FLOW UNDER A FUNCTION AND
DISCRETE DECOMPOSITION OF PROPERLY
INFINITE $W^*$-ALGEBRAS

by

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ABSTRACT

The aim of this thesis is to generalize the classical flow under a function construction to non-abelian $W^*$-algebras. We obtain existence and uniqueness theorems for this generalization. As an application we show that the relationship between a continuous and a discrete decomposition of a properly infinite $W^*$-algebra is that of generalized flow under a function. Since continuous decompositions are known to exist for any properly infinite $W^*$-algebra, this leads to generalizations of Connes' results on discrete decomposition.
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Introduction

The purpose of this thesis is to generalize the classical "flow under a function" construction to non-abelian $W^*$-algebras. That is, given a $W^*$-algebra $N$, an automorphism $\theta$ of $N$ and a positive self-adjoint operator $\phi$ affiliated to the centre of $N$ we show in analogy with the abelian case, how to construct a continuous action $\alpha$ of the reals on a $W^*$-algebra $M$ (see Definition 1.1). The covariant system $\{M,\alpha\}$ (see Definition 1.1) is called the flow built on $\{N,\theta\}$ under the ceiling $\phi$.

We obtain "existence" and "uniqueness" theorems for the representation of a given covariant system over the reals as a flow built under a ceiling. As an application we obtain Connes' discrete decomposition theorems ([1] théorème 4.4.1, théorème 5.3.1 and théorème 5.4.2) from Takesaki's continuous decomposition theorems ([2] theorem 8.1, lemma 8.2 and corollary 8.4) thereby elucidating the connection between these two results.

In section 1 we state the main results of Takesaki's duality theory for crossed products. In section 2 we define flow built under a ceiling and give necessary and sufficient conditions for a covariant system over the reals to be isomorphic to a flow built under a ceiling. Section 3 deals with the uniqueness problem. That is, we show the relationship between $\{N_1,\theta_1,\phi_1\}$ and $\{N_2,\theta_2,\phi_2\}$ when the corresponding flows are isomorphic. In section 4 we deal with the uniqueness problem in case the flows are only weakly equivalent (see definition 1.5). In section 5 we derive discrete decomposition theorems for properly infinite $W^*$-algebras using Takesaki's continuous decomposition theorem and our
results on flow built under a ceiling. The appendix consists of a proof of the results of section 2 in the special case where the ideas of [3] and [4] may be applied.
1. **Crossed Products**

The purpose of this section is to collect some results on crossed products over abelian groups. Except for the proof of proposition 1.10, these results and proofs may be found in [2], [5] and [6].

In the following let $G$ denote a locally compact abelian group with dual group $\hat{G}$. Haar measure in $G$ is denoted simply by $dt$.

We begin with some measure theoretical results.

A function $\xi$ from $G$ into a Hilbert space $H$ is called strongly (or Bourbaki) measurable iff it satisfies the conditions:

(i) $t \rightarrow (\xi(t), \eta)$ is measurable for $\eta \in H$.

(ii) for each compact set $K \subset G$, there is a separable subspace $H_1$ of $H$ such that $\xi(t) \in H_1$ for almost every $t \in K$.

$L^2(G;H)$ denotes the vector space of strongly measurable functions $\xi: G \rightarrow H$ which satisfy:

$$\int_G ||\xi(t)||^2 dt < \infty.$$ 

By identifying elements of $L^2(G;H)$ which are equal almost everywhere and by defining

$$(\xi, \eta) = \int_G (\xi(t), \eta(t)) dt$$

we obtain the Hilbert space $L^2(G;H)$. $L^2(G;\mathbb{C})$ is denoted by $L^2(G)$.

$L^2(G;H)$ is identified with $H \otimes L^2(G)$ by mapping $\xi \otimes \eta$ to the function $t \rightarrow \eta(t)\xi$.

$L^\infty(G)$ denotes the vector space of measurable functions $f: G \rightarrow \mathbb{C}$.
which satisfy \( \sup\{|f(t)|: t \in G\} < \infty \). By identifying elements of \( L^\infty(G) \) which are equal locally almost everywhere, we obtain \( L^\infty(G) \).

We make no distinction between elements of \( L^\infty(G) \) and the corresponding multiplication operator on \( L^2(G) \). Operators on \( L^2(G;H) \) are usually defined by displaying an operator on \( L^2(G;H) \). For example, if \( t \to x(t) \) is a function from \( G \) to \( B(H) \) which satisfies:

(i) for each \( \xi \in H \), \( t \to x(t)\xi \) is strongly measurable

(ii) \( \sup\{||x(t)||: t \in G\} < \infty \),

then \( t \to x(t) \) defines an operator on \( L^2(G;H) \) by the formula:

\[
(x\xi)(t) = x(t)\xi(t) \quad \text{for} \quad \xi \in L^2(G;H).
\]

We now begin the definition of crossed products.

Definition 1.1. A continuous action \( \alpha \) of \( G \) on a \( \mathcal{W} \)-algebra \( M \) is a homomorphism \( t \to \alpha_t \) of \( G \) into the group of automorphisms of \( M \) such that for each \( x \in M \), the map \( t \to \alpha_t(x) \) is \( \sigma \)-strong* continuous. The pair \( \{M,\alpha\} \) is called a covariant system over \( G \).

We have the usual notion of homomorphism (imbedding, isomorphism) \( \kappa: \{M,\alpha\} \to \{N,\beta\} \). That is, \( \kappa \) is a continuous \( \mathcal{W} \)-algebra homomorphism (imbedding, isomorphism) of \( M \) into \( N \) such that \( \kappa \circ \alpha_t = \beta_t \circ \kappa \), \( t \in G \).

Typical examples of covariant systems can be obtained in the following way: suppose \( M \) is a \( \mathcal{W} \)-algebra acting on a Hilbert space \( H \) and \( t \to U(t) \) is a strongly continuous unitary representation of
G on \( H \) such that for all \( t \in G \), \( U(t)M U(t)^* = M \). Then \( \alpha_t(x) = U(t)xU(t)^* \) for \( t \in G \) and \( x \in M \) defines a continuous action of \( G \) on \( M \) (see [2] proposition 3.2).

Given a covariant system \( \{M, \alpha\} \) over \( G \) with \( M \) acting on a Hilbert space \( H \), we can define a continuous imbedding \( \pi_\alpha \) of \( M \) into the bounded operators on the Hilbert space \( L^2(G;H) \) by:

\[
\pi_\alpha(x)\xi(t) = \alpha^{-t}(x)\xi(t).
\]

We also have a strongly continuous unitary representation of \( G \) on \( L^2(G;H) \) defined by:

\[
\lambda_\alpha(s)\xi(t) = \xi(t-s).
\]

Note that \( \lambda_\alpha(s)\pi_\alpha(x)\lambda_\alpha(s)^* = \pi_\alpha(\alpha_s(x)) \) for all \( s \in G \) and for all \( x \in M \).

**Definition 1.2.** Given a covariant system \( \{M, \alpha\} \) over \( G \) with \( M \) acting on \( H \), the crossed product \( \mathcal{W}\{M, \alpha\} \) is the \( \mathcal{W}^* \)-algebra on \( L^2(G;H) \) generated by \( \pi_\alpha(M) \) and \( \lambda_\alpha(G) \).

In [2] proposition 3.4, it is shown that the definition of \( \mathcal{W}^*\{M, \alpha\} \) is independent of the Hilbert space \( H \) on which \( M \) acts.

More precisely, if \( \kappa \) is an isomorphism of \( \{M, \alpha\} \) onto \( \{N, \beta\} \) then there is an isomorphism \( \overline{\kappa} \) of \( \mathcal{W}^*\{M, \alpha\} \) onto \( \mathcal{W}^*\{N, \beta\} \) such that

\[
\overline{\kappa} \pi_\alpha(x) = \pi_\beta(\kappa x), \text{ for all } x \in M \text{ and}
\]

\[
\overline{\kappa} \lambda_\alpha(s) = \lambda_\beta(s), \text{ for all } s \in G.
\]
We can relate this definition of crossed product to the original definition (see for example [7] chapitre 1, section 9.2) in the following way: assume that $\alpha$ is unitarily implemented on the Hilbert space $H$, that is, there is a strongly continuous unitary representation $t \rightarrow U(t)$ of $G$ in $H$ such that $\alpha_t(x) = U(t)xU(t)^*$ for all $t \in G$ and all $x \in M$. We define a unitary operator $\hat{U}$ on $L^2(G;H)$ by:

$$(\hat{U}\xi)(t) = U(t)\xi(t) \quad \text{for} \quad \xi \in L^2(G;H).$$

Then, with the usual identification of $L^2(G;H)$ with $H \otimes L^2(G)$ we have:

$$U\pi_\alpha(x)U^* = x \otimes 1, \quad \text{for all} \quad x \in M$$

$$U\lambda_\alpha(s)U^* = U(s) \otimes \lambda(s), \quad \text{for all} \quad s \in G$$

where $(\lambda(s)\xi)(t) = \xi(t-s)$ for $\xi \in L^2(G)$. The $\hat{W}^*$-algebra generated by $M \otimes 1$ and $\{U(s) \otimes \lambda(s) : s \in G\}$ is the "original" definition of crossed product.

Using the unitary implementation we can give some operators in the commutant $\hat{W}^*\{M,\alpha\}'$. Namely,

$$M' \otimes 1 \in \hat{W}^*\{M,\alpha\}' \quad \text{and}$$

$$U(s)' \otimes \lambda(s) \in \hat{W}^*\{M,\alpha\}' \quad \text{for} \quad s \in G.$$ 

We shall use this observation later.

The first non-trivial result concerning crossed products deals with the dual action. We can define a continuous action of $\hat{G}$ on $\hat{W}^*\{M,\alpha\}$ by looking at the characters $\chi_p(t) = \langle p, t \rangle$ for $p \in \hat{G}$. We define a
strongly continuous unitary representation \( p \mapsto \mu_\alpha(p) \) of \( \hat{G} \) on \( L^2(G;H) \) by the formula:

\[
\mu_\alpha(p)\xi(t) = \langle p, t \rangle \xi(t), \text{ for } \xi \in L^2(G;H).
\]

Then

\[
\mu_\alpha(p)\pi_\alpha(x)\mu_\alpha(p)^* = \pi_\alpha(x), \text{ for all } x \in M
\]

\[
\mu_\alpha(p)\lambda_\alpha(s)\mu_\alpha(p)^* = \langle p, s \rangle \lambda_\alpha(s), \text{ for all } p \in \hat{G}, s \in G.
\]

Hence \( \hat{\alpha}_p(y) = \mu_\alpha(p)y\mu_\alpha(p)^* \) for \( y \in W^*(M,\alpha) \) defines a continuous action of \( \hat{G} \) on \( W^*(M,\alpha) \).

**Definition 1.3.** Given a covariant system \( \{M,\alpha\} \) over \( G \), the dual covariant system \( \{W^*(M,\alpha),\hat{\alpha}\} \) is defined by:

\[
\hat{\alpha}_p(y) = \mu_\alpha(p)y\mu_\alpha(p)^* \text{ for } p \in \hat{G} \text{ and } y \in W^*(M,\alpha).
\]

It is clear that \( \pi_\alpha(M) \subset \{y \in W^*(M,\alpha): \hat{\alpha}_y(y) = y, \text{ for all } p \in \hat{G}\} \), but the converse is not so obvious. To state the next result we use the notation

\[
M^\alpha = \{x \in M: \alpha_t(x) = x, \text{ for all } t \in G\}.
\]

**Theorem 1.4** (the uniqueness theorem). For a covariant system \( \{M,\alpha\} \) over \( G \) we have

\[
W^*(M,\alpha)^\alpha = \pi_\alpha(M).
\]

**Proof.** The following proof is due to Landstad [6]. We have
\[ W^*\{M,\alpha\}^\hat{\alpha} = W^*\{M,\alpha\} \cap \mu_\alpha(\hat{G})' \] so
\[ W^*\{M,\alpha\}^\hat{\alpha} = [W^*\{M,\alpha\}' \cup \mu_\alpha(\hat{G})]' . \]

We assume that \( \alpha \) is unitarily implemented on \( H \) by \( t \to U(t) \). By our previous remark
\[ M' \otimes 1 \cup \{U(t)^* \otimes \lambda(t) : t \in G\} \subset W^*\{M,\alpha\}' . \]

Hence it suffices to show
\[ [M' \otimes 1 \cup \{U(t)^* \otimes \lambda(t) : t \in G\} \cup \mu_\alpha(\hat{G})]' \subset \pi_\alpha(M) . \]

Using the operator \( U \) as previously defined, it suffices to show that
\[ [UM' \otimes 1 U^* \otimes \lambda_\alpha(G) \cup \mu_\alpha(\hat{G})]' \subset M \otimes 1 . \]

Since \( \lambda_\alpha(G) \) and \( \mu_\alpha(\hat{G}) \) generate \( 1 \otimes B(L^2(G)) \) it suffices to show that if \( y \in B(H) \) and \( y \otimes 1 \in [UM' \otimes 1 U^*]' \) then \( y \in M \). Since
\[ (Ux \otimes 1 U^* \xi)(t) = U(t)xU(t)^* \xi(t) \text{ for } \xi \in L^2(G;H) \text{ and } x \in B(H) \]
it is clear that \( y \) must be in \( M \) if \( y \otimes 1 \) commutes with \( U M' \otimes 1 U^* \).

This theorem may be used to deduce the relationship between \( \{M,\alpha\} \) and \( \{N,\beta\} \) under the assumption that the dual covariant systems are isomorphic. This relationship leads to the definition:

**Definition 1.5.** Let \( \{M,\alpha\} \) and \( \{N,\beta\} \) be covariant systems over \( G \).

(i) an \( \alpha \) cocycle is a \( \sigma \)-strong \( ^* \) continuous map \( t \to u_t \) of \( G \) into the unitary group of \( M \) which satisfies:
\[ u_{t+s} = u_t a_t(u_s), \text{ for all } s, t \in G. \]

(ii) \([M,\alpha]\) and \([N,\beta]\) are said to be weakly equivalent iff there is an isomorphism \(\kappa\) of \(M\) onto \(N\) and a \(\beta\) cocycle \(t \mapsto u_t\) such that

\[ \kappa \alpha_t(x) = u_t \beta_t(\kappa x) u_t^*, \text{ for all } t \in G, x \in M. \]

Proposition 1.6. The covariant systems \([M,\alpha]\) and \([N,\beta]\) over \(G\) are weakly equivalent iff the dual covariant systems \([W^*\{M,\alpha\},\hat{\alpha}]\) and \([W^*\{N,\beta\},\hat{\beta}]\) are isomorphic.

Proof: see [2] proposition 3.5 and corollary 3.6 for the implication \(\Rightarrow\). The converse is easy in view of the uniqueness theorem.

We now give a (weak) characterization of crossed products:

Proposition 1.7. Let \([N,\beta]\) be a covariant system over \(\hat{G}\). Let \(M\) be a \(W^*\)-subalgebra of \(N\) and \(t \mapsto u(t)\) a strongly continuous unitary representation of \(G\) in \(N\) such that

(i) \(N\) is generated by \(M\) and \(\{u(t): t \in G\}\)

(ii) \(u(t)M u(t)^* = M\) for all \(t \in G\)

(iii) \(M \subset N^\beta\) and \(\beta(p u(t)) = <p, t> u(t), \text{ for all } t \in G, p \in \hat{G}.\)

We denote by \(\alpha\), the continuous action of \(G\) on \(M\) given by

\[ \alpha_t(x) = u(t)x u(t)^* \text{ for } t \in G \text{ and } x \in M. \]

Then there is an isomorphism \(\kappa\) of \([N,\beta]\) with \([W^*\{M,\alpha\},\hat{\alpha}]\) such that:
\[ \kappa x = \pi_\alpha(x), \text{ for all } x \in M \]

\[ \kappa u(t) = \lambda_\alpha(t), \text{ for all } t \in G. \]

**Proof:** This proof is essentially due to Takesaki. We assume \( N \) acts on \( H \). Let \( F \) be the Fourier transform mapping \( L^2(G) \) onto \( L^2(G) \).

Then

\[ x \rightarrow 1 \otimes F \pi_\beta(x) 1 \otimes F^* \]

is an isomorphism of \( N \) into the bounded operators on \( L^2(G;H) \). We have

\[ 1 \otimes F \pi_\beta(x) 1 \otimes F^* = x \otimes 1, \text{ for all } x \in M \]

\[ 1 \otimes F \pi_\beta(u(t)) 1 \otimes F^* = u(t) \otimes \lambda(t), \text{ for all } t \in G. \]

Now, let \( U \) be the unitary operator on \( L^2(G;H) \) given by

\[ (U\xi)(t) = u(t)\xi(t) \text{ for } \xi \in L^2(G;H). \]

Then

\[ U^* 1 \otimes F \pi_\beta(x) 1 \otimes F^* U = \pi_\alpha(x) \text{ for } x \in M \]

\[ U^* 1 \otimes F \pi_\beta(u(t)) 1 \otimes F^* U = \lambda_\alpha(t) \text{ for } t \in G. \]

Set \( \kappa(y) = U^* 1 \otimes F \pi_\beta(y) 1 \otimes F^* U \) for \( y \in N \). We have

\[ \kappa \hat{\beta}_p(y) = \hat{\alpha}_p(\kappa y), \text{ for all } y \in N, \ p \in \hat{G}. \]

Proposition 1.7 yields the bidual theorem.

**Theorem 1.8 (The Bidual Theorem).** Let \( \{M, \alpha\} \) be a covariant system
over $G$. Let $\beta$ be the continuous action of $G$ on $M \otimes B(L^2(G))$
defined by $\beta_t = \alpha_t \otimes \text{ad}\lambda(-t)$ (Here $\text{ad}_u(x) = uxu^*$). Then
\[ \{W\{M,\alpha\},\alpha\} \text{ is isomorphic to } \{M \otimes B(L^2(G)),\beta\} . \]

Proof: Since $\mu_\alpha(G)$ and $\lambda_\alpha(G)$ generate $1 \otimes B(L^2(G))$ it follows
that $\pi_\alpha(M)$, $\lambda_\alpha(G)$ and $\mu_\alpha(G)$ together generate $M \otimes B(L^2(G))$.
That is, $W\{M,\alpha\}$ and $\mu_\alpha(G)$ generate $M \otimes B(L^2(G))$. We also have
\[
\beta_t(x) = x, \text{ for all } t \in G, x \in W\{M,\alpha\} \\
\beta_t(\mu_\alpha(p)) = \langle p,t \rangle \lambda_\alpha(p), \text{ for all } p \in \hat{G}, t \in G .
\]
Moreover, $p \mapsto \text{ad}\mu_\alpha(p)$ gives a continuous action of $\hat{G}$ on $W\{M,\alpha\}$.
Namely,
\[
\hat{\alpha}_p(x) = \mu_\alpha(p)x\mu_\alpha(p)^*, \text{ for all } x \in W\{M,\alpha\}, p \in \hat{G} .
\]
By proposition 1.7, $\{M \otimes B(L^2(G)),\beta\}$ is isomorphic to
\[
\{W\{W\{M,\alpha\},\hat{\alpha}\},\hat{\alpha}\} .
\]

Remark: The Bidual theorem says that the bidual covariant system is
weakly equivalent to
\[
\{M \otimes B(L^2(G)), \alpha \otimes \text{id}\} .
\]
In case $M$ is properly infinite and $G$ is second countable, Takesaki
has shown (see [2] lemma 4.7) that $\{M \otimes B(L^2(G)), \alpha \otimes \text{id}\}$ is weakly
equivalent to $\{M,\alpha\}$.

Theorem 1.9. Let $\{M,\alpha\}$ be a covariant system over a second countable
group \( G \). If \( M \) is properly infinite then the bidual covariant system \( \{W^*W^*\{M,\alpha\},\alpha\} \) is weakly equivalent to \( \{M,\alpha\} \).

We now prepare for the strong version of the characterization of crossed products.

Proposition 1.10. Let \( \{N,\beta\} \) be a covariant system over \( \hat{G} \). Suppose \( t \rightarrow u(t) \) is a strongly continuous unitary representation of \( G \) in \( N \) such that \( \beta_p(u(t)) = \langle p, t \rangle u(t) \), for all \( t \in G, \ p \in \hat{G} \). Then \( N \) is generated by \( N^\beta \) and \( u(G) \).

