THE MOORE SPECTRAL SEQUENCE FOR PRINCIPAL FIBRATIONS

by

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ABSTRACT

A proof of the Moore theorem which in the case of a principal fibration gives a spectral sequence converging to the homology of the base space is given. Also computed is the algebra structure of the homology of the Grassmannians, using Hopf algebra techniques and the cohomology of Grassmannians. Finally, it is shown that a spectral sequence for regular covering which was constructed earlier is a special case of the Moore Theorem.
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1. Differential Graded Algebras, Modules and Tor [5], [11]

A graded module is a collection of modules \( \{A_n\}_{n \geq 0} \) over a commutative ring \( K \) with unit. A differential graded module or a chain complex (DG module for short) \( \{A_n, d_n\}_{n \geq 0} \) is a graded module \( \{A_n\}_{n \geq 0} \) together with morphisms of modules \( d_n : A_n \to A_{n-1} \) such that \( d_{n-1}d_n = 0 \).

A morphism \( f : \{A_n\}_{n \geq 0} \to \{B_n\}_{n \geq 0} \) of graded modules is a collection \( \{f_n\}_{n \geq 0} \) of morphisms of modules \( f_n : A_n \to B_n \). A morphism \( f : \{A_n, d_n\} \to \{B_n, d'_n\} \) of DG modules is a collection \( \{f_n\}_{n \geq 0} \) of morphisms of modules such that \( d'_n f_n = f_{n-1} d_n \).

If \( A = \{A_n, d_n\} \) and \( B = \{B_n, d'_n\} \) are DG modules then a new DG module \( A \otimes B \) is defined by \( (A \otimes B)_n = \sum_{i+j=n} A_i \otimes B_j \) and \( d_n (a \otimes b) = d_i a \otimes b + (-1)^i a \otimes d'_j b \) for \( a \in A_i, b \in B_j \). The ring \( K \) itself can be considered as a DG module by \( k_o = K \) and \( k_n = 0 \) for \( n > 0 \) and \( d_n = 0 \) for all \( n \). Then we have \( A \otimes K = K \otimes A = A \) for any DG module \( A \).

A DG module \( A \) is called a DG algebra if there are morphisms \( \phi : A \otimes A \to A \) and \( \eta : K \to A \) of DG modules, such that the following diagrams commute.

![Diagram](https://via.placeholder.com/150)

Let \( A \) be a DG algebra, a DG module \( M \) is called a DG left \( A \)-module if
there is a morphism \( \phi_M : A \otimes M \to M \) of DG modules such that the following diagrams commute:

A DG right \( A \)-module is defined similarly. If \( M, N \) are two DG left \( A \)-modules, a morphism \( f : M \to N \) of DG modules is called a morphism of DG left \( A \)-modules if the following diagram commutes.

A morphism of DG right \( A \)-modules is defined similarly.

Let \( M, M', M'' \) be graded modules. A diagram \( M' \xrightarrow{f} M \xrightarrow{g} M'' \) of morphisms of graded modules is called exact if each \( M' \xrightarrow{f_n} M_n \xrightarrow{g_n} M''_n \) is exact as morphisms of modules (i.e. \( \ker g_n = \text{Im} f_n \)).
A diagram $\begin{array}{c} M' \xrightarrow{f} M \xrightarrow{g} M'' \end{array}$ of DG modules and their morphisms is called proper exact if $M', M, M''$ and $Z(M'), Z(M), Z(M''), Z(M), Z(M''), Z(M''), Z(M'), Z(M), Z(M''), Z(M''), Z(M'')$ are all exact as graded modules, where $Z$ denotes the graded module of cycles and $H$ the graded module of the homology of a DG module. Exactness for sequences of DG modules is defined similarly.

A graded module $P$ is called projective if any diagram

$\begin{array}{c} P \xrightarrow{f} M \xrightarrow{\pi} N \xrightarrow{\gamma} 0 \end{array}$

of morphisms of graded modules, where $M \xrightarrow{\pi} N \xrightarrow{\gamma} 0$ is exact, can be completed (i.e. there exists a morphism $g : P \to M$ such that $\pi \circ g = f$).

A DG module $P$ is called a proper projective DG module if any diagram of DG modules and DG morphisms with a proper exact row

$\begin{array}{c} P \xrightarrow{f} M \xrightarrow{\pi} N \xrightarrow{\gamma} 0 \end{array}$

can be completed.

A DG left $A$-module $P$ is called a proper projective left $A$-module if any diagram

$\begin{array}{c} P \xrightarrow{f} M \xrightarrow{\pi} N \xrightarrow{\gamma} 0 \end{array}$

of left $A$-modules and their morphisms, where $M \xrightarrow{\pi} N \xrightarrow{\gamma} 0$ is proper exact can
be completed.

For any graded module (DG module or DG left A-module) M there is a projective (proper projective DG or proper projective DG left A) module P and a morphism \( f : P \to M \) of graded modules (DG modules or DG left A-modules respectively) such that \( P \xrightarrow{f} M \xrightarrow{0} \) is exact in case of graded modules and proper exact in case of DG or DG left A-modules. [5], [11].

Let M be a DG left A-module for some DG algebra A. A proper projective resolution \( X = \{X^n, e^n\} \) for M is a collection \( \{X^n\}_{n \geq 0} \) of proper projective DG left A-modules, together with morphisms of DG left A-modules \( e^n : X^n \to X^{n-1}, \quad n \geq 0, \quad e^0 : X^0 \to M \) such that the sequence

\[
0 \to X^n \xrightarrow{e^n} X^{n-1} \to \ldots \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^0} X^0 \xrightarrow{e^0} M \to 0
\]

is proper exact.

The statement above insures the existence of a proper projective DG left A-module \( X^0 \) and a morphism \( \varepsilon^0 : X^0 \to M \) of DG left A-modules such that \( X^0 \to M \to 0 \) is proper exact. Since the kernel of \( \varepsilon^0 : X^0 \to M \) is also a DG left A-module, it follows by using the homology exact sequence of the exact sequence \( 0 \to \ker \varepsilon^0 \to X^0 \xrightarrow{\varepsilon^0} M \to 0 \) of DG modules that \( 0 \to \ker \varepsilon^0 \to X^0 \) is a proper exact sequence of DG left A-modules. Let \( X^1 \) be a proper projective DG left A-module and let \( \varepsilon^1 : X^1 \to \ker \varepsilon^0 \) be a morphism of DG left A-modules such that \( X^1 \xrightarrow{\varepsilon^1} \ker \varepsilon^0 \to 0 \) is proper exact. Then let \( \varepsilon : X^1 \to \ker \varepsilon^0 \subseteq X^0 \). Thus \( X^1 \xrightarrow{\varepsilon^1} X^0 \xrightarrow{\varepsilon^0} M \) is proper exact and repetition of this process shows the existence of proper projective resolutions for any DG left A-module M.

Also for any proper projective DG module P, and a DG algebra A, \( A \otimes P \) is a proper projective DG left A-module, and for any DG left A-module M, there exists a proper projective DG module P and a morphism \( \pi = A \otimes P \to M \) of DG left A-modules, such that \( A \otimes P \xrightarrow{\pi} M \xrightarrow{0} \) is proper exact. This, combined with the previous construction of a proper projective resolution, shows we can
For any DG left A-module M, a proper projective resolution $X = \{X^n, e^n\}$ such that $X^n = A \otimes P^n$ for some proper projective DG module $P^n$.

For a DG left A-module N and a DG right A-module M, we can construct a DG module $M \otimes N$ as the cokernel of $M \otimes \phi_N - \phi_M \otimes N : M \otimes A \otimes N \to M \otimes N$.

Let $A$ be an algebra with trivial differentials. Let $M$ be a DG right $A$ module also with trivial differentials. Then we can construct a proper projective resolution $X = \{X^n, e^n\}$ such that each $X^n$ has trivial differentials [9]. Let $N$ be a DG left $A$-module also with trivial differentials. Let $Y = \{Y^n, e^n\}$ be such a proper projective resolution. Define $(X \otimes Y)^n = \{(X \otimes Y)_i, n \geq 0\}$ by $(X \otimes Y)^n = \sum_{i+j} X^i \otimes Y^j$, $\eta^n(x \otimes y) = e^i(x) \otimes y + (-1)^j x \otimes e^j y$ for $x \in X^i$, $y \in Y^j$. Then $\eta^{n-1} \eta^n = 0$, and each $(X \otimes Y)^n$ has trivial differentials.

Then define $\text{tor}_A(M, N)$ to be $H(X \otimes Y)$ as a bigraded module. The same line of proof as in the case of ordinary modules shows this definition is independent of resolutions chosen and also $\text{tor}_A(M, N) = H(X \otimes N) = H(M \otimes Y)$.

