TANGENTIAL CO-ORDINATES

by

Ralph Duncan James

A Thesis submitted for the Degree of

MASTER OF ARTS

in the Department

of

MATHEMATICS

The University of British Columbia

APRIL, 1930

TABLE OF CONTENTS

TANGENTIAL CO-ORDINATES.

1. Introduction. This paper outlines certain developments in the theory of Analytic Geometry, employing tangential or line co-ordinates. It supplements the work of Master's Theses written by Valgardsson of the University of Manitoba, and L. W. Heaslip of the University of British Columbia. In particular it treats of diameters of conics, evolutes, envelopes, loci of points, transformations between point and line co-ordinates, polar reciprocals of curves, and applications of the Calculus to line co-ordinate Geometry.

2. Important Relations From Earlier Theses. The relations listed below, (a) to(k) from Valgardsson's thesis, and (l) to (n) from Heaslip's, are required in the development of this paper.

(a) The co-ordinates of a line in rectangular line co-ordinates are defined as the reciprocals of the intercepts of the line on a set of rectangular axes.

(b) A point, other than the origin, whose point co-ordinates are (X, Y), has in line co-ordinates the equation,

$$Xx + Yy - 1 = 0 \qquad (1)$$

Note. Throughout this paper large letters are used to denote point co-ordinates, and small letters, line co-ordinates.

(c) The equation of the point of intersection of two lines, (x_1, y_1) and (x_2, y_2) is,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (2)$$

(d) The angle θ, between the negative extension of the y - axis and the line joining the origin to a point P, is called the vectorial angle of the point P, and

$$\tan\theta = -\frac{X}{Y} = \frac{dy}{dx} \qquad (3)$$

(e) The equation of the midpoint of the line joining two points P_1 and P_2 whose equations are,

$$X_1 x + Y_1 y - 1 = 0$$

and

$$X_2 x + Y_2 y - 1 = 0,$$

respectively, is,

$$(X_1 x + Y_1 y - 1) + (X_2 x + Y_2 y - 1) = 0 \qquad (4)$$

(f) Two lines (x_1, y_1) and (x_2, y_2) are

(i) parallel if $x_1 y_2 - x_2 y_1 = 0$

(ii) perpendicular if $x_1 x_2 + y_1 y_2 = 0$ (5)

(g) A curve may be considered as the envelope of a variable line, and its equation in line co-ordinates is called its line or tangential equation.

(h) The equation of the point of intersection of the tangents at the points where the line (x_1, y_1) cuts the conic,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is

$$(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0 \qquad (6)$$

(i) The equation of the pole of the line (x_1, y_1) with respect to the circle

$$x^2 + y^2 - 1 = 0$$

is
$$x_1 x + y_1 y - 1 = 0 \qquad (7)$$

(j) The general equation of the second degree represents a conic section. The nature of the conic is shown in the following table, where

$$\Theta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \ \xi = c(a+b) - (f^2 + g^2), \text{ and } \Delta = h^2 - ab$$

$\Theta \neq 0$ $c \neq 0$	Ellipse, if $\xi < 0$ and c and Θ agree in sign Imaginary locus if $\xi > 0$ and c and Θ agree in sign Hyperbola, if c and Θ differ in sign, while ξ has any value.
$\Theta \neq 0$ $c = 0$	Parabola, ξ is always negative.
$\Theta = 0$ $c \neq 0$	Two real points, if $\xi < 0$. Two imaginary points if $\xi > 0$.
$\Theta = 0$ $c = 0$ $\xi \neq 0$	Two distinct real points, one of which is infinitely distant, and the other is in a finite region of the plane.
$\Theta = 0$ $c = 0$ $\xi = 0$	Two infinitely distant real points if $\Delta > 0$ Two coincident and infinitely distant points if $\Delta = 0$ Two imaginary points on the line at infinity if $\Delta < 0$

(k) The equation of the centre of a central conic given by the general equation is,

$$gx + fy + c = 0 \qquad (8)$$

(l) The co-ordinates of a line in polar line co-ordinates are (ρ, θ), where ρ is the perpendicular distance from the pole to the line, and θ is the vectorial angle between the polar axis and this perpendicular.

(m) Any curve, whose equation in polar line co-ordinates is

$$f(\rho, \theta) = 0$$

has as its first positive pedal the curve given by the same equation

$$f(P, \Theta) = 0$$

in polar point co-ordinates.

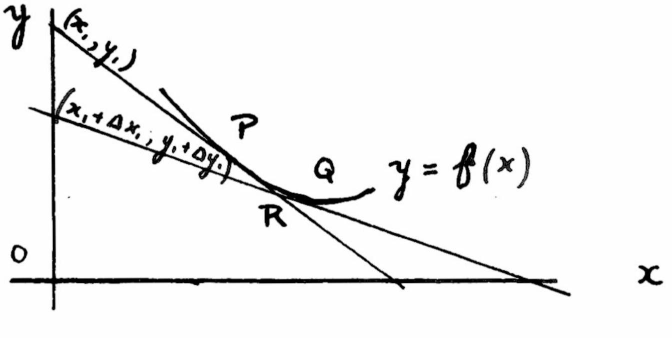
(n) If the equation of a curve is given in rectangular line co-ordinates, the equation of its first positive pedal in point co-ordinates may be found by replacing

$$x \text{ by } \frac{X}{X^2 + Y^2}, \text{ and } y \text{ by } \frac{Y}{X^2 + Y^2}.$$

