NORMAL FUNCTIONS OF PRODUCT VARIETIES

by

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Abstract

The work of this thesis is to motivate the following:

Statement: The Hodge conjecture holds for products of varieties

\[ Z = X \times C \] where (i) \( X \) is smooth, projective of dimension \( 2m-1 \), (ii) \( C \) is a smooth curve.

The basic setting of this thesis is depicted by the following diagram:

\[ \overrightarrow{k}^{-1}(U) = Z^0 \rightarrow \overrightarrow{Z} \quad \overset{k}{\longrightarrow} \quad \overrightarrow{k} \quad \overset{\Sigma}{\longrightarrow} \quad \mathbb{P}^1 \]

where (i) \( \overrightarrow{k}^{-1}(t) = Z_t = \{ X_t \} \quad \text{a Lefschetz pencil of hyperplane sections of } X \)

(ii) \( \Sigma \) is the singular set of \( k \), i.e., \( k = \overrightarrow{k} \bigg|_{\overrightarrow{Z}^0} \) is smooth and proper.

Corresponding to this diagram are the extended Hodge bundle \( \bigcup_{t \in \mathbb{P}^1} H^{2m-1}(Z_t, \mathbb{C}) \) with integrable connection \( \overrightarrow{\nabla} \), and the family of intermediate Jacobians \( \bigcup_{t \in \mathbb{P}^1} J(Z_t) \) with corresponding normal functions \( \bar{v} : \mathbb{P}^1 \longrightarrow \bigcup_{t \in \mathbb{P}^1} J(Z_t) \). Now \( \overrightarrow{\nabla} \) induces an operator (also denoted by \( \overrightarrow{\nabla} \)) on the normal functions, and those normal functions \( \bar{v} \) satisfying the differential equation \( \overrightarrow{\nabla} \bar{v} = 0 \) are labeled horizontal, which includes those normal functions arising from the primitive algebraic cocycles.
in $H^{2m}(Z)$. Now the known generalization of Lefschetz's techniques state that every primitive integral class of type $(m,m)$ in $H^{2m}(Z)$ comes from a horizontal normal function in some natural way, so that what's needed to prove the above statement is some way of converting a normal function to an algebraic cocycle. We motivate this statement by proving some results about the group of normal functions, in particular our main result:

**Theorem:** The group of normal functions are horizontal.

To prove this theorem, we exhibit $\tilde{v}_v$ as a global section of some holomorphic vector bundle over $\mathbb{P}^1$, and then show that there are no non-zero global sections of this vector bundle. The main idea is to compare the quasi-canonical extensions of certain holomorphic vector bundles with integrable connection with those extensions arising from algebra (hypercohomology), by calculating certain periods of growth. Once this comparison is made precise, we apply a vanishing theorem statement about the global sections of the algebraic extensions to our geometric extensions, thus concluding the proof of the theorem.
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Introduction

The modern techniques used in the work of the Hodge conjecture stems from the ideas introduced by Poincaré and Lefschetz, where Poincaré originally introduced the notion of a normal function, and Lefschetz subsequently used it later to solve the Hodge conjecture for surfaces. Poincaré's motivation for using normal functions came from his study of curves on a complex, algebraic surface S, where he associated algebraic families of curves with normal functions and hence was able to determine the dimension of the "now called" Picard variety, Pic(S) of S, by means of some linear data, namely \( \dim \text{Pic}(S) = \dim \mathbb{H}^1(S, \theta_S) \). We remark that this result is not entirely valid in characteristic \( p > 0 \).

In higher dimensions, the Hodge conjecture is only known for very special cases, such as for hypersurfaces \( Z \) in \( \mathbb{P}^5 \) of degree \( \leq 5 \) ([2] and [30]), and Fermat varieties of certain dimensions and degrees ([24]), where the knowledge of the geometry of such \( Z \) is required in proving the conjecture.

The state of art in attempting to prove the Hodge conjecture for a given \( Z \) of dimension \( 2m \) is to use a generalized form of Lefschetz's techniques, and to do this one first associates to every primitive, integral cohomology class \( \gamma \in \text{Prim}^{m,m}(Z, \mathbb{Z}) \) a complex, analytic object, namely a normal function \( \nu \), and secondly find some way of transforming \( \nu \) into an algebraic cycle.

As mentioned above, when \( m = 1 \) this conjecture unfolds, and is
due to the Jacobi inversion theorem for curves (cf (0.29)). In higher dimensions there is no longer necessarily such a strong relationship between a variety and its Hodge structure as in the case of a curve, and consequently we are left with the need to study the normal functions in more detail, in particular in terms of their infinitesimal properties (cf [10]).

The work of this thesis is to extend some of the known generalizations of Lefschetz's techniques (such as in [28]) to the case of the product varieties \( Z = X \times C \), where \( X \) is a smooth, projective variety of dimension \( 2m-1 \), and \( C \) is a smooth curve. We focus on a particular fibering of \( Z \), namely a pencil of divisors of the form \( \{ Z_t = X_t \times C \}_{t \in \mathbb{P}^1} \), where \( \{ X_t \}_{t \in \mathbb{P}^1} \) is a Lefschetz pencil of hyperplane sections of \( X \). Associated to this pencil are the extended\(^1\) Hodge bundle \( \mathcal{H}^{2m-1}(Z_t, C) \) with extended connection \( \nabla \), the Hodge filtration subbundles \( \mathcal{F}_p H^{2m-1}(Z_t, C) \) \((p \geq 0)\), their duals \( \mathcal{F}^p H^{2m-1}(Z_t, C) \), and the corresponding family of intermediate Jacobians \( \mathcal{J}(Z_t, C) \), where generically \( \mathcal{J}(Z_t) = \mathcal{F}_m H^{2m-1}(Z_t, C) / H^{2m-1}(Z_t, \mathcal{I}) \).

Now the connection \( \nabla \) does not preserve the Hodge filtration above, but rather satisfies a so called infinitesimal period relation. As a consequence of this relation and the known explicit description of this connection, \( \nabla \) induces an operator (still denoted by \( \nabla \)) on the holomorphic cross sections (normal functions) \( \nu : \mathbb{P}^1 \rightarrow \bigcup_{t \in \mathbb{P}^1} \mathcal{J}(Z_t) \). In chapter

\(^1\)Extended referring to a quasi-compactification over the singular fibres.
4, we prove that the Abel-Jacobi mapping (defined in chapter 0) is meromorphic, and as a consequence it can be shown that a primitive, algebraic cocycle \( \gamma \in \text{Prim}^{m,m}(Z,\mathbb{Z}) \) induces a normal function \( v : \mathbb{P}^1 \to \cup_t J(Z_t) \). Now it is a standard fact that such \( v \) satisfy the differential equation \( \nabla v = 0 \) (the horizontality condition), and we prove in this thesis that all our normal functions satisfy this horizontality condition, thus resembling the situation of a Lefschetz pencil of hyperplane sections of \( Z \) ([28]).

The plan of this thesis is roughly as follows:

In chapter 1 we check that the important part of the integral cohomology \( \text{Prim}^{m,m}(Z,\mathbb{Z}) \) lies in the image of the cohomology classes of the normal functions.

In chapter 2, we rigorously define the generalized intermediate Jacobians \( J(Z_t) \) and identify a subgroup lying in the image of the Abel-Jacobi morphism. For a certain restricted class of \( Z \), this subgroup is all of \( J(Z_t) \) so that using the techniques in chapter 4, we are able to mimic a modern form of Lefschetz's proof to obtain the Hodge conjecture for such \( Z \). Although the knowledge of the conjecture for such \( Z \) is not significantly newer than what's already known, the known techniques for proving the conjecture for these \( Z \) requires the understanding of the geometry of \( X \), whereas in this thesis the emphasis is on the geometry of the hyperplane sections \( X_t \) of \( X \).

In chapter 3, we deduce our horizontality statement on the normal functions from a vanishing theorem result, thus providing a step in the study of the normal functions in terms of their infinitesimal
properties. In addition to this, we arrive at an explicit description of the kernel of the cohomology class homomorphism \( \delta : \text{normal functions} \rightarrow \text{cohomology classes} \) (defined in chapter 2).

The ideas on proving the horizontality statement for the normal functions come from a close study of [28, §4], where the differences in some proofs are accounted for by the difference in local monodromy between the case \( \{ Z_t \}_{t \in \mathbb{P}^1} \) and that of a Lefschetz pencil ([28]). We also utilize the knowledge of the irreducibility of the monodromy group action on the rational, vanishing cohomology associated to the \( \{ X_t \}_{t \in \mathbb{P}^1} \), and exploit this fact to its fullest to obtain the desired horizontality result.

Finally, in chapter 4 we summarize our results on the normal functions, and make some general remarks in this direction.

A summary of the logical sequence of steps in this thesis is given in the chart below.
Explicitly describe the \( E_{2}^{1,2m-1}(\bar{k}) \) term, which contains the cohomology classes of the normal functions.

Construct the generalized intermediate Jacobians \( J(Z_{t}) \) and compute a subgroup \( J_{A}(Z_{t}) \) in \( J(Z_{t}) \) which lies in the image of the Abel-Jacobi morphism.

Every normal function with values in the \( J_{A}(Z_{t}) \) gives rise to an algebraic cocycle. This corresponds to the Hodge conjecture for case 1 in ch. 4.

(i) Explicitly describe the kernel of \( \delta : \) normal fxns \( \rightarrow \) cohomology classes in \( E_{2}^{1,2m-1}(\bar{k}) \).

(ii) Corollaries.

Deduce that all the normal functions are horizontal from a vanishing theorem result.

Prove the meromorphicity of the Abel-Jacobi mapping.

(i) Discuss those normal functions which do not take their values in the \( J_{A}(Z_{t}) \). The difficulty in proving the Hodge conjecture for \( Z \) lies here. The techniques in this thesis are insufficient to prove the conjecture at this stage.

(ii) Some conjectural statements made.
Chapter 0. The Preliminaries.

The subject of this thesis is centered around an analytic result about a special class of manifolds, namely the Hodge conjecture for products of varieties based on certain inductive assumptions. The general statement of the conjecture is for all projective, algebraic manifolds. The striking phenomena about such manifolds $Z$ is the relationship between their analytic and topological properties, formally expressed in the following way:

\[(0.1) \quad H^m(Z, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(Z, \mathbb{C}), \quad m = 0, 1, 2, \ldots\]

where $H^m(Z, \mathbb{C})$, $H^{p,q}(Z, \mathbb{C})$ are the deRham, respectively Dolbeault cohomology with complex coefficients. This relationship holds more generally for compact, complex manifolds $Z$ which are known to be Kähler ([11]).

In doing analysis on such manifolds, one is frequently confronted with both the topological and analytic aspects of the subject, as reflected by the above relationship (0.1). In this thesis, the topological aspects will be the weak and strong Lefschetz theorems, the primitive decomposition theorem, Lefschetz pencils, the Leray spectral sequence, whereas for the analytic aspect we have the Hodge filtrations subbundles, the Gauss-Manin connection and the regularity theorem, and hypercohomology. Both these aspects play a role in reducing the problem of the Hodge conjecture to something more manageable as will be discussed below.
The statement of the Hodge conjecture for projective, algebraic manifolds $Z$ is the following (let $\mathbb{Q} =$ rational numbers):

(0.2) $\text{Hodge}^{p,p}(Z,\mathbb{Q})$: The fundamental class homomorphism, $\mathbb{Q} \otimes \mathbb{C}^p(Z) \to H^{p,p}(Z,\mathbb{Q})$ is an epimorphism, where $\mathbb{C}^p(Z)$ is the group of codimension $p$ algebraic cycles in $Z$ and $H^{p,p}(Z,\mathbb{Q}) = H^{p,p}(Z,\mathbb{C}) \cap H^{2p}(Z,\mathbb{Q})$.

(0.3) equivalently: (the group generated by $\{c_p(V) | V$ is a holomorphic vector bundle over $Z\}) \otimes \mathbb{Q}$.

$= H^{p,p}(Z,\mathbb{Q})$, where $c_p(V)$ is the $p^{th}$ Chern class of $V$.

The equivalence of these two statements can be found in [9, Theorem Q].

We should remark that the original statement of the Hodge conjecture involved integral coefficients ($\text{Hodge}^{p,p}(Z,\mathbb{Z})$) for Kähler manifolds $Z$. Both formulations ((i) integer coefficients, (ii) Kähler manifolds) are now known to be false, and one can find a suitable counterexample for (ii) in [30], and a comment on (i) in [13].

The main body of theorems required in this thesis will be stated now. We first fix an embedding $i: Z \subset \mathbb{C} \to Z$ of a smooth, ample divisor $Z_t$ in a projective manifold $Z$, and denote $n = \dim Z$. Also we let $[Z_t] \in H^{1,1}(Z,\mathbb{Z})$ be the fundamental class of $Z_t$, and $[Z_t]^r = \cup \text{ product (}$ of $[Z_t]$ with itself $r$ times. We introduce the following:
(0.4) **Definition.** Let \( m \leq n \). The primitive \( m \) cohomology, \( \text{Prim}^m(Z) \) is defined to be the kernel of the cup product homomorphism
\[
\bigwedge [Z_t]^{n-m+1} : H^m(Z) \longrightarrow H^{2n-m+2}(Z).
\]

The following are true:

(0.5) **Theorem (weak Lefschetz).** \( i_* : H^r(Z, \mathbb{Z}) \longrightarrow H^r(Z, \mathbb{Z}) \) is an isomorphism for \( r < n - 1 \), and an epimorphism for \( r = n - 1 \).

(0.6) **Theorem (strong Lefschetz).** \( \bigwedge [Z_t]^r : H^{n-r}(Z, \mathbb{C}) \longrightarrow H^{n+r}(Z, \mathbb{C}) \)
is an isomorphism for all \( 0 \leq r \leq n \).

(0.7) **Theorem (primitive decomposition).**
\[
H^r(Z, \mathbb{C}) \cong \bigoplus_{\ell \geq \max\{0, r-n\}} [Z_t]^{\ell} \cap \text{Prim}^{r-2\ell}(Z, \mathbb{C}).
\]

We remark that due to the bidegree property of \( \bigwedge [Z_t] \), i.e., \( \bigwedge [Z_t] : H^{p,q}(Z, \mathbb{C}) \longrightarrow H^{p+1,q+1}(Z, \mathbb{C}) \), the primitive decomposition theorem holds for \( H^{p,q}(Z, \mathbb{C}) \), and also for \( H^{p,p}(Z, \mathbb{C}) \). More precisely:

(0.8) \[
H^{p,q}(Z, \mathbb{C}) \cong \bigoplus_{\ell \geq \max\{0, p+q-n\}} [Z_t]^{\ell} \cap \text{Prim}^{p-\ell,q-\ell}(Z, \mathbb{C})
\]

\[
H^{p,p}(Z, \mathbb{C}) \cong \bigoplus_{\ell \geq \max\{0, 2p-n\}} [Z_t]^{\ell} \cap \text{Prim}^{p-\ell,p-\ell}(Z, \mathbb{C}).
\]

If one wants to prove Hodge \( \mathbb{C}^{p,p}(Z, \mathbb{C}) \) inductively for all \( p \geq 0 \) and all \( Z \), then the following deductions can be made:

(i) The weak and strong Lefschetz theorems imply that the conjecture need only be proven for \( \dim Z = n = 2m \) and \( p = m \).

(ii) Using the fact that the composite:
\[
H^{2m-2}(Z) \xrightarrow{i_*} H^{2m-2}(Z_t) \xrightarrow{i_*} H^{2m}(Z)
\]is precisely the cup product with...
[\mathcal{Z}_t], the primitive decomposition theorem implies that \( H^{2m}(\mathcal{Z}) = \text{Prim}^{2m}(\mathcal{Z}) \oplus i_* H^{2m-2}(\mathcal{Z}_t) \). Therefore it suffices to prove the following:

(0.9) \( \text{Prim}^{m,m}(\mathcal{Z}, \mathbb{Q}) \) is generated over \( \mathbb{Q} \) by algebraic cocycles.

The basic setting of this thesis is depicted by the following diagram:

\[
\begin{array}{c}
\mathcal{Z} \\
\downarrow k \\
\mathbb{P}^1 - \Sigma = U \\
\mathbb{P}^1 = \text{complex, projective 1 space.}
\end{array}
\]

where

(i) \( \bar{k} \) is a morphism, \( \dim \mathcal{Z} = 2m \)

(ii) \( \Sigma \) = singular set of \( \bar{k} \) (i.e., the finite set of points in \( \mathbb{P}^1 \) where \( \bar{k} \) is not smooth)

(iii) the cohomology of \( \mathcal{Z} \) differs from that of \( \mathcal{Z} \) by the cohomology of a variety of dimension \( \leq n-2 \).

We remark that \( k \) is smooth and proper over \( U \).

Now it is known that the fibers of \( k \) are all topologically equivalent (in the \( C^\infty \) sense) so that the Leray cohomology sheaf \( R^{2m-1}k_* \mathcal{C} \) over \( U \) associated to the presheaf: \( V \subset U \) open

\[
\lim_{\substack{\text{open} \\ V}} H^0(V, R^{2m-1}k_* \mathcal{C}) = H^{2m-1}(k^{-1}(V), \mathcal{C})
\]

In the situation of (0.10), there is the Leray spectral sequence for \( \bar{k} \) which abuts to \( H^\bullet(\mathcal{Z}, \mathcal{C}) \), and whose \( E_2 \) terms are:

\[
E_2^{p,q}(k) = H^p(\mathbb{P}^1, R^qk_* \mathcal{C}).
\]
sequence degenerates at the $E_2$ terms, i.e., $E_2^{P,q}(k) \rightarrow H^{P+q}(Z,\mathbb{C})$, and this for example occurs when (0.10) arises from a Lefschetz pencil. The definition of a Lefschetz pencil is given in (1.1) and the existence of such pencils is proven in [6, exposé XVII].

Due to the nature of the techniques in this paper, it becomes necessary to utilize the sheaf of invariant cycles defined by $j_* R^{2m-1} k^* \mathbb{C}$ where $K$ is the integers, rationals, or complex numbers. The cohomology of main interest in this thesis is $E_2^{1,2m-1}(k) = H^1(P^1, R^{2m-1} k^* \mathbb{C})$ which can be compared with the more naturally arising cohomology $H^1(P^1, j_* R^{2m-1} k^* \mathbb{C})$, by means of the local cycle invariant property which states the following:

(0.11) Theorem ([31, (15.12)]). The restriction homomorphism $R^{2m-1} k^* \mathbb{C} \rightarrow j_* R^{2m-1} k^* \mathbb{C}$ is an epimorphism.

We actually prove that the above is an isomorphism for our particular morphism $\bar{k}$, however it should be remarked that the kernel of the above epimorphism has zero dimensional support, hence $H^1(P^1, R^{2m-1} k^* \mathbb{C}) \simeq H^1(P^1, j_* R^{2m-1} k^* \mathbb{C})$.

(0.12) Focusing on the analytic aspects of (0.10), there is a holomorphic vector bundle over $U$ with corresponding locally free sheaf $F = \theta_U \otimes R^{2m-1} k^* \mathbb{C}$ and integrable connection $\nabla = \bar{\partial} \otimes 1$ which can also be derived algebraically ([19]) via the relation:

$R^{2m-1} k^* \mathbb{C} \otimes \Omega^\cdot_{Z/U} \simeq \theta_U \otimes R^{2m-1} k^* \mathbb{C}$, where $R^\cdot$ is the right derived

\footnote{We define $\theta_U$ to be the sheaf of germs of holomorphic functions on $U$.}
hypercohomology coming from the spectral sequence of a suitable
bicomplex ([12, Chapter 0]). The complex of relative differentials
\( \Omega^\cdot_{Z/\cal U} \) is defined in the following way:

\[
\begin{align*}
(0.13) & \quad \Omega^1_{Z/\cal U} \text{ is the cokernel sheaf in the short exact sequence:} \\
& \quad 0 \to \Omega^1_{U, \cdot} \xrightarrow{k^*} \Omega^1_{\cdot, \cal U} \to \Omega^1_{\cdot, \cal U} \to 0 \quad \text{and} \quad \Omega^r_{\cdot, \cal U} = \Lambda^r \Omega^1_{\cdot, \cal U} \ . \end{align*}
\]

There is a differential \( \partial : \Omega^r_{\cdot, \cal U} \to \Omega^{r+1}_{\cdot, \cal U} \) induced from the differential on
\( \Omega^r_{Z/\cal U} \), and one checks that \( \partial \) is \( \cal U \)-linear, making \( \Omega^\cdot_{\cal U} \) into a
complex. There is a filtration on \( \Omega^\cdot_{\cal U} \) defined by:

\[
(0.14) \quad \text{This defines a filtration} \ F = F^0 \supset F^1 \supset F^2 \supset \ldots \supset F^{2m-1} \supset 0
\]

by Hodge subbundles, where \( F^p = \mathbb{R}^{2m-1} k^*_y F^p \Omega^\cdot_{\cal U} \).

(0.15) The infinitesimal period relation holds: \( \forall F^p \subset \Omega^1_{\cal U} \otimes F^{p-1} \) for
all \( p \geq 0 \) ([8]).

(0.16) There is a bilinear form \( Q(-,-) \) defined on \( H^r(Z,\cal O) \) in the
following way: Let \( \xi, \eta \in H^r(Z,\cal O) \) and from (0.7) express \( \xi, \eta \) as
\( \xi = \bigoplus \{ \xi^\ell \} \in \cal H^\cdot \ell \), \( \eta = \bigoplus \{ \eta^\ell \} \in \cal H^\cdot \ell \). Define \( Q(\xi, \eta) = \)
Given the Hodge decomposition $H^r(Z,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(Z,\mathbb{C})$, define $\text{Pr}^{p,q} : H^r(Z,\mathbb{C}) \to H^{p,q}(Z,\mathbb{C})$ to be the canonical projection. Then the Weil operator

$$C : H^r(Z,\mathbb{C}) \to H^r(Z,\mathbb{C})$$

is defined as:

$$C = \sum_{p+q=r} \text{Pr}^{p,q}.$$

It is a standard fact ([11]) that the bilinear form $\langle \xi, \eta \rangle = \mathcal{Q}(\xi, \mathcal{C}_D)$ is positive definite Hermitian.

(0.17) Let $L$ be a relatively flat ample divisor in $Z$, i.e., induces an ample divisor on the fibers $k^{-1}(t)$ over $P^1$. Then (0.6) and (0.7) generalize to.

(0.18) Let $L : j_* R^{2m-1-r}k_* C \to j_* R^{2m-1+r}k_* C$ be an isomorphism

$$j_* R^r k_* C \cong \bigoplus_{\ell \geq \max\{0, r-n\}} L^\ell \bigoplus_{r \geq \max\{0, r-n\}} j_* R^{r-2\ell}k_{\text{Prim}}^{r-2\ell}k_* C.$$

In particular, (0.18) holds over $U$, and the constructions in (0.16) generalize here.

(0.19) The Hodge filtration\footnote{We are still working over $U$.} in (0.14) is isotropic, that is $(H^F_p)^\perp = F^{2m-p}$, where $(H^F_p)^\perp = \{ u \in F | \mathcal{Q}(u, H^F_p) = 0 \}$. This naturally identifies the dual bundle $F_p^\ast$ with $F/F^{2m-p}$ (see (2.12)).