Let $X = \{X^n, e^n\}$ be a proper projective resolution for a DG left $A$-module $M$. Define $T(X)$ by

$$T(X)_n = \sum_{i+j=n} X^i_j, \quad d_n(x) = e^i_j(x) + (-1)^i d_j(x) \quad \text{for} \quad x \in X^i_j.$$ 

Then $d_{n-1}d_n = 0$ and $T(X)$ becomes a DG left $A$-module. The maps $e^n$ induce a morphism of DG left $A$-modules $\epsilon : T(X) \to M$ which induces an isomorphism $\epsilon^* : H(T(X)) \to H(M)$.

Let $M$ be a DG right $A$-module, $N$ a DG left $A$-module, $X$ a proper projective resolution for $M$, $Y$ a proper projective resolution for $N$. Define $\text{Tor}_A(M, N) = H(T(X) \otimes T(Y))$. The same line of argument as in the
case of ordinary modules shows that this definition is independent of the resolutions involved and also:

\[ \text{Tor}_A(M,N) = H(T(X) \otimes T(Y)) = H(T(X) \otimes N) = H(M \otimes T(Y)) \, . \]

Also if \( A, M, N \) all have trivial differentials we obtain

\[ \text{Tor}_A(M,N)_n = \sum_{s+t=n} \text{tor}_A(M,N)_{s,t} \, . \]

Example: Let \( A = E_{Z_2} [x_1, x_2, \ldots] \) be the exterior algebra on generators \( x_i \) of degree \( i \), with trivial differentials. \( E_{Z_2} [x_1, x_2, \ldots] = \bigotimes_{i \geq 1} E_{Z_2} [x_i] \) as algebras. Let \( i^n x = E_{Z_2} [x_1] \otimes Z_2^i y_n \) \( \deg i^n y_n = ni, \, i^0 y_0 = 1, \)

\[ i^n \epsilon (1 \otimes i^n y_n) = x_1 \otimes i^n y_{n-1} \text{ for } n \geq 1. \]

Then \( \{i^n x, i^n \epsilon\} \geq 0 \) is a proper projective resolution for the DG left \( E_{Z_2} [x_1] \) module \( Z_2 \). Therefore

\[ \{X^n, \epsilon^n\} \geq 0 \text{ where} \]

\[ X^n = \bigotimes_{\sum_{j=1}^s n_j = n} \, i_j x_1^j, \, \epsilon^n (x_1 \otimes \ldots \otimes x_i) = \bigotimes_{i=1}^s x_1^i \otimes x_1^i \otimes \ldots \otimes x_1^i + \ldots + x_1^i \otimes \ldots \otimes \epsilon x_1^i, \text{is a proper projective resolution for} \]

the DG left \( A \)-module \( Z_2 \). Then \( T(X)_k = A \otimes \bigotimes_{\sum_{j=1}^s n_j = n} i^n y_n^1 \ldots Z_2^i y_n^s \), where \( k = \sum_{j} (n_j i_j + n_j) \), and \( H(Z_2 \otimes T(X)) = \text{Tor}_A(Z_2, Z_2) \). The elements \( z_{i,n} = 1 \otimes 1 \otimes i^n y_n, \, n > 0 \) are cycles but not boundaries and dimension \( z_{i,n} = (i+1)n \). They generate \( \text{Tor}_A(Z_2, Z_2) \) as an algebra and

\[ \text{Tor}_A(Z_2, Z_2) = \Gamma_{Z_2} [z_1, z_2, z_3, 1', \ldots] = \Gamma_{Z_2} [z_1, z_2, \ldots] \text{ the divided} \]
polynomial algebra on the generators $z_i$ of degree $i+1$ \[11\].

**Theorem** \[9\],[11]\[

Let $A$ be a DG algebra, $M$ a DG right $A$-module, $N$ a DG right module. Then there is a first quadrant spectral sequence $\{E^r, d^r\}_{r \geq 2}$ such that

a) $E^2_{s,t} = \text{tor}_{H(A)} (H(M), H(N))_{s,t}$

where $H(A)$ is a DG algebra, $H(M)$, $H(N)$ are DG right and left $H(A)$ modules respectively with trivial differentials;

b) $E^\infty = G \text{Tor}_A (M, N)$ the associated graded module for some filtration of $\text{Tor}_A (M, N)$.

**Proof:** Let $X = \{X^n, e^n\}_{n \geq 0}$ be a proper projective resolution for the DG right $A$-module $N$, such that for each $n$, $X^n = A \otimes P^n$ for some proper projective DG module $P^n$. Then $\text{Tor}_A (M, N) = H(M \otimes T(X))$. Filter $T(X)$ by resolution degree:

$$F_s T(X) = \sum_{i \leq s} X^i_j.$$

This filtration is increasing, bounded below ($F_{-\infty} T(X) = 0$) and convergent ($[F_n T(X)]_n = T(X)_n$), also each $F_s T(X)$ is a DG left $A$-submodule. Filter $M \otimes T(X)$ by:

$$F_s (M \otimes T(X)) = M \otimes F_s T(X).$$

This filtration is also increasing, bounded below and convergent. Therefore we obtain a convergent 0-spectral sequence $\{E^r, d^r\}_{r \geq 0}$ of $K$ modules, where:
and $d^0$ is the induced differential and $d^1_{s,t} = H^s_{s+t} (F_s/F_{s-1}) \to H^s_{s+t-1}(F_{s-1}/F_{s-2})$ is the boundary morphism in the homology long exact sequence of the triple $(F_{s-2}, F_{s-1}, F_s)$ [12].

But $E^0_{s,t} = [F_s(M \otimes T(X))]_{s+t} / [F_{s-1}(M \otimes T(X))]_{s+t} = (M \otimes X^S)$ since $F_s T(X) = F_{s-1} T(X) \oplus X^S$ as $A$-modules, and since the resolution degree also contributes to the gradation of $T(X)$. Also $d^0$ coincides with the differential on $M \otimes X^S$. Hence $E^1_{s,t} = H^t_{s-t}(M \otimes X^S)$. But $H(M \otimes X^S) = H(M) \otimes H(X^S)$ by [9], since $X^S$ is a proper projective DG left $A$-module.

Tracing the definition of the boundary homomorphism shows that

$$d^1_{s,t} : E^1_{s,t} = [H(M) \otimes H(X^S)]_{t} \to [H(M) \otimes H(X^{S-1})]_{t} = E^1_{s-1,t}$$

coincides with

$$H(M) \otimes c^S_\star : H(M) \otimes H(X^S) \to H(M) \otimes H(X^{S-1})$$

Finally $H(X^S) = H(A \otimes P^n) = H(A) \otimes H(P^n)$ by [9], since $P^n$ is proper projective DG module. Hence $H(X^S)$ is a proper projective DG left $H(A)$-module both with trivial differentials ([5], [9]). Since $c^S_\star : H(X^S) \to H(X^{S-1})$ is a morphism of left $H(A)$-modules and since

$$c^S_\star : H(X^S) \to H(X^{S-1}) \to \ldots \to H(X^0) \to H(M) \to 0$$

is exact, $H(X) = \{H(X^n), c^S_\star\}_{n \geq 0}$ is a proper projective resolution for DG left $H(A)$-module $H(M)$, $H(M)$ with trivial differentials. Therefore
\[ E_{s,t}^2 = H_{s,t}(H(M), H(X)) = \text{tor}_{H(A)}(H(M), H(N))_{s,t}. \]

Also as observed before:

\[ [\text{Tor}_{H(A)}(H(M), H(N))]_{n} = \sum_{s+t=n} \text{tor}_{H(A)}(H(M), H(N))_{s,t} = \sum_{s+t=n} E_{s,t}^2. \]

Finally \( E_{s,t}^0 = [M \otimes X^s]_t \) also shows that this spectral sequence is actually a first quadrant spectral sequence.

Let \( M \) be a DG left \( A \)-module for some DG algebra \( A \). Let \( X = \{X^n, \epsilon^n\}_{n \geq 0} \) be a proper projective resolution for \( M \). On \( T(X) \) consider the following filtration, called the skeletal filtration:

\[ F_{s} T(X) = \bigoplus_{i \leq s} \phi(A \otimes T(X)_i) = \bigoplus_{i \leq s} A \cdot T(X)_i. \]

The skeletal filtration is increasing, convergent \( (F_{s} T(X)_n = T(X)_n) \) and bounded below since \( F_{-1} T(X) = 0 \). Also each \( F_{s} T(X) \) is a DG left \( A \) submodule of \( T(X) \).