Similarly, if the equation of the curve is given in rectangular point co-ordinates, the equation of its first negative pedal in line co-ordinates may be found by replacing

$$X \text{ by } \frac{x}{x^2 + y^2}, \text{ and } Y \text{ by } \frac{y}{x^2 + y^2}.$$

3. <u>Points of Contact.</u>



Let (x_1, y_1) be the co-ordinates of the tangent touching the curve $y = f(x)$ at the point P. Let $(x_1+\Delta x_1, y_1+\Delta y_1)$ be a neighboring tangent touching the curve at Q. The equation of the point of intersection R of the lines (x_1, y_1) and $(x_1+\Delta x_1, y_1+\Delta y_1)$, by equation (2), is,

$$\frac{y - y_1}{x - x_1} = \frac{\Delta y_1}{\Delta x_1} .$$

As Q approaches P along the curve, Δx_1 and Δy_1 approach zero. In the limit R and Q coincide with P. Therefore

$$\frac{y - y_1}{x - x_1} = \underset{\Delta x_1 \to 0}{L}\left(\frac{\Delta y_1}{\Delta x_1}\right) = \left(\frac{dy}{dx}\right)_1 \qquad (9)$$

is the equation of the point of contact, P', of the tangent (x_1, y_1) to the curve $y = f(x)$. The equation may also be written in the form

$$\frac{p_1}{p_1 x_1 - y_1}x - \frac{1}{p_1 x_1 - y_1}y - 1 = 0 \qquad (10)$$

4. <u>The Condition that a Point Shall Lie on a Conic.</u> Let the equations of the point and conic be

$$Xx + Yy - 1 = 0 \qquad (11)$$

and

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (12)$$

respectively. Substituting for y from (11) in (12), we obtain

$$(aY^2 - 2hXY + bX^2)x^2 + 2(hY - bX + gY^2 - fXY)x \qquad (13)$$
$$+ (b + 2fY + cY^2) = 0$$

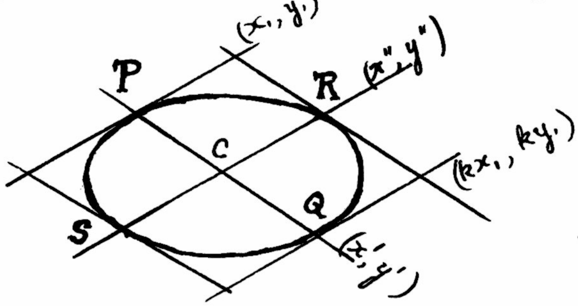
The point lies on the conic if and only if (13) has equal roots. The condition for equal roots is the vanishing of the discriminant $(hY - bX + gY^2 - fXY)^2 - (aY^2 - 2hXY + bX^2)(b + 2fY + cY^2)$. Accordingly

$$(f^2 - bc)X^2 + 2(ch - fg)XY + (g^2 - ac)Y^2$$
$$+ 2(fh - bg)X + 2(gh - af)Y + h^2 - ab = 0 \qquad (14)$$

is the condition that the point shall lie on the conic.

5. Diameters. Conjugate Diameters.

(i) The diameter of a central conic is defined as the line passing through the points of contact of parallel tangents to the conic.



Let (x_1, y_1) and (kx_1, ky_1) be parallel lines tangent to the conic $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

at the points P and Q respectively. The equations of P and Q are $\quad (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0$

and $\quad \{k(ax_1 + hy_1) + g\}x + \{k(hx_1 + by_1) + f\}y + k(gx_1 + fy_1) + c = 0$

respectively. On solving, we obtain the co-ordinates of the diameter through P and Q, viz.,

$$x' = -\frac{\begin{vmatrix} gx_1 + fy_1 + c, & hx_1 + by_1 + f \\ k(gx_1 + fy_1) + c, & k(hx_1 + by_1) + f \end{vmatrix}}{\begin{vmatrix} ax_1 + hy_1 + g, & hx_1 + by_1 + f \\ k(ax_1 + hy_1) + g, & k(hx_1 + by_1) + f \end{vmatrix}} = \frac{(ch - fg)x_1 + (bc - f^2)y_1}{(af - gh)x_1 + (fh - bg)y_1}$$

$$y' = -\frac{\begin{vmatrix} ax_1 + hy_1 + g, & gx_1 + fy_1 + c \\ k(ax_1 + hy_1) + g, & k(gx_1 + fy_1) + c \end{vmatrix}}{\begin{vmatrix} ax_1 + hy_1 + g, & hx_1 + by_1 + f \\ k(ax_1 + hy_1) + g, & k(hx_1 + by_1) + f \end{vmatrix}} = \frac{(g^2 - ac)x_1 + (fg - ch)y_1}{(af - gh)x_1 + (fh - bg)y_1} \tag{15}$$

On trial it is seen that

$$gx' + fy' + c = 0$$

and thus, equation (8), every diameter passes through the centre.

The diameter passing through the points of contact of tangents parallel to (x',y') is said to be conjugate to the diameter (x',y').