(0.20) The holomorphic bundles $\mu_{r,2m-1-p} \overset{\text{def'n}}{=} F^p/F^{p+1}$ are not holomorphic subbundles of $F$, however if we let $\varepsilon_U$ be the sheaf of $C^\infty$ functions on $U$, and define $\varepsilon_p = \varepsilon_U \otimes F^p$, $\varepsilon_{p,2m-1-p} = \varepsilon_U \otimes F^p_{2m-1-p}$, then there is a $C^\infty$ direct sum decomposition:
\( \epsilon = \epsilon^0 = \sum_{p+q=2m-1} \epsilon_{p,q}. \) We remark that \( Q(-,-) \) is defined on \( \epsilon \), and

\[ \epsilon_{p,2m-1-p} = \epsilon_p \cap \epsilon \bar{2m-1-p} = \epsilon_p \cap \epsilon_{p+1} \perp \] (- means complex conjugation).

(0.21) There are the Riemann-Hodge bilinear relations associated to the bundles \( \epsilon_{p,q} \) and \( Q(-,-) \), which are mentioned in [8, p. 133].

According to the results in [8], the data (0.12) \( \to \) (0.21) define a polarizable variation of Hodge structure over \( U \). This terminology will be useful in Chapter 4. It should be remarked that not all variations of Hodge structures necessarily come from the situation (0.10) above. Those that do are said to arise from a geometric situation. For more on this see [8].

In Chapter 2 we discuss the quasi-canonical extensions of the sheaves \( \{ F^p \}_{p \geq 0} \) over \( U \) to locally free sheaves \( \{ \bar{F}^p \}_{p \geq 0} \) over \( \mathbb{P}^1 \).

The Gauss-Manin connection also extends to an operator \( \bar{V} \) on the \( \bar{F}^p \). There is the regularity theorem ([20]) associated to the \( \bar{F}^p \) and \( \bar{V} \), which implies the following result:

(0.22) **Theorem.** The underlying vector bundles associated to the sheaves \( \{ F^p \}_{p \geq 0} \) are algebraic.

A basic corollary to (0.22) is that we may also view the sheaves \( \{ \bar{F}^p \}_{p \geq 0} \) as algebraic sheaves; and by the results of Serre ([23]), there is no distinction between these points of view in terms of their cohomology. Thus we will think of \( \{ \bar{F}^p \}_{p \geq 0} \) primarily as analytic sheaves, and whenever convenient, as algebraic sheaves.
There are a few technical points one should know before reading this thesis, in particular Chapter 3.

(0.23) Let $S$ be a projective, algebraic manifold of dimension $r$. A divisor $D \subset S$ is said to have normal crossings if for every $z_0 \in D$, there is a polydisk $\Delta^r \subset S$ centered at $z_0 = (0, \ldots, 0)$ with coordinates $\{z_1, \ldots, z_r\}$, such that $D \cap \Delta^r = \{z = (z_1, \ldots, z_r) \in \Delta^r \text{ satisfying the relation } z_1 \cdots z_t = 0\}$ for some $1 \leq t \leq r$. An example of such a divisor $D$ is in the case $S = \mathbb{P}^2$ and $D = \text{node}.$

(0.24) We define the log deRham complex $\Omega_S^r(\log D)$ (only when $D$ is a divisor with normal crossings) in the following way: Let $z_0 \in S$ and $\Delta^r$ as above with $\Delta^r \cap D = \{z = (z_1, \ldots, z_r)\}$ satisfying the relation $z_1 \cdots z_t = 0$. Define $\Omega_S^1(\log D)$ to be the locally free sheaf which is generated over $\Delta^r$ by $\{\frac{dz_1}{z_1}, \ldots, \frac{dz_t}{z_t}, dz_{t+1}, \ldots, dz_r\}$, and set $\Omega_S^p(\log D) = \wedge^p \Omega_S^1(\log D)$. There is the standard differential $\partial : \Omega_S^r(\log D) \rightarrow \Omega_S^{r+1}(\log D)$ making $\Omega_S^r(\log D)$ into a complex. There is a bicomplex associated to the complex, with corresponding hypercohomology spectral sequence abutting to $\mathbb{H}^r(S, \Omega_S^r(\log D)) = \mathbb{H}^r(S-D, \mathbb{C})$. We also have a Hodge filtration $\{\mathbb{F}_p \Omega_S^r(\log D)\}_{p \geq 0}$ of $\Omega_S^r(\log D)$ which defines a Hodge filtration $\mathbb{F}^p \mathbb{H}^r(S-D, \mathbb{C}) = \mathbb{H}^r(S, \mathbb{F}_p \Omega_S^r(\log D))$ on $\mathbb{H}^r(S-D, \mathbb{C})$.

(0.25) We can arrive at $\mathbb{F}^p \mathbb{H}^r(S-D, \mathbb{C})$ in an entirely different way as follows: There is the complex of meromorphic differentials $\Omega_S^r(*D)$ with poles of arbitrary (finite) order, and an order of pole...
filtration $G^*_{S}(*D)$ on this complex, which is defined in Chapter 3. We obtain the relationship:

$$F^*_{H^*}(S-D, \mathcal{O}) = H^*_{(S, F^*_{\mathcal{O}} S (\log D))} = H^*_{(S, G^*_{S}(*D))}.$$ 

It should be remarked that for $G^*_{S}(*D)$, the divisor $D$ need not have normal crossings or be nonsingular. Further information on this can be found in [11].

(0.26) Given $S$ as above with $D \subset S$ any smooth divisor, there is the Gysin sequence:

$$\ldots \rightarrow H^{r-2}(D) \rightarrow H^{r}(S) \rightarrow H^{r}(S-D) \rightarrow R H^{r-1}(D) \rightarrow H^{r+1}(S) \rightarrow \ldots.$$ 

The homomorphism $R$ is called the residue homomorphism. More generally if $D$ is a smooth subvariety in $S$ of codimension $\ell$, then the corresponding Gysin sequence is the following:

$$\ldots \rightarrow H^{*}(S) \rightarrow H^{*}(S-D) \rightarrow R H^{*+2\ell+1}(D) \rightarrow H^{*+1}(S) \rightarrow \ldots.$$ 

The homomorphism $R$ is called the generalized residue homomorphism. A more explicit description of $R$ can be found in [17] and [7].

As a conclusion to this section we outline two proofs of the Hodge conjecture for surfaces $Z (m = 1)$. We in fact prove $H_{m, 1}^{1, 1}(Z, \mathbb{Z})$, the integer coefficient version of the conjecture.

(0.27) The first proof readily adapts itself to version (0.3) of the conjecture, and we proceed as follows: One checks that the exponential short exact sequence: $0 \rightarrow \mathbb{Z} \rightarrow \theta^{*}_{Z} \rightarrow \exp \theta^{*}_{Z} \rightarrow 0$ yields the following diagram:
\[ (0.28) \]

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1(Z, \theta_Z) & \rightarrow & H^1(Z, \theta_Z^*) & \rightarrow \frac{c_1}{1st} H^2(Z, \mathbb{Z}) & \rightarrow H^2(Z, \theta_Z) \\
\downarrow H^1(Z, \mathbb{Z}) & & & & \text{homomorphism} & \\
\text{homomorphism} & & & & & \\
\end{array}
\]

\[
\text{\textit{Dolbeault isomorphism}}
\]

\[
0 \rightarrow \text{Pic}^0(Z) \rightarrow \text{Pic}(Z) \rightarrow H^2(Z, \mathbb{Z}) \rightarrow H^0,2(Z, \mathbb{C})
\]

\[
\text{Picard group of } Z
\]

It is now obvious from (0.28) that \( c_1(\text{Pic}(Z)) = \text{ker} \cdot \text{Pic}^0,2 \)

\[ = H^2(Z, \mathbb{Z}) \cap H^{1,1}(Z, \mathbb{C}) = H^{1,1}(Z, \mathbb{C}) . \]

\[
(0.29) \text{ For the second approach, we give a sketch of Lefschetz's proof}
\]

\text{of the conjecture for surfaces. Version (0.2) of the Hodge conjecture}

\text{is more useful here. Lefschetz's proof consists mainly of two steps:}

\text{\textbf{Step I. Construct a pencil (Lefschetz pencil) of hyperplane}

\text{sections } \{Z_t\} \text{ of } Z \text{, with corresponding family of (generalized)

\text{Jacobians } \bigcup_{t \in \mathbb{P}^1} J(Z_t) \text{. Lefschetz proves that every class}

\gamma \in \text{Prim}^{1,1}(Z, \mathbb{Z}) \text{ is the cohomology class (in some natural way) of}

\text{a holomorphic cross section } v : \mathbb{P}^1 \rightarrow \bigcup_{t \in \mathbb{P}^1} J(Z_t) \text{. These holomorphic}

\text{cross sections are called in our modern day language, normal functions.}

\text{\textbf{Step II. By a process of integration, there is the Abel-Jacobi}

\text{morphism } \phi : S^g(Z_t) \rightarrow J(Z_t) \text{ which is birational when } Z_t \]
is smooth (Jacobi inversion theorem), where \( g \) is the genus of \( Z_t \) and \( S^g(Z_t) \) is the \( g \)th symmetric product of \( Z_t \). Now let

\[ \gamma \in \text{Prim}^{1,1}(Z,\mathbb{Z}) \]

with corresponding normal function \( v \). For \( t \in \mathbb{P}^1 \), \( v(t) \) defines a divisor on \( Z_t \) via the Jacobi inversion theorem, and as \( t \) varies in \( \mathbb{P}^1 \), the \( v(t) \) trace out a divisor \( D \subset Z \). One checks that the fundamental class of \( D \) is precisely \( \gamma \).

We want to state which parts of Lefschetz's techniques above generalize to higher dimensions.

(0.30) First of all, Step I completely generalizes and two proofs can be found in [28] and [31].

(0.31) In Step II we can still obtain the Abel-Jacobi morphism:

Let \( C_m^0(Z_t) \) be the group of codimension \( m \) algebraic cycles in \( Z_t \) which are homologous to \( 0 \). Then there is a homomorphism

\[ \phi: C_m^0(Z_t) \rightarrow J(Z_t) \]

defined by a process of integration. We remark that no analogue of the Jacobi inversion theorem exists ([7, §13]) which is the essential difficulty in adapting Lefschetz's proof to higher dimensions.

(0.32) It is easy to show how an algebraic class \( \gamma \in \text{Prim}^{m,m}(Z,\mathbb{Z}) \) gives rise to a normal function \( v: \mathbb{P}^1 \rightarrow \bigcup_{t \in \mathbb{P}^1} J(Z_t) \). Given such a \( \gamma \), it follows from the definition of primitive cohomology and the weak Lefschetz theorem, that \( \gamma \cap Z_t \) is homologous to \( 0 \) in \( Z_t \). Thus there exists \( \zeta_t \) a real \( 2m-1 \) cycle in \( Z_t \) with boundary,
\[ \partial_t \cdot \gamma \cap Z_t, \text{ and by integrating } \zeta_t \text{ with a basis of forms in } \]
\[ F^m H^{2m-1}(Z_t, \mathbb{C}) \text{ (precisely the Abel-Jacobi morphism), we are able to } \]
\[ \text{define a holomorphic cross section } v : P^1 \to \bigcup_{t \in P^1} J(Z_t). \]

\((0.33)\) Finally we wish to introduce some useful terminology.

\((0.34)\) **Definition.** (i) The invertible part of \( J(Z_t) \) denoted by \( J(Z_t)_{\text{inv}} \), is defined to be \( \phi(C^m_{2m-1}(Z_t)) \), \( \phi \) being defined in \((0.31)\) as the Abel-Jacobi morphism.

(ii) A normal function \( v : P^1 \to \bigcup_{t \in P^1} J(Z_t) \) is said to be invertible if \( v(t) \in J(Z_t)_{\text{inv}} \) for generic \( t \in P^1 \).

\((0.35)\) **Definition.** Given the variety \( Z \) with diagram \((0.10)\), we say that there is an inversion theorem for the group \( G \) of normal functions \( v : P^1 \to \bigcup_{t \in P^1} J(Z_t = k^{-1}(t)) \) if there exists a subgroup \( G_{\text{inv}} \) of \( G \) satisfying the following two properties:

(i) Every \( v \in G_{\text{inv}} \) is invertible

(ii) The group generated by the cohomology classes of \( v \), as \( v \) ranges in \( G_{\text{inv}} \), is precisely \( \text{Prim}^{m,m}(Z, \mathbb{Z}) \).

It is hoped that such an inversion theorem holds for our special varieties \( Z = X \times \mathbb{C}, \) and in some cases we can be more specific \((4.44)\).

We now introduce some notation which will be assumed throughout this thesis.

(i) \( X, \mathbb{C} \) are smooth, projective varieties of dimensions \( 2m-1, 1 \) respectively. We will denote \( Z = X \times \mathbb{C} \).
(ii) All integral cohomology is intended modulo torsion.

(iii) \( \mathbb{Z} \) is the ring of integers, \( \mathbb{Q} \) is the rational numbers, \( \mathbb{C} \) is the complex numbers, and \( \mathbb{P}^N \) denotes complex, projective \( N \) space.
Chapter 1. The Hodge Structure on $E^{1,2m-1}_{2}(k)$.

A pencil of subvarieties covering $Z$ is equivalent to a non-
constant rational map $\tilde{k} : Z \to P^1$, and the purpose of this section is
to show that for a particular morphism $\tilde{k}$, the important part of
$\text{Prim}^{2m}(Z)$ is contained in $E^{1,2m-1}_{2}(k)$, where by definition
$E^{1,2m-1}_{2}(k) = H^1(P^1, R^{2m-1}k^*C)$. The module $E^{1,2m-1}_{2}(k)$ is easily seen
to be the $E_2$ term of the Leray spectral sequence for $\tilde{k}$, abutting
to $H^{2m}(Z,C)$. It also contains the image of the normal functions
defined in Chapter 2, which justifies the importance of this section.

(1.1) Definition. A Lefschetz pencil of hyperplane sections
$
\{X_t\}_{t \in \mathbb{P}^1}$ of $X$ is a pencil satisfying three conditions:

(i) for generic $t \in \mathbb{P}^1$, $X_t$ is smooth

(ii) a singular section has only a single ordinary double point

(iii) the base locus $D$ of the pencil is smooth.

Matters can be arranged so that $X_0$ and $X_\infty$ are both smooth, and we
will assume this throughout the rest of this paper. Given a pencil
as above, we define $\bar{X} = B_D(X) = \text{blow up of } X \text{ along } D$.

There is a morphism $\bar{f} : \bar{X} \to \mathbb{P}^1$ with fibers $\bar{f}^{-1}(t) = X_t$,
and a commutative diagram:

\[
\begin{array}{ccc}
\bar{X} - \bar{f}^{-1}(\Sigma) & \cong & \bar{X} \\
\downarrow f & & \downarrow \bar{f} \\
\mathbb{P}^1 - \Sigma & \cong & \mathbb{P}^1 \\
\end{array}
\]

(1.2)

where $\Sigma = \text{singular set of } \bar{f}$, i.e., $\mathbb{P}^1 - \Sigma$ is the set for which $\bar{f}$ is
smooth (and proper).
To arrive at a degeneration result for the Leray spectral sequence for \( \widetilde{k} \), we first prove:

\[(1.3)\] **Proposition.** For all integers \( q \), the restriction homomorphism \( R^q f_* \mathbb{C} \to j_* R^q f_* \mathbb{C} \) is an isomorphism.

**Proof.** This is (6.3.1) of theorem 6.3 in [6, p. 319]. One way to see this is first to note that for \( q \neq 2m-1, 2m-2 \) the proposition is clear [6, p. 195], and consider the following argument: Localize over a disk \( \Delta \subset \mathbb{P}^1 \) with \( \Delta - \{0\} = \Delta^* \subset U, \Delta \cap \Sigma = 0 \), \( X_0 = \) singular fiber, and \( X_t = \) smooth fiber \((t \in \Delta^*)\). There is a local vanishing cycle \( \delta \) in \( H_{2m-2}(X_t, \mathbb{C}) \) and a long exact sequence ([6, p. 196]):

\[(1.4)\]  
\[0 \to H^{2m-2}(X_0, \mathbb{C}) \to H^{2m-2}(X_t, \mathbb{C}) \xrightarrow{(, \delta)} \to H^{2m-1}(X_0, \mathbb{C}) \to H^{2m-2}(X_t, \mathbb{C}) \to 0\]

Now the results in [6, p. 196] imply that \((, \delta)\) is an epimorphism so that \( H^{2m-1}(X_0, \mathbb{C}) = H^{2m-1}(X_t, \mathbb{C}) \) and furthermore by the Picard-Lefschetz formula, the invariant subspace of \( H^{2m-2}(X_t, \mathbb{C}) \) is a subspace of codimension 1. Therefore \( R^{2m-2}f_* \mathbb{C} \simeq j_* R^{2m-2}f_* \mathbb{C} \) and

\( R^{2m-1}f_* \mathbb{C} \simeq j_* R^{2m-1}f_* \mathbb{C} \), hence the above assertion is proven.

Recall that \( Z = X \times \mathbb{C} \). Now define \( \widetilde{Z} = \mathbb{X} \times \mathbb{C} \). There is a morphism \( \tilde{k} : \widetilde{Z} \to \mathbb{P}^1 \) with fibers \( \tilde{k}^{-1}(t) = X_t \times \mathbb{C} \), and a diagram analogous to (1.2) above:

\[(1.5)\]

From (1.3) we obtain the following useful lemma:

\(1\) The stalks of \( j_* R^{2m-2}f_* \mathbb{C} \) consists of those cocycles which are invariant (via deformation) under the Picard-Lefschetz transformations (see (2.10)). Also \( H^0(\Delta, R^{2m-2}f_* \mathbb{C}) \simeq H^{2m-2}(X_0, \mathbb{C}) \).
(1.6) **Lemma.** For all integers \( q \), the restriction homomorphism
\[ R^q k_* \mathcal{C} \to j_* R^q k_* \mathcal{C} \]
is an isomorphism.

**Proof.** Use (1.3) applied to the Kunneth formula decomposition of \( H^q(X \times C) \).

(1.7) **Corollary.** The Leray spectral sequences for \( \bar{f} \) and \( \bar{k} \) degenerate at the \( E_2 \) terms.

**Proof.** Let \( L \) be a very ample divisor class which induces on the fibers of \( \bar{f} \) (resp. \( \bar{k} \)) a very ample divisor. For all \( \ell \geq 0 \), there is a commutative diagram for \( \bar{f} \) (resp. \( \bar{k} \)):

\[
\begin{array}{ccc}
\bigoplus [L] & \downarrow \sim \uparrow & j_* R^{2m-2-\ell} f_* \mathcal{C} \\
(1.3) & \uparrow = & j_* R^{2m-2+\ell} f_* \mathcal{C} \\
(1.8) & \uparrow \sim & j_* R^{2m-2+\ell} f_* \mathcal{C}
\end{array}
\]

where the bottom row induced isomorphism is the unique one making the diagram commutative. We now apply Deligne's criterion ([5]).

We now set out to identify the primitive cohomology of \( Z \).

More precisely we have the following:

(1.9) **Proposition.** \( \text{Prim}^{2m}(Z) = \text{Prim}^{2m-1}(X) \otimes H^1(C) \).

\[ \otimes ([X_t x C] - [X x C_t]) \sim \text{Prim}^{2m-2}(X) \otimes H^0(C) \]

where \( C_t \) is a hyperplane section of \( C \), and \([ \ ]\) denotes the Poincaré dual.
Proof. Via the Segre embedding:

\[
\begin{array}{c}
Z = X \times \mathbb{C} \\
\downarrow{i} \\
P_1 \quad P_2 \\
\downarrow{\pi} \\
X \quad \mathbb{C}
\end{array}
\]

, the pullback of the hyperplane class,

\[i^* \theta \big|_{\mathbb{P}^N} (l)\] is equal to \(P_1^* \theta_X (l) \otimes P_2^* \theta_C (l)\), which corresponds to the very ample divisor \(X \times \mathbb{C} + X \times \mathbb{C}\). Denote \(L_1 = X \times \mathbb{C}, L_2 = X \times \mathbb{C}\), \(L = L_1 + L_2\)

\((\theta_Z (l) \simeq i^* \theta \big|_{\mathbb{P}^N} (l))\). Now \(\cup L\) acts on \(H^{2m}(Z)\) in the following way:

\[
\begin{align*}
\cup L = \cup L_1 & : H^{2m-2}(X) \otimes H^2(C) \to H^{2m}(X) \otimes H^2(C) \\
\cup L = \cup L_1 & : H^{2m-1}(X) \otimes H^1(C) \to H^{2m+1}(X) \otimes H^1(C) \\
\cup L = \cup (L_1 + L_2) & : H^{2m}(X) \otimes H^0(C) \to H^{2m+2}(X) \otimes H^0(C) \otimes H^{2m}(X) \otimes H^2(C)
\end{align*}
\]

Now let \([\gamma_1] \in H^{2m-2}(X), [\gamma_2] \in H^{2m}(X)\). Then \(L \cup ([\gamma_1] \otimes [C]) = [\gamma_1 \cap X_t] \otimes [C_t]\) and \(L \cup ([\gamma_2] \otimes [C]) = [\gamma_2 \cap X_t] \otimes [C_t] \otimes [\gamma_2] \otimes [C_t]\).

Now \(L \cup ([\gamma_1] \otimes [C_t]) = L \cup ([\gamma_2] \otimes [C])\) if and only if \([\gamma_2] = [\gamma_1] \cup [X_t]\) and \([\gamma_2] \cup [X_t] = 0\), thus \([\gamma_1] \in \text{Prim}^{2m-2}(X)\). It is easy to check that the above argument implies (1.9).

Remark. Since the cohomology of main interest in this thesis will be \(H^{2m-1}(X) \otimes H^1(C)\), there is no loss in generality if we assume \(\text{Prim}^{2m}(Z) = \text{Prim}^{2m-1}(X) \otimes H^1(C)\).

To arrive at the main result of this section, the following lemmas will be needed. Throughout this paper we will denote \(Z_t = X_t \times \mathbb{C}\). We first introduce the following inclusion morphisms:

\[
\begin{align*}
(i) & \quad i_o : \overline{Z^0} \hookrightarrow \overline{Z}, \quad j_o : \overline{X^0} \hookrightarrow \overline{X} \\
(ii) & \quad i_1 : Z_t \hookrightarrow \overline{Z}, \quad j_1 : X_t \hookrightarrow \overline{X}
\end{align*}
\]
(iii) \( i_2 : D \times C \rightarrow Z_t \), \( j_2 : D \hookrightarrow X_t \)

(iv) \( i_3 : Z_t \rightarrow Z \), \( j_3 : X_t \hookrightarrow X \)

1.11 Lemma.

(i) \( H^\ast(X) \simeq H^\ast(X) \oplus H^\ast(-2)(D) \)

(ii) there is a commutative diagram:

\[
\begin{array}{ccc}
H^{2m}(Z) & \xrightarrow{i_*} & H^{2m}(Z_t) \\
\downarrow && \downarrow \sim \downarrow \\
H^{2m}(Z) \oplus H^{2m-2}(D \times C) & \xrightarrow{i_3^* \oplus i_2^*} & \end{array}
\]

(iii) the Gysin homomorphism \( i_1^* : H^{2m-2}(Z_t) \rightarrow H^{2m}(Z) \) becomes:

\[
H^{2m-2}(Z_t) \xrightarrow{i_3^* \oplus (i_2^*)} H^{2m}(Z) \oplus H^{2m-2}(D \times C)
\]

Proof. (i) follows from [6, p. 272]. By applying the Künneth formula to \( H^\ast(Z = \bar{x} \times \bar{C}) \), the assertions (ii) and (iii) are a trivial consequence of proposition 5.1.1 in [6, p. 279].