Proof: We may assume that \( \beta \) is unitarily implemented by \( \rho \rightarrow V(\rho) \) on the Hilbert space \( H \) for \( N \). We have \( V(\rho)u(t)V(\rho)^* = \langle \rho, t \rangle u(t) \), for all \( t \in G, \ p \in \hat{G} \). We want to show that

\[
N = \{[N \cap V(\hat{G})'] \cup u(G)\}^\prime
\]

Equivalently,

\[
N' = [N' \cup V(\hat{G})]^\prime \cap u(G)' .
\]

Now, \( x \rightarrow u(t)xu(t)^* \), defines a continuous action \( \delta \) of \( G \) on the \( W^* \)-algebra \( [N' \cup V(\hat{G})]^\prime \) such that

\[
\delta_t(x) = x, \text{ for all } x \in N', \ t \in G
\]

\[
\delta_t(V(\rho)^*) = \langle \rho, t \rangle V(\rho)^*, \text{ for all } \rho \in \hat{G}, \ t \in G .
\]

Moreover, \( x \rightarrow V(\rho)^*xV(\rho) \) defines a continuous action \( \alpha \) of \( \hat{G} \) on \( N' \). We apply proposition 1.7 to conclude that \( \{W^*\{N',\alpha\},\hat{\alpha}\} \) is isomorphic to \( \{[N' \cup V(\hat{G})]^\prime,\delta\} \). The isomorphism carries \( \pi_\alpha(x) \) to
\[ x \text{ for } x \in N' \text{ and } \lambda_\alpha(p) \text{ to } V(p)^* \text{ for } p \in \hat{G} \]. By theorem 1.4 (the uniqueness theorem)

\[ W^*\{N', \alpha\}^G = \pi_\alpha(N') \]

or

\[ [N' \cup V(\hat{G})]^'' \cap u(G)' = N' \].

The characterization theorem for crossed products now follows.

\textbf{Theorem 1.11 (The characterization theorem).} Let \( \{N, \delta\} \) be a covariant system over \( \hat{G} \). Suppose that \( t \rightarrow u(t) \) is a strongly continuous unitary representation of \( G \) in \( N \) such that \( \beta_p(u(t)) = \langle p, t \rangle u(t) \), for all \( t \in G, \ p \in \hat{G} \). Let \( M = N^\delta = \{x \in N : \beta_p(x) = x, \text{ for all } p \in \hat{G}\} \). Then \( \alpha_t(x) = u(t)xu(t)^* \) for \( x \in M, \ t \in G \), defines a continuous action of \( G \) on \( M \) and there is an isomorphism \( \kappa \) of \( \{N, \delta\} \) with \( \{W^*\{M, \alpha\}, \hat{\alpha}\} \) such that

\[ \kappa x = \pi_\alpha(x), \text{ for all } x \in M \]

\[ \kappa u(t) = \lambda_\alpha(t), \text{ for all } t \in G \].

\textbf{Proof:} In view of propositions 1.7 and 1.10 we need only verify that \( u(t)Mu(t)^* = M \) for all \( t \in G \). But for \( x \in M, \ t \in G \) and \( p \in \hat{G} \)

\[ \beta_p(u(t)xu(t)^*) = \beta_p(u(t))x\beta_p(u(t)^*) \]

\[ = \langle p, t \rangle u(t)xu(t)^* \langle p, t \rangle \]

\[ = u(t)xu(t)^* \].

We have the following special case as a corollary.
Corollary 1.12. Let \( \{M,\alpha\} \) be a covariant system over \( G \). Suppose that \( p \to u(p) \) is a strongly continuous unitary representation of \( \hat{G} \) in the centre of \( M \) such that \( \alpha_t(u(p)) = \langle p, t \rangle u(p) \) for all \( t \in G \) and all \( p \in \hat{G} \). Let \( \chi_p \), for \( p \in \hat{G} \), denote the character in \( L^\infty(G) \) given by \( \chi_p(s) = \langle p, s \rangle \). Let \( t \to \sigma_t \) denote the continuous action of \( G \) on \( L^\infty(G) \) given by \( (\sigma_t(f))(s) = f(s-t) \) for \( f \in L^\infty(G) \). Then there is an isomorphism \( \kappa \) of \( \{M,\alpha\} \) with \( \{M^\alpha \otimes L^\infty(G), \text{id} \otimes \sigma\} \) such that

\[
\kappa x = x \otimes 1 \text{ for } x \in M^\alpha
\]

\[
\kappa u(p) = 1 \otimes \chi_p \text{ for } p \in \hat{G}.
\]

Proof: Since \( u(p) \) is in the centre of \( M \), the action \( \beta \) of \( \hat{G} \) on \( M^\alpha \) given by \( u(p) \) is trivial. By theorem 1.11 there is an isomorphism \( \kappa_1 \) of \( \{M,\alpha\} \) onto \( \{W^*(M^\alpha,\hat{\beta}),\hat{\beta}\} \) such that

\[
\kappa_1 x = \pi_\beta(x) \text{ for all } x \in M^\alpha,
\]

\[
\kappa_1 u(p) = \lambda_\beta(p) \text{ for all } p \in \hat{G}.
\]

Since \( \beta \) is the trivial action, \( \pi_\beta(x) = x \otimes 1 \) for \( x \in M^\alpha \). Moreover

\[
(\lambda_\beta(p)\xi)(q) = \xi(q-p) \text{ for } \xi \in L^2(\hat{G};H)
\]

(as usual, \( H \) is the Hilbert space of \( M \)). Using the Fourier transform mapping \( L^2(\hat{G}) \to L^2(G) \) we obtain an isomorphism \( \kappa \) of \( M \) into \( M^\alpha \otimes L^\infty(G) \) such that

\[
\kappa x = x \otimes 1, \text{ for all } x \in M^\alpha
\]

\[
\kappa u(p) = 1 \otimes \chi_p, \text{ for all } p \in \hat{G}.
\]
Since \( \{ \chi_p : p \in \hat{G} \} \) generates \( L^\infty(G) \), \( \kappa \) is onto. Moreover, 
\[
\kappa \circ \sigma_t \circ \kappa^{-1} = \text{id} \circ \sigma_t \quad \text{for} \quad t \in G.
\]

||
2. Flow built under a ceiling

We first review the classical "flow under a function construction (see [3] and [4]).

Let $(\omega, \Lambda, \mu)$ be a complete $\sigma$-finite measure space. Let $T: \omega \rightarrow \omega$ be a bimeasurable bijection such that $\mu \circ T$ is equivalent to $\mu$. Let $\phi: \omega \rightarrow [0, \infty)$ be $\Lambda$ measurable and assume that there is a partition

$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ of $\omega$ into $T$ invariant measurable sets and numbers $\varepsilon_n > 0$

for each $n$ such that $\phi(\omega) \geq \varepsilon_n$ for $\omega \in \Omega_n$ and $n = 1, 2, 3, \ldots$.

We denote by $(\mathbb{R}, L, m)$, Lebesgue measure on $\mathbb{R}$. Set

$\Omega_\phi = \{(\omega, s) \in \omega \times \mathbb{R}: 0 \leq s < \phi(\omega)\}$.

Let $(\Omega_\phi, \Lambda_\phi, \mu_\phi)$ be the completion of the restriction of $\Lambda \times L$ to $\Omega_\phi$ with respect to $\mu \times m$. Note that for each $\omega \in \Omega$ and $r \geq 0$, there is a unique integer $n \geq 0$ such that

$\phi(\omega) + \phi(T\omega) + \ldots + \phi(T^{n-1}\omega) \leq r < \phi(\omega) + \phi(T\omega) + \ldots + \phi(T^n\omega)$.

Similarly, for each $\omega \in \Omega$ and $r < 0$, there is a unique integer $n < 0$ such that

$-\phi(T^{n-1}\omega) - \ldots - \phi(T^n\omega) \leq r < -\phi(T^{n-1}\omega) - \ldots - \phi(T^{n+1}\omega)$.

So, if we set

$\phi_n(\omega) = \begin{cases} 
\phi(\omega) + \ldots + \phi(T^{n-1}\omega) & \text{for } n > 0 \\
0 & \text{for } n = 0 \\
-\phi(T^{n-1}\omega) - \ldots - \phi(T^n\omega) & \text{for } n < 0
\end{cases}$

then for each $(\omega, r) \in \Omega \times \mathbb{R}$, there is a unique integer $n$ such that
This observation allows us to define, for each $t \in \mathbb{R}$, a mapping $W^T_t$ of $\Omega_\phi$ by:

$$W^T_t(\omega, s) = (T^n_\omega, s + t - \phi_n(\omega))$$

where $n$ is the unique integer such that $(T^n_\omega, s + t - \phi_n(\omega)) \in \Omega_\phi$.

The mappings $W^T_t$ satisfy the following properties: (See [3] and [4]):

1) for each $t \in \mathbb{R}$, $W^T_t$ is a bimeasurable bijection of $\Omega_\phi$ such that $\mu_\phi \circ W^T_t$ is equivalent to $\mu_\phi$

2) for each $s,t \in \mathbb{R}$, $W^T_{t+s} = W^T_t \circ W^T_s$

3) if we equip $\Omega_\phi \times \mathbb{R}$ with the completion of $A_\phi \times L$ with respect to $\mu_\phi \times m$ then $(\omega, s, t) \rightarrow W^T_t(\omega, s)$ is a measurable mapping.

The action of $\mathbb{R}$ on $\Omega_\phi$ given by $t \rightarrow W^T_t$ is called the flow built on the transformation $T$, under the ceiling function $\phi$. Theorem 2 of [3] and theorem 3.1 of [4] give conditions under which an action of $\mathbb{R}$ on a measure space may be realized in this way. In the appendix we show that the arguments of [3] and [4] are valid under weaker conditions.

Our purpose in this section is firstly to recognize the "flow under a function" construction in operator algebraic terms, secondly to generalize the construction to non-abelian $W^*$-algebras and thirdly, to give necessary and sufficient conditions under which a covariant system over $\mathbb{R}$ may be realized in this way.

Note that properties 1), 2) and 3) of $W^T_t$ allow us to define a continuous action $\alpha$ of $\mathbb{R}$ on the abelian $W^*$ algebra $L^\infty(\mu_\phi)$ by:
\[ \alpha_t(f) = f \circ W_{-t}^T, \text{ for } f \in L^\infty(\mu_\phi). \]

We also have the action \( \theta \) of \( \mathbb{Z} \) (the integers) on the abelian algebra \( L^\infty(\mu) \) defined by:

\[ \theta(f) = f \circ T^{-1}, \text{ for } f \in L^\infty(\mu). \]

We would like to view the covariant system \( \{L^\infty(\mu_\phi), \alpha\} \) as arising from \( \{L^\infty(\mu), \theta\} \) without reference to the measure spaces. For this we introduce some notation. Let \( t \to \sigma_t \) denote the continuous action of \( \mathbb{R} \) on \( L^\infty(m) (= L^\infty(\mathbb{R})) \) given by

\[ (\sigma_t f)(s) = f(s-t), \text{ for } f \in L^\infty(m). \]

For \( s \in \mathbb{R} \), let \( \chi_s \) be the character \( \chi_s(r) = e^{irs} \) in \( L^\infty(m) \).

**Lemma 2.1.** There is a (unique) automorphism \( \tilde{\theta} \) of \( L^\infty(\mu) \otimes L^\infty(m) \) satisfying:

1. \( \tilde{\theta}(x \otimes l) = \theta x \otimes l, \text{ for all } x \in L^\infty(\mu) \)
2. \( \tilde{\theta}(1 \otimes \chi_s) = \theta(e^{is\phi}) \otimes \chi_s, \text{ for all } s \in \mathbb{R}. \)

The covariant system \( \{L^\infty(\mu_\phi), \alpha\} \) is isomorphic to the restriction of \( \text{id} \otimes \sigma \) to the fixed subalgebra of \( L^\infty(\mu) \otimes L^\infty(m) \) under \( \tilde{\theta} \).

**Proof:** Let \( (\Omega \times \mathbb{R}, A \times L, \mu \times m) \) denote the completion of \( A \times L \) with respect to the measure \( \mu \times m \). Define a mapping \( \tilde{T} \) of \( \Omega \times \mathbb{R} \) by

\[ \tilde{T}(\omega, r) = (T\omega, r - \phi(\omega)). \]
Then \( \tilde{T} \) is a bimeasurable bijection and \( \mu \times \tilde{m} \circ \tilde{T} \) is equivalent to \( \mu \times m \). This gives an automorphism \( \tilde{\theta} \) of \( L^\infty(\mu \times m) \) by

\[
\tilde{\theta} f = f \circ \tilde{T}^{-1} \quad \text{for } f \in L^\infty(\mu \times m).
\]

Identifying \( L^\infty(\mu \times m) \) with \( L^\infty(\mu) \otimes L^\infty(m) \) yields properties (i) and (ii) of \( \tilde{\theta} \). Note that the sets \( \tilde{T}^n \Omega \) for \( n \in \mathbb{Z} \) are disjoint and their union is \( \Omega \times \mathbb{R} \). Hence \( L^\infty(\mu) \) is isomorphic to the fixed subalgebra of \( L^\infty(\mu \times m) \) under \( \tilde{\theta} \). Moreover, under this identification, \( \alpha_t \) corresponds to the restriction of \( \text{id} \circ \sigma_t \).

We now propose to take the conclusion of lemma 2.1 as the definition of "flow under a function" for non-abelian \( \mathbb{W} \) algebras. For the existence of \( \tilde{\theta} \) we need

**Lemma 2.2.** Let \( G \) and \( H \) be locally compact abelian groups. Let \( \{M, \alpha\} \) be a covariant system over \( G \) and let \( (g, q) \rightarrow v(g, q) \) be a strongly continuous map from \( G \times \hat{H} \) into the unitary operators in the centre of \( M \) satisfying:

1. \[ v(g_1 + g_2, q) = v(g_1, q) \alpha_{g_1} (v(g_2, q)) \quad \text{for all } g_1, g_2 \in G \] and \( q \in \hat{H} \).

2. \[ v(g, q_1 + q_2) = v(g, q_1) v(g, q_2) \quad \text{for all } g \in G \] and \( q_1, q_2 \in \hat{H} \).

Then there is a (unique) continuous action \( \tilde{\alpha} \) of \( G \) on \( M \otimes L^\infty(H) \) such that
\[ \tilde{a}_g(x \otimes 1) = a_g \cdot x \otimes 1, \text{ for all } g \in G, \ x \in M \]
\[ \tilde{a}_g(1 \otimes \chi_q) = v(g,q) \theta \chi_q, \text{ for all } g \in G, \ q \in \hat{H} \]

(where \( \chi_q \) is the character \( \chi_q(h) = \langle q, h \rangle \)). Moreover, if \( L(H) \) denotes the \( W^* \)-algebra on \( L^2(H) \) generated by shift then there is a strongly continuous map \( g \rightarrow v_g \) of \( G \) into the unitary operators in the centre of \( M \otimes L(H) \) such that

\[ v_{g_1+g_2} = v_{g_1} \theta \id(v_{g_2}), \text{ for all } g_1, g_2 \in G \]
\[ \tilde{a}_g(x) = v_g \theta \id(x) v_g^*, \text{ for all } g \in G, \ x \in M \otimes L^\infty(H). \]

**Proof:** \( \hat{L}(H) \) is the \( W^* \)-algebra on \( L^2(\hat{H}) \) generated by the unitary operators \( \lambda(q) \) for \( q \in \hat{H} \) where

\[ (\lambda(q)\xi)(p) = \xi(p-q) \text{ for } \xi \in L^2(\hat{H}). \]

The Fourier transform \( F : L^2(H) \rightarrow L^2(\hat{H}) \) carries \( \chi_q \) to \( \lambda(q) \), i.e. \( F\chi_q F^* = \lambda(q) \). We also have \( FL(H)F^* = L^\infty(\hat{H}) \). Hence, it suffices to exhibit a strongly continuous map \( g \rightarrow w_g \) of \( G \) into the unitaries in the centre of \( M \otimes \hat{L}(H) \) such that

\[ w_{g_1+g_2} = w_{g_1} \theta \id(w_{g_2}), \text{ for all } g_1, g_2 \in G \]
\[ w_g(1 \otimes \lambda(q))w_g^* = v(g,q) \theta \lambda(q), \text{ for all } g \in G, \ q \in \hat{H}. \]

For this we assume that \( M \) acts on \( H \). Then \( M \otimes \hat{L}(H) \) acts on \( L^2(\hat{H};H) \). Since \( (g,q) \rightarrow v(g,q) \) is strongly continuous we can define \( \tilde{w}_g \) in the centre of \( M \otimes \hat{L}(H) \) by:

\[ (w_g \xi)(p) = v(g,p)\xi(p) \text{ for } \xi \in L^2(\hat{H},H). \]
It follows that $w_g$ is unitary and $g \to w_g$ is strongly continuous. The properties stated above for $w_g$ are satisfied.

We now define "flow under a function" in general.

**Definition 2.3.**

1) Let $\theta$ be an automorphism of a $\mathcal{W}^*$ algebra $N$. Let $\phi$ be a positive self-adjoint operator affiliated to the centre of $N$.

$\phi$ is called a $\theta$ ceiling (or just a ceiling if $\theta$ is understood) iff there is a partition of unity $\{e_i : i \in I\}$ in the centre of $N$ and numbers $\varepsilon_i > 0$ for $i \in I$ such that

$$\theta(e_i) = e_i, \quad \text{for all } i \in I$$

$$\phi e_i \geq \varepsilon_i e_i, \quad \text{for all } i \in I.$$

2) If $\phi$ is a $\theta$ ceiling, let $\tilde{\theta}$ be the automorphism of $N \otimes L^\infty(\mathbb{R})$ (given by lemma 2.2) which satisfies:

$$\tilde{\theta}(x \otimes 1) = \theta x \otimes 1, \quad \text{for all } x \in N$$

$$\tilde{\theta}(1 \otimes x_s) = \theta(e^{i s \phi}) \theta x_s, \quad \text{for all } s \in \mathbb{R}.$$ 

Set $M = [N \otimes L^\infty(\mathbb{R})]^\theta$ (the fixed algebra) and for $x \in M$ set

$$\alpha_t(x) = \text{id} \theta \sigma_t(x), \quad \text{for all } t \in \mathbb{R}.$$ 

The continuous action $\alpha$ of $\mathbb{R}$ on $M$ is called the flow built on the automorphism $\theta$ under the ceiling $\phi$.

We next show that $\{N, \theta\}$ is determined by $\{M, \alpha\}$ together with a map
s, t \mapsto \nu(s, t)\) of \([0, 2\pi) \times \mathbb{R}\) into the unitaries in the centre of \(M\).
For this we introduce the following notation: \(\delta\) is the automorphism of \(\ell^\infty(\mathbb{Z})\) defined by \((\delta f)(n) = f(n-1)\) for \(f \in \ell^\infty(\mathbb{Z})\). For \(0 \leq s < 2\pi\), \(\nu_s\) is the character (in \(\ell^\infty(\mathbb{Z})\)) defined by \(\nu_s(n) = e^{ins}\).

Lemma 2.4 (Reversal lemma). Suppose \(\{M, \alpha\}\) is the flow built on \(\{N, \theta\}\) under the ceiling \(\phi\). Let \(\tilde{\phi}\) be the automorphism of \(N \otimes \ell^\infty(\mathbb{R})\) as in definition 2.3. There exists a strongly continuous unitary representation \(s \mapsto u_s\) of (the group) \([0, 2\pi)\) in the centre of \(N \otimes \ell^\infty(\mathbb{R})\) and a family \(\{h_t: t \in \mathbb{R}\}\) of self-adjoint operators affiliated to the centre of \(M\) such that

(i) \(\tilde{\phi}(u_s) = e^{-is}u_s\), for all \(s \in [0, 2\pi)\)

(ii) \(\text{spec } h_t \subseteq \mathbb{Z}\) for all \(t \in \mathbb{R}\) and \(h_t \geq 0\) if \(t \geq 0\)

(iii) \(\text{id } \circ (u_s)^{ish} = e^{it}u_s\), for all \(s \in [0, 2\pi)\), \(t \in \mathbb{R}\).