Let the co-ordinates of the two tangents parallel to (x',y') be $(k_1 x',k_1 y')$ and $(k_2 x',k_2 y')$ respectively. Let the points of contact of the tangents be R and S respectively. The respective equations of these points are $\{k_1(ax'+hy')+g\}x+\{k_1(hx'+by')+f\}y+k_1(gx'+fy')+c=0$ and $\{k_2(ax'+hy')+g\}x+\{k_2(hx'+by')+f\}y+k_2(gx'+fy')+c=0$

Solving, we obtain the co-ordinates of the diameter conjugate to (x',y'), viz.,

$$\left.\begin{aligned} x'' &= \frac{(ch-fg)x'+(bc-f^2)y'}{(af-gh)x'+(fh-bg)y'} \\ y'' &= \frac{(g^2-ac)x'+(fg-ch)y'}{(af-gh)x'+(fh-bg)y'} \end{aligned}\right\} \qquad (16)$$

Substituting x' and y' from equation (15), these reduce to

$$x'' = -\frac{cx_1}{gx_1+fy_1}, \quad y'' = -\frac{cy_1}{gx_1+fy_1} \qquad (17)$$

It is seen, equation (5 i), that the diameter (x'',y'') is parallel to the tangent (x_1,y_1), and by trial,

$$gx''+fy''+c = 0$$

Dividing x'' by y'' in equations (16), we obtain

$$\frac{x''}{y''} = \frac{(ch-fg)x'+(bc-f^2)y'}{(g^2-ac)x'+(fg-ch)y'}$$

or $$(g^2-ac)x'x''+(fg-ch)(x'y''+x''y')+(f^2-bc)y'y''=0 \qquad (18)$$

which is the condition that two diameters shall be conjugate. In the case of the ellipse

$$a^2x^2 + b^2y^2 - 1 = 0$$

this reduces to

$$\frac{x'x''}{y'y''} = -\frac{b^2}{a^2}$$

(ii) The parabola may be considered as the limiting case of a central conic, given by the general equation of the second degree, as the parameter c approaches zero. From this point of view, the diameter of the general parabola may be defined as the limiting position of the line passing through the points of contact of parallel tangents (x_1, y_1) and (kx_1, ky_1) to the general conic as c and k simultaneously approach zero. Accordingly, by reference to the equations (15) obtained under (i) of this section, the co-ordinates of the diameter passing through the point of contact of a tangent (x_1, y_1) to the general parabola are as follows,

$$\left.\begin{aligned} x' &= -\frac{f(gx_1+fy_1)}{(af-gh)x_1+(fh-bg)y_1} \\ y' &= \frac{g(gx_1+fy_1)}{(af-gh)x_1+(fh-bg)y_1} \end{aligned}\right\} \quad (19)$$

6. <u>The **Axes** of the General Conic.</u> Perpendicular conjugate diameters are the axes of the conic. Let the co-ordinates of conjugate diameters be (x', y') and (x'', y''). For perpendicularity, equation (5 ii), we have

$$\left(\frac{x'}{y'}\right)\left(\frac{x''}{y''}\right) = -1$$

Replacing $\frac{x''}{y''}$ by its value from equation (18), we obtain

$$(fg - ch)x'^2 + (f^2 - bc - g^2 + ac)x'y' + (ch - fg)y'^2 = 0$$

Since a diameter passes through the centre of the conic, we have $\quad gx' + fy' + c = 0$

On solving these two equations we obtain the co-ordinates of the axes, viz.,

$$\bar{x} = \frac{-2g(fg-ch) - f(f^2-g^2) - cf(a-b) \pm f\sqrt{\{(f^2-g^2)+c(a-b)\}^2 + 4(fg-ch)^2}}{2\{h(f^2-g^2)+fg(a-b)\}}$$

$$\bar{y} = \frac{2f(fg-ch) - g(f^2-g^2) - cg(a-b) \mp g\sqrt{\{(f^2-g^2)+c(a-b)\}^2 + 4(fg-ch)^2}}{2\{h(f^2-g^2)+fg(a-b)\}} \qquad (20)$$

7. <u>Properties of Diameters.</u> Let the chord (x_2, y_2) cut the conic $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

at the point P whose equation is

$$Xx + Yy + 1 = 0$$

Since the point P lies on the conic and also on the line, we must have the relations, (equation (14)),

$$(f^2-bc)X^2 + 2(ch-fg)XY + (g^2-ac)Y^2 + 2(fh-bg)X + 2(gh-af)Y + h^2 - ab = 0 \qquad (21)$$

and

$$Xx_2 + Yy_2 - 1 = 0 \qquad (22)$$

Substituting for Y from (22) in (21), we obtain

$$\begin{aligned} &\{(g^2-ac)x_2^2 - 2(ch-fg)x_2y_2 + (f^2-bc)y_2^2\}X^2 \\ &+2\{(ch-fg)y_2 - (g^2-ac)x_2 + (fh-bg)y_2^2 - (gh-af)x_2y_2\}X \\ &+\{(g^2-ac) + 2(gh-af)y_2 + (h^2-ab)y_2^2\} = 0 \end{aligned} \qquad (23)$$

and similarly,

$$\begin{aligned} &\{(g^2-ac)x_2^2 - 2(ch-fg)x_2y_2 + (f^2-bc)y_2^2\}Y^2 \\ &+2\{(ch-fg)x_2 - (f^2-bc)y_2 - (fh-bg)x_2y_2 + (gh-af)x_2^2\}Y \\ &+\{(f^2-bc) + 2(fh-bg)x_2 + (h^2-ab)x_2^2\} = 0 \end{aligned} \qquad (24)$$

