

$$J_0(\hbar,g,\lambda;x) \geq \tfrac{\delta}{4}|x|^2 + C_1x + C_0, \text{ for } (\hbar,g,\lambda,x) \in [0,\gamma) \times \overline{C}_\varepsilon \times [0,1] \times R^{2m} \tag{4.5}$$

and hence

$$J_\mu(\hbar,g,\lambda;x) - C_0 \geq \frac{\delta}{4}|x|^2 + (C_1-\mu)x \geq \frac{\delta}{4}|x|^2 - (|\mu| + |C_1|)|x| \quad .$$

It follows that

$$J\ (\hbar,g,\lambda;x) - C_0 \geq 0 \quad \text{if} \quad |x| \geq 4\delta^{-1}(|\mu| + |C_1|) \ . \tag{4.6}$$

Since $J_\mu(\hbar,g,\lambda;0) - C_0 = 0$ we have $\min\limits_x J_\mu(\hbar,g,\lambda;x) - C_0 \leq 0$ for every $\mu \in R^{2m}$ and it suffices to show that $\frac{\partial}{\partial x_i} J_0(\hbar,g,\lambda;x) = \mu_i$ $(i=1,\ldots,2m)$ has at most one root in $\{x: |x| \leq 4\delta^{-1}(|\mu| + |C_1|)\}$. By eqn. (4.5) ,

$$\det\left[\left.\frac{\partial^2}{\partial x_i \partial x_j}\right|_{x=0} J_0(\hbar,g,\lambda;x)\right] \geq (\frac{\delta}{4})^{2m} \ , \text{ for } (\hbar,g,\lambda) \in [0,\ \gamma) \times \overline{C}_\varepsilon \times [0,\ 1] .$$

Using this and the fact that derivatives of J_0 with respect to x are uniformly bounded in $(\hbar,g,\lambda) \in [0,\ \gamma) \times \overline{C}_\varepsilon \times [0,\ 1]$ and choosing γ and ε smaller if necessary, there is an $a' > 0$ such that

$$\det\left[\frac{\partial^2}{\partial x_i \partial x_j} J_0(\hbar,g,\lambda;x)\right] \geq \frac{1}{2}(\frac{\delta}{4})^{2m} \quad \text{for all} \quad (\hbar,g,\lambda,x) \in [0,\gamma) \times \overline{C}_\varepsilon \times [0,1] \times D_{a'} \tag{4.7}$$

One can argue using (4.7), the fact that derivatives of J_0 are uniformly bounded and an adaptation of the proof of the inverse function theorem, that there is an $a > 0$ such that $\nabla J_0(\hbar,g,\lambda;\cdot) : D_a \to R^{2m}$ is one-one and hence $\nabla J_0(\hbar,g,\lambda;x) = \mu$ has at most one solution in D_a , for any $\mu \in R^{2m}$. Let $2\rho = a\delta/4$. Then for $\mu \in D_\rho$ and γ small enough that $|C_1| < \rho$, $\{x : |x| \leq 4\delta^{-1}(|\mu| + |C_1|)\} \subset D_a$ and so J_μ has a uniquely attained global minimum.

Call the location of the global minimum $\xi(\hbar,g,\mu,\lambda)$. Then ξ is C^∞ in $(\hbar,g,\mu,\lambda) \in [0,\gamma) \times \overline{C}_\varepsilon \times D_\rho \times [0, 1]$ by eqn. (4.7) and the implicit function theorem [Warn 71] . Finally, eqn. (4.2) follows from the fact that $\xi(\hbar,g,\mu,\lambda)$ can be made arbitrarily close to zero by choosing γ and ρ sufficiently small by eqn. (4.6), and hence the coefficients of K can be made arbitrarily close to those of J_0 . □

From now on we take $r = r_0$.

Theorem 4.3: $\hbar T_{2m}(\hbar,g,\mu,\lambda)$ is C^∞ in $(\hbar,g,\mu,\lambda) \in [0,\gamma) \times \overline{C}_\varepsilon \times D_\rho \times [0, 1]$.

Proof: Translating x by ξ in eqn. (3.2) and then scaling by $\hbar^{\frac{1}{2}}$ gives

$$\hbar T_{2m}(\hbar,g,\mu,\lambda) = -J_\mu(\hbar,g,\lambda;\xi(\hbar,g,\mu,\lambda)) + \hbar\, \ell n\left[\frac{\int e^{-\frac{1}{\hbar}K(\hbar,g,\mu,\lambda;\hbar^{\frac{1}{2}}x)}dx}{\int e^{-\frac{1}{2}xC(\lambda)^{-1}x}dx}\right], \tag{4.8}$$

where J_μ and K_μ are given by (4.1) and (4.2) respectively.

The first term on the right side of eqn. (4.8) has the required smoothness, by Lemma 4.2. Using the bound of eqn. (4.2) and Lebesgue's Dominated Convergence Theorem, the second term can be differentiated under the integral sign with respect to $(t=\hbar^{\frac{1}{2}},g,\mu,\lambda) \in [0,\gamma^{\frac{1}{2}}) \times \overline{C}_\varepsilon \times D_\rho \times [0, 1]$. The only thing to check is that odd t derivatives vanish at $t = 0$, i.e.

$$\frac{\partial^k}{\partial t^k} D_g^\alpha D_\mu^\beta \frac{\partial^\ell}{\partial\lambda^\ell} [t^2 T_{2m}(t^2,g,\mu,\lambda)] \to 0 \quad \text{as} \quad t \downarrow 0 \text{ , if } k \text{ is odd.} \tag{4.9}$$

To see this, note that by eqn. (4.2)

$$\frac{1}{t^2} K(t^2,g,\mu,\lambda;tx) \geq c|x|^2 \quad \text{for all} \quad (t,g,\mu,\lambda,x) \in (-\gamma^{\frac{1}{2}},\gamma^{\frac{1}{2}})\times\overline{C}_\varepsilon\times D_\rho\times[0,1]\times R^{2m} ,$$

so that in fact the second term on the right side of eqn. (4.8) is C^∞ in $(t,g,\mu,\lambda) \in (-\gamma^{\frac{1}{2}}, \gamma^{\frac{1}{2}}) \times \bar{C}_\varepsilon \times D_\rho \times [0, 1)$. But by scaling

$$\int e^{-\frac{1}{t^2}K(t^2,g,\mu,\lambda;tx)} dx = t^{-2m} \int e^{-\frac{1}{t^2}K(t^2,g,\mu,\lambda;x)} dx .$$

Therefore the second term on the right side of eqn. (4.8) is invariant under $t \to -t$, and eqn. (4.9) follows. □

Remark: A continuum version of Theorem 4.3 would be more difficult to prove than the lattice version, because in a continuum theory with space time dependent coupling constants the function ξ which removes the linear term from the interaction satisfies a non-linear partial differential equation. Also, smoothness properties of T_{2m} with respect to g and μ would be in terms of functional differentiation in the continuum theory rather than the partial differentiation in the lattice theory.

In the next lemma, we use the notation

$$\langle \cdot \rangle_{\hbar,g,\mu,\lambda} = \frac{\int \cdot \, e^{-\frac{1}{\hbar}:I_\mu(\hbar,g,x):} d\gamma_{\hbar C(\lambda)}}{\int e^{-\frac{1}{\hbar}:I_\mu(\hbar,g,x):} d\gamma_{\hbar C(\lambda)}} .$$

Lemma 4.4: The following limits are uniform in $g, \mu, \lambda \in \bar{C}_\varepsilon \times D_\rho \times [0, 1]$.

(i) $$\lim_{\hbar \downarrow 0} \hbar \, T_{2m}(\hbar,g,\mu,\lambda) = -J_\mu(0,g,\lambda;\xi(0,g,\mu,\lambda))$$

(ii) $$\lim_{\hbar \downarrow 0} \langle x_i \rangle_{\hbar,g,\mu,\lambda} = \xi_i(0,g,\mu,\lambda)$$

(iii) $\lim_{h\downarrow 0} h^{-1}\langle x_i;x_j\rangle_{h,g,\mu,\lambda} = (M^{-1})_{ij}$, where $M_{ab} = \left.\frac{\partial}{\partial x_a \partial x_b}\right|_{x=0} K(0,g,\mu,\lambda;x)$

is invertible (in fact, positive definite) by eqn. (4.2).

