DERIVED CATEGORIES AND FUNCTORS

by

Donald Doo Fuey Loo

B.Sc., University of British Columbia, 1970

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics

We accept this thesis as conforming to the
required standard

THE UNIVERSITY OF BRITISH COLUMBIA

September, 1971

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

Date Sept. 30, 1971

ABSTRACT

For each abelian category A, there is a category D(A), called the derived category of A, whose objects are complexes of objects of A, and whose morphisms are formal fractions of homotopy classes of complex morphisms having as denominators homotopy classes inducing isomorphisms in cohomology.

If $F : A \longrightarrow B$ is an additive functor between abelian categories, then under suitable conditions on A, there is a functor $RF : D(A) \longrightarrow D(B)$ with the property that if objects X of A are considered as complexes concentrated at degree 0, then there are isomorphisms $H^n RF(X) \simeq R^n F(X)$ for all n, where $R^n F$ is the ordinary n^{th} right derived functor of F. RF is called the derived functor of F, and one may look upon it as a kind of extension of F.

CONTENTS

Acknowledgements

I would like to thank Dr. K. Hoechsmann for his guidance and assistance during the writing of this thesis.

In addition, I am grateful to the National Research Council for their financial support.

CHAPTER I

INTRODUCTION

Let F: $\underline{A} \longrightarrow \underline{B}$ be an additive functor between abelian categories where $\underline{A}$ has enough injectives. The "right derived functor" of F is an exact δ functor or an exact connected sequence $\{R^nF, \delta^n: n \geq 0\}$ of functors from $\underline{A}$ to $\underline{B}$ together with a natural map $\eta: F \longrightarrow R^0F$.

Its definition is as follows:

For X $\in$ Object $\underline{A}$, take an injective resolution

$$I^*: 0 \longrightarrow X \xrightarrow{i_X} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I \longrightarrow \cdots$$

and set $R^nF(X) = H^nF(I^*)$.

$\eta : F \longrightarrow R^0F$ is induced by the universality of Ker $d^0 = R^0F$.

$$\begin{array}{ccccccc} 0 \longrightarrow & F(X) & \xrightarrow{F(i_X)} & F(I^0) & \xrightarrow{F(d^0)} & F(I^1) & \longrightarrow \\ & & \searrow & \nearrow & & & \\ & & & \text{Ker } F(d^0) = R^0F(X) & & & \end{array}$$

Given a map $X \xrightarrow{f} Y$ the nice mapping properties of injectives yield a morphism $\alpha: I^* \longrightarrow J^*$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\ & & \downarrow f & & \downarrow \alpha^0 & & \downarrow \alpha^1 & & \\ 0 & \longrightarrow & Y & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \end{array}$$

of resolutions, which is unique up to homotopy of complexes. Since the cohomology functors H^n do not distinguish homotopic morphisms, this produces a well-defined map

$$R^nF(f) : R^nF(X) \longrightarrow R^nF(Y) \quad .$$

The connecting morphisms δ^n are defined via the " Snake Lemma " from the fact that every short exact sequence in $\underline{A}$ can be injected into a short exact sequence

$$0 \longrightarrow I^* \longrightarrow J^* \longrightarrow K^* \longrightarrow 0$$

of injective resolutions and the latter stays exact after an application of F .

If F is left exact, η is an isomorphism and $\{ R^nF, \delta^n \}$ coincides with the sequence of "right satellites" of F -- i.e. an exact connected sequence of functors whose 0^{th} term is isomorphic to F.

The disadvantage of the theory outlined above is its heavy reliance on injectives. But in case F is left exact, R^nF can be computed from any F-acyclic resolution ($X \in Ob\ \underline{A}$ is F-acyclic if $R^nF(X) = 0$ for $n \geq 1$) for in this situation, if

$$X^* : \ 0 \longrightarrow A \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$$

is an F-acyclic resolution of A, then

$$R^nF(A) \approx H^nF(X^*)$$

and even if injectives are not available, the right satellites of F can be defined in terms of F-acyclic resolutions if there is a functorial way of assigning such resolutions to objects of $\underline{A}$ and F is exact on these resolutions. This is usually done in group-cohomology with coinduced modules and in ordinary sheaf-cohomology with flabby sheaves.

In all cases the following pattern emerges:
We have a class $\underline{I}$ of objects of $\underline{A}$ with the following properties:

(AC1) Each object of $\underline{A}$ admits a monomorphism into an object of $\underline{I}$.

(AC2) $\underline{I}$ is closed under finite direct sums and cokernels of monomorphisms.

(AC3) Short exact sequences of objects in $\underline{I}$ remain exact after an application of F.

$\underline{I}$ is called a class of F-acyclic objects in $\underline{A}$. It is not unique in general but there is always a maximal one.

In order to define something like a "right derived functor" we seem to need a functorial way of starting with F-acyclic resolutions. One of the points to be made in the sequel is that it is not necessary. The existence of any class $\underline{I}$ with properties AC1 - AC3 will enable us to construct a derived functor.

Outline:

Let a quiso (quasi-isomorphism), denoted by a double arrow, be a morphism $X^* \Longrightarrow Y^*$ of co-chain complexes inducing isomorphisms in cohomology. Thus, in cohomology, a quiso has an inverse.

Starting with the category K(A̲) of co-chain complexes and homotopy classes of co-chain maps of A̲, we form a category D(A̲), whose objects are objects of A̲ but whose morphisms, called "quasi-morphisms" and denoted by broken arrows X $\dashrightarrow$ Y, are formal fractions of morphisms of K(A̲) with quisos as denominators. Every quasi-morphism from X to Y induces a well-defined map in cohomology.

An F-acyclic resolution can be viewed as a quiso of complexes.

$$\begin{array}{ccccccccccccc}
A & : & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\Downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
I^* & : & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow \cdots
\end{array}$$

Given a map $X \xrightarrow{f} Y$ and resolutions $X \Longrightarrow I^*$, $Y \Longrightarrow J^*$, we obtain the diagram

$$\begin{array}{ccc}
X & \Longrightarrow & I^* \\
{\scriptstyle f}\downarrow & & \\
Y & \Longrightarrow & J^*
\end{array}$$

which establishes a quasi-morphism $I^* \dashrightarrow J^*$ from I^* to J^*. As a consequence of the exactness of F on I̲, we know that F turns quasi-morphisms between I̲ - complexes into quasi-morphisms between their images. Taking the map induced in cohomology by the quasi-morphism $F(I^*) \dashrightarrow F(J^*)$, we have:

$$H^nF(I^*) \longrightarrow H^nF(J^*)$$

Procuring the long exact sequence from this kind of "derived" functor is a more subtle task, but it too, can be done.

We note that this procedure of defining the "R^nF's" make hardly any distinction between objects X of $\underline{A}$ and complexes X^* of such objects. In general we will work on the level of the complexes.

Evidently any complex X^* of objects of $\underline{A}$ admits a quiso $X^* \Longrightarrow I^*$ into a complex I^* of objects of $\underline{I}$, and any morphism $f : X^* \longrightarrow Y^*$ induces a quasi-morphism of their respective "F-acyclic" resolutions

$$\begin{array}{ccc} X^* & \Longrightarrow & I^* \\ f \downarrow & & \\ Y^* & \Longrightarrow & J^* \end{array}$$

The functor $\underline{R}F : D(\underline{A}) \longrightarrow D(\underline{B})$, is defined by setting $\underline{R}F(X^*) = F(I^*)$.

The $\underline{R}^nF$ can then be recovered from $\underline{R}F$ by taking n^{th} cohomology. Thus $\underline{R}F$ will be referred to as the "derived" functor of F and we may look upon it as a kind of extension of F from $K(\underline{A})$ to $D(\underline{A})$. This policy of staying on the level of the complexes has the advantage of circumventing the usual spectral sequences in the study of composite functors. Indeed, given $\underline{A} \xrightarrow{F} \underline{B} \xrightarrow{G} \underline{C}$, we obtain a natural map $\gamma : \underline{R}(G \cdot F) \longrightarrow \underline{R}G \cdot \underline{R}F$ which is an isomorphism under favourable conditions.

Dually, what has been said about right derived functors applies equally well to left derived functors after reversing the appropriate arrows, interchanging the terms "left" and "right", "projective" and "injective", etc.

In conclusion we sum up some of the strengths and weaknesses of this approach:

(1) We can define derived functors whenever we have a class I of F-acyclic objects as described above; appropriately dualized for left-derived functors. An example of a non-trivial application is the problem of defining TOR for sheaves where we have plenty of flats but hardly any projectives.

(2) Derived functors for complexes -- hypercohomology, are handled just as easily as those for objects.

(3) The spectral sequence relating to the derived functors of a composite is supplanted by a simple composition of derived functors. However, for the more delicate information extractable from a spectral sequence, it seems that this version will usually be too crude, though the usual spectral sequences can always be set up.

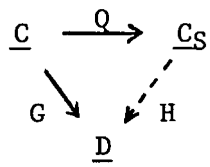
CHAPTER II

CALCULAS OF LEFT FRACTIONS

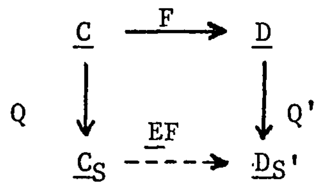
§1. Definitions of Localisations and Extensions

Let S be a class of morphisms in a category $\underline{C}$. We would like to "invert" the elements of S. In other words, find the "smallest" category such that every morphism of S is invertible. The familiar situation of a ring localised at a multiplicative system gives us a clue as how to define such a category.

Definition. The localisation of $\underline{C}$ with respect to S should be a pair $(\underline{C}_S, Q)$ where $Q : \underline{C} \longrightarrow \underline{C}_S$ is a functor with $Q(s)$ being an isomorphism for all $s \in S$ and is universal with respect to this property, i.e. given any other similar pair $(\underline{D}, G)$, there is a unique functor $H : \underline{C}_S \longrightarrow \underline{D}$ such that $H \cdot Q = G$.



As in the case of extending a homomorphism between two rings to their localisations, we will study the conditions under which a given functor $F : \underline{C} \longrightarrow \underline{D}$ will extend to the localisations $(\underline{C}_S, Q)$ and $(\underline{D}_{S'}, Q')$. That is, when we can complete the diagram



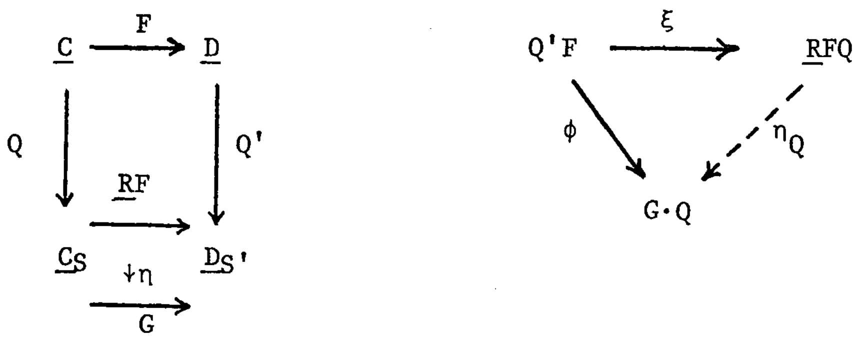
Obviously, we have:

Proposition 1. A necessary and sufficient condition for F to extend to $\underline{C}_S$ is that $Q'F(s)$ be an isomorphism in $\underline{D}_{S'}$ $\forall s \in S$.

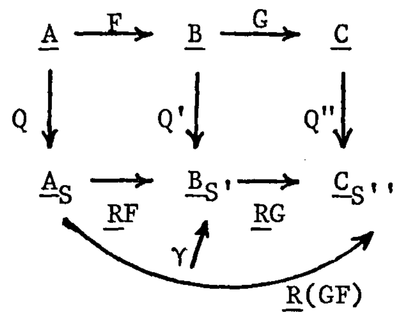
F will be called (S,S') exact.

In case an extension does not exist we consider the weaker notion of an approximate extension.

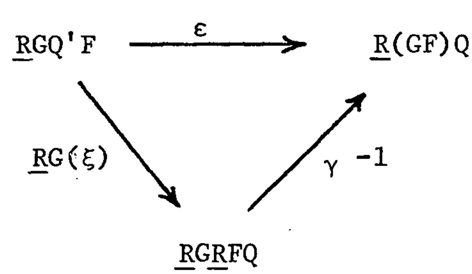
Definition. A right approximate extension of F is a pair $(\underline{R}F, \xi)$, where $\underline{R}F : \underline{C}_S \longrightarrow \underline{D}_{S'}$ and $\xi : Q'F \longrightarrow \underline{R}FQ$ a natural map satisfying the universal property : for any other pair (G, ϕ) with $G : \underline{C}_S \longrightarrow D_{S'}$ and $\phi : Q'F \longrightarrow G \cdot Q$ there is a unique map $\eta : \underline{R}F \longrightarrow G$ such that $\eta_Q \cdot \xi = \phi$.



In conjunction with extensions we consider composite functors $\underline{A} \xrightarrow{F} \underline{B} \xrightarrow{G} \underline{C}$ and right approximate extensions $(\underline{R}F,\xi)$, $(\underline{R}G,\xi')$ and $(\underline{R}(GF),\xi'')$. Applying the universality of the pair $(\underline{R}(GF), \xi'')$ to $(\underline{R}G\cdot\underline{R}F, \underline{R}G(\xi\_).\xi'_{F\_})$, we obtain a (unique) natural map $\gamma : \underline{R}(GF) \longrightarrow \underline{R}G\cdot\underline{R}F$ satisfying the relation $\gamma_X\cdot \xi''_X = \underline{R}G(\xi_X)\cdot \xi'_{F(X)}$.



In case γ is an isomorphism, there is a natural map $\varepsilon : \underline{R}G\cdot Q'F \longrightarrow \underline{R}(GF)Q$ which we will call the edge morphism, and is defined by $\varepsilon_X = \gamma_X^{-1}\cdot \underline{R}G(\xi_X)$ where $\xi : F \longrightarrow \underline{R}F\cdot Q$ is the natural map.