1.12 Lemma. \( E_2^{1,2m-1}(k) \simeq \ker i_1^* : H^{2m}(Z) \rightarrow H^{2m}(Z_t) \) \( / \ker i_0^* : H^{2m}(Z) \rightarrow H^{2m}(Z^0) \).

Proof. This is (3.6) in [28, p. 194]. Note that this induces a Hodge structure on \( E_2^{1,2m-1}(k) \). An intrinsic definition of the Hodge structure on \( E_2^{1,2m-1}(\bar{k}) \) is given in [31].

We have been leading up to our main result, namely:
(1.13) Theorem. $E_{2}^{1,2m-1}(k) \cong \text{Prim}^{2m}(Z) \oplus H^{2m-2}(DxC)_{v}$ where

$H^{2m-2}(DxC)_{v} = \ker i_{2,*}: H^{2m-2}(DxC) \longrightarrow H^{2m}(Z_{t}) (t \in U)$.  

Proof. $\cup I_{1} = [Z_{t}]$ can be described as a composite of the following homomorphisms ($p = \text{Poincaré duality}$):

(1.14) \[ H^{2m}(Z) \xrightarrow{i_{3}^{*}} H^{2m}(Z_{t}) \xrightarrow{\partial} H_{2m-2}(Z_{t}) \xrightarrow{i_{3},*} H^{2m}(Z) \]

Applying the Künneth formula, the following commutative diagram is obtained:

\[
\begin{array}{cccccc}
H^{2m}(Z) &=& H^{2m}(X) \otimes H^{0}(C) &+& H^{2m-1}(X) \otimes H^{1}(C) &+& H^{2m-2}(X) \otimes H^{2}(C) \\
\downarrow i_{3}^{*} & & \downarrow i_{3}^{*} & & \downarrow i_{3}^{*} & & \downarrow i_{3}^{*} (\text{weak Lefschetz}) \\
H^{2m}(Z_{t}) &=& H^{2m}(X_{t}) \otimes H^{0}(C) &+& H^{2m-1}(X_{t}) \otimes H^{1}(C) &+& H^{2m-2}(X_{t}) \otimes H^{2}(C) \\
\downarrow \sim p & & \downarrow \sim p & & \downarrow \sim p & & \downarrow \sim p \\
H_{2m-2}(Z_{t}) &=& H_{2m-4}(X_{t}) \otimes H^{2}(C) &+& H_{2m-3}(X_{t}) \otimes H_{1}(C) &+& H_{2m-2}(X_{t}) \otimes H_{0}(C) \\
\downarrow i_{3},* & & \downarrow i_{3},* (\text{weak Lef.}) & & \downarrow i_{3},* (\text{weak Lef.}) & & \downarrow i_{3},* (\text{weak Lef.}) \\
H_{2m-2}(Z) &=& H_{2m-4}(X) \otimes H^{2}(C) &+& H_{2m-3}(X) \otimes H_{1}(C) &+& H_{2m-2}(X) \otimes H_{0}(C) \\
\end{array}
\]

Note that $H^{2m-2}(X_{t}) = i_{3}^{*}H^{2m-2}(X) \oplus \ker j_{3,*}([28, p. 200])$, so that

$i_{3,*}$ is injective on the image of $i_{3}^{*}$. Therefore $\ker i_{3}^{*}$ in (1.14) is equal to $\ker \bigcup I_{1} = \oplus_{q=0}^{2m} H_{0}^{2m-q}(X) \otimes H^{q}(C)$, where $H_{0}^{*}(X) = \ker \bigcup h : q=0$

$H^{*}(X) \longrightarrow H^{*+2}(X)$, $h$ being a hyperplane class of $X$. Therefore

(1.15) $\ker i_{3}^{*} = H_{0}^{2m}(X) \otimes H^{0}(C) \oplus \text{Prim}^{2m}(Z)$. 

There is a commutative diagram \((t \in U)\):

\[
\begin{array}{cccc}
H^{2m-2}(Z_t) & \cong & H^{2m-2}(X_t) \otimes H^0(C) & \oplus H^{2m-3}(X_t) \otimes H^1(C) & \oplus H^{2m-4}(X_t) \otimes H^2(C) \\
& \downarrow i_3,* & \downarrow i_3,* & \downarrow i_3,* & \downarrow i_3,* \\
H^{2m}(Z) & \cong & H^{2m}(X) \otimes H^0(C) & \oplus H^{2m-1}(X) \otimes H^1(C) & \oplus H^{2m-2}(X) \otimes H^2(C)
\end{array}
\]

(1.16)

If we denote \(X_o = \text{singular section } (o \in E)\), then from \([6, \text{p. 196}]\) we have \(H^q(X_t) \cong H^q(X_o)\) for all \(q \neq 2m-2\). Also \(H^{2m-2}(X_t) \cong j_3^* H^{2m-2}(X) \otimes H^{2m-2}(X_t)_V\), where \(H^{2m-2}(X_t)_V = \ker j_3,*\), which is the subgroup of \(H^{2m-2}(X_t)\) generated by the vanishing cocycles \((j_3 H^{2m-2}(X)\) corresponds to the fixed part of a variation of Hodge structure related to the Hodge bundle over \(U\) with fibers \(H^{2m-2}(X_t)\)). From the above discussion and (1.16) it is clear that the image of \(R^{2m-2-k_0}C\) (via \(i_3,*\)) in \(H^{2m}(Z)\) is a constant system.

There is the analogous diagram to (1.16) above:

\[
\begin{array}{cccc}
H^{2m-2}(Z_t) & \cong & H^{2m-2}(X_t) \otimes H^0(C) & \oplus H^{2m-3}(X_t) \otimes H^1(C) & \oplus H^{2m-4}(X_t) \otimes H^2(C) \\
& \downarrow i_2^* & \downarrow i_2^* & \downarrow i_2^* & \downarrow i_2^* \\
H^{2m-2}(Dxc) & \cong & H^{2m-2}(D) \otimes H^0(C) & \oplus H^{2m-3}(D) \otimes H^1(C) & \oplus H^{2m-4}(D) \otimes H^2(C)
\end{array}
\]

(1.17)

To show that \(i_2^* H^{2m-2}(Z_t)\) is invariant in \(t \in P^1\), i.e., the image of \(R^{2m-2-k_0}C\) (via \(i_2^*\)) is a constant system in \(H^{2m-2}(Dxc)\), it suffices to
consider the image of \( j_2^*: H^{2m-2}(X_t) \rightarrow H^{2m-2}(D) \) using a similar reasoning as above.

There is a commutative diagram:

\[
\begin{array}{ccc}
H^{2m-2}(X_t) & \xrightarrow{\theta} & H^{2m-2}(X) \\
j_2^* & \downarrow & j_3\ H^{2m-2}(X) \\
H^{2m-2}(D) & \cong & H^{2m-4}(D)
\end{array}
\]

\[\text{(1.18)}\]

By a simple diagram chase, it is easy to see that \( j_2^* H^{2m-2}(X_t) = 0. \)

From (1.11) part (iii) we conclude that the image (via \( i_1^* \)) of \( \mathbb{R}^{2m-2} \setminus \mathbb{C} \) in \( H^{2m}(Z) \) is a constant system so that via the Gysin sequence we have:

\[\text{(1.19)} \quad \ker i^*: H^{2m}(Z) \rightarrow H^{2m}(Z^0) = \text{Im } i_{1^*}: H^{2m-2}(Z_t) \rightarrow H^{2m}(Z)\]

for any \( t \in U. \)

There is a commutative diagram:

\[
\begin{array}{ccc}
H^{2m-2}(X_t) & \xrightarrow{\text{[hyperplane class]}} & H^{2m-4}(X_t) \\
j_2^* & \downarrow & j_2^* \\
H^{2m-2}(D) & \xrightarrow{\text{[hyperplane class]}} & H^{2m-4}(D)
\end{array}
\]

\[\text{(1.20)}\]

and it is easy to see from this that \( j_2^*: H^{2m-2}(X_t) \rightarrow H^{2m-2}(D) \) is
surjective. Also \( H^{2m-3}(D) \cong j^* H^{2m-3}(X_t) \oplus H^{2m-3}(D)_v \), therefore from (1.17) we get \((-i_2^*) H^{2m-2}(Z_t) \cap H^{2m-2}(D_v) = 0\).

There is an analogous diagram to (1.20):

\[
\begin{array}{ccc}
H^{2m-2}(X_t) \cup \text{[hyperplane class]} & \cong & H^{2m-4}(X_t) \\
j_3,* \text{ (weak Lef.)} & \downarrow & j_3,* \\
H^{2m}(X) \cup \text{[hyperplane class]} & \cong & H^{2m-2}(X) \text{ (strong Lef.)} \\
\end{array}
\]

so that \( j_3,* : H^{2m-4}(X_t) \rightarrow H^{2m-2}(X) \) is surjective. Note also that

\[ H^{2m-1}(X) \cong j_3,* H^{2m-3}(X_t) \oplus \text{Prim}^{2m-1}(X). \]

Combining these results with (1.11), (1.16) and (1.17), it is easy to check that the expression in (1.12) becomes:

(1.22) \[ E_2^{1,2m-1}(k) \cong \text{Prim}^{2m}(Z) \oplus H^{2m-2}(D_v). \]

(1.23) Corollary. \[ H^{m,m}(Z) \cap E_2^{1,2m-1}(k) \cong \text{Prim}^{m,m-1}(X) \oplus H^{0,1}(C) \]

\[ \oplus \text{Prim}^{m-1,m}(X) \otimes H^{1,0}(C) \]

\[ \oplus H^{m-1,m-1}(D_v). \]

Proof. Apply (1.9) to (1.13).
Chapter 2. The Intermediate Jacobian $J(Z_t)$.

We will define $J(Z_t)$ and show that it admits a non-trivial Abel-Jacobi morphism. The image of the Abel-Jacobi morphism will be called the invertible part of $J(Z_t)$. As the terminology suggests, we will be interested in inverting normal functions so that the cohomology classes in $\text{Prim}_m^m(Z, \mathbb{Q})$ will come from normal functions whose cohomology classes are rational multiples of algebraic cocycles. The ultimate attempt is to find an inversion theorem ((0.35)), which is only known in certain cases (for example, see (4.44)).

There is a Hodge filtration on $H^{2m-1}(Z_t)$ defined by:

\begin{equation}
F^p H^{2m-1}(Z_t, \mathbb{C}) = H^{2m-1,0}(Z_t) \oplus \ldots \oplus H^p,2m-1-p(Z_t) \quad \text{where} \quad p \geq 0
\end{equation}

and $t \in U$.

\begin{equation}
F^p,^* H^{2m-1}(Z_t, \mathbb{C}) = H^{2m-1}(Z_t, \mathbb{C}) / F^p H^{2m-1}(Z_t, \mathbb{C})
\end{equation}

For $J(Z_t)$, we are mainly interested in $F^m,^* H^{2m-1}(Z_t, \mathbb{C}) = H^{m-1,m}(Z_t) \oplus \ldots \oplus H^{0,2m-1}(Z_t)$. Via projection, $H^{2m-1}(Z_t, \mathbb{Z})$ embeds itself as a lattice in $F^m,^* H^{2m-1}(Z_t, \mathbb{C})$. We now have the following:

\begin{equation}
\text{(2.3) Definition. For} \quad t \in U, \quad \text{the intermediate jacobian} \quad J(Z_t) \quad \text{is defined to be:} \quad J(Z_t) = F^m,^* H^{2m-1}(Z_t, \mathbb{C}) / H^{2m-1}(Z_t, \mathbb{Z})
\end{equation}

Remarks: (i) this definition appears elsewhere ([11, p. 331]).

(ii) $J(Z_t)$ is not in general an abelian variety ([27]).
The Künneth formula provides us with an inclusion $H^{2m-2}(X_t) \otimes H^1(C) \hookrightarrow H^{2m-1}(Z_t)$. Let $\{\gamma\} \in H_{2m-2}(X_t, \mathbb{Z})$, where $\gamma$ is a $2m-2$ chain. Then $\gamma$ defines a cylinder homomorphism $\rho_\gamma : H^1(C, \mathbb{Z}) \longrightarrow H^{2m-1}(Z_t, \mathbb{Z})$ obtained from the correspondence $c \in C \mapsto \gamma \circ c$ in $X_t \times C$. The graph of the cylinder map is a real $2m$ cycle $T \in C \times Z_t$ and via Poincaré duality we obtain $[T] \in H^{2m}(C \times Z_t, \mathbb{Z})$. Via the Künneth formula, the projection of $[T]$ in $H^1(C, \mathbb{Z}) \otimes H^{2m-1}(Z_t, \mathbb{Z})$ defines a homomorphism: (* denotes dual space)

\[
(2.4) \quad H^1(C, \mathbb{Z}) \cong H^1(C, \mathbb{Z})^* \longrightarrow H^{2m-1}(Z_t, \mathbb{Z}) \cong H_{2m-1}(Z_t, \mathbb{Z})
\]

which recovers the cylinder homomorphism $\rho_\gamma$. Thinking of $[T] \in H^{2m}(C \times Z_t, \mathbb{C})$ and projecting $[T]$ into $H^{1,0}(C) \otimes (H^{m-1,m}(Z_t) \otimes \cdots \otimes H^{0,2m-1}(Z_t)) = H^{1,0}(C) \otimes \mathbb{C}^{m,\ast} H^{2m-1}(Z_t, \mathbb{C})$ we obtain a complex homomorphism: ( denotes conjugate)

\[
(2.5) \quad \rho_\gamma : H^{0,1}(C) \longrightarrow H^{m,\ast} H^{2m-1}(Z_t, \mathbb{C})\quad \text{which is induced by } \rho \circ \rho_\gamma.
\]

Therefore $\rho_\gamma$ defines a morphism $\phi_\gamma : J(C) \longrightarrow J(Z_t)$ which is a transcendental analogue to the Abel-Jacobi morphism defined in [27]. (Note: $J(C)$ is the usual Jacobian of the curve $C$.)

We now prove:

(2.6) **Theorem.** (i) Every class in $H^{2m-2}(X_t, \mathbb{C}) \otimes H^1(C, \mathbb{C}) \cap J(Z_t) (\text{if} U)$ lies in the subgroup generated by $\phi_\gamma(J(C))$ where $\gamma$ ranges through $H^{2m-2}(X_t, \mathbb{Z})$. 


Moreover (ii) if $\gamma$ is an algebraic cycle, then $\phi_\gamma$ is the usual Abel-Jacobi morphism$^1$.

(iii) The image generated by $\phi_\gamma(J(C))$ as $\gamma$ ranges in $H^{m-1,m-1}(X_t,\mathbb{Z}) = H^{2m-2}(X_t,\mathbb{Z}) \cap H^{m-1,m-1}(X_t,\mathbb{C})$ is precisely $H^{m-1,m-1}(X_t,\mathbb{Z}) \otimes H^0,1(C) \cap J(Z_t)$.

(2.7) Corollary. If the Hodge $(m-1,m-1)$ conjecture is true for $X_t$, then the image of the (algebraic) Abel-Jacobi morphism contains $H^{m-1,m-1}(X_t,\mathbb{Z}) \otimes H^0,1(C) \cap J(Z_t)$.

Proof of (2.6). (i) Let $[x]_J \in J(Z_t) \cap H^{2m-2}(X_t,\mathbb{C}) \otimes H^1(C,\mathbb{C})$. Then $[x]_J$ comes from $[x] \in H^{2m-2}(Z_t,\mathbb{C}) \otimes H^1(C,\mathbb{C})$. Since $H^{2m-1}(Z_t,\mathbb{C}) \cong H^{2m-1}(Z_t,\mathbb{Z}) \otimes \mathbb{C}$, we may assume that $[x] = \sum r_i[y_i] \otimes [z_i]$ where $r_i \in \mathbb{C}$, $[y_i] \in H^{2m-2}(X_t,\mathbb{Z})$, $[z_i] \in H^1(C,\mathbb{Z})$. Now each $\gamma_i$ defines a cylinder homomorphism $\rho_{\gamma_i} : H_1(C,\mathbb{Z}) \to H^{2m-1}(Z_t,\mathbb{Z})$, and hence an Abel-Jacobi morphism $\phi_{\gamma_i} : J(C) \to J(Z_t)$. Furthermore, $\rho_{\gamma_i}(\zeta_i) = \gamma_i \otimes \zeta_i$. There is a commutative diagram:

\[
\begin{array}{ccc}
H_1(C,\mathbb{Z}) & \xrightarrow{\rho_{\gamma_i}} & H^{2m-1}(Z_t,\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^0,1(C) & \xrightarrow{\rho_{\gamma_i}C} & F^m,1 \cdot H^{2m-1}(Z_t,\mathbb{C}) \\
\downarrow \text{(mult. by } r_i) & & \downarrow \text{(mult. by } r_i) \\
H^0,1(C) & \xrightarrow{\rho_{\gamma_i}C} & F^m,1 \cdot H^{2m-1}(Z_t,\mathbb{C})
\end{array}
\]

$^1$"Usual" referring to the definition of the Abel-Jacobi morphism, involving algebraic cycles (see (A.1)).
which simply says that $\rho_{\gamma_i}^C$ is a complex homomorphism. It is now obvious from (2.8) that $\rho_{\gamma_i}^C (r_i [\zeta_i]) = r_i [\gamma_i] \otimes [\zeta_i]$ so that

$$\sum_{i=1}^N \gamma_i (J(C)) \text{ contains } [x]_J.$$ Finally (ii) and (iii) are obvious.

Remark. If $H^{2m-2}(X_t, \mathcal{O})$ is generated by algebraic cocycles, then it is expected that the following statement holds:

(2.9) Statement. Every holomorphic cross section (normal function) of the family $U J(Z_t)$ is in the image of the (algebraic) Abel-Jacobi mapping, hence it is invertible. Therefore the Hodge conjecture holds for $\text{Prim}^{m,m}(Z, \mathcal{O})$. (This statement is verified in Chapter 4).

In order to discuss normal functions, we need to rigorously define $J(Z_t)$ for $t \in \Sigma$. This is done in terms of the nearby fibers of $Z_t$ in (1.5). The definition of $J(Z_t)$ will involve the extension of some locally free sheaf over $U$ to one over $\mathbb{P}^1$, therefore we will now sheafify everything.

Over $U$ we have the Leray cohomology sheaves

$$F = \Theta_U^C \mathcal{R}^{2m-1}\kappa_\mathcal{C} \sim \mathcal{R}^{2m-1}\kappa_\mathcal{C}^{\ast \Omega} \otimes \mathcal{O}^{\mathbb{Z}_/U}$$

with Hodge filtration subbundles

$$F^p = \mathcal{R}^{2m-1}\kappa_\mathcal{C}^{p\Omega} \otimes \mathcal{O}^{\mathbb{Z}_/U},$$

where $\Omega$ is the complex of relative differential forms, $F^p$ is the $p^{th}$ Hodge filtration, and $\mathcal{R}^*$ denotes the right derived hypercohomology. By the monodromy theorem, the local monodromy (Picard-Lefschetz) transformations $T$ satisfy the identity
\((T^N - I)q = 0\) for \(N, q \geq 0\) (i.e., \(T\) is quasi-unipotent). We can be more precise though:

(2.10) **Proposition.** The local monodromy transformations \(T\) satisfy the identity \(T^2 = I\).

**Proof.** We localize the family in (1.5) over a disk \(\Delta \subset \mathbb{P}^1\) with \(\Delta \cap \Sigma = 0 \in \Delta\). Applying the Kunneth formula to \(H^{2m-1}(Z_t)\) \((t \in \Delta^*)\), it suffices to consider the local monodromy transformation \(T\) associated to \(H^{2m-2}(X_t)\) about \(O \in \Delta\). Let \(\gamma \in H^{2m-2}(X_t)\). From [6, p. 196] we have the Picard-Lefschetz formula:

(2.11) \(T(\gamma) = \gamma + (-1)^{m-1}((e(T)-1)/2)(\gamma, \delta)\delta\) where \(e(T) = -1\), \(e : \pi_1(\Delta^*) \rightarrow \{-1, 1\}\) being a character, and \(\delta = \) local vanishing cocycle.

Therefore \(T^2(\gamma) = T(\gamma) + (-1)^{m-1}((e(T)-1)/2)(\gamma, \delta)T(\delta)\). It is easy to compute \(T(\delta) = \delta + (-1)^{m-1}((e(T)-1)/2)(\delta, \delta)\delta\) where \((\delta, \delta) = (-1)^{m-1}2\), i.e., \(T(\delta) = \delta + 2((e(T)-1)/2)\delta = e(T)\delta\).

Therefore \(T^2\gamma = T(\gamma) - (-1)^{m-1}((e(T)-1)/2)(\gamma, \delta)\delta = \gamma, \) i.e., \(T^2(\gamma) = \gamma\).

(2.12) We now define \(F^{p,*} = F/F^{2m-p}\). There is a short exact sequence:

(2.13) \[0 \rightarrow R^{2m-1}k_*\mathbb{Z} \rightarrow F^{m,*} \rightarrow J \rightarrow 0\] over \(U\), where the cokernel sheaf \(J\) can be interpreted as the sheaf of holomorphic cross sections of the family \(U \cup J(Z_t)\) of jacobians over \(U\). We wish to extend (2.13) over \(\mathbb{P}^1\), and we choose the so called "quasi-canonical" extension.
This extension is actually a local procedure, so we localize over a disk $\Delta \subset \mathbb{P}^1$ with $\Delta \cap \Sigma = 0 \in \Delta$. This procedure of extension is discussed in [31, \S 6], however since the monodromy is so simple, a simplified approach will suffice. We first extend $F$ to $\mathbb{P}^1$. Via the Künneth formula it suffices to consider the Leray cohomology sheaves $F_f$ with Hodge filtration subsheaves $F^p_f$ associated to (1.2), where $F_f = \mathcal{O} \otimes \mathcal{R}^{2m-2}f_*L$.

Let $t \in \Delta^*$, and choose $\{v_1, \ldots, v_N\} \in H^{2m-2}(X_t, \mathcal{O})$ such that $(v_i, \delta) = 0$ for all $i = 1, \ldots, N$ ($\delta$ = local vanishing cocycle), and so that $\{v_1, \ldots, v_N, \delta\}$ is a basis of $H^{2m-2}(X_t, \mathcal{O})$. Now $F_f$ defines a flat bundle over $\Delta^*$, so by parallel translation we can extend $\{v_1, \ldots, v_N, \delta\}$ to horizontal multivalued sections of $F_f$ over $\Delta^*$ (horizontal with respect to an integrable connection - the Gauss-Manin connection, defined in (2.14)). By the Picard-Lefschetz formula, the $\{v_1, \ldots, v_N\}$ are single valued, and $\delta$ is the "eigenvector" section associated to the eigenvalue $-1$ of the local monodromy transformation $T$, i.e., $T\delta = -\delta([6, p. 196])$. Define $v(t) = \exp((\log(t))/2)\delta$. Then $v(t)$ is a single valued section of $F_f$ over $\Delta^*$. The quasi-canonical extension of $F_f$ over $\Delta$ is defined to be the sheaf generated by $\{v_1, \ldots, v_N, v(t)\}$. One checks that this sheaf is in fact free over $\Delta$, and that $v(t)$ vanishes at the origin. By considering the double covering $s = t^2 : \Delta^* \to \Delta^*$, it is easy to see that the local monodromy transformation associated to $s^*F_f$ is trivial, a fortiori unipotent. Therefore $s^*F_f$ extends trivially to a
free sheaf over $\Delta$ and this is known as the canonical extension of $s^*F_\ell$, which is the extension associated to unipotent local monodromy transformations (see [3, p. 91] for more information).