In particular, $\lim_{\hbar\downarrow 0} \hbar^{-1}\langle x_i;x_j\rangle_{\hbar,g,0,\lambda} = C(\lambda)_{ij}$.

<u>Proof</u>: (i) The result follows from eqns. (4.8) and (4.2).

(ii) Differentiating eqn. (4.8) with respect to μ_i , the left side gives $\langle x_i\rangle_{\hbar,g,\mu,\lambda}$, while

$$\frac{\partial}{\partial\mu_i} J_\mu(\hbar,g,\lambda;\xi(\hbar,g,\mu,\lambda)) = \frac{\partial}{\partial\mu_i}[J_0(\hbar,g,\lambda;\xi(\hbar,g,\mu,\lambda)) - \mu\xi(\hbar,g,\mu,\lambda)]$$

$$= \sum_{j=1}^{2m} \frac{\partial}{\partial\xi_j} J_0(\hbar,g,\lambda;\xi) \frac{\partial\xi_j}{\partial\mu_i} - \xi_i - \sum_{j=1}^{2m} \mu_j \frac{\partial\xi_j}{\partial\mu_i} = -\xi_i \ .$$

The result then follows since $\frac{\partial}{\partial\mu_i}$ applied to the second term on the right side of eqn. (4.8) is $O(\hbar)$ by eqn. (4.2).

$$\text{(iii)}\quad \hbar^{-1}\langle x_i,x_j\rangle_{\hbar,g,\mu,\lambda} = \frac{\hbar^{-1}\int (x_i,x_j) e^{-\frac{1}{\hbar}J_\mu(\hbar,g,\lambda;x)}dx}{\int e^{-\frac{1}{\hbar}J_\mu(\hbar,g,\mu;x)}dx}$$

$$= \frac{\hbar^{-1}\int ((x_i+\xi_i);(x_j+\xi_j)) e^{-\frac{1}{\hbar}K(\hbar,g,\mu,\lambda;x)}dx}{\int e^{-\frac{1}{\hbar}K(\hbar,g,\mu,\lambda;x)}dx}$$

$$= \frac{\hbar^{-1}\int (x_i;x_j) e^{-\frac{1}{\hbar}K(\hbar,g,\mu,\lambda;x)} dx}{\int e^{-\frac{1}{\hbar}K(\hbar,g,\mu,\lambda;x)} dx}$$

$$= \frac{\int (x_i;x_j) e^{-\frac{1}{\hbar}K(\hbar,g,\mu,\lambda;h^{\frac{1}{2}}x)} dx}{\int e^{-\frac{1}{\hbar}K(\hbar,g,\mu,\lambda;h^{\frac{1}{2}}x)} dx}$$

$$\to \frac{\int (x_i;x_j) e^{-\frac{1}{2}xMx} dx}{\int e^{-\frac{1}{2}xMx} dx} ,$$

using eqn. (4.2) in the last step. Since M is a symmetric positive definite quadratic form, the last integral is a Gaussian integral that can be evaluated explicitly to give $(M^{-1})_{ij}$. □

<u>Lemma 4.5</u>: $\lim_{h \downarrow 0} \mu(\hbar,g,\lambda) = 0$ uniformly in $(g,\lambda) \in \overline{C}_\varepsilon \times [0, 1]$.

<u>Proof</u>: To simplify the notation, let $f(\hbar,\mu) = \hbar\, T_{2m}(\hbar,g,\mu,\lambda)$ and $f(\mu) = -J_\mu(0,g,\lambda;\xi(0,g,\mu,\lambda))$. By Lemma 4.4(i), $\lim_{\hbar \downarrow 0} f(\hbar,\mu) = f(\mu)$ uniformly in g and λ . By eqn. (3.5) $f(\hbar,\cdot)$ is convex, so the same is true of f . Also, f is smooth for small $|\mu|$ and $f(\mu) \geq 0$ with $f(\mu) = 0$ only if $\mu = 0$. Let $\varepsilon \in (0,\rho)$ and set

$\alpha = \min_{\substack{s=\pm\varepsilon \\ |\hat{\mu}|=1}} |\frac{\partial}{\partial s} f(s\hat{\mu})|$. Then $\alpha > 0$ and for any $|\hat{\mu}| = 1$,

$$\frac{\partial}{\partial s} f(s\hat{\mu}) \begin{cases} \leq -\alpha & s = -\varepsilon \\ \geq \alpha & s = +\varepsilon \end{cases}$$

. But $\frac{\partial}{\partial s} f(\hbar, s\hat{\mu}) = \hat{\mu} \langle x \rangle_{h,g,s\hat{\mu},\lambda}$ and

$$\frac{\partial}{\partial s} f(s\hat{\mu}) = \frac{-\partial}{\partial s}[J_0(0,g,\lambda;\xi(0,g,s\hat{\mu},\lambda) - s\hat{\mu}\xi(0,g,s\hat{\mu},\lambda)]$$

$$= \hat{\mu}\xi(0,g,s\hat{\mu},\lambda) \quad .$$

Therefore $\left|\frac{\partial}{\partial s} f(\hbar,s\hat{\mu}) - \frac{\partial}{\partial s} f(s\hat{\mu})\right| = \left|\hat{\mu} \langle x \rangle_{\hbar,g,s\hat{\mu},\lambda} - \hat{\mu}\xi(0,g,s\hat{\mu},\lambda)\right|$

$$\leq \left| \langle x \rangle_{\hbar,g,s\hat{\mu},\lambda} - \xi(0,g,s\hat{\mu},\lambda)\right| \quad . \tag{4.10}$$

The right hand side of eqn. (4.10) goes to zero as $\hbar \downarrow 0$ uniformly in g,λ, $s = \pm\varepsilon$ and $|\hat{\mu}| = 1$ by Lemma 4.4 (ii).

Therefore there exists a $\delta > 0$ such that $\left|\frac{\partial}{\partial s} f(\hbar,s\hat{\mu}) - \frac{\partial}{\partial s} f(s\hat{\mu})\right| < \frac{\alpha}{2}$ for all $(\hbar,g,s,\hat{\mu},\lambda) \in (0,\delta) \times \overline{C}_\varepsilon \times \{-\varepsilon,+\varepsilon\} \times \{\hat{\mu} : |\hat{\mu}| = 1\} \times [0, 1]$, and

so $\frac{\partial}{\partial s} f(\hbar,s\hat{\mu}) \begin{cases} \leq -\frac{\alpha}{2}, & s = -\varepsilon \\ \geq \frac{\alpha}{2}, & s = +\varepsilon \end{cases}$ for all $\hbar < \delta$, $|\hat{\mu}| = 1$. It follows that the minimum of $f(\hbar,\mu)$ is attained at some point $s(\hbar)\hat{\mu}(\hbar)$ with $|\hat{\mu}(\hbar)| = 1$ and $\hat{s}(\hbar) < \varepsilon$. □

<u>Theorem 4.6</u>: $\Gamma_{2m}(\hbar,g,\lambda)$ is C^∞ in $(\hbar,g,\lambda) \in [0,\gamma) \times \overline{C}_\varepsilon \times [0, 1]$.

<u>Proof</u>: We first show smoothness of $\Gamma_{2m}(\hbar,g,\lambda) = -\hbar\, T_{2m}(\hbar,g,\mu(\hbar,g,\lambda),\lambda)$ in the open set $(0,\gamma) \times C_\varepsilon \times (0,1)$. By Lemma 4.5, $\mu(\hbar,g,\lambda) \in D_\rho$ for

$\hbar < \gamma$ sufficiently small. Therefore by Lemma 4.4 (iii),

$$\det\left[\left.\frac{\partial^2}{\partial\mu_i\partial\mu_j}\right|_{\mu(\hbar,g,\lambda)} \hbar\, T_{2m}(\hbar,g,\mu,\lambda)\right] = \det\left[\hbar^{-1}\left\langle x_i, x_j\right\rangle_{\hbar,g,\mu,\lambda}\right] \geq C > 0 \tag{4.11}$$

uniformly in $\hbar$, g and λ if ε and γ are sufficiently small. By eqn. (4.11) and the implicit function theorem [Warn 71] , $\mu(h,g,\lambda)$ is C^∞ in $(\hbar,g,\lambda) \in (0,\gamma) \times C_\varepsilon \times (0,1)$.