(iv) if \(\tilde{\alpha}\) denotes the continuous action of \(\mathbb{R}\) on \(M \otimes \ell^\infty(\mathbb{Z})\) given by lemma 2.2 applied to \(\{M, \alpha\}\) and the map \(t, s \mapsto e^{it}\), then there is an isomorphism \(\pi\) of \(N \otimes \ell^\infty(\mathbb{R})\) with \(M \otimes \ell^\infty(\mathbb{Z})\) such that

\[\pi(x) = x \otimes 1, \text{ for all } x \in M\]

\[\pi(u_s) = 1 \otimes \nu_s, \text{ for all } s \in [0, 2\pi)\]

\[\pi \circ \tilde{\phi} \circ \pi^{-1} = \text{id } \circ \delta\]

\[\pi \circ \text{id } \circ \sigma_t \circ \pi^{-1} = \tilde{\alpha}_t, \text{ for all } t \in \mathbb{R}\, .\]

(v) \(\{N, \theta\}\) is isomorphic to the restriction of \(\text{id } \circ \delta\) to the fixed subalgebra under \(\tilde{\alpha}\) .
Proof: We first show that it suffices to prove the existence of $s \to u_s$ and $t \to h_t$ satisfying (i), (ii) and (iii) in case the centre of $N$ is $\sigma$-finite. To see this, note that if $p$ is a $\sigma$-finite projection in the centre of $N$ then $q = \bigvee_{n \in \mathbb{Z}} \theta^n(p)$ is $\sigma$-finite and $\theta$ invariant. Hence $q \theta 1$ is in the centre of $M$ and is $\sigma$ invariant. Moreover, the restriction of $\alpha$ to $Mq \theta 1$ is the flow built on the restriction of $\theta$ to $Nq$ under the ceiling $\phi q$. So, by choosing a partition of unity in the centre of $N$ consisting of $\sigma$-finite $\theta$ invariant projections, the general case follows from the $\sigma$-finite case.

Now suppose that the centre $Z$ of $N$ is $\sigma$-finite. We can find a complete $\sigma$-finite measure space $(\Omega, A, \mu)$ and a bimeasurable bijection $T$ of $\Omega$ with $\mu \circ T$ equivalent to $\mu$ so that $Z$ is isomorphic to $L^\infty(\mu)$ and under this identification $\theta$ is given by

$$\theta f = f \circ T^{-1} \text{ for } f \in L^\infty(\mu).$$

We may also assume that $\phi$ corresponds to a measurable function $\omega \to \phi(\omega)$ and there is a partition $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ of $\Omega$ into measurable sets and numbers $\epsilon_n > 0$ such that $\phi(\omega) \geq \epsilon_n$ for $n \in \Omega_n$.

As in the proof of lemma 2.1, $Z \theta L^\infty(R)$ is identified with $L^\infty(\mu \times m)$ and $\tilde{\theta}$ is given by $\tilde{T}: \Omega \times R \to \Omega \times R$ where

$$\tilde{T}(\omega, r) = (T\omega, r - \phi(\omega)).$$

Now set $\Omega_\phi = \{ (\omega, r) \in \Omega \times R: 0 \leq r < \phi(\omega) \}$. Then

$$\Omega \times R = \bigcup_{n \in \mathbb{Z}} T^n(\Omega_{\phi}).$$
is a partition of $\Omega \times \mathbb{R}$ into $A \times L$ measurable sets. For $0 \leq s < 2\pi$, $k \in \mathbb{Z}$ and $(\omega, r) \in \tilde{T}^k(\Omega_{\phi})$ set

$$u_s(\omega, r) = e^{iks}.$$ 

For $t \in \mathbb{R}$, $k, n \in \mathbb{Z}$, $(\omega, r) \in \tilde{T}^k(\Omega_{\phi})$ and $(\omega, r-t) \in \tilde{T}^{k+n}(\Omega_{\phi})$ set

$$h_t(\omega, r) = n.$$ 

Note that $u_s \cdot \tilde{T}^{-1} = e^{-is}u_s$, for all $s \in [0, 2\pi)$. Also, $h_t$ is integer valued, non negative if $t \geq 0$ and $h_t \cdot \tilde{T}^{-1} = h_t$ for all $t \in \mathbb{R}$. We have, for all $s \in [0, 2\pi)$, $t \in \mathbb{R}$ and $\omega, r \in \Omega \times \mathbb{R}$

$$u_s(\omega, r-t) = e^{-ish_t(\omega, r)}u_s(\omega, r).$$ 

In terms of automorphisms we have

$$\tilde{\theta}(u_s) = e^{-is}u_s,$$ for all $s \in [0, 2\pi)$

and

$$\text{id} \sigma_t(u_s) = e^{ish_t}u_s,$$ for all $s \in [0, 2\pi)$, $t \in \mathbb{R}$. 

Hence, parts (i), (ii) and (iii) are proven. Part (iv) now follows by applying corollary 1.12 to $N \otimes L^\infty(\mathbb{R})$ with the action $\tilde{\theta}$ and the representation $s \mapsto u_s$. Part (v) simply states that

$$[N \otimes L^\infty(\mathbb{R})]^\text{id} \sigma = N \otimes 1.$$ 

Remark: This lemma shows that centre $[N \otimes L^\infty(\mathbb{R})]^{\tilde{\theta}}$ equals

$$(\text{centre } N) \otimes L^\infty(\mathbb{R})^{\tilde{\theta}}$$ since both these subalgebras correspond to

(centre $M$) $\otimes 1$ under $\pi$.

We now use lemma 2.4 to prove a "lifting lemma".
Lemma 2.5. Let \( \{M, \alpha\} \) be a covariant system over \( \mathbb{R} \). Let \( M_1 \) be a \( \mathcal{W}^* \)-subalgebra of \( M \) such that \( \alpha_t(M_1) = M_1 \), for all \( t \in \mathbb{R} \) and centre \( M_1 \subset \text{centre} \ M \). Set \( \alpha_t^1(x) = \alpha_t(x) \), for all \( t \in \mathbb{R} \), \( x \in M_1 \).

Suppose \( \{M_1, \alpha^1\} \) is isomorphic to the flow built on \( \{N_1, \theta_1\} \) under \( \phi_1 \). Then there is an imbedding \( \kappa \) of \( \{N_1, \theta_1\} \) into a covariant system \( \{N, \theta\} \) over \( \mathbb{Z} \) such that centre \( \kappa N_1 \subset \text{centre} \ N \) and \( \{M, \alpha\} \) is isomorphic to the flow built on \( \{N, \theta\} \) under \( \phi = \kappa \phi_1 \).

Proof: Using lemma 2.4 we obtain a family \( \{h_t: t \in \mathbb{R}\} \) of self adjoint operators affiliated to the centre of \( M_1 \) with \( \text{spec} \ h_t \subset \mathbb{Z} \) and an isomorphism \( \pi_1 \) of \( N_1 \Theta L^\infty(\mathbb{R}) \) with \( M_1 \Theta L^\infty(\mathbb{E}) \) such that, in the notation of lemma 2.4:

\[
\pi_1 \circ \theta_1 \circ \pi_1^{-1} = \text{id} \Theta \delta
\]

\[
\pi_1 \circ \text{id} \Theta \sigma_t \circ \pi_1^{-1} = \alpha_t^1, \text{ for all } t \in \mathbb{R}.
\]

Since centre \( M_1 \subset \text{centre} \ M \) and since the map \( t, s \rightarrow e^{ish} \) satisfies the conditions of lemma 2.2, we can extend \( \alpha^1 \) to \( \tilde{\alpha} \) on \( M \Theta L^\infty(\mathbb{E}) \) satisfying:

\[
\tilde{\alpha}_t(x \Theta l) = \alpha_t(x) \Theta l, \text{ for all } x \in M
\]

\[
\tilde{\alpha}_t(l \Theta \nu_s) = e^{ish} \Theta \nu_s, \text{ for all } t \in \mathbb{R}, s \in [0, 2\pi).
\]

Let \( N = [M \Theta L^\infty(\mathbb{E})]^{\tilde{\alpha}} \) and for \( x \in N \) set

\[
\theta(x) = \text{id} \Theta \delta(x).
\]

Set \( \kappa x = \pi_1(x \Theta l)^e \) for \( x \in N_1 \). Then \( \kappa \) is an imbedding of \( \{N_1, \theta_1\} \) and
centre $\kappa N_1 \subset [(centre M_1) \otimes L^\infty(\mathbb{Z})]^\sim \subset [(centre M) \otimes L^\infty(\mathbb{Z})]^\sim \subset centre N$.

To conclude the proof, set $v_r = \pi_\alpha (1 \otimes \chi_r)$ for $r \in \mathbb{R}$. Then $r \mapsto v_r$ is a strongly continuous unitary representation of $\mathbb{R}$ in the centre of $M \otimes L^\infty(\mathbb{Z})$ and

$$\tilde{\alpha}_t (v_r) = e^{-irt} v_r, \text{ for all } t, r \in \mathbb{R}$$

$$\text{id } \delta(v_r) = \kappa \circ \theta_\alpha (e^{ir\phi}) v_r, \text{ for all } r \in \mathbb{R}.$$

By corollary 1.12 there is an isomorphism $\pi$ of $M \otimes L^\infty(\mathbb{Z})$ with $N \otimes L^\infty(\mathbb{R})$ such that:

$$\pi \circ \tilde{\alpha}_t = \pi^{-1} = \text{id } \sigma_t, \text{ for all } t \in \mathbb{R}$$

$$\pi \circ \text{id } \delta = \pi^{-1} = \theta$$

where

$$\tilde{\theta}(x \otimes 1) = \theta x \otimes 1, \text{ for all } x \in N$$

$$\tilde{\theta}(1 \otimes \chi_r) = \theta (e^{ir\phi}) \otimes \chi_r, \text{ for all } r \in \mathbb{R}.$$

Thus, $\{M, \alpha\}$ is isomorphic to the restriction of $\text{id } \sigma$ to the fixed subalgebra of $N \otimes L^\infty(\mathbb{R})$ under $\tilde{\theta}$.

The main result of this section is

**Theorem 2.6.** A covariant system $\{M, \alpha\}$ over $\mathbb{R}$ is isomorphic to a flow built under a ceiling iff the restriction of $\alpha$ to the centre of $M$ is nowhere trivial (i.e. if $e$ is a projection in the centre of $M$ such that $\alpha_t (xe) = xe$ for all $x$ in the centre of $M$ then $e = 0$).
Proof: Assume that \( \{M, \alpha\} \) is isomorphic to the flow built on \( \{N, \theta\} \) under the ceiling \( \phi \). The existence of an isomorphism of \( N \otimes L^\infty(R) \) with \( M \otimes L^\infty(E) \) as in lemma 2.4 shows that the restriction of \( \alpha \) to the centre of \( M \) is isomorphic to the flow built on the restriction of \( \theta \) to the centre of \( N \) under the ceiling \( \phi \). Hence it suffices to show that when \( N \) is abelian, the flow built on \( \{N, \theta\} \) under \( \phi \) is nowhere trivial. Now let \( e \) be a projection in \( N \otimes L^\infty(R) \) such that \( \theta(e) = e \) and \( \text{id} \otimes \sigma_t(xe) = xe \) for all \( x \) in \( [N \otimes L^\infty(R)]^\theta \). Then \( e \) is of the form \( f \theta 1 \) for \( f \in N^\theta \) and the flow built on the restriction of \( \theta \) to \( N_f \) under \( \phi^f \) is trivial. Hence it suffices to show that when \( N \) is abelian the flow built on \( \{N, \theta\} \) under \( \phi \) is not the trivial flow.

Similarly, it suffices to show that when \( N \) is abelian and \( \sigma \)-finite, the flow built on \( \{N, \theta\} \) under \( \phi \) is not trivial. In this case, we choose a complete \( \sigma \)-finite measure space \( (\Omega, \mathcal{A}, \mu) \) and a bi-measurable bijection \( T \) of \( \Omega \) with \( \mu \circ T \) equivalent to \( \mu \) such that \( N \) is isomorphic to \( L^\infty(\mu) \) and \( \theta \) is given by \( \theta f = f \circ T^{-1} \) for \( f \in L^\infty(\mu) \). Lemma 2.1 now shows that the flow built on \( \{N, \theta\} \) under \( \phi \) is isomorphic to the flow given by the action \( t \to W_t^\phi \) of \( R \) on \( \Omega^\phi = \{ (\omega, s) \in \Omega \times R: 0 \leq s < \phi(\omega) \} \). It is easy to see that this flow is non-trivial.

For the converse, lemma 2.5 shows that it suffices to assume that \( M \) is abelian. We may also assume that \( M \) is \( \sigma \)-finite. To see this, let \( p \) be a \( \sigma \)-finite projection in \( M \). Then

\[
e = \bigvee_{t \in \mathbb{Q}} \alpha_t(p) \quad (\mathbb{Q} \text{ is the rationals})
\]

is \( \sigma \)-finite and \( \alpha \) invariant. Now let \( \{e_i: i \in I\} \) be a partition
of unity in $M$ consisting of $\sigma$-finite $\alpha$ invariant projections. We assume that for each $i \in I$, the restriction of $\alpha$ to $M_{e_i}$ is isomorphic to the flow built on $\{N_i, \theta_i\}$ under $\phi_i$. Set

$$N = \sum_{i \in I} N_i$$

$$\theta x = \sum_{i \in I} \theta_i(x_i) \text{ for } x = \sum_{i \in I} x_i \in N$$

$$\phi = \sum_{i \in I} \phi_i$$

Then $(M, \alpha)$ is isomorphic to the flow built on $\{N, \theta\}$ under $\phi$.

To conclude the proof we refer to the appendix where the result is proven for the case $M$ abelian and $\sigma$-finite.

We now characterize those actions which may be realized as a flow under a constant ceiling.

**Theorem 2.7.** A covariant system $(M, \alpha)$ over $R$ is isomorphic to a flow built under a constant ceiling of height $c$ iff there is a unitary $u$ in the centre of $M$ such that

$$\alpha_t(u) = e^{\frac{-it}{c}} u, \text{ for all } t \in R.$$

**Proof:** Let $\theta$ be an automorphism of $N$ and let $\tilde{\theta}$ be the automorphism of $N \otimes L^\infty(R)$ satisfying

$$\tilde{\theta}(x \otimes 1) = \theta x \otimes 1, \text{ for all } x \in N$$

$$\tilde{\theta}(1 \otimes \chi_s) = e^{isc} \theta \chi_s, \text{ for all } s \in R.$$

Then $u = 1 \otimes \chi_{2\pi}$ is fixed by $\tilde{\theta}$ and
\[
\text{id } \sigma_t(u) = e^{\frac{-2\pi i t}{c}} u, \text{ for all } t \in \mathbb{R}.
\]

For the converse, let \( M_1 \) be the \( \mathcal{W}^* \)-subalgebra of \( M \) generated by \( u \).

Then \( \alpha_t(M_1) = M_1 \), for all \( t \in \mathbb{R} \) and \( M_1 \subseteq \text{centre } M \). Hence, by lemma 2.5 it suffices to show that the restriction of \( \alpha \) to \( M_1 \) is isomorphic to a flow built under the constant ceiling \( c \). Since \( \alpha_c(u) = u \), we get a continuous action \( \beta \) of the group \( [0,c) \) (mod \( c \)) on \( M_1 \) by

\[
\beta_t(x) = \alpha_t(x), \text{ for all } x \in M_1, \quad 0 \leq t < c.
\]

By corollary 1.12 there is an isomorphism of \( M_1 \) with \( L^\infty[0,c) \) which carries \( u \) to the function \( v(s) = e^{is2\pi/c} \) and carries \( \beta \) to the usual action of \( [0,c) \) on \( L^\infty[0,c) \). But \( L^\infty[0,c) \) is isomorphic to the fixed subalgebra of \( L^\infty(\mathbb{R}) \) under \( \sigma_c \). This composite isomorphism carries \( M_1 \) to \( L^\infty(\mathbb{R})^{\sigma_c} \) and carries the restriction of \( \alpha \) to the restriction of \( \sigma \). Now, take \( N_1 = \mathfrak{c} \), \( \theta_1 = \text{identity} \). Then the restriction of \( \alpha \) to \( M_1 \) is isomorphic to the flow built on \( \{N_1, \theta_1\} \) under the constant ceiling \( c \).
3. Uniqueness of Flow Under a Ceiling

Let \( \{M, \phi\} \) be the flow built on \( \{N, \theta\} \) under the ceiling \( \phi \).

In this section we investigate the extent to which the isomorphism class of \( \{M, \phi\} \) determines \( \{N, \theta\} \) and \( \phi \). We first exhibit two ways of modifying \( \{N, \theta\} \) and \( \phi \) so that the resulting flows are isomorphic. Next we state and prove our uniqueness result. Finally, we give a uniqueness result in the special case of a constant ceiling.

**Lemma 3.1.** Let \( \theta \) be an automorphism of \( N \) and \( \theta \) a \( \theta \) ceiling.

Suppose \( f \) is a self-adjoint operator affiliated to the centre of \( N \) such that \( \psi = \phi + \theta(f) - f \) is also a ceiling operator. Then the flow built on \( \{N, \theta\} \) under \( \phi \) is isomorphic to the flow built on \( \{N, \theta\} \) under \( \psi \).

**Proof.** Let \( \tilde{\theta} \) be the automorphism of \( N \Theta L^\infty(R) \) as in definition 2.3.

Set \( v_s = \theta(e^{isf}) \Theta \chi_s \) for \( s \in R \). Since

\[
\text{id} \circ \sigma_t(v_s) = e^{-ist}v_s, \text{ for all } s, t \in R,
\]

by corollary 1.12, there is an isomorphism \( \kappa \) of \( N \Theta L^\infty(R) \) with \( N \Theta L^\infty(R) \) such that

\[
\kappa(x \Theta 1) = x \Theta 1, \text{ for all } x \in N
\]

\[
\kappa(v_s) = 1 \Theta \chi_s, \text{ for all } s \in R.
\]

In particular

\[
\kappa \circ \text{id} \circ \sigma_t \circ \kappa^{-1} = \text{id} \circ \sigma_t, \text{ for all } t \in R
\]

\[
\kappa \circ \tilde{\theta} \circ \kappa^{-1}(x \Theta 1) = \theta x \Theta 1, \text{ for all } x \in N
\]
\[ \kappa \circ \theta \circ \kappa^{-1}(1 \otimes \chi_s) = \theta(e^{i s \psi}) \otimes \chi_s, \text{ for all } s \in \mathbb{R}. \]

Hence, \( \kappa \) gives an isomorphism between the flow built on \( \{N, \theta\} \) under \( \phi \) and the flow built on \( \{N, \theta\} \) under \( \psi \).

Our second modification deals with "cut down" automorphisms. For this we introduce the notion of recurrent projection.

**Definition 3.2.** Let \( \theta \) be an automorphism of \( N \) and let \( e \) be a projection in the centre of \( N \). \( e \) is said to be recurrent iff

\[ e \leq \bigvee_{n<0} \theta^n(e) \text{ and } e \leq \bigvee_{n>0} \theta^n(e). \]

There is a canonical way to partition \( e \) as \( e = \sum_{n=1}^{\infty} e_n \) where each \( e_n \) is in the centre of \( N \) and satisfies

\[ \theta(e_n) \leq e \]

for \( n \geq 2 \) \( \theta^j(e_n)e = 0 \) for \( j = 1, 2, \ldots, n - 1 \) and \( \theta^n(e_n) \leq e \).

We also have \( e = \sum_{n=1}^{\infty} \theta^n(e_n) \) (see [1] definition 5.4.1). It follows from the properties of \( \{e_n : n = 1, 2, \ldots\} \) that \( \{\theta^j(e_n) : n = 1, 2, \ldots ; j = 0, 1, 2, \ldots, n - 1\} \) and \( \{\theta^j(e_n) : n = 1, 2, \ldots, j = 1, 2, \ldots, n\} \) are orthogonal families and

\[ \bigvee_{n \in \mathbb{Z}} \theta^n(e) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \theta^j(e_n) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \theta^j(e_n). \]

We can define an automorphism \( \theta_e \) of \( N_e \) by

\[ \theta_e(x) = \sum_{n=1}^{\infty} \theta^n(x e_n) \text{ for } x \in N_e. \]
\( \theta_e \) is called the cut down or reduction of \( \theta \) to \( N_e \).

We can also define the cut down or reduction of a \( \theta \) ceiling \( \phi \) by

\[
\phi_e = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n-1} \theta^{-m}(\phi) \right) e_n.
\]

It is easy to see that \( \phi_e \) is a \( \theta_e \) ceiling.

**Lemma 3.3.** Let \( \theta \) be an automorphism of \( N \) and let \( \phi \) be a \( \theta \) ceiling. Let \( e \) be a recurrent projection in the centre of \( N \) with \( \sqrt{\theta^n(e)} = 1 \). Then the flow built on \( \{N, \theta\} \) under \( \phi \) is isomorphic to the flow built on \( \{N_e, \theta_e\} \) under \( \phi_e \).