If each \( X^n \) is of the form \( X^n = A \otimes P^n \) for some proper projective DG module \( P^n \), in the spectral sequence corresponding to the skeletal filtration we have:

\[ E_{s,t}^0 = F_{s} T(X)_{s+t} / F_{s-1} T(X)_{s+t} = A_t \otimes \bigoplus_{i \leq s} P^i_j \]

and \( d_{s,t}^0 = d_A \otimes \sum P^i_j \). Since \( P^i_j \) are projective modules we see \( E_{s,t}^1 = H_t(A) \otimes \bigoplus_{i+j=s} P^i_j \). Since each \( F_{s} T(X) \) is a DG left \( A \)-module each \( E_{s,t}^0 \) is a left \( A_0 \) module and the differentials are morphisms of left \( A_0 \) modules. Hence each \( E_{s,t}^1 \) is a left \( H_0(A) \) module and \( d_{s,t}^1 \) are morphisms of left \( H_0(A) \) modules. Also:
as bigraded modules. Further

\[
\begin{array}{c}
\begin{array}{c}
H_t(A) \bigotimes_{H_0(A)} E^1_{s,0} = H_t(A) \bigotimes \{ H_0(A) \bigotimes \sum_{i+j=s} p^i \} \\
= H_t(A) \bigotimes \sum_{i+j=s} p^i \\
= E^1_{s,t}
\end{array}
\end{array}
\]

2. Cubical Singular Homology [4], [10]:

Let \( I^N = [0,1]^N \) the n-dimensional standard cube, \( I^0 = \) one point. For a topological space \( X \), let \( Q_n(X) \) denote the free abelian group generated by continuous maps \( c : I^N \to X \) for \( n \geq 0 \). On the graded group \( \{Q_n(X)\}_{n \geq 0} \) define maps:

\[
\lambda^\varepsilon_i : Q_n(X) \to Q_{n-1}(X) \text{ by } (\lambda^\varepsilon_i c)(x_1, \ldots, x_{n-1}) = c(x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1})
\]

where \( \varepsilon = 0 \) or 1, and \( 1 \leq i \leq n \). Define \( d_n : Q_n(X) \to Q_{n-1}(X) \) for \( n > 0 \), by \( d_n(c) = \sum_{i=0}^{n} (-1)^i(\lambda^\varepsilon_i c - \lambda_{i+1} c) \), \( d_0 = 0 \). Then \( d_{n-1}d_n = 0 \). Let
\(Q(X) = \{Q_n(X), d_n \}_{n \geq 0}\) be the resulting chain complex. This construction is natural with respect to continuous maps, i.e. any continuous map \(f : X \to Y\) gives rise to a morphism \(Q(f) : Q(X) \to Q(Y)\) of chain complexes.

We also have \(s_i : Q_n(X) \to Q_{n+1}(X)\) defined by \((s_i c)(x_1, \ldots, x_{i+1}) = c(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1})\) for \(0 \leq i \leq n+1\). Let \(D_n(X) \subset Q_n(X)\) be the subgroup generated by the elements \(\{s_i c\}, c : I^{n-1} \to X, 1 \leq i \leq n\). We have \(d(D_n(X)) \subset D_{n-1}(X)\). For any continuous map \(f : X \to Y\), \(Q(F)(D_n(X)) \subset D_n(Y)\) for all \(n \geq 0\). Hence we can define a functor \(C(X)\) from the category of topological spaces and continuous maps to the category of chain complexes and their morphisms by \(C(X) = Q(X)/D(X)\). If \(X = \text{point}\), then \(C_0(X) = Z\), \(c_o : I^0 \to X\) the only map and \(C_n(X) = 0 \ n > 0\), i.e. as DG modules \(C(\text{pt}) = Z\).

Let \(G\) be an abelian group. Define \(C_n(X,G) = G \otimes C_n(X)\) and \(\tilde{d}_n = G \otimes d_n : C_n(X,G) \to C_{n-1}(X,G)\), then \(\{C_n(X,G), \tilde{d}_n\}\) is a chain complex. Also for a continuous map \(f : X \to Y\), \(G \otimes C(f)\) defines a morphism between \(C(X,G)\) and \(C(Y,G)\), turning \(C(G,-)\) into a natural functor.

Let \(X, Y\) be topological spaces, there is a natural morphism \(\bar{\rho}\) of chain complexes:

\[
\bar{\rho} : Q(X) \otimes Q(Y) \to Q(X \times Y)
\]

defined by \(\bar{\rho}(c^p \otimes c^q) = c^p \times c^q\) where \(c^p : I^p \to X, c^q : I^q \to Y\) and \(c^p \times c^q : I^{p+q} = I^p \times I^q \to X \times Y\) \((c^p \times c^q)(t_1, \ldots, t_{p+q}) = (c^p(t_1, \ldots, t_p), c^q(t_{p+1}, \ldots, t_{p+q}))\). Also \(\bar{\rho}\) is associative, i.e.
For 1 \leq i \leq p+1, \bar{\rho}(s_i c^p \otimes c^q) = s_i \bar{\rho}(c^p \otimes c^q) and for 1 \leq j \leq q+1
\bar{\rho}(c^p \otimes s_j c^q) = s_{p+j} \bar{\rho}(c^p \otimes c^q).

Hence \bar{\rho} defines \rho: C(X) \bigotimes C(Y) \to C(X \times Y), which is also associative.

For a commutative ring K with unit, we can define
\rho: C(X,K) \bigotimes C(Y,K) \to C(X \times Y,K) by \rho(kc^p \otimes k'c^q) = kk' \rho(c^p \otimes c^q), and

a similar diagram for associativity commutes. Since point \times point = point
C(point, K) becomes a DG algebra and we have C(point, K) = K as DG algebras.

3. Local Coefficients \[4\], \[10\]
Let X be a topological space. A local system on X is a collection
of groups G_x for each x \in X, such that for every path \omega : I \to X with
\omega(0) = a, \omega(1) = b there is a given isomorphism T_\omega : G_a \to G_b which only
depends on the homotopy class of \omega and also T_{\omega \ast \omega'} = T_\omega \ast T_{\omega'} for another
path \omega' with \omega'(0) = \omega(1) where:

\[(\omega \ast \omega')(t) = \begin{cases} 
\omega(2t) & \text{for } 0 \leq t \leq 1/2 \\
\omega'(2t-1) & \text{for } 1/2 \leq t \leq 1.
\end{cases}\]
If $X$ is path connected all the groups $G_x$ are isomorphic. We can choose a base point $b$, and for any other point $x$, a fixed path $\omega_x$ with $\omega_x(0) = b$, $\omega_x(1) = x$. This allows us to identify each $G_x$ with $G = G_b$ via $T_{\omega_x}$.

We form a chain complex by setting $C_n(X,G) = G \otimes C_n(X)$ and by defining

$$d_n(g \otimes c^n) = \sum_{i=0}^{n} (g \otimes \lambda_i^0 c^n - T(c^n, i)g \otimes \lambda_i^1 c^n)$$

where $T(c^n, i)$ is the automorphism of $G = G_b$ defined by the path $t \to c^n(0,0,\ldots,0,t,0,\ldots)$ and $t$ appears at the $i^{th}$ place. Using the fact $T$ only depends on the homotopy class of the path, it follows that $d_{n-1}d_n = 0$.

4. Principal Fibrations [9]

A continuous map $p : E : B$ is called a fibration if it has homotopy lifting property for finite simplicial complexes, i.e. for any finite simplicial complex $A$ and any commutative diagram of continuous maps:

```
\begin{tikzcd}
A \times 0 \arrow{r}{f} \arrow{d} \arrow{r}{h} & E \arrow{d}{p} \\
A \times I \arrow{r}{h} & B
\end{tikzcd}
```

admits $H : A \times I \to E$ such that $H_0 = f$ and $p \cdot H = h$.

Let $X$ be a topological space and $\mu : X \times X \to X$ a continuous map, $e \in E$,
then \((X, \mu, e)\) is called a topological monoid if the following diagrams commute:

\[
\begin{array}{ccc}
X \times X \times X & \xrightarrow{X \times \mu} & X \times X \\
\mu \times X & \downarrow & \downarrow \mu \\
X \times X & \xrightarrow{\mu} & X
\end{array}
\]

\[
\begin{array}{ccc}
e \times X & \xrightarrow{\mu} & X \\
p_2 & & \downarrow \mu \\
X & & X
\end{array}
\]

\[
\begin{array}{ccc}
X \times e & \xrightarrow{\mu} & X \\
p_1 & & \downarrow \mu \\
X & & X
\end{array}
\]

where \(p_1\) and \(p_2\) are projections onto the first and second factors respectively.