Let the roots of equations (23) and (24) be X_1, X_2 and Y_1, Y_2 respectively. The equations of the points in which the line (x_2, y_2) cuts the conic are therefore

$$X_1 x + Y_1 y - 1 = 0 \tag{25}$$

and

$$X_2 x + Y_2 y - 1 = 0 \tag{26}$$

The equation of the midpoint of the line joining these two points, (equation (4)), is

$$(X_1 + X_2)x + (Y_1 + Y_2)y - 2 = 0$$

which, because of equations (23) and (24), becomes

$$\begin{aligned} &\left\{(ch-fg)y_2 - (g^2-ac)x_2 + (fh-bg)y_2^2 - (gh-af)x_2 y_2\right\} x \\ &+ \left\{(ch-fg)x_2 - (f^2-bc)y_2 - (fh-bg)x_2 y_2 + (gh-af)x_2^2\right\} y \\ &+ \left\{(g^2-ac)x_2^2 - 2(ch-fg)x_2 y_2 + (f^2-bc)y_2^2\right\} = 0 \end{aligned} \tag{27}$$

Let the line (x_2, y_2) be parallel to (x_1, y_1), in which case $x_2 = kx_1$ and $y_2 = ky_1$. Equation (27) thus becomes

$$\begin{aligned} &\left\{(ch-fg)y_1 - (g\ -ac)x_1 + (fh-bg)ky_1^2 - (gh-af)kx_1 y_1\right\} x \\ &+ \left\{(ch-fg)x_1 - (f^2-bc)y_1 - (fh-bg)kx_1 y_1 + (gh-af)kx_1^2\right\} y \\ &+ \left\{(g^2-ac)kx_1^2 - 2(ch-fg)kx_1 y_1 + (f^2-bc)ky_1^2\right\} = 0 \end{aligned}$$

By trial we see that the co-ordinates (x', y') of the diameter passing through the point of contact of the tangent (x_1, y_1) satisfy this equation. Hence the diameter bisects all chords parallel to the tangent (x_1, y_1).

The equation of the point of intersection of the tangents to the conic at the extremities of the chord (kx_1, ky_1), (which is parallel to (x_1, y_1)), by equation (6), is

$$\left\{k(ax_1 + hy_1) + g\right\} x + \left\{k(hx_1 + by_1) + f\right\} y + k(gx_1 + fy_1) + c = 0$$

On trial it is seen that the co-ordinates of the diameter (x', y') satisfy this equation. Therefore the tangents at the extremities of any chord parallel to a tangent (x_1, y_1) intersect on the diameter passing through the point of contact of the tangent (x_1, y_1).

The co-ordinates of the axes of the general parabola are obtained by putting $c = 0$ in equations (20). They are

$$\bar{x} = 0, \qquad -\frac{f(f^2+g^2)}{h(f^2+g^2)+fg(a-b)} . \tag{28}$$

$$\bar{y} = 0, \qquad \frac{g(f^2+g^2)}{h(f^2+g^2)+fg(a-b)} .$$

Comparing equations (19) and (28), we see that the diameters of a parabola are parallel to the axis.

8. <u>The Evolute of a Curve.</u> The evolute of a curve is defined as the envelope of the normals to the curve. Let the equation of the curve be

$$y = f(x) \tag{29}$$

Let (x_1, y_1) be any tangent to the curve and (x_2, y_2) the corresponding normal. Since the normal is perpendicular to the tangent and passes through its point of contact we have the relations

$$x_1 x_2 + y_1 y_2 = 0$$

and
$$\frac{p_1}{p_1 x_1 - y_1} x_2 - \frac{1}{p_1 x_1 - y_1} y_2 - 1 = 0$$

Solving for x_2 and y_2 and dropping the subscripts, we have

$$x = \frac{y_1(p_1 x_1 - y_1)}{x_1 + p_1 y_1}, \qquad y = -\frac{x_1(p_1 x_1 - y_1)}{x_1 + p_1 y_1} \tag{30}$$

It follows that the equation of the evolute is obtained on eliminating x_1 and y_1 between equations (29) and (30). If the equation of the curve is given in the form

$$f(x,y) = 0$$

then $\frac{dy}{dx} = -\frac{f_x}{f_y}$, and equations (30) may be written

$$x = \frac{y_1(x_1 f_{x_1} + y_1 f_{y_1})}{y_1 f_{x_1} - x_1 f_{y_1}}, \quad y = -\frac{x_1(x_1 f_{x_1} + y_1 f_{y_1})}{y_1 f_{x_1} - x_1 f_{y_1}}. \tag{31}$$

Example. To find the equation of the evolute of the ellipse

$$a^2x^2 + b^2y^2 - 1 = 0$$

$$f_{x_1} = 2a^2x_1 \quad , \quad f_{y_1} = 2b^2y_1 .$$

Therefore, equation (31),

$$x = \frac{y_1(2a^2x_1^2 + 2b^2y_1^2)}{(2a^2x_1y_1 - 2b^2x_1y_1)} = \frac{1}{(a^2-b^2)x_1}$$

$$y = -\frac{x_1(2a^2x_1^2 + 2b^2y_1^2)}{(2a^2x_1y_1 - 2b^2x_1y_1)} = -\frac{1}{(a^2-b^2)y_1}$$

and

$$\frac{a^2}{(a^2-b^2)^2x^2} + \frac{b^2}{(a^2-b^2)^2y^2} = a^2x_1^2 + b^2y_1^2 = 1$$