The extension of smoothness to $[0,\gamma) \times \overline{C}_\varepsilon \times [0, 1]$ poses no difficulty since derivatives of Γ_{2m} can be seen to be uniformly bounded in $(\hbar,g,\lambda) \in (0,\gamma) \times C_\varepsilon \times (0,1)$ using eqn. (4.11) and the fact that derivatives of T_{2m} are uniformly bounded (by Theorem 4.3). □

§5. Irreducibility

In this section we give the proof of Theorem 1.4.3(b). The first theorem of this section allows us to analyze the graphs occurring in $D_1^N \Gamma_{2N}(0,g,\lambda)$ instead of those in $D_1^N E(0)$.

Theorem 5.1: For $N \geq 2$, $-D_1^N \Gamma_{2N}(0,g,\lambda)$ is given by a finite linear combination of graphs which is topologically identical to the sum of graphs equal to $-D_1^N E(0)$ (as given in Theorem 1.3), with the following rules of evaluation:

1. Whereas a vertex in $-D_1^N E(0)$ takes a factor $-q_{kj}{:}\phi^k(R^2){:}$, a vertex in $-D_1^N \Gamma(0,g,\lambda)$ takes a factor $-\sum_{i=1}^{2N} q_{kj} g_{kji} {:}x_i^k{:}$.

2. No vertex is fixed — all are summed over the lattice.

3. A line joining x_i to x_j contributes $C(\lambda)_{ij}$.

Proof: Since

$$\Gamma_{2N}(\hbar,g,\lambda) = -\hbar \,\ell n \int e^{-\sum_{i=1}^{2N} [\sum_{k=2}^{n} \sum_{j=0}^{\frac{n}{2}} q_{kj} \hbar^{j+\frac{k}{2}-1} g_{kji} :x_i^k: - \hbar^{-\frac{1}{2}} \mu_i(\hbar,g,\lambda) x_i]} d\gamma_{C(\lambda)}$$

and $$E(\hbar) = -\hbar \frac{1}{|\Lambda|} \lim_{\Lambda} \ell n \int e^{-\int_\Lambda [\sum_{k=2}^{n} \sum_{j=0}^{\frac{n}{2}} q_{kj} \hbar^{j+\frac{k}{2}-1} : \phi^k : - \hbar^{-\frac{1}{2}} \mu(\hbar)\phi]} d\mu_{C_1} ,$$

differentiation of Γ with respect to $t = \hbar^{\frac{1}{2}}$ is formally very similar to differentiation of E with respect to $t = \hbar^{\frac{1}{2}}$, and with the rules 1-3 above, yields graphs of the form (1.10) with f replaced by $b(t,g,\lambda) \equiv t^{-1}\mu(t^2,g,\lambda)$. However a different mechanism is responsible for hooking the graphs $D_1^{\ell} b(t,g,\lambda)$ onto the corresponding legs, as we now explain. For $b \in R^{2N}$, let

$$z(t,g,b,\lambda) = T(t^2,g,tb,\lambda) = \ell n \int e^{-\sum_{i=1}^{2N} [\sum_{k=2}^{\frac{n}{2}} q_{kj} t^{2j+k-2} g_{kji} :x_i^k: - b_i x_i]} d\gamma_{C(\lambda)} . \quad (5.1)$$

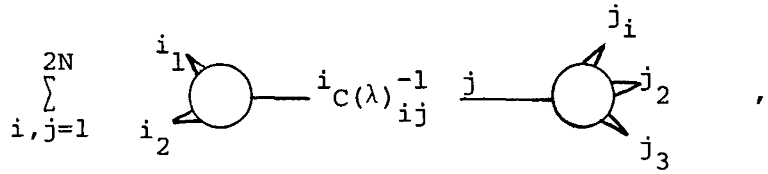
Then $b(t,g,\lambda)$ is characterized by $\frac{\partial}{\partial b_i} z(t,g,b(t,g,\lambda),\lambda) = 0$, $(i=1,\ldots,2N)$. Simplifying the notation by denoting differentiation by subscripts and using an implied summation convention, differentiating eqn. (5.1) with respect to t gives

$$z_{bt} + z_{bb} b_t = 0 \text{ , and so } b_t = -z_{bb}^{-1} z_{bt} .$$

Since $z_{bb} = t^2 T_{\mu\mu}$, it follows from Lemma 4.4 (iii) with $t^2 = \hbar$

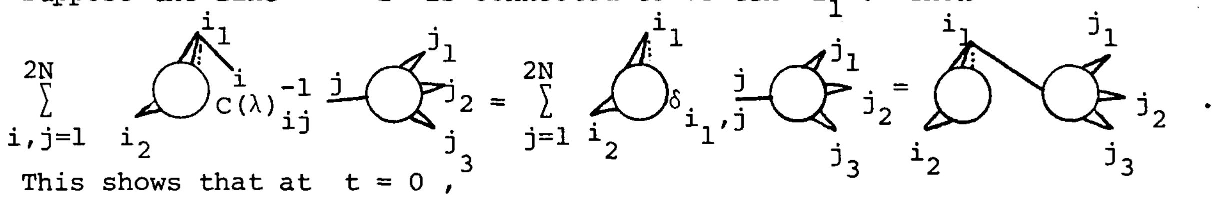
that z_{bb} is invertible if t is sufficiently small. In fact, $\lim_{t\downarrow 0} z^{-1}_{b_i b_j} = C(\lambda)^{-1}_{ij}$. The matrix inverse z^{-1}_{bb} plays the role here of the denominator $D_2^2\zeta(t,f(t))$ in Theorem 1.3.

To see that z^{-1}_{bb} hooks things up in the right way, consider for example $z_{ttb}z^{-1}_{bb}z_{bttt}$ at $t = 0$, i.e., a linear combination of terms of the form

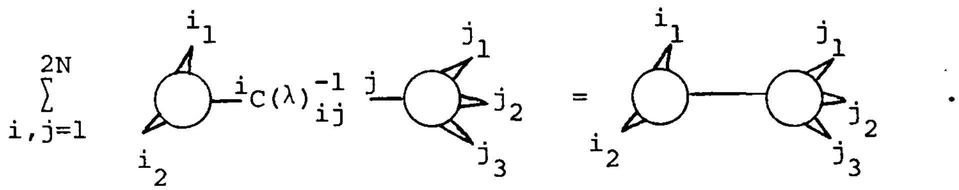


where a vertex denoted k is fixed at x_k .

Suppose the line ——•i is connected to vertex i_1 . Then



This shows that at $t = 0$,



□

<u>Corollary 5.2</u>: $D_1^N\Gamma_{2N}(0,1,\lambda) = \sum_{|\alpha|\le N} \frac{1}{\alpha!} D_2^\alpha D_1^N\Gamma_{2N}(0,0,\lambda)$,

where α is a multi-index with $2N(\frac{n}{2}+1)(n-1)$ components.

<u>Proof</u>: By Theorem 5.1, $D_1^N\Gamma_{2N}(0,g,\lambda)$ is a polynomial in g of degree N , so the Corollary follows by Taylor's Theorem. □

The following Lemma shows that when $g = 0$ the interaction (defined in (3.1)) occurring in the lattice pressure $T(\hbar,g,\mu(\hbar,g,\lambda),\lambda)$ vanishes.

<u>Lemma 5.3</u>: $\mu(\hbar,0,\lambda) = 0$ for $(\hbar,\lambda) \in [0,\gamma) \times [0,1]$.

<u>Proof</u>: It was shown in the proof of Theorem 3.4 that

$\hbar\, T_{2N}(\hbar,0,\mu,\lambda) = \hbar\, \ln \int e^{\frac{1}{\hbar}\mu x}\, d\gamma_{\hbar C(\lambda)}$ is strictly convex as a function of μ .