A fibration \(p : E \to B\) is called a principal fibration if for some point \(b \in B\), \(G = p^{-1}(b)\) \(e \in G\), there is a map \(\mu_E : G \times E \to E\) such that \(p \circ (g, x) = p(x)\) and \((G, \mu = \mu_E|_{G \times G}, e)\) is a topological monoid and the following diagrams commute:

\[
\begin{array}{ccc}
G \times G \times E & \xrightarrow{G \times \mu_E} & G \times E \\
\mu_G \times E & \downarrow & \downarrow \mu_E \\
G \times E & \xrightarrow{\mu_E} & E
\end{array}
\]

\[
\begin{array}{ccc}
e \times E & \xrightarrow{\mu_E} & E \\
p_2 & & \downarrow \mu_E \\
E & & E
\end{array}
\]

where \(p_2(e, x) = x\).
Therefore when we pass to the singular cubical chains over a commutative ring $K$ with unit we obtain, for a principal fibration with fiber $G$, morphisms of DG modules:

$$\phi : C(G,K) \otimes C(G,K) \rightarrow C(G \times G,K) \xrightarrow{\rho} C(G,K)$$

an associative multiplication with unit:

$$\eta : K = C(\text{point}, K) \xrightarrow{e_*} C(G,K)$$

so that $C(G,K)$ becomes a DG algebra and

$$\phi_E : C(G,K) \otimes C(E,K) \rightarrow C(G \times E,K) \xrightarrow{\rho} C(E,K)$$

turns $C(E,K)$ into a DG left $C(G,K)$ module.

5. The action of $\pi_1(B)[8]$

Let $p : E \rightarrow B$ be a fibration. Let $\omega : I \rightarrow B$ a path from $a$ to $b$. We can construct $\tilde{T}_\omega : C(p^{-1}(a),K) \rightarrow C(p^{-1}(b),K)$ as follows:

Let $c : I^0 \rightarrow p^{-1}(a)$ be a 0-dimensional cube, let $S_c$ be a lifting of $\omega$ which starts at $c(0)$. Proceeding inductively, when $S_c$ is defined on all $n$-dimensional cubes, for an $(n+1)$-dimensional cube $c$ consider the following commutative diagram:

\[
\begin{array}{ccc}
\partial I^{n+1} \times I \cup I^{n+1} \times 0 & \rightarrow & E \\
\downarrow & & \\
I^{n+1} \times I & \xrightarrow{f} & B \\
\end{array}
\]
where
\[ \tilde{f} \mid \lambda_1^c (\text{id}_{I^m+1}) I^n \times I = S \] and \[ \tilde{f} \mid I^{n+1} \times 0 = c \]

\[ f(x_1, \ldots, x_{n+1}, t) = \omega(t) \]. Since \( I^{n+2} \) is a deformation retract of \( I^{n+1} \), there is a lifting \( S_c \) of \( f \), using the fact that \( p \) is a fibration. Define \((T_c)(x) = S_c(x,1)\) for any singular cube \( c \). By construction \( dT_c = T_c d \). Also any other lifting gives rise to a map which is chain homotopic to \( T_c \) and if \( \omega, \omega' \) have the same endpoints and are homotopic to each other by a homotopy leaving the endpoints fixed, the resulting maps \( T_{c1} \) and \( T_{c2} \) are chain homotopic. Hence for each homotopy class of paths joining \( a \) to \( b \) and any abelian group \( G \)

\[ T_\omega : H(p^{-1}(a), G) \to H(p^{-1}(b), G) \]

which satisfies \( T_{\omega \times \omega'} = T_\omega T_{\omega'} \). Also \( T_\omega = 1 \) for the constant path \( \omega \).

Hence \( T_\omega \) is an isomorphism for any path \( \omega \), since \( \omega \times \omega^{-1} \) and \( \omega^{-1} \times \omega \) are homotopic to the constant path, where \( \omega^{-1}(t) = \omega(1-t) \). Thus \( H_q(F, G) \) is a local system on \( B \) for any \( q \geq 0 \).

A local system defines homomorphisms \( \omega_n : \pi_1(B) \to \text{Aut} (H_n(F, G)) \).

Proposition 1: Let \( p : E \to B \) be principal fibration with fiber \( G \), which is path connected. Then \( \pi_1(B) \) acts trivially on \( H(F) \), \( \omega_n \pi_1(B) \) = 1 for all \( n \geq 0 \).

Proof: Let \( \omega : S^1 \to B \) be a loop. It will be shown that we can choose \( T_\omega : C(G) \to C(G) \) to be the identity map. Let \( p_1 : E' \to S^1 \) be the pullback fibration, which is also a principal fibration with fiber \( G \). In the homotopy exact sequence of this fibration \( p_1 \times : \pi_1(E') \to \pi_1(S^1) \) is onto since \( G \) is connected. Hence there exists \( s' : S^1 \to E' \) such that \( p_1 s' \) is homotopic to the identity map on \( S^1 \). Using the homotopy lifting
property we can lift this homotopy to \( E' \), obtaining a section \( s \) of \( p_1 : E' \to S^1 \) with \( s(1) = e \). Now define \( F : G \times S^1 \to E' \) by \( F(g,x) = (x,\mu(g,s(x))) \). For a singular cube \( c : I^n \to G \), define \( S_\omega c : I^n \times I \to E \) by \( S_\omega c(x,t) = \mu(c(x),p_2 \tilde{s}(t)) \), where \( \tilde{s} : I \to S^1 \not\rightarrow E' \) and \( p_2 : E' \to E \).

These maps satisfy all the properties required in the definition of the action. But then \( T_\omega c(x) = S_\omega c(x,1) = S_\omega c(x,0) = c(x) \) since \( p_2 \tilde{s}(1) = p_2 \tilde{s}(0) = e \). Therefore \( \pi_1(B) \) acts trivially on \( H(G) \).

**Proposition 2:** Let \( p : E \to B \) be a principal fibration with fiber \( G \). Then \( \partial : \pi_1(B) \to \pi_0(G) \) in the homotopy exact sequence of the fibration preserves the multiplication.

**Proof:** Let \( \omega_0 \) and \( \omega_1 \) be two loops in \( B \) starting and ending at \( b = p(e) \). Let \( \tilde{\omega}_0, \tilde{\omega}_1 \) be liftings of \( \omega_0 \) and \( \omega_1 \) starting at \( e \). Then \( \tilde{\omega} : I \to E \) defined by \( \tilde{\omega}(t) = \tilde{\omega}_0(2t) \) for \( 0 \leq t \leq 1/2 \) and \( \tilde{\omega}(t) = \mu(\tilde{\omega}_0(1), \tilde{\omega}_1(2t-1)) \) for \( 1/2 \leq t \leq 1 \), is a lifting of \( \omega_0 \ast \omega_1 \) starting at \( e \). Since \( \tilde{\omega}(1) = \mu(\tilde{\omega}_0(1), \tilde{\omega}_1(1)) \), we have \( \partial[\omega_0 \ast \omega_1] = [\mu(\tilde{\omega}_0(1), \tilde{\omega}_1(1))] = \mu([\tilde{\omega}_0(1)], [\tilde{\omega}_1(1)]) = \mu(\partial[\omega_0], \partial[\omega_1]). \)

**Corollary 1:** Under the same conditions of the proposition 2, \( \ker \partial \) acts trivially in \( H(G) \).

**Proof:** Let \( \omega \) be a loop in \( B \) such that \( \partial[\omega] = 0 \). Let \( p_1 : E' \to S^1 \) be the pullback fibration as in proposition 1. Then the homotopy exact sequences of \( p \) and \( p_1 \) give rise to the following commutative diagram:
Hence for $[S^1] \in \pi_1(S^1)$ we have $\partial [S^1] = \partial \omega \ast [S^1] = \partial [\omega] = 0$. Therefore $[S^1] \in \pi_1(S^1)$ is in the image of $p_1 \ast \omega$. Then the same construction as in proposition 1 shows that $[\omega]$ acts trivially on $H(G)$.

Corollary 2: If in addition to the conditions in proposition 2, E is path connected, then $\pi^o_0(G)$ is a group.

Proof: Since E is path connected $\partial : \pi_1(B) \to \pi^o_0(G)$ is onto and $\partial$ preserves the multiplication by proposition 2.

Proposition 3: Let $p : E \to B$ be a principal fibration with fiber G, then the action of $\pi_1(B,b)$ on $H(G)$ is given by $[\omega] \cdot x = (\tilde{\omega}(1)) \ast (x)$ where $\omega$ is a loop in B, $x \in H(G)$, $\tilde{\omega}$ a lifting of $\omega$ starting at e and $(\tilde{\omega}(1)) \ast$ is the map induced by $g \mapsto \mu(g, \tilde{\omega}(1))$.