The equation of the evolute is

$$b^2x^2 + a^2y^2 = (a^2-b^2)^2x^2y^2.$$

9. <u>The Family of Curves $\lambda f_1(x,y) + \mu f_2(x,y) = 0$.</u> Let

$$f_1(x,y) = 0 \tag{32}$$

$$f_2(x,y) = 0 \tag{33}$$

be the tangential equations of two curves. Then

$$\lambda f_1(x,y) + \mu f_2(x,y) = 0 \tag{34}$$

is the equation of any curve touching the common tangents of the curves (32) and (33). This follows, since equation (34) is satisfied by the values of x and y for which both (32) and (33) vanish, and so is satisfied by the co-ordinates of the tangents, real or imaginary, which are common to the curves (32) and (33).

10. Envelopes of One-parameter Families of Curves. Loci of Points. The equation

$$f(x,y,\alpha) = 0 \tag{35}$$

defines a one-parameter family of curves or points. Consecutive curves or points of such a family are defined as curves which correspond to two consecutive values of the parameter.

The envelope of a family of curves given by equation (35) is defined as the envelope of the limiting position of common tangents to consecutive curves. Let α and $\alpha+\Delta\alpha$ be two consecutive values of the parameter. Then

$$f(x,\ y,\ \alpha\ ) = 0 \tag{36}$$

and

$$f(x,\ y,\ \alpha+\Delta\alpha\ ) = 0 \tag{37}$$

are consecutive curves, and

$$\frac{f(x,\ y,\ \alpha+\Delta\alpha) - f(x,\ y,\ \alpha\ )}{\Delta\alpha} = 0$$

is the equation of a curve touching the common tangents of the curves (36) and (37). Accordingly,

$$\underset{\Delta\alpha\to 0}{L}\left\{\frac{f(x,\ y,\ \alpha+\Delta\alpha) - f(x,\ y,\ \alpha\ )}{\Delta\alpha}\right\}$$

$$= f_\alpha(x,\ y,\ \alpha\ ) = 0 \tag{38}$$

is an equation satisfied by the co-ordinates of the common tangent to (36) and (37) in its limiting position. It follows that the equation of the envelope is obtained on eliminating α between equations (35) and (38).

The equation of a point involves in general two parameters. A point, however, may be restricted in position, through being constrained to satisfy some geometric condition. In that case the totality of points satisfying the condition constitutes a locus. Any condition imposed upon the point may be expressed as a relation between the parameters. It is thus possible to eliminate one of the parameters from the equation of the point. The problem of finding the equation of the locus of a point, satisfying a certain geometric condition, therefore reduces to that of finding the equation of the locus of a point whose equation involves a single parameter. Such an equation is linear in x and y and has the form (35) above. For a fixed value of α,

$$f(x, y, \alpha) = 0 \tag{39}$$

and

$$f(x, y, \alpha + \Delta\alpha) = 0$$

are consecutive points, P_1 and P_2 of the locus. The equation

$$\frac{f(x, y, \alpha + \Delta\alpha) - f(x, y, \alpha)}{\Delta\alpha} = 0$$

is the equation of a point on the line passing through P_1 and P_2. Accordingly

$$\underset{\Delta\alpha \to 0}{L} \left\{ \frac{f(x, y, \alpha + \Delta\alpha) - f(x, y, \alpha)}{\Delta\alpha} \right\}$$

$$= f_\alpha(x, y, \alpha) = 0 \tag{40}$$

is an equation which is satisfied by the coordinates of the limiting position of the line $P_1 P_2$. The locus is clearly the envelope of these limiting lines determined by letting α vary. The co-ordinates of any such line satisfy both

(39) and (40), and so must satisfy the equation obtained on eliminating α between (39) and (40). This resultant equation is, accordingly, that of the locus. It is thus seen that the method of determining the locus of a point is identical with that of finding the envelope of a one-parameter family of curves.

A simpler treatment is possible when the equation of the point, for fixed values of x and y, is algebraic in terms of the parameter α.

(i) Consider the equation of a point

$$\alpha f_1(x, y) + f_2(x, y) = 0 \qquad (41)$$

where α is a variable parameter, and $f_1(x, y) = 0$, $f_2(x, y) = 0$ are linear in x and y and independent of α. Equation (41) is satisfied by the co-ordinates of the line passing through the two points whose equations are

$$f_1(x, y) = 0$$

$$f_2(x, y) = 0,$$

respectively. In other words, the locus of a point is a straight line when the equation of the point contains a single variable parameter which enters to the first degree only.

(ii) Consider the equation of a point

$$\alpha^2 f_1(x, y) + \alpha f_2(x, y) + f_3(x, y) = 0 \qquad (42)$$

where α, f_1, f_2, f_3 have meanings as above. For a fixed line, (x_1, y_1) there are two values of α for which (42) is satisfied. If (x_1, y_1) is tangent to the locus of (42), the two values become coincident. The values of x and y for which (42) has equal roots are given by equating the discriminant to zero. Thus

$$\left\{f_2(x,\ y)\right\}^2 - 4\left\{f_1(x,\ y)\right\}\left\{f_3(x,\ y)\right\} = 0$$

is the equation satisfied by all values of x and y which are the co-ordinates of tangents to (42). It is therefore the equation of the locus, and since it is of the second degree, the locus must be a conic section.