§2. Existence and Description

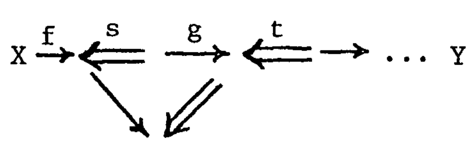
We assume S contains identities and is closed under composition. For brevity elements of S will be denoted by double arrows $\Longrightarrow$. $Ob\underline{C}_S = Ob\underline{C}$. Arrows of $\underline{C}_S$ are equivalence classes of reduced words $fs^{-1}gt^{-1}$ ..... that is, finite diagrams of the form $X \xrightarrow{f} \overset{s}{\Longleftarrow} \xrightarrow{g} \overset{t}{\Longleftarrow} \cdots\; Y$ with the equivalence relation:

$$X \overset{s}{\Longrightarrow} Y \overset{s}{\Longleftarrow} X \text{ replaced by } 1_X$$

$$Y \overset{s}{\Longleftarrow} Y \overset{s}{\Longrightarrow} Y \text{ replaced by } 1_Y$$

The composition of any two reduced words $X \to \Longleftarrow \to \Longleftarrow ..Y$ and $Y \to \Longleftarrow \to \Longleftarrow ..Z$ is $X \to \Longleftarrow \to \Longleftarrow \ldots Y \to \Longleftarrow \to \Longleftarrow \ldots Z$. The functor $Q : \underline{C} \longrightarrow \underline{C}_S$ is defined by $Q(X \xrightarrow{f} Y) = X \xrightarrow{f} Y$ with $Q(s)^{-1} = X \overset{s}{\Longleftarrow} Y$ for s in S. If $G : \underline{C} \longrightarrow \underline{D}$ is any functor with the property that G(s) is an isomorphism in $\underline{D}$ for every s in S, then the functor $H : \underline{C}_S \longrightarrow \underline{D}$ satisfying the relation $H \cdot Q = G$ is uniquely determined by $H(X \xrightarrow{f} \overset{s}{\Longleftarrow} \xrightarrow{g} \overset{t}{\Longleftarrow} \ldots Y) = G(X) \xrightarrow[G(f)]{} \xrightarrow{G(s)^{-1}} \xrightarrow[G(g)]{} \xrightarrow{G(t)^{-1}} \ldots G(Y)$.

<u>Remark</u>. This is too academic and cumbersome to be of much use. We would like to simplify the sequence of arrows $X \xrightarrow{f} \overset{s}{\Longleftarrow} \xrightarrow{g} \overset{t}{\Longleftarrow}$ ......Y by successive "push-outs" of each pair $\Longleftarrow \to$



and thus reducing the words to the form $X \to \Longleftarrow Y$.

In order to ensure that pairs $\xrightarrow{f}\xleftarrow[]{s}$ and $\xrightarrow{g}\xleftarrow[]{t}$ resulting from the same "pushout" diagram are identified we require the additional hypothesis on S that if pairs of arrows $\overset{\alpha}{\underset{\beta}{\rightrightarrows}}$ are equalized by an element $s \in S$, then they are coequalized by some $t \in S$. More generally this leads us to consider an axiomization of S.

<u>Definition</u>. The class S is called a right multiplicative system if:

(FR0). S contains all identities and is closed under composition.

(FR1). Any diagram

$$\begin{array}{ccc} X & & \\ s \Uparrow & & \\ X' & \xrightarrow{f} & Y \end{array}$$

with $s \in S$ can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ s \Uparrow & & \Uparrow t \\ X' & \xrightarrow{f} & Y \end{array}$$

with $t \in S$.

(FR2) Given a pair of maps $X \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Y$ if there is a $s \in S$ such that $\alpha s = \beta s$, then there is a $t \in S$ such that $t\alpha = t\beta$.

Dually we can define a left-multiplicative system FR0°, FR1°,FR2° by reversing all the arrows. A class of morphisms which is both a left and right multiplicative system in <u>A</u> is called a multiplicative system in <u>A</u>.

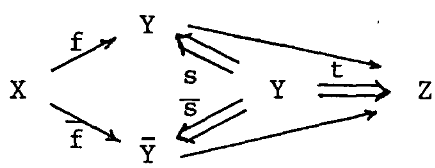
Proposition 2. Let $\underline{C}$ be a category and S a right multiplicative system in $\underline{C}$, then the localisation of $\underline{C}$ with respect to S exists and is unique up to isomorphism.

Proof. Appendix.

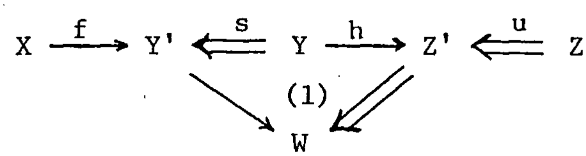
Granted the existence of a right multiplicative system S in $\underline{C}$ we construct the localisation $(\underline{C}_S, Q)$ by setting:

$$\mathrm{Ob}\underline{C}_S = \mathrm{Ob}\underline{C}$$

For any X, $Y \in \mathrm{Ob}\underline{C}_S$, a morphism from X to Y, denoted by broken arrows X ----> Y, and called a quasi-morphism from X to Y, is an equivalence class of morphisms $X \xrightarrow{f} Y' \xLeftarrow{s} Y$ where the equivalence relation between two pairs $X \xrightarrow{f} Y' \xLeftarrow{s} Y$ and $X \xrightarrow{\bar{f}} \bar{Y} \xLeftarrow{\bar{s}} Y$ holds if there is a diagram



such that the two inner triangles are commutative and the outer edges form a commutative square. The composition of two quasi-morphisms X ----> Y = $X \xrightarrow{f} Y' \xLeftarrow{s} Y$ and Y ----> Z = $Y \xrightarrow{h} Z \xLeftarrow{u} Z$ is defined by the diagram



With (1) obtained by applying (FR1) to $Y' \overset{s}{\Longleftarrow} Y \overset{h}{\longrightarrow} Z'$.

The functor $Q : \underline{C} \longrightarrow \underline{C}_S$ is defined by $Q(X) = X$ and $Q(X \overset{f}{\longrightarrow} Y) = X \overset{f}{\longrightarrow} Y \overset{1_Y}{\Longleftarrow} Y$ with $Q(s)^{-1} = Y \overset{s}{\Longleftarrow} X$ for $s \in S$.

Alternately one can also describe the category $\underline{C}_S$ by:

<u>Proposition 2'.</u> $\mathrm{Hom}_{\underline{C}_S}(X,Y) = \varinjlim_{Y' \in \mathrm{Ob} J_Y} \mathrm{Hom}_{\underline{C}}(X,Y')$ where J_Y is the filtered category of diagrams $Y \overset{s}{\Longrightarrow} Y'$ and whose morphisms are commutative diagrams

$$\begin{array}{ccc} Y & \overset{s}{\Longrightarrow} & Y' \\ t \Downarrow & \swarrow f & \\ Y'' & & \end{array}$$

We now go on with the problem of constructing right approximate extensions. Let $\underline{I} \subseteq \underline{C}$ be a full subcategory sucy that:

(RAC1). $F|\underline{I}$ is $(S_\circ, S')$ exact where $S_\circ = AR\underline{I} \cap S$.

(RAC2). For every object X of $\underline{C}$, there is a $s \in S$ and an object I of $\underline{I}$ such that $X \overset{s}{\Longrightarrow} I$.

(RAC2) immediately implies that $S_\circ$ is a right multiplicative system in $\underline{I}$, and moreover, that $\underline{I}_{S_\circ}$ is a full subcategory of $\underline{C}_S$. By Proposition 1, $F|\underline{I}$ has an extension $\underline{E}(F|\underline{I})$ to $\underline{I}_{S_\circ}$. We next note that the inclusion functor $i : \underline{I}_{S_\circ} \longrightarrow \underline{C}_S$ is not only fully faithful, but also yields an equivalence of categories. To see this, we assign to each object X of $\underline{C}_S$ a morphism $\nu_X : X \Longrightarrow r(X)$, $r(X) \in \mathrm{ob}\underline{I}$. This defines a functor $r : \underline{C}_S \longrightarrow \underline{I}_{S_\circ}$ and an isomorphism $\nu : Id_{\underline{C}_S} \approx i \cdot r$ which is natural since any morphism $X \longrightarrow Y$ sets up the diagram

$$\begin{array}{ccc} X & \Longrightarrow & r(X) \\ \downarrow & & \\ Y & \Longrightarrow & r(Y) \end{array}$$

representing a well-defined quasi-morphism $r(X) \dashrightarrow r(Y)$. Upon calling $\underline{I}$ a right F-acyclic subcategory of $\underline{C}$, we have

Proposition 3. Let $F : \underline{C} \longrightarrow \underline{D}$ be a covariant functor and S,S' right multiplicative system in $\underline{C}$,$\underline{D}$ respectively. Let $\underline{I}$ be a right F-acyclic subcategory of $\underline{C}$, then the functor

$$\underline{C}_S \underset{r}{\approx} \underline{I}_{S_\circ} \xrightarrow{E(F|\underline{I})} \underline{D}_{S'}$$
$$\underline{R}F : \underline{C}_S \longrightarrow \underline{D}_{S'}$$

together with the natural map

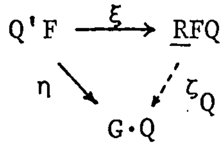
$$\xi_X : F(X) \xrightarrow{F(\nu_X)} F(rX) = \underline{R}F(X)$$

is a right derived functor of F. Moreover, $(\underline{R}F, \xi)$ is unique up to isomorphism of functors.

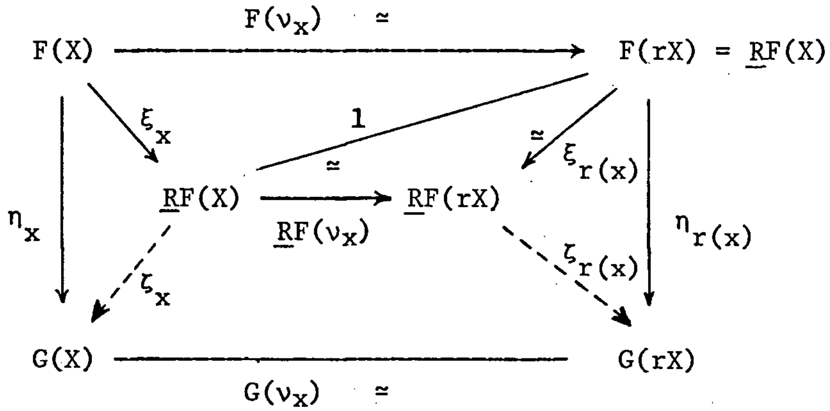
Proof. We must prove universality. Let (G,η) be given.

$$G : \underline{C}_S \longrightarrow \underline{D}_{S'} \qquad \eta : Q'F \longrightarrow GQ$$

Want: a unique $\zeta : \underline{R}F \longrightarrow G$ such that $\zeta_Q \cdot \xi = \eta$.



First, to see the uniqueness of ζ, for any object X of $\underline{C}$, pick $X \overset{\nu_X}{\Longrightarrow} r(X)$.



Note that $G(\nu_X)$ is an isomorphism since ν_X is one in $\underline{C}_S$ and because $\xi_{r(x)}$ is an isomorphism there is only one possible choice for $\zeta_{r(x)}$ as shown (namely, $\zeta_{r(x)} = \eta_{r(x)} \cdot \xi^{-1}_{r(x)}$).

But the lower trapezoid has isos on top and base, thus there is only one possible choice for ζ_X

$$\zeta_X = G(\nu_X)^{-1} \cdot \zeta_{r(x)} \cdot \underline{R}F(\nu_X)$$

or since $F(rX) = \underline{R}F(X)$, we also have

$$\zeta_X = G(\nu_X)^{-1} \cdot \eta_{r(x)} \cdot$$

Naturality:

$$\zeta_X : F(rX) \xrightarrow{\eta_{r(x)}} G(rX) \xrightarrow{G(\nu_X)^{-1}} G(X)$$

is clearly natural with respect to morphisms $X \longrightarrow Y$ in $\underline{C}$.

Next by extension, we observe that formally there is a bijection between the sets

$$\text{Hom}(\underline{R}FQ, G\cdot Q) \approx \text{Hom}(\underline{R}F, G).$$

Hence ζ is natural with respect to quasi-morphisms $X \longrightarrow Y' \Longleftarrow Y$ in $\underline{C}_S$.

Finally to show that the pair $(\underline{R}F,\xi)$ is unique, assume another pair $(\underline{R}F',\xi')$ having the same properties, then the universality of $(\underline{R}F,\xi)$ guarantees the existence of a unique $\eta : \underline{R}F \longrightarrow \underline{R}F'$ satisfying $\eta_Q\cdot\xi = \xi'$. Similarly, there is also a unique $\eta' : \underline{R}F' \longrightarrow \underline{R}F$ with $\eta'_Q\cdot\xi' = \xi$. But $\eta_Q\cdot\eta'_Q\cdot\xi = \xi$; $\eta'_Q\eta_Q\cdot\xi' = \xi'$. Therefore $\eta_Q\cdot\eta'_Q = \text{Id}$; $\eta'_Q\cdot\eta_Q = \text{Id}$.

Remark. A left approximate extension of F is a functor $\underline{L}F : \underline{C}_S \longrightarrow \underline{D}_S$ together with a natural map $\sigma : \underline{L}F\cdot Q \longrightarrow Q'F$ with the property that for any similar pair (G,τ) there is a unique $\omega : G \longrightarrow \underline{L}F$ satisfying $\sigma\cdot\omega_Q = \tau$. By appropriately reversing the arrows proposition 3 can also be proved for left F-acyclic subcategories and left approximate extensions. And although the notion of an acyclic subcategory do leave us much freedom of choice, we will show in the appendix that there is, in fact, always a maximal such subcategory.

Next suppose we have two functors

$$F : \underline{C} \longrightarrow \underline{D}$$

$$G : \underline{D} \longrightarrow \underline{E}$$

and right multiplicative systems S,S',S'' in these respective categories. Suppose further there is a right F-acyclic subcategory $\underline{I}$ of $\underline{C}$ and a right G-acyclic subcategory $\underline{I}'$ of $\underline{D}$. We then have right approximate extensions $\underline{R}F$, $\underline{R}G$, $\underline{R}(GF)$ and

Proposition 4. If $F(\underline{I}) \subseteq \underline{I}'$, then the canonical morphism

$$\gamma : \underline{R}(GF) \longrightarrow \underline{R}G \cdot \underline{R}F$$

is an isomorphism.