Proof: Define $S^c_\omega : I^n \times I \to E$ for $c : I^n \to G$ by $S^c_\omega(x,t) = \mu(c(x), \tilde{\omega}(t))$, which satisfies all the conditions required in the definition of the action. But then $T^c \omega(x) = S^c_\omega(x,1) = \mu(c(x), \tilde{\omega}(1))$ which is exactly the morphism $C(G) \to C(G)$ induced by the map $G \to G$ defined by $g \mapsto \mu(g, \tilde{\omega}(1))$.

6. Serre's Spectral Sequence [10], [12]

In his paper "Homologie Singulière Espaces Fibre" (Ann of Math., 1951) Serre proved that for a fibration $p : E \to B$ with fiber F where E is path connected, and for any commutative ring $K$, there is a first quadrant 1-spectral sequence $\{E^r_s, d^r_s\}_{r \geq 1}$ of $K$ modules, such that:

1) $E^1_{s,t} = H_t(F,K) \otimes C_s(B)$ such that $d^1_{s,t} : E^1_{s,t} \to E^1_{s-1,t}$ is defined by $d_B^1$ on $C(B)$ and the fact that $H^t_t(F,K)$ is a system of local coefficients as in section 3. Hence $E^\infty_{s,t} = H^\infty_s(B; H^t_t(F,K))$ with local coefficients.

2) $E^\infty = GH(E,K)$ the associated graded module corresponding to the filtration of $H(E,K)$. 
More explicitly this spectral sequence is obtained by filtering

\[ C(E,K) = K \otimes C(E) \]

where \( C(E) \) is the normalized cubical singular chains of \( E \) whose vertices lie in \( F = p^{-1}(b), b \in B \) the base point. This filtration is defined by:

\[ F_s C(E,K) \]

the submodule generated by cubes \( c : I^n \to E \), such that \( pc : I^n \to B \) is independent of the first \( n \)-s coordinates.

Let \( p : E \to B \) be a principal fibration with fiber the topological monoid \( G \). By proposition 3 in section 5 the action of \( \pi_1(B) \) is given by multiplication by an element of \( H^1(G) \). Hence

\[
\begin{align*}
    d^1_{s,t} : H_t(G,K) \otimes C_s(B) &\to H_t(G,K) \otimes C_{s-1}(B) \\
    d^1(x \otimes c) &= \sum_{i=1}^s (-1)^i (x \otimes \lambda_i^0 c - T_i x \otimes \lambda_i^1 c)
\end{align*}
\]

where \( T_i : H_t(G,K) \to H_t(G,K) \) is the map induced by multiplication by \( \tilde{\omega}_i(1) \) where \( \tilde{\omega}_i \) is a lifting of the loop \( \omega_i : I \to B \), \( \omega_i(t) = c(0, \ldots, t, \ldots, 0) \), where \( t \) appears at the \( i^{th} \) place, starting at \( e \). In case of a principal fibration with fibre \( G \), the multiplication

\[
H_t(G,K) \otimes \mathbb{E}_s^1 \to \mathbb{E}_s^1
\]

induces an isomorphism

\[
H_t(G,K) \otimes H^1(G,K) \otimes \mathbb{E}_s^1 \to \mathbb{E}_s^1
\]

such that
commutes by inspecting $d_{s,t}^1$ using proposition 3 in section 5.

7. The Moore Theorem [6], [9]

Let $p : E \to B$ be a principal fibration with path connected total space, and the fiber a topological monoid $G$. Let $K$ be a commutative ring with unit. Let also $C(E,K)$ be the cubical singular chains of $B$ with all vertices mapped onto $b = p(G)$. As mentioned in section 4 $C(G,K)$ is a DG algebra and $C(E,K)$ is a DG left $C(E,K)$ module. Let $X = \{X^n, e^n\}_{n \geq 0}$ be a proper projective resolution of $C(E,K)$ as a DG left $C(G,K)$ module such that $X^n = C(G,K) \otimes P^n$, where $P^n$ is a proper projective DG module. Then $T(X) = C(G,K) \otimes T(P)$. The augmentation $\varepsilon^0 : X^0 \to C(E,K)$ induces a morphism $\varepsilon : T(X) \to C(E,K)$ of DG left $C(G,K)$ modules such that $\varepsilon_* : H(T(X)) \to H(E,K)$ is an isomorphism.

Consider the skeletal filtration on $T(X)$ and the Serre filtration on $C(E,K)$. $\varepsilon$ preserves these filtrations since for any $c : I^p \to E$ and $c' : I^q \to G$, $p(c' \times c) : I^{p+q} \to B$ is independent of the first $q$ coordinates, therefore $c' \times c$ lies in $F C(E,K)$. Therefore $\varepsilon$ induces a morphism $\{\varepsilon^r\}_{r \geq 1} : \{E^r T(X), d^r\} \to \{E^r C(E,K), d^r\}$ of the corresponding
first quadrant spectral sequences of $K$-modules. The isomorphism $\varepsilon_* : H(T(X)) \to H(E,K)$ induces also an isomorphism of the corresponding filtrations $\varepsilon_* \big| : F^s_s H(T(X)) \to F^s_s H(E,K)$, and hence an isomorphism $\varepsilon^\infty : E^\infty(T(X)) \to E^\infty(C(E,K))$. Also $\varepsilon : E^1T(X) \to E^1C(E,K)$ is a morphism of bigraded left $H(G,K)$ modules and the following diagram commutes since in both filtrations the submodules are submodules of $C(G,K)$ modules:

$$
\begin{array}{ccc}
H^s_t(G,K) \otimes E^s_0 T(X) & \xrightarrow{\sim} & E^1_{s,t} T(X) \\
\downarrow \quad \downarrow H^s_t(G,K) \otimes \varepsilon^1_{s,0} & \quad & \downarrow \quad \varepsilon^1_{s,t} \\
H^s_t(G,K) \otimes E^s_0 C(E,K) & \xrightarrow{\sim} & E^1_{s,t} C(E,K)
\end{array}
$$

Since $T(X)_n$ and $C_n(E,K)$ are left $C_0(G,K)$ modules for all $n$ and the differentials are morphisms of $C_0(G,K)$ modules, therefore $\{E^rT(X), d^r\}_{r \geq 1}$ and $\{E^rC(E,K), d^r\}_{r \geq 1}$ are spectral sequences of $H_0(G,K)$ modules and $\{\varepsilon^r\}_{r \geq 1}$ is a morphism of $H_0(G,K)$ spectral sequences.

Proposition 1: Under the conditions above if $\varepsilon^2_{s,0}$ is an isomorphism for $s < n$ and an epimorphism for $s = n$, then the same is true for $\varepsilon^2_{s,t}$ for any $t$.

Proof: $E^1_{s,0} C(E,K) = H^0_0(G,K) \otimes C_s(B)$ is a free $H^0_0(G,K)$ module since $C_s(B)$ is a free abelian group. $E^1_{s,0} T(X) = H^0_0(G,K) \otimes \bigoplus_{i+j=s} P^i_j$ is a projective $H^0_0(G,K)$ module since $\bigoplus_{i+j=s} P^i_j$ is a projective $K$-module. (Since $\bigoplus_{i+j=s} P^i_j$ is a projective it is a direct summand of a free $K$-module $F$, then $H^0_0(G,K) \otimes \bigoplus_{i+j=s} P^i_j$ is a direct summand of the free $H^0_0(T,K)$ module $H^0_0(G,K) \otimes F$.)
Therefore the mapping cylinder ([5], [8]) $M$ of

$\varepsilon_{*,0}^1 : E_{*,0}^1 T(X) \to E_{*,0}^1 C(E,K)$ is a complex of projective left $H^*_0(G,K)$ modules ($M_n = E_{n,0}^1 C(E,K) \oplus E_{n-1,0}^1 T(X)$), and the long exact sequence for $\varepsilon_{*,0}^1 ([5], [8])$ the hypothesis for $\varepsilon_s^2$ correspond to $H_s(M) = 0$ for $s \leq n$. Since $M_n$ are all projective, we can then construct a 'partial' contracting homotopy ([12]), i.e. morphisms $\tau_i : M_i \to M_{i+1}$ $-1 \leq i \leq n$

$d_{i+1} \tau_i + \tau_{i-1} d_i = M_i$ exactly as in the case of acyclic complexes [12].

Now consider $\bar{M}$, the mapping cylinder of

$H_*(G,K) \otimes E_{*,0}^1 T(X) \to H_*(G,K) \otimes E_{*,0}^1 C(E,K)$

Then $\bar{M} = H_*(G,K) \otimes M$, therefore $H_*(G,K) \otimes \tau_i$ gives a 'partial' contracting homotopy for $\bar{M}$, therefore $H_s(\bar{M}) = 0$ for $s \leq n$, therefore $H_*(G,K) \otimes \varepsilon_{*,0}^1$ induces isomorphism up to dimension $n$, and an epimorphism on dimension $n$. And the following commutative diagram

$\xymatrix{
\varepsilon_{*,t}^1 : E_{*,t}^1 T(X) \ar[r] & E_{*,t}^1 C(E,K) \\
H_*(G,K) \otimes E_{*,0}^1 T(X) \ar[r] \ar[d] & H_*(G,K) \otimes E_{*,0}^1 C(E,K) \ar[d] \\
E_{*,t}^1 T(X) \ar[r] & E_{*,t}^1 C(E,K)
}$

shows that $\varepsilon_{s,t}^2$ is an isomorphism for $s < n$ and an epimorphism for $s = n$ for any $t$.