(iii) Consider the equation of a point

$$f(x,\ y,\alpha) = \alpha^n f_1(x,\ y) + \ldots\ldots\ldots + f_{n+1}(x,\ y) = 0 \qquad (43)$$

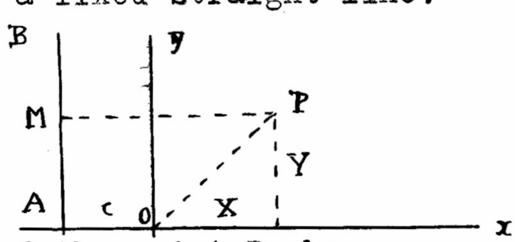
The equation of the locus of (43) is obtained by equating to zero the discriminant of (43) considered as an equation in α. This discriminant is found by eliminating α between (43) and $f_\alpha(x,\ y,\alpha) = 0$, which is precisely the method given in a previous paragraph for finding the equation of an envelope.

Example 1. The locus of the point whose equation is

$$\alpha^2 x - \alpha y - 1 = 0$$

is clearly $\qquad y^2 = 4x.$

Example 2. To find the locus of a point which moves so that its distance from a fixed point is always a constant times its distance from a fixed straight line.



Let the equation of the point P be

$$Xx + Yy - 1 = 0 \tag{44}$$

Take the fixed point at the origin, and let AB, (-c, 0), be the fixed straight line. From

$$|OP| = e.|PM| \qquad \text{(e, a constant), we have}$$

$$X^2 + Y^2 = e^2(X + c)^2$$

or

$$Y^2 = (e^2 - 1)X + 2e^2cX + e^2c^2 \tag{45}$$

Substituting in equation (44), we obtain

$$\left\{(e^2-1)X^2 + 2e^2cX + e^2c^2\right\} y^2 = (1 - Xx)^2$$

This equation cannot be put in the form of equation (42) where f_1, f_2 and f_3 are independent of X and linear in x and y, and thus we use the general method. Differentiating equation (45) with respect ot X, we have

$$X = - \frac{y^2e^2c + x}{e^2x^2 - x^2 - y^2}$$

Making this substitution, equation (45) reduces to

$$e^2c^2(x^2+y^2) + 2e^2cx - (1 - e^2) = 0 \tag{46}$$

The method of obtaining this equation shows that every point on the locus lies on the curve (46). We must now show that every point on the curve (46) is on the locus. Suppose that the point P is on the curve (46), but not on the locus. Thus we have

$$X_2^2 + Y_2^2 = e^2(X_2 + c)^2 \tag{47}$$

and

$$|OP_2| \neq e\,|P_2M|$$

From equation (47) it follows that

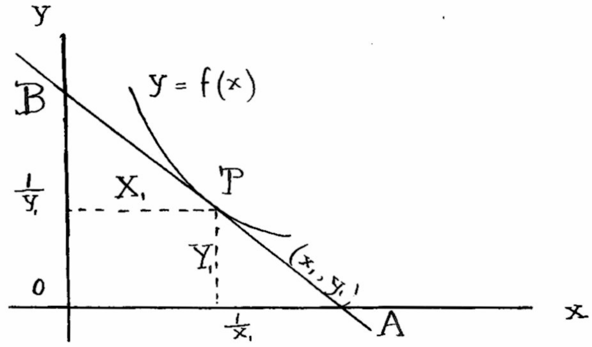
$$\pm\sqrt{X_2^2 + Y_2^2} = e\,(X_2 + c)$$

whence

$$|OP_2| = e\,|P_2M|$$

in contradiction to our hypothesis. Accordingly, every point on the curve (46) is on the locus.

11. Transformations Between Point and Line Co-ordinates.



Let (x_1, y_1) be the co-ordinates of the tangent AB to the curve $y = f(x)$ at the point P. Let the point co-ordinates of P be (X_1, Y_1). The equation of P is

$$X_1 x + Y_1 y - 1 = 0$$

However, since P is the point of contact of thetangent (x_1, y_1) its equation is

$$\frac{p_1}{p_1 x_1 - y_1} x - \frac{1}{p_1 x_1 - y_1} y - 1 = 0$$

where $p_1 = \left(\frac{dy}{dx}\right)_1$.

Comparing these equations, it is seen that

$$X = \frac{p_1}{p_1 x_1 - y_1}, \quad Y = -\frac{1}{p_1 x_1 - y_1}$$

The equation of the line AB in point co-ordinates is

$$x_1 X + y_1 Y - 1 = 0$$

Since AB is tangent to the curve, its equation is

$$Y - Y_1 = P_1(X - X_1)$$

where $P_1 = \left(\frac{dY}{dX}\right)_1$.

This may be written

$$\frac{P_1}{P_1 X_1 - Y_1} X - \frac{1}{P_1 X_1 - Y_1} Y - 1 = 0$$

whence we have

$$x = \frac{P_1}{P_1 X_1 - Y_1}, \quad y = -\frac{1}{P_1 X_1 - Y_1}.$$

Hence, dropping primes, the equations

$$X = \frac{p}{p\,x - y}, \quad Y = -\frac{1}{p\,x - y} \qquad (48), \qquad x = \frac{P}{P\,X - Y}, \quad y = -\frac{1}{P\,X - Y} \qquad (49)$$

give the transformations from point to line co-ordinates, and from line to point co-ordinates respectively.