Proof. For X in $Ob\underline{C}_S$, to obtain $\underline{R}G \cdot \underline{R}F$ we have to apply G to I' in the situation

$$\begin{array}{ccc} F(X) \xrightarrow{\xi_X} & \underline{R}F(X) & = F(rX) \\ & \Downarrow & \\ & I' & \end{array}$$

This gives

$$\begin{array}{ccc} GF(X) \longrightarrow & GF(rX) & = \underline{R}(GF)(X) \\ & \downarrow \gamma & \\ & G(I') & = \underline{R}G \cdot \underline{R}F(X). \end{array}$$

Because $F(rX) \in \underline{I}'$, γ is now an isomorphism in $\underline{E}_{S''}$

Finally, under the conditions of Proposition 4, we can define an edge morphism $\varepsilon : \underline{R}G \cdot F \longrightarrow \underline{R}(GF)$ by the commutative diagram

$$\begin{array}{ccc} \underline{R}G \cdot F & \xrightarrow{RG(\xi_{-})} & \underline{R}G \cdot \underline{R}F \\ & \searrow^{\varepsilon} & \downarrow \gamma^{-1} \\ & & \underline{R}(GF) \end{array} \quad .$$

CHAPTER III

PSEUDO SPLIT SEQUENCES AND MAPPING CONES

Let $\underline{A}$ be an abelian category. We denote by $C(\underline{A})$ the abelian category of all co-chain complexes over $\underline{A}$ and by $K(\underline{A})$ the homotopy category thereof. For each object X of $C(\underline{A})$ ($K(\underline{A})$), define the complex $T(X^*)$ by $T(X^*)^n = X^{n+1}$, $d^n_{T(X^*)} = -d^n_{X^*}$, then $T : C(\underline{A}) \longrightarrow C(\underline{A})$ (resp. $K(\underline{A}) \longrightarrow K(\underline{A})$) is an automorphism and is called the translation automorphism. We will often write $X^*[1]$ instead of $T(X^*)$ and $X^*[n]$ instead of $T^n(X^*)$.

To every short exact sequence

$$\Sigma : 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0 \qquad \text{in } C(\underline{A}),$$

there corresponds a long exact cohomology sequence

$$\cdots \longrightarrow H^n(X_0^*) \xrightarrow{H^n(i)} H^n(X_1^*) \xrightarrow{H^n(p)} H^n(X_2^*) \xrightarrow{\delta^n_\Sigma} H^{n+1}(X_0^*) \longrightarrow \cdots$$

with the connecting morphism δ^n_Σ furnished by the Snake Lemma.

We consider the following conditions on short exact sequences under which more readily accessible descriptions of the connecting morphisms are available:

<u>Definition</u>. We call a short exact sequence

$$\Sigma : 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0$$

in C(A) quasi or pseudo split if it splits in each dimension,

i.e. $$0 \longrightarrow X_0^n \longrightarrow X_1^n \longrightarrow X_2^n \longrightarrow 0$$

splits for each n.

Proposition 5. If Σ quasi-splits, then one can extract from the differential operator $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} d_{X_0^*} & \beta \\ 0 & d_{X_2^*} \end{pmatrix}$ of X_1^*, a morphism β: $X_2^* \longrightarrow X_0^*[1]$ called the twist of Σ. β has the most important property that in cohomology the connecting morphism δ_Σ^n: $H^n(X_2^*) \longrightarrow H^{n+1}(X_0^*)$ one obtains from the Snake Lemma is induced by the twist. In other words, the long exact cohomology sequence of Σ is obtained from the sequence of complexes

$$\cdots \longrightarrow X_1^*[n] \longrightarrow X_2^*[n] \xrightarrow{T^n(\beta)} X_0^*[n+1] \xrightarrow{T^n(i)} X_1^*[n+1] \longrightarrow \cdots$$

by taking 0-cohomology.

Furthermore β measures the degree of splitness of Σ in the sense that it is homotopic to zero if and only if the sequence splits.

Proof. Appendix

Remark: As to their availability, we would like to "track" down all the quasi-split sequences and will show that up to homotopy isomorphism--an isomorphism of short exact sequences in $K(\underline{A})$, every quasi-split sequence is a "truncated mapping cone" sequence.

We next construct a quasi-split sequence * whose twist is the homotopy class of a given map $f : X^* \longrightarrow Y^*$. First, the mapping cone of f is the co-chain complex C_f^* defined by

$$C_f^n = Y^n \oplus X^{n+1}$$

$$\partial^n_{C_f^*} = \begin{pmatrix} \partial^n_y & f^{n+1} \\ 0 & -\partial^{n+1}_x \end{pmatrix}$$

$$\text{or} \qquad \partial_{C_f^*} = \begin{pmatrix} \partial_y & T(f) \\ 0 & T(\partial_x) \end{pmatrix}$$

Then the sequence

$$* : 0 \longrightarrow Y^* \xrightarrow{i} C_f^* \xrightarrow{p} X^*[1] \longrightarrow 0$$

quasi-splits with twist T(f), which is equivalent to f in the homotopy category. By a mapping cone sequence we shall mean a sequence of the form

$$** : X^* \xrightarrow{f} Y^* \longrightarrow C_f^* \longrightarrow X^*[1] \xrightarrow{T(f)} Y^*[1] \longrightarrow C_f^*[1] \xrightarrow{T(p)} X^*[2] \longrightarrow \cdots$$

From the long exact cohomology sequence derived from *, or applying 0-cohomology to ** we immediately have

Proposition 6. $f : X^* \longrightarrow Y^*$ induces isomorphisms in cohomology if and only if the cohomology of C_f^* is trivial.

Proposition 7. Given any homotopy commutative diagram

$$\begin{array}{ccc} X_1^* & \xrightarrow{f} & X_2^* \\ a_1 \downarrow & & \downarrow a_2 \\ Y_1^* & \xrightarrow[g]{} & Y_2^* \end{array}$$

there is a map $a : C_f^* \longrightarrow C_g^*$ such that each of the squares in

$$\begin{array}{ccccc} X_2^* & \longrightarrow & C_f^* & \longrightarrow & X_1^*[1] \\ \downarrow a_2 & & \downarrow a & & \downarrow T(a_1) \\ Y_2^* & \longrightarrow & C_g^* & \longrightarrow & Y_1^*[1] \end{array}$$

commutes.

Proof. In matrix notation, $a = \begin{pmatrix} a_2 & h \\ 0 & a_1 \end{pmatrix}$ where $h^n : X_1^{n+1} \longrightarrow Y_2^n$ is the homotopy of the original square.

Commutativity is trivial.

In order to establish a relation between quasi-split and mapping cone sequences, we first compare any short exact sequence $\Sigma : 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0$ with a truncated mapping cone sequence * :

$$\begin{array}{lccccccccc}
\Sigma : & 0 \longrightarrow & X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & \longrightarrow 0 & & \\
 & & {\scriptstyle -1}\uparrow & & {\scriptstyle 1}\uparrow & & & & & \\
* : & 0 \longrightarrow & X_0^* & \xrightarrow{-i} & X_1^* & \longrightarrow & C_{-i}^* & \longrightarrow (& X_0^*[1] \longrightarrow \cdots &)
\end{array}$$

By Proposition 7 there is a morphism $\pi : C_{-i}^* \longrightarrow X_2^*$ rendering the diagram

$$\begin{array}{ccccccc}
X_0^* & \xrightarrow{-i} & X_1^* & \longrightarrow & C_{-i}^* & \longrightarrow & X_0^*[1] \\
\downarrow & & {\scriptstyle p}\downarrow & & \downarrow {\scriptstyle \pi = (p,0)} & & \downarrow \\
0 & \longrightarrow & X_2^* & \longrightarrow & X_2^* & \longrightarrow & 0
\end{array}$$

commutative. Applying cohomology and the 5 Lemma,

$$\begin{array}{ccccccccccc}
\to H^n(X_0^*) & \xrightarrow{H^n(i)} & H^n(X_1^*) & \xrightarrow{H^n(p)} & H^n(X_2^*) & \xrightarrow{\delta_\Sigma^n} & H^{n+1}(X_0^*) & \longrightarrow & H^{n+1}(X_1^*) & \longrightarrow \\
{\scriptstyle -1}\uparrow & & {\scriptstyle 1}\uparrow & & \uparrow {\scriptstyle H^n(\pi)} & & \uparrow & & \uparrow & \\
\to H^n(X_0^*) & \xrightarrow[H^n(-i)]{} & H^n(X_1^*) & \longrightarrow & H^n(C_{-i}^*) & \longrightarrow & H^n(X_0^*[1]) & \longrightarrow & H^n(X_1^*[1]) & \longrightarrow
\end{array}$$

we conclude that π induces an isomorphism in cohomology. Hence up to homotopy and cohomology, every short exact sequence $\Sigma \; : 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0$ is a truncated mapping cone sequence. That is, the diagram

$$\begin{array}{ccccc} X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* \\ {\scriptstyle -1}\uparrow & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle \pi} \\ X_0^* & \xrightarrow{-i} & X_1^* & \longrightarrow & C_{-i}^* \end{array}$$

represents an isomorphism of complexes in some suitable homotopy category where morphisms inducing isomorphisms in cohomology are invertible. Moreover:

<u>Proposition</u>. If Σ quasi-splits with twist β and section s, then $\pi : C_{-i}^* \longrightarrow X_2^*$ has a homotopy inverse given by $\begin{pmatrix} s \\ \beta \end{pmatrix}$.

Thus we have the very important result that in the homotopy category $K(\underline{A})$ every quasi-split sequence is isomorphic to a truncated mapping cone sequence. The morphism π associated with Σ measures the deviation of the sequence from being quasi-split and is always an isomorphism in cohomology.

<u>Remark</u>. There is a dual story. Suppose once again we are given $\Sigma : 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0.$ Then there exists a morphism $\iota : X_0^*[1] \longrightarrow C_p^*$ (again evoking proposition 7) such that

$$\begin{array}{ccccccc}
X_0^* & \longrightarrow & 0 & \longrightarrow & X_0^*[1] & \xrightarrow{1} & X_0^*[1] \\
{\scriptstyle i}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \iota=\binom{0}{i}} & & \big\downarrow{\scriptstyle T(i)} \\
X_1^* & \xrightarrow{p} & X_2^* & \longrightarrow & C_p^* & \longrightarrow & X_1^*[1]
\end{array}$$

commutes. By the 5 lemma ι also induces an isomorphism in cohomology.

<u>Proposition.</u> π and ι are related by the homotopy commutative diagram

$$\begin{array}{ccccc}
 & & C_{-i}^* & & \\
 & \swarrow^{\pi} & & \searrow & \\
X_2^* & & & & X_0^*[1] \\
 & \searrow & & \swarrow_{\iota} & \\
 & & C_p^* & &
\end{array}$$

<u>Proof</u>. The maps $C_{-i}^* \rightrightarrows C_p^*$ in matrix notation appears as $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$. Hence, $h = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is the required homotopy.

Now it is clear that:

Proposition. The functors $\mathrm{Hom}_{K(\underline{A})}(-,Z^*)$ and $\mathrm{Hom}_{K(\underline{A})}(Z^*,-)$ when applied to any mapping cone sequence

$$X^* \xrightarrow{f} Y^* \xrightarrow{j} C_f^* \longrightarrow X^*[1] \xrightarrow{T(f)} Y^*[1] \longrightarrow C_f^*[1] \longrightarrow \cdots\cdot$$

yield long exact sequences

$$\mathrm{Hom}_{K(\underline{A})}(Z^*,X^*) \xrightarrow{f_\#} \mathrm{Hom}_{K(\underline{A})}(Z^*,Y^*) \xrightarrow{j_\#} \mathrm{Hom}_{K(\underline{A})}(Z^*,C_f^*) \longrightarrow \cdots\cdot$$

$$\mathrm{Hom}_{K(\underline{A})}(X^*,Z^*) \xleftarrow{f^\#} \mathrm{Hom}_{K(\underline{A})}(Y^*,Z^*) \xleftarrow{j^\#} \mathrm{Hom}_{K(\underline{A})}(C_f^*,Z^*) \longleftarrow \cdots\cdot$$

Although $\underline{A}$ is assumed to be an abelian category, the category $K(\underline{A})$ is in general not abelian. However, in view of the developments in this chapter, we can axiomize all the structures that carry enough information for our purposes by the notion of a triangulated category, where a triangulated category is a triple $(\underline{C}, \Delta, T)$ such that:

(1) T is an automorphism of $\underline{C}$.

(2) Δ is a collection of sextuples (X,Y,Z, u,v,w) called triangles of $\underline{C}$, where in each triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

A morphism of triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & W & \xrightarrow{w} & T(X) \\ f\downarrow & & g\downarrow & & h\downarrow & & T(f)\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & W' & \xrightarrow{w'} & T(X') \end{array} .$$

The triple ($\underline{C}$, Δ, T) satisfies the axioms:

(TR1). Δ is closed under isomorphisms.

Every morphism $u : X \longrightarrow Y$ can be embedded in a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.

For any object X in $\underline{C}$, the sextuple $(X,X,0,id_X,0,0)$ is a triangle.

(TR2). (X,Y,Z,u,v,w) is a triangle iff $(Y,Z,T(X),v,w,-T(u))$ is a triangle.

(TR3). Given a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ f\downarrow & & g\downarrow & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X') \end{array}$$

where the first square is commutative and the rows are triangles, there exists $h : Z \longrightarrow Z'$ such that (f,g,h) is a mapping of triangles.

In the case of $K(\underline{A})$, $T : K(\underline{A}) \longrightarrow K(\underline{A})$ is the translation automorphism and a triangle in $K(\underline{A})$ is any sextuple isomorphic to a mapping cone sequence of the form

$$X^* \xrightarrow{f} Y^* \xrightarrow{i} C_f^* \longrightarrow X^*[1].$$

CHAPTER IV

THE DERIVED CATEGORY

We will now follow the programme outlined in Chapter II for the construction of the derived functor $\underline{R}F$ of any functor $F : \underline{A} \longrightarrow \underline{B}$.

Let $\underline{A}$ be abelian. We denote $C^*(\underline{A})$ for the category $C(\underline{A})$ or any one of its full subcategories $C^+(\underline{A})$, $C^-(\underline{A})$, and $C^b(\underline{A})$ whose objects are complexes of $\underline{A}$ bounded above, bounded below, and bounded on both sides respectively, and similarly denote $K^*(\underline{A})$, $K^+(\underline{A})$, $K^-(\underline{A})$ and $K^b(\underline{A})$ for their corresponding homotopy categories.