(2.14) There is the natural integrable "Gauss-Manin" connection $\nabla$ defined on $F_\ell = \Theta \otimes R^{2m-2}f_\ell^*g$, which is of the form $\Theta \otimes 1$. Over $P^1$, $F_\ell$ extends to $\tilde{F}_\ell$, and $\nabla$ extends to $\tilde{\nabla}$. Now locally over the disk $\Delta$ mentioned above, $\tilde{\nabla} = (v(t)/2)d\log(t)$, so that $F_\ell^d \subseteq \Omega^1_{P^1}(\log E) \otimes \tilde{F}_\ell^d$. The extensions $\tilde{F}_\ell^d$ of $F_\ell^d$ are defined as:

$$\tilde{F}_\ell^d = j_*F_\ell^d \cap \tilde{F}_\ell^d.$$  

Recall the morphism $s = t^2$ defined above. Then the following is true:

(2.15) Proposition. $H^0(\Delta, \tilde{F}_\ell^d) = \{ \sigma \in H^0(\Delta, s^*\tilde{F}_\ell^d) \mid \sigma(t) = \sigma(-t) \}$ for all $t \in \Delta$.

Proof. Let $\sigma \in H^0(\Delta, \tilde{F}_\ell^d)$. Then $s^*\sigma = \sigma(t^2)$ which lies in the RHS of (2.15). Conversely if $\sigma$ is a member of the RHS then $\sigma(\sqrt{s})$ is well defined for all $\sqrt{s} \in \Delta$. We need to prove that $\sigma(\sqrt{s}) \in H^0(\Delta, \tilde{F}_\ell^d)$. Now $\sigma(\sqrt{s}) = \sum_{i=1}^N g_i(\sqrt{s})v_i + g_0(\sqrt{s})\delta$, where

$\left\{v_1, \ldots, v_N, \delta\right\}$ is the horizontal multivalued basis of $H^0(\Delta, \tilde{F}_\ell^d)$. Since $\left\{v_1, \ldots, v_N\right\}$ is a single valued set of horizontal independent sections, clearly $g_i(\sqrt{s})$ must be holomorphic in $s$ in $\Delta^*$ for $i = 1, \ldots, N$, and since each $g_i$ is bounded on $\Delta$, $g_i(\sqrt{s})$ must be holomorphic in $s$ over $\Delta$ for $i = 1, \ldots, N$. Since $g_0(\sqrt{s})$ is a holomorphic function times $v(s)$, it is obvious that the above assertion holds.

2 A further discussion of this appears in (A.2).
(2.16) **Corollary.** (i) $\tilde{F}^P_F$ is completely characterized by its growth of periods near $\Sigma$.

(ii) $\tilde{F}^P_F$ is uniquely characterized by the property that $\tilde{F}^P_F / \tilde{F}^P_F$ is locally free.

**Proof.** This is a result of the above discussion, (2.15) and the properties of the canonical extension ([28, p. 190], also [3, p. 91]).

(2.17) The infinitesimal period relation holds: \( \mathbb{V} \mathbb{F}^P_F \subset \Omega^1_{\rho}(\log \Sigma) \otimes \tilde{F}^{-1}_F \).

Everything carries over to the $\tilde{F}^P$ associated to (1.5). The short exact sequence of (2.13) extends over $\mathbb{P}^1$ to:

\[
0 \to j_* \mathbb{R}^{2m-1}_{k_\mathbb{Z}} \to \tilde{\mathbb{F}}^* \to \mathbb{J} \to 0,
\]

with coboundary homomorphisms $\delta : H^0(\mathbb{P}^1, \mathbb{J}) \to H^1(\mathbb{P}^1, j_* \mathbb{R}^{2m-1}_{k_\mathbb{Z}})$, and commutative diagram:

\[
\begin{array}{ccc}
H^0(\mathbb{P}^1, \mathbb{J}) & \xrightarrow{\delta} & H^1(\mathbb{P}^1, j_* \mathbb{R}^{2m-1}_{k_\mathbb{Z}}) \\
\downarrow{\delta} & & \downarrow{\delta} \\
E^{1,2m-1}(k) & & E^{1,2m-1}(k)
\end{array}
\]

It should be remarked that $\ker \delta_C / \ker \delta = T_0$, where $T_0$ is a torsion subgroup of $H^0(\mathbb{P}^1, \mathbb{J}) / \ker \delta$. This follows from the fact that $H^1(\mathbb{P}^1, j_* \mathbb{R}^{2m-1}_{k_\mathbb{Z}})$ may have torsion, i.e. via $\delta$, $T_0$ embeds in $H^1(\mathbb{P}^1, j_* \mathbb{R}^{2m-1}_{k_\mathbb{Z}})$ so that if we divide out the torsion in $H^1(\mathbb{P}^1, j_* \mathbb{R}^{2m-1}_{k_\mathbb{Z}})$, then $T_0$ is also removed. For the purpose of simplifying the statements of the results in this thesis, we assume $H^1(\mathbb{P}^1, j_* \mathbb{R}^{2m-1}_{k_\mathbb{Z}})$ is torsionless, so that $T_0 = 0$ thus $\ker \delta_C = \ker \delta$.

---

3 A precise interpretation of this statement can be found in (A.2).
We introduce the following basic definition:

(2.19) Definition. A normal function is an element of $H^0(P^1, J)$. 

The Gauss-Manin connection defines a homomorphism:

(2.20) $\nabla : \tilde{F}^m, * = \tilde{F}/\tilde{F}^m \rightarrow \Omega^1_p (\log Z) \otimes F/\tilde{F}^{m-1}$, and since $\nabla$ kills $j_2^{2m-1}K$, it defines a homomorphism:

(2.21) $\nabla : \tilde{F} \rightarrow \Omega^1_p (\log Z) \otimes F/\tilde{F}^{m-1}$.

(2.22) Definition. The horizontal normal functions, written $H^0(P^1, J)_h$, are defined to be the kernel of $\nabla : H^0(P^1, J) \rightarrow$

$H^0(P^1, \Omega^1_p (\log Z) \otimes F/\tilde{F}^{m-1})$.

Remark. It will later be shown (Chapter 3) that $H^0(P^1, J)_h = H^0(P^1, J)$.

We now prove the following:

(2.23) Proposition. (i) $j_2^{2m-1}K \simeq \otimes^2 j_2^{2m-1-q}f_2K \otimes H^2(C, K)_{q=0}$

where $K = \mathbb{Z}, \mathbb{Q},$ or $\mathbb{C}$.

(ii) $\tilde{F}^m, * \simeq \tilde{F}^m, * \otimes H^0(C) \oplus \tilde{F}^{m-1}, * \otimes H^1, 0(C) \oplus \tilde{F}^{m-1}, * \otimes H^2(C)$.

(iii) $\tilde{F}^{m-1}, * \simeq \tilde{F}^{m-1} / \tilde{F}^m \oplus \tilde{F}^m, *$ if and only if the cup product pairing defined in (0.20) induces a holomorphic cup product pairing:

Proof. (i), (ii), and ($\Rightarrow$) of (iii) are obvious. Therefore we prove only

(iii) ($\Leftarrow$). There is a short exact sequence:
0 \rightarrow \mathcal{F}_x^{m} \rightarrow \mathcal{F}_x^{m-1} \rightarrow \mathcal{F}_x^{m-1,*} \rightarrow 0, \text{ and an inclusion of sheaves:}

\mathcal{F}_x^{m} \subseteq \mathcal{F}_x^{m-1} \subseteq \mathcal{F}_x^{m-1,*}, \text{ so that } \mathcal{F}_x^{m-1} / \mathcal{F}_x^{m} \text{ is a holomorphic subsheaf of } \mathcal{F}_x^{m-1,*}.

We now localize over a disk \( \Delta \). Let \( \theta_{\Delta, t} \) be the localization of \( \theta_{\Delta} \) at \( t \in \Delta \), and let \( \{w_1, \ldots, w_r\} \) be a basis of \( H^0(\Delta, \mathcal{F}_x^{m-1,*}) \), which we can assume generates the stalks \( \mathcal{F}_x^{m-1,*}_t = \mathcal{F}_x^{m-1,*} \otimes \theta_{\Delta, t} \) as an \( \theta_{\Delta, t} \) module. Note that \( N = \dim C^{m-1,2m-2}(X_t, \mathcal{O}) \) for \( t \in \Delta \cap U \).

Let \( r = \dim C^{m-1, m-1}(X_t, \mathcal{O}) \). There exists a basis \( \{s_1, \ldots, s_r\} \) of \( H^0(\Delta, \mathcal{F}_x^{m-1}) \) which generates the stalks \( \mathcal{F}_x^{m-1} / \mathcal{F}_x^{m} \) for all \( t \in \Delta \). There is no loss of generality in assuming that \( s_1 = w_1, \ldots, s_r = w_r \). A typical \( \alpha \in H^0(\Delta, \mathcal{F}_x^{m-1,*}) \) is of the form \( \alpha = \sum_{j=1}^{N} a_j w_j \) where \( a_j \in H^0(\Delta, \theta_{\Delta}) \). Also \( \alpha \cup w_i = \sum_{j=1}^{N} a_j w_j \cup w_i \). Define \( c_{ij} = w_i \cup w_j \in H^0(\Delta, \mathcal{F}_x^{m-1,*} \cup \mathcal{F}_x^{m-1,*}) \).

Then \( c_{ij} \) is a holomorphic section of the bundle \( U H^{2m-2,2m-2}(X_t, \mathcal{O}) \), which is in fact a trivial holomorphic line bundle over \( \Delta \). Therefore \( c_{ij} \) can be identified as an element of \( H^0(\Delta, \theta_{\Delta}) \). Define

\[ T = \{ \alpha \in H^0(\Delta, \mathcal{F}_x^{m-1,*}) \mid \sum_{j=1}^{N} a_j c_{ji} = 0 \text{ for all } i = 1, \ldots, r \} \]. There is a short exact sequence:
(2.24) \[ 0 \rightarrow T \rightarrow \frac{F_{m}}{F_{m}},^* \xrightarrow{\psi} \frac{\Theta_{\Delta}^r}{\Theta_{\Delta}^r} \rightarrow 0, \text{ with } \text{res } \psi : \frac{F_{m}}{F_{m}} / \frac{F_{m}}{F_{m}} \rightarrow \Theta_{\Delta}^r \]

injective. Note also that \( T \subseteq \frac{F_{m}}{F_{m}},^* \). Let \( K \) be the sheaf of meromorphic functions on \( \Delta \). Tensoring (2.24) with \( ^* \otimes K \), we see that the morphism \( \text{res } \psi : \frac{F_{m}}{F_{m}} / \frac{F_{m}}{F_{m}} \otimes K \rightarrow K^r \) is an isomorphism, and also

\[ T \otimes K = \frac{F_{m}}{F_{m}},^* \otimes K. \]

Let \( M \) be the sheaf which makes the following a short exact sequence:

(2.25) \[ 0 \rightarrow \frac{F_{m}}{F_{m}},^* \rightarrow \frac{F_{m}}{F_{m}} / \frac{F_{m}}{F_{m}} \rightarrow \Theta_{\Delta}^r \rightarrow M \rightarrow 0. \] \( M \) is in fact torsionless.

Then tensoring (2.25) with \( K \) and looking at the corresponding short exact sequence, we obtain:

\[ 0 \rightarrow \frac{F_{m}}{F_{m}},^* \otimes K \rightarrow \frac{F_{m}}{F_{m},^*} \otimes K \rightarrow M \otimes K \rightarrow 0. \]

Therefore \( M \otimes K = 0 \), hence \( M = 0 \) so that \( \psi \) in (2.25) is an isomorphism. This immediately implies that \( \frac{F_{m}}{F_{m}},^* = T \otimes \frac{F_{m}}{F_{m}},^* \), so that via the projection \( \frac{F_{m}}{F_{m}},^* \rightarrow \frac{F_{m}}{F_{m}},^* \), \( T = \frac{F_{m}}{F_{m}},^* \).

(2.26) \textbf{Corollary.} \begin{itemize}
\item[(i)] \( \oplus_{q=0}^{1,2m-1}(k) \sim \oplus_{q=0}^{2} \Theta^1 H^1(P^1, j_* R^{2m-1}q_{*} \mathcal{F}) \otimes H^q(C) \)
\item[(ii)] \[ \oplus_{q=0}^{1} H^i(P^1, j_* R^{2m-1}k_{*} \mathcal{Z}) \sim \oplus_{q=0}^{2} \Theta^1 H^1(P^1, j_* R^{2m-1}q_{*} \mathcal{F}_{*} \mathcal{Z}) \]
\item[(iii)] \[ \oplus_{q=0}^{3} \Theta^1 H^1(P^1, j_* R^{2m-1}q_{*} \mathcal{F}) \otimes H^q(C) \oplus \frac{F_{m}}{F_{m}},^* \otimes H^0,1(C) \]
\end{itemize}

the splitting of \( \frac{F_{m}}{F_{m}},^* \) in (2.23) (iii) holds.
Proof. Obvious.

We now define $\tilde{J}'$ to be the image of $\tilde{F}_m ^* \otimes H^1(C)$ in $\tilde{J}$, via the composite $\tilde{F}_m ^* \otimes H^1(C) \subset \tilde{F}_m ^* \longrightarrow \tilde{J}$, whenever $\tilde{F}_m ^* = \tilde{F}_m ^/ F_m \oplus F_m ^*$. We now prove the following which basically says that we can discard all horizontal normal functions which take their values in a particular Hodge level in the family of intermediate Jacobians.

(2.27) Proposition. $H^0(\mathbb{P}^1, \tilde{J}')_h \subset \ker \delta_{\mathbb{C}}$.

Proof. Let $\sigma \in H^0(\mathbb{P}^1, \tilde{J}')_h$. Then $\sigma$ is equivalent to the data

$$\sigma = \{\sigma_a, U_a\}_{a \in I}, \sigma_a - \sigma_\beta \in H^0(U_{a\beta} = U_a \cap U_\beta, j_* R^{2m-1} k_* \mathbb{Z}),$$

where $\sigma_a \in H^0(U_a, \tilde{F}_m ^* \otimes H^1(C)$ and $\nabla \sigma_a = 0$. We can write

$$\sigma = \sum_{j=1}^{2g} \gamma_a \otimes v_j$$

where $\{v_1, \ldots, v_{2g}\}$ is a basis of $H^1(C, \mathbb{Z})$, $g$ being the genus of $C$.

Now $(\delta \sigma) = \sigma - \sigma_\beta = \sum_{j=1}^{2g} (\gamma_a - \gamma_\beta) \otimes v_j$, where

$$\gamma_a - \gamma_\beta \in H^0(U_{a\beta}, j_* R^{2m-2} f_* \mathbb{Z}).$$

Now define $\sigma_a^1 = \gamma_a \otimes v_1$. Then

$$\sigma_a^1 = \{\sigma_a^1, U_a\}_{a \in I}$$

satisfies the properties $\sigma_a^1 - \sigma_\beta^1 \in H^0(U_{a\beta}, j_* R^{2m-1} k_* \mathbb{Z})$,

$$(\nabla \sigma_a^1 = 0), \text{ and } \sigma_a^1 \in H^0(U_a, \tilde{F}_m ^* \otimes H^1(C), \text{ and } \nabla \sigma_a^1 = 0. \text{ That is } \sigma_a^1 \text{ defines a horizontal normal function. From [31, §9], } \delta \sigma_a^1 \text{ is of type } (m,m).$$
Now it can be checked that no class of type \((m,m)\) can be represented in the form \(\gamma \otimes v_1\) unless it is zero. Such an exercise will be left to the reader, so that the above (2.27) is proven.

**Remark.** The theorem we used in the reference mentioned above, [31, §9] is the optimal version of the theorem on normal functions which states the following:

(2.28) **Theorem.** All the integral classes of type \((m,m)\) in \(E^{1,2m-1}(k)\) coincide with the cohomology classes of the horizontal normal functions.

In the next section we prove our main result on the normal functions, namely that all the normal functions are horizontal.

Knowing this enables us to prove a stronger result than (2.27), namely \(H^0(P^1, J') \subseteq \ker d_C\). In fact we can do much better by explicitly describing \(\ker d_C\) (3.58).
Chapter 3. A Vanishing Theorem and its Corollaries.

In this chapter an explicit description of \( \ker \delta : H^0(P^1, J) \rightarrow H^1(P^1, j_* R^{2m-1-k_*} \mathbb{Z}) \) is given, and as a corollary to the general machinery developed we deduce our main theorem on the normal functions. The description of \( \ker \delta \) comes from a vanishing theorem, and at this point we first establish the machinery to provide the correct setting for the statement of the theorem. We first prove a few lemmas.

(3.1) **Lemma.** \( H^{2m}(Z) / i_3, * H^{2m-2}(Z_t) \simeq \text{Prim}^{2m}(Z) \).

**Proof.** There is a commutative diagram:

\[
\begin{array}{cccccc}
H^{2m-2}(Z_t) & \simeq & H^{2m-4}(X_t) \otimes H^2(C) \otimes H^{2m-3}(X_t) \otimes H^1(C) \otimes H^{2m-2}(X_t) \otimes H^0(C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{2m}(Z_t) & \simeq & H^{2m}(X_t) \otimes H^0(C) \otimes H^{2m-1}(X_t) \otimes H^1(C) \otimes H^{2m-2}(X_t) \otimes H^2(C) \\
\downarrow i_3,* & & \downarrow i_3,* & & \downarrow i_3,* & & \downarrow (\text{weak Lef.}) i_3,* \\
H^{2m}(Z) & \simeq & H^{2m}(X) \otimes H^0(C) \otimes H^{2m-1}(X) \otimes H^1(C) \otimes H^{2m-2}(X) \otimes H^2(C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(Z) & \simeq & H^{2m-2}(X) \otimes H^0(C) \otimes H^{2m-1}(X) \otimes H^1(C) \otimes H^{2m}(X) \otimes H^0(C)
\end{array}
\]

for which \( H^{2m-1}(X) / j_3,* H^{2m-3}(X_t) \simeq \text{Prim}^{2m-1}(X) \) by the primitive decomposition theorem, and there is a commutative diagram:
where \( h \) is a hyperplane class which induces a hyperplane class on \( X_t \).

From (3.2) and (3.3) we see that \( H^{2m}(Z) / i_3^* H^{2m-2}(Z) \simeq \text{Prim}^{2m-1}(X) \otimes H^1(C) \), which is defined to be \( \text{Prim}^{2m}(Z) \) (see the remark on p. 23).

(3.4) Lemma. \( H^{2m-1}(Z_t) \simeq H^{2m-1}(Z_t) \otimes i_3^* H^{2m-1}(Z) \).

Proof. There is a commutative diagram:

\[
\begin{array}{ccc}
H^{2m-1}(Z) & \simeq & H^{2m-1}(X) \otimes H^0(C) \otimes H^{2m-2}(X) \otimes H^1(C) \otimes H^{2m-3}(X) \otimes H^2(C) \\
i_3^* & \downarrow & i_3^* \\
H^{2m-1}(Z_t) & \simeq & H^{2m-1}(X_t) \otimes H^0(C) \otimes H^{2m-2}(X_t) \otimes H^1(C) \otimes H^{2m-3}(X_t) \otimes H^2(C) \\
& \downarrow & \downarrow \\
i_3^*, \text{ weak Lef.} & \simeq & i_3^*, \text{ weak Lef.} \\
H^{2m+1}(Z) & \simeq & H^{2m+1}(X) \otimes H^0(C) \otimes H^{2m}(X) \otimes H^1(C) \otimes H^{2m-1}(X) \otimes H^2(C) \\
i_3^*, \text{ weak Lef.} & \downarrow & \downarrow \\
H^{2m+2}(X_t) & \simeq & j_3^* H^{2m-2}(X) \otimes H^{2m-2}(X_t) \otimes H^2(C) \\
\end{array}
\]

Also \( H^{2m-2}(X_t) \simeq j_3^* H^{2m-2}(X) \otimes H^{2m-2}(X_t) \). Therefore the middle column of (3.5) satisfies image \( i_3^* \cap \ker i_3^* = 0 \). For the other two columns we use the following commutative diagram:
which implies that \( j_{3,*} : H^{2m-3}(X_t) \otimes H^2(C) \to H^{2m-1}(X) \otimes H^2(C) \) is injective, and therefore image \( i^* \cap \ker i_{3,*} = 0 \) for all three columns, i.e., \( H^{2m-1}(Z_t) \approx H^{2m-1}(Z_t) \vee i_{3,*} H^{2m-1}(Z) \).

Remark. It should be noted that in general \( i_{3,*} : H^{2m-1}(Z) \to H^{2m-1}(Z_t) \) is not injective. An easy example of this is when \( X \) is a hypersurface. In this case \( H^{2m-1}(X_t) = 0 \), so that \( i^* : H^{2m-1}(X) \otimes H^0(C) \to H^{2m-1}(X) \otimes H^0(C) \) is not injective. Referring to (3.5) we see that \( i_{3,*} : H^{2m-1}(Z) \to H^{2m-1}(Z_t) \) is obviously not injective. From this it is also clear that \( Z_t \) is not ample in \( Z \) for generic \( X \).

From (3.1), the Gysin sequence for the pair \((Z, Z_t)\) provides us with the short exact sequence:

\[
(3.7) \quad 0 \to \text{Prim}^2(Z) \to H^2(Z - Z_t) \to H^{2m-1}(Z_t)^\vee \to 0 .
\]

(3.8) The complements\(^1\). There is a commutative diagram:

\[^1\text{This discussion evolves from the ideas in [28, § 4].}\]
The short exact sequence (3.7) sheafifies to:

\[(3.10) \quad 0 \longrightarrow \text{Prim}^{2m}(Z) \longrightarrow R^2h_*\mathcal{C} \longrightarrow R^{2m-1}k_*\mathcal{C}_v \longrightarrow 0 \quad \text{over } U^2 \quad \text{(where Prim}^{2m}(Z) \text{ is the constant sheaf over } U).\]

Tensoring with $\theta_U \otimes -$ we obtain:

\[(3.11) \quad 0 \longrightarrow \theta_U \otimes \text{Prim}^{2m}(Z) \longrightarrow \theta_U \otimes R^2h_*\mathcal{C} \longrightarrow \theta_U \otimes R^{2m-1}k_*\mathcal{C}_v \longrightarrow 0 .\]

From (3.4) there is a natural splitting of $F$ (over $U$) into

\[F \simeq F_v \otimes i_3^*H^{2m-1}(Z) \otimes \theta_U , \text{ which extends to } F \simeq F_v \otimes i_3^*H^{2m-1}(Z) \otimes \theta_{P^1} .\]

There is a Hodge filtration defined on $\zeta = \theta_U \otimes R^{2m}h_*\mathcal{C} \simeq \mathbb{R}^{2m}pr_*\Omega^*_{Z\times U/U}(\log Z^o)$ over $U$, i.e. $\zeta^p = \mathbb{R}^{2m}pr_*\Omega^*_{Z\times U/U}(\log Z^o)$,

which provides us with the following short exact sequence:

\[(3.12) \quad 0 \longrightarrow \theta_U \otimes F^{m+1}\text{Prim}^{2m}(Z) \longrightarrow \zeta^{m+1} \longrightarrow \text{residue} \longrightarrow F^m_v \longrightarrow 0 .\]

(3.13) **Lemma.** In terms of the quasi-canonical extensions, the short exact sequence (3.12) extends to:

\[(3.14) \quad 0 \longrightarrow \theta_{P^1} \otimes F^{m+1}\text{Prim}^{2m}(Z) \longrightarrow \zeta^{m+1} \longrightarrow \tilde{F}^m_v \longrightarrow 0 \quad \text{over } P^1 .\]

$^2h,k$ are the restrictions of $\tilde{h},\tilde{k}$ over $U$. 