**Proof:** Let \( \tilde{\theta} \) be the automorphism of \( N \otimes L^\infty(R) \) as in definition 2.3. Then \( e \otimes 1 \) is recurrent for \( \tilde{\theta} \). Moreover, if \( e = \sum_{n=1}^{\infty} e_n \) is the canonical partition of \( e \) then \( e \otimes 1 = \sum_{n=1}^{\infty} e_n \otimes 1 \) is the canonical partition of \( e \otimes 1 \). For \( x \in N_e \) we have

\[
\tilde{\theta}_{e \otimes 1}(x \otimes 1) = \sum_{n=1}^{\infty} \tilde{\theta}^n(x_n \otimes 1) = \theta_e(x) \otimes 1
\]

and for \( s \in R \) we have

\[
\tilde{\theta}_{e \otimes 1}(e \otimes x_s) = \sum_{n=1}^{\infty} \tilde{\theta}^n(e_n \otimes x_s) = \theta_e(e^{is\phi}) \otimes x_s.
\]

Hence the flow built on \( \{N_e, \theta_e\} \) under \( \phi_e \) is isomorphic to the restriction of \( \text{id} \otimes \sigma \) to the fixed subalgebra of \( N_e \otimes L^\infty(R) \) under \( \tilde{\theta}_{e \otimes 1} \).

Now, if \( x \in [N \otimes L^\infty(R)]^{\tilde{\theta}} \) then \( x(e \otimes 1) \in [N_e \otimes L^\infty(R)]^{\tilde{\theta}_{e \otimes 1}} \).

Furthermore, if \( x(e \otimes 1) = 0 \) then \( \tilde{\theta}^n(x(e \otimes 1)) = x\theta^h(e) \otimes 1 = 0 \) for
all $n \in \mathbb{Z}$. Since $\bigvee_{n \in \mathbb{Z}} \theta^n(e) = 1$, it follows that $x = 0$.

Hence, the map $x \mapsto x(e \circ 1)$ is an isomorphism of $[N \circ L^\infty(\mathbb{R})]_e$ into $[N \circ L^\infty(\mathbb{R})]_{e \circ 1}$. To show that this map is onto, let $x \in [N \circ L^\infty(\mathbb{R})]_{e \circ 1}$. Since

$$1 = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \theta^k(e_n)$$

we set

$$y = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \theta^k(xe_n \circ 1).$$

Then $y(e \circ 1) = x$ and

$$\tilde{\theta}(y) = \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \tilde{\theta}^k(xe_n \circ 1)$$

$$= \sum_{n=1}^{\infty} \tilde{\theta}^n(xe_n \circ 1) + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \tilde{\theta}^k(xe_n \circ 1)$$

$$= \theta_{e \circ 1}(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \theta^k(xe_n \circ 1)$$

$$= x + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \theta^k(xe_n \circ 1)$$

$$= y.$$

Thus, $x \mapsto x(e \circ 1)$ is onto. Since this isomorphism intertwines the restrictions of $id \circ \sigma$ the lemma is proven. ||

The main result of this section is

**Theorem 3.4.** The flow built on $\{N_1, \theta_1\}$ under $\phi_1$ is isomorphic to the flow built on $\{N_2, \theta_2\}$ under $\phi_2$ iff there exist

(i) recurrent projections $e_j$ in the centre of $N_j$ with $\bigvee_{n \in \mathbb{Z}} \theta_j^n(e_j) = 1$ for $j = 1, 2$. 


(ii) an isomorphism \( \kappa \) of \( \{(N_1)e_1, (\theta_1)e_1\} \) with \( \{(N_2)e_2, (\theta_2)e_2\} \)

(iii) a self-adjoint operator \( g \) affiliated to the centre of \( (N_1)e_1 \) such that
\[
(\phi_1)e_1 = \kappa(\phi_2)e_2 + (\theta_1)e_1(g) - g.
\]

Proof: Lemmas 3.1 and 3.3 show that the stated conditions imply that the flows are isomorphic.

For the converse we begin with

Lemma 3.5. The flow built on \( \{N_1, \theta_1\} \) under \( \phi_1 \) is isomorphic to the flow built on \( \{N_2, \theta_2\} \) under \( \phi_2 \) iff there is a \( \mathcal{W}^* \)-algebra \( Q \) with commuting automorphisms \( \gamma_1, \gamma_2 \) such that:

1. there are imbeddings \( \pi_j \) of \( \{N_j, \theta_j\} \) into \( \{Q, \gamma_j\} \) for \( j = 1, 2 \).
2. \( \pi_1(N_1) = \gamma_2 \) and \( \pi_2(N_2) = \gamma_1 \).
3. the centre of \( \pi_j(N_j) \) is contained in the centre of \( Q \) for \( j = 1, 2 \).
4. there is a strongly continuous unitary representation \( r \to v_r \) of \( \mathbb{R} \) in the centre of \( Q \) such that
\[
\gamma_1(v_r) = \pi_1(\theta_1 e^{	ext{i}r \phi_1})v_r, \text{ for all } r \in \mathbb{R}
\]
\[
\gamma_2(v_r) = \pi_2(\theta_2 e^{-\text{i}r \phi_2})v_r, \text{ for all } r \in \mathbb{R}.
\]

Proof: Assuming the conditions are satisfied, let \( \tilde{\gamma}_j \) (for \( j = 1, 2 \)) be the automorphism of \( Q \otimes L^\infty(\mathbb{R}) \) satisfying:
\[ \gamma_j(x \theta 1) = \gamma_j x \theta 1, \text{ for all } x \in Q \]
\[ \gamma_j(1 \theta x_r) = \pi_j(\theta_j e^{-rj}) \theta x_r, \text{ for all } r \in \mathbb{R} \].

Set \( u_r = \nu_r \theta x_r \) for \( r \in \mathbb{R} \). Then
\[ \gamma_1(u_r) = u_r, \text{ for all } r \in \mathbb{R} \]
\[ \gamma_2 \theta \text{id}(u_r) = \pi_2(\theta_2 e^{-rj}) u_r, \text{ for all } r \in \mathbb{R} \].

By corollary 1.12 there is an automorphism \( \pi \) of \( Q \theta L^\infty(\mathbb{R}) \) such that
\[ \pi(x \theta 1) = x \theta 1, \text{ for all } x \in Q \]
\[ \pi(1 \theta x_r) = u_r, \text{ for all } r \in \mathbb{R} \].

In particular,
\[ \pi^{-1} \cdot \text{id} \theta \sigma_t \cdot \pi = \text{id} \theta \sigma_t, \text{ for all } t \in \mathbb{R} \]
\[ \pi^{-1} \cdot \gamma_1 \cdot \pi = \gamma_1 \theta \text{id} \]
\[ \pi^{-1} \cdot \gamma_2 \theta \text{id} \cdot \pi = \gamma_2 \].

Now, the restriction of \( \text{id} \theta \sigma \) to \( [(Q \theta L^\infty(\mathbb{R}))^\perp]_1 \) is isomorphic to the flow built on \( \{N_1, \theta_1\} \) under \( \phi_1 \). Similarly, the restriction of \( \text{id} \theta \sigma \) to \( [(Q \theta L^\infty(\mathbb{R}))^\perp]_2 \) is isomorphic to the flow built on \( \{N_2, \theta_2\} \) under \( \phi_2 \). Hence, \( \pi \) gives an isomorphism between the two flows.

For the converse, let \( \tilde{\theta}_j \) be the automorphism of \( N_j \theta L^\infty(\mathbb{R}) \) which satisfies:
\[ \tilde{\theta}_j(x \theta 1) = \theta_j x \theta 1, \text{ for all } x \in N_j \]
\[ \tilde{\theta}_j(1 \theta x_r) = \theta_j(e^{-rj}) \theta x_r, \text{ for all } r \in \mathbb{R} \].
We apply the reversal lemma (lemma 2.4) to \([N_2, \theta_2]\) and \(\phi_2\) to obtain an isomorphism \(\pi_0\) of \([N_1 \theta L^\infty(R)]^\theta_1 \theta \ell^\infty(Z)\) with \(N_2 \theta L^\infty(R)\) and a family \(\{h_t : t \in R\}\) of self-adjoint operators affiliated to the centre of \([N_1 \theta L^\infty(R)]^\theta_1\) with \(\text{spec } h_t \subset Z\) for all \(t \in R\) and \(h_t > 0\) for \(t > 0\) such that

\[
\pi_0^{-1} \circ \theta_2 \circ \pi_0 = \text{id} \oplus \delta
\]

\[
\pi_0^{-1} \circ \text{id} \circ \sigma_t \circ \pi_0(x \oplus l) = (\text{id} \circ \sigma_t(x)) \oplus l,
\]

for all \(x \in [N_1 \theta L^\infty(R)]^\theta_1, t \in R\)

\[
\pi_0^{-1} \circ \text{id} \circ \sigma_t \circ \pi_0(1 \oplus \nu_s) = e^{i t \theta \nu_s}, \text{ for all } s \in [0, 2\pi), t \in R.
\]

Since centre \([N_1 \theta L^\infty(R)]^\theta_1 = [\text{(centre } N_1) \theta L^\infty(R)]^\theta_1\) we can use the map \(t, s \rightarrow e^{i t \theta \nu_s}\) to define a continuous action \(\beta\) of \(R\) on \(N_1 \theta L^\infty(R) \oplus l^\infty(Z)\) (lemma 2.2) satisfying:

\[
\beta_t(x \oplus l) = (\text{id} \oplus \sigma_t(x)) \oplus l, \text{ for all } x \in N_1 \theta L^\infty(R), t \in R
\]

\[
\beta_t(1 \oplus \nu_s) = e^{i t \theta \nu_s}, \text{ for all } s \in [0, 2\pi), t \in R.
\]

Note that \(\beta_t\) commutes with both \(\theta_1 \oplus \text{id}\) and \(\text{id} \oplus \text{id} \oplus \delta\) for all \(t \in R\). We set

\[
Q = [N_1 \theta L^\infty(R) \oplus l^\infty(Z)]^\beta.
\]

For \(x \in N_1\), set \(\pi_1(x) = x \oplus l \oplus l\). For \(x \in N_2\), set \(\pi_2(x) = \pi_0(x \oplus l)\).

Let \(\gamma_1\) be the restriction of \(\theta_1 \oplus \text{id}\) to \(Q\) and let \(\gamma_2\) be the restriction of \(\text{id} \oplus \text{id} \oplus \delta\) to \(Q\). Finally, for \(r \in R\) set

\[
\nu_r = \pi_0(1 \oplus \chi_r^*)1 \oplus \chi_r \oplus l.
\]
It is easy to verify that the conditions of the lemma are satisfied. 

Now, define $\pi, \gamma_1$ and $\gamma_2$ as in the proof of lemma 3.5. As previously noted, the restriction of $\id \otimes \sigma$ to $[Q^1 \otimes L^\infty(\mathbb{R})]$ is isomorphic to the flow built $\{N_2, \theta_2\}$ under $\phi_2$. Hence, by the reversal lemma (lemma 2.4) there is a strongly continuous unitary representation $s \mapsto u_s$ of $[0, 2\pi)$ in the centre of $Q^1 \otimes L^\infty(\mathbb{R}) = (centre Q) \gamma_1 \otimes L^\infty(\mathbb{R})$ such that $\gamma_2(u_s) = e^{-is} u_s$ for all $s \in [0, 2\pi)$. Moreover, for each $t \in \mathbb{R}$ and $s \in [0, 2\pi)$, $\id \otimes \sigma_t(u_s) = e^{ish} u_s$, where $h_t$ is a self-adjoint operator affiliated to the centre of $Q^1 \otimes L^\infty(\mathbb{R})$ with $\spec h_t \subset \mathcal{E}$ and $h_t \geq 0$ if $t \geq 0$. We therefore have:

\[
\gamma_2 \otimes \id(\pi(u_s)) = e^{-is} \pi(u_s), \quad \text{for all } s \in [0, 2\pi) \\
\gamma_1(\pi(u_s)) = \pi(u_s), \quad \text{for all } s \in [0, 2\pi) \\
\id \otimes \sigma_t(\pi(u_s)) = \pi(e^{ish}_t) \pi(u_s), \quad \text{for all } t \in \mathbb{R}, s \in [0, 2\pi).
\]

Now, let $k$ be the self-adjoint operator with $\spec k \subset \mathcal{E}$ affiliated to the centre of $Q \otimes L^\infty(\mathbb{R})$ such that $\pi(u_s) = e^{isk}$, for all $s \in [0, 2\pi)$. Then

\[
\gamma_2 \otimes \id(k) = k - 1 \\
\gamma_1(k) = k \\
\id \otimes \sigma_t(k) \geq k \quad \text{for } t \geq 0.
\]

We shall show that there is a self-adjoint operator $k_0$ affiliated to the centre of $Q$ with $\spec k_0 \subset \mathcal{E}$ such that
\[ \gamma_2(k_0) = k_0 - 1 \]
\[ \gamma_1(k_0) \geq k_0 . \]

To show this we let \( \alpha \) be the automorphism of \( Q \otimes L^\infty(R) \) which satisfies

\[ \alpha(x \otimes 1) = x \otimes 1, \text{ for all } x \in Q \]
\[ \alpha(1 \otimes \chi_r) = \pi_1(e^{-ir\phi}) \otimes \chi_r, \text{ for all } r \in R. \]

Now, \( \alpha \) may be approximated by automorphisms \( \tilde{\alpha} \) of the form

\[ \alpha x = \sum_{n=1}^{\infty} \text{id} \otimes \sigma_n (\pi_1(e_n) \otimes lx) \]

where \( \{e_n : n = 1,2,\ldots\} \) is a partition of unity in the centre of \( N_1 \) and \( t_n > 0 \). Hence

\[ \alpha(k) \geq k. \]

Since \( \gamma_1 \otimes \text{id} = \alpha \tilde{\gamma}_1 \) it follows that

\[ \gamma_1 \otimes \text{id}(k) \geq k. \]

To show the existence of \( k_0 \) we may assume as usual that the centre of \( Q \) is \( \sigma \)-finite. In this case we can find a complete \( \sigma \)-finite measure space \( (\Omega, \Lambda, \mu) \) and commuting bimeasurable bijections \( T_1 \) and \( T_2 \) of \( \Omega \) with \( \mu \circ T_j \) equivalent to \( \mu \) for \( j = 1,2 \) such that the centre of \( Q \) is isomorphic to \( L^\infty(\mu) \) and under this identification \( \gamma_j(f) = f \circ T_j^{-1} \) for \( f \in L^\infty(\mu) \), \( j = 1,2 \). Hence (centre \( Q \)) \( \otimes L^\infty(R) \) is identified with \( L^\infty(\mu \times m) \) (where \( R,L,m \)) is Lebesgue measure on \( R \). Under this identification \( \gamma_j \otimes \text{id} \) is given by the mapping \( \overline{T_j}(\omega,r) = (T_j \omega, r) \) for \( j = 1,2 \), and \( (\omega,r) \in \Omega \times R \). We choose an
A measurable integer valued function \( k : (\omega, r) \rightarrow k(\omega, r) \) which represents the operator \( k \). Then

\[
k(T_{1}^{-1} \omega, r) = k(\omega, r) - 1 \text{ for } \text{a.e. } (\omega, r),
\]

\[
k(T_{2}^{-1} \omega, r) \geq k(\omega, r) \text{ for } \text{a.e. } (\omega, r).
\]

We choose an \( r_{0} \in \mathbb{R} \) so that for a.e. \( \omega \)

\[
k(T_{2}^{-1} \omega, r_{0}) = k(\omega, r_{0}) - 1
\]

\[
k(T_{1}^{-1} \omega, r_{0}) \geq k(\omega, r_{0}).
\]

Then \( k_{0}(\omega) = k(\omega, r_{0}) \) satisfies the conditions

\[
\gamma_{2}(k_{0}) = k_{0} - 1
\]

\[
\gamma_{1}(k_{0}) \geq k_{0}.
\]

Now set \( u_{s}^{2} = \frac{\text{isk}_{0}}{2} \) for \( s \in [0, 2\pi) \). Since \( \gamma_{2}(u_{s}^{2}) = e^{-is}u_{s}^{2} \), there is a positive, self-adjoint operator \( h_{1} \) affiliated to the centre of \( N_{1} \) with \( \text{spec } h_{1} \subset \mathbb{Z} \) such that

\[
\gamma_{1}(u_{s}^{2}) = \pi_{1}(e^{-i\text{sh}h_{1}})u_{s}^{2}, \text{ for all } s \in [0, 2\pi).
\]

Since the roles of \( \gamma_{1} \) and \( \gamma_{2} \) are interchangable we have

**Lemma 3.6.** In the situation of lemma 3.5 there are, for \( j = 1, 2 \), strongly continuous unitary representations \( u_{s}^{j} \) of \([0, 2\pi)\) in the centre of \( Q \) and positive self-adjoint operators \( h_{j} \) affiliated to the centre of \( N_{j} \) with \( \text{spec } h_{j} \subset \mathbb{Z} \) such that
\[ \gamma_1(u_s^1) = e^{-is}u_s^1, \quad \gamma_2(u_s^1) = \pi_2(e^{-i\theta}u_s^1) \text{ for } s \in [0,2\pi) \]
\[ \gamma_2(u_s^2) = e^{-is}u_s^2, \quad \gamma_1(u_s^2) = \pi_1(e^{-i\theta}u_s^2) \text{ for } s \in [0,2\pi). \]

Using this lemma, we shall show that it suffices to assume that \( \theta_1 \) and \( \theta_2 \) are conservative.

Recall that an automorphism \( \theta \) on \( N \) is called conservative iff there are no non-zero central projections \( p \) in \( N \) such that \( \{\theta^n(p) : n \in \mathbb{Z}\} \) is an orthogonal family. By a maximality argument we can find a central projection \( p \) in \( N_1 \) such that \( \{\theta^n(p) : n \in \mathbb{Z}\} \) is an orthogonal family and if \( q_1 \) denotes \( \bigvee_{n \in \mathbb{Z}} \theta^n(p) \) then \( \theta_1 \) restricted to \( (N_1)_{1-q_1} \) is conservative. It follows that \( \theta_1 \) restricted to \( (N_1)_{q_1} \) is isomorphic to \( \text{id} \otimes \theta^{\infty}(Z) \).

Hence, there is a self-adjoint operator \( h \) affiliated to the centre of \( (N_1)_{q_1} \) with \( \text{spec} h \subset \mathbb{Z} \) such that
\[ h_{q_1} = h - \theta_1(h). \]

Since \( \pi_1(q_1) \) is both \( \gamma_1 \) and \( \gamma_2 \) invariant in the centre of \( Q \), there exists a \( \theta_2 \) invariant projection \( q_2 \) in the centre of \( N_2 \) such that \( \pi_1(q_1) = \pi_2(q_2) \). Now set \( u_s^2 = \pi_1(e^{ish})u_s^2 \) for \( s \in [0,2\pi) \). Then
\[ \gamma_1(u_s^2) = u_s^2, \quad \text{for all } s \in [0,2\pi) \]
\[ \gamma_2(u_s^2) = e^{-is}u_s^2, \quad \text{for all } s \in [0,2\pi). \]

Hence, there exists \( s \to u_s \) in the centre of \( (N_2)_{q_2} \) such that
\[ \pi_2(u_s) = u_s^2, \quad \text{for all } s \in [0,2\pi). \]
In particular \( \theta_2(u_s) = e^{-is}u_s \), for all \( s \in [0,2\pi) \). So \( \theta_2 \) restricted to \((N_2)_q^2\) is isomorphic to \( \theta \delta \) on \((N_2)_q^2 \). Hence, there is an isomorphism \( \kappa \) of the restriction of \( \theta_1 \) to \((N_1)_q^1 \) with the restriction of \( \theta_2 \) to \((N_2)_q^2 \). To show that the ceilings \( \phi_1q_1 \) and \( \phi_2q_2 \) correspond in the right way we note that if \( \phi \) is a self-adjoint operator affiliated to the centre of \( \mathfrak{A} \otimes \ell^\infty(\mathfrak{L}) \) (\( \mathfrak{A} \) is any \( \mathfrak{W}^\star \)-algebra) then there is a self-adjoint operator \( g \) affiliated to the centre of \( \mathfrak{A} \otimes \ell^\infty(\mathfrak{L}) \) such that \( \phi = g - \text{id} \delta(g) \).

The same reasoning applied to the restriction of \( \theta_2 \) to \((N_2)_1-q_2\) shows that this restriction is conservative. Hence, for the rest of the proof we assume that \( \theta_1 \) and \( \theta_2 \) are conservative. The property of conservative automorphisms which we will use is that every non-zero central projection is recurrent.