Proposition 9. In $K^*(\underline{A})$ the class of morphisms which induces isomorphisms in cohomology, called the class of quasi-isomorphisms or quisos, form a multiplicative system.

Proof. Appendix.

For brevity, quisos will be denoted by double arrows $\Longrightarrow$.

Definition. The derived category $D(\underline{A})$ of $\underline{A}$ is the localisation of $K(\underline{A})$ with respect to the class of all quisos. Similarly, there are localisations $D^+(\underline{A})$, $D^-(\underline{A})$, and $D^b(\underline{A})$ of $K^+(\underline{A})$, $K^-(\underline{A})$ and $K^B(\underline{A})$ respectively. As a comparison of their relative sizes we have:

Proposition: Each of the functors

$$\begin{array}{ccccccc} & & & & D^{+}(\underline{A}) & & \\ & & & \nearrow & & \searrow & \\ A & \longrightarrow & D^{b}(\underline{A}) & & & & D(\underline{A}) \\ & & & \searrow & & \nearrow & \\ & & & & D^{-}(\underline{A}) & & \end{array}$$

is a full embedding.

<u>Proof</u>. Appendix.

Continuing with the programme of Chapter II, we next look for "suitable" acyclic subcategories.

If $\underline{I}$ is any subcategory of $\underline{A}$ such that every object of $\underline{A}$ admits a monic into an object of $\underline{I}$ then for X^{*} in $C^{+}(\underline{A})$ trivial in negative dimensions, we can construct a quiso

$$\begin{array}{ccccccccccccccccccc} \cdots \to & 0 & \to & X^{0} & \to & X^{1} & \to & X^{2} & \to & \cdots & \to & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^{n} & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\ & \downarrow & & \downarrow s^{0} & & \downarrow s^{1} & & \downarrow s^{2} & & & & \downarrow s^{n-1} & & \Big\downarrow & & & & \\ \cdots \to & 0 & \to & I^{0} & \to & I^{1} & \to & I^{2} & \to & \cdots & \to & I^{n-1} & \text{push-out} & \Big\downarrow & & & & \\ & & & & & & & & & & & \Big\downarrow & & \Big\downarrow & & & & \\ & & & & & & & & & & & \operatorname{cokd}_{I}^{n-2} & \longrightarrow & P^{n} & & & & \end{array}$$

by induction and the facts that in any pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \beta \downarrow & & \downarrow \beta' \\ C & \xrightarrow{\alpha'} & P^{n} \end{array} \qquad \beta \text{ induces an epimorphism } \operatorname{Ker}\alpha \twoheadrightarrow \operatorname{Ker}\alpha'$$

and β' induces a monomorphism cok $\alpha \rightarrowtail$ cok α'. Thus:

Proposition 11. Let $\underline{I}$ be a full-subcategory of $\underline{A}$ such that every object of $\underline{A}$ admits a mono into an object of $\underline{I}$, then every object of $C^+(\underline{A})$ admits a quiso into an object of $C^+(\underline{I})$.

Proof. Appendix.

Now let $\underline{B}$ be another abelian category, and $F : \underline{A} \longrightarrow \underline{B}$ be an additive functor. F induces functors $C^*(\underline{A}) \longrightarrow C^*(\underline{B})$ $K^*(\underline{A}) \longrightarrow K^*(\underline{B})$ which will still be denoted by F. We recall that if F preserves short exact sequences of objects of $\underline{I}$ and $\underline{I}$ is closed under cokernels of monomorphisms, then F takes short exact sequences of $C^+(\underline{I})$ into exact sequences of $C^+(\underline{B})$. Moreover, if $\underline{I}$ is closed under direct sums, then for a quiso $I_1^* \overset{s}{\Longrightarrow} I_2^*$ of objects of $C^+(\underline{I})$,

$$F(C_s^*) = F(I_1^*[1] \oplus I_2^*) = F(I_1^*[1]) \oplus F(I_2^*) = C_{F(s)}^*$$

and the exactness of F on objects of $C^+(\underline{I})$ implies that $F(I_1^*) \overset{F(s)}{\Longrightarrow} F(I_2^*)$ is a quiso, so F maps quisos into quisos, and is in particular, (quiso,quiso) exact. Summarizing, we have

Proposition 12. Let $\underline{I}$ be a full subcategory of $\underline{A}$ which is closed under direct sums and cokernels of monomorphisms, further, assume F preserves short exact sequences of objects of $\underline{I}$, then the restriction of F to $K^*(\underline{I})$ maps quisos into quisos.

In view of Propositions 11 and 12, we say that a full subcategory $\underline{I}$ of $\underline{A}$ is right F-acyclic if

(AC1). Every object X in $\underline{A}$ admits a mono into an object of $\underline{I}$.

(AC2). $\underline{I}$ is closed under (finite) direct sums and cokernels of monos.

(AC3). F preserves short exact sequences of objects of $\underline{I}$.

Then, $K^+(\underline{I})$ will be a right F-acyclic subcategory of $K^+(\underline{A})$ in the sense of Chapter II (RAC1, RAC2). Hence

<u>Theorem 1:</u> If $\underline{A}$ contains a right F-acyclic subcategory $\underline{I}$, then the right approximate extension $(\underline{R}F,\xi)$ of F to $D^+(\underline{A})$ exists and is given by

$$D^+(\underline{A}) \xrightarrow[\simeq]{r} D^+(\underline{I}) \xrightarrow{\underline{E}(F|\underline{I})} D(\underline{B})$$

as in Proposition 3.

$(\underline{R}F,\xi)$ is called the <u>right derived functor</u> of F.

Granted the presence of a right F-acyclic subcategory $\underline{I}$ of $\underline{A}$, $\underline{R}F(X^*)$ is found on any object X^* of $K^+(\underline{A})$ by first taking a quiso $X^* \xRightarrow{\nu_{X^*}} I^*$ into an object of $C^+(\underline{I})$ and setting

$$\underline{R}F(X^*) = F(I^*) \text{ and}$$

$$\xi_x = F(\nu_x) \quad .$$

Dually the presence of a left F-acyclic subcategory $\underline{P}$ of $\underline{A}$ guarantees the existence of a left derived functor $(\underline{L}F, \sigma)$ with $\underline{L}F : D^{-}(\underline{A}) \longrightarrow D(\underline{B})$. In principle it is of course possible to speak of the right approximate extension of F to $D(\underline{A})$ or the left approximate extension of F to $D(\underline{A})$. For its existence one would need:

(a). Stronger conditions on $\underline{I}$ as to obtain the appropriate generalization of Proposition 11, and

(b). Stronger exactness conditions of F on $\underline{I}$.

For our purposes we will focus our attention only on the right derived functor of F on D^{+}, and the left derived functor on D^{-}.

For the rest of this chapter, $F: \underline{A} \longrightarrow \underline{B}$ will always denote an additive functor, $\underline{I}$ a right F-acyclic subcategory of $\underline{A}$.

The higher order cohomology functors can be obtained from $\underline{R}F$ by setting $\underline{R}^{n}F(X^{*}) = H^{n}\underline{R}F(X^{*})$ and there are natural maps

$$\xi^{n} : H^{n}F \longrightarrow \underline{R}^{n}F \qquad \text{defined by}$$

$$\xi^{n} : H^{n}(\xi_{X}) : H^{n}F(X^{*}) \longrightarrow H^{n}\underline{R}F(X^{*}) = \underline{R}^{n}F(X^{*}) .$$

We next construct the long exact sequence of the derived functors.

Let $\Sigma : 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0$ be any short exact sequence in $C^*(\underline{A})$. Up to isomorphisms in cohomology, Σ is essentially a truncated mapping cone sequence

$$\begin{array}{ccccccccccc} & & & & & & C^*_{-i} & & & & \\ & & & & & \overset{\pi}{\swarrow\!\!\swarrow} & & \searrow & & & \\ 0 \to X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & & & & X_0^*[1] \longrightarrow X_1^*[1] & \longrightarrow & \cdots \\ & & & & & \searrow & & \underset{\iota}{\swarrow\!\!\swarrow} & & & \\ & & & & & & C_p^* & & & & \end{array}$$

More precisely, in the derived category $D^*(\underline{A})$, the morphisms π and ι are isomorphisms, and the diagrams

$$(*)\qquad \begin{array}{ccccccc} X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & \overset{d_\Sigma}{\dashrightarrow} & X_0^*[1] \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \Downarrow{\scriptstyle \iota} \\ X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & \longrightarrow & C_p^* \end{array}$$

and

$$\begin{array}{ccccccc} X_0^* & \xrightarrow{-i} & X_1^* & \longrightarrow & C^*_{-i} & \longrightarrow & X_0^*[1] \\ {\scriptstyle -1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle \pi}\Downarrow & & \downarrow \\ X_0^* & \xrightarrow{i} & X_1^* & \longrightarrow & X_2^* & \overset{d'_\Sigma}{\dashrightarrow} & X_0^*[1] \end{array}$$

are isomorphisms of sequences in $D^*(\underline{A})$. Moreover, these isomorphisms are natural in the sense of:

<u>Lemma 1</u>. For a given morphism of short exact sequences

$$\begin{array}{lcccccccccc}
\Sigma & : & 0 & \longrightarrow & X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & \longrightarrow & 0 \\
 & & & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\
\Sigma' & : & 0 & \longrightarrow & Y_0^* & \xrightarrow{i'} & Y_1^* & \xrightarrow{p'} & Y_2^* & \longrightarrow & 0
\end{array}$$

the diagram

$$\begin{array}{ccccccc}
X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & \overset{d_\Sigma}{\dashrightarrow} & X_0^*[1] \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow \\
Y_0^* & \xrightarrow{i'} & Y_1^* & \xrightarrow{p'} & Y_2^* & \overset{d_{\Sigma'}}{\dashrightarrow} & Y_0^*[1]
\end{array}$$

is commutative.

Proof. By Proposition 7 there is a morphism $a : C_p^* \longrightarrow C_{p'}^*$, rendering square (1) of the diagram

$$\begin{array}{ccccccc}
d_\Sigma & : & X_2^* & \longrightarrow & C_p^* & \overset{\iota}{\Longleftarrow} & X_0^*[1] \\
 & & \downarrow f_2 & (1) & \downarrow a & (2) & \downarrow \\
d_{\Sigma'} & : & Y_2^* & \longrightarrow & C_{p'}^* & \overset{\iota'}{\Longleftarrow} & Y_0^*[1]
\end{array}$$

commutative. Square (2) is clearly commutative.

Now the quiso resolution of the mapping cone sequence (*) can be chosen again to be a mapping cone sequence (**), or equivalently, under the isomorphism of categories $D(\underline{I}) \underset{r}{\simeq} D(\underline{A})$, mapping cone sequences are preserved.

$$
\begin{array}{lcccccccccccccc}
 & X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & \overset{d_\Sigma}{\dashrightarrow} & X_0^*[1] & \longrightarrow & X_1^*[1] & \longrightarrow & X_2^*[1] & \longrightarrow & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \Downarrow \iota & & \downarrow & & \downarrow & & \\
 & X_0^* & \xrightarrow{i} & X_1^* & \xrightarrow{p} & X_2^* & \longrightarrow & C_p^* & \longrightarrow & X_1^*[1] & \longrightarrow & X_2^*[1] & \longrightarrow & \cdots \\
 & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
(**) & I_0^* & \xrightarrow{\alpha} & I_1^* & \xrightarrow{\beta} & I_2^* & \longrightarrow & C_\beta^* & \longrightarrow & I_1^*[1] & \longrightarrow & I_2^*[1] & \longrightarrow & \cdots
\end{array}
$$

Since F preserves mapping cone sequences of $K^+(\underline{I})$, an application of it to (**) yields the long mapping cone sequence

$$F(I_1^*) \xrightarrow{F(\beta)} F(I_2^*) \longrightarrow F(C_\beta^*) \longrightarrow F(I_1^*[1]) \longrightarrow \cdots$$

Finally, taking 0-cohomology yields the long <u>exact</u> sequence

$$\cdots \longrightarrow H^0F(I_1^*) \longrightarrow H^0F(I_2^*) \longrightarrow H^0F(C_\beta^*) \longrightarrow H^0F(I_1^*[1]) \longrightarrow \cdots$$

which is precisely

$$\cdots \longrightarrow \underline{R}^0F(X_1^*) \longrightarrow \underline{R}^0F(X_2^*) \longrightarrow \underline{R}^1F(X_0^*) \longrightarrow \underline{R}^1F(X_1^*) \longrightarrow \cdots$$

and we have

<u>Theorem 2</u>. $\{\underline{R}^nF , \delta^n\}$ is exact on short exact sequences

$$\Sigma : \ 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0 \quad \text{of } C^*(\underline{A})$$

where $\delta_\Sigma^n = \underline{R}^nF(d_\Sigma)$.

Comparison with Classical Theory.

Classically, one is interested in $\underline{R}^n F(X)$ for X in $\underline{A}$. By considering X as a complex concentrated at degree 0 the quiso resolution $X \Longrightarrow I^*$ can always be chosen to be the form

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

with the sequence $0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow$ exact.

We always have $\underline{R}^n F(X) = 0$ for $n<0$.

The long exact cohomology sequence for $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ takes the form $0 \longrightarrow \underline{R}^0 F(X) \longrightarrow \underline{R}^0 F(Y) \longrightarrow \underline{R}^0 F(Z) \longrightarrow \underline{R}^1 F(X) \longrightarrow \cdots$

Hence:

(1) The map $\xi^\circ : F \longrightarrow \underline{R}^0 F$ is an isomorphism if and only if F is left exact. In that case, the $\underline{R}^n F$'s are the right satellites of F.

(2) For any X in ob($\underline{I}$), $\underline{R}^n F(X) = 0$ for $n>0$.

(3) If X is acyclic (ξ_X is an isomorphism) then $\underline{R}^n F(X) = 0$ for all $n>0$.

And in case $\underline{A}$ has enough injectives, the $\underline{R}^n F$'s are isomorphic to the classical derived functors.