---

\[(3.9) \quad \overset{\bar{z}}{Z} \overset{\delta}{\longrightarrow} Z \times P^1 \overset{\delta}{\longleftarrow} \mathcal{C} \quad \text{where } \mathcal{C} = Z \times P^1 - \bar{z} \quad \bar{h}^{-1}(t) = z - Z_t \quad \text{pr is the projection}\]
Proof. It should be noted that the remarks concerning the quasi-canonical extension of $F$ apply similarly to $\zeta$. In particular (2.16) holds for $\zeta$. We first localize over a disk $\Delta$ with $\Delta^* \subset \mathbb{U}$.

Now over $\Delta$ we have $\zeta \sim \theta_{\Delta} \otimes \text{Prim}^{2m}(Z) \oplus \frac{F}{v}$. I claim that

$$\zeta^{m+1} \sim \theta_{\Delta} \otimes F^{m+1}_\text{Prim}^{2m}(Z) \oplus \frac{F}{v}.$$  

To prove this we just simply note that

$$\zeta/(\theta_{\Delta} \otimes F^{m+1}_\text{Prim}^{2m}(Z) \oplus \frac{F}{v}) \sim \theta_{\Delta} \otimes F^{m}_\text{Prim}^{2m}(Z) \oplus \frac{F}{v}/\frac{F}{v}$$

which is free. We now invoke (2.16) to conclude the above claim. It is now clear that (3.12) extends locally, hence globally as well.

Tensoring (3.14) with $\Omega^1_p$ - we obtain:

$$0 \longrightarrow \begin{array}{c} \Omega^1_p \otimes F^{m+1}_\text{Prim}^{2m}(Z) \longrightarrow \Omega^1_p \otimes \zeta^{m+1} \longrightarrow \Omega^1_p \otimes \frac{F}{v} \longrightarrow 0 \end{array}$$

hence the long exact sequence:

$$\cdots \longrightarrow H^1(p^1, \Omega^1_p \otimes \zeta^{m+1}) \longrightarrow H^1(p^1, \Omega^1_p \otimes \frac{F}{v}) \longrightarrow 0.$$  

As a preliminary to stating the main vanishing theorem, we prove the following:

(3.17) Proposition. (i) $H^{2m}(Z - Z_t) \simeq H^{2m-1}(X - X_t) \otimes H^1(C)$

(ii) $F^{m+1}H^{2m}(Z - Z_t) \simeq F^mH^{2m-1}(X - X_t) \otimes H^{0,1}(C)$

$$\oplus F^{m+1}H^{2m-1}(X - X_t) \otimes H^{0,1}(C)$$
where $F^p_\mathbb{H}^2m(Z - Z_t) = \mathbb{H}^2m(Z, F^p_\mathbb{H}^Z(\log Z_t))$ and where $\mathbb{H}^\cdot$ denotes hypercohomology.

**Proof.** (i) $Z - Z_t = X \times C - X_t \times C = \{X - X_t\} \times C$. Therefore via the Künneth formula $\mathbb{H}^2m(Z - Z_t) \simeq \mathbb{H}^2m(X - X_t) \otimes \mathbb{H}^0(C) \otimes \mathbb{H}^{2m-1}(X - X_t) \otimes \mathbb{H}^1(C) \otimes \mathbb{H}^{2m-2}(X - X_t) \otimes \mathbb{H}^2(C)$.

Now $\mathbb{H}^2m(X - X_t) = \mathbb{H}^2m(X, \Omega^\cdot_X(*X_t))$, the hypercohomology of the complex of meromorphic differentials with poles of arbitrary (finite) order. The $E_1$ terms of this hypercohomology spectral sequence abutting to $\mathbb{H}^2m(X - X_t)$ are of the form $E_1 = \mathbb{H}^{2m-p}(X, \Omega^p_X((p+1)X_t))$ for $p \geq 0$. By Serre duality, $E_1 = \mathbb{H}^{p-1}(X, \Omega^{2m-1-p}_X(-(p+1)X_t))$ which is zero by Nakano's generalization of Kodaira's vanishing theorem. Therefore $\mathbb{H}^2m(X - X_t) = 0$, and one can argue from this result that $\mathbb{H}^{2m-2}(X - X_t)$ is zero also.

(ii) is a trivial consequence of (i) and its proof will be omitted.

**Remark.** (3.17) can also be proven from the following Gysin sequences:
(3.18) \[ H^{2m-2}(X_t) \to H^{2m}(X) \to H^{2m}(X - X_t) \to H^{2m-1}(X_t) \to H^{2m+1}(X) \]

(weak Lef.)

which implies that \( H^{2m}(X - X_t) = 0 \).

(3.19) \[ H^{2m-4}(X_t) \xrightarrow{i'} H^{2m-2}(X) \to H^{2m-2}(X - X_t) \xrightarrow{r} H^{2m-3}(X_t) \xrightarrow{i''} H^{2m-1}(X). \]

Now a quick inspection of (3.3) will reveal that \( i' \) is surjective, hence \( r \) is injective. Similarly (3.6) implies that \( i'' \) is injective. A combination of these two facts immediately implies that \( H^{2m-2}(X - X_t) = 0 \).

There is a diagram analogous to (3.9):

(3.20) \[
\begin{array}{ccc}
X & \xrightarrow{f} & X \times \mathbb{P}^1 \\
\downarrow{\bar{f}} & & \downarrow{pr}
\end{array}
\]

\[ g \quad \text{where} \quad \bar{C}_f = X \times \mathbb{P}^1 - X \]

\[ g^{-1}(t) = X - X_t \]

As before, we define \( \bar{\zeta}_f = \theta_U \otimes R^{2m-2} g_* \mathbb{C} \) over \( U \), with Hodge filtration subsheaves \( \bar{\zeta}_f^p \) over \( U \), and extend to \( \bar{\zeta}_f^p \) over \( \mathbb{P}^1 \).

From (3.17) part (ii) we obtain:

(3.21) **Corollary.** \( \bar{\zeta}^{m+1} \cong \bar{\zeta}_f^m \otimes H^{1,0}(C) \otimes \bar{\zeta}_f^{m+1} \otimes H^{0,1}(C) \).

We now state our main vanishing theorem:
Theorem. \( H^{p} (p, \Omega \otimes \mathcal{E}) = 0 \) for all \( p \geq m+1 \).

The proof of the above theorem will be obtained through a series of lemmas, propositions, and other theorems. This treatment is the same as in [28, § 4] so that only an outline will be given. The only real difference between this treatment and the one in [28] is that the hyperplane sections \( X_t \) of \( X \) are even dimensional.

Now over \( U \), \( \varphi_f \cong \mathbb{R}^{2m-1} \mathcal{E} \log X^O \) on \( X \times U/U \).

The order of pole filtration on the complex \( \Omega^* \mathcal{E} X \times U/U \) is defined as:

\[
\begin{align*}
G_{X \times U/U}^p (\Omega^* \mathcal{E} X) &= \begin{cases} \\
\Omega^q \mathcal{E} X \times U/U((q-p+1) \mathcal{E} X^O) & \text{if } q \geq p \\
0 & \text{if } q < p
\end{cases}
\end{align*}
\]

Over \( U \) we have:

\[
\varphi_f \cong \mathbb{R}^{2m-1} \mathcal{E} \mathcal{G} \mathcal{E} \mathcal{E} X \times U/U (\mathcal{E} X^O)
\]
Now there is an obvious extension of $R^{2m-1}_{pr_*G^P\Omega^*_{XxU/U}}(\pi^O)$ over $U$ to $P^1$, namely:

\[(3.25) \quad \zeta'_P = R^{2m-1}_{pr_*G^P\Omega^*_{XxP/P}}(\pi^O)\]

We will later compare $\zeta'_P$ to $\zeta_P$ but for now we will prove the following:

\[(3.26) \quad \text{Proposition.} \quad H^1(P^1, \Omega^1_P \otimes R^{2m-1}_{pr_*G^P\Omega^*_{XxP/P}}(\pi^O)) = 0\]

**Proof.** We will proceed in three steps.

**Step I:** Lemma. (i) $\pi \subseteq XxP^1$ is a hyperplane section under the Segre embedding of $P^N x P^1$ in $P^{2N-1} (X \subseteq P^N)$

therefore (ii) $\pi$ is linearly equivalent in $XxP^1$ to the divisor $X x P^1 + Xx\{\infty\}$.

**Proof.** This is proven in (4.43) in [28].

Now denote $\bar{\pi} = X x x P^1 + Xx\{\infty\}$. Then it is easy to prove:

\[(3.27) \quad \text{Proposition.} \quad H^1(P^1, \Omega^1_P \otimes R^{2m-1}_{pr_*G^P\Omega^*_{XxP/P}}(\pi^O)) = 0\]

**Proof.** This is (4.44) of [28], however since the proof is easy we will provide it here.

Let $Px : X x P^1 \to X$ be the canonical projection. Using algebraic differentials, the complex $G^P\Omega^*_{XxP/P}(\pi^O)$ takes the form:
0 \rightarrow P^\bullet X_p^1 (X_{\infty}) \otimes pr^* \theta_1 (1) \rightarrow P^\bullet X_p^1 (2X_\infty) \otimes pr^* \theta_1 (2) \rightarrow \cdots

This complex clearly contains the subcomplex \( L' \):

0 \rightarrow P^\bullet X_p^1 (X_{\infty}) \otimes pr^* \theta_1 (1) \rightarrow P^\bullet X_p^1 (2X_\infty) \otimes pr^* \theta_1 (1) \rightarrow \cdots

Now it is obvious that \( R^{2m-1} pr^* L' = H^{2m-1} (X, G^p \Omega' (X_{\infty})) \otimes \theta_1 (1) \)

so that \( H^1 (p^1, \Omega_p^1 \otimes R^{2m-1} pr^* L') = 0 \). It is easy to check that the cokernel of the inclusion \( \Omega_p^1 \otimes R^{2m-1} pr^* L' \subset \Omega_p^1 \otimes R^{2m-1} pr^* G^p \)

\( \Omega' X \times P^1 / P^1 \) has support over \( pr^{-1} (\infty) \), therefore since \( H^1 (\ast) \) is right exact on the above inclusion of sheaves, we must have

\[ H^1 (p^1, \Omega_p^1 \otimes R^{2m-1} pr^* G^p \Omega' X \times P^1 / P^1 (\ast Y)) = 0. \]

**Step II**: We would like to replace \( \bar{Y} \) by \( \bar{X} \) in (3.27). For this we need the following version of the semicontinuity theorem:

(3.28) **Theorem.** Let \( f : V \rightarrow S \) be a proper, flat morphism of Noetherian schemes, and let \( K' \) be a complex of sheaves on \( V \), whose terms \( K^i \) are coherent \( \theta_V \) - modules that are flat over \( S \), and whose differentials are linear over \( f^* \theta_S \). For each \( p \geq 0 \), the function:

\[ s \mapsto \dim_k (s) H^p (V_s, K'^s) \]

is upper semicontinuous on \( S \).
Proof. This is (4.45) of [28].

The relation for linear equivalence of the two divisors $\bar{x}$ and $\bar{y}$ in the above lemma (ii) is given by a divisor $N \subset X \times P^1 \times P^1$ where $N_0 = b_1^{-1}(0) = \bar{x}$ and $N_\infty = b_1^{-1}(\infty) = \bar{y}$, $b_1, b_2$ being the canonical projections in:

(3.29)

Now introduce the complex $K' = G^P \Omega^* \rightarrow X \times P^1 \times P^1$. We need the following:

(3.30) **Lemma.** $R^{2m}b_{2,*}K^* = 0$.

Proof. This is (4.46) of [28].

(3.31) **Corollary.** For all $s$ in $P^1$, $R^{2m}pr_*K_s = 0$.

(3.32) **Corollary.** $H^1(P^1, \Omega^1_{P^1} \otimes R^{2m-1}pr_*K'_s) = H^{2m}(X \times P^1, pr^* \Omega^1_{P^1} \otimes K_s')$.

Proof. Identify the two $E_2$ terms for $H^{2m}(X \times P^1, pr^* \Omega^1_{P^1} \otimes K'_s)$ and use (3.31).
Step III: Now define $d(s) = \dim H^{2m}(\mathbb{X} \times \mathbb{P}^1, \text{pr}_*\mathbb{P}^1_\mathbb{L} \otimes K_s)$. It is easy to see that $d(\infty) = 0$ ((3.27)), so that by (3.28), $d(s) = 0$ in a Zariski neighbourhood of $\infty$. Therefore (3.26) is a consequence of the following:

(3.33) Lemma. $d(s)$ is closed under specialization. \(^3\)

Proof. This follows from (4.49) in [28].

To arrive at a proof of the vanishing theorem (3.22), it will be necessary to give a description of the local dual and vanishing cycles associated to the Lefschetz pencil $\{X_t\}$ of $X$. A thorough treatment of this can be found in [7, §15]. We localize the family in (1.2) over a disk $\Delta$ with $\Delta \cap \Xi = 0 \in \Delta$. Let $X_0$ be the corresponding singular fiber with singular point $z_0 \in X_0$. We consider an open polydisk (radius $c$) neighbourhood in $\tilde{X}$ centered at $z_0$, so that a local description of $X_t$ near $z_0$ in suitable coordinates $z = (z_1, \ldots, z_{2m-1})$ is precisely:

\[
\{z = (z_1, \ldots, z_{2m-1}) \mid z \cdot z = \sum_{j=1}^{2m-1} z_j^2 = t \text{ and } |z|^2 \leq c, \text{ where } t \in \Delta\}.
\]

We can write $z = x + iy$ where $x = (x_1, \ldots, x_{2m-1})$ and $y = (y_1, \ldots, y_{2m-1})$.

\(^3\) Specialization in this context refers to the notion of continuity, i.e., $d(s) = 0$ for all $s \in \mathbb{P}^1$. 

are real. Now define for $c > \varepsilon > 0$, the following cycle:

\begin{equation}
\delta_\varepsilon = \{z = x + iy \mid x \cdot x = \varepsilon, y = 0\} \subset X_\varepsilon.
\end{equation}

Also we will denote $U_\varepsilon = X_\varepsilon \cap \text{above polydisk}$. Note that for all $z \in U_\varepsilon$, $x \neq 0$ otherwise $z \cdot z = \varepsilon < 0$. Therefore $U_\varepsilon$ retracts onto $\delta_\varepsilon$ via the retraction map $z = x + iy \mapsto (\varepsilon/x \cdot x)^{1/2}x$. Note also that $\delta_\varepsilon = 2m-2$ sphere. Therefore we obtain the following:

\begin{equation}
H_q(U_\varepsilon, \mathcal{L}) = \begin{cases} 
\mathbb{Z} & \text{for } q = 2m-2, 0 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

It is not hard to check that $\delta_\varepsilon$ is the local vanishing cycle in $H_{2m-2}(X_\varepsilon, \mathcal{L})$ associated to $z_0 \in X_0$.

**The relative dual cycle:** From (3.35) and Poincaré-Lefschetz duality, we obtain:

\begin{equation}
H_q(U_\varepsilon, \partial U_\varepsilon, \mathcal{L}) = 0 \text{ for } q \neq 2m-2, 4m-4; \text{ and } H_{2m-2}(U_\varepsilon, \partial U_\varepsilon, \mathcal{L}) \sim \mathbb{Z}.
\end{equation}

We exhibit the generator of $H_{2m-2}(U_\varepsilon, \partial U_\varepsilon, \mathcal{L})$ as the relative dual cycle $\theta_\varepsilon$.

Define $\theta_\varepsilon$ to be the $2m-2$ cell in $U_\varepsilon$ defined by:
\[
\theta_\varepsilon = \begin{cases} 
  y = (y_1, \ldots, y_{2m-2}, 0) & \quad y \cdot y \leq (c - \varepsilon)/2 \\
  x = (0, \ldots, \sqrt{\varepsilon} + y') 
\end{cases}
\]

Then \( \delta_\varepsilon \cap \theta_\varepsilon = (0, \ldots, \sqrt{\varepsilon}) \) and \( \partial \theta_\varepsilon \) is a 2m-3 sphere in \( \partial U_\varepsilon \).

There is a short exact sequence:

\[
(3.38) \quad 0 \rightarrow H_{2m-2}(U_\varepsilon, \mathbb{Z}) \rightarrow H_{2m-2}(X_\varepsilon, \mathbb{Z}) \rightarrow H_{2m-2}(X_\varepsilon, U_\varepsilon, \mathbb{Z}) \rightarrow 0
\]

which by duality defines a short exact sequence:

\[
(3.39) \quad 0 \rightarrow H_{2m-2}(X_\varepsilon - U_\varepsilon, \mathbb{Z}) \rightarrow H_{2m-2}(X_\varepsilon, \mathbb{Z}) \rightarrow H_{2m-2}(U_\varepsilon, \partial U_\varepsilon, \mathbb{Z}) \rightarrow 0
\]

where \( r(\gamma) = (\gamma, \delta_\varepsilon) \theta_\varepsilon \).

From (3.38) and (3.20), there is a commutative diagram (we denote \( U \) to be a polydisk in \( X \times P \) centered at \( z_0 \) and inducing the polydisk described above in \( \bar{X} \)):

\[
\begin{array}{ccc}
0 & \rightarrow & H_{2m-2}(U_\varepsilon, \mathbb{Z}) \\
\uparrow & & \uparrow r_1 \\
0 & \rightarrow & H_{2m-2}(X_\varepsilon, \mathbb{Z}) \\
\uparrow & & \uparrow r_2 \\
H_{2m-1}(U \cap g^{-1}(\varepsilon), \mathbb{Z}) & \rightarrow & H_{2m-1}(g^{-1}(\varepsilon), \mathbb{Z}) \\
\uparrow & & \uparrow r_3 \\
0 = H_{2m-1}(U \cap pr^{-1}(\varepsilon), \mathbb{Z}) & \rightarrow & H_{2m-1}(X, \mathbb{Z}) \\
\end{array}
\]
Note that the homomorphisms $r_1$ are inverse to the "tube over cycle" homomorphism. A simple diagram chase will reveal that $i'$ is injective and $r'$ is surjective. Note also that $r_1$ is an isomorphism. Therefore combining (3.39) with (3.40) we obtain the following: $(t_1 = \text{tube over cycle homomorphisms})$

\[
\begin{array}{c}
0 \longrightarrow H_{2m-2}(X^n - U^n, \mathbb{Z}) \longrightarrow H_{2m-2}(X^n, \mathbb{Z}) \xrightarrow{r} H_{2m-2}(U^n, \mathbb{Z}) \longrightarrow 0 \\
\downarrow t_3 \quad \quad \downarrow t_2 \quad \quad \sim \quad \downarrow t_1 \\
0 \longrightarrow H_{2m-1}(g^{-1}(\epsilon) - U, \mathbb{Z}) \longrightarrow H_{2m-1}(g^{-1}(\epsilon), \mathbb{Z}) \longrightarrow H_{2m-1}(g^{-1}(\epsilon) \cap U, \mathbb{Z}) \longrightarrow 0
\end{array}
\]

Note that the image of $H_{2m-1}(g^{-1}(\epsilon) - U, \mathbb{C})$ in $H_{2m-1}(g^{-1}(\epsilon), \mathbb{C})$ is a codimension 1 subspace.

We make the following remark before finishing off the proof of (3.22):

(3.42) Remark. As in [28], we assume that the torsion of $\xi_{\mathfrak{p}}^P$ (which can only be supported on $\Sigma$) is divided out so that $\xi_{\mathfrak{p}}^P$ is torsionless. This will in no way affect the vanishing theorem arguments to follow.

We need the following useful result:

\[4\] A discussion of the "tube over cycle" homomorphism appears in [7, §3].
Proposition. (i) \( \frac{\zeta_*^P}{f} \leq \frac{\zeta_*^P}{f} \) for \( p \geq m+1 \).

(ii) \( \frac{\zeta_*^m-k+1}{f} \leq \frac{\zeta_*^m-k+1}{f} \) for \( 1 \leq k \leq m+1 \).

Proof. The proof is almost the same as in [28], except that in this case \( n = 2m-1 \) is odd. Therefore whenever necessary, some estimates have to be checked. Using Dolbeault resolutions to compute the hypercohomology and localizing over \( \Delta \), a local section of \( \frac{\zeta_*^P}{f} \) is a relatively closed \( C^\infty \) \( 2m-1 \) form \( \phi \) on \( X_\Delta - \bar{X} \) with a pole along \( \bar{X} \) of order at most \( v = 2m-p \). We determine the period growth of \( \phi_t = \text{restriction of } \phi \text{ to } X - X_t \), around \( 0 \in \Delta \), and this is done by integrating \( \phi_t \) over the integral cycles coming from \( H_{2m-1}(\tilde{g}^{-1}(t), \mathbb{Z}) \) (\( t \in \Delta^* \)). Note that the fibers \( \tilde{g}^{-1}(t) - U \) are topologically the same for all \( t \in \Delta \), so that cycles in \( H_{2m-1}(\tilde{g}^{-1}(t) - U, \mathbb{Z}) \) can be kept uniformly away from \( \bar{X} \) as \( t \) varies in \( \Delta \). Therefore using (3.41) it is clear that \( \int \phi_t \) is uniformly bounded over cycles in \( H_{2m-1}(\tilde{g}^{-1}(t), \mathbb{Z}) \) which come from \( H_{2m-1}(\tilde{g}^{-1}(t) - U, \mathbb{Z}) \). Therefore it suffices to check \( \int \phi_t \) over the tube of the dual cycle described by the homomorphism \( t_2 \) in (3.41). Since taking residues is dual to taking tubes, it suffices to evaluate \( \int \text{res } \phi_t \), where \( \theta_t \) is the relative portion of the dual cycle. There is no loss of generality in assuming \( t \) is real and positive.
The explicit description of $\theta_t$ is given in (3.37). We utilize the following results from [28]:

\[ (3.44) \text{Lemma.} \]

\[
\left| \int_{\theta_t} \text{res } \varphi \right| \leq A \int_0^{\sqrt{1-t}} (t + r^2)^{p-2m+\frac{1}{2}} r^{2m-3} \, dr
\]

where $A$ is some constant.