We next reduce to the case where the operator \( h_1 \) is one to one. Let \( e_1 \) be the support of \( h_1 \) and set \( \tilde{e}_1 = \phi_n^1(e_1) \). We first show that \( \tilde{e}_1 = 1 \). There is a \( \theta_2 \) invariant projection \( f \) in the centre of \( N_2 \) such that \( \pi_1(\tilde{e}_1) = \pi_2(f) \). Moreover

\[
\gamma_1(u_s^2\pi_1(1-\epsilon)) = u_s^2\pi_1(1-\epsilon), \quad \text{for all} \quad s \in [0,2\pi)
\]

\[
\gamma_2(u_s^2\pi_1(1-\epsilon)) = e^{-is}u_s^2\pi_1(1-\epsilon), \quad \text{for all} \quad s \in [0,2\pi).
\]

Hence, there exists \( s \to u_s \) in the centre of \((N_2)_1-f\) such that \( \pi_2(u_s) = u_s^2\pi_1(1-\epsilon) \) for all \( s \in [0,2\pi) \). In particular

\[
\theta_2(u_s) = e^{-is}u_s, \quad \text{for all} \quad s \in [0,2\pi).
\]
This contradicts the assumption that $\theta_2$ is conservative unless $\bar{e}_1 = 1$.

We now cut down to $\pi_1(e_1)$. The map $x \mapsto \pi_1(x)$ is am imbedding of $\{(N_1)e_1, (\theta_1)e_1\}$ into $\{\pi_1(e_1), (\gamma_1)e_1\}$ and since $\gamma_1^n(\pi_1(e_1)) = 1$, the map $x \mapsto \pi_2(x)\pi_1(e_1)$ is am imbedding of $\{N_2, \theta_2\}$ into the restriction of $\gamma_2$ to $\pi_1(e_1)$. We also have:

$$(\gamma_1)_{\pi_1}(e_1)(v_{\pi_1}(e_1)) = \pi_1((\theta_1)e_1^{ir(\phi_1)e_1})v_{\pi_1}(e_1), \text{ for all } r \in \mathbb{R}$$

$$\gamma_2(v_{\pi_1}(e_1)) = [\pi_2(\theta_2)e^{ir\phi_2})\pi_1(e_1)]v_{\pi_1}(e_1), \text{ for all } r \in \mathbb{R}$$

$$(\gamma_1)_{\pi_1}(e_1)(u_{\pi_1}(e_1)) = \pi_1(e^{ish})u_{\pi_1}(e_1), \text{ for all } s \in [0, 2\pi]$$

where $h$ is positive self-adjoint and $1 - 1$. Hence it suffices to prove the theorem under the assumption that in the situation of lemma 3.6 $h_1$ is $1 - 1$.

We next reduce to the case where $h_1 = 1$. Let $p$ be the pro-

jection in the centre of $Q$ such that

$$u_{s}^2 = \sum_{n \in \mathbb{Z}} e^{ins} \gamma_2(p), \text{ for all } s \in [0, 2\pi].$$

We have $\gamma_1^{-1}(u_{s}^2) = \pi_1(\theta_1^{-1}e^{ish_1})u_{s}^2$, for all $s \in [0, 2\pi]$. Let

$$\theta_1^{-1}(h_1) = \sum_{n \geq 1} ne_n$$

be the spectral resolution of $\theta_1^{-1}(h_1)$. It follows that

$$\gamma_1^{-1}(p) = \sum_{n \geq 1} \pi_1(e_n)\gamma_2^n(p).$$

Similarly, for $m \geq 1$, $\gamma_1^{-m}(u_{s}^2) = \pi_1(\theta_1^{-m}(e^{-ish_1})\theta_1^{-1}(e^{ish_1}))u_{s}^2$ for $s \in [0, 2\pi]$. We let $\theta_1^{-m}(h_1) + \ldots + \theta_1^{-1}(h_1) = \sum_{n \geq 1} ne_n^{(m)}$ be the spectral resolution. Then
\[ \gamma_1^{-m}(p) = \sum_{n \geq 1} \pi_1(e_n^{(m)}) \gamma_2^n(p). \]

The projections \( e_n^{(m)} \) have the properties

(i) \( e_n^{(m)} = 0 \) for \( n < m \) (since \( h_1 \geq 1 \))

(ii) \( e_n^{(m)}(j) = 0 \) for \( m < j \) and \( n \geq k \) (since \( h_1 \geq 1 \)).

Since \( \gamma_1^{-m}(p) p = 0 \), for all \( m \geq 1 \), it follows that \( \{ \gamma_1^k(p) : k \in \mathbb{Z} \} \) is an orthogonal family. Let \( f \) be the projection in the centre of \( N_2 \) such that

\[ \pi_2(f) = \sum_{n \in \mathbb{Z}} \gamma_1^n(p). \]

Note that \( \theta_2^n(f) = 1 \) (since \( \pi_2(f) \geq p \)). We shall show that the cut down \( (\gamma_2^*)_{\pi_2(f)} \) satisfies

\[ (\gamma_2^*)_{\pi_2(f)}(p) = \gamma_1^{-1}(p). \]

For this, define projections \( f_n \leq f \) in the centre of \( N_2 \) by

\[ \pi_2(f_n) = \sum_{k \in \mathbb{Z}} \gamma_1^k(\pi_1(e_n^{(1)})p). \]

Since \( \sum_{n \geq 1} e_n^{(1)} = 1 \) it follows that \( f = \sum_{n \geq 1} f_n \). We claim that this is the canonical partition of \( f \). First,

\[ \gamma_2(\pi_2(f_1)) = \sum_{k \in \mathbb{Z}} \gamma_1^k(\pi_1(e_n^{(1)})\gamma_2(p)). \]

Since \( \pi_1(e_n^{(1)})\gamma_2(p) \leq \pi_2(f) \) we have \( \theta_2(f_1) \leq f \). Next, for \( n > 1 \) and \( 1 \leq j \leq n - 1 \) we have

\[ \gamma_2^j(\pi_2(f_n)) = \sum_{k \in \mathbb{Z}} \gamma_1^k(\pi_1(e_n^{(1)})\gamma_2^j(p)). \]
But \( \pi_2(f) \gamma_2^j(p) = \sum_{\ell=1}^j \pi_1(e_\ell^j(p)) \gamma_2^j(p) \) and \( e_\ell^j = 0 \) for \( \ell = 1, \ldots, j \) and \( 1 \leq j \leq n - 1 \). Hence \( \theta_2^j(f) = 0 \) for \( 1 \leq j \leq n - 1 \). Finally

\[
\gamma_2^n(\pi_2(f)) = \sum_{k \in \mathbb{Z}} \gamma_1^k(\pi_1(e_1^n)) \gamma_2^j(p) \quad \text{and} \quad \pi_1(e_1^n) \gamma_2^j(p) \leq \pi_2(f).
\]

Hence \( \theta_2^n(f) = f \). This shows that \( f = \sum_{n \geq 1} f_n \) is the canonical partition of \( f \). We now compute

\[
(\gamma_2^j) \pi_2(f)(p) = \sum_{n \geq 1} \gamma_2^n(\pi_2(f_n)p) = \sum_{n \geq 1} \gamma_2^n(\pi_1(e_1^n)p) = \gamma_1^{-1}(p).
\]

Now set \( u_s = \sum_{n \in \mathbb{Z}} e^{-ins} \gamma_1^n(p) \). Then

\[
\gamma_1(u_s) = e^{is} u_s, \quad \text{for all} \ s \in [0,2\pi)
\]

and

\[
(\gamma_2^j) \pi_2(f)(u_s) = e^{-is} u_s, \quad \text{for all} \ s \in [0,2\pi).
\]

Hence, as before, it suffices to prove the theorem under the assumption that in lemma 3.6 \( h_1 = 1 \).

In this case, by corollary 1.12, we may assume that \( Q = N_1 \otimes \ell^\infty(\mathbb{Z}) \), \( \pi_1(x) = x \otimes 1 \) for all \( x \in N_1 \), \( \gamma_1 = \theta_1 \otimes \delta^{-1} \) and \( \gamma_2 = \text{id} \otimes \delta \).

If we regard \( N_1 \otimes \ell^\infty(\mathbb{Z}) \) as bounded functions \( x : n \rightarrow x_n \) from \( \mathbb{Z} \) to \( N_1 \), we see that \( \gamma_1 \) consists of those operators \( x \) satisfying

\[
x_n = \theta_1^{-n} x_0.\]

Hence \( \kappa : x \rightarrow \pi_2(x) \) is an isomorphism of \( N_2 \) with \( N_1 \) such that \( \kappa \circ \theta_2 \circ \kappa^{-1} = \theta_1 \). So for \( x \in N_2 \), \( \pi_2(x) = \theta_1^{-n}(\kappa x) = \kappa(\theta_2^{-n})x \) for all \( n \in \mathbb{Z} \).

Now, choose a self-adjoint operator \( h \) affiliated to the centre of \( N_1 \otimes \ell^\infty(\mathbb{Z}) \) such that
\[ v_r = e^{rh}, \text{ for all } r \in \mathbb{R}. \]

Then

\[ \gamma_1(h) = h + \pi_1(\theta_1\phi_1) \]
\[ \gamma_2(h) = h - \pi_2(\theta_2\phi_2). \]

If \( n \to h_n \) represents \( h \) we have

\[ \theta_1(h_{n+1}) = h_n + \theta_1\phi_1, \text{ for all } n \in \mathbb{Z} \]
\[ h_{n-1} = h_n - \theta_1^{-n}(\theta_2\phi_2), \text{ for all } n \in \mathbb{Z}. \]

Choose \( n = 0 \) in the first equation and \( n = 1 \) in the second to obtain

\[ \theta_1(h_1) = h_0 + \theta_1\phi_1 \]
\[ h_0 = h_1 - \kappa\phi_2. \]

That is

\[ h_0 = \theta_1^{-1}(h_0) + \phi_1 - \kappa\phi_2. \]

Now take \( g = \theta_1^{-1}(h_0) \), then

\[ \phi_1 = \kappa\phi_2 + \theta_1g - g. \]

This concludes the proof of theorem 3.4.

For the constant ceiling case we have

Theorem 3.7. The flow built on \( \{N_1, \theta_1\} \) under the constant ceiling \( \phi_1 = c \) is isomorphic to the flow built on \( \{N_2, \theta_2\} \) under the constant ceiling \( \phi_2 = c \) iff \( \{N_1, \theta_1\} \) is isomorphic to \( \{N_2, \theta_2\} \).

Proof: If \( \{N_1, \theta_1\} \) is isomorphic to \( \{N_2, \theta_2\} \) then the flows are isomorphic.
For the converse, we apply lemma 3.5 to obtain \( Q, \gamma_1, \gamma_2, \pi_1, \pi_2 \) and \( r \rightarrow v_r \). We have

\[
\gamma_1(v_r) = e^{ir_c}v_r, \quad \text{for all } r \in \mathbb{R}
\]

\[
\gamma_2(v_r) = e^{-ir_c}v_r, \quad \text{for all } r \in \mathbb{R}.
\]

Hence, \( v_{2\pi} \) is fixed by both \( \gamma_1 \) and \( \gamma_2 \). Choose a self-adjoint operator \( k \) affiliated to the centre of \( Q \) fixed by both \( \gamma_1 \) and \( \gamma_2 \) such that

\[
i^{2\pi}k
\frac{e}{c} = v_{2\pi}.
\]

Set \( w_s = e^{-is\frac{k}{c}}v_{s/c} \) for \( s \in \mathbb{R} \). Then \( w_{2\pi} = 1 \) so \( s \rightarrow u_s = w_s \) for \( s \in [0,2\pi) \) is a strongly continuous unitary representation of \([0,2\pi)\) in the centre of \( Q \) such that

\[
\gamma_1(u_s) = e^{is}u_s, \quad \text{for all } s \in [0,2\pi)
\]

\[
\gamma_2(u_s) = e^{-is}u_s, \quad \text{for all } s \in [0,2\pi).
\]

As in the proof of theorem 3.5, \( \{N_1, \theta_1\} \) is isomorphic to \( \{N_2, \theta_2\} \).
4. Flow under a ceiling and weak equivalence

In this section we show that the uniqueness results of section 3
hold with weak equivalence replacing isomorphism. We shall need

**Proposition 4.1.** Let \( \{M, \alpha\} \) be a covariant system over a locally
compact abelian group \( G \). Let \( t \mapsto u_t \) be an \( \alpha \) cocycle in \( M \)
(see definition 1.5). If there is a strongly continuous unitary
representation \( p \mapsto v_p \) of \( \hat{G} \) in the centre of \( M \) such that

\[
\alpha_t(v_p) = \langle t, p \rangle v_p, \quad \text{for all } p \in \hat{G}, \quad t \in G
\]

then there is a unitary \( u \in M \) such that

\[
u_t = u\alpha_t(u^*), \quad \text{for all } t \in G.
\]

**Proof:** Let \( F_2 \) be the \( 2 \times 2 \) matrices over \( \mathbb{C} \) with matrix units
\( \{e_{ij} : i, j = 1, 2\} \). Define a continuous action \( \beta \) of \( G \) on \( M \otimes F_2 \)
by:

\[
\beta_t(\sum_{i,j=1,2} x_{ij} \otimes e_{ij}) = \alpha_t(x_{11}) \otimes e_{11} + u_t\alpha_t(x_{21}) \otimes e_{21} + \alpha_t(x_{21})u_t^* \otimes e_{12} + u_t\alpha_t(x_{22})u_t^* \otimes e_{22}
\]

where \( x = \sum_{i,j=1,2} x_{ij} \otimes e_{ij} \) is in \( M \otimes F_2 \) and \( t \in G \). Note that
the projections \( e = 1 \otimes e_{11} \) and \( f = 1 \otimes e_{22} \) are in \( (M \otimes F_2)^\beta \) and
they are equivalent relative to \( M \otimes F_2 \).

We claim that it suffices to show that \( e \) and \( f \) are equivalent
relative to \( (M \otimes F_2)^\beta \). For if \( w \) is a partial isometry in \( (M \otimes F_2)^\beta \)
such that
then there is a unitary $u$ in $M$ such that $w = u \otimes e_{21}$. Since $\beta_t(w) = w$ for all $t \in G$ we have

$u \otimes e_{21} = u_t \alpha_t(u) \otimes e_{21}$ for all $t \in G$.

Hence

$u_t = u \alpha_t(u^*)$ for all $t \in G$.

We now show that $e$ and $f$ are equivalent relative to $(M \otimes F_2)^\beta$.

Using the map $p \rightarrow v_p$, we apply corollary 1.12 to deduce that $M \otimes F_2$ is isomorphic to $(M \otimes F_2)^\beta \otimes L^\infty(G)$. We conclude the proof with the following lemma.

Lemma 4.2. Let $e$ and $f$ be projections in a $W^*$-algebra $M$. Let $A$ be a finite $W^*$-algebra and suppose that $e \otimes 1$ and $f \otimes 1$ are equivalent projections in $M \otimes A$. Then $e$ and $f$ are equivalent in $M$.

Proof: We may assume that $A$ is $\sigma$-finite since the support of any normal finite trace is a $\sigma$-finite central projection.

Since $M_e \otimes A$ is isomorphic to $M_f \otimes A$ we can reduce to the case where $e$ and $f$ are finite or properly infinite.

By the comparison theorem ([7] p. 218 théorème 1) we may assume that $e \leq f$. In case both $e$ and $f$ are finite, let $\tau$ be a normal trace on $M_f$ and let $\phi$ be a normal finite trace on $A$ with $\phi(1) \neq 0$. Then $\tau \otimes \phi(e \otimes 1) = \tau \otimes \phi(f \otimes 1)$ since $e \otimes 1 \sim f \otimes 1$.

Hence $\tau(e) = \tau(f)$. Since $e \leq f$ we conclude that $e \sim f$.

We now assume that both $e$ and $f$ are properly infinite. We
show that there is a central projection \( g \) such that \( 0 \not\leq ge \sim gf \).

Since \( e \) is infinite we can find an infinite mutually orthogonal family of equivalent \( \sigma \)-finite projections in \( M_e \). By [7] p. 218 corollaire 2, there is a central projection \( g_1 \) and a mutually orthogonal infinite family \( \{e_i = i \in I\} \) of equivalent \( \sigma \)-finite projections such that

\[
0 \not\leq g_1 e = \sum_{i \in I} e_i.
\]

By the same result, there is a central projection \( g_2 \) and a mutually orthogonal family \( \{f_j : j \in J\} \) of equivalent projections such that \( I \subset J \), for each \( i \in I \), \( f_j \sim e_i g_2 \) and

\[
0 \not\leq g_2 f = \sum_{j \in J} f_j.
\]

Since \( e_i \leq g_1 \) for all \( i \in I \) and since \( f_j \sim e_i g_2 \) for all \( i \in I \) we have \( f_j \leq g_1 \) for all \( j \in J \). Hence \( g = g_1 g_2 \not= 0 \) and

\[
0 \not\leq ge = \sum_{i \in I} ge_i
\]

\[
0 \not\leq gf = \sum_{j \in J} f_j.
\]

Now, using the equivalence \( e \Theta f \sim f \Theta 1 \) we obtain a mutually orthogonal family \( \{e_i : i \in I\} \) of equivalent \( \sigma \)-finite projections in \( M_f \Theta A \) such that

\[
\sum_{j \in J} f_j \Theta 1 = (gf) \Theta 1 = \sum_{i \in I} e_i.
\]

Since each \( gf_j \) is \( \sigma \)-finite and each \( e_i \) is \( \sigma \)-finite, the cardinality of \( I \) is the same as the cardinality of \( J \) ([7] p. 224 lemma 6). Hence \( ge \sim gf \).
Now let \( \{g_k : k \in K\} \) be a maximal family of mutually orthogonal central projections in the centre of \( \mathcal{M} \) such that \( g_k e \circ g_k f \neq 0 \). Set \( g_0 = \sum_{k \in K} g_k \). If \((1-g_0)f \neq 0\) then \((1-g_0)\epsilon \neq 0\) since \( \epsilon \) and \( f \) have the same central support (due to \( \epsilon \circ 1 \sim f \circ 1 \)). In this case we can repeat the above argument to contradict the maximality of \( \{g_k : k \in K\} \). Hence \( g_0f = f \) and \( g_0\epsilon = \epsilon \) so that \( f \sim \epsilon \).

This concludes the proof of proposition 4.1.

We apply proposition 4.1 to flow under a ceiling:

**Lemma 4.3.** Let \( \{M, \alpha\} \) be the flow built on \( \{N, \theta\} \) under \( \phi_0 \). Let \( w \) be a unitary in \( N \) and set \( \tilde{\theta}_1(x) = w\theta(x)w^* \) for \( x \in N \). Then \( \{M, \alpha\} \) is weakly equivalent to the flow built on \( \{N, \tilde{\theta}_1\} \) under \( \phi \).

**Proof:** Let \( \tilde{\theta} \) and \( \tilde{\theta}_1 \) be the automorphisms of \( N \otimes L^\infty(\mathbb{R}) \) which satisfy

\[
\tilde{\theta}(x \otimes 1) = \theta x \otimes 1, \quad \tilde{\theta}_1(x \otimes 1) = \theta_1 x \otimes 1, \quad \text{for all } x \in N
\]

\[
\tilde{\theta}(1 \otimes x) = \theta e^{i\phi} \theta x, \quad \tilde{\theta}_1(1 \otimes x) = \theta_1 e^{i\phi} \theta_1 x, \quad \text{for all } s \in \mathbb{R}.
\]

Then \( \tilde{\theta}_1(x) = w \otimes 1 \tilde{\theta}(x)(w \otimes 1)^* \), for all \( x \in N \otimes L^\infty(\mathbb{R}) \). By the reversal lemma (lemma 2.4) we know that \( N \otimes L^\infty(\mathbb{R}) \) is isomorphic to \( M \otimes L^\infty(\mathbb{Z}) \) in such a way that \( \tilde{\theta} \) corresponds to \( \text{id} \otimes \delta \). Hence, by proposition 4.1, there is a unitary \( u \) in \( N \otimes L^\infty(\mathbb{R}) \) such that

\[
w \otimes 1 = u \tilde{\theta}(u^*) .
\]

Set \( u_t = u^* \text{id} \circ_t(u) \) for \( t \in \mathbb{R} \). Then \( u_t \) is fixed by \( \tilde{\theta} \), so \( u_t \in M \), for all \( t \in \mathbb{R} \). Moreover, \( t \mapsto u_t \) is an \( \alpha \) cocycle.
Let \( \pi \) be the automorphism of \( N \otimes L^\infty(R) \) given by:

\[
\pi(x) = uxu^*, \text{ for all } x \in N \otimes L^\infty(R).
\]

Then

\[
\pi \circ \theta \circ \pi^{-1} = \theta_1
\]

\[
\pi \circ (ad_{u_t} \circ \text{id } \sigma_t) \circ \pi^{-1} = \text{id } \sigma_t, \text{ for all } t \in R.
\]

(Here \( ad_{u_t}(x) = u_t xu_t^* \), for all \( x \in N \otimes L^\infty(R) \)). Hence, the flow built on \( \{N, \theta_1\} \) under \( \phi \) is isomorphic (by \( \pi \)) to the flow \( t \rightarrow ad_{u_t} \circ \text{id } \sigma_t \) on \( M \). That is, the flows are weakly equivalent.