This result using the Comparison Theorem, which is proved in
section 8, guarantees that $\varepsilon_{s,t}^2$ is an isomorphism for all $s$ and $t$. In particular $\varepsilon_{s,o}^2$ is an isomorphism for all $s$, i.e.

$$(\varepsilon_{s,o}^1) : \{E_{s,o}^1 T(X), d_{s,o}^1 \} \rightarrow \{E_{s,o}^1 C(E,K), d_{s,o}^1 \}$$

induces isomorphisms between their homologies. Therefore the same proof as in proposition 1 proves that:

$$K \bigotimes_{H^*_o(G,K)} E_{s,o}^1 T(X) \rightarrow K \bigotimes_{H^*_o(G,K)} E_{s,o}^1 C(E,K)$$

induces an isomorphism between their homologies. But

$$K \bigotimes_{H^*_o(G,K)} E_{s,o}^1 T(X) = K \bigotimes_{H^*_o(G,K)} \{H^*_o(G,K) \bigotimes \sum P^1_j \} = K \bigotimes_{H^*_o(G,K)} C(G,K)$$

and

$$K \bigotimes_{H^*_o(G,K)} E_{s,o}^1 C(E,K) = C(B,K)$$

as DG modules, hence

$$\text{Tor}_r (K,C(E,K)) = H(B,K) \otimes C(G,K)$$

Theorem: (Moore [6], [9])

Let $p : E \rightarrow B$ be a principal fibration with fibre the topological monoid $G$. Then for any commutative ring $K$ with unit, there is a first quadrant 0-spectral sequence $\{E^r, d^r\}_{r \geq 0}$ of $K$ modules such that:

1) $E_{s,t}^2 = \text{tor}_{H^*_o(G,K)}(K,H(E,K))_{s,t}$

2) $E^\infty = \text{GH}(B,K)$ the associated graded module corresponding to some filtration of $H(B,K)$.

Proof: The theorem is just the last statement combined with the spectral sequence for Tor, in section 1.
Corollary: Let $p : EG \to BG$ be a classifying fibration, i.e. a principal fibration with fiber $G$, such that $EG$ is contractible. Then

$$H(BG, K) = \text{Tor}_{C(G,K)}(K, K)$$

for any commutative ring $K$ with unit.

Proof: Consider the augmentation $\varepsilon : C(EG, K) \to K$ which is a morphism of DG left $C(G, K)$-modules and induces an isomorphism between their homologies, since $EG$ is contractible. Then $\varepsilon$ induces a morphism of spectral sequences $\{E^r_s, d^r_s\}_{r \geq 0}$ and $\{P^r_s, d^r_s\}_{r \geq 0}$ which converges to $\text{Tor}_{C(G,K)}(K,C(EG,K))$ and $\text{Tor}_{C(G,K)}(K,K)$ respectively. But

$$E^2_{s,t} = \text{tor}_{H(G,K)}(K,H(EG,K))_{s,t} \quad \text{and} \quad E^2_{s,t} = \text{tor}_{H(G,K)}(K,K)_{s,t}$$

since $\varepsilon_\ast$ is an isomorphism, $\varepsilon^2$ is therefore an isomorphism. Therefore $\varepsilon^\infty$ is an isomorphism. This proves

$$\text{Tor}_{C(G,K)}(K,\varepsilon) : \text{Tor}_{C(G,K)}(K,C(EG,K)) \to \text{Tor}_{C(G,K)}(K,K)$$

is an isomorphism using the Five Lemma. Hence

$$\text{Tor}_{C(G,K)}(K,K) = H(BG, K).$$
these filtrations.

a) If $f : K \to L$ is an epimorphism and if $\tilde{f}_k : K_k/K_{k-1} \to L_k/L_{k-1}$ is not an epimorphism, then $\tilde{f}_j : K_j/K_{j-1} \to L_j/L_{j-1}$ is not a monomorphism for some $j > k$.

b) If $f : K \to L$ is a monomorphism and if $\tilde{f}_j : K_j/K_{j-1} \to L_j/L_{j-1}$ is not a monomorphism, then $\tilde{f}_k : K_k/K_{k-1} \to L_k/L_{k-1}$ is not an epimorphism for some $k < j$.

Proof: a) Let $j$ be the largest number such that $\tilde{f}_j$ is not an isomorphism, then $j > k$. By the Five Lemma and induction $f_j : K_j \to L_j$ is an epimorphism. Therefore $\tilde{f}_j$ is an epimorphism, this proves $j > k$. Since $\tilde{f}_j$ is not an isomorphism it can not be a monomorphism.

b) Let $k$ be the smallest number such that $f_k$ is not an isomorphism, then $k < j$ by the Five Lemma. The Five Lemma also proves that $\tilde{f}_k$ is a monomorphism, hence $k < j$. $\tilde{f}_k$ can not be an epimorphism because this would by the Five Lemma force $f_k$ to be an isomorphism.

Notation: Let $\{E^r_{s,t}, d^r_{s,t}\}_{r \geq 0}$ be a first quadrant spectral sequence. Define $[n,p,s] = d^{p-s}\left(E^{p,s}_{n-p,s,n-1-s}\right) \subseteq E^{p-s}_{s,n-1-s}$ for $p-s \geq 2$. Then $[n,p,s] = 0$ unless $s \geq 0$ and $n > s$. Let $[n,p,s] = 0$ for $p-s < 2$. Let also $B^r_{s,t}$ and $Z^r_{s,t}$ be the submodules of $E^2_{s,t}$ which correspond to the boundaries and cycles of $E^r_{s,t}$ respectively for $r \geq 2$ (since $E^r_{s,t}$ for $r > 2$ is a subquotient of $E^2_{s,t}$). Then we get a filtration of $E^2_{s,t} : 0 \subset B^2_{s,t} \subset B^3_{s,t} \subset \ldots \subset B^{t+1}_{s,t} \subset Z_{s,t}^s \subset \ldots \subset Z^3_{s,t} \subset Z^2_{s,t} \subset E^2_{s,t}$ with the associated graded modules $[s+t+1, s+2, s], [s+t+1, s+3, s], [s+t+1, s+t+1, s], E_{s,t}^\infty, [s+t, s, 0], [s+t, s, 1], \ldots, [s+t, s, s-2]$.

Let $f : E \to \bar{E}$ be a morphism of spectral sequences where $E$ and $\bar{E}$ are first quadrant spectral sequences. Suppose $f^\infty : E^\infty \to \bar{E}^\infty$ is an isomorphism.
Lemma 2: [3], [15] In addition to the conditions above suppose also $f^2_{s,t}$ is an isomorphism for $s < p$ and $t < n-1-s$ and $f^2_{p,t}$ is an epimorphism for $t < n-1-s$. Then $f : [n,p,s] \rightarrow [n,p,s]$ is an isomorphism.

Proof: The claim is trivially valid unless $2 < s+2 < p < n$.

Suppose $f : [n,p,s] \rightarrow [n,p,s]$ is not an epimorphism for $2 < s+2 < p < n$.

Consider:

$$0 \subseteq B^2 \subseteq B^3 \subseteq B^{n-p+1} \subseteq Z^p \subseteq \ldots \subseteq Z^2 \subseteq E^2_{p,n-p}$$

with the associated graded modules $[n+1, p+2, p], \ldots, [n+1, n+1, p], E^\infty_{p,n-p}, [n, p, 0], \ldots [n, p, p-2]$. Since $p > s+2$, we have $n-p < n-1-s$. Therefore $f^2_{p,n-p}$ is an epimorphism. Then by Lemma 1a $f : [n,p,s] \rightarrow [n,p,s]$ is not a monomorphism for some $s < s^1 < p-2$. Now consider:

$$0 \subseteq B^2 \subseteq B^{n-s^1} \subseteq Z^s \subseteq \ldots \subseteq Z^2 \subseteq E^2_{s^1,n-l-s^1}$$

with the associated graded modules $[n, s^1+2, s^1], [n, s^1+3, s^1], \ldots [n, n, s^1], E^\infty_{s^1, n-l-s^1}, [n-1, s^1, 0], \ldots, [n-1, s^1, s^1-2]$. By hypothesis $f^2_{s^1,n-l-s^1}$ is an isomorphism, therefore by Lemma 1b there exists $p^1$ with $s^1+2 < p^1 < p$ such that $f : [n, p^1, s^1] \rightarrow [n, p^1, s^1]$ is not an epimorphism. Now we can repeat the same argument obtaining integers

$$\ldots > s_2 > s_1 > s \text{ and } p > p^1 > p_2 > \ldots$$

satisfying $p_1 > s_1+2$, which is impossible. This contradiction proves that $f : [n,p,s] \rightarrow [n,p,s]$ is an epimorphism.