Now the expression on the RHS of (3.44) is bounded above by

\[ (3.45) \]

\[
A \int_0^{\sqrt{1-t}} (t + r^2)^{p-m-1} \, dr. \quad \text{Therefore for } p \geq m+1, \text{ (3.44) is obviously bounded as } t \to 0. \text{ This proves (i) of (3.43).}
\]

We now make a change of variables by setting $w = r/\sqrt{t}$. Then $dr = \sqrt{t} \, dw$ and (3.45) becomes:

\[ (3.46) \]

\[
(A \sqrt{t}) t^{p-m-1} \int_0^{(t^{1/2} - 1)^{1/2}} (1 + w^2)^{p-m-1} \, dw.
\]

Now assume $0 \leq p \leq m$. Then it is clear that $(1 + w^2)^{p-m-1} \leq (1 + w^2)^{-1}$, so that the following holds:
\[
\int_0^{(t^{-1}-1)^{\frac{1}{2}}}(1+w)^{m-1} dw \leq \int_0^{(t^{-1}-1)^{\frac{1}{2}}}(1+w^2)^{-1} dw = \tan^{-1}(t^{-1})^{\frac{1}{2}}
\]

which is bounded at \( t \to 0 \).

Therefore (3.46) is bounded by \((B\sqrt{t})/t^{m+1-p}\) where \( B \) is some constant.

By setting \( p = m-k+1 \) (for \( 1 \leq k \leq m+1 \)) we obtain part (ii) of (3.43).

(3.48) **Corollary.** Let \( 1 \leq k \leq m+1 \). Then \( H^1(P^1, \Omega^1_p \otimes \mathcal{Z}^{m-k-1}(k\Sigma)) = 0 \).

**Proof.** There is a short exact sequence:

\[
0 \rightarrow \Omega^1_p \otimes \mathcal{Z}^{P} \rightarrow \Omega^1_p \otimes \mathcal{Z}^{P} (k\Sigma) \rightarrow L' \rightarrow 0 \text{ where } p = m-k+1
\]

and \( L' \) has 0 dimensional support. Since \( H^1(P^1, \Omega^1_p \otimes \mathcal{Z}'P_f) = 0 = H^1(P^1, L') \), the assertion is obvious.

We now have the following:

(3.50) **Main Vanishing Theorem.** \( H^1(P^1, \Omega^1_p \otimes \mathcal{Z}^{P_f}) = 0 \) for all \( p \geq m+1 \).

**Proof.** There is a short exact sequence:

\[
0 \rightarrow \Omega^1_p \otimes \mathcal{Z}^{P_f} \rightarrow \Omega^1_p \otimes \mathcal{Z}^{P_f} \rightarrow L'' \rightarrow 0 \text{ where } L'' \text{ has 0 dimensional support. We now proceed as in (3.48).}
\]

This concludes the proof of our first main result. The rest of this section will be devoted to giving corollaries to (3.50), in particular our main theorem on the normal functions.
(3.52) **Corollary.** \( H^1(p^1, \Omega^1_{p^1} \otimes \tilde{z}^{m+1}_{\tilde{f}}) \simeq H^1(p^1, \Omega^1_{p^1} \otimes (\tilde{z}^m_{\tilde{f}} / \tilde{z}^{m+1}_{\tilde{f}})) \otimes H^{1,0}(C). \)

**Proof.** This follows from (3.21), (3.50) and the following short exact sequence:

(3.53) \[
0 \longrightarrow \tilde{z}^{m+1}_{\tilde{f}} \longrightarrow \tilde{z}^m_{\tilde{f}} \longrightarrow \tilde{z}^m_{\tilde{f}} / \tilde{z}^{m+1}_{\tilde{f}} \longrightarrow 0.
\]

(3.54) **Corollary.** \( H^0(p^1, \tilde{F}^m_{v}, \tilde{f}^{m, \ast}_v) \simeq H^0(p^1, \tilde{F}^{m-1}_{v} / \tilde{F}^m_{v}) \otimes H^{0,1}(C). \)

**Proof.** There is a commutative diagram:

(3.55)

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1_{p^1} \otimes \tilde{z}^{m+1}_{\tilde{f}} \otimes H^{1,0}(C) & \longrightarrow & \Omega^1_{p^1} \otimes \tilde{z}^m_{\tilde{f}} \otimes H^{1,0}(C) & \longrightarrow & \Omega^1_{p^1} \otimes \tilde{z}^m_{\tilde{f}} / \tilde{z}^{m+1}_{\tilde{f}} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \Omega^1_{p^1} \otimes \tilde{F}^m_{v} \otimes H^{1,0}(C) & \longrightarrow & \Omega^1_{p^1} \otimes \tilde{F}^{m-1}_{v} \otimes H^{1,0}(C) & \longrightarrow & \Omega^1_{p^1} \otimes \{(\tilde{F}^{m-1}_{v} / \tilde{F}^m_{v}) \otimes H^{1,0}(C) \} \\
& & & & & & \longrightarrow 0
\end{array}
\]

Now one easily checks that (3.50) implies that \( H^1(p^1, \Omega^1_{p^1} \otimes \tilde{z}^{m+1}_{\tilde{f}} \otimes H^{1,0}(C)) \)
Since \( \mathfrak{F}_v \) is made up of \( \mathfrak{F}_v^m \) and \( \mathfrak{F}_v^{m-1} \), we can combine (3.55), (3.52) and (3.16) to obtain the following commutative diagram:

\[
\begin{array}{ccc}
H^1(P^1, \Omega^1 \otimes \mathfrak{F}_v^m) & \xrightarrow{\sim} & H^1(P^1, \Omega^1 \otimes \mathfrak{F}_v^{m-1} / \mathfrak{F}_v^m) \otimes H^1,0(C) \\
\downarrow & & \downarrow \\
H^1(P^1, \Omega^1 \otimes \mathfrak{F}_v^{m-1}) & \xrightarrow{\sim} & H^1(P^1, \Omega^1 \otimes \mathfrak{F}_v^m / \mathfrak{F}_v^{m-1}) \otimes H^1,0(C)
\end{array}
\]

Now the bundle \( \mathfrak{F}_v^{m-1} / \mathfrak{F}_v^m \) is self dual, so that Serre duality provides us with the desired result.

A good consequence of the above result is that we can give a more explicit description of the ker \( \delta \), defined in (2.18). More precisely, we obtain:

(3.57) Corollary. The ker \( \delta \) is equal to the image of

\[
H^0(P^1, \mathfrak{F}_v^{m-1} / \mathfrak{F}_v^m) \otimes H^0,1(C) \oplus J^{2m-1}(Z) \text{ in } H^0(P^1, \mathfrak{J}), \text{ where }
\]

\( J^{2m-1}(Z) \) is defined as: \( J^{2m-1}(Z) = \mathfrak{F}_v^{m} \wedge H^{m-1}(Z) / H^{2m-1}(Z, \mathcal{Z}) \).

Proof. Use (3.55), the splitting defined just preceding (3.12) and the short exact sequence (2.17).
Corollary. 5  \( H^0(P^1, J') \) lies in the image of \( J^{2m-1}(Z) \).

(3.58) **Corollary.** \( H^1(P^1, \Omega^1_{P^1} \otimes F_P) = 0 \) for all \( p \geq m+1 \).

**Proof.** The two decompositions:

\[
\tilde{F}_P = \tilde{F}_P \otimes H^0,1(C) \otimes \tilde{F}_P \otimes H^1,0(C)
\]

are related in an obvious way via the residue homomorphism. We now apply (3.50), using an analogous sequence to (3.16).

(3.60) **Corollary.** \( H^0(P^1, \Omega^1_{P^1} \otimes F_P, *) = 0 \) for all \( p \geq m+1 \).

**Proof.** We first note that \( \tilde{F}_P \cong \tilde{F}_P \otimes M \) where \( M = i^*_{P^1} H^{2m-1}(Z) \) so that \( \tilde{F}_P, * \cong \tilde{F}_P, * \otimes M \). Now clearly \( H^0(P^1, \Omega^1_{P^1} \otimes M) = 0 \), and since \( \Omega^1_{P^1} = \theta_{P^1}(-2) \), it is obvious that \( H^0(P^1, \Omega^1_{P^1} \otimes \tilde{F}_P, *) \rightarrow H^0(P^1, \tilde{F}_P, *) \) is injective. But \( H^0(P^1, \tilde{F}_P, *) \cong H^1(P^1, \Omega^1_{P^1} \otimes \tilde{F}_P) \) which is 0 by (3.59). Therefore we have:

\[
H^0(P^1, \Omega^1_{P^1} \otimes \tilde{F}_P, *) \cong H^0(P^1, \Omega^1_{P^1} \otimes \tilde{F}_P, *) \otimes H^0(P^1, \Omega^1_{P^1} \otimes M) = 0.
\]

We now arrive at our main result for the normal functions:

---

5 The proof of (3.58) can be found in (A.3).
Theorem. \( H^0(F^1, \bar{J}) = H^0(F^1, \bar{J})_h \).

Remark. (3.58) becomes: \( H^0(F^1, \bar{J}) \) lies in the image of \( J^{2m-1}(Z) \).

Proof of (3.61). We first localize \( F \) over a disk \( \Delta \) with
\[ \Delta \cap \Sigma = 0 \in \Delta. \]

Let \( N = T-I \), where \( T \) is the local monodromy transformation.
Then \( N \) acts on the sheaf \( F \) so as to obtain a subsheaf \( \bar{W} = \ker N \) of \( \bar{F} \), which defines a flat subbundle of dimension \( b_{2m-1} - 2g \) (where we assume for simplicity that \( H^{2m-1}(Z) = H^{2m-2}(X_t) \otimes H^1(C) \)).

\[ b_{2m-1} = \dim H^{2m-1}(Z_t), \text{ and } g = \text{genus of } C. \]

We let \( \theta_{\Delta,0} \) be the localization of \( \theta_{\Delta} \) at \( 0 \in \Delta \). Define \( \bar{F}_0 \) (respectively \( \bar{W}_0 \)) to be the stalk of \( \bar{F} \) (respectively \( \bar{W} \)) over \( 0 \in \Delta \), i.e., \( \bar{F}_0 = \bar{F} \otimes \theta_{\Delta,0} \).

Let \( \psi: \bar{W}_0 \longrightarrow \bar{F}_0^{m,*} \) be the morphism obtained from the composite
\[ \bar{W}_0 \hookrightarrow \bar{F}_0 \longrightarrow (\bar{F}_0/\bar{F}_0^m) = \bar{F}_0^{m,*}. \]
We remark that a horizontal basis for \( \bar{W} \) over \( \Delta \) is given by \( \{v_1, \ldots, v_N\} \otimes \{\text{basis of } H^1(C)\} \), where the \( \{v_i\} \) are orthogonal (under the cup product pairing) to the local vanishing cocycle \( \delta \), hence \( \bar{W} \) is naturally self dual under the cup product pairing. Also if \( \delta \in H^{m-1,m-1}(X_t, \mathcal{Q}) \) via horizontal displacement for generic \( t \in \Delta^* \), then in fact the irreducibility of the \( \Pi_1(U) \) action on the vanishing cohomology associated to the Lefschetz
pencil (due to the Picard Lefschetz formula, and the fact that $\pi_1(U)$ acts transitively on the vanishing cocycles) implies that $H^{2m-2}(X_t, \mathcal{O}_V) \cong H^{m-1,m-1}(X_t, \mathcal{O})$ for all $t \in U$. Now let $v$ be any normal function. Then from the explicit description of $\nabla$ in (2.14), and the infinitesimal period relation (2.20), it is clear that $\nabla v = 0$ in this case. Therefore it suffices to assume that $\delta$ is not of type $(m-1,m-1)$ for generic $t \in \Delta^\star$.

We first justify the claim:

\[ F^m \subset F^m_{\bar{f},0} \] implies that $H^{2m-2}(X_t, \mathcal{O}) \subset H^{m-1,m-1}(X_t, \mathcal{O})$ for $t \in U, \quad f,o \quad f,o$

where $F^m_{\bar{f},0}$ is the localization of $F^m$ at $0 \in \Delta$ (resp. $F^m_{\bar{f},0}$).

Any class $\sigma \in F^m_{\bar{f},0}$ is represented in the form $\sigma = \sum_{i=1}^{N} g_i v_i + g_0 v(t)$ where $v(t) = \delta \sqrt{t}$ and $\{ g_i \}$ are holomorphic functions around 0.

Now the relation $F^m \subset \mathcal{W}$ implies that for all such $\sigma$ above, $N \sigma = 0$, i.e. $0 = -2g_0 v(t)$ which implies $g_0 \equiv 0$. This immediately implies that $\delta \subset H^{2m-2}(X_t)$ = 0 for $t$ in a neighbourhood of $0 \in \Delta$. Therefore $\delta$ remains of type $(m-1,m-1)$ under horizontal displacement in $U$, thus justifying the above claim from the properties of the $\pi_1(U)$ action on $H^{2m-2}(X_t, \mathcal{O})$.

Now utilizing the above claim and the above assumption that $\delta$ is not of type $(m-1,m-1)$ for generic $t \in \Delta^\star$, we deduce that $F^m_{\bar{f},0} \cap \mathcal{W} \subset F^m_{\bar{f},0}$, and therefore applying Nakayama's lemma to the unique maximal ideal $m \subset \mathcal{O}$, we obtain:

\[ (\bar{W} \cap F^m_{\bar{f},0}) / m F^m_{\bar{f},0} \subset F^m_{\bar{f},0} / m F^m_{\bar{f},0} \]

Now taking into account the description of $F^m_{\bar{f},0}$ in terms of $F^{m-1}_{\bar{f},0}$ and $F^{m-2}_{\bar{f},0}$,
we can deduce (using the symmetry of any given basis of $H^1(\mathbb{C})$) that $\dim_{\mathbb{C}} \{F^m \cap W \}/ m F \subset \mathbb{C} = b_{2m-1}/2 - 2g$. Since $\dim_{\theta , \Delta , o} W = b_{2m-1} - 2g$ and $\dim_{\theta , \Delta , o} F^m , ^* = b_{2m-1}/2$, we conclude that there exists a commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & F^m & \rightarrow & F^m & \rightarrow & 0 \\
\downarrow \text{mod } m & & \downarrow b_{2m-1}/2 & & \downarrow b_{2m-1} & & \downarrow b_{2m-1}/2 \\
0 & \rightarrow & W \cap F^m & \rightarrow & W & \rightarrow & 0 \\
\end{array}
$$

for which it is clear by a suitable diagram chase (and Nakayama's lemma) that $\psi (\bar{W}_o ) = \bar{F}^m , ^* . $

(3.62) The upshot of the result $\psi (\bar{W}_o ) = \bar{F}^m , ^*$ is that given a section $\sigma \in H^0(\Delta, F^m , ^* ),$ $\sigma$ can be represented by a section $\bar{\sigma} \in H^0(\Delta, \bar{F})$ with holomorphic derivative with respect to the Gauss-Manin connection. That is $\nabla \sigma \in H^0(\Delta, \Omega^1_{\mathbb{P}^1} \otimes F^{m+1} , ^* ).$ Therefore if $v$ is any normal function, we have $\nabla v \in H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \otimes F^{m+1} , ^* ).$ We now invoke (3.60) to conclude the proof of the theorem.

Remark. As a by-product of the results of this section, the following is easy to prove and will be left to the reader:

(3.63) Proposition. $H^0(\mathbb{P}^1, F^{m} , ^*) = i^{*} F^P , ^* H^{2m-1}(\mathbb{Z}, \mathbb{C})$ for all $p \geq m+1.$

This concludes chapter 3.
Chapter 4. A Summary on the Normal Functions.

The purpose of this chapter is twofold, namely:

(i) Prove the meromorphicity of the Abel-Jacobi mapping and apply it to the cases where an inversion theorem for the group of normal functions is known to hold (cf (0.35)).

(ii) Discuss a somewhat deeper aspect of the normal functions not already mentioned in chapter 3.

In regard to (i), we remark that the meromorphicity is a significantly stronger result than that attainable in the case of a Lefschetz pencil ([28,(4.58)]), for in [28] the meromorphicity is guaranteed provided the family of intermediate Jacobians locally embeds (over $\mathbb{P}^1$) in a Kähler manifold which is proper over $\mathbb{P}^1$ ([30,.p.202]). This criterion has been verified in a few geometric cases ([30]). The reason for the stronger assertion in (i) as opposed to that in [28] is entirely due to the differences in the local monodromy ($T^2=I$ rather than $(T-I)^2=0$). We remark that the purpose of proving the meromorphicity of the Abel-Jacobi mapping is to be able to construct algebraic cycles from normal functions which are known to be invertible. It is an easily verifiable fact that the knowledge of (i) implies the equivalence of the following two statements:

(A) $\text{Prim}_{m,m}^m(Z,\mathbb{Q})$ is generated by algebraic cocycles.

(B) Every class $\gamma \in \text{Prim}_{m,m}^m(Z,\mathbb{Z})$ is the cohomology class of an invertible normal function $v$.

The difficulty in proving (B) is the lack of knowledge of the image of the Abel-Jacobi morphism in $J(Z_t)$ (for $t \in U$).
In chapter 2, we identified a subgroup of $J(Z_t)$ (denoted by $J_A(Z_t)$ below) which from inductive assumptions is contained in the image of the Abel-Jacobi morphism. In this chapter we identify the part of $E_2^{1,2m-1}(\bar{k})$ that lies in the image of the normal functions which take their values in $\bigcup_t J_A(Z_t)$ ((4.30) and (4.32)) to be the classes of type $(m,m)$ in the cohomology of a certain sheaf, namely $H^1(F^1,L_k)$ where $L_k = L_f \otimes H^1(C,\mathcal{Z})$ and $L_f$ is defined in (4.9) part (i). The sheaf describing these normal functions is defined in (4.9) part (iv), and is denoted by $\mathcal{I}$. We then deduce from (i) and inductive assumptions that those classes in $E_2^{1,2m-1}(\bar{k}) \cap H^0(P^1,\mathcal{I})$ are in fact algebraic ((4.43)).

To discuss (ii), we remark that from the above discussion it is clear that the non-trivial part of proving (B) would be to study those normal functions $\nu$ which are not elements of $H^0(P^1,\mathcal{I})$. We indicate a direction for studying such $\nu$ by introducing the terms $E_2^1(p)$ in the remark following (4.29). The basic philosophy is the following: The cohomology $F^m E_2^{1,2m-1}(\bar{k})$ projects into $E_2^{1,m-1} \otimes H^1(C)$, and those normal functions which induce the zero cohomology class in $E_2^{1,m-1} \otimes H^1(C)$ are precisely those normal functions which are elements of $H^0(P^1,\mathcal{I}) + \ker \delta$. Now the terms $E_2^1(p)$ are expressed partly in terms of the Gauss-Manin connection and the general idea is that those normal functions $\nu / \not\in H^0(P^1,\mathcal{I})$ for which $\delta \nu \neq 0$ in $E_2^{1,2m-1}(\bar{k})$ will have a non-trivial cohomology class representation in $E_2^{1,m-1} \otimes H^1(C)$ written in terms of $\nabla$, thus reflecting on the infinitesimal properties of $\nu$. It is believed that the knowledge of (B) lies in a full understanding of the infinitesimal properties of the normal functions.
In the situation of our Lefschetz pencil \( \{X_t\}_{t \in \mathbb{P}^1} \), it can be shown that for "general enough" \( X \), the sheaf \( L_t \), and hence \( \mathcal{I} \) are constant over \( \mathbb{P}^1 \) (see (4.10) part(iii)). This would imply that \( \delta H^0(\mathbb{P}^1, \mathcal{I}) = 0 \) in \( E_2^{1,2m-1}(\bar{k}) \). To get an overall picture of what is happening, we have the following cases that can occur for a given \( Z \):

**Case 1** \( E_2^{m-1} = 0 \). This is implied\(^1\) by the cohomological condition

\[
F^m H^{2m-2}(X_t)_v = 0 \quad \text{for} \ t \in U, \text{and in this case,} \ H^1(\mathbb{P}^1, L_k \otimes \mathcal{O}) = E_2^{1,2m-1}(\bar{k}).
\]

We prove that \( \delta H^0(\mathbb{P}^1, \mathcal{I}) \) in \( E_2^{1,2m-1}(\bar{k}) \) is equal to all the \( Z \) classes of type \((m,m)\) in \( E_2^{1,2m-1}(\bar{k}) \) ((4.30)). The results in this chapter are the strongest in this case, for if we assume that the Hodge \((m-1,m-1)\) conjecture holds for \( X \) (for generic \( t \in U \)), which is equivalent to:

\[
H^{2m-2}(X_t, \mathcal{O})_v \text{ is generated by algebraic cocycles, when } F^m H^{2m-2}(X_t)_v = 0,
\]

then it is proven that the Hodge \((m,m)\) conjecture holds for \( Z \).

**Case 2** \( E_2^{m-1} \neq 0 \) and \( \delta H^0(\mathbb{P}^1, \mathcal{I}) \neq 0 \) in \( E_2^{1,2m-1}(\bar{k}) \). We remark that in this case \( F^m H^{2m-2}(X_t)_v \neq 0 \), and as a consequence, \( H^1(\mathbb{P}^1, L_k) = 0 \) and \( \delta H^0(\mathbb{P}^1, \mathcal{I}) = 0 \) in \( E_2^{1,2m-1}(\bar{k}) \). This is the most difficult case to handle and the techniques in this chapter are insufficient in proving the Hodge conjecture for such \( Z \).

**Case 3** \( E_2^{m-1} \neq 0 \) and \( \delta H^0(\mathbb{P}^1, \mathcal{I}) = 0 \) in \( E_2^{1,2m-1}(\bar{k}) \). We also remark here that \( F^m H^{2m-2}(X_t)_v \neq 0 \), and consequently \( H^1(\mathbb{P}^1, L_k) = 0 \). Now this case occurs precisely when \( \text{Prim}^{m,m}(Z, \mathcal{O}) \otimes H^{m-1,m-1}(Dx C, \mathcal{O})_v = 0 \) (assuming \( E_2^{m-1} \neq 0 \)), and examples of \( Z \) for which this occurs are when \( X \subset \mathbb{P}^{2m} \) is a generic hypersurface of degree \( \geq 2 + 3/(m-1) \), where \( m \geq 3 \) ([7, §13]). It would be nice to be able to detect from the knowledge of the normal functions

---

\(^1\)It can be shown that if \( E_2^{1,2m-1}(\bar{k}) \neq 0 \), then \( E_2^{m-1} = 0 \) iff \( F^m H^{2m-2}(X_t)_v = 0 \); see (A.5) for the proof.
when such a situation occurs for a given $Z = X \times C$, and we give some comments in this direction ((4.47) and (4.48)).

We now exhibit some non-trivial examples of cases 1 and 2 so as to verify that all 3 cases are not vacuous. We also work in the case $m = 2$ so that $X$ is a body (threefold). The easiest class of bodies $X$ studied were introduced by Fano and satisfy the embracing conditions: $\text{Pic}(X) = \mathcal{L}$ and $H^{0,3}(X) = 0$. From these conditions we see that every line bundle of positive degree on $X$ is ample, and that the canonical divisor $K_X$ has negative degree. Consequently some integral multiple $N$ of the positive generator of $\text{Pic}(X)$ is equivalent to $\theta_X(-K_X)$, and this $N$ is called the index of the Fano body. If we choose our Lefschetz pencil $\{X_t\}_{t \in \mathbb{P}^1}$ so that $\theta_X(X_t)$ generates $\text{Pic}(X)$, then the case $N > 2$ provides examples for which case 1 holds. The more interesting examples for which case 1 holds occur when $N = 2$, and such $X$ are of the following 3 forms ([27, p.43]):

1. a cubic hypersurface in $\mathbb{P}^4$
2. the intersection of 2 quadrics in $\mathbb{P}^5$
3. the intersection of the Grassmannian of lines in $\mathbb{P}^4$ with 3 hyperplanes in $\mathbb{P}^9$.