Since $\hbar\, T_{2N}(\hbar,0,-\mu,\lambda) = \hbar\, T_{2N}(\hbar,0,\mu,\lambda)$, it follows that

$\inf_{\mu \in R^{2N}} \hbar\, T_{2N}(\hbar,0,\mu,\lambda)$ occurs at $\mu(\hbar,0,\lambda) = 0$. $\square$

To simplify the notation for derivatives with respect to components of g , given indices k_ℓ, j_ℓ, i_ℓ we write $g_\ell = g_{k_\ell j_\ell i_\ell}$, and denote derivatives with respect to g_ℓ with a subscript ℓ , e.g.,

$\Gamma_{12\ldots N} = \dfrac{\partial^N}{\partial g_i \cdots \partial g_N}\Gamma$ and we drop the subscript $2N$ from Γ_{2N} and T_{2N} .

The following lemma is the first step in identifying the graphs contributing to $-\Gamma_{12\ldots N}(\hbar,0,\lambda)$.

<u>Lemma 5.4</u>: For $\hbar < \gamma$, $-\Gamma_{12\ldots N}(\hbar,0,\lambda)$ is a finite sum of graphs with the N vertices $-q_{k_\ell j_\ell}\hbar^{j_\ell+\frac{1}{2}k_\ell-1} :x_{i_\ell}^{k_\ell}:$ $(\ell=1,\ldots,N)$ and lines $C(\lambda)$. No self-lines can appear. Graphs enter the sum with either a plus sign or a minus sign, but all those with minus signs are 1-PR . Furthermore, every 1-PI graph with the mentioned vertices enters the sum with a plus sign. The combinatorial factor of a 1-PI graph is the same as for $T_{12\ldots N}(\hbar,0,0,\lambda)$.

Lemma 5.4 will be improved in Theorem 5.6 where it will be shown that all the 1-PR graphs in $-\Gamma_{12\ldots N}(h,0,\lambda)$ cancel, leaving only the 1-PI graphs.

Proof of Lemma 5.4: The variables $\hbar$ and λ play no significant role in the proof so we drop them from the notation. Derivatives are denoted by subscripts and an implicit summation convention is used. In the following, all derivatives of T are evaluated at $(g,\mu(g))$.

Differentiating the equation $-\Gamma(g) = T(g,\mu(g))$ with respect to g_1 gives $-\Gamma_1 = T_1 + T_\mu \mu_1 = T_1$, since $T_\mu = 0$. Note that in T_1 the g dependence of μ is not differentiated. Differentiating $-\Gamma_1 = T_1$ with respect to g_2 gives

$$-\Gamma_{12} = T_{12} + T_{1\mu}\mu_2 .$$

To compute μ_i , differentiate the equation $T_\mu = 0$ with respect to g_i to obtain $T_{\mu i} + T_{\mu\mu}\mu_i = 0$, i.e.,

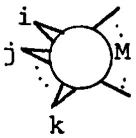
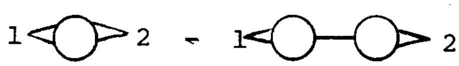
$$\mu_i = -T_{\mu\mu}^{-1} T_{\mu i} , \tag{5.2}$$

where the inverse on the right side is a matrix inverse. Therefore

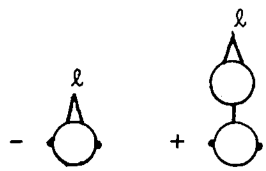
$$-\Gamma_{12} = T_{12} - T_{1\mu} T_{\mu\mu}^{-1} T_{\mu 2} . \tag{5.3}$$

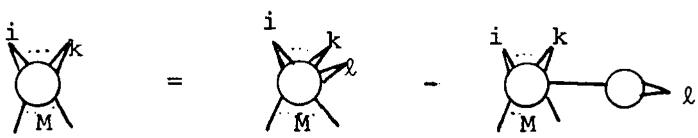
Note that when $g = 0$ $(g,\mu(g)) = (0,0)$ by Lemma 5.3 and we have a free theory. Using the lattice analogue of the formula (1.3.3) for the derivative of the logarithm of a partition function and the definition of T in eqn. (3.2), a derivative of the form $T_{ij\ldots k\underbrace{\mu\ldots\mu}_{M}}$ at $g = 0$ is the sum of all connected graphs with fixed vertices as specified by the g_ℓ's , and M fixed one-legged vertices because of the μ derivatives. As shown in the proof of Theorem 5.1, $T_{\mu\mu}^{-1}$ serves to link up graphs in a free theory.

We use a graph notation for the derivatives as follows. Denote

$\underbrace{T_{ij\ldots k\mu\ldots\mu}}_{M}$ by  and μ_i by (-1)  , where the dot on the $\mu_i = -T^{-1}_{\mu\mu}T_{\mu i}$ graph indicates that a $T^{-1}_{\mu\mu}$ has amputated a leg that was brought down by differentiation with respect to μ . When $g = 0$  or  is given by a sum of connected lattice graphs without self-lines. (In particular, at $g = 0$  $= 0$) . In this notation, eqn. (5.3) becomes $-\Gamma_{12} =$  .

The theorem now follows by repeated differentiation of eqn. (5.3) using the following facts:

$$\frac{\partial}{\partial g_\ell} T^{-1}_{\mu\mu} = -T^{-1}_{\mu\mu}(T_{\mu\mu\ell} + T_{\mu\mu\mu}\,\mu_\ell)T^{-1}_{\mu\mu} = $$ 

$$\frac{\partial}{\partial g_\ell} \underbrace{T_{ij\ldots k\mu\ldots\mu}}_{M} = \frac{\partial}{\partial g_\ell}$$  .

Clearly all graphs occurring in $-\Gamma_{12\ldots N}(0)$ with a minus sign are 1-PR , because a minus sign is introduced with every factor of $T^{-1}_{\mu\mu}$ (and in no other way) and a factor of $T^{-1}_{\mu\mu}$ corresponds to a line whose removal disconnects the graph. Furthermore $-\Gamma_{12\ldots N}(0)$ contains the term $+T_{12\ldots N}(0,0)$ which is the sum of all connected graphs (with combinatorial factors) having vertices as in the statement of the Lemma, and hence contains as a subset all 1-PI graphs. □

The following theorem is the key to obtaining the cancellation of all

1-PR graphs in $-\Gamma_{12\ldots N}(\hbar,0,\lambda)$. Is is inspired by [CFR 81].

Theorem 5.5: Given $g_\ell = g_{k_\ell j_\ell i_\ell}$ $(i=1,\ldots,N)$, if at least one i_ℓ is an element of $\{1,\ldots,N\}$ and at least one i_ℓ is an element of $\{N+1,\ldots,2N\}$ then for all $\hbar \in [0,\gamma)$

$$D^s\Gamma_{12\ldots N}(\hbar,0,0) = 0 \ , \ s = 0,\ 1 \ .$$

Proof: Since $\hbar$ plays no role in the proof it is omitted.

Beginning with the case $s = 0$, since $C(0)^{-1} = m_1^4 \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix}$ does not mix $\{x_1,\ldots,x_N\}$ and $\{x_{N+1},\ldots,x_{2N}\}$ we can write

$$T(g,\mu,0) = S_{(1)}(g(1),\mu(1)) + S_{(2)}(g(2),\mu(2))$$

where $\mu(1)$ and $g(1)$ (respectively $\mu(2)$ and $g(2)$) consist of those μ_i and g_{kji} with $i \in \{1,\ldots,N\}$ (respectively $i \in \{N+1,\ldots,2N\}$), and

$$S_{(1)}(g(1),\mu(1)) = \ln \int e^{-\sum\limits_{i=1}^{N}[\sum\limits_{k=2}^{n}\sum\limits_{j=0}^{\frac{n}{2}} q_{kj}g_{kji}:x_i^k:-\mu_i x_i]} \, d\gamma_{m_1^{-4}R_1}(x_1,\ldots,x_N)$$

$$S_{(2)}(g(2),\mu(2)) = \ln \int e^{-\sum\limits_{i=N+1}^{2N}[\sum\limits_{k=2}^{n}\sum\limits_{j=0}^{\frac{n}{2}} q_{kj}g_{kji}:x_i^k:-\mu_i x_i]} \, d\gamma_{m_1^{-4}R_2}(x_{N+1},\ldots,x_{2N}) \ .$$

Therefore
$$\frac{\partial}{\partial\mu_i} T(g,\mu,0) = \begin{cases} \dfrac{\partial}{\partial\mu_i} S_{(1)}(g(1),\mu(1)) & i \in \{1,\ldots,N\} \\[2ex] \dfrac{\partial}{\partial\mu_i} S_{(2)}(g(2),\mu(2)) & i \in \{N+1,\ldots,2N\} \ . \end{cases}$$

It follows that $\mu_i(g,0) = \begin{cases} \mu_i^{(1)}(g(1),0) & i \in \{1,\dots,N\} \\ \mu_i^{(2)}(g(2),0) & i \in \{N+1,\dots,2N\} \end{cases}$,

and hence $\Gamma(g,0) = -T(g,\mu(g,0),0)$

$$= -S_{(1)}(g(1),\mu^{(1)}(g(1),0),0) - S_{(2)}(g(2),\mu^{(2)}(g(2),0),0) ,$$

and the theorem follows in the case $s = 0$.