Let the tangential equation of a curve be

$$y = f(x) \qquad (50)$$

The substitutions of equations (48) give

$$-\frac{1}{PX - Y} = f\left(\frac{P}{PX - Y}\right)$$

or

$$F(X, Y, P) = 0 \qquad (51)$$

In general, P appears in this equation to a degree higher than the first, and thus for any given X and Y, P will not be single-valued. However, to every tangent (x, y) satisfying equation (50) there is a point of contact (X, Y) and one and only one corresponding value of the slope P, such that X, Y, and P satisfy equation (51). Those values of X and Y for which equation (51) gives equal roots for P are those satisfying the P-discriminant relation. This relation contains the singular solution of the differential equation (51).[1] Thus we see that the

---

[1] See D. A. Murray, Differential Equations, § 33, page 42.

equation of the curve (50) in point co-ordinates is the singular solution of (51), that is, the envelope of the general solution of (51).

If (50) is an algebraic equation, than on clearing of fractions, equation (51) will be a polynomial in $(PX-Y)$, with coefficients functions of P. Its solutions are of the form

$$PX - Y = f(P) \qquad (52)$$

This equation is a Clairaut equation,[1] and has for solution

$$CX - Y = f(C) \qquad (53)$$

which represents a family of straight lines. The singular solution of equation (52) is the C - discriminant relation of equation (53). That is, the required equation in point co-ordinates is the envelope of the family of straight lines (53).

Obviously the above argument and indicated procedure also applies, on a change of notation, when passing from point to line co-ordinates.

Example 1. An ellipse, with centre at the origin, and axes coinciding with the co-ordinate axes, has in line co-ordinates the equation

$$a^2x^2 + b^2y^2 = 1,$$

where a and b are the semi-major and semi-minor axes respectively. Applying the transformations of equations (49), we obtain

$$(a^2 - X)P^2 + 2XYP + (b^2 - Y) = 0$$

The P - discriminant relation is

[1] See: D. A. Murray, Differential Equations, § 33, page 42

$$X^2Y^2 - (a^2 - X)(b^2 - Y) = 0$$

or

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

which is the equation of the ellipse in point co-ordinates.

Example 2. The equation of the astroid in point co-ordinates is

$$X^{\frac{2}{3}} + Y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

Applying the transformations (48), we have

$$p^{\frac{2}{3}} + 1 = a^{\frac{2}{3}}(px - y)^{\frac{2}{3}} \tag{54}$$

Differentiating with respect to p, we obtain

$$p^{-\frac{1}{3}} = a^{\frac{2}{3}}x(px - y)^{-\frac{1}{3}}$$

or

$$a^2px^3 = px - y$$

Making this substitution, equation (54) becomes

$$p^{\frac{2}{3}} = \frac{1}{a^2x^2 - 1} \tag{55}$$

Eliminating p between equations (54) and (55), we obtain

$$x^2 + y^2 = a^2x^2y^2, \tag{56}$$

the tangential equation of the astroid.

The astroid may be defined as the envelope of a line, the sum of the squares of whose intercepts on the axes is a constant.

Accordingly, its tangential equation is

$$\left(\frac{1}{x}\right)^2 + \left(\frac{1}{y}\right)^2 = a^2$$

or

$$x^2 + y^2 = a^2x^2y^2$$

The transformations (49) give

$$(P^2+1)(PX-Y)^2 = a^2P^2 \tag{57}$$

Differentiating with respect to P, we have

$$2(P^2+1)(PX-Y)X+2P(PX-Y)^2=2a^2P \tag{58}$$

From equation (57) we obtain

$$(PX-Y)^2=\frac{a^2P}{P^2+1}$$

On making this substitution, equation (58) becomes

$$P^2+1=\frac{a^{\frac{2}{3}}}{X^{\frac{2}{3}}} \tag{59}$$

Eliminating P between equations (58) and (59), we have

$$X^{\frac{2}{3}}+Y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

the point equation of the astroid.

12. Polar Reciprocals. Let the point equation of a curve be

$$f(X, Y)=0 \tag{60}$$

The question arises, what locus does

$$f(x, y)=0 \tag{61}$$

represent in line co-ordinates? Evidently to every point (X_1, Y_1) satisfying equation (60) there corresponds a tangent $(x_1 = X_1,\ y_1 = Y_1)$ satisfying equation (61). Since this tangent has intercepts $\frac{1}{x_1}=\frac{1}{X_1}$ and $\frac{1}{y_1}=\frac{1}{Y_1}$, its point equation is

$$X_1X+Y_1Y-1=0$$

This is clearly the equation of the polar of the point (X_1, Y_1) with respect to the circle

$$X^2+Y^2-1=0$$

Thus we see that the curve (61) is the envelope of the polars of points on the curve (60). That is, it is the polar reciprocal[1] of (60). Therefore a curve in rectangular point co-ordinates and its polar reciprocal in rectangular line co-ordinates have identical equations. By reference to the footnote it is easily seen that a curve in rectangular line co-ordinates and its polar reciprocal in point co-ordinates have identical equations. Hence the transformations (48) and (49) when applied to the equation of any curve give its polar reciprocal,. the equation of the curve and that of its polar reciprocal being in the same system of co-ordinates.