We conclude with a note on composite functors $\underline{A} \xrightarrow{F} \underline{B} \xrightarrow{G} \underline{C}$. If F sends the right F-acyclic subcategory $\underline{I}$ of $\underline{A}$ into a right G-acyclic subcategory $\underline{J}$ of $\underline{B}$, then we have an isomorphism

$$\gamma : \underline{R}(G\cdot F)|_{\underline{A}} \longrightarrow (\underline{R}G\cdot\underline{R}F)|_{\underline{A}}$$

which gives rise to an "edge" morphism

$$\varepsilon : \underline{R}G|_{\underline{B}}\cdot F \longrightarrow \underline{R}(G\cdot F)|_{\underline{A}}$$

defined by the commutative diagram

$$\begin{array}{ccc}
\underline{R}G|_{\underline{B}}\cdot F & \xrightarrow{\quad\varepsilon\quad} & \underline{R}(G\cdot F)|_{\underline{A}} \\
{\scriptstyle \underline{R}G(\xi)}\searrow & & \nearrow{\scriptstyle \gamma^{-1}} \\
& (\underline{R}G\cdot\underline{R}F)|_{\underline{A}} &
\end{array}$$

and is obtained by applying G to

$$\begin{array}{ccc}
F(X^*) & \xrightarrow{F(\nu_x)} & F(I^*) \\
\Searrow & & \\
& J^* &
\end{array}$$

yielding

$$\begin{array}{ccc}
GF(X^*) & \longrightarrow & GF(I^*) = \underline{R}(G\cdot F)(X^*) \\
\approx\ \searrow{\scriptstyle \gamma} & & \nearrow{\scriptstyle \varepsilon_{X^*}} \\
& \underline{R}G\cdot F(X^*) = G(J^*) &
\end{array}$$

This replaces the usual spectral sequence arguments concerning composite functors.

CHAPTER V

EXT

We conclude with an example of how the Ext - functors work out in the language of derived functors. First we note

Lemma a. Any quiso $s : I^* \Longrightarrow Y^*$ where I^* is an injective in $C^+(\underline{A})$ has a homotopy inverse.

Hence every morphism in $D(\underline{A})$ of a complex X^* to a complex of injectives bounded below is represented by an actual morphism of complexes, and if $\underline{A}$ has enough injectives, there is a canonical equivalence of categories:

$$K^+(\underline{I}) \simeq D^+(\underline{A}) \quad .$$

Next, we observe that the Hom-functors of $\underline{A}$ can be extended to a bifunctor

$$\mathrm{Hom}^* : C(\underline{A})^{opp} \times C(\underline{A}) \longrightarrow C(\underline{Ab})$$

by setting
$$\mathrm{Hom}^n(X^*, Y^*) = \prod_{p \in \mathbb{Z}} \mathrm{Hom}_{\underline{A}}(X^p, Y^{p+n})$$

$$d^n = \prod_{p \in \mathbb{Z}} (d_x^{p-1} + (-1)^{n+1} d_y^{p+n})$$

Under this definition, the n-cycles of the complex $\mathrm{Hom}^*(X^*, Y^*)$ are in a one-to-one correspondence with morphisms of X^* to $Y^*[n]$ and the n-boundaries corresponds to those morphisms which are homotopic to zero. Thus one has a natural isomorphism

$$(*) \qquad H^n(\mathrm{Hom}^*(X^*,Y^*)) \simeq \mathrm{Hom}_{K(\underline{A})}(X^*,Y^*[n])$$

which, together with Lemma a gives

Lemma b. For each injective I^* in $C^+(\underline{A})$ the functor $X^* \longrightarrow \mathrm{Hom}^*(X^*,I^*)$ preserves quisos in $C(\underline{A})$.

Therefore, assuming that $\underline{A}$ has enough injectives, one defines the Ext-groups by

$$\mathrm{Ext}^n(X^*,Y^*) = H^n(\mathrm{Hom}^*(X^*,I^*))$$

for X^* in $C(\underline{A})$ and an injective resolution $Y^* \longrightarrow I^*$.

For objects X, Y of $\underline{A}$ and an injective resolution $Y \Longrightarrow I^*$ of Y, $\mathrm{Ext}^n(X,Y) = H^n(\mathrm{Hom}^*(X,I^*)) = H^n(\mathrm{Hom}(X,I^*))$. Thus the $\mathrm{Ext}^n(X,Y)$ defined is the usual Ext.

Finally for any quiso $s : Y^* \Longrightarrow I^*$ of Y^* into a complex of injectives, by Lemma a

$$\mathrm{Hom}_{D(\underline{A})}(X^*,I^*[n]) \simeq \mathrm{Hom}_{K(\underline{A})}(X^*,I^*[n])$$

and (*)

$$\mathrm{Hom}_{K(\underline{A})}(X^*,I^*[n]) \simeq H^n(\mathrm{Hom}^*(X^*,Y^*))$$

one gets natural isomorphisms

$$\mathrm{Ext}^n(X^*,Y^*) \simeq \mathrm{Hom}_{D(\underline{A})}(X^*,Y^*[n])$$

for X^* in $C(\underline{A})$, Y^* in $C^+(\underline{A})$.

A P P E N D I X

1. CALCULAS OF LEFT - FRACTIONS

We use the axioms:

(FR0) S contains all identities and is closed under composition.

(FR1) Any $\begin{array}{ccc} X & & \\ s\Uparrow & & \\ X' & \longrightarrow & Y \end{array}$ with $s \in S$ can be completed

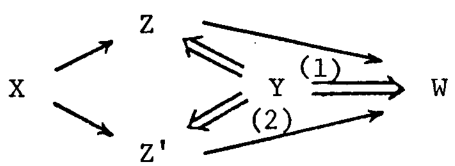
to a commutative square $\begin{array}{ccc} X & \longrightarrow & Y' \\ s\Uparrow & & \Uparrow t \\ X' & \longrightarrow & Y \end{array}$ with $t \in S$.

(FR2) Given $X \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Y$, suppose $\exists$ $s \in S$ with $\alpha s = \beta s$, then there is $t \in S$ such that $t\alpha = t\beta$.

<u>Definition</u>. A quasi-arrow from X to Y is a diagram

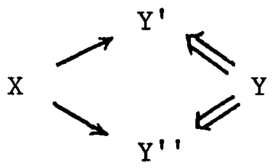
$$X \longrightarrow Z \Longleftarrow Y.$$

<u>Definition</u>. A kite in <u>C</u> is a diagram of the form



with triangles (1) and (2) commuting.

[1] Any diagram

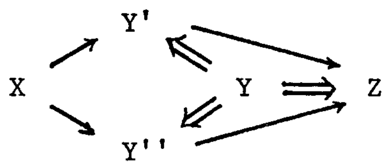
(*) 

can be completed to a kite.

Proof. An application of FR1 to $\begin{array}{ccc} Y'' & & \\ \Uparrow & & \\ Y & \Rightarrow & Y' \end{array}$

yields a commutative square $\begin{array}{ccc} Y'' & \longrightarrow & Z \\ \Uparrow & & \Uparrow \\ Y & \Longrightarrow & Y' \end{array}$.

Add on to (*)

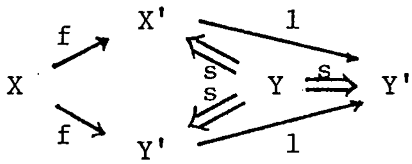


Definition. Two quasi-arrows are said to be "~"

if they fit into a kite whose edges form a commutative "square".

[2] "~" is an equivalence relation on the set of all quasi-arrows from X to Y.

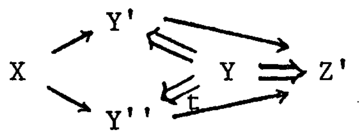
Proof. Symmetry is trivial; reflexivity follows from

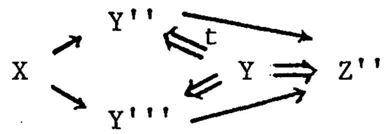


Transitivity: Assume $(X \rightarrow Y' \Leftarrow Y) \sim (X \rightarrow Y'' \overset{t}{\Leftarrow} Y)$

and $(X \rightarrow Y'' \overset{t}{\Leftarrow} Y) \sim (X \rightarrow Y''' \Leftarrow Y)$

then there are kites with equal edges:





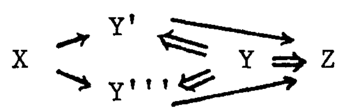
By FR1) $\exists$ a commutative square

$$\begin{array}{ccc} Z' & \longrightarrow & W \\ \Uparrow & & \Uparrow \\ Y & \Longrightarrow & Z'' \end{array} .$$

But since the maps $Y \overset{t}{\Longrightarrow} Y'' \rightrightarrows \begin{smallmatrix} Z' \\ Z'' \end{smallmatrix} \rightrightarrows W$ are equal,

$Y'' \rightrightarrows \begin{smallmatrix} Z' \\ Z'' \end{smallmatrix} \rightrightarrows W \Longrightarrow Z$ are also equal (for some Z by FR2).

It follows that the outer edges of

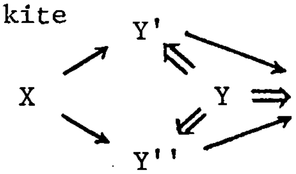


are equal.

<u>Definition</u>. An equivalence class of quasi-arrows for X to Y is called a quasi-morphism from X to Y.

We will denote quasi-morphisms by broken arrows $X \dashrightarrow Y$.

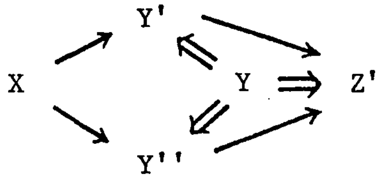
[3] Equality can be tested on any kite. That is, given a kite



with $(X \to Y' \Leftarrow Y) \sim (X \to Y'' \Leftarrow Y)$

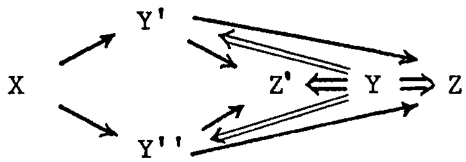
there is $Z \Rightarrow W$ equalising its outer edges.

Proof. By definition of equivalence, there is a kite



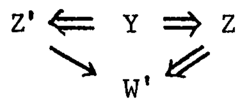
whose edges are equal, and we have

the diagram

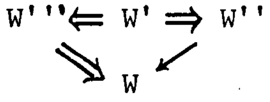


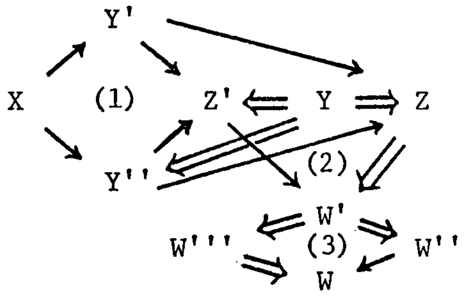
with the square on the left commutative.

By FR1 we get a commutative diagram

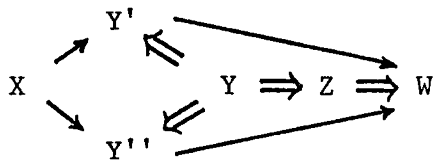


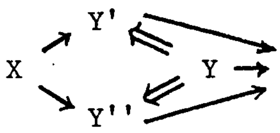
Since the morphism $Y \Rightarrow Y'$ equalizes $Y' \rightrightarrows {Z' \atop Z} \rightrightarrows W'$, there is a $W' \Rightarrow W''$ rendering the rows of $Y' \rightrightarrows {Z' \atop Z} \rightrightarrows W' \Rightarrow W''$ equal. Correspondingly the equalisation of $Y \Rightarrow Y''$ on the rows $Y'' \rightrightarrows {Z' \atop Z} \rightrightarrows W'$ induces a morphism $W' \Rightarrow W'''$ coequalising them; and an application of FR1 to $W''' \Leftarrow W' \Rightarrow W''$ yields

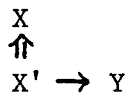

. Finally from

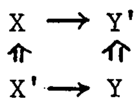
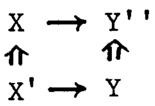
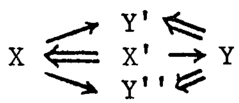


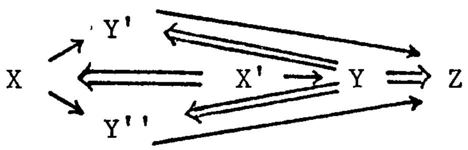
where (1) (2) and (3) are commutative, it follows that the kite


has equalised outer edges.

Note. [3] works for pseudo-kites as well. That is, kite-like diagrams  in which Y → W is arbitrary.

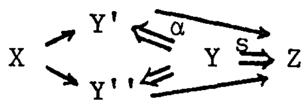
[4]. Any two quasi-arrows making a fixed  into commutative squares (by FR1) are equivalent.

Proof. Assuming  and  commute, we use [1] to fit  into a kite

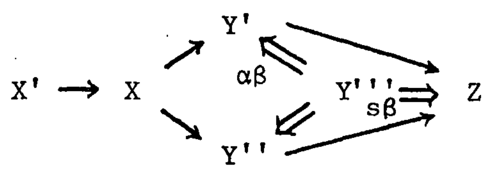


whose outer edges are equalised by $X' \Rightarrow X$; hence also by some $Z \Rightarrow W$.

<u>[5]</u>. If the quasi-arrows $X \begin{smallmatrix} \nearrow Y' \Leftarrow \\ \searrow Y'' \Leftarrow \end{smallmatrix} Y$ are equivalent, so are $X' \to X \begin{smallmatrix} \nearrow Y' \Leftarrow \\ \searrow Y'' \Leftarrow \end{smallmatrix} Y \overset{\beta}{\Leftarrow} Y'''$.

<u>Proof</u>. Any kite  whose outer edges are

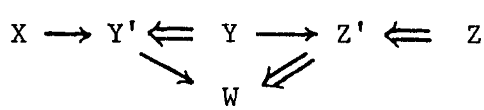
equalised produces by composition a kite



whose edges are trivially equalised.

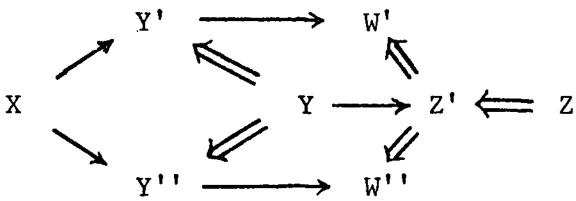
<u>[6]</u>. <u>Composition of quasi-morphisms</u>.