To prove the theorem in the case $s = 1$, we begin by noting that

$$D_2\Gamma(g,0) = -\frac{d}{d\lambda}\bigg|_{\lambda=0} T(g,\mu(g,\lambda),\lambda) = -D_2T(g,\mu(g,0),0)D_2\mu(g,0) - D_3T(g,\mu(g,0),0)$$

$$= -D_3T(g,\mu(g,0),0) , \tag{5.4}$$

since $D_2T(g,\mu(g,\lambda),\lambda) = 0$. Denoting expectations with respect to $d\gamma_{C(\lambda)}$ by $[\cdot]_\lambda$ and expectations with respect to $\dfrac{e^{-:I_\mu(g,x):}\,d\gamma_{C(\lambda)}}{\int e^{-:I_\mu(g,x):}\,d\gamma_{C(\lambda)}}$

by $\langle\cdot\rangle_{g,\mu,\lambda}$, we have

$$D_3T(g,\mu,0) = \frac{\partial}{\partial\lambda}\bigg|_0 \ln[e^{-:I_\mu(g,x):}]_\lambda = [e^{-:I_\mu(g,x):}]_0^{-1} \frac{\partial}{\partial\lambda}\bigg|_0 [e^{-:I_\mu(g,x):}]_\lambda$$

$$= [e^{-:I_\mu(g,x):}]_0^{-1}\Big([\frac{\partial}{\partial\lambda}\bigg|_0(-:I_0(g,x):_{C(\lambda)})e^{-:I_\mu(g,x):}]_0$$

$$+ [e^{-:I_\mu(g,x):};\ -\frac{1}{2}\sum_{i,j=1}^{2N} x_i DC(0)_{ij}^{-1}x_j]_0\Big) \tag{5.5}$$

$$= -\Big\langle\frac{\partial}{\partial\lambda}\bigg|_0(:I_0(g,x):_{C(\lambda)})\Big\rangle_{g,\mu,0} - \frac{1}{2}\sum_{i,j=1}^{2N} DC(0)_{ij}^{-1}(\langle x_i x_j\rangle_{g,\mu,0} - [x_i x_j]_0) .$$

By eqns. (5.4) and (5.5),

$$D_2\Gamma(g,0) = \left\langle \frac{\partial}{\partial\lambda}\bigg|_0 :I_0(g,x):_{C(\lambda)} \right\rangle_{g,\mu(g,0),0}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{2N} DC(0)^{-1}_{ij} \left(\langle x_i x_j \rangle_{g,\mu(g,0),0} - [x_i x_j]_0 \right) \qquad (5.6)$$

Now differentiate eqn. (5.6) with respect to g_a and g_b where $i_a \in \{1,2,\ldots,N\}$ and $i_b \in \{N+1,N+2,\ldots,2N\}$. Since

$$:x_i^k:_{C(\lambda)} = \sum_{j=0}^{[\frac{k}{2}]} c_{kj}(-1)^j C(\lambda)^j_{ii} x_i^{k-2j} ,$$

$$\frac{\partial}{\partial\lambda}\bigg|_0 :x_i^k:_{C(\lambda)} = \sum_{j=0}^{[\frac{k}{2}]} c_{kj}(-1)^j jC(0)^{j-1}_{ii} DC(0)_{ii} x_i^{k-2j} .$$

Therefore $\frac{\partial}{\partial\lambda}\big|_0 :I_0(g,x):_{C(\lambda)}$ is a sum of two polynomials: one in $x_1,\ldots,x_N$ depending only on g_ℓ with $i_\ell \in \{1,\ldots,N\}$, and one in $x_{N+1},\ldots,x_{2N}$ depending only on g_ℓ with $i_\ell \in \{N+1,\ldots,2N\}$. Since as was seen in the proof of the $s = 0$ case the measure $\langle\cdot\rangle_{g,\mu(g,0),0}$ factors into a product of probability measures in $x_1,\ldots,x_N$ and $x_{N+1},\ldots,x_{2N}$ depending only on g_ℓ with $i_\ell \in \{1,\ldots,N\}$ and g_ℓ with $i \in \{N+1,\ldots,2N\}$ respectively, $\frac{\partial^2}{\partial g_a \partial g_b} \left\langle \frac{\partial}{\partial\lambda}\big|_0 :I_0(g,x):_{C(\lambda)} \right\rangle_{g,\mu(g,0),0} = 0$.

Next, observe that the term involving $[x_i x_j]_0$ on the right side of eqn. (5.6) does not depend on g at all and hence vanishes after taking g derivatives. It remains only to show that

$$\frac{\partial^2}{\partial g_a \partial g_b} \left\langle x_i x_j \right\rangle_{g,\mu(g,0),0} = 0 . \tag{5.7}$$

Consider the case where both i and j are in $\{1,\ldots,N\}$. Then by factorization of the measure $\langle x_i x_j \rangle_{g,\mu(g,0),0}$ depends only on the g_ℓ with $i_\ell \in \{1,\ldots,N\}$ and eqn. (5.7) holds since $i_b \in \{N+1,\ldots,2N\}$. The case where both i and j are in $\{N+1,\ldots,2N\}$ is similar. Now consider the case where exactly one of i, j lies in $\{1,\ldots,N\}$. Then by factorization of the measure,

$$\langle x_i x_j \rangle_{g,\mu(g,0),0} = \langle x_i \rangle_{g,\mu(g,0),0} \cdot \langle x_j \rangle_{g,\mu(g,0),0} .$$

Each factor on the right side of the above equation vanishes by definition of $\mu(g,0)$. This completes the proof of eqn. (5.7) and hence of the Lemma. $\square$

We now show that all 1-PR graphs occurring in $-\Gamma_{12\ldots N}(\hbar,0,\lambda)$ cancel, and identify explicitly the remaining 1-PI graphs. As in the statement of Theorem 1.4.3 we write $d(a) = \frac{-1}{4\pi} \log \frac{U_0''(a)}{m^2}$.

<u>Theorem 5.6</u>: The derivative $-\Gamma_{12\ldots N}(\hbar,0,\lambda)$ is a polynomial in $\hbar$ where the coefficient of $\hbar^m$ is the sum of all $d(a)$-renormalized m loop 1-PI graphs with vertices $-(P^{(k_\ell)}(a)/k_\ell!)\,x_{i_\ell}^{k_\ell}$ $(\ell=1,\ldots,N)$ and $C(\lambda)$ lines with self-lines allowed. Note that the vertices are fixed. Each graph takes the same combinatorial factor that it has in $T_{12\ldots N}(\hbar,0,0,\lambda)$.

Proof: We first show that $-\Gamma_{12\ldots N}(\hbar,0,\lambda)$ can be written as a sum of 1-PI graphs having $\hbar$ dependent vertices. Part of the work was done in Theorem 5.4, from which it follows that we can write

$$-\Gamma_{12\ldots N} = \sum_{k=1}^{K} I_k(\hbar,\lambda) + \sum_{m=1}^{M} R_m(\hbar,\lambda) - \sum_{\ell=1}^{L} N_\ell(\hbar,\lambda) , \tag{5.3}$$

where the three sums on the right side of eqn. (5.3) are respectively the sum of all 1-PI graphs made of $C(\lambda)$-lines and vertices $-q_{k_\ell j_\ell} \hbar^{j_\ell+\frac{1}{2}k_\ell-1} :x_{i_\ell}^{k_\ell}:$ (having the same combinatorial factor as in $\Gamma_{12\ldots N}(\hbar,0,0,\lambda))$, the sum of all 1-PR graphs occurring in the expansion of Theorem 5.4 with a plus sign, and the sum of all 1-PR graphs occurring in the expansion with a minus sign. We now use Theorem 5.5 to show that the last two sums cancel.