Suppose $f : [n,p,s] \rightarrow [n,p,s]$ is not a monomorphism. Consider

$$0 \subseteq B^2 \subseteq \ldots \subseteq B^{n-s} \subseteq Z^s \subseteq \ldots \subseteq Z^2 \subseteq E^2_{s,n-l-s}$$

with the associated graded modules $[n, s+2, s], [n, s+3, s], \ldots [n,n,s]$. 
\[ E^\infty_s, n-1-s, [n-1, s, 0], \ldots, [n-1, s, s-2]. \] Since \( f^2 \) is an isomorphism by hypothesis, Lemma 1b provides us with \( p_1, s+2 \leq p_1 < p \) such that \( f : [n,p_1,s] \to [n,p_1,s] \) is not an epimorphism. Now consider

\[ 0 \subseteq B^2 \subseteq \ldots \subseteq B^{n-p_1+1} \subseteq Z^1 \subseteq \ldots \subseteq Z^2 \subseteq E^2_{p_1, n-p_1} \]

with the associated graded modules \([n+1, p_1+2, p_1], [n+1, p_1+3, p_1], \ldots, [n+1, n+1, p_1], E^\infty_{p_1, n-p_1}, [n, p_1, 0], \ldots, [n, p_1, s], \ldots, [n, p_1, p_1-2]\).

Since \( f^2_{p_2, n-p_1} \) is an isomorphism by hypothesis, Lemma 1a provides us with \( s_1, p_1-2 \geq s_1 > s \) such that \( f : [n,p_1,s_1] \to [n,p_1,s_1] \) is not a monomorphism. Repeating the argument shows the existence of integers \( p_1, s_1 \) such that:

\[
p > p_1 > p_2 > \ldots \quad \text{and} \quad s < s_1 < s_2 < \ldots
\]

such that \( p_1 > s_1+2 \) which is impossible. This contradiction proves that \( f : [n,p,s] \to [n,p,s] \) is also a monomorphism, hence an isomorphism.

Proof of the Comparison Theorem: By virtue of the property P it suffices to prove \( f^2_{s,0} \) is an isomorphism for all \( s > 0 \). Hence it suffices to prove for arbitrary \( n \) the statement

\[
I_n : f^2_{s,0} \text{ is an isomorphism for } s < n \quad \text{and} \quad f^2_{n,0} \text{ is an epimorphism.}
\]

We proceed by induction on \( n \).

Since both \( E \) and \( \bar{E} \) are first quadrant spectral sequences \( E^2_{0,0} = E^\infty_{0,0} \), \( E^2_{0,0} = \bar{E}^\infty_{0,0} \), \( E^2_{1,0} = E^\infty_{1,0} \) and \( \bar{E}^2_{1,0} = \bar{E}^\infty_{1,0} \), hence \( f^2_{0,0} \) and \( f^2_{1,0} \) are isomorphisms. Therefore \( I_1 \) holds.

Suppose \( I_n \) has been proven. To show \( I_{n+1} \) is true it suffices to show \( f^2_{n,0} \) is a monomorphism and \( f^2_{n+1,0} \) is an epimorphism. Consider

\[
0 \subseteq Z^n \subseteq \ldots \subseteq Z^2 \subseteq E^2_{n,0}
\]
with associated graded modules $E^\infty_{n,o}$, $[n,n,0], [n,n,1], \ldots, [n,n,n-2]$. By induction hypothesis $f^2_{s,o}$ is an isomorphism for $s < n$ and $f^2_{n,o}$ is an epimorphism. This, using the property $P$ shows that $f^2_{s,t}$ is an isomorphism for $s < n$ and $f^2_{n,t}$ is an epimorphism for any $t$. This, by Lemma 2 shows that $f : [n,n,s] \to [n,n,s]$ is an isomorphism for any $s$. Therefore using the Five Lemma repeatedly shows that $f^2_{n,o}$ is actually an isomorphism.

To prove $f^2_{n+1,o}$ is an epimorphism consider:

$$0 \subset Z^{n+1} \subset \cdots \subset Z^2 \subset E^2_{n+1,o}$$

with associated graded modules $E^\infty_{n+1,o}$, $[n+1,n+1,0], \ldots, [n+1,n+1,n-1]$. Again by the Five Lemma it suffices to show $f^2_{n+1,o}$ induces epimorphisms on the associated graded modules. Suppose $f : [n+1,n+1,p] \to [n+1,n+1,p]$ is not an epimorphism for some $0 < p < n-1$, consider:

$$0 \subset B^2 \subset \cdots \subset B^{n-p+1} \subset Z^p \subset \cdots \subset Z^2 \subset E^2_{p,n-p}$$

with associated graded modules $[n+1,p+2,p], \ldots, [n+1,n+1,p]$, $E^\infty_{p,n-p}$, $[n,p,0], \ldots, [n,p,p-2]$. Since $f^2_{p,n-p}$ is an isomorphism, Lemma 1a provides us with $s$, $0 < s < p-2$ such that $f : [n,p,s] \to [n,p,s]$ is not an monomorphism contrary to Lemma 2. This contradiction shows that $f^2_{n+1,o}$ is an epimorphism.

9. Examples

Let $G = 0 = \bigcup_{n \geq 0} 0(n)$. $G$ can be given a cell complex structure such that the cellular chains $C(0,Z_2) = K \otimes E_{Z_2}[x_1,x_2,\ldots]$ where $E_{Z_2}[x_1,x_2,\ldots]$ is the exterior algebra on generators $x_i$ of dimension 1 and $K$ is the ring $Z_2[t]/(1+t^2)$ considered as a graded algebra over
$\mathbb{Z}_2$, $\tilde{C}(0, \mathbb{Z}_2)$ has trivial differentials [13]. Then by the example in section 1:

\[ \text{Tor}_A^\wedge(\mathbb{Z}_2, \mathbb{Z}_2) = \Gamma_{\mathbb{Z}_2}^\wedge[y_1, y_2, \ldots] \]\n
the divided polynomial algebra on $y_i$, 

$\dim y_i = i+1$, where $A = \mathbb{Z}_2[x_i]$. Also \( \text{Tor}_K^\wedge(\mathbb{Z}_2, \mathbb{Z}_2) = \Gamma_{\mathbb{Z}_2}^\wedge[y] \dim y = 1. \)

Therefore $H(B\mathbb{O}, \mathbb{Z}_2) = \Gamma_{\mathbb{Z}_2}^\wedge[z_1] \dim z_1 = i$, $i = 1, 2, \ldots$ as graded vector spaces. Similarly for $SO = \bigcup_{n \geq 0} SO(n)$ we have $H(BSO, \mathbb{Z}_2) = \Gamma_{\mathbb{Z}_2}^\wedge[z_2, z_3, \ldots]$ as graded vector spaces, but not as algebras [14].

$B\mathbb{O}$, the Grassmannians, is a topological monoid and $H^\wedge(B\mathbb{O}, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_2, \ldots]$ the polynomial algebra generated by the Stiefel-Whitney classes $\omega_i$ of dimension $i$ [8]. The comultiplication in $H^\wedge(B\mathbb{O}, \mathbb{Z}_2)$, induced by the multiplication on $B\mathbb{O}$, by Cartan's formula, is given by

\[ \Delta(\omega_i) = \sum_{i=0}^{n} \omega_i \otimes \omega_{n-i}, \quad \omega_0 = 1. \]

These operations turn $H^\wedge(B\mathbb{O}, \mathbb{Z}_2)$ into a Hopf algebra [7]. Since $\mathbb{Z}_2$ is a field $H^\wedge(B\mathbb{O}, \mathbb{Z}_2)$ is the dual Hopf algebra.

Proposition: $H^\wedge(B\mathbb{O}, \mathbb{Z}_2)$ as an algebra, is a polynomial algebra on generators $x_i$ of dimension $i$, $i = 1, 2, \ldots$.

Proof: It suffices to prove that in $H^\wedge(B\mathbb{O}, \mathbb{Z}_2)$ there is only one primitive element in each dimension, since this, using the duality of $P(H^\wedge(B\mathbb{O}, \mathbb{Z}_2))$ and $Q(H^\wedge(B\mathbb{O}, \mathbb{Z}_2))$ [7], forces $H^\wedge(B\mathbb{O}, \mathbb{Z}_2)$ to have only one indecomposable element in each dimension and for dimensional reasons they cannot have any relations.