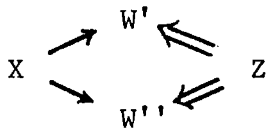
Given $X \to Y' \Leftarrow Y$, $Y \to Z' \Leftarrow Z$, we define their composite by applying FR1 to $Y' \Leftarrow Y \to Z'$

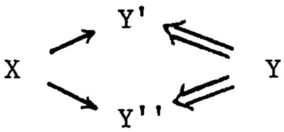


By [4], [5] this does not depend on W.

Dependence on Y': Suppose $X \longrightarrow Y' \Longleftarrow Y$ and $X \longrightarrow Y'' \Longleftarrow Y$ are equivalent with compositions

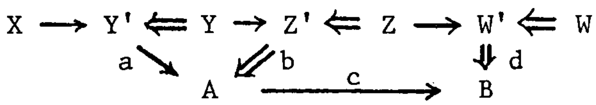


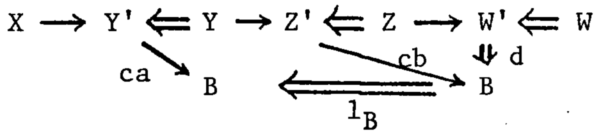
Then any kite on  gives a pseudo-kite on

 . The latter can be equalized by some $s \in S$.

Dependence on Z' can be analogously proved. Hence composition of quasi-morphisms is well defined.

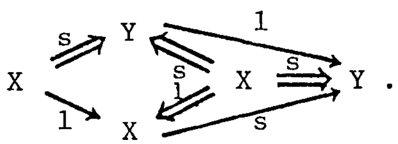
Associativity follows from the diagrams





The quasi-morphism $X \dashrightarrow X = X \overset{1}{\Rightarrow} X \overset{1}{\Leftarrow} X$ is the identity.

Note that for $s : X \Rightarrow Y$, $X \overset{s}{\Rightarrow} Y \overset{s}{\Leftarrow} X$ is also the identity

by 

<u>Definition</u>. We form the category $\underline{C}_S$:

$$\text{Ob } \underline{C}_S = \text{Ob } \underline{C}$$

$$\text{Hom}_{\underline{C}_S}(X,Y) = \text{set of all quasi-morphisms } X \dashrightarrow Y.$$

and define $Q : \underline{C} \longrightarrow \underline{C}_S$ by $Q(X \xrightarrow{f} Y) = X \xrightarrow{f} Y \overset{1}{\Longleftarrow} Y$. Then $Q(1_x) = 1_{Q(x)}$ and the diagram

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & Y & \overset{1}{\Longleftarrow} & Y & \xrightarrow{g} & Z & \overset{1}{\Longleftarrow} & Z \\ & & & g \searrow & & \swarrow\!\!\swarrow 1 & & & \\ & & & & Z & & & & \end{array}$$

shows that $Q(g.f) = Q(g).Q(f)$.

<u>[7]</u>. $Q(s)$ is an isomorphism for all $s \in S$.

<u>Proof</u>. Given $X \overset{s}{\Longrightarrow} Y$, put $t = Y \xrightarrow{1} Y \overset{s}{\Longleftarrow} X$.

$$Q(s)\cdot t = 1_Y \quad : \quad \begin{array}{ccccccccc} Y & \xrightarrow{1} & Y & \overset{s}{\Longleftarrow} & X & \overset{s}{\Longrightarrow} & Y & \overset{1}{\Longleftarrow} & Y \\ & & & 1 \searrow & & \swarrow\!\!\swarrow 1 & & & \\ & & & & Y & & & & \end{array}$$

$$t\cdot Q(s) = 1_X \quad \begin{array}{ccccccccc} X & \overset{s}{\Longrightarrow} & Y & \overset{1}{\Longleftarrow} & Y & \xrightarrow{1} & Y & \overset{s}{\Longleftarrow} & X \\ & & & 1 \searrow & & \swarrow\!\!\swarrow 1 & & & \\ & & & & Y & & & & \end{array}$$

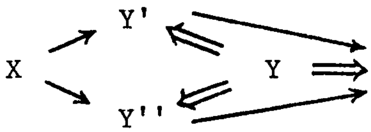
<u>[8]</u>. ($\underline{C}_S$, Q) is universal.

<u>Proof.</u> Given $G : \underline{C}_S \longrightarrow \underline{D}$ with $G(s)$ iso for all $s \in S$.

define $H: \underline{C}_S \longrightarrow \underline{D}$ by $H(X) = G(X)$. On quasi-morphism

$$X \dashrightarrow Y = X \longrightarrow Y' \overset{s}{\Longleftarrow} Y, \quad H(X \dashrightarrow Y) = G(X) \longrightarrow G(Y') \xrightarrow{G(s)^{-1}} G(Y) .$$

H is well-defined since equivalent quasi-arrows can be fitted into

a kite 

whose outer edges are equalised.

Suppose $H' : \underline{C}_S \longrightarrow \underline{D}$ is another functor satisfying $H' \cdot Q = G$, then H and H' agree on objects of $\underline{C}_S$. To see that they agree on quasi-morphisms $X \dashrightarrow Y = X \xrightarrow{f} Y' \overset{s}{\Longleftarrow} Y$, we have

$$\begin{aligned} H(X \xrightarrow{f} Y' \overset{s}{\Longleftarrow} Y) &= G(s)^{-1} \cdot G(f) = [H' \cdot Q(s)]^{-1} \cdot H'Q(f) \\ &= H'(Q(s)^{-1}) \cdot H'(Q(f)) \\ &= H'(Q(s)^{-1} \cdot Q(f)) \\ &= H'(X \xrightarrow{f} Y' \overset{s}{\Longleftarrow} Y) . \end{aligned}$$

Thus the existence of the localisation $(\underline{C}_S, Q)$ is established. As for its uniqueness, if $(\underline{C}'_S, Q')$ is another localisation then by the universality of $(\underline{C}_S, Q)$ there is a unique $H : \underline{C}_S \longrightarrow \underline{C}'_S$ satisfying $H \cdot Q = Q'$; similarly, there is a unique $H' : \underline{C}'_S \longrightarrow \underline{C}_S$ such that $H' \cdot Q' = Q$. Hence $H' \cdot H = 1_{\underline{C}_S}$ and $H \cdot H' = 1_{\underline{C}'_S}$.

<u>Remark</u>. Dually using the calculas of right fractions and FR0°, FR1°, FR2° , one can define quasi-morphisms $X \Longleftarrow Y' \longrightarrow Y$ and show that in case S is a left multiplicative system, the localisation $(\underline{C}_S, Q)$ also exists.

<u>Definition</u>. A category <u>I</u> is said to be filtered if :

L1). Every pair of objects of <u>I</u> can be embedded in a diagram

$$\begin{matrix} X & \searrow & \\ & & Z \\ Y & \nearrow & \end{matrix} \quad .$$

L2). Given $X \begin{matrix} \nearrow Y \\ \searrow Y' \end{matrix}$ in <u>I</u>, there exists $\begin{matrix} Y \searrow \\ Y' \nearrow \end{matrix} Z$ such that the square $X \begin{matrix} \nearrow Y \searrow \\ \searrow Y' \nearrow \end{matrix} Z$ commutes.

L3). Given a diagram $X \rightrightarrows Y$, there exists a map $Y \longrightarrow Z$ such that the two maps obtained by composition are the same.

If <u>I</u> is filtered, then <u>I</u> behaves as well as an inductive system for taking limits (<u>Grothendieck Topologies</u>, Chapter I).

[9]. For each object Y of <u>C</u>, we define a category I_Y:

Objects of I_Y are morphisms $Y \overset{s}{\Longrightarrow} X$ with $s \in S$.

A morphism in I_Y between two objects $Y \overset{s}{\Longrightarrow} X$ and $Y \overset{t}{\Longrightarrow} X'$ is a morphism $f : X \longrightarrow X'$ such that the diagram $\begin{matrix} X & \xrightarrow{f} & X' \\ {}_s\nwarrow & Y & \nearrow_t \end{matrix}$ commutes. We claim:

(1). I_Y is filtered.

(2). For objects X,Y in <u>C</u>, $\mathrm{Hom}_{\underline{C}_S}(X,Y) = \varinjlim_{Y' \in \mathrm{Ob}\, \underline{I}_Y} \mathrm{Hom}_{\underline{C}}(X,Y')$

(3). If $\underline{C}$ is additive, so is $\underline{C}_S$.

Proof.

(L1). For two objects $Y \Rightarrow X$ and $Y \Rightarrow X'$ apply (FR1) to $\begin{array}{ccc} X' & & \\ \Uparrow & & \\ Y & \Rightarrow & X \end{array}$

to get the commutative diagram $\begin{array}{ccc} X' & \longrightarrow & Z \\ \Uparrow & & \Uparrow \\ Y & \Rightarrow & X \end{array}$.

(L2). Given $\begin{array}{ccccc} & s \swarrow & Y & \searrow s' & \\ X & & \overset{f}{\underset{g}{\rightrightarrows}} & & X' \end{array}$ with $fs = gs = s'$ by (FR2) there exists

$X' \overset{h}{\Longrightarrow} Z$ such that $hf = hg$ $\quad \begin{array}{ccccc} & s \swarrow & Y & \searrow s' & \\ X & & \overset{f}{\underset{g}{\rightrightarrows}} & & X \end{array} \overset{h}{\Longrightarrow} Z$

therefore (L2) is satisfied.

(L3). Given $\begin{array}{ccc} Y & \Rightarrow & X' \\ \Downarrow & ② \searrow ① & \uparrow \\ X'' & \longleftarrow & X \end{array}$ with triangles (1) (2) commuting. Complete

$X \begin{array}{c} \nearrow X' \nwarrow \\ \Longleftarrow \\ \searrow X'' \swarrow \end{array} Y$ to a kite $X \begin{array}{c} \nearrow X' \nwarrow \\ \Longleftarrow \\ \searrow X'' \swarrow \end{array} Y \Rightarrow W$ and note that the

morphism $Y \Rightarrow X$ equalizes the pair of composites $X \begin{array}{c} \nearrow X' \searrow \\ \searrow X'' \nearrow \end{array} W$.

Therefore by (FR2) there is $W \Rightarrow Z$ such that the top and bottom

rows of $X \begin{array}{c} \nearrow X' \searrow \\ \searrow X'' \nearrow \end{array} W \Rightarrow Z$ are equal. Finally the diagram

$$Y \begin{array}{c} \nearrow\!\!\!\nearrow \; X' \; \searrow \\ \Rightarrow \; \overset{\uparrow}{\underset{\downarrow}{X}} \; \rightarrow \\ \searrow\!\!\!\searrow \; X'' \; \nearrow \end{array} Z$$

shows that (L3) is satisfied.

(2) is clear from the definition of $\varinjlim$

(3) Because $\mathrm{Hom}_{\underline{C}}(X,Y')$ is an abelian group for each Y',

$$\mathrm{Hom}_{\underline{C}_S}(X,Y) = \varinjlim_{I_{Y'}} \mathrm{Hom}_{\underline{C}}(X,Y') \text{ is also an abelian group.}$$

Remark.

If S is a left multiplicative system in $\underline{C}$, one can also calculate $\mathrm{Hom}_{\underline{C}_S}(X,Y)$ as $\varinjlim_{X' \in \mathrm{ob} J_X} \mathrm{Hom}_{\underline{C}}(X',Y)$ where J_X is the filtered category of objects $X' \Longrightarrow X$ and morphisms

$$\begin{array}{ccc} & X & \\ \nearrow\!\!\!\nearrow & & \nwarrow\!\!\!\nwarrow \\ X' & \longrightarrow & X'' \end{array} .$$

2. A NOTE ON F-ACYCLIC SUBCATEGORIES

The notion of a F-acyclic subcategory appears to leave much freedom of choice. In general, it is not necessarily unique. However, we show that there is always a maximal such subcategory.

Definition.

Let $F : \underline{C} \longrightarrow \underline{D}$ and $(\underline{R}F,\xi)$ the right approximate extension of F. An object of X of $\underline{C}$ is said to be right F-acyclic if and only if $\xi_X : F(X) \longrightarrow \underline{R}F(X)$ is an isomorphism.

Lemma. The full subcategory $\underline{I}^*$ of $\underline{C}$ consisting of all right F-acyclic objects is a right F-acyclic subcategory of $\underline{C}$ containing $\underline{I}$.

Proof.

(RAC1) Set $F^* = F|\underline{I}^*$, $S^* = S \cap A\underline{RI}^*$, $F_0 = F|\underline{I}$

For objects X, Y in $\underline{I}^*$ and a morphism $X \overset{s}{\Longrightarrow} Y$ in S^*.

We have

$$\begin{array}{ccc} X & \overset{\nu_X}{\Longrightarrow} & r(X) \\ {\scriptstyle s}\Downarrow & & \Big\downarrow{\scriptstyle \simeq\, \alpha} \\ Y & \overset{\nu_Y}{\Longrightarrow} & r(Y) \end{array} \qquad \begin{array}{ccccl} F(X) & \underset{\simeq}{\overset{\xi_X}{\dashrightarrow}} & F(rX) & = & \underline{R}F(X) \\ {\scriptstyle Q'F^*(s)}\Big\downarrow & & \Big\downarrow & {\scriptstyle F(\alpha)} & = \underline{E}F_0(\alpha) \\ F(Y) & \underset{\xi_Y}{\overset{\simeq}{\dashrightarrow}} & F(rY) & = & \underline{R}F(Y) \end{array}$$

$\underline{E}F_0(\alpha)$ is invertible in $\underline{D}_{S'}$, since α is invertible in $\underline{C}_S$. Hence $Q'F^*(s)$ is invertible in $\underline{D}_{S'}$, so F^* is (S^*, S') exact.

(RAC2) In order to show each X in $\underline{C}$ admits $X \overset{s}{\Longrightarrow} I$ with $s \in S^*$ and $I \in ob\underline{I}^*$, it suffices to show $\underline{I} \subset \underline{I}^*$ since $\underline{I}$ has this property. But for X in $\underline{I}$, and $X \overset{\nu_X}{\Longrightarrow} r(X)$ with $\nu_X \in S_0 = S \cap AR\underline{I}$, F_0 is (S_0, S') exact. Therefore, $\xi_X = Q'(F_0(\nu_X))$ is an iso in $\underline{D}_{S'}$, and $\underline{I} \subset \underline{I}^*$.