In fact, treating $i_1,\ldots,i_N$ as free variables, it follows from Theorem 5.5 that $D_3^s\Gamma_{12\ldots N}(\hbar,0,0) = 0$, $s = 0, 1$ for any admissible imposition of $i_1\ldots i_N$ on the lattice L_{2N} of 2N points. On the other hand $\sum_{k=1}^{K} D_2^s I_k(\hbar,0) = 0$, $s = 0, 1$ for any AI , by Lemma 2.2. It follows from eqn. (5.3) that

$$\sum_{m=1}^{M} D_2^s R_m(\hbar,0) = \sum_{\ell=1}^{L} D_2^s N_\ell(\hbar,0) , \quad s = 0,\ 1, \text{ for any A.I.} \tag{5.4}$$

We now show that this implies that $\sum_{m=1}^{M} R_m(\hbar,\lambda)$ consists of exactly the same graphs as $\sum_{\ell=1}^{L} N_\ell(\hbar,\lambda)$.

For a graph G with vertices as in R_m or N_ℓ , denote by $\overline{G}$ the graph obtained from G by cancelling all factors

$-q_{kj}\hbar^{j+\frac{1}{2}-1}$. Since R_1 is reducible and has N vertices, it can be imposed on L_{2N} by choosing $i_1,\ldots,i_N$ in such a way that a line of reducibility of R_1 (i.e., a line whose removal disconnects R_1) joins x_1 to x_{N+1}, and no other line joins $\{x_1,\ldots,x_N\}$ to $\{x_{N+1},\ldots,x_{2N}\}$. This imposition of R_1 on L_{2N} of course also imposes the other R_m's and N_ℓ's on L_{2N}. Since all these graphs are connected, at least one line crosses from $\{x_1,\ldots,x_N\}$ to $\{x_{N+1},\ldots,x_{2N}\}$ for each graph. But $\frac{d}{d\lambda}\overline{R}_m(0)$ or $\frac{d}{d\lambda}\overline{N}_\ell(0)$ is zero if and only if more than one line makes the crossing from $\{x_1,\ldots,x_N\}$ to $\{x_{N+1},\ldots,x_{2N}\}$. Hence for the above imposition

$$\sum_{m=1}^{M}\frac{d}{d\lambda}\overline{R}_m(0)=\sum_{\text{one line}}\frac{d}{d\lambda}\overline{R}_m(0)=\sum_{\text{one line}}\frac{d}{d\lambda}\overline{N}_\ell(0) \tag{5.5}$$

where $\sum_{\text{one line}}\frac{d}{d\lambda}\overline{G}_i$ denotes the sum over those i for which G_i has a single line joining $\{x_1,\ldots,x_N\}$ to $\{x_{N+1},\ldots,x_{2N}\}$. But because of the form of $C(\lambda)$ (eqn. (2.1), for a graph $\overline{N}_\ell$ on $\overline{R}_m$ with exactly one line joining $\{x_1,\ldots,x_N\}$ to $\{x_{N+1},\ldots,x_{2N}\}$, $\frac{d}{d\lambda}\overline{N}_\ell(0)$ on $\frac{d}{d\lambda}\overline{R}_m(0)$ is r multiplied by a product of r_{ij}'s ($1\le i,j\le N$ or $N+1\le i,j\le 2N$), because it is only when the line joining $\{x_1,\ldots,x_N\}$ to $\{x_{N+1},\ldots,x_{2N}\}$ is differentiated that the result is non-zero. It follows that the second equality in eqn. (5.5) is an equality of polynomials in the r_{ij} ($1\le i,j\le N$ or $N+1\le i,j\le 2N$), and so the coefficients of these polynomials must agree. However these coefficients characterize the graphs topologically. To see this, note that the r_{ij} are in a one-one correspondence with lines joining x_i to x_j. Thus a product of r_{ij}'s characterizes the parts of the graph sitting in each of the sublattices $\{x_1,\ldots,x_N\}$ and $\{x_{N+1},\ldots,x_{2N}\}$. Because there will be only one vertex

x_{i_p} in each sublattice that does not have its full quota k_p of lines provided by the sublattice graphs, there is one and only one way that the line crossing from $\{x_1,\ldots,x_N\}$ to $\{x_{N+1},\ldots,x_{2N}\}$ can join the two sublattices, and the graph is uniquely determined. Therefore

$$\sum_{\text{one line}} \overline{R}_m = \sum_{\text{one line}} \overline{N}_\ell \tag{5.6}$$

with exactly the same graphs occurring on each side of the equation. Now discard the graphs contributing to eqn. (5.6) from eqn. (5.4) and repeat the above procedure until none of the R_m remain. We now show that no graphs N_ℓ remain, arguing by contradiction. Discarding all R_m graphs and the corresponding N_ℓ graphs from eqn. (5.4) leaves $0 = \sum' D_2^s N_\ell(h,0)$ $s = 0,1$, for every AI, where $\sum'$ denotes the sum over the remaining graphs. Therefore $0 = \sum' \frac{d}{d\lambda}\overline{N}_\ell(0)$, $s = 0, 1$, for every AI . Each term in $\sum' \frac{d}{d\lambda}\overline{N}_\ell(0)$ is nonnegative, and since $\overline{N}_\ell$ is 1-PR, for a given ℓ_0 the $i_1,\ldots,i_N$ can be chosen in such a way as to make $\frac{d}{d\lambda}\overline{N}_{\ell_0}(0) > 0$. But this contradicts $0 = \sum' \frac{d}{d\lambda}\overline{N}_\ell(0)$ and hence there can be no N_ℓ remaining. The end result is that $\sum_{m=1}^{M} R_m(\hbar,\lambda) = \sum_{\ell=1}^{L} N_\ell(\hbar,\lambda)$, with exactly the same graphs on each side of the equation, and hence

$$-\Gamma_{12\ldots N}(\hbar,0,\lambda) = \sum_{k=1}^{K} I_k(\hbar,\lambda) . \tag{5.7}$$

To identify the graphs contributing to the right side of eqn. (5.7) as those stated in the theorem, we begin by obtaining an explicit formula for q_{kj} . By definition (eqn. (1.6)), $q_{kj} = \frac{1}{j!}D^j q_k(0)$, where q_k is defined in eqn. (3.1.9) by the requirement $\sum_{k=3}^{n} a_k{:}\phi^k{:}_{\hbar C} = \sum_{k=0}^{n} q_k(\hbar){:}\phi^k{:}_{\hbar C_1}$.

Let $\tilde{a}_k = \begin{cases} a_k, & 3 \le k \le n \\ 0, & \text{otherwise} \end{cases}$, and extend the definition of

$c_{kj} = \dfrac{k!}{2^j j!(k-2j)!}$ by setting $c_{kj} = 0$ if $j > [\frac{k}{2}]$. Then by Lemma 2.2.1,

$$\sum_{k=3}^{n} a_k :\phi^k:_{\hbar C} = \sum_{k=0}^{n} \tilde{a}_k :\phi^k:_{\hbar C} = \sum_{k=0}^{n} \tilde{a}_k \sum_{j=0}^{[\frac{k}{2}]} c_{kj} (\hbar d)^j :\phi^{k-2j}:_{\hbar C_1}$$

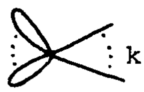
$$= \sum_{j=0}^{\frac{n}{2}} \sum_{k=0}^{n} \tilde{a}_k c_{kj} (\hbar d)^j :\phi^{k-2j}:_{\hbar C_1}$$

$$= \sum_{j=0}^{\frac{n}{2}} \sum_{\ell=0}^{n} \tilde{a}_{\ell+2j} c_{\ell+2j,j} (\hbar d)^j :\phi^{\ell}:_{\hbar C_1}$$

$$= \sum_{\ell=0}^{n} [\sum_{j=0}^{\frac{n}{2}} \tilde{a}_{\ell+2j} c_{\ell+2j,j} d^j \hbar^j] :\phi^{\ell}:_{\hbar C_1} \quad . \tag{5.8}$$

Therefore $q_{kj} = \tilde{a}_{k+2j}\, c_{k+2j,j} d^j$, and a vertex

$$-q_{kj}\, \hbar^{j+\frac{k}{2}-1} :x_i^k: = -\tilde{a}_{k+2j}\, \hbar^{j+\frac{k}{2}-1}\, c_{k+2j,j} d^j :x_i^k:$$

can be interpreted as j  k where each closed loop takes a factor of d , the combinatoric factor $c_{k+2j,j}$ counts the number of ways of choosing j pairs from $k+2j$ lines, each half-line takes a factor $\hbar^{\frac{1}{2}}$, and the vertex takes the factor $\frac{-1}{\hbar}\tilde{a}_{k+2j}$. This means that there is a one-one correspondence between I-PI graphs having vertices

$-q_{kj}\hbar^{j+\frac{1}{2}k-1} :x_i^k:$ and no self-lines, and d-renormalized 1-PI graphs having vertices $\frac{-1}{\hbar}\tilde{a}_{k+2j}x_i^{k+2j}$ with self-lines allowed and each line taking a factor $\hbar$.