Let $p_n$, $n \geq 1$ be defined inductively by

\[ p_n = p_{n-1} \omega_1 + \ldots + p_1 \omega_{n-1} \quad \text{if } n \text{ is even} \]

\[ p_n = \omega_n + p_{n-1} \omega_1 + \ldots + p_1 \omega_{n-1} \quad \text{if } n \text{ is odd}. \]
Claim: $p^n, n \geq 1$ are the only primitive elements in $H^*(BO, \mathbb{Z}_2)$.

Proof: $p_1 = \omega_1$ is clearly primitive. Suppose by induction it has been proven that $p_1, \ldots, p_{n-1}$ are primitive. If $n$ is even:

$$p_n = p_{n-1} \omega_1 + \cdots + p_1 \omega_{n-1}$$

$$\Delta(p_n) = \Delta(p_{n-1} \omega_1) + \cdots + \Delta(p_1 \omega_{n-1}) = \Delta(p_{n-1}) \Delta(\omega_1) + \cdots + \Delta(p_1) \Delta(\omega_{n-1})$$

$$= (1 \otimes p_{n-1} + p_{n-1} \otimes 1)(1 \otimes \omega_1 + \omega_1 \otimes 1) + \cdots + (1 \otimes p_1 + p_1 \otimes 1)(\sum_{i=0}^{n-1} \omega_i \otimes \omega_{n-1-i})$$

$$= 1 \otimes p_n + p_n \otimes 1 + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \omega_i \otimes p_j \omega_{n-i-j} + \sum_{j=1}^{n-1} \sum_{i=0}^{n-j} \omega_j \otimes p_i \omega_{n-j-i}$$

But

$$\sum_{j=1}^{n-i} \omega_j \otimes p_i \omega_{n-i-j} = \omega_i \sum_{j=1}^{n-i} p_j \omega_{n-i-j} = \begin{cases} \omega_i \otimes \omega_{n-i} & \text{i odd} \\ 0 & \text{i even} \end{cases}$$

and

$$\sum_{i=0}^{n-j-1} \omega_i \otimes p_{n-j} \otimes \omega_j = \sum_{i=0}^{n-j-1} \omega_i \otimes p_{n-j} \otimes \omega_j = \begin{cases} \omega_{n-j} \otimes \omega_j & \text{j odd} \\ 0 & \text{j even} \end{cases}$$

Therefore all undesired terms cancel proving $p_n$ is primitive. If $n$ is odd:

$$\Delta(p_n) = \Delta(\omega_n + p_1 \omega_{n-1} + \cdots + p_{n-1} \omega_1) = \Delta(\omega_n) + \Delta(p_1 \omega_{n-1}) + \cdots + \Delta(p_{n-1}) \Delta(\omega_1)$$

$$= \sum_{i=0}^{n-1} \omega_i \otimes \omega_{n-i} + (1 \otimes p_{n-1} + p_{n-1} \otimes 1)(1 \otimes \omega_1 + \omega_1 \otimes 1) + \cdots + (1 \otimes p_1 + p_1 \otimes 1)(\sum_{i=0}^{n-1} \omega_i \otimes \omega_{n-1-i})$$

$$= 1 \otimes p_n + p_n \otimes 1 + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \omega_i \otimes p_j \omega_{n-i-j} + \sum_{j=1}^{n-1} \sum_{i=0}^{n-j} \omega_j \otimes p_i \omega_{n-j-i}$$
But
\[ \sum_{j=1}^{n-i} \omega_i \otimes p_j \omega_{n-i-j} = \omega_1 \otimes \sum_{j=1}^{n-i} p_j \omega_{n-i-j} = \begin{cases} \omega_i \otimes \omega_{n-i} & \text{i even} \\ 0 & \text{i odd} \end{cases} \]

and
\[ \sum_{i=0}^{n-j-1} \omega_i p_{n-i-j} \otimes \omega_j = (\sum_{i=0}^{n-j-1} \omega_i p_{n-i-j}) \otimes \omega_j = \begin{cases} \omega_{n-j} \otimes \omega_j & \text{j even} \\ 0 & \text{j odd} \end{cases} \]

Therefore all undesired terms cancel proving \( p_n \) is primitive.

To prove the uniqueness, consider the exact sequence [7]:

\[ 0 \to P(\xi H^*(BO, \mathbb{Z}_2)) \to P(H^*(BO, \mathbb{Z}_2)) \to Q(H^*(BO, \mathbb{Z}_2)) \]

where \( \xi H^*(BO, \mathbb{Z}_2) \) is the subalgebra of \( H^*(BO, \mathbb{Z}_2) \) generated by the squares.

Hence \( \xi H^*(BO, \mathbb{Z}_2)_n \) = 0 for \( n \) odd. Since \( QH^*(BO, \mathbb{Z}_2) \) has only one nonzero element \( [\omega_n] \) in each degree \( n \) and for odd \( n \), \( p_n \to [\omega_n] \), there can not be any other primitive element of odd degree. If \( p_{2n} \) is a primitive element of degree \( 2n \), then \( p'_{2n} \) is decomposable. The reason for this is that if \( p_{2n} \) in its expansion in terms of \( \omega_i \)'s contain \( \omega_{2n} \) it can not be primitive, since in \( \Delta(p_{2n}) \) the term \( \omega_n \otimes \omega_n \) would appear and it cannot be cancelled by any other term because \( \Delta(\omega_n^2) \) does not contain \( \omega_n \otimes \omega_n \). Suppose \( p'_{2n} \) is another primitive element, then both \( p_{2n} \) and \( p'_{2n} \) lie in \( \xi H^*(BO, \mathbb{Z}) \) because of the above exact sequence, hence they have square roots \( x_1 \) and \( y_1 \) respectively. But in a polynomial algebra over \( \mathbb{Z}_2 \) the square root of a primitive element is primitive. If \( \deg x_1 = \deg y_1 = n \) is still even we can repeat the same argument until we get \( x_k, y_k \) such that \( \deg x_k = \deg y_k \) odd, \( x_k^2 = p_{2n} \) and \( y_k^2 = p'_{2n} \), \( x_k, y_k \) both primitive. But the argument in the beginning of the
uniqueness proof shows that \( x_k = y_k \), hence \( p_{2n} = p_{2n}' \), proving the uniqueness.

Remark: In [1] the following theorem has been proved:

Theorem: Let \( p : E \to B \) a regular covering, let \( \pi = \pi_1(B)/p_\ast \pi_1(E) \).
Then there is a first quadrant 2-spectral sequence \( \{ E^r, d^r \}_{r \geq 2} \) such that:

1) \( E^2_{s,t} = H_s(\pi; H_t(E)) \) where \( \pi \) acts on \( E \) as covering transformations turning \( H_s(E) \) into a \( \pi \)-module, and \( H(\pi; H_s(E)) \) is the group homology with coefficients in \( \pi \) [2].

2) \( E^\infty = CH(B) \) the associated graded group for some filtration of \( H(B) \).

Theorem: This spectral sequence is the same as the Moore spectral sequence of the principal fibration \( p : E \to B \) with fiber the discrete group \( \pi \).

Proof: The spectral sequence is obtained by filtering \( C \otimes C(E) \)

by resolution degree, where \( C = \{ C_n, d_n \}_{n \geq 0} \) and

\[
C_n \xrightarrow{d_n} C_{n-1} \to \ldots \to C_1 \xrightarrow{d_1} C_0 \to Z \to 0
\]

is exact and \( \pi \) acts on \( C_n \) on the left, \( d_n \) preserves the action of \( \pi \) and each \( C_n \) is \( \pi \)-free [2]. \( C \otimes C(E) \) is the quotient of \( C \otimes C(E) \) by the submodule generated by \( \{ a \otimes sb - as \otimes b \} \).

If we identify \( \pi \) with the fiber \( G \), we see \( C(G,Z) = Z(\pi) \) where

\( C(G,Z) = C(G) \) the normalized cubical chains of \( G \). With this terminology a \( \pi \)-free abelian group is nothing but a free \( Z(\pi) \)-module. Therefore \( C \) is a proper projective resolution for \( C(G,Z) = Z(\pi) \)-module \( Z \). Also

\( C \otimes C(E) = C \otimes C(E) = C \otimes C(E) \). Since the filtration is obtained

\( Z(\pi) \) \( C(G,Z) \)
by the resolution degree this spectral sequence coincides with the Moore spectral sequence. We also obtain:

\[ \text{tor}_{H(G,Z)}(Z,H(B,Z))_{s,t} = H_{t}(\pi;H_{s}(B,Z)) \]
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