3. QUASI-SPLIT SEQUENCES

A short exact sequence

$$\Sigma \ : 0 \longrightarrow X^* \xrightarrow{i} C^* \xrightarrow{p} Y^* \longrightarrow 0$$

in $C(\underline{A})$ is called pseudo or quasi split if each of the short exact sequences $0 \longrightarrow X^n \longrightarrow C^n \longrightarrow Y^n \longrightarrow 0$ splits i.e., $C^n \simeq X^n \oplus Y^n$.

[1] . Let us see how the complex C^* in this case may be described in terms of X^* and Y^*.

The maps i and p can be written as matrices:

$$i = \begin{pmatrix} 1_X \\ 0 \end{pmatrix} \qquad p = \begin{pmatrix} 0, 1_Y \end{pmatrix}$$

and the differential operator ∂_C of C^* may be represented by

$$\partial_C^n = \begin{pmatrix} \alpha^n & \beta^n \\ \gamma^n & \delta^n \end{pmatrix} : X^n \oplus Y^n \longrightarrow X^{n+1} \oplus Y^{n+1}$$

where $\alpha^n : X^n \longrightarrow X^{n+1}$ $\qquad \beta^n : Y^n \longrightarrow X^{n+1}$

$\gamma^n : X^n \longrightarrow Y^{n+1}$ $\qquad \delta^n : Y^n \longrightarrow Y^{n+1}$

From $\partial_C i = i\partial_X$, $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1_X \\ 0\end{pmatrix} = \begin{pmatrix}1_X \\ 0\end{pmatrix}\partial_X$ We get

$$\begin{pmatrix}\alpha \\ \gamma\end{pmatrix} = \begin{pmatrix}\partial_X \\ 0\end{pmatrix}.$$

From $p\partial_C = \partial_Y p$, $(0,1_Y)\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \partial_Y(0,1_Y)$ we get $(\gamma,\delta) = (0,\partial_Y)$.

Therefore, ∂_C has the form $\partial_C = \begin{pmatrix}\partial_X & \beta \\ 0 & \partial_Y\end{pmatrix}$

Moreover from $\partial_C^2 = 0$, $\begin{pmatrix}\partial_X & \beta \\ 0 & \partial_Y\end{pmatrix}\begin{pmatrix}\partial_X & \beta \\ 0 & \partial_Y\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$

We get $\partial_X\beta + \beta\partial_Y = 0.$

Thus β is a complex map $Y^* \longrightarrow X^*[1]$.

β depends not only on the given quasi-split sequence Σ , but also on the sections $S^n : Y^n \longrightarrow C^n$ used to decompose C^n. In our matrix notation, we have used $S^n = \begin{pmatrix}0 \\ 1\end{pmatrix} : Y^n \longrightarrow C^n$ and

$$\partial_C S - S\partial_Y = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} - \begin{pmatrix}0 \\ 1\end{pmatrix}\partial_Y = \begin{pmatrix}\beta \\ \delta\end{pmatrix} - \begin{pmatrix}0 \\ \partial_Y\end{pmatrix} = \begin{pmatrix}\beta \\ 0\end{pmatrix}$$

Now, any other section must be of the form

$$S' = \begin{pmatrix}h \\ 1\end{pmatrix} \quad \text{where } h^n : Y^n \longrightarrow X^n$$

and induces $\beta' : Y^* \longrightarrow X^*[1]$ also satisfying

$$\begin{pmatrix} \beta' \\ 0 \end{pmatrix} = \partial_C S' - S'\partial_Y$$

$$\begin{pmatrix} \beta' - \beta \\ 0 \end{pmatrix} = \partial_C(S' - S) - (S' - S)\partial_Y = \begin{pmatrix} \partial_X h - h\partial_Y \\ 0 \end{pmatrix}$$

Hence $\beta - \beta' = h\partial_Y - \partial_X h$

Taking into account of the different signs of the differentials of X^* and $X^*[1]$, we see that

$$\beta : Y^* \longrightarrow X^*[1]$$

is determined up to homotopy by the given quasi-split sequence.

<u>Definition.</u> The homotopy class of maps $\beta : Y^* \longrightarrow X^*[1]$ associated with a quasi-split sequence

$$\Sigma : \quad 0 \longrightarrow X^* \longrightarrow C^* \longrightarrow Y^* \longrightarrow 0$$

in the manner described is called the twist of the sequence.

<u>[2]</u>. If $\Sigma : 0 \longrightarrow X^* \xrightarrow{i} C^* \xrightarrow{p} Y^* \longrightarrow 0$ quasi-splits with twist $\beta : Y^* \longrightarrow X^*[1]$, then applying 0-cohomology to the sequence

$$0 \to X^* \xrightarrow{i} C^* \xrightarrow{p} Y^* \xrightarrow{\beta} X^*[1] \xrightarrow{T(i)} C^*[1] \xrightarrow{T(p)} Y^*[1] \xrightarrow{T(\beta)} X^*[2] \to$$

and noting that the diagram

$$\begin{array}{ccccccccccc}
\cdots\to H^n(X^*) & \xrightarrow{H^n(i)} & H^n(C^*) & \xrightarrow{H^n(p)} & H^n(Y^*) & \xrightarrow{\delta^n_\Sigma} & H^{n+1}(X^*) & \xrightarrow{H^{n+1}(i)} & H^{n+1}(C^*) & \to \cdots \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
\cdots\to H^0(X^*[n]) & \xrightarrow[H^0(T^n i)]{} & H^0(C^*[n]) & \xrightarrow[H^0(T^n p)]{} & H^0(Y^*[n]) & \xrightarrow[H^0(T^n \beta)]{} & H^0(X^*[n+1]) & \xrightarrow[H^o(T^{n+1} i)]{} & H^0(C^*[n+1]) & \to \cdots
\end{array}$$

is commutative for every integer n, and since all the isomorphisms are natural, we conclude that the connecting morphism δ^n_Σ in cohomology is induced by the twist.

<u>[3]</u> Given $\Sigma : 0 \longrightarrow X^* \xrightarrow{i} C^* \xrightarrow{p} Y^* \longrightarrow 0$ quasi-split with twist β, then:

(a) β is homotopic to 0 iff the sequence splits, in this case, the section is $\begin{pmatrix} -h \\ 1 \end{pmatrix}$ where $h : \beta \sim 0$.

(b) If X^* is contractible with $h : 1_X \sim 0$ then p has a homotopy inverse given by $p^{-1} = \begin{pmatrix} -h\beta \\ 1 \end{pmatrix}$.

<u>Proof</u>.

(a) Let $h : \beta \sim 0, \quad -\partial^n_X h^n + h^{n+1}\partial^n_Y = \beta^n$ for all n.

Define $$S^n = \begin{pmatrix} -h^n \\ 1_{Y^n} \end{pmatrix} : Y^n \longrightarrow C^n .$$

Then $(p\cdot S)^n = 1_{Y^n}$ and $S : Y^* \longrightarrow C^*$ is a chain map since

$$\begin{pmatrix} \partial_X^n & \beta^n \\ 0 & \partial_Y^n \end{pmatrix}\begin{pmatrix} -h^n \\ 1 \end{pmatrix} = \begin{pmatrix} -\partial_X^n h^n + \beta^n \\ \partial_Y^n \end{pmatrix} = \begin{pmatrix} -h^{n+1}\partial_Y^n \\ \partial_Y^n \end{pmatrix}$$

$$= \begin{pmatrix} -h^{n+1} \\ 1 \end{pmatrix}\partial_Y^n .$$

Conversely if Σ quasi-splits with section $S = \begin{pmatrix} -h \\ 1 \end{pmatrix}$, then

from $$\begin{pmatrix} \partial_X^n & \beta^n \\ 0 & \partial_Y^n \end{pmatrix}\begin{pmatrix} -h^n \\ 1 \end{pmatrix} = \begin{pmatrix} -h^{n+1} \\ 1 \end{pmatrix}\partial_Y^n$$

we get $-\beta^n = h^{n+1}\partial_Y^n - \partial_X^n h^n$ i.e. $h : \beta \simeq 0$.

(b). From the homotopy commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^* & \xrightarrow{i} & C^* & \xrightarrow{p} & Y^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow 1 & & \downarrow 1 & & \\ 0 & \longrightarrow & 0 & \longrightarrow & C^* & \xrightarrow{p} & Y^* & \longrightarrow & 0 \end{array}$$

we see that p is an isomorphism in $K(\underline{A})$.

Because $h : 1_X \simeq 0$, $-1_{X^n} = \partial_X^n h^{n+1} + h^n \partial_X^{n+1}$

$$-\beta^n = \partial_X^n h^{n+1}\beta^n + h^n\partial_X^{n+1}\beta^n$$

Substituting $\partial_X^{n+1}\beta^n = -\beta^{n+1}\partial_Y^n$

we get $-\partial_X^n h^{n+1} \beta^n + h^{n+2} \beta^{n+1} \partial_Y^n = \beta^n$

Hence $h\cdot\beta$ is a homotopy from β to 0, and by (a), $p^{-1} = \binom{-h\beta}{1}$.

[4]. For any complex X^*, the mapping cone $\pm\, id_{X^*}$ is contractible with homotopy $h = \begin{pmatrix} 0 & 0 \\ \pm 1_{X^*} & 0 \end{pmatrix}$.

Proof.

$$\begin{pmatrix} 0 & 0 \\ \pm 1_{X^*} & 0 \end{pmatrix}\begin{pmatrix} \partial_{X^*} & \pm 1_{X^*} \\ 0 & -\partial_{X^*} \end{pmatrix} + \begin{pmatrix} \partial_{X^*} & \pm 1_{X^*} \\ 0 & -\partial_{X^*} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \pm 1_{X^*} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1_{X^*} & 0 \\ \mp\partial_X & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pm\partial_X & 1_{X^*} \end{pmatrix} = \begin{pmatrix} 1_{X^*} & 0 \\ 0 & 1_{X^*} \end{pmatrix}$$

[5]. If $\Sigma : 0 \longrightarrow X_0^* \xrightarrow{i} X_1^* \xrightarrow{p} X_2^* \longrightarrow 0$ quasi-splits with twist β and section $S = \binom{0}{1}$ then $\pi : C_{-i}^* \longrightarrow X_2^*$ has a homotopy inverse given by $\rho = \binom{S}{\beta}$.

Proof.

$$X_1^n = X_0^n \oplus X_2^n \qquad \partial_1^n = \begin{pmatrix} \partial_0^n & \beta^n \\ 0 & \partial_2^n \end{pmatrix}$$

$$i^n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : X_0^n \longrightarrow X_1^n$$

$$C_{-i}^n = X_0^n \oplus X_2^n \oplus X_0^{n+1}$$

$$\partial_{C_{-i}^*}^n = \begin{pmatrix} \partial_0^n & \beta^n & -1_{X_0^{n+1}} \\ 0 & \partial_2^n & 0 \\ 0 & 0 & \partial_0^{n+1} \end{pmatrix}$$

Set $\alpha : X_2^*[-1] \longrightarrow C_{-id_{X_0^*}}^*$ $\qquad \alpha^n = \begin{pmatrix} \beta^{n-1} \\ 0 \end{pmatrix}$.

Then $C_\alpha^n = X_0^n \oplus X_0^{n+1} \oplus X_2^n$

$$\partial_{C_\alpha^*}^n = \begin{pmatrix} \partial_0^n & -id_{X_0^{n+1}} & \beta^n \\ 0 & -\partial_0^{n+1} & 0 \\ 0 & 0 & \partial_2^n \end{pmatrix}$$

and the map $\xi : C_{-i}^* \longrightarrow C_\alpha^*$ defined by $\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is an isomorphism with inverse $\xi^{-1} = \xi$. We have a commutative diagram

$$C_{-i}^* \xrightarrow[(p,0)]{\pi} X_2^*, \qquad C_{-i}^* \xrightarrow[\cong]{\xi} C_\alpha^* \xrightarrow{\omega} X_2^*$$

where ω is defined by

$$0 \longrightarrow C^*_{-id_{X_0^*}} \longrightarrow C^*_\alpha \xrightarrow{\omega} \underset{\displaystyle X_2^*}{\underset{\|}{X_2^*[-1]([1])}} \longrightarrow 0$$

$C^*_{-1_{X_0}}$ is contractible with homotopy $h = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$: $1d_{C^*_{-X_0}} \sim 0$

By [3b] we have $\omega^{-1} = \begin{pmatrix} -h\alpha \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 1 \end{pmatrix}$. Thus $\pi^{-1} = \xi^{-1} \cdot \omega^{-1}$ $= \begin{pmatrix} 0 \\ 1 \\ \beta \end{pmatrix} = \begin{pmatrix} S \\ \beta \end{pmatrix}$.

4 QUISOS AND THE DERIVED CATEGORY

A quiso or quasi-isomorphism is a co-chain map or homotopy class of one , which induces an isomorphism in cohomology.

[1]. The class of quisos in $K^*(\underline{A})$ form a right and left multiplicative system.

Proof.

(FR0) is clear.

(FR1): Given
$$\begin{array}{ccc} X^* & & \\ s\Uparrow & & \\ Y^* & \xrightarrow{f} & Z^* \end{array}$$
, make

$$\begin{array}{ccc} C_i^* \simeq X^*[1] & & C_g^* \\ \Uparrow T(s) & & t\uparrow \\ Y^*[1] & \xrightarrow{T(f)} & Z^*[1] \\ i\uparrow & & \uparrow\ T(f)i = g \\ C_s^* & \xrightarrow{1} & C_s^* \\ \uparrow & \nearrow & \\ X^* & & \end{array}$$

By proposition 7(or TR3) there exists a morphism $h : X^*[1] \longrightarrow C_g^*$

completing the square
$$\begin{array}{ccc} X^*[1] & \xrightarrow{h} & C_g^* \\ T(s)\Uparrow & & \uparrow t \\ Y^*[1] & \xrightarrow{T(f)} & Z^*[1] \end{array}$$

From the long exact sequence

$$\ldots \to H^n(C_s^*) \xrightarrow{H^n(g)} H^n(Z^*[1]) \xrightarrow{H^n(t)} H^n(C_g^*) \longrightarrow H^n(C_s^*[1]) \longrightarrow \ldots$$

and the fact that $H^n(C_s^*) = 0$, for all n, we see that t is a quiso.