It remains only to identify the overall power of $\hbar$ of a graph. An unrenormalized graph has a power of $\hbar$ given by $\underline{P} = I - V + 1$, where I is the number of lines of the graph, V is the number of vertices, and the extra $+1$ comes from the $\hbar$ in $-\Gamma = \hbar T$. But $I - V + 1$ is exactly the number of loops in the unrenormalized graph. □

In conclusion we combine the results of this chapter to prove Theorem 1.4.3(b). By eqn. (1.1) and Corollary 1.2 we need only show that for $N \geq 2$,

$$-v_N(a) = \frac{-1}{N!} D_1^N V(0,a) = -\frac{1}{N!} D^N q_1(0) - \frac{1}{N!} D_1^N E(0) \tag{5.9}$$

is the appropriate sum of graphs. The first term on the right side of eqn. (5.9) was identified in eqn. (1.2) to be the $d(a)$-renormalized single vertex N-loop diagram. By Theorem 5.1 the second term is a sum of graphs topologically identical to the L_{2N}-graphs whose sum is $-\frac{1}{N!}D_1^N\Gamma(0,1,1)$, where Γ is the L_{2N} Legendre transform (evaluated at the classical field equals zero). By Corollary 5.2,

$$-\frac{1}{N!} D_1^N\Gamma(0,1,1) = -\frac{1}{N!} \sum_{|\alpha|\leq N} \frac{1}{\alpha!} D_2^\alpha D_1^N \Gamma(0,0,1) = \frac{-1}{N!} \left.\frac{d^N}{dh^N}\right|_0 \sum_{|\alpha|\leq N} \frac{1}{\alpha!} D_2^\alpha \Gamma(h,0,1) . \tag{5.10}$$

But by Theorem 5.6 the right side of eqn. (5.10) is exactly the desired sum of graphs: the different terms in the sum over α give the N-loop graphs with different kinds of vertices.

Finally we show that the combinatorial factors are as indicated in

Remark 1 under Theorem 1.4.3. By Theorem 5.6 the combinatorial factor of a graph contributing to $D_2^{\alpha}(\hbar,0,1)$ is the same as for $D_2^{\alpha}T(\hbar,0,0,1)$, namely the factor associated with the graph by Wick's Theorem. The factor of $\frac{1}{\alpha!}$ occurring on the right side of eqn. (5.10) provides the factor $\frac{1}{\Pi V_{jk}!}$ appearing in Remark 1. Since the $\frac{1}{N!}$ on the right side of (5.10) is cancelled by an $N!$ brought down by $\frac{d^N}{dh^N}$, the combinatorial factor of a graph in $-\frac{1}{N!}D_1^N\Gamma(0,1,1)$, and hence in $-v_N(a)$, is as stated in Remark 1.

CHAPTER 6: FAILURE OF THE 1-PI LOOP EXPANSION

§1. An Asymptotic Connected Loop Expansion

Until now we have discussed the asymptotics for the effective potential when $a \notin B$. The set $B_2 \subset B$ is not very interesting because it corresponds to a massless theory which will be divergent when $\hbar = 0$. The set $B_1 \subset B$ is more interesting and has recently received some attention in the literature [FOS 83], [BC 83], [CF 83] for the case of a double-well potential. Consider the classical potential $U_0(a) = (a^2 - \frac{1}{8})^2$, with $m = 1$ (so $P(a) = (a^2 - \frac{1}{8})^2 - \frac{1}{2}a^2)$, for which $B = B_1 = [-\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}]$. (The constant $\frac{1}{8}$ appearing in U_0 is arbitrary and can be replaced by any positive constant provided that m is also changed so as to agree with the curvature of U_0 at its minima). It is clear that the loop expansion must break down at least for the interval $|a| < \frac{\sqrt{6}}{12}$ where $U_0''(a) < 0$, since in that interval $\gamma(a)$ and the graphs contributing to $v_n(a)$ as given by Theorem 1.4.3 are divergent. In the above references the authors take both minima of the classical potential into account for $|a| < \frac{1}{\sqrt{8}}$ and conclude that for $|a| < \frac{1}{\sqrt{8}}$ the $O(\hbar)$ approximation to the effective potential is the straight line interpolation of $U_0(a) - \hbar\gamma(a)$, $|a| > \frac{1}{\sqrt{8}}$.

In this section we give a rigorous proof that this picture is correct, showing how it is a consequence of the definition of the effective potential using the Fenchel transform and the occurrence of a phase transition [GJS 76]. Moreover we find that the N^{th} order contribution is the sum of all N-loop <u>connected</u> graphs, for $N \geq 2$.

<u>Theorem 1.4.5</u>: Let $V(\hbar,a)$ denote the effective potential for $m = 1$ and $P(a) = (a^2 - \frac{1}{8})^2 - \frac{1}{2}a^2$. Then for $|a| < \frac{1}{\sqrt{8}}$ $D_1V(0,a) = -\gamma(\frac{1}{\sqrt{8}}) = 0$, and for $N \geq 2$, $\frac{-1}{N!} D_1^N V(0,a)$ is given by the sum of all N-loop <u>connected</u> graphs with no self-lines, with three- and four-legged vertices taking factors $\frac{-1}{3!} P^{(3)}(\frac{1}{\sqrt{8}}) = -\sqrt{2}$ and $\frac{1}{4!} P^{(4)}(\frac{1}{\sqrt{8}}) = -1$ respectively, and lines corresponding to the free covariance of mass one. Graphs take combinatorial factors as per Remark 1 under Theorem 1.4.3.

<u>Proof</u>: Let $P(x) = U_0(x) - \frac{1}{2}x^2 = x^4 - \frac{3}{4}x^2 + \frac{1}{64}$, and let

$$p(\hbar,\mu) = \hbar \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ln \int e^{-\frac{1}{\hbar}\int_\Lambda [:P(\phi):-\mu\phi]} d\mu_{\hbar C},$$

where $C = (-\Delta+1)^{-1}$. The boundary conditions on C can be chosen as a matter of convenience since p is independent of a wide variety of boundary conditions [GRS 76]. By taking C to have periodic boundary conditions and appealing to Lemma 2.2.2, translation of ϕ by $\frac{\pm 1}{\sqrt{8}}$ gives

$$p(\hbar,\mu) = \hbar \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ell n \int e^{-\frac{1}{\hbar}\int_\Lambda [:\phi^4:\pm\sqrt{2}:\phi^3:-\mu(\phi\pm\frac{1}{\sqrt{8}})]} d\mu_{\hbar C} .$$

By Lemma 2.2.4 we obtain

$$p(\hbar,\mu) = \hbar \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ell n \int e^{-\int_\Lambda [\hbar:\phi^4:\pm\sqrt{2}\,\hbar^{\frac{1}{2}}:\phi^3:-\mu(\hbar^{-\frac{1}{2}}\phi\pm\hbar^{-1}\frac{1}{\sqrt{8}})]} d\mu_C$$

$$= \pm \frac{1}{\sqrt{8}}\mu + \hbar \lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ell n \int e^{-\int_\Lambda [\hbar:\phi^4:\pm\sqrt{2}\,\hbar^{\frac{1}{2}}:\phi^3:-\mu\hbar^{-\frac{1}{2}}\phi]} d\mu_C \qquad (1.1)$$

We will apply results of [GJS 76] which use free BC , so for the remainder of this section we take C to have free boundary conditions. As usual, Wick dots in an integrand match the measure.