A "converse" of lemma 4.3 is:

**Lemma 4.4.** Suppose \( \{M_1, \alpha^1\} \) is weakly equivalent to the flow built on \( \{N, \theta\} \) under \( \phi \). Then there is a unitary \( w \) in \( N \) such that \( \{M_1, \alpha^1\} \) is isomorphic to the flow built on \( \{N, adw \circ \theta\} \) under \( \phi \).

**Proof:** Let \( \tilde{\theta} \) be the automorphism of \( N \otimes L^\infty(R) \) as in definition 2.3. There is an \( \text{id } \sigma \circ \text{id } \sigma_t \) cocycle \( t \rightarrow u_t \) in \([N \otimes L^\infty(R)]^{\tilde{\theta}}\) such that \( \alpha^1 \) is isomorphic to the restriction of \( t \rightarrow ad_{u_t} \circ \text{id } \sigma_t \) to \([N \otimes L^\infty(R)]^{\tilde{\theta}}\).

By proposition 4.1, there is a unitary \( u \) in \( N \otimes L^\infty(R) \) such that

\[
u_t = u^* \text{id } \sigma_t(u), \text{ for all } t \in R.
\]

Hence \( u \tilde{\theta}(u^*) \) is fixed by \( \text{id } \sigma \), so there is a unitary \( w \in N \) with

\[
w \theta 1 = u \tilde{\theta}(u^*).
\]

Let \( \pi = adu \), then

\[
\pi \circ \tilde{\theta} \circ \pi^{-1} = \text{ad}(w \theta 1) \circ \tilde{\theta}
\]

\[
\pi \circ ad_{u_t} \circ \text{id } \sigma_t \circ \pi^{-1} = \text{id } \sigma_t, \text{ for all } t \in R.
\]
Hence \( \{M_1, \alpha^{1}\} \) is isomorphic to the flow built on \( \{N, \text{ad} \circ \theta\} \) under \( \phi \).

The main result of this section is:

\textbf{Theorem 4.5.} The flow built on \( \{N_1, \theta_1\} \) under \( \phi_1 \) is weakly equivalent to the flow built on \( \{N_2, \theta_2\} \) under \( \phi_2 \) iff there exist

(i) recurrent projections \( e_j \) in the centre of \( N_j \) with
\[
\sqrt{\theta_j^n(e_j)} = 1 \quad \text{for} \quad j = 1,2
\]

(ii) an isomorphism \( \kappa \) of \( \left( N_1 e_1 \right) \) with \( \left( N_2 e_2 \right) \)

(iii) a unitary \( u \) in \( \left( N_2 e_2 \right) \) such that
\[
\kappa \cdot (\theta_j^-) e_1 = \text{ad} u \cdot (\theta_j^-) e_2.
\]

(iv) a self-adjoint operator \( g \) affiliated to the centre of \( \left( N_2 e_2 \right) \) such that
\[
\kappa(\phi_1^-) e_1 = (\phi_2^-) e_2 + (\theta_2^-) e_2 (g) - g.
\]

\textbf{Proof:} If the conditions are satisfied then theorem 3.4 and lemma 4.3 show that the flows are weakly equivalent. Conversely, if the flows are weakly equivalent then by lemma 4.4 we can find a unitary \( w \) in \( N_2 \) such that the flow built on \( \{N_2, \text{ad} \circ \theta_2\} \) under \( \phi_2 \) is isomorphic to the flow built on \( \{N_1, \theta_1\} \) under \( \phi_1 \). Now, apply theorem 3.4 to obtain recurrent projections \( e_j \) in the centre of \( N_j \) with
\[
\sqrt{\theta_j^n(e_j)} = 1 \quad \text{for} \quad j = 1,2, \quad \text{an isomorphism} \quad \kappa \quad \text{of} \quad \{(N_1 e_1, (\theta_1 e_1)\}
\]
with \( \{(N_2 e_2, (\text{ad} \circ \theta_2 e_2)\} \) and a self-adjoint operator \( g \) affiliated
to the centre of \( (N_2)_e \) such that
\[
\kappa(\phi_1)_e = (\phi_2)_e + (\text{ad}w \theta_2)_e(g) - g.
\]
Since \((\text{ad}w \theta_2)_e = \text{ad}(w_2) \circ (\theta_2)_e\) we let \( u = w_2 \). The conditions of the theorem are satisfied.

For the case of a constant ceiling we have

Theorem 4.6. The flow built on \( \{N_1, \theta_1\} \) under the constant ceiling \( c \) is weakly equivalent to the flow built on \( \{N_2, \theta_2\} \) under \( c \) iff \( \{N_1, \theta_1\} \) is weakly equivalent to \( \{N_2, \theta_2\} \).

Proof: Lemma 4.3 shows that if \( \{N, \theta_1\} \) is weakly equivalent to \( \{N_2, \theta_2\} \) then the resulting flows are weakly equivalent. For the converse, lemma 4.5 shows that there is a unitary \( w \) in \( N_2 \) such that the flow built on \( \{N_2, \text{ad}w \theta_2\} \) under \( c \) is isomorphic to the flow built on \( \{N_1, \theta_1\} \) under \( c \). Theorem 3.7 shows that \( \{N_2, \text{ad}w \theta_2\} \) is isomorphic to \( \{N_1, \theta_1\} \). That is, \( \{N_2, \theta_2\} \) is weakly equivalent to \( \{N_1, \theta_1\} \).

We conclude this section with a result connecting \( W^*(N, \theta) \) to \( W^*(M, \alpha) \).

Proposition 4.7. Let \( \{M, \alpha\} \) be the flow built on \( \{N, \theta\} \) under \( \phi \). Then \( M \) is properly infinite iff \( N \) is properly infinite. In this case \( W^*(N, \theta) \) is isomorphic to \( W^*(M, \alpha) \).

Proof: In the notation of the reversal lemma (lemma 2.4), there is an isomorphism \( \pi \) of \( N \theta L^\infty(R) \) with \( M \theta L^\infty(Z) \) such that
\[ \pi \circ \theta \circ \pi^{-1} = \text{id} \circ \delta \]

\[ \pi \circ \text{id} \circ \sigma_t \circ \pi^{-1} = \alpha_t, \text{ for all } t \in \mathbb{R}. \]

Hence \( M \) is properly infinite iff \( N \) is. For the rest of the proof we assume that \( M \) and \( N \) are properly infinite. Consider the continuous action \( \beta \) of \( \mathbb{Z} \times \mathbb{R} \) on \( N \theta L^\infty(\mathbb{R}) \) defined by

\[ \beta(n,t) = \text{id} \circ \sigma_t \circ \sigma_n, \text{ for all } (n,t) \in \mathbb{Z} \times \mathbb{R}. \]

We shall show that \( W\{N \theta L^\infty(\mathbb{R}),\beta\} \) is isomorphic to \( W\{N,\theta\} \). Now, \( W\{N \theta L^\infty(\mathbb{R}),\beta\} \) is generated by the operators

\[ \{\pi_\beta(x): x \in N \theta L^\infty(\mathbb{R})\}, \{\lambda_\beta(1,0)\} \text{ and } \{\lambda_\beta(0,t): t \in \mathbb{R}\}. \]

Moreover, there are commuting actions \( s \mapsto \gamma^1_s \) of \([0,2\pi)\) and \( p \mapsto \gamma^2_p \) of \( \mathbb{R} \) on \( W\{N \theta L^\infty(\mathbb{R}),\beta\} \) such that

\[ \hat{\beta}(s,p) = \gamma^1_s \circ \gamma^2_p, \text{ for all } (s,p) \in [0,2\pi) \times \mathbb{R}. \]

Let \( P \) be the \( \hat{W} \) algebra generated by \( \{\pi_\beta(x): x \in N \theta L^\infty(\mathbb{R})\} \) and \( \{\lambda_\beta(0,t): t \in \mathbb{R}\} \). Since

\[ \gamma^2_p(\pi_\beta(x)) = \pi_\beta(x), \text{ for all } x \in N \theta L^\infty(\mathbb{R}), \ p \in \mathbb{R} \]

\[ \gamma^2_p(\lambda_\beta(0,t)) = e^{-ipt}\lambda_\beta(0,t), \text{ for all } t \in \mathbb{R}, \ p \in \mathbb{R} \]

it follows, by theorem 1.11 (the characterization theorem), that \( P \) is isomorphic to the crossed product of \( N \theta L^\infty(\mathbb{R}) \) by \( \text{id} \circ \sigma \). Another application of theorem 1.11 shows that this crossed product is isomorphic to \( N \theta B(L^2(\mathbb{R})) \). The composite isomorphism carries
\[ \pi_\beta(x) \text{ to } x \text{ for } x \in N \oplus L^\infty(R) \]
\[ \lambda_\beta(0,t) \text{ to } 1 \oplus \lambda_t \text{ for } t \in R. \]

(Here \((\lambda_t, \xi)(s) = \xi(s-t) \) for \( \xi \in L^2(R) \)). Since
\[ \gamma_s^1(y) = y, \text{ for all } y \in P, \quad s \in [0,2\pi) \]
\[ \gamma_s^1(\lambda_\beta(1,0)) = e^{-is} \lambda_\beta(1,0), \text{ for all } s \in [0,2\pi) \]

it follows by theorem 1.11, that \( W^*\{N \oplus L^\infty(R) , \beta \} \) is isomorphic to the crossed product of \( P \) by \( \text{ad} \lambda_\beta(1,0) \).

In summary, \( W^*\{N \oplus L^\infty(R) , \beta \} \) is isomorphic to the crossed product of \( N \oplus B(L^2(R)) \) by the automorphism \( \theta_1 \) which satisfies
\[ \theta_1(x \oplus 1) = \theta x \oplus 1, \text{ for all } x \in N \]
\[ \theta_1(1 \oplus \chi_r) = e^{ir\phi} \oplus \chi_r, \text{ for all } r \in R \]
\[ \theta_1(1 \oplus \lambda_t) = 1 \oplus \lambda_t, \text{ for all } t \in R. \]

According to lemma 2.2, \( \theta_1 \) is weakly equivalent to \( \theta \oplus \text{id} \). Since \( N \) is properly infinite and since \( B(L^2(R)) \) is isomorphic to \( B(l^2(\mathbb{C})) \)
it follows that \( \theta_1 \) is weakly equivalent to \( \theta \) (see the remark following theorem 1.8). Hence, by proposition 1.6, \( W^*\{N \oplus L^\infty(R) , \beta \} \) is isomorphic to \( W^*\{N, \theta \} \). A similar analysis of \( M \oplus l^\infty(\mathbb{C}) \) with the action
\[ (n,t) \rightarrow \text{id} \oplus \delta^n \circ \tilde{\alpha}_t, \text{ for all } (n,t) \in \mathbb{Z} \times R \]
shows that \( W^*\{N \oplus L^\infty(R) , \beta \} \) is also isomorphic to \( W^*\{M, \alpha \} \). ||
5. Application to Properly Infinite $\mathcal{W}^*$-algebras

We shall apply our results on flow under a ceiling to the situation given by the following theorem of Takesaki.

Theorem 5.1 ([2] Theorem 8.1, lemma 8.2 and corollary 8.4). Let $\mathcal{P}$ be a properly infinite $\mathcal{W}^*$ algebra. Then there is a covariant system $\{\mathcal{M}, \alpha\}$ over $\mathbb{R}$ with the properties:

(i) $\mathcal{M}$ is properly infinite and semi-finite
(ii) there is a faithful, normal, semi-finite (abbreviated f.n.s-f) trace $\tau$ on $\mathcal{M}$ such that
$$\tau \circ \alpha_t = e^{-t}\tau, \text{ for all } t \in \mathbb{R}$$
(iii) $\mathcal{W}\{\mathcal{M}, \alpha\}$ is isomorphic to $\mathcal{P}$.

Moreover, $\{\mathcal{M}, \alpha\}$ is unique up to weak equivalence. ||

We shall refer to $\{\mathcal{M}, \alpha\}$ as a continuous decomposition of $\mathcal{P}$.

Note that the covariant system consisting of the restriction of $\alpha$ to the centre of $\mathcal{M}$ is unique up to isomorphism.

We first examine the implications of property (ii) in case $\{\mathcal{M}, \alpha\}$ is the flow built on $\{\mathcal{N}, \theta\}$ under $\phi$.

Proposition 5.2. Let $\{\mathcal{M}, \alpha\}$ be the flow built on $\{\mathcal{N}, \theta\}$ under $\phi$.

Then (a) $\mathcal{M}$ is semifinite iff $\mathcal{N}$ is semifinite. (b) There is a f.n.s-f trace $\tau_1$ on $\mathcal{M}$ such that $\tau_1 \circ \alpha_t = e^{-t}\tau_1$ for all $t \in \mathbb{R}$ iff there is a f.n.s-f trace $\tau_2$ on $\mathcal{N}$ such that $\tau_2 \circ \theta = \tau_2(e^{\phi})$ (see [8] for an explanation of the notation).
Proof: In the notation of the reversal lemma (lemma 2.4), there is an isomorphism \( \pi \) of \( N \otimes L^\infty(\mathbb{R}) \) with \( M \otimes \ell^\infty(\mathbb{Z}) \) such that

\[
\pi \circ \theta \circ \pi^{-1} = \text{id} \otimes \delta
\]

\[
\pi \circ \text{id} \otimes \sigma_t = \pi^{-1} = \tilde{\alpha}_t, \text{ for all } t \in \mathbb{R}.
\]

Here, \( M = [N \otimes L^\infty(\mathbb{R})]^\otimes \) and for \( x \in M, t \in \mathbb{R} \)

\[
\alpha_t(x) = \text{id} \otimes \sigma_t(x)
\]

\[
\pi(x) = x \otimes 1.
\]

Thus, part (a) is proven.

Let \( \tau_3 \) and \( \tau_4 \) be f.n.s-f traces on \( N \) and \( M \) respectively. Let \( m \) and \( n \) be the usual traces on \( L^\infty(\mathbb{R}) \) and \( \ell^\infty(\mathbb{Z}) \) respectively. Let \( \rho \) (respectively \( \rho_t, t \in \mathbb{R} \)) be the positive self-adjoint 1-1 operator affiliated to the centre of \( N \) (respectively \( M \)) such that \( \tau_3 \circ \theta = \tau_3(\rho \cdot) \) (respectively \( \tau_4 \circ \alpha_t = \tau_4(\rho_t \cdot) \)) for \( t \in \mathbb{R} \). We have

\[
\tau_3 \circ \theta \circ m \circ \text{id} \circ \sigma_t = \tau_3 \circ \theta \circ m, \text{ for all } t \in \mathbb{R}
\]

\[
\tau_4 \circ \theta \circ n \circ \text{id} \circ \delta = \tau_4 \circ \theta \circ n.
\]

By approximating \( \tilde{\theta} \) by automorphisms \( \tilde{\theta} \) of the form

\[
\tilde{\theta}(x) = \sum_{n=1}^\infty \theta \circ \sigma \circ \left( e_n \otimes 1x \right), \text{ for } x \in N \otimes L^\infty(\mathbb{R})
\]

where \( \{e_n : n = 1,2,\ldots\} \) is a partition of unity in the centre of \( N \) we see that

\[
\tau_3 \circ \theta \circ m \circ \tilde{\theta} = \tau_3 \circ \theta \circ m = \tau_3 \circ \theta \circ m(\rho \otimes 1 \cdot).
\]

Similarly
\( \tau_4 \circ n \circ \alpha_t = (\tau_4 \circ \alpha_t) \circ n = \tau_4 \circ n(\rho_t \circ 1), \) for all \( t \in \mathbb{R}. \)

Now, set \( \tau_5 = \tau_3 \circ m \) and \( \tau_6 = \tau_4 \circ n \circ \pi, \) then

\[
\begin{align*}
(1) \quad & \tau_5 \circ \pi = \tau_5(\rho \circ 1), \quad \tau_5 \circ \text{id} \circ \sigma_t = \tau_5, \quad \text{for all} \ t \in \mathbb{R} \\
(2) \quad & \tau_6 \circ \pi = \tau_6, \quad \tau_6 \circ \text{id} \circ \sigma_t = \tau_6(\rho_t \circ 1), \quad \text{for all} \ t \in \mathbb{R}.
\end{align*}
\]

Let \( k \) be the positive self-adjoint \( 1 \)-\( 1 \) operator affiliated to the centre of \( N \circ L^\infty(\mathbb{R}) \) such that

\[ \tau_5 = \tau_6(k \circ 1). \]

A computation using (1) and (2) shows

\[ \widetilde{\theta}^{-1}(k) = k(\rho \circ 1) \]
\[ \text{id} \circ \sigma_t^{-1}(k)(\circ 1) = k^{-1}(\rho_t), \quad \text{for all} \ t \in \mathbb{R}. \]

Now, let \( h \) be the self-adjoint operator affiliated to the centre of \( N \circ L^\infty(\mathbb{R}) \) such that \( e^{irh} = 1 \circ \chi_r, \) for all \( r \in \mathbb{R}. \) Then

\[ \text{id} \circ \sigma_t^{-1}(e^h) = e^{t^h}, \quad \text{for all} \ t \in \mathbb{R} \]
\[ \widetilde{\theta}^{-1}(e^h) = (e^{\phi \circ 1} e^{-h}). \]

Set \( k_1 = k e^{-h}. \) Then

\[ \widetilde{\theta}^{-1}(k_1) = k_1(\rho e^{\phi \circ 1}) \]
\[ \text{id} \circ \sigma_t^{-1}(k_1)(\circ 1) = k_1^{-1}(\rho_t e^t), \quad \text{for all} \ t \in \mathbb{R}. \]

Now suppose \( \tau_4 \circ \alpha_t = -e^t \tau_4, \) for all \( t \in \mathbb{R}. \) That is \( \rho_t = e^t \) for all \( t \in \mathbb{R}. \) Then \( \text{id} \circ \sigma_t^{-1}(k_1) = k_1, \) for all \( t \in \mathbb{R}. \) Hence, there is a positive self-adjoint \( 1 \)-\( 1 \) operator \( k_2 \) affiliated to the
centre of $N$ such that $k_1 = k_2 \theta 1$. Since $\tilde{\theta}^{-1}(k_1) = k_1(\rho e^\phi \theta 1)$ we see that

$$\tilde{\theta}^{-1}(k_2) = k_2\rho e^\phi .$$

Set $\tau_2 = \tau_3(k_2^{-1} \cdot)$. A computation shows

$$\tau_2 \circ \theta = \tau_2(e^{-\phi \cdot} \cdot) .$$

Conversely, if $\tau_3 \circ \theta = \tau_3(e^{-\phi \cdot} \cdot)$ then

$$\tilde{\theta}^{-1}(k_1) = k_1 .$$

Thus, $k_1$ is affiliated to the centre of $M$. Since

$$\text{id} \circ \tau_3^{-1}(k_1^{-1}) = k_1^{-1}(\rho_t e^t), \text{ for all } t \in \mathbb{R}$$

we have

$$\alpha_{-t}(k_1^{-1}) = k_1^{-1}(\rho_t e^t), \text{ for all } t \in \mathbb{R} .$$

Set $\tau_4 = \tau_4(k_1 \cdot)$. A computation shows

$$\tau_4 \circ \alpha_t = e^{-t} \tau_4 .$$

Due to proposition 5.2 we make the following definition:

**Definition 5.3.** A discrete decomposition of a properly infinite $W^*$-algebra $P$ is a covariant system $\{N, \theta\}$ over $\mathbb{Z}$ with the properties

(i) $N$ is properly infinite and semi-finite

(ii) there is a f.n.s-f trace $\tau$ on $N$ and a $\theta$ ceiling operator $\phi$ such that

$$\tau \circ \theta = \tau(e^{-\phi \cdot} \cdot) .$$

(iii) $W^*\{N, \theta\}$ is isomorphic to $P$. 