The example of the cubic $X$ is the most well studied, and in this case it is known that:

1. $\dim \mathbb{H}^3(X) = 10$, therefore $H^1(\mathbb{P}^4, L_\varphi \otimes \mathcal{C}) = H^1(\mathbb{P}^4, R^2 F_* \mathcal{C}) = H^3(X) \neq 0$.

2. If we denote $F(X)$ to be the variety of lines in $X$ (as a subvariety of the Grassmannian of lines in $\mathbb{P}^4$), then it is proven in [27, p.17] that there exists a curve $C \subseteq F(X)$ for which $H^{2,2}((X \times C) \cap$
(3) Consequently (since $J_A(Z_t) = J(Z_t)$) there exists normal functions $\nu$ taking their values in $\bigcup_{t \in P^1} J_A(Z_t)$ for which $\delta \nu \neq 0$ in $E_{Z_t}^{1,2m-1}(\Omega)$. Although the Hodge conjecture is already known to hold for $Z = X \times C$, for bodies $X$ with $N \geq 2$ (for example see [30]), the techniques of proof require some knowledge of the geometry of $X$ (existence of a covering family of lines), whereas from the techniques of this thesis, only a knowledge of the geometry of $X_t$ is required. In particular, we need only check that the geometric genus of $X_t$ is zero.

Some examples of Fano bodies $X$ satisfying case 2 are those $X$ where $N = 1$, $X$ a complete intersection, and where $X$ embeds as a hyperplane section in a smooth fourfold. The most well known example of such $X$ is the quartic hypersurface in $P^4$. It follows from [27,p.42] that there exists a curve $C$ for which $H^{2,2}(X \times C) \cap \{ H^3(X, \Omega) \otimes H^1(C, \Omega) \} \neq 0$, and hence there exists a normal function $\nu$ which does not take its values in $\bigcup_{t \in P^1} J_A(Z_t)$ (i.e. $\nu \notin H^0(P^1, \Omega))$, and for which $\delta \nu \neq 0$. It is also a standard fact that the Hodge conjecture holds for $Z = X \times C$ where $C$ is any curve and $X$ is any complete intersection Fano body of index 1 satisfying the above embedding condition.

Finally, one of the intentions in the process of writing up chapter 4 is to give somewhat more general proofs (whenever possible) of the results in this chapter than needed for the case $\{X_t\}_{t \in P^1}$ a Lefschetz pencil, with the hope of writing future papers stemming from these results.
Due to the product structure on $Z_t$, the Jacobian $J(Z_t)$ is reducible as a principal torus ([27, p. 10]). More explicitly:

\[(4.1) \quad J(Z_t) \approx (F^{m,*} H^{2m-1}(X_t)) \otimes \mathcal{O}^o(C) / H^{2m-1}(X_t, \mathbb{Z}) \otimes \mathcal{O}^o(C, \mathbb{Z})
\]

\[\otimes F^{m,*} (H^{2m-2}(X_t) \otimes \mathcal{O}^1(C)) / H^{2m-2}(X_t, \mathbb{Z}) \otimes \mathcal{O}^1(C, \mathbb{Z})\]

\[\otimes (F^{m-1,*} H^{2m-3}(X_t)) \otimes \mathcal{O}^2(C) / H^{2m-3}(X_t, \mathbb{Z}) \otimes \mathcal{O}^2(C, \mathbb{Z})\]

\[(4.2) \quad \text{We define } J = F^{m,*} (H^{2m-2}(X_t) \otimes \mathcal{O}^1(C)) / H^{2m-2}(X_t, \mathbb{Z}) \otimes \mathcal{O}^1(C, \mathbb{Z})\]

Since the main interest is $\text{Prim}^{2m}(Z) \sim \text{Prim}^{2m-1}(X) \otimes \mathcal{O}^1(C)$, we need only focus on $J$. In view of (2.23), we may assume the following:

\[(4.3) \quad \text{(i)} \quad F^{m,*} \sim F^{m-1,*} \otimes \mathcal{O}^0(C) \otimes F^{m,*} \otimes \mathcal{O}^1(C)\]

\[(\text{ii}) \quad j_* R^{2m-1} k_* \mathbb{Z} = j_* R^{2m-2} f_* \mathbb{Z} \otimes \mathcal{O}^1(C, \mathbb{Z}), \text{ i.e., } H^{2m-1}(Z_t, \mathbb{Z}) \]

\[= H^{2m-2}(X_t, \mathbb{Z}) \otimes \mathcal{O}^1(C, \mathbb{Z})\]

\[(\text{iii}) \quad \text{and } \bar{J} \text{ modified accordingly.}\]

All constructions in Chapters 1, 2, and 3 carry over to this case.

Assuming only the Hodge $(m-1,m-1)$ conjecture for $X_t$, we see that only a restricted part of $J$ is known to be invertible. In fact we can describe this part very precisely:
Lemma. (i) \( H^{m-1,m-1}(X_t, \mathbb{Z}) \otimes H^1(C, \mathbb{Z}) \) is a lattice in
\[ H^{m-1,m-1}(X_t, \mathbb{Z}) \otimes \mathbb{Z}^0,1(C, \mathbb{C}) \]

(ii) \( J_A(Z_t) = \frac{H^{m-1,m-1}(X_t, \mathbb{Z}) \otimes \mathbb{Z}^0,1(C, \mathbb{C})}{(H^{m-1,m-1}(X_t, \mathbb{Z}) \otimes H^1(C, \mathbb{Z}))} \)
is an abelian variety.

Proof. (i) is obvious.; (ii)—the induced Hermitian inner product on \( J_A \) from \( J \) has the required properties for \( J_A \) to be an abelian variety (verification left to reader).

There are a few preliminary remarks that we will need before proving the main results of this section, which will be stated now.

Let \( F^m_\mu \) be the horizontal subsheaf of \( F^m \) (with respect to the Gauss-Manin connection).

There is a short exact sequence:

\[
0 \longrightarrow \left( \frac{F^m_\mu}{F^m} \right) \otimes H^0,1(C) \longrightarrow F^m_\mu \longrightarrow F^m_\mu \otimes H^1(C) \longrightarrow 0
\]

Also the inclusion \( F^m_\mu \hookrightarrow F^m_\mu \) defines a homomorphism:
(4.7) \( f^*_h \rightarrow f^*_m \otimes H^1(C) \). We will use this homomorphism later in Chapter 4.

We consider the embeddings:

\[
\begin{align*}
&j_* R^2 m^{-2} f_* \mathbb{Z} \hookrightarrow f^{-m,1,*} \\
&f^{-m-1} \bigg/ f^{-m} \quad \hookrightarrow f^{-m,1,*}
\end{align*}
\]

In this context define the following sheaves over \( P^1 \):

(4.9) (i) \( L_f = j_* R^2 m^{-2} f_* \mathbb{Z} \cap (f^{-m-1} \big/ f^{-m}) \)

(ii) \( L_k = L_f \otimes H^1(C, \mathbb{Z}) \)

(iii) \( \mathcal{F} = L_f \otimes H^0,1(C, \mathcal{O}) \otimes \Theta \)

(iv) \( \mathcal{I} = \mathcal{F}/L_k \)

(4.10) Remarks. (i) Via the stalks over \( U \), the stalk of \( L_f \) over \( t \) is precisely equal to \( \{ v \in H^m,1(X_t, \mathbb{Z}) \mid v \text{ remains of type (m-1,m-1) under local horizontal displacement in } U \} \)

(ii) \( \mathcal{I} \) should be interpreted as the sheaf of normal functions taking their values in \( U P^1 J_A (Z_t) \).
(iii) For our Lefschetz pencil \( \{ X_t \}_{t \in \mathbb{P}^1} \),

\( L_t \cong H^{m-1,m-1}(X, \mathbb{Z}) \) is the constant sheaf if \( H^{m-1-r,m-1+r}(X_t)_v \neq 0 \)

for generic \( t \) and some \( r \geq 1 \). This follows from the fact that

\( \Pi_1(U) \) acts irreducibly on \( H^{2m-2}(X_t)_v \).

(iv) \( L_t \) is locally constant over \( U \).

There is a commutative diagram of short exact sequences:

\[
\begin{array}{cccccc}
0 & \rightarrow & L & \rightarrow & F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & j_* \mathbb{R}^{2m-1} & \rightarrow & \mathbb{F}^{m,*} & \rightarrow & 0
\end{array}
\]

yielding the following commutative diagram of long exact sequences:

\[
\begin{array}{cccccc}
\cdots & \rightarrow & H^0(P^1, I) & \rightarrow & H^1(P^1, L) & \rightarrow & H^1(P^1, F) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & i' & & \\
0 & \rightarrow & \ker \delta & \rightarrow & H^0(P^1, J) & \rightarrow & H^0(P^1, j_* \mathbb{R}^{2m-1} \mathbb{Z}) & \rightarrow & H^1(P^1, \mathbb{F}^{m,*}) & \rightarrow & \cdots
\end{array}
\]

Now suppose that some integral class \( v \) of type \((m,m)\) in

\[ E^1_{2m-1}(k) \cong H^1(P^1, j_* \mathbb{R}^{2m-1} \mathbb{Z}) \otimes \mathbb{C} \]

comes from \( H^1(P^1, L) \), and let \( \sigma \)

be any normal function in \( H^0(P^1, J) \) for which \( \delta \sigma = v \) (this is possible

\[ 2 \cdot H^{m-1,m+1}(X, \mathbb{Z}) \] is interpreted as the constant sheaf on \( P^1 \), and the

isomorphism is obtained via \( j_3^* \).
by (2.28). If we prove the injectivity of \( i' \), then a diagram chase will reveal the existence of \( \tilde{\sigma} \in H^0(P^1,\mathcal{J}) \) coming from \( H^0(P^1,\mathcal{I}) \), such that \( \tilde{\sigma} - \sigma \in \ker \delta \). It will later be proven that \( \tilde{\sigma} \) is invertible so that \( \delta \tilde{\sigma} = \nu \) is algebraic over \( \mathbb{Q} \). We now set out to verify the remaining details.

(4.13) **Lemma.** \( i' \) is injective.

**Proof.** The short exact sequence:

\[
0 \rightarrow \mathcal{F}^m \rightarrow \mathcal{F} \rightarrow \mathcal{F}^m_* \rightarrow 0,
\]

yields the long exact sequence:

\[
\cdots \rightarrow H^1(P^1,\mathcal{F}^m) \rightarrow H^1(P^1,\mathcal{F}) \rightarrow H^1(P^1,\mathcal{F}^m_*) \rightarrow \cdots
\]

The cup product induces natural isomorphisms: (via primitive decomposition)

\[(4.16) \quad (i) \quad \mathcal{F}^* \simeq \mathcal{F}^m_* \quad (ii) \quad \mathcal{F}^* \simeq \mathcal{F}^m_\ast\]

There is a composite of morphisms:

\[
\mathcal{F} \xrightarrow{\text{incl}} \mathcal{F}^m_* \xrightarrow{\text{incl}^*} \mathcal{F}^* \xrightarrow{\text{incl}^*} \mathcal{F}^m_* \simeq \mathcal{F},
\]

which implies that \( \mathcal{F} \) is split with direct summand \( \mathcal{F}^m_* \). Therefore \( \text{incl}^*: H^1(P^1,\mathcal{F}) \rightarrow H^1(P^1,\mathcal{F}^m_*) \).
is injective. Now there is a commutative diagram:

\[
\begin{array}{ccc}
& & H^1(P^1, F) \\
\text{incl}_* & \downarrow & \\
H^1(P^1, F) & \overset{i'}{\longrightarrow} & H^1(P^1, F^m, *)
\end{array}
\]

and since \( H^1(P^1, F) \cap H^1(P^1, F^m) = 0 \) in \( H^1(P^1, F) \), it is obvious by (4.15) that \( i' \) is injective.

We now introduce some further notation:

(4.18) Denote

(i) \( V_f = \text{cokernel sheaf making the following a short exact sequence: } 0 \to L_f \to j_* R^{2m-2} f_* \mathbb{Z} \to V_f \to 0 \)

(ii) \( V_k = \text{cokernel sheaf making the following a short exact sequence: } 0 \to L_k \to j_* R^{2m-1} k_* \mathbb{Z} \to V_k \to 0 \). Note that \( V_k \simeq V_f \otimes H^1(C, \mathbb{Z}) \).

(iii) \( L_f^K = L_f \otimes K, L_k^K = L_k \otimes K \), where \( K = \mathcal{O}, \text{or } C \)

(iv) \( V_f^K = V_f \otimes K, V_k^K = V_k \otimes K \)

We consider the sheaves \( L_f^K, L_k^K, V_f^K, V_k^K \) over \( U \) and state the following:
(4.19) **Theorem.** (i) \( L^0_f, L^0_k, V^0_f, V^0_k \) define underlying polarizable variations of Hodge structures over \( U \).

(ii) The short exact sequence diagrams:

(a) \[
\begin{array}{c}
0 \longrightarrow L^C_f \longrightarrow R^{2m-2} L^C_f \longrightarrow V^C_f \longrightarrow 0 \\
\downarrow \quad \downarrow U \\
0 \quad 0
\end{array}
\]

(b) \[
\begin{array}{c}
0 \longrightarrow L^C_k \longrightarrow R^{2m-1} L^C_k \longrightarrow V^C_k \longrightarrow 0 \\
\downarrow \quad \downarrow U \\
0 \quad 0
\end{array}
\]

are exact sequences of variations of Hodge structures over \( U \).

**Proof.** One proves this by going through the requirements for a variation of Hodge structure defined in [8], and this will be left to the reader. Note that the infinitesimal period relation is trivially satisfied for filtrations of \( L^C_f \otimes \theta_U, L^C_k \otimes \theta_U \), hence also for \( V^C_f \otimes \theta_U, V^C_k \otimes \theta_U \).

(4.20) **Corollary.** There are intrinsically defined Hodge structures
on $H^1(P^1, L_f^1)$, $H^1(P^1, L_K^1)$, $H^1(P^1, j_*^R R^{2m-2} f_*^* C)$, $H^1(P^1, j_*^R R^{2m-1} K^* C)$, $H^1(P^1, V^C_f)$, $H^1(P^1, V^C_K)$. Furthermore, the following long exact sequences are exact sequences of Hodge structures:

(a) $\cdots \rightarrow H^1(P^1, L_f^1) \rightarrow H^1(P^1, j_*^R R^{2m-1} K^* C) \rightarrow H^1(P^1, V^C_f) \rightarrow \cdots$

(b) $\cdots \rightarrow H^1(P^1, L_f^1) \rightarrow H^1(P^1, j_*^R R^{2m-2} f_*^* C) \rightarrow H^1(P^1, V^C_f) \rightarrow \cdots$

Proof. This is a direct consequence of (4.19) and the results in [31, §7].

We remark that from the morphism defined in (4.7) and the inclusion $j_*^R R^{2m-1} K^* \mathbb{C} \hookrightarrow \mathbb{C}^m$, we get a homomorphism:

(4.21) $\nu : j_*^R R^{2m-1} K^* \mathbb{C} \rightarrow \mathbb{C}^m, \otimes H^1(C)$.

It is trivial to verify the following:

(4.22) Proposition. (i) $\ker \nu = L_f^1$

hence (ii) $\text{Im } \nu = V^C_f$

We also have the following which will aid in computing a particular Hodge level of $H^1(P^1, V^C_K)$:

(4.23) Proposition. The Gauss-Manin connection defines a short exact sequence:

$$0 \rightarrow \mathbb{C}^{m, \ast} \rightarrow \mathbb{C}^{m+1, \ast} \rightarrow \Omega^1_{p^1} \otimes \mathbb{C}^{m+1, \ast} \rightarrow 0$$
Proof. This follows from a similar argument to the proof of (3.62), combined with (3.12) of [28].

(4.24) Corollary. (i) \( H^1(P^1, F_{m}^*) \rightarrow H^1(P^1, F_{m}^*) \) is injective, hence (ii) there is a commutative diagram:

\[
\begin{array}{c}
\text{injection} \\
\downarrow \\
H^1(P^1, F_{m}^*) \otimes H^1(C)
\end{array}
\]

Proof. (i) From (3.60) we have \( H^0(P^1, \Omega^1_{P^1} \otimes F_{m+1}^*) = 0 \), therefore (i) follows from (4.23).

(ii) is obvious from (i).

There is a commutative diagram: (Note: \( \overline{J}_{L}, \overline{J}_{V} \) are by definition cokernel sheaves)

\[\text{See (A.4) for more details.}\]
Remark. Since \( V_f, V_k \) define variations of Hodge structures of weights \( 2m-2, 2m-1 \) respectively, it follows from [31, §7] that

\[
H^1(P^1, V_f^C) \text{ and } H^1(P^1, V_k^C) \cong H^1(P^1, V_f^C) \otimes H^1(C, C) \text{ have Hodge structures of weights } 2m-1, 2m \text{ respectively, as is readily verifiable in our case of the Lefschetz pencil } \{X_t\}_{t \in P^1}.
\]

We now utilize some results from [31, §9].

(4.27) **Theorem.**

(i) \( H^1(P^1, \overline{F^p}) \subseteq F^p E^1_{2, m-2}(f) \) for all \( p \leq 2m-2, \)

(ii) \( F^{p+1, k, m-2}(f) = H^1(P^1, \overline{F^p}) \).

(iii) \( H^1(P^1, \overline{F^m}) = F^{m+1, k, m-2}(f) = F^m E^1_{2, m-2}(f) \).
where $E^{1,2m-2}_{2}(f) = H^{1}(P^{1}, j_{*}R^{2m-2}f_{*}\mathbb{C})$.

**Proof.** Using the notation of [31], (i) and (ii) follow from the fact that the complex $\varphi_{P,h} \rightarrow \varphi_{f} \rightarrow \varphi_{Q} \subset \Omega^{1}_{P}(\log \Sigma) \otimes \varphi_{f}$ is filtered quasi-isomorphic to the complex $\sigma^{*}_{(2)}$ (this is (9.1) in [31]). Since the above Hodge filtrations in (i) and (ii) come from $H^{1}(P^{1}, \mathcal{P}^{*}_{\Sigma})$ and $H^{1}(P^{1}, \mathcal{P}^{*}_{\Sigma})$, using the above quasi-isomorphism and computing the $E^{2}_{2}$ terms we see that (i) and (ii) are obvious. Now (iii) follows from a similar reasoning as above plus the following fact: The Hodge filtration on $\varphi_{P}^{C}(V^{C}_{f}) = \text{canonical extension of } \Omega^{1}_{\mathcal{C}} \otimes V^{C}_{f}$ over $U$ to $P^{1}$, denoted by $\varphi^{P}(V^{C}_{f})$, is the unique one induced from $\varphi^{P}_{f} \rightarrow \varphi^{P}(V^{C}_{f})$.

Since $\varphi^{m,*}_{f} = \varphi^{m,*}(V^{C}_{f})$, the result follows.

We need one more result to prove our next theorem in this section, namely:

(4.28) **Proposition.**

$$H^{1}(P^{1}, j_{*}R^{2m-2}f_{*}\mathbb{Z}) \cap H^{1}(P^{1}, \varphi^{m-1}_{f,h}) = H^{1}(P^{1}, L_{f}).$$

**Proof.** There is a short exact sequence:
where \( d(x,y) = x-y \) in \( \mathbb{F}_{f,h} \), for all \( x \in \mathbb{L}_{f} \mathbb{R}^{2m-2}_{f} \mathbb{Z} \), and \( y \in \mathbb{R}^{m-1} \). Now it is easy to see that \( \mathbb{L}_{f} \mathbb{R}^{2m-2}_{f} \mathbb{Z} + \mathbb{R}^{m-1} = T \), therefore modulo possible torsion, \( H^{1}(P, T) \rightarrow H^{1}(P, \mathbb{F}_{f,h}) \) is injective. The intersection defined in (4.28) is actually an intersection in \( H^{1}(P, \mathbb{F}_{f,h}) \), and taking cohomology we get:

\[
H^{1}(P, L_{f}) \rightarrow H^{1}(P, \mathbb{L}_{f} \mathbb{R}^{2m-2}_{f} \mathbb{Z}) \oplus H^{1}(P, \mathbb{R}^{m-1}) \xrightarrow{d_{*}} H^{1}(P, \mathbb{F}_{f,h}) .
\]

Now it is easy to check that the \( \ker d_{*} \) is the LHS of (4.28). One checks that this is precisely the statement of (4.28).

We now make the following remark:

Remark. \( \mathbb{F}_{f,h}^{P} E_{2}^{1,2m-2}(f) / H^{1}(P, \mathbb{F}_{f,h}^{P}) = E_{2}(p) \), where \( E_{2}(p) \) is the second \( E_{2} \) term of the spectral sequence mentioned in the proof of (4.27). Moreover, one checks (using the results in [31, §9]) that

\[
E_{2}(p) = H^{0}(P, (j_{*}(\Omega^{1}_{U} \otimes \mathbb{F}_{f}^{P-1})) \cap \mathbb{V} \mathbb{F}_{f}^{P}) / \mathbb{V} \mathbb{F}_{f}^{P} ,
\]

involving the infinitesimal behaviour of \( \mathbb{F}_{f}^{P} \). We also remark that in our case of the Lefschetz pencil \( \{ X_{t} \}_{t \in \mathbb{P}^{1}} \), either \( H^{1}(P, \mathbb{F}_{f,h}^{P}) = 0 \) or \( E_{2}(p) = 0 \) (for \( p=m-1 \), see the proof in (A.5)). The general case is similar).

It is easy to prove the following:
Theorem. (i) The integral classes of type \((m,m)\) in 
\[ H^1(P^1, F^{m-1}) \otimes H^1(C) \]
coincide with those of type \((m,m)\) in 
\[ H^1(P^1, L_k), \]
moreover, if \(E_2(m-1) = 0\) then (ii) the integral classes
of type \((m,m)\) in \(E_2^{1,2m-1}(k)\) coincide with those of type \((m,m)\) in 
\[ H^1(P^1, L_k). \]

Proof. (i) Follows from (4.28); (ii) follows from (i) and the above
discussion.

Remark. We can now complete our story concerning the Hodge structures
defined in (4.27) and the diagram (4.26). If we look at the cohomology
of (4.26) and apply our known results to it, we obtain the following
summarizing diagram:

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & E_{m,m}^m(k, \mathbb{Z}) \cap \{H^1(P^1, F^{m-1}) \to H^1(P^1, L_k) \to E_{m-1,m}^m(f) \otimes H^0,1(C) \\ \otimes H^1(C) \}
& E_2(k, \mathbb{Z}) \to F_{m+1,*}E_2^{1,2m-1}(k) \\
\downarrow & \downarrow & \downarrow \\
0 & E_2(k, \mathbb{Z}) \cap \{E_{m-1,m}^m(V_f^c) \to H^1(P^1, V_k) \to (F_{m+1,*}E_2^{1,2m-2}(\bar{k})) \}
& \otimes E_{m-1,m}^m(V_f^c) \otimes H^1(C) \\
& \downarrow & \downarrow & \downarrow \\
& 0 & 0 & 0
\end{array}
\]
where 

(i) $E^{m, m}(k, \mathbb{Z}) = \text{integral classes of type } (m, m) \text{ in } E_2^{1, 2m-1}(k)$

(ii) $E_2(k, \mathbb{Z}) = \text{integral classes in } E_2^{1, 2m-1}(k)$

(iii) $E^{p, q}(\bar{\mathcal{E}}) = \text{Hodge } (p, q) \text{ level of } E_2^{1, 2m-2}(\bar{\mathcal{E}})$

(iv) $E^{p, q}(\nu^C) = \text{Hodge } (p, q) \text{ level of } \mathcal{H}^{1}(p^{1}, \nu^C)$

As a corollary to the results in this section so far, we obtain the following:

(4.32) **Corollary.** Let $\sigma \in H^0(p^1, \mathcal{J})$ and assume $\delta \sigma = 0$ in $H^1(p^1, \mathcal{V}_k)$. Then $\sigma$ decomposes into $\sigma = \sigma_o + \sigma_1$, where $\delta \sigma_1 = 0$ and $\sigma_0(t) \in J_A(Z_t)$ for all $t$.