In Theorem 2.2 of [GJS 76], for sufficiently small $\hbar$ and μ the one-point function corresponding to the pressure $p(\hbar,\mu)$ is controlled using a low temperature cluster expansion. It follows from their results that

$$|D_2^{\pm}p(\hbar,0) - (\pm\frac{1}{\sqrt{8}})| = O(\hbar^2) ,$$

as perturbation theory and eqn. (1.1) would suggest. (In the notation of [GJS 76] the + version of the above equation is $\lambda\langle\phi(x)\rangle = \lambda\xi_+ + O(\lambda^4)$ where $\lambda = \hbar^{\frac{1}{2}}$ and $\xi_+ = (8\hbar)^{-\frac{1}{2}}$ is the location of the minimum occurring on the positive axis of $\hbar^{-1}U_0(\hbar^{\frac{1}{2}}a) = \hbar a^4 - \frac{1}{4}a^2 + (64\hbar)^{-1})$. Therefore for any $|a| < \frac{1}{\sqrt{8}}$ there is a $\delta(a) > 0$ such that $a \in [D_2^-p(\hbar,0),\ D_2^+p(\hbar,0)]$

for all $\hbar < \delta(a)$, and hence (see Figure 2 on p. 19)

$$V(\hbar,a) = \sup_{\mu}[\mu a - p(\hbar,\mu)] = -p(\hbar,0) \ , \ \hbar < \delta(a) \ . \tag{1.2}$$

In [GJS 76] an infinite volume theory corresponding to the interaction $\hbar x^4 + \sqrt{2}\,\hbar^{\frac{1}{2}} x^3$ and covariance C is obtained. In §6 of [GJS 76] it is shown that the perturbation series in $\hbar^{\frac{1}{2}}$ for a generalized Schwinger function of this theory is asymptotic. The pressure is not discussed, but it is straightforward to use the estimates of [GJS 76] to show that perturbation theory is also asymptotic for $p(\hbar,0)$, as we now show.

Let $t = \hbar^{\frac{1}{2}}$ and $\zeta_\Lambda(t) = \frac{1}{|\Lambda|} \ln \int e^{-\int_\Lambda [t^2:\phi^4:+\sqrt{2}t:\phi^3:]} d\mu_C$.

Then $p(t^2,0) = t^2\zeta(t)$, where $\zeta(t) = \lim_{\Lambda} \zeta_\Lambda(t)$. By the Fundamental Theorem of Calculus,

$$\zeta_\Lambda(t) - \zeta_\Lambda(s) = \int_s^t D\zeta_\Lambda(x)\,dx = \frac{1}{|\Lambda|}\int_s^t \left\langle 2x:\phi^4(\Lambda): + \sqrt{2}:\phi^3(\Lambda):\right\rangle_{x,\Lambda} dx \ , \ s,t \geq 0 \tag{1.3}$$

where $$\langle\cdot\rangle_{x,\Lambda} = \frac{\int \cdot\, e^{-\int_\Lambda [x^2:\phi^4:+\sqrt{2}\,x:\phi^3:]} d\mu_C}{\int e^{-\int_\Lambda [x^2:\phi^4:+\sqrt{2}\,x:\phi^3:]} d\mu_C}$$.

By Theorem 6.2 of [GJS 76], $\frac{1}{|\Lambda|}\left|\left\langle 2x{:}\phi^4(\Lambda){:}+\sqrt{2}\,{:}\phi^3(\Lambda){:}\right\rangle_{x,\Lambda}\right|$ is bounded uniformly in Λ and small $x \geq 0$. Taking the limit $\Lambda \uparrow R^2$ in eqn. (1.3) and using Lebesgue's Dominated Convergence Theorem gives

$$\zeta(t) - \zeta(s) = \int_s^t \left\langle 2x{:}\phi^4(0){:}+\sqrt{2}{:}\phi^3(0){:}\right\rangle_x dx \ , \quad s,\ t \geq 0 \ ; \ s,\ t \ \text{small},$$

where $\langle\cdot\rangle_x = \lim\limits_{\Lambda} \langle\cdot\rangle_{x,\Lambda}$. Since $\left\langle 2x{:}\phi^4(0)+\sqrt{2}{:}\phi^3(0){:}\right\rangle_x$ is C^∞ in small $x \geq 0$ by Theorem 6.1 of [GJS 76], it follows that

$$D\zeta(t) = \left\langle 2t{:}\phi^4(0) + \sqrt{2}{:}\phi^3(0){:}\right\rangle_t \quad t \geq 0 \ , \ t \ \text{small}. \tag{1.4}$$

In Theorem 6.1 of [GJS 76] it is also shown that the derivatives of the right side of (1.4) at $t = 0$ are given by perturbation theory. Since $p(t^2,0) - t^2\zeta(t)$, the same is true of $p(t^2,0)$. The odd order derivatives of $p(t^2,0)$ vanish at $t = 0$ (because of the $t \to -t$ symmetry of ζ), and the derivatives with respect to t of order $2n$ at $t = 0$ correspond to derivatives with respect to $\hbar$ of order n at $\hbar = 0$. It follows that $p(\hbar,0)$ is C^∞ in small $\hbar = t^2$, including $\hbar = 0^+$.

Since $p(\hbar,0) = \hbar\zeta(\hbar^{\frac{1}{2}})$, $D_1 p(0,0) = \zeta(0) = 0$. But by definition of γ in eqn. (3.1.6), $\gamma(\frac{1}{\sqrt{8}}) = 0$, and hence $-D_1 V(0,a) = -\gamma(\frac{1}{\sqrt{8}}) = 0$ by (1.2). Moreover for $N \geq 2$, $-\frac{1}{N!}D_1^N V(0,a) = \frac{1}{N!}D_1^N p(0,0)$ is given by perturbation theory to be the sum of all connected n-loop graphs with 3 and 4 legged vertices taking factors $-\sqrt{2}$ and -1 respectively and with legs corresponding to $(-\Delta+1)^{-1}$, with the usual combinatoric factors and no closed loops. $\square$

Theorem 1.4.5 shows that the asymptotic behaviour of the effective potential corresponding to the double-well classical potential $U_0(a) = (a^2-\frac{1}{8})^2$ is quite different for $|a| < \frac{1}{\sqrt{8}}$ and $|a| > \frac{1}{\sqrt{8}}$. For $|a| > \frac{1}{\sqrt{8}}$ Theorem 1.4.3 gives an asymptotic one-particle irreducible loop expansion for $V(\hbar,a)$ with graphs having three- and four-legged vertices taking factors $-4a$ and -1 respectively and lines of mass $12a^2 - \frac{1}{2}$. However for $|a| < \frac{1}{\sqrt{8}}$ Theorem 1.4.5 gives an asymptotic connected loop expansion for $V(\hbar,a)$ graphs having three and four-legged vertices taking factors $-4(\frac{1}{\sqrt{8}}) = -\sqrt{2}$ and -1 respectively and lines of mass $12(\frac{1}{\sqrt{8}})^2 - \frac{1}{2} = 1$, i.e., the vertex factors and lines are calculated at an endpoint of $B = [-\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}]$. The asymptotics for $V(\hbar,a)$ are independent of a when $|a| < \frac{1}{\sqrt{8}}$ because for $|a_0| < \frac{1}{\sqrt{8}}$ and $\hbar$ sufficiently small $V(\hbar,\cdot)$ is linear with slope zero near a_0 and approaches the linear portion of $\mathrm{conv}U_0$ at an a_0-independent rate.

The mechanism responsible for the fact that connected graphs rather than 1-PI graphs occur for $|a| < \frac{1}{\sqrt{8}}$ is clear from the proof of Theorem 1.4.5: the supremum in the definition of V is attained at a point independent of a and small $\hbar$ and the cancellation of reducible graphs provided by $\mu(\hbar,a)$ when $a \in B^c$ does not take place. Thus in some sense Theorem 1.4.5 shows what is lost by defining the effective potential to be linear when there is a phase transition. It is an interesting open question whether $V(\hbar,a)$ can be defined in B by an analytic continuation from B^c, so that V itself might have a double-well structure.

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