Proposition 5.2 and theorem 5.1 yield the following connection between discrete and continuous decompositions:

**Theorem 5.4.** Let \( P \) be a properly infinite \( W^* \)-algebra, \( \{M, \alpha\} \) a covariant system over \( \mathbb{R} \), \( \{N, \theta\} \) a covariant system over \( \mathbb{Z} \) and \( \phi \) a \( \theta \) ceiling operator. Then any two of the following imply the third:

(i) \( \{M, \alpha\} \) is a continuous decomposition of \( P \)

(ii) \( \{N, \theta\} \) is a discrete decomposition of \( P \) with \( \tau \circ \theta = \tau(e^{-\phi \cdot}) \)

for some f.n.s-f trace \( \tau \) on \( N \)

(iii) \( \{M, \alpha\} \) is weakly-equivalent to the flow built on \( \{N, \theta\} \) under \( \phi \).

**Proof.** We first prove that (i) and (ii) imply (iii). Let \( \{M, \alpha^1\} \) be the flow built on \( \{N, \theta\} \) under \( \phi \). Proposition 5.2 and proposition 4.7 show that \( \{M, \alpha^1\} \) is a continuous decomposition of \( P \). Theorem 5.1 shows that \( \{M, \alpha\} \) and \( \{M, \alpha^1\} \) are weakly equivalent.

Now assume that \( \{M, \alpha\} \) is weakly equivalent to \( \{M, \alpha^1\} \) and \( \{M, \alpha\} \) is a continuous decomposition of \( P \). Then, there is a f.n.s-f trace \( \tau_1 \) on \( M_1 \) such that \( \tau_1 \circ \alpha^1_t = e^{-t \tau_1} \), for all \( t \in \mathbb{R} \).

Proposition 5.2 shows that there is a f.n.s-f trace \( \tau \) on \( N \) such that \( \tau \circ \theta = \tau(e^{-\phi \cdot}) \). Proposition 4.7 shows that \( W^* \{N, \theta\} \) is isomorphic to \( W^* \{M_1, \alpha^1\} \) which is isomorphic to \( W^* \{M, \alpha\} \) (by weak equivalence) and so \( W^* \{N, \theta\} \) is isomorphic to \( P \). Thus (ii) holds.

Finally we show that (ii) and (iii) imply (i). Proposition 5.2 shows that there is a f.n.s-f trace \( \tau_1 \) on \( M_1 \) such that \( \tau_1 \circ \alpha^1_t = e^{-t \tau_1} \), for all \( t \in \mathbb{R} \). By weak equivalence, the same is true...
of \{M,\alpha\}. Proposition 4.7 shows that $W^*\{M_1,\alpha_1\}$ is isomorphic to $W^*\{N,\theta\}$ which is isomorphic to $P$. Hence $W^*\{M,\alpha\}$ is isomorphic to $P$. Thus (i) holds.

Theorems 5.1, 2.6 and 5.4 give a generalization of [1] théorème 5.3.1.

Corollary 5.5. A properly infinite $W^*$ algebra $P$ has a discrete decomposition iff for any (and hence all) continuous decompositions $\{M,\alpha\}$, the restriction of $\alpha$ to the centre of $M$ is nowhere trivial.

Proof: Let $\{M,\alpha\}$ be a continuous decomposition of $P$ such that the restriction of $\alpha$ to the centre of $M$ is nowhere trivial. Theorem 2.6 shows that there exists $\{N,\theta\}$ and $\phi$ such that $\{M,\alpha\}$ is isomorphic to the flow built on $\{N,\theta\}$ under $\phi$. By theorem 5.4, $\{N,\theta\}$ is a discrete decomposition of $P$. Conversely, let $\{N,\theta\}$ be a discrete decomposition of $P$. Let $\{M,\alpha\}$ be the flow built on $\{N,\theta\}$ under $\phi$. Theorem 2.6 shows that the restriction of $\alpha$ to the centre of $M$ is nowhere trivial and theorem 5.4 shows that $\{M,\alpha\}$ is a continuous decomposition of $P$.

Theorems 4.5, 5.1 and 5.4 give a generalization of [1] théorème 5.4.2.

Corollary 5.6. For $j = 1,2$, let $\{N_j,\theta_j\}$ be a discrete decomposition of $P_j$. Then $P_1$ is isomorphic to $P_2$ iff for $j = 1,2$ there are recurrent projections $e_j$ in the centre of $N_j$ with $\bigvee_{n \in \mathbb{Z}} \theta_j^n(e_j) = 1$ such that the reductions $\{(N_1)_e, (\theta_1)^e_1\}$ and $\{(N_2)_e, (\theta_2)^e_2\}$ are weakly equivalent.
Proof: Assume that $P_1$ is isomorphic to $P_2$. Theorems 5.1 and 5.4 show that the flow built on $\{N_1, \theta_1\}$ under $\phi_1$ is weakly equivalent to the flow built on $\{N_2, \theta_2\}$ under $\phi_2$. By theorem 4.5 the conclusion of the theorem holds. Conversely, for $j = 1, 2$, let $\tau_j$ be the f.n.s-f trace on $N_j$ such that $\tau_j \circ \theta_j = \tau_j(e^{-\phi_j j})$. Then for $j = 1, 2$

$$\tau_j \circ (\theta_j)_{e_j} = \overline{\tau_j(e^{-\phi_j j})}$$

(where $\overline{\tau_j}$ is the restriction of $\tau_j$ to $(N_j)_{e_j}$). Let $\kappa$ be an isomorphism of $(N_1)_{e_1}$ with $(N_2)_{e_2}$ and $u$ a unitary in $(N_2)_{e_2}$ such that

$$\kappa \circ (\theta_1)_{e_1} \circ \kappa^{-1} = adu \circ (\theta_2)_{e_2}.$$

Let $f$ be the self-adjoint operator affiliated to the centre of $(N_2)_{e_2}$ such that

$$\tau_1 \circ \kappa^{-1} = \tau_2(e^{-f j}).$$

A computation shows

$$\kappa(\phi_1)_{e_1} = (\phi_2)_{e_2} + (\theta_2)^{-1}(f) - f.$$

Thus, by theorem 4.5, the flow built on $\{N_1, \theta_1\}$ under $\phi_1$ and the flow built on $\{N_2, \theta_2\}$ under $\phi_2$ are weakly equivalent. These covariant systems give continuous decompositions of $P_1$ and $P_2$ (theorem 5.4). Hence $P_1$ is isomorphic to $P_2$. ||

The constant ceiling case gives rise to a generalization of [1] théorème 4.4.1.

Corollary 5.7. Let $P$ be a properly infinite $W^*$ algebra and $c$ a
positive real number. \( P \) has a discrete decomposition \( \{N, \theta\} \) such that
\( N \) has a f.n.s-f trace \( \tau \) with \( \tau \circ \theta = e^{-ct} \) iff for any (and hence all) continuous decompositions \( \{M, \alpha\} \) of \( P \), there is a unitary \( u \) in the centre of \( M \) with \( \alpha_t(u) = e^{\frac{-it2\pi}{c}} u \), for all \( t \in \mathbb{R} \).

**Proof:** Suppose we have \( \{M, \alpha\} \) and \( u \) with \( \alpha_t(u) = e^{\frac{-it2\pi}{c}} u \), for all \( t \in \mathbb{R} \). Theorem 2.7 shows that there exists \( \{N, \theta\} \) such that \( \{M, \alpha\} \) is isomorphic to the flow built on \( \{N, \theta\} \) under \( c \). By theorem 5.4, \( \{N, \theta\} \) is a discrete decomposition of \( P \) and \( N \) has a f.n.s-f trace \( \tau \) such that \( \tau \circ \theta = e^{-ct} \). Conversely, assuming we have \( \{N, \theta\} \) and \( \tau \) such that \( \tau \circ \theta = e^{-ct} \), we let \( \{M, \alpha\} \) be the flow built on \( \{N, \theta\} \) under \( c \). Theorem 2.7 shows that there is a unitary \( u \) in the centre of \( M \) such that \( \alpha_t(u) = e^{\frac{-it2\pi}{c}} u \), for all \( t \in \mathbb{R} \). Theorem 5.4 shows that \( \{M, \alpha\} \) is a discrete decomposition of \( P \).

The corresponding uniqueness result is:

**Corollary 5.8.** For \( c > 0 \), \( j = 1,2 \), let \( \{N_j, \theta_j\} \) be a discrete decomposition of \( P_j \) such that \( N_j \) has a f.n.s-f trace \( \tau_j \) with \( \tau_j \circ \theta_j = e^{-ct} \tau_j \). Then \( P_1 \) is isomorphic to \( P_2 \) if \( \{N_1, \theta_1\} \) is weakly equivalent to \( \{N_2, \theta_2\} \).

**Proof:** Assume that \( P_1 \) is isomorphic to \( P_2 \). Theorems 5.1 and 5.4 show that the flows built on \( \{N_j, \theta_j\} \) under \( c \) for \( j = 1,2 \) are weakly equivalent. By theorem 4.6, the covariant systems \( \{N_1, \theta_1\} \) and \( \{N_2, \theta_2\} \) are weakly equivalent. Conversely, if \( \{N_1, \theta_1\} \) and \( \{N_2, \theta_2\} \) are weakly equivalent then the corresponding flows are weakly equivalent (theorem 4.6). These covariant systems give continuous decompositions of \( P_1 \) and \( P_2 \) (Theorem 5.4). Hence \( P_1 \) is isomorphic to \( P_2 \).
References


Appendix

In this appendix we give a proof of:

**Theorem 1.** Let \( \{M, \alpha\} \) be a covariant system over \( \mathbb{R} \) with \( M \) abelian and \( \sigma \)-finite. If \( \alpha \) is nowhere trivial then there is an abelian \( W^* \)-algebra \( N \) with an automorphism \( \theta \) and a \( \theta \) ceiling \( \phi \) such that \( \{M, \alpha\} \) is isomorphic to the flow built on \( \{N, \theta\} \) under \( \phi \).

The proof consists of showing that the arguments of [3] and [4] are valid under weaker conditions.

We shall need the following measure theoretical notions.

**Definition 2.** Let \( (\Omega, A, \mu) \) be a complete \( \sigma \)-finite measure space.

(i) An automorphism \( T \) of \( (\Omega, A, \mu) \) is a bimeasurable bijection of \( \Omega \) such that \( \mu \circ T \) is equivalent to \( \mu \).

(ii) If \( T \) is an automorphism of \( (\Omega, A, \mu) \), a \( T \) ceiling is a measurable function \( \phi: \Omega \rightarrow [0, \infty) \) such that there is a measurable partition \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \) of \( \Omega \) into \( T \) invariant sets and numbers \( \varepsilon_n > 0 \) for each \( n \) such that \( \phi(\omega) \geq \varepsilon_n \) for \( \omega \in \Omega_n \) and \( n = 1, 2, \ldots \).

(iii) A measurable action of a locally compact \( \sigma \)-compact (for convenience) abelian group \( G \) on \( (\Omega, A, \mu) \) is a family \( \{W_t: t \in G\} \) of automorphisms of \( (\Omega, A, \mu) \) which satisfies

(a) \( W_{t+s} = W_t \circ W_s \), for all \( s, t \in G \)

(b) if \( (G, L, m) \) denotes Haar measure on \( G \) and we equip \( \Omega \times G \) with the completion \( \widetilde{A \times L} \) of \( A \times L \) with respect to \( \mu \times m \) then the map \( (\omega, t) \rightarrow W_t(\omega) \) is measurable.
(iv) Measurable actions \( \{W_t : t \in G\} \) and \( \{\bar{W}_t : t \in G\} \) of \( G \) on \( (\Omega, \mathcal{A}, \mu) \) and \( (\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mu}) \) respectively are said to be isomorphic iff there are invariant measurable sets \( \Omega_0 \subseteq \Omega \) and \( \bar{\Omega}_0 \subseteq \bar{\Omega} \) with \( \mu(\Omega \setminus \Omega_0) = 0 \), \( \bar{\mu}(\bar{\Omega} \setminus \bar{\Omega}_0) = 0 \) and a bimeasurable bijection \( S : \Omega_0 \to \bar{\Omega}_0 \) such that

\[
S \circ W_t \big|_{\Omega_0} = \bar{W}_t \circ S
\]

and such that \( \bar{\mu} \circ S \) is equivalent to \( \mu \big|_{\Omega_0} \).

(v) If \( T \) is an automorphism of \( (\Omega, \mathcal{A}, \mu) \) and \( \phi \) is a \( T \) ceiling we refer to the measurable action \( t \to \bar{W}_t^T, \phi \) of \( \mathbb{R} \) on \( (\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mu}) \) constructed in section 2 as the flow built on the automorphism \( T \) under the function \( \phi \).

Note that if \( T \) is a bijection of a set \( \Omega \) and \( \phi : \Omega \to [\epsilon, \infty) \) for some \( \epsilon > 0 \) we can consider the action \( t \to \bar{W}_t^T, \phi \) of \( \mathbb{R} \) on the set \( \Omega_\phi = \{(\omega, s) \in \Omega \times \mathbb{R} : 0 \leq s < \phi(\omega)\} \). Our first result is a generalization of theorem 1 of [3]; the proof is almost the same.

Lemma 3. Let \( T \) be a bijection of a set \( \Omega \) and let \( \phi : \Omega \to [\epsilon, \infty) \) for some \( \epsilon > 0 \). Suppose that there is a \( \sigma \)-algebra \( B \) on \( \Omega_\phi = \{(\omega, s) : 0 \leq s < \phi(\omega)\} \) and a measure \( \nu \) on \( B \) such that

(i) \( (\Omega_\phi, B, \nu) \) is a complete \( \sigma \)-finite measure space

(ii) \( t \to \bar{W}_t^T, \phi \) is a measurable action

(iii) the functions \( F \) and \( G \) defined below are \( B \) measurable,

\[
F(\omega, s) = \phi(\omega) \text{ for } (\omega, s) \in \Omega_\phi
\]

\[
G(\omega, s) = s \text{ for } (\omega, s) \in \Omega_\phi.
\]
Then, there is a complete \( \sigma \)-finite measure \( \mu \) on a \( \sigma \)-algebra \( A \) of subsets of \( \Omega \) such that \( T \) is an automorphism of \( (\Omega, A, \mu) \), \( \phi \) is \( A \) measurable, \( B = A_\phi \) and \( \nu \) is equivalent to \( \nu_\phi \).

**Proof:** As in the proof of theorem 1 of [3] we set

\[ A = \{ E \in \Omega: E \times [0,\varepsilon) \in B \} . \]

Since \( G \) is \( B \) measurable, \( \Omega \times [0,\varepsilon) \in B \) so that \( \Omega \in A \) and \( A \) is a \( \sigma \)-algebra. Note that \( \phi \) is \( A \) measurable because \( F \) is \( B \) measurable.

For \( 0 \leq s < \varepsilon \), define bijections \( V_s \) of \( \Omega \times [0,\varepsilon) \) by:

\[
V_s(\omega, t) = \begin{cases} 
(\omega, t+s), & \text{if } 0 \leq t + s < \varepsilon \\
(\omega, t-s), & \text{if } t + s \geq \varepsilon .
\end{cases}
\]

Then

\[
V_s(\omega, t) = \begin{cases} 
W_s^{T,\phi}(\omega, t), & \text{if } 0 \leq G \cdot W_s^{T,\phi}(\omega, t) < \varepsilon \\
W_s^{T,\phi}(\omega, t), & \text{if } G \cdot W_s^{T,\phi}(\omega, t) \geq \varepsilon .
\end{cases}
\]

Since \( s \rightarrow W_s^{T,\phi} \) is a measurable action of \( \mathbb{R} \) it follows that \( s \rightarrow V_s \) is a measurable action of the group \([0,\varepsilon)\) on the reduction

\[
(\Omega \times [0,\varepsilon), B_{\varepsilon}^{\Omega \times [0,\varepsilon)}, \nu_{\varepsilon}^{\Omega \times [0,\varepsilon)}) .
\]

We adopt the notation \((\Omega \times [0,\varepsilon), B_{\varepsilon}^{\Omega \times [0,\varepsilon)}, \nu_{\varepsilon}^{\Omega \times [0,\varepsilon)})\) for this measure space.

Similarly \(([0,\varepsilon), L_{\varepsilon}^{0}, m_{\varepsilon}^{0})\) denotes the reduction of Lebesgue measure \((\mathbb{R}, L, m)\) to \([0,\varepsilon)\). Now, for \( f \) a bounded \( B \) measurable function on \( \Omega \times [0,\varepsilon) \), the function
\[(\omega,t,s) \mapsto \int_{v_s(\omega,t)}\]

on \(\Omega \times [0,\varepsilon) \times [0,\varepsilon)\) is \(B_\varepsilon \times L_\varepsilon\) measurable. So for \(v\) a.e. \((\omega,t) \in \Omega \times [0,\varepsilon)\) the function

\[s \mapsto \int_{v_s(\omega,t)}\]

on \([0,\varepsilon)\) is \(L_\varepsilon\) measurable. In fact the set of all \((\omega,t) \in \Omega \times [0,\varepsilon)\) for which this is \(L_\varepsilon\) measurable is a \(V_s\) invariant set of the form \(\Omega_f \times [0,\varepsilon)\) where \(\Omega_f \in A\) and \((\Omega \setminus \Omega_f) \times [0,\varepsilon)\) is null. Set

\[f(\omega,t) = \begin{cases} \frac{1}{\varepsilon} \int_0^\varepsilon V_s(\omega,t)ds, & \text{if } \omega \in \Omega_f \\ 0, & \text{otherwise.} \end{cases}\]

Then \(f \circ V_s = \tilde{f}\) for all \(0 \leq s < \varepsilon\). Hence, there is an \(A\) measurable function \(\tilde{f}\) such that

\[\tilde{f}(\omega) = \tilde{f}(\omega,t), \text{ for all } (\omega,t) \in \Omega \times [0,\varepsilon).\]

Define a measure \(\nu'\) on \(B_\varepsilon\) by:

\[\int f d\nu' = \frac{1}{\varepsilon} \int_0^\varepsilon f d\nu_\varepsilon\text{ for } f \text{ bounded and } B_\varepsilon \text{ measurable.}\]

Then \(\nu'\) is equivalent to \(\nu_\varepsilon\) (by Fubini's theorem) and

\[\nu' \circ V_s = \nu', \text{ for all } s \in [0,\varepsilon).\]

Define a measure \(\mu\) on \(A\) by

\[\mu(E) = \nu'(E \times [0,\varepsilon))( = \nu(E \times [0,\varepsilon)))\]

for \(E \in A\). Then \((\Omega,A,\mu)\) is a complete \(\sigma\)-finite measure space.
Note that since $G$ is $\mathcal{B}$ measurable, $A \times L^0_\varepsilon \subset \mathcal{B}_\varepsilon$ where $L^0_\varepsilon$ is the $\sigma$-algebra on $[0,\varepsilon)$ generated by the intervals. Since

$$\int fd\nu' = \int \tilde{f}d\nu'$$

for $f$ bounded and $\mathcal{B}_\varepsilon$ measurable, it follows that for $E \in A \times L^0_\varepsilon$

$$\nu'(E) = \mu \times m_\varepsilon(E).$$

Hence $A \times L_\varepsilon \subset \mathcal{B}_\varepsilon$ and $\nu'(E) = \mu \times m_\varepsilon(E)$ for $E \in A \times L_\varepsilon$. We now show that $T$ is an automorphism. For this we consider the sets

$$E_{n,k} = \{\omega \in \Omega: k2^{-n} \leq \phi(\omega) < (k+1)2^{-n}\}$$

for $n = 1,2,\ldots$ and $k = 1,2,\ldots$. Then $\Omega = \bigcup_{k=1}^{\infty} E_{n,k}$ is a partition of $\Omega$ into disjoint sets and since $F$ is $\mathcal{B}$ measurable we have $E_{n,k} \in A$ for all $n$ and $k$. For $E \subset \Omega$ we have, for sufficiently large $n$,

$$\{(T\omega,s): \omega \in E \cap E_{n,k}, 0 \leq s < \varepsilon - 2^{-n}\}$$

$$\subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{W_{k2^{-n}}((E \cap E_{n,k}) \times [0,\varepsilon]) \cap \Omega \times [0,\varepsilon]\}.$$ 

Hence

$$\{(T\omega,s): \omega \in E, 0 \leq s < \varepsilon - 2^{-n}\}$$

$$\subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{W_{k2^{-n}}((E \cap E_{n,k}) \times [0,\varepsilon]) \cap \Omega \times [0,\varepsilon]\}.$$ 

So

$$TE \times [0,\varepsilon) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{W_{k2^{-n}}((E \cap E_{n,k}) \times [0,\varepsilon]) \cap \Omega \times [0,\varepsilon]\}.$$ 

This shows that for $E \in A$, $TE \in A$ and $\mu(E) = 0$ implies $\mu(TE) = 0$. Similarly we can show that for $E \in A$, $T^{-1}E \in A$. Hence $T$ is an automorphism of $(\Omega,A,\mu)$. 
We next show that \( A \times L_\varepsilon = B_\varepsilon \). For this we consider the \( \sigma \)-algebra \( B_\varepsilon \times L_\varepsilon \). That is, the completion of \( B_\varepsilon \times L_\varepsilon \) with respect to \( v' \times m_\varepsilon \). We also consider the maps

\[
R(\omega, t, s) = (V_s(\omega, t), s)
\]

and

\[
S(\omega, t, s) = (\omega, t, G \cdot V_s(\omega, t))
\]

for \((\omega, t, s) \in \Omega \times [0, \varepsilon) \times [0, \varepsilon)\). Using addition mod \( \varepsilon \) we have

\[
R(\omega, t, s) = (\omega, t+s, s) \text{ for } (\omega, t, s) \in \Omega \times [0, \varepsilon) \times [0, \varepsilon)
\]

\[
S(\omega, t, s) = (\omega, t, s+t) \text{ for } (\omega, t, s) \in \Omega \times [0, \varepsilon) \times [0, \varepsilon).
\]

Since \( s \rightarrow V_s \) is a measurable action and since \( G \) is \( B \) measurable it follows that \( R \) and \( S \) are automorphisms of \( \Omega \times [0, \varepsilon) \times [0, \varepsilon) \). In fact \( R \) and \( S \) preserve \( v' \times m_\varepsilon \).