(FR2): Given $X^* \overset{s}{\Longrightarrow} Y^* \underset{g}{\overset{h}{\rightrightarrows}} Z^*$ with $hs = gs$. To prove FR2 we consider the morphism $f = h - g$ and reduce to showing that for any $X^* \overset{s}{\Longrightarrow} Y^* \xrightarrow{f} X^*$ $fs = 0$, there is a quiso t such that $tf = 0$. Because $\mathrm{Hom}_{K(\underline{A})}(-, Z^*)$ is exact on the sequence

$$X^* \overset{s}{\Longrightarrow} Y^* \longrightarrow C_s^* \longrightarrow X^*[1] \longrightarrow \ldots$$

and $fs = 0$, there exists a morphism $g : C_s^* \longrightarrow Z^*$ completing (1) of the diagram

$$\begin{array}{ccccc} X^* & \overset{s}{\Longrightarrow} & Y^* & \xrightarrow{f} & Z^* \\ & & {\scriptstyle j}\searrow & (1) & \nearrow{\scriptstyle g} \\ & & & C_s^* & \end{array}$$

Now take $t : Z^* \longrightarrow C_g^*$. Then from the exact sequence

$$\ldots \to H^n(C_s^*) \xrightarrow{H^n(g)} H^n(Z^*) \xrightarrow{H^n(t)} H^n(C_g^*) \longrightarrow H^n(C_s^*[1]) \longrightarrow \ldots$$

it follows that t is a quiso.

Dually we can prove FR0° - FR2° .

We define the derived category $D(\underline{A})$ of $\underline{A}$ as the localisation of $K(\underline{A})$ with respect to quisos. Similarly, $D^*(\underline{A})$, $D^+(\underline{A})$, $D^-(\underline{A})$, $D^b(\underline{A})$ are the localisations of $K^*(\underline{A})$, $K^+(\underline{A})$, $K^-(\underline{A})$ and $K^b(\underline{A})$

respectively. [1] gives us a good hold on $D^*(\underline{A})$ in the sense that it can be handled as a category of left or right fractions (using FR0-FR2 or FR0°-FR2°).

[2].

(a). Let S be a multiplicative system in a category $\underline{C}$. Let $\underline{D}$ be a full subcategory of $\underline{C}$ and assume that $S \cap \mathrm{Mor}\underline{D}$ is a multiplicative system in $\underline{D}$, assume further that one of the following two conditions hold:

i). Whenever $X' \overset{s}{\Longrightarrow} X$ with $s \in S$ and $X \in \mathrm{Ob}\ \underline{D}$, there exists $X'' \xrightarrow{f} X'$ such that $sf \in S$ and $X' \in \mathrm{Ob}\ \underline{D}$.

ii). Whenever $X' \overset{s}{\Longrightarrow} X$ with $s \in S$ and $X' \in \mathrm{Ob}\ \underline{D}$, there exists $X \xrightarrow{f} X''$ such that $fs \in S$ and $X'' \in \mathrm{Ob}\ \underline{D}$.

then $\underline{D}_{S \cap \mathrm{Mor}\ \underline{D}}$ can be identified with a full subcategory of $\underline{C}_S$.

(b). Each of the functors

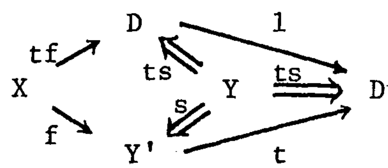
$$\begin{array}{ccccccc} & & & \nearrow & D^+(\underline{A}) & \searrow & \\ \underline{A} & \longrightarrow & D^b(\underline{A}) & & & & D(\underline{A}) \\ & & & \searrow & D^-(\underline{A}) & \nearrow & \end{array}$$

is a full embedding.

Proof.

(a): We prove only for right multiplicative systems and ii). The other cases are similar.

Given a quasi-morphism $X \xrightarrow{f} Y' \overset{s}{\Longleftarrow} Y$ with $X, Y \in \mathrm{Ob}\ \underline{D}$.
By hypothesis, there exists $t : Y' \longrightarrow D$ such that $ts \in S$ and $D \in \mathrm{Ob}\ \underline{D}$. From the kite



We conclude that $(X \longrightarrow Y' \Longleftarrow Y) \in \mathrm{Hom}_{\underline{D}_{S \cap \mathrm{Mor}\underline{D}}}(X,Y)$.

(b) (i) $D^+(\underline{A}) \longrightarrow D(\underline{A})$ is a full embedding.

Let $X^* \overset{s}{\Longrightarrow} Y^*$ be a quiso with $X^* \in K^+(\underline{A})$ and $m \in \mathbb{Z}$ satisfying $X^n = 0$ for all $n < m$.

Set : $Z^* = (\rightarrow 0 \rightarrow \mathrm{im}\,d_Y^{m-1} \rightarrow Y^m \xrightarrow{d_Y^m} Y^{m+1} \xrightarrow{d_Y^{m+1}} Y^{m+2} \rightarrow ..)$

$f : Y^* \longrightarrow Z^* \qquad f^n = \begin{cases} 1_{Y^n} & n \geq m \\ d_Y^{n-1} & n = m-1 \\ 0 & n < m \end{cases}$

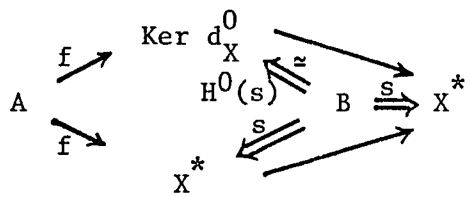
$$\begin{array}{cccccccccccc}
X^* & : & .. \rightarrow 0 & \longrightarrow & 0 & \longrightarrow & X^m & \xrightarrow{d_X^m} & X^{m+1} & \xrightarrow{d_X^{m+1}} & X^{m+2} & \longrightarrow .. \\
\Downarrow s & & \downarrow 0 & & \downarrow 0 & & \downarrow s^m & & \downarrow s^{m+1} & & \downarrow s^{m+2} & \\
Y^* & : & .. \rightarrow Y^{m-2} & \rightarrow & Y^{m-1} & \rightarrow & Y^m & \rightarrow & Y^{m+1} & \rightarrow & Y^{m+2} & \longrightarrow .. \\
\downarrow f & & \downarrow & & \downarrow & & \downarrow d_Y^{m-1} & & \downarrow 1 & & \downarrow 1 & \\
Z^* & : & .. \rightarrow 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{im}\,d_Y^{m-1} & \rightarrow & Y^{m+1} & \rightarrow & Y^{m+2} & \longrightarrow ..
\end{array}$$

Thus $H^n(fs) : H^n(X^*) \simeq H^n(Z^*)$ for all n; $fs \in S$, and by (a), we are done.

(ii) $\underline{A} \longrightarrow D^+(\underline{A})$

Let $A \xrightarrow{f} X^* \overset{s}{\Longleftarrow} B$ be a quasi-morphism where $A, B \in Ob\underline{A}^*$, $X^* \in Ob\ D^b(\underline{A})$. Since s is a quiso we may assume X^* is truncated and looks like $.. \longrightarrow 0 \longrightarrow X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \rightarrow ..$ with $H^0(s) : B \simeq Ker\ d_X^0$.

Now from



we see that $X \xrightarrow{f} X^* \overset{s}{\Longleftarrow} B$ can be replaced by $A \xrightarrow{f} Ker\ d_X^0 \xleftarrow{\simeq} B$, which is a morphism in $\underline{A}$.

[3] The following is a substitute for "Cartan-Eilenberg" resolutions :

Let $\underline{I}$ be a full subcategory of $\underline{A}$ such that every object of $\underline{A}$ admits a monomorphism into an object of $\underline{I}$, then every object of $C^+(\underline{A})$ admits a quiso into an object of $C^+(\underline{I})$.

Proof.

We may assume $X^* \in C^+(\underline{A})$ is trivial in negative dimensions.

$$X^* = .. \longrightarrow 0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow ...$$

Define $I^n = 0$, $s^n = 0$, and $d_I^n = 0$ $\forall\, n<0$, and imbed $X^0 \overset{s^0}{\rightarrowtail} I^0$ with $I^0 \in ob\underline{I}$.

Assume I^n, s^n exist with

$d_I^n\, d_I^{n-1} = 0 \;\forall\, n \leq k-1$ and $s^n d_X^{n-1} = d_I^{n-1} s^{n-1}$ $\forall\, n \leq k$.

$$\begin{array}{ccccccccccccccc}
\cdots\to & 0 & \to & X^0 & \to & \cdots\cdots & \to & X^{k-2} & \to & X^{k-1} & \xrightarrow{d_X^{k-1}} & X^k & \xrightarrow{d_X^k} & X^{k+1} & \to \cdots \\
 & \downarrow & & \downarrow s^0 & & & & \downarrow s^{k-2} & & \downarrow s^{k-1} & & \downarrow s^k & & & \\
\cdots\to & 0 & \to & I^0 & \to & \cdots\cdots & \to & I^{k-2} & \to & I^{k-1} & \xrightarrow[d_I^{k-1}]{} & I^k & & &
\end{array}$$

For $n = k+1$ let $(P^{k+1}, \beta^k, \alpha^{k+1})$ be defined by the push-out diagram

$$\begin{array}{ccccccc}
 & X^k & \xrightarrow{d_X^k} & X^{k+1} & \longrightarrow & X^{k+2} & \to \cdots \\
 & s^k\downarrow & \text{pushout} & & \searrow \alpha^{k+1} & & \\
 & I^k & & & & & \\
 & \pi^k \downarrow & & & & & \\
I^k/\text{im } d_I^{k-1} = & \text{cok } d_I^{k-1} & \xrightarrow{\beta^k} & P^{k+1} & & &
\end{array}$$

Now imbed $P^{k+1} \overset{\gamma^{k+1}}{\rightarrowtail} I^{k+1}$ and set $s^{k+1} = \gamma^{k+1}\alpha^{k+1}$, $d_I^k = \gamma^{k+1}\beta^k \cdot \pi^k$.

Then, $d_I^k d_I^{k-1} = 0$, $s^{k+1} d_x^k = \gamma^{k+1} \cdot \alpha^{k+1} d_X^k = \gamma^{k+1} \cdot \beta^k \cdot \pi^k \cdot s^k = d_I^k \cdot s^k$.

Hence, by induction we obtain $s : X^* \longrightarrow I^*$.

To see that s is a quiso, we recall that if:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \beta\downarrow & & \downarrow\beta' \\ C & \xrightarrow{\alpha'} & D' \end{array}$$

is a pushout in $\underline{A}$, then 1) β induces an epi : $\ker\alpha \twoheadrightarrow \ker\alpha'$.

2) β' induces a mono : $\operatorname{cok}\alpha \rightarrowtail \operatorname{cok}\alpha'$.

In our situation we have for each k

1) $\pi^k s^k$ induces epi's

$$Z^k(X^*) = \ker d_X^k \twoheadrightarrow \ker\beta^k \simeq \ker\gamma^{k+1}\beta^k \simeq \ker d_I^k \big/ \operatorname{im} d_I^{k-1} = H^k(I^*).$$

2) α^{k+1} induces monos:

$$\operatorname{cok} d_X^k \rightarrowtail \operatorname{cok}\beta^k \simeq \operatorname{cok}\beta^k\pi^k \rightarrowtail \operatorname{cok}(\gamma^{k+1}\beta^k\pi^k) = \operatorname{cok} d_I^k.$$

From the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^k(X^*) & \longrightarrow & \operatorname{cok} d_X^k & \longrightarrow & \operatorname{coim} d_X^{k+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^k(I^*) & \longrightarrow & \operatorname{cok} d_I^k & \longrightarrow & \operatorname{coim} d_I^{k+1} & \longrightarrow & 0 \end{array}$$

We see that $H^k(X^*) \rightarrowtail H^k(I^*)$ is monic.

Hence, $H^k(X^*) \longrightarrow H^k(I^*)$ is iso for all k.

5. EXT

[1] Let $\underline{A}$ be an abelian category and $f : X^* \longrightarrow I^*$ be a morphism in $C(\underline{A})$. Assume:

1) X^* is acyclic i.e. $H^n(X^*) = 0 \; \forall n$,
2) each I^n is injective,
3) I^* is bounded below,

then f is homotopic to zero.

Proof. Well known, (by induction).

[2] Let $\underline{A}$ be an abelian category and $s : I^* \longrightarrow Y^*$ a morphism in $C(\underline{A})$. Assume:

1) s is a quasi-isomorphism,
2) each I^n is injective,
3) I^* is bounded below,

then s has a homotopy inverse.

Proof. (from R. Hartshorne's Residues and Duality)

The mapping cone C_s^* is acyclic. The morphism $\nu = (1d,0) : C_s^* \longrightarrow I^*[1]$ satisfies [1] and is therefore homotopic to 0.

Let $H \equiv (k,t) : I^*[1] \oplus Y^* \longrightarrow I^*$ be the homotopy operator.

$$\nu = (1d,0) = (k,t)\partial_{C_S^*} + \partial_I(k,t)$$

Separating components, we have

$$1d_{I^*[1]} = T(\partial_I)k + kT(\partial_I) + t\cdot T(s)$$

$$\text{and} \quad \partial_Y t - t\partial_Y = 0.$$

Thus $t : Y^* \longrightarrow I^*$ is a morphism of complexes and $1d_{I^*}$ is homotopic to $t\cdot s$, so t is a homotopy inverse of s.

References

Artin, M. Grothendieck Topologies, mimeographed seminar notes, Harvard University, 1962.

Bredon, G.E., Sheaf Theory, McGraw-Hill, New York, 1967.

Cartan, H. and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, N.J., 1956.

Gabriel, P., and M. Zisman, Calculas of Fractions and Homotopy Theory, Springer-Verlag New York Inc., 1967.

Gamst, J. and K. Hoechsmann, Products in Sheaf Cohomology I, II.

Hartshorne, R., Residues and Duality, Lecture notes in mathematics, no. 20, Springer-Verlag, 1966.

Hoechsmann, K., Notes on Derived Categories and Functors, (unpublished.)

MacLane, S., Homology, Academic Press, New York, 1963.

Mitchell, B., Theory of Categories, Academic Press, New York, 1965.

Spanier, E.H., Algebraic Topology, McGraw-Hill, New York, 1966.

Swan, R., The Theory of Sheaves, The University of Chicago Press, Chicago, 1964.