13. Geometric Interpretation of Certain Integrals.

An interpretation can now be given for any integral in line co-ordinates. For, from the previous section, it is seen that the usual interpretation applies to the polar reciprocal of the given curve and not to the curve itself. Thus $\int \sqrt{1 + \left(\frac{dY}{dX}\right)^2}\,.\,dX$ gives the length of arc of a curve in point co-ordinates, while $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.\,dx$ gives the

---

1 The polar reciprocal of a curve, with respect to the circle $X^2 + Y^2 = 1$, may be defined as the envelope of the polars of points on the curve, or, as the locus of the poles of tangents to the curve.

length of arc of the polar reciprocal of a curve whose equation is in line co-ordinates. The integral $\int Y . dX$ gives the area under a curve in point co-ordinates, while $\int y . dx$ gives the area under the polar reciprocal of a curve whose equation is in line co-ordinates.

Again, (§2, (m)), any curve with equation $f(\rho, \theta) = 0$ in polar line co-ordinates has as its first positive pedal the curve given by the same equation $f(P, \theta) = 0$ in polar point co-ordinates. Hence the interpretation of an integral in polar line co-ordinates is applied to the first positive pedal of the curve.

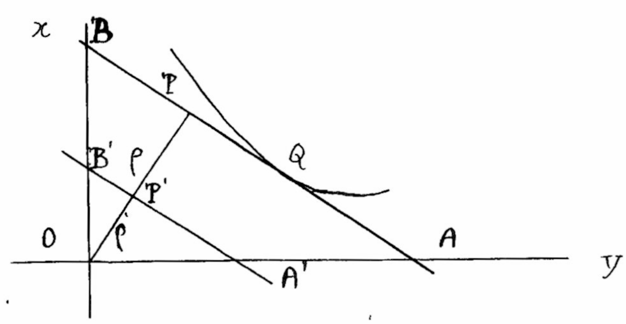
14. Pedal Curves. Let the tangential equation of a curve be $f(x, y) = 0$. The equation of its polar reciprocal is found in point co-ordinates by replacing x by X and y by Y. The equation of its first positive pedal is found in point co-ordinates by replacing x by $\frac{X}{X^2 + Y^2}$ and y by $\frac{Y}{X^2 + Y^2}$. Thus we see that the equation of the first positive pedal of any curve in point co-ordinates may be obtained from the equation of the polar reciprocal of the curve in the same system of co-ordinates by replacing X by $\frac{X}{X^2 + Y^2}$ and Y by $\frac{Y}{X^2 + Y^2}$. Accordingly the first positive pedal of any curve is the inverse of the polar reciprocal of the curve.

Similarly, the tangential equation of the polar reciprocal of any curve in point co-ordinates is obtained by replacing X by x and Y by y.

The equation of the first positive pedal of the polar reciprocal is found by replacing x by $\frac{X}{X^2+Y^2}$, and y by $\frac{Y}{X^2+Y^2}$. Therefore, the first positive pedal of the polar reciprocal of any curve is the inverse of the curve.

15. The Transformation $x = \frac{x}{x^2+y^2}$, $y = \frac{y}{x^2+y^2}$.

By means of the transformation $X = \frac{X}{X^2+Y^2}$, $Y = \frac{Y}{X^2+Y^2}$, we may write the equation of the inverse of any curve. The question arises whether the transformation $x = \frac{x}{x^2+y^2}$, $y = \frac{y}{x^2+y^2}$ has a geometrical significance. As the figure shows, let AB be a tangent (x, y) to the curve $f(x, y) = 0$. Let A' B' be the line $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ and $OP=\rho$ and $OP'=\rho'$ be the perpendiculars from the origin on the lines AB and A' B' respectively.



AB is parallel to A'B' since $\frac{x}{y} = \frac{\frac{x}{x^2+y^2}}{\frac{y}{x^2+y^2}}$. Whence we have

$$\frac{OA}{OA'} = \frac{OP}{OP'} = \frac{\rho}{\rho'}$$

or

$$\frac{\frac{1}{x}}{\frac{x}{x^2+y^2}} = \frac{\rho}{\rho'}$$

so that

$$\rho\rho' = 1$$

Hence P and P' are inverse points. From this it follows that $A'B'$ is the polar of the point P. Therefore, the envelope of $A'B'$ is the polar reciprocal of the locus of P, that is, of the first positive pedal of the curve $f(x,y) = 0$. Hence the transformations

$$x = \frac{X}{X^2 + Y^2}, \qquad y = \frac{Y}{X^2 + Y^2} \tag{62}$$

when applied to the tangential equation of a curve, give the equation of the polar reciprocal of the first positive pedal of the curve.

Applying (62) to the equation of a point P

$$Xx + Yy - 1 = 0 \tag{63}$$

we have

$$x^2 + y^2 - Xx - Yy = 0 \tag{64}$$

which, § 2 (j), is the equation of a parabola. By means of equation (14), we see that the inverse point of P

$$\frac{X}{X^2 + Y^2}x + \frac{Y}{X^2 + Y^2}y - 1 = 0$$

lies on the parabola, and also, equations (28), lies on the axis of the parabola. It is therefore the vertex. It follows that the equations (62) transform a point into a parabola whose vertex is the inverse point of the given point.