Writing \(-t\) for the additive inverse of \( t \) mod \( \varepsilon \) we have:

\[
R \circ S^{-1} \circ R(\omega, t, s) = (\omega, s, -t)
\]

\[
S \circ R^{-1} \circ S(\omega, t, s) = (\omega, -s, t)
\]

for \((\omega, t, s) \in \Omega \times [0, \varepsilon) \times [0, \varepsilon)\). Now consider

\[
C = \{ E \in B_\varepsilon \times L_\varepsilon : \{ (\omega, s) : (\omega, t, s) \in E \} \in A \times L_\varepsilon \text{ for a.e. } t \in [0, \varepsilon) \}.
\]

We shall show that \( B_\varepsilon \times L_\varepsilon = C \).

Note that if \( E \in B_\varepsilon \times L_\varepsilon \) then for a.e. \( s \in [0, \varepsilon) \)

\[
\{ (\omega, t) : (\omega, t, s) \in E \} \in B_\varepsilon.
\]

For \( E \in B_\varepsilon \), we apply this to \( R^{-1} \circ S \circ R^{-1} E \times [0, \varepsilon) \). We have:
\{(w,t): (w,t,s) \in R^{-1} \circ S \circ R^{-1} E \times [0,\varepsilon) \}

= \{(w,t): (w,s,-t) \in E \times [0,\varepsilon) \}

= \{w: (w,s) \in E \times [0,\varepsilon) \}.

Hence, for \( E \in \mathcal{B}_\varepsilon \) and for a.e. \( t \in [0,\varepsilon) \)

\{w: (w,t) \in E\} is \( A \) measurable.

Thus, for \( E \in \mathcal{B}_\varepsilon \) and \( F \in \mathcal{L}_\varepsilon \) and for a.e. \( t \)

\{(w,s): (w,t,s) \in E \times F\} = \{w: (w,t) \in E\} \times F \in A \times \mathcal{L}_\varepsilon.

Since \( \mathcal{C} \) is clearly a \( \sigma \)-algebra we have \( \mathcal{B}_\varepsilon \times \mathcal{L}_\varepsilon \subseteq \mathcal{C} \). To show that
\( \mathcal{B}_\varepsilon \times \mathcal{L}_\varepsilon = \mathcal{C} \) we need only show that \( \mathcal{C} \) is complete with respect to
\( \nu' \times m_\varepsilon \). For this we assume that \( E \subseteq F \subseteq \mathcal{C} \) and \( \nu' \times m_\varepsilon(F) = 0 \).

Hence \( \nu' \times m_\varepsilon(\mathcal{R}^{-1} E \times [0,\varepsilon)) = 0 \). So for a.e. \( s \)

\{(w,t): (w,t,s) \in \mathcal{R}^{-1} E \times [0,\varepsilon) \} \subseteq \mathcal{B}_\varepsilon \times \mathcal{L}_\varepsilon \) measurable.

That is, for a.e. \( s \) \( \{(w,t): (w,-s,t) \in F\} \) is \( \nu' \) null.

That is, for a.e. \( t \) \( \{(w,s): (w,t,s) \in F\} \) is \( \nu' \) null. Since
\( \{(w,s): (w,t,s) \in E\} \subseteq \{(w,s): (w,t,s) \in F\} \), the former set is
\( A \times \mathcal{L}_\varepsilon \) measurable.

This shows that \( \mathcal{C} = \mathcal{B}_\varepsilon \times \mathcal{L}_\varepsilon \). Now for \( E \in \mathcal{B}_\varepsilon \) we have
\( \mathcal{R}^{-1} E \subseteq \mathcal{B}_\varepsilon \times [0,\varepsilon) \subseteq \mathcal{C} \). That is, for a.e. \( t \)

\{(w,s): (w,t,s) \in \mathcal{R}^{-1} E \times [0,\varepsilon) \}

is \( A \times \mathcal{L}_\varepsilon \) measurable. That is, for a.e. \( t \)

\{(w,s): (w,s,-t) \in E \times [0,\varepsilon) \} = E
is $A \times L_{\phi}$ measurable. This shows that $B_{\phi} = A \times L_{\phi}$. To conclude
the proof we first note that $W_{T,\phi}^{n}$ is a measurable flow relative to
$(\Omega_{\phi}, A \times L_{\phi}, \mu \times m_{\phi})$. For $n = 1, 2, \ldots$ set

$$E_{n} = \{(\omega, t) \mid (n-1)e < \theta_{\omega, t} < ne\}.$$

Then for any set $E$,

$$E = \bigcup_{n=1}^{\infty} E \cap E_{n}$$

$$= \bigcup_{n=1}^{\infty} \left[ W_{T,\phi}^{n} E \cap E_{n} \right].$$

But $W_{T,\phi}^{n} E \cap E_{n} \subseteq \Omega \times (0, e)$. Thus $E$ is $B$-measurable iff $E$ is
$A \times L_{\phi}$ measurable. In the same way $\nu(E) = 0$ iff $\mu \times m(E) = 0$.

Our next result is a generalization of theorem 2 of [3]. The proof was
gleaned from the proofs of theorem 2 of [3] and theorem 3.1 of [4].

**Theorem 4.** A nowhere trivial measurable action of $\mathbb{R}$ is isomorphic to
a flow built under a function. (Here, nowhere trivial means the con-
tinuous action of $\mathbb{R}$ on $L^{\infty}$ is nowhere trivial).

**Proof:** Let $(X, \mathcal{B}, \nu)$ be a complete $\sigma$-finite measure space and $t \rightarrow W_{t}$
a nowhere trivial measurable action of $\mathbb{R}$ on $(X, \mathcal{B}, \nu)$. Since the
action is non-trivial we can find a measurable set $E$ and a positive
number $t_{0}$ such that $\nu((W_{t_{0}}E) \setminus E) \neq 0$. Since the action is measurable,
for $\nu$ a.e. $x$ the function

$$t \rightarrow 1_{E} \circ W_{t}(x)$$
is Lebesgue measurable. In fact the set $Y$ of all $x$ such that
$t \to 1_E \circ W_t(x)$ is Lebesgue measurable is $W_t$ invariant and
$\nu(X \setminus Y) = 0$. For each number $a > 0$ we define

$$\psi_a(x) = \begin{cases} \frac{1}{a} \int_a^0 1_E \circ W_t(x) dt, & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

$\psi_a$ is measurable by Fubini's theorem. Since the action is measurable
$f \to f \circ W_{-t}$ defines a continuous action of $\mathbb{R}$ on $L^\infty(\nu)$ and so
$\psi_a \to 1_E$ $\sigma$-weakly. Hence, we may choose a small enough so that

$$\nu(E_1 \cap W_t E_2) \neq 0$$

where

$$E_1 = \{x \in X: \psi_a(x) < 1/4\}$$

$$E_2 = \{x \in X: \psi_a(x) > 3/4\}$$

Note that $s \to \psi_a \circ W_s(x)$ is continuous for each $x$. In fact for
t, $s \in \mathbb{R}$ and $x \in X$,

$$|\psi_a(W_t(x)) - \psi_a(W_s(x))| \leq \frac{2}{a} |t-s| .$$

Define extended real value functions $\tilde{\chi}$ and $\chi$ on $X$ by:

$$\tilde{\chi}(x) = \begin{cases} \sup\{u: W_u x \in E_1 \cap W_t (E_2)\} \\ -\infty, \text{ if the set is empty} \end{cases}$$

$$\chi(x) = \begin{cases} \inf\{u: W_u x \in E_1 \cap W_t (E_2)\} \\ +\infty, \text{ if the set is empty} \end{cases}$$
Then \( \bar{x} \) and \( \bar{y} \) are measurable for by the continuity of \( \psi_a \), \( \bar{x} \) and \( \bar{y} \) are the sup and inf respectively of the measurable extended real valued functions \( \bar{x}_u \) and \( \bar{y}_u \) for \( u \) rational where

\[
\bar{x}_u(x) = \begin{cases} 
 u, & \text{if } W_u(x) \in E \cap W_{t_0} (E_2) \\
 -\infty, & \text{otherwise}
\end{cases}
\]

\[
\bar{y}_u(x) = \begin{cases} 
 u, & \text{if } W_u(x) \in E \cap W_{t_0} (E_2) \\
 +\infty, & \text{otherwise}.
\end{cases}
\]

Define measurable sets

\[
X_1 = \{x \in X: \bar{x}(x) = \infty, \bar{y}(x) = -\infty\}
\]

\[
X_2 = \{x \in X: \bar{x}(x) = \infty, \bar{y}(x) > -\infty\}
\]

\[
X_3 = \{x \in X: -\infty < \bar{x}(x) < \infty\}
\]

\[
X_4 = \{x \in X: \bar{x}(x) = -\infty\}
\]

These sets are measurable, disjoint and they cover \( X \). Since \( E \cap W_{t_0} (E_2) \subset X_1 \cup X_2 \cup X_3 \), one of \( X_1, X_2 \) or \( X_3 \) is non-null. Since each \( X_i \) \( i = 1, \ldots, 4 \) is \( W \) invariant we shall show that the reduction of \( t \rightarrow W_t \) to each \( X_i \) \( i = 1, 2, 3 \) is isomorphic to a flow build under a function.

For \( x \in X_1 \), the set \( \{u: W_u(x) \in E \cap W_{t_0} (E_2)\} \) contains arbitrarily large positive and negative numbers \( u \). Set

\[
\Omega = \{x \in X_1: \psi_a(x) = \frac{1}{2} \text{ and } \psi_a W_t(x) > \frac{1}{2} \text{ for } 0 < t < \frac{a}{8}\}.
\]

As in the proof of theorem 2 of [3], for each \( x \in X_1 \) the trajectory
\{W^x: t \in \mathbb{R}\} intersects \(\Omega\) for arbitrarily large positive and negative numbers \(t\). For \(\omega \in \Omega\) set

\[\phi(\omega) = \inf\{t > 0: W^t_\omega(\omega) \in \Omega\}\.\]

Then \(\phi > \alpha/8\). For \(\omega \in \Omega\), set \(T_\omega = W^s_\omega(\omega)\) where \(s = \inf\{t > 0: W^t_\omega(\omega) \in \Omega\}\). We map \(\Omega_\phi\) onto \(X_1\) by:

\[S(\omega, t) = W^t_\omega(\omega), \text{ for } (\omega, t) \in \Omega_\phi\.\]

Note that \(S \circ W^t_\phi = W^t_\omega \circ S\), for all \(t \in \mathbb{R}\). Using \(S\) we obtain a complete \(\sigma\)-finite measure on \(\Omega_\phi\) for which \(W^t_\phi\) is a measurable action. As in the proof of theorem 2 of [3], the functions \(F\) and \(G\) of lemma 3 are measurable. Hence, by lemma 3, the reduction to \(X_1\) is isomorphic to a flow built under a function.

We next show that the reduction to \(X_2\) is isomorphic to a flow built under a function. For \(x \in X_2\), \(-\infty < \chi(x) < \infty\) and \(\chi(W^t_\omega x) = \chi(x) - t\) for all \(t \in \mathbb{R}\). Set

\[\Omega = \bigcup_{n = -\infty}^{\infty} \{x \in X_2: \chi(x) = n\}\]

\[\phi(\omega) = 1 \text{ for } \omega \in \Omega\]

\[T_\omega = W^1_\omega(\omega) \text{ for } \omega \in \Omega\.\]

Define a bijection \(S\) of \(\Omega_\phi\) onto \(X_2\) by

\[S(\omega, t) = W^t_\omega(\omega), \text{ for } (\omega, t) \in \Omega_\phi\.\]

Then \(S \circ W^t_\phi = W^t_\omega \circ S\), for all \(t \in \mathbb{R}\). Using \(S\) we obtain a complete \(\sigma\)-finite measure on \(\Omega_\phi\) such that \(t \mapsto W^t_\phi\) is a measurable action. The functions \(F\) and \(G\) of lemma 3 are measurable so by
lemma 3 the reduction of $t \to W_t$ to $X_2$ is isomorphic to a flow built under a function. For $X_3$ we proceed as for $X_2$ using $\chi$.

Now select a maximal disjoint family of non-null, measurable, $W$ invariant sets such that the reduction to each is isomorphic to a flow built under a function. Since the sets are disjoint and non-null the family is countable. Since the flow is nowhere trivial the first part of the proof shows that the complement of the union of this family is null. By taking the "direct sum" of the flows built under functions we obtain an isomorphism of $W_t$ with a flow built under a function.

The following result is well known in the "separable" case. We include a proof for the sake of completeness.

Lemma 5. Let $(M, \alpha)$ be a covariant system over a locally compact, $\sigma$-compact, abelian group $G$. Suppose that $M$ is abelian and $\sigma$-finite. Then there is a complete $\sigma$-finite measure space $(\Omega, \mathcal{A}, \mu)$, a measurable action $t \to W_t$ of $G$ on $(\Omega, \mathcal{A}, \mu)$ and an isomorphism $\kappa$ of $L^\infty(\mu)$ with $M$ such that for all $f \in L^\infty(\mu)$ and all $t \in G$

$$\kappa(f \circ W_{-t}) = \alpha_t(\kappa f).$$

Proof: Let $N$ be the subset of $M$ consisting of all $x$ for which $t \to \alpha_t(x)$ is norm continuous. $N$ is a $C^*$-subalgebra of $M$ containing $1$. For $f$ a continuous, compactly supported function on $G$ and for $x \in M$

$$y = \int_G f(t)\alpha_t(x)dt$$
is in $N$. Hence $N$ is $\sigma$-dense in $M$.

We let $\Omega$ be the spectrum of $N$ and let $\pi: C(\Omega) \rightarrow N$ be the inverse of the Gelfand transformation. Since $\alpha_t$ preserves $N$, we can find homeomorphisms $W_t$ for $t \in G$ of $\Omega$ such that

$$\pi(f \circ W_t) = \alpha_t(\pi(f))$$

for each $f \in C(\Omega)$ and $t \in G$. It follows that $t \rightarrow W_t$ is an action of $G$ on $\Omega$. That is, $W_{t+s} = W_t \circ W_s$, for all $t, s \in G$.

Since $t \rightarrow f \circ W_t$ is norm continuous for $f \in C(\Omega)$ it follows that $\omega, t \rightarrow f \circ W_t(\omega)$ is product measurable where we take the Borel $\sigma$-algebra on $\Omega$ and let $(G, L, m)$ be Haar measure on $G$. Hence, the same is true for a Borel measure function $f$ on $\Omega$.

Now, let $\tau$ be a faithful normal state on $M$ (since $M$ is $\sigma$-finite we can find such a state) and let $(\Omega, A, \mu)$ be the Radon measure on $\Omega$ which satisfies

$$\int f d\mu = \tau(\pi(f)), \text{ for all } f \in C(\Omega).$$

Using the G.N.S. construction for $\tau$, we may assume that $M$ acts on a Hilbert space $H$ which contains a cyclic and separating vector $\xi_0$ with

$$\tau(x) = (x\xi_0, \xi_0), \text{ for all } x \in M.$$

Since the support of $\mu$ is $\Omega$, $C(\Omega)$ imbeds in $L^2(\mu)$ and the map

$$f \mapsto \pi(f)\xi_0$$

extends to a unitary operator $U: L^2(\mu) \rightarrow H$. For $f \in L^\infty(\mu)$ we
identify \( f \) with the multiplication operator \( \xi \rightarrow f \xi \) for \( \xi \in L^2(\mu) \).

Under this identification we have

\[
U_f U^* = \pi(f), \quad \text{for all } f \in C(\Omega).
\]

Hence \( \kappa: f \rightarrow U_f U^* \) is an isomorphism of \( L^\infty(\mu) \) with \( M \).

We now show that each \( W_t \) is an automorphism of \( (\Omega, A, \mu) \).

Since \( W_t \) is a homeomorphism it suffices to show that if \( E \subset \Omega \) is a Borel set with \( \mu(E) = 0 \) then \( \mu(W_t E) = 0 \). For any compact set \( K \subset W_t(E) \), \( W_{-t}(K) \) is compact and \( W_{-t}(K) \subset E \). We can find a sequence \( f_n, n = 1, 2, \ldots \) of continuous functions such that

\[
1_{W_{-t}(K)} \leq f_n \leq 1
\]

and

\[
\int f_n \, d\mu \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

That is, \( \tau(\pi(f_n)) \rightarrow 0 \) as \( n \rightarrow \infty \) and \( 0 \leq \pi(f_n) \leq 1 \) for all \( n \).

Hence, \( \pi(f_n) \rightarrow 0 \) \( \sigma \)-weakly as \( n \rightarrow \infty \) and so

\[
\tau(\alpha_t \pi(f_n)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

That is

\[
\int f_n \circ W_{-t} \, d\mu \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

Since \( 1_K \leq f_n \leq 1 \) and since \( f_n \circ W_{-t} \) is continuous, we conclude that \( \mu(K) = 0 \). Hence \( \mu(W_t E) = 0 \).

We next show that \( t \rightarrow W_t \) is a measurable action of \( G \) on \( (\Omega, A, \mu) \). For this it suffices to show that if \( E \subset \Omega \) is Borel with \( \mu(E) = 0 \) then
\[ F = \{ (\omega, t) : \mathcal{W}_t \omega \in E \} \]

(which is product measurable) is \( \mu \times m \) null. But

\[
\mu \times m(F) = \int \left[ \int_{\Omega} l_{F}(\omega, t) d\mu(\omega) \right] dt
\]
\[
= \int \left[ \int_{\Omega} l_{E \circ \mathcal{W}_t (\omega)} d\mu(\omega) \right] dt
\]
\[
= \int \mu(W_{-t}E) dt
\]
\[
= 0 .
\]

Finally, we have \( \kappa(f \circ W_{-t}) = \alpha_t(\kappa f) \) for \( f \in C(\Omega) \). Since \( C(\Omega) \) is \( \sigma \)-dense in \( L^\infty(\mu) \) and since \( \mathcal{W}_t \) defines a continuous action of \( G \) on \( L^\infty(\mu) \) we have

\[
\kappa(f \circ W_{-t}) = \alpha_t(\kappa f), \text{ for all } f \in L^\infty(\mu) .
\]

Proof of theorem 1: Use lemma 5 to represent \( \alpha \) as a measurable action of \( \mathbb{R} \) on a complete \( \sigma \)-finite measure space. Apply theorem 4 to conclude that this measurable action is isomorphic to a flow built under a function. Lemma 2.1 now shows that \( \{ M, \alpha \} \) is isomorphic to a flow built under a ceiling operator.