(4.33) One of the tasks involved in inverting normal functions is to prove that the Abel-Jacobi mapping is meromorphic. Before doing so, we make a base change so as to trivialize the Picard-Lefschetz transformations. We first check things out locally. Let $\Delta_1, \Delta$ be small open disks with $\Sigma \cap \Delta = 0$, and let $s = t^2: \Delta_1 \to \Delta$ be onto.

There is a pullback diagram:
(4.34) $\overline{X}_\Delta \times \Delta_1 = \overline{X}_{\Delta_1} \longrightarrow \overline{X}_\Delta = \overline{X}$ restricted to $\Delta$.

with respective monodromy transformations $T_1 = I$ and $T$ with $T^2 = I$. Note that $\overline{X}_{\Delta_1}$ is singular, otherwise the local cycle invariant property would fail to hold (also note that $s = t^2$ is not smooth at 0).

To globalize (4.34) consider the hyperelliptic Riemann surface $M$ obtained from the affine plane curve in $\mathbb{C}^2$ defined by:

(4.35) $\{(w,z) \in \mathbb{C}^2 \mid w^2 - \Pi(z - x) = 0\}$, by first completing and then desingularizing. We obtain the corresponding global diagram for $\overline{Z}$:

(4.36) $M \times \mathbb{P}^1 \overline{Z} \overleftarrow{\overline{Z}_M} \overrightarrow{\overline{Z}} \overrightarrow{\overline{Z}}$.

By a desingularization process, we may assume that $\overline{Z}_M$ is smooth.

There is a corresponding diagram to (1.5):
Note that all the local monodromy transformations associated to (4.37) over $U_M$ are all trivial. In a rather obvious fashion the family of intermediate jacobians $U J(Z_t)$ extends over $M$ with compact fibers $U_M$ over the singular set $E_M$.

Let $W$ be the family of relative codimension $m$ algebraic cycles in $\bar{Z}$ over $\mathbb{P}^1$, which are homologous to zero fiberwise. Then $W$ pulls back to $W_M = W \times_{\mathbb{P}^1} M$ over $M$. A normal function $\sigma \in H^0(\mathbb{P}^1, J)$ will pull back to $\pi^* \sigma : M \rightarrow UJ(Z_t)$. If the cohomology class of $\pi^* \sigma$, $\delta_M(\pi^* \sigma) = q^* \text{algebraic cocycle}$ (for $q \in \mathbb{Q}$), then

$$\delta \sigma = \left(\frac{q}{2}\right) \cdot \pi^* \sigma$$

(algebraic cocycle). In view of the results in [30], we need only check that the Abel-Jacobi morphism defined over $U_M$ extends to a meromorphic map $\phi_M : W_M \rightarrow UJ(Z_t)$ over $M$.

(4.38) Now we will show that in fact $\phi_M$ extends continuously across the fibers of $W_M$ over $E_M$, therefore we can actually deduce that $\phi_M$
is analytic. Assume that $\phi_M$ is meromorphic and let $\sigma \in H^0(\mathbb{P}^1, \mathcal{J})$ have the property that $\sigma(t) \in$ invertible part of $J(Z_t)$ for generic $t \in \mathbb{P}^1$. There is a diagram:

\begin{equation}
\begin{array}{ccc}
\Gamma & \xrightarrow{\ell} & Y \\
\downarrow & & \downarrow \\
W_M & \xrightarrow{\phi_M} & U_M^*J(Z_t) \\
\downarrow & \xrightarrow{Pr} & \downarrow \\
M & & M
\end{array}
\end{equation}

where $\Gamma = \text{graph of } \phi_M$ and $Y$ is some projective variety dominating $\Gamma$ (it exists since $\Gamma$ is a Moishezon space [11]). Now $\pi^*\sigma(M)$ is a curve in $U_M^*J(Z_t)$ which pulls back to $\ell^{-1}\pi^*\sigma(M)$ in $Y$. Choose a curve $D \subset \ell^{-1}\pi^*\sigma(M)$ which maps onto $M$ via $\text{Pr}^o\ell$. $D$ also maps into $W_M$ and determines an algebraic cycle $D_M$ on $\overline{Z}_M$. It is easy to check that $(1/2\deg(\text{Pr}^o\ell|_D))\overline{n}_*[D_M]$ is the algebraic cocycle corresponding to $\sigma$.

(4.40) We now verify (4.38): Since we are dealing with a family over $M$ with trivial Picard-Lefschetz transformations, the extended Leray cohomology sheaf $\overline{F}_M$ over $M$ is described precisely as

$$\overline{F}_M = \theta_M \otimes j_M^*R^{2m-1}k_M^*\mathcal{C} = \{j_M^*R^{2m-1}k_M^*\mathcal{C}\} \cap \overline{Z}_M^{\mathcal{O}_M/U_M} \cap \overline{F}_M.$$
Since by definition $\bar{\mathcal{F}}_M = \bar{\mathcal{J}}_M \ast \mathcal{F}^P \cap \bar{\mathcal{F}}_M$, we must have

$\bar{\mathcal{F}}_M = \{ \bar{\mathcal{J}}_M \ast R^{2m-1}_{k_M} \ast \mathcal{F}^P \Omega^* \} \cap \bar{\mathcal{F}}_M.$

There is the local cycle invariant property:

$\bar{\mathcal{F}}_M = \{ \bar{\mathcal{J}}_M \ast R^{2m-1}_{k_M} \ast \mathcal{F}^P \Omega^* \} \cap \bar{\mathcal{F}}_M.$

(4.42) The restriction homomorphism $\text{res}: R^{2m-1}_{k_M} \ast \mathcal{C} \longrightarrow \bar{\mathcal{J}}_M \ast R^{2m-1}_{k_M} \ast \mathcal{C}$
is an epimorphism.

Therefore we have $\theta \otimes R^{2m-1}_{k_M} \ast \mathcal{C} \longrightarrow \bar{\mathcal{F}}_M \leftarrow \bar{\mathcal{F}}^m_M$. Now localize over

a disk $\Delta \subset M$ with $\Delta \cap \Sigma_M = 0 \in \Delta$. By the above discussion, a local

section $v \in H^0(\Delta, \bar{\mathcal{F}}^m_M)$ can be represented as a $C^\infty$ form over

$k^m_M(\Delta)$, hence it is obvious that the Abel-Jacobi map extends across the

singular fibers of $\mathcal{W}_M$ over $\Sigma_M$.

Summary. Looking over the above discussion, we are able to deduce the following:

(4.43) Theorem: Given $Z = XX \mathcal{C}$ where (i) $X$ is smooth, projective

dimension $2m-1$, and (ii) $\mathcal{C}$ is a smooth curve.

Assume there exists a Lefschetz pencil $\{ X_t \}_{t \in \mathbb{P}^1}$ of hyperplane

sections of $X$ such that the Hodge $(m-1, m-1)$ conjecture holds for the

generic $X_t$. Then if a normal function $v : \mathbb{P}^1 \longrightarrow U_{\mathbb{P}^1} J(Z_t)$
satisfies the property that $\delta v = 0$ in $H^1(P^1, V)$, then up to a rational scalar, $\delta v$ is the fundamental class of an algebraic cycle in $\mathbb{Z}$.

(4.44) We remark that in view of (4.30) part(ii), the above theorem has its greatest strength when $E_2(m-1) = 0$, as mentioned in case 1 of the introduction to this chapter. In case 3, the above theorem becomes trivial and consequently all the primitive $2m$ algebraic cocycles of such $Z = X \times \mathbb{C}$ must lie in $H^{m,m}(X,\mathbb{Q}) \otimes H^0(C,\mathbb{Q}) \oplus H^{m-1,m-1}(X,\mathbb{Q}) \otimes H^2(C,\mathbb{Q})$ (cf (1.9)).

We now put together our facts on the normal functions $H^0(P^1, J)$.

Now $J'$ was defined in chapter 2 to be the image of $F^m \otimes H^1(C)$ in $J'$, provided we have the splitting $\frac{F^m}{F^m} = \frac{F^{m-1}}{F^{m-1}} \oplus \frac{F^m}{F^m}$, and we have $J_L$ defined in (4.26), $J_L = \text{image of } (\frac{F^{m-1}}{F^m}) \otimes H^0,^1(C)$ in $J$.

Our summary on the normal functions is given by the following theorem:

(4.45) **Theorem.** (i) All the normal functions are horizontal.

(ii) $H^0(P^1, J') \subseteq J^{2m-1}(Z)$ (we are assuming for simplicity that $\text{ker } \delta = \text{ker } \delta'$, see the discussion following (2.18)).

(iii) If $\sigma \in H^0(P^1, J_L)$ and the $(m-1,m-1)$ conjecture holds for the generic $X_L$, then $\delta \sigma$ is a rational multiple of an algebraic cocycle.

(iv) $\text{ker } \delta = \left\{ H^0(P^1, \frac{F^{m-1}}{F^m}) \otimes H^0,^1(C) / F, v \right\} \otimes H^1(C, \mathbb{Z}) \oplus J^{2m-1}(Z)$. 

Moreover, if $H^1(P^1, L_k) = 0$, e.g. $L_k$ is the constant sheaf over $P^1$, such as in cases 2 and 3 of the introduction to this chapter, then:

$$(V) \quad H^0(P^1, \mathcal{J}_L) = H^0(P^1, \mathcal{F}_{\mathcal{F}^m} / \mathcal{F}^m) \otimes H^0,1(C) / H^0(P^1, L_k).$$

(4.46) Recall the situation of a generic hypersurface $X \subset \mathbb{P}^{2m}$ of degree $\geq 2 + 3/(m-1)$, $(m \geq 3)$, where we have $\text{Prim}_{m,m}(Z, \mathcal{Q}) \otimes H^{m-1,m-1}(\mathbb{D} \times \mathbb{C}, \mathcal{Q}) = 0$. Then the following is true:

(4.47) Corollary. Given $Z = X \times \mathbb{C}$ where $X$ is defined in (4.46). Then

$H^0(P^1, \mathcal{J}) = \ker \delta = H^0(P^1, \mathcal{F}_{\mathcal{F}^m} / \mathcal{F}^m) \otimes H^0,1(C) / H^0(L_k, \mathcal{J}) \otimes J^{2m-1}(Z).$

One could make the following guess:

(4.48) Conjecture. (4.47) holds for all (smooth) hypersurfaces $X \subset \mathbb{P}^{2m}$ $(m \geq 3)$ of degree $\geq 2 + 3/(m-1)$, i.e., all the normal functions $\sigma$ decompose into $\sigma = \sigma_L + \sigma'$ where $\sigma_L \in H^0(P^1, \mathcal{J}_L)$ and $\sigma' \in J^{2m-1}(Z)$. 
More specifically we can conjecture the following:

\begin{equation}
(4.49) \text{Prim}^{m,m}(Z,\mathbb{Q}) = 0 \text{ for all hypersurfaces } X \subset \mathbb{P}^{2m} \text{ of degree } \geq 2 + 3/(m-1).
\end{equation}

Now (4.49) would imply the Hodge conjecture trivially for such products $Z = X \times C$, and hence for all Fermat fourfolds, since Fermat varieties are rationally dominated by varieties of the form $Z = X \times C$ ([25]). We remark that given a Fermat variety $Z' \subset \mathbb{P}^{2m+1}$, there exists a dominating morphism $\xrightarrow{\alpha} Z'$, where $X \subset \mathbb{P}^m$ is a Fermat variety, $C$ is a Fermat curve, and $E$ is a Fermat subvariety of $Z = X \times C$. If one can prove in this case that \text{Prim}^{m,m}(Z,\mathbb{Q}) = 0, then it is easy to see that all the cocycles in $H^{m,m}(Z',\mathbb{Q})$ come from blowups of cycles from $E, X \times C$ where $X_S$ is a Fermat subvariety of $X$. This adds weight to the following conjecture (see [24, §4]).

(4.50) Conjecture. Let $Z''$ be a Fermat variety of dimension $2m$. Then $H^{m,m}(Z'',\mathbb{Q})$ is generated by algebraic cocycles arising from Fermat surfaces $E$ and products of Fermat curves $C \times C$.

---

$B_E(X \times C)$ is defined to be the blow up of $X \times C$ along $E$. 

Bibliography


Appendix

(A.1) The Abel-Jacobi morphism. We give a brief outline of the construction of the Abel-Jacobi morphism, and refer the reader to [8, p. 165] for further details.

We let (i) $Y$ be a smooth, projective variety of dimension $2m-1$
(ii) $W$ be the group of codimension $m$ algebraic cycles on $Y$ (modulo rational equivalence) which are homologous to zero
(iii) $\{w_1, \ldots, w_n\}$ be a basis of $\tilde{F}_{H}^{m,2m-1}(Y, \mathbb{C})$.

The Abel-Jacobi morphism is a map $\phi : W \to J(Y)$ defined in the following way:

Let $D \in W$. Then by definition of $W$, there exists a real dimensional $2m-1$ cycle $\gamma$ with $H\gamma = D$. We define $\phi(D) = \left( \int_{\gamma} w_1, \ldots, \int_{\gamma} w_n \right)$ as a class in $J(Y)$. One checks that $\phi$ is well defined and is indeed a morphism from the Hilbert scheme $W$ to $J(Y)$.

(A.2) Some remarks on the sheaves $\{F^p\}_{p \geq 0}$. Associated to the diagram (1.5) in chapter 1 is the period mapping (defined in [8, p. 156]) which associates to every point $t \in U$ the corresponding Hodge filtration $\{F_{p}^{H}Z_{t}, C\}_{p \geq 0}$ as a point in a classifying space $E$ of Hodge structures (modulo the action of a given monodromy group on $E$). Now by the construction of $E$, there is a natural embedding of $E$ in a product of Grassmannians, so that the bundles $\{F^p\}_{p \geq 0}$ are pullbacks of universal bundles by certain maps (which are in fact algebraic). These maps extend to $\mathbb{P}^1$ (since $\mathbb{P}^1$ is a curve), giving the bundles $\{F^p\}_{p \geq 0}$.

Now the sheaf $j_*F$ over $\mathbb{P}^1$ can be described as the sheaf of
holomorphic sections of $F$ over $U$ with arbitrary growth near $\Sigma$. One can easily check from our sheaves $\{F_p\}_{p\geq 0}$ defined above that $F = j_* F \cap F$ (as an intersection in $j_* F$), and from this it is easy to deduce part (ii) in (2.16).

If we localize the sheaves over a disk $\Delta \subset \mathbb{P}^1$ with $\Delta \cap \Sigma = 0 \in \Sigma$, and assume the notation on page 34, then from prop. 5.2 in [3,p. 91] ((i) and (ii) of prop. 5.2 are equivalent in our case since we are extending a Hodge bundle which is self dual under $\cup$) the canonical extension $s_\ast F$ is completely characterized by the growth of sections about 0 with respect to any given horizontal multivalued basis (coefficients grow like powers of $\log |t|$). Consequently we obtain part (i) of (2.16) from (2.15).

(A.3) We wish to verify (3.58) which says that $H^0(\mathbb{P}^1, J') \subset J^{2m-1}(Z)$.

Let $\sigma \in H^0(\mathbb{P}^1, J')$. Then $\sigma$ is equivalent to the following data: $\{a, U\}_{a \in I}$ where (i) $a \in H^0(U, F^m, *) \otimes H^1(C)$

(ii) $U_{a} \cap U_{b} \subset U$

(iii) $\sigma_{a} - \sigma_{b} \in H^0(U_{a} \cap U_{b}, R^{2m-2} F_{*} \mathbb{Z} \cap F^m, *)$

$\otimes H^1(C, \mathbb{Z})$

where $R^{2m-2} F_{*} \mathbb{Z}$ and $F^m, *$ are viewed as subsheaves of $F^{m-1, *}$.

Now such $\sigma$ give rise to integral classes $v \in F^{m} H^{2m-2}(X_t) \otimes F^m, H^{2m-2}(X_t)$ which remain in this Hodge level under local horizontal displacement around $t$ in $U$. It is easy to check that such a $v$ will remain in the above Hodge level under horizontal displacement in $U$. We use the irreducibility of the $\pi_1(U)$ action on $H^{2m-2}(X_t, \mathbb{Q})$ to conclude that there are two cases to consider:

case (i) $\sigma_{a} - \sigma_{b} \not\in H^{2m-1}(Z)$ for some $a, b \in I$. Then
\[ H^{2m-2}(X_t)_v \subset F^{m-2}(X_t) \otimes F^m H^{2m-2}(X_t) \] and one checks that (3.58) is obvious from (3.57).

\[ \text{case (ii) } \sigma_\alpha - \sigma_\beta \in i^*_3 H^{2m-1}(Z) \text{ for all } \alpha, \beta \in I. \] Then (3.58) follows from a simple argument using (3.63).

\[ (A.4) \text{ Some remarks on (4.23). } \] To prove (4.23), we argue in a similar fashion as (3.61) that \( \bar{\Omega} \setminus \{ \Omega \cap \Omega^{-1} \} \) and then deduce similarly that \( \bar{\Omega} \setminus \{ \Omega \cap \Omega^{-1} \} \). The surjectivity of \( \bar{\nabla} \) in (4.23) follows from the fact that \( \Omega^{1} \otimes \bar{\nabla} \bar{\nabla} \) (analogous argument to [28,(3.12)]).

\[ (A.5) \text{ We prove that if } E_{2}^{1,2m-2}(\bar{\nabla}) \neq 0, \text{ then } E_{2}(m-1) = 0 \iff \]
\[ F^{m-2}(X_t)_v = 0 \] (Recall the definition of \( E_{2}^{1,2m-2}(\bar{\nabla}) \) in (4.27)).

**Proof.** If \( F^{m-2}(X_t)_v = 0 \), then \( j_*(\Omega^{1} \otimes \bar{\nabla} \bar{\nabla}) \cap \bar{\nabla} \bar{\nabla} \bar{\nabla} = 0 \), so that \( E_{2}(m-1) = 0 \) (we don't need the added hypothesis \( E_{2}^{1,2m-2}(\bar{\nabla}) \neq 0 \) in this case); conversely, suppose that \( E_{2}(m-1) = 0 \). Then \( F^{m-2}(X_t)_v = 0 \) implies that \( H^{1}(P^1, \bar{\nabla}) \). I claim that if \( F^{m-2}(X_t)_v \neq 0 \), then \( \bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{\nabla} = i^*_3 \bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{\nabla} \) is the constant sheaf over \( P^1 \).

To see this, we simply note that a local section \( \sigma \) of \( \bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{\nabla} \) always extends to a global multivalued section \( \sigma \) of \( \bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{\nabla} \). Now the hypothesis \( F^{m-2}(X_t)_v \neq 0 \) implies that for generic \( t \in U \), the vanishing cocycles \( \{ \delta_1, \ldots, \delta_N \} \) of \( H^{2m-2}(X_t)_v \) are not of type \( (m-1,m-1) \). Now if we consider the horizontal section \( \sigma \) near a singular point in \( X \)
corresponding to \( \delta_1 \) say (where we may assume WLOG that \( \sigma \cup \delta_1 \neq 0 \)),
then for \( \sigma \) to be a multivalued section of \( \mathcal{F}^{m-1} \), it must induce classes in \( \mathcal{F}^{m-1}H^{2m-2}(X_t) \) for all \( t \in U \), and hence by the Picard-Lefschetz formula, \( \sigma + \left( (-1)^{m-1} \sigma \cup \delta_1 \right) \) must also induce classes in \( \mathcal{F}^{m-1}H^{2m-2}(X_t) \) (all \( t \in U \)). Since \( \delta_1 \) is integral, it must be of Hodge type \((m-1,m-1)\) over \(U\), and hence so are the \{\delta_1, \ldots, \delta_N\}, contradicting the above statement: for generic \( t \in U \), \{\delta_1, \ldots, \delta_N\} are not of type \((m-1,m-1)\).

This justifies the above claim.

Now using the claim, it is easy to see that given \( E_2(m-1) = 0 \),
then either \( F^mH^{2m-2}(X_t)_V = 0 \) or else \( F^mH^{2m-2}(X_t)_V \neq 0 \) and \( F^{m-1}E_2^{1,2m-2} \) is surjective. It follows from (3.54) that it suffices to prove the following:

(A.6) Referring to the construction of \( \overline{X}_{\Delta_1} \) in (4.34), it should be noted that the singularity of \( \overline{X}_{\Delta_1} \) is precisely a single ordinary double point (verification left to the reader).

(A.7) A remark on the proof of (4.13):

From the s.e.s. \( 0 \rightarrow F^m \rightarrow \overline{F} \rightarrow F^{m,*} \rightarrow 0 \), we obtain the L.E.S.:

\[
H^0(p^1, \overline{F}) \rightarrow H^0(p^1, F^{m,*}) \rightarrow H^1(p^1, \overline{F}) \rightarrow H^1(p^1, F) \rightarrow H^1(p^1, \overline{F}^{m,*})
\]

Now I claim that \( H^0(p^1, \overline{F}) \rightarrow H^0(p^1, F^{m,*}) \) is surjective. It follows from (3.54) that it suffices to prove the following:
\[ H^0(P^1, \overline{F}^{m-1}) \rightarrow H^0(P^1, \overline{F}^{m-1} / \overline{F}^m) \text{ is surjective.} \]

From the s.e.s. \[ 0 \rightarrow \overline{F}^m \rightarrow \overline{F}^{m-1} \rightarrow \overline{F}^{m-1} / \overline{F}^m \rightarrow 0 \]

plus (3.60) and Serre duality, it is clear that the above claim is justified.

Using the above claim, it is not hard to check that there exists an isomorphism: \[ H^1(P^1, \overline{F}) \cong H^1(P^1, \overline{F}^m) \oplus H^1(P^1, \overline{F}^m, *) \]

such that the following diagram commutes:

\[
\begin{array}{ccc}
H^1(P^1, \overline{F}) & \xrightarrow{=} & H^1(P^1, \overline{F}^m) \oplus H^1(P^1, \overline{F}^m, *) \\
\uparrow \text{incl}^* & & \uparrow \text{proj} \\
H^1(P^1, \overline{F}) & \xrightarrow{i'} & H^1(P^1, \overline{F}^m, *) \\
\end{array}
\]

Now we should also remark that utilizing the properties of our Lefschetz pencil \( \{ X_t \}_{t \in \mathbb{P}^1} \), one can give a case by case argument ((4.10) part (iii)) to verify the injectivity of \( i' \).