A GROUP ANALYSIS OF NONLINEAR DIFFERENTIAL EQUATIONS

by

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ABSTRACT

A necessary and sufficient condition is established for the existence of an invertible mapping of a system of nonlinear differential equations to a system of linear differential equations based on a group analysis of differential equations. It is shown how to construct the mapping, when it exists, from the invariance group of the nonlinear system. It is demonstrated that the hodograph transformation, the Legendre transformation and Lie's transformation of the Monge-Ampère equation are obtained from this theorem. The equation $(u_x)^p u_{xx} - u_{yy} = 0$ is studied and it is determined for what values of p this equation is transformable to a linear equation by an invertible mapping.

Many of the known non-invertible mappings of nonlinear equations to linear equations are shown to be related to invariance groups of equations associated with the given nonlinear equations. A number of such examples are given, including Burgers' equation $u_{xx} + uu_x - u_t = 0$, a nonlinear diffusion equation $(u^{-2}u_x)_x - u_t = 0$, equations of wave propagation $\{v_y - w_x = 0,\ v_y - avw - bv - cw = 0\}$, equations of a fluid flow $\{w_y + v_x = 0,\ w_x - v^{-1}w^p = 0\}$ and the Liouville equation $u_{xy} = e^u$.

As another application of group analysis, it is shown how conservation laws associated with the Korteweg-deVries equation, the cubic Schrödinger equation, the sine-Gordon equation and Hamilton's field equation are related to the

invariance groups of the respective equations.

All relevant background information is in the thesis, including an appendix on the known algorithm for computing the invariance group of a given system of differential equations.

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INTRODUCTION.

A variety of transformations arises in the study of differential equations. Two important classes of transformations are integral transformations, which include the Laplace and Fourier transformations, and geometrical transformations, which include contact and point transformations. In this work we are concerned with transformations of the latter type. One of the important aspects of geometrical transformations is that their formulations generally do not depend on the linearity of differential equations while those of integral transformations depend critically on it. For this particular reason, in connection with recent developments in nonlinear physics, there has arisen a revived interest in various geometrical theories of differential equations [1].

In this thesis we focus our attention on a group analysis of differential equations [2-9] and its use for the study of relationships between differential equations. More specifically, we are interested in answering the question <u>when a given system of nonlinear differential equations can be transformed into a system of linear differential equations</u>. There is a good reason to believe that a group analysis of differential equations is helpful in answering this question.

Before we elaborate the motivation, we need to review some basic ideas of geometrical transformations important for the study of differential equations. In order to keep the geometrical picture simple we only consider the case involving one dependent variable u and one independent variable x.

Let w be a vector space with coordinates $(x,z,z_1,z_2,\dots)$ where $x\in R$, $z\in R$, $z_k\in R$ and consider a mapping $w\to w$:

$$x'=\bar{x}'(x,z,z_1,\dots,z_p),\quad z'=\bar{z}'(x,z,z_1,\dots,z_q),\quad z'_k=\bar{z}'_k(x,z,z_1,\dots,z_{q_k}). \tag{0.1}$$

where $\bar{x}'$, $\bar{z}'$ and $\bar{z}'_k$ are sufficiently differentiable functions of their arguments and $k=1,2,3,\dots$. Let $u(x)$ be a function $R\to R$, sufficiently differentiable in the domain of interest and $u_k(x)=(d/dx)^k u(x)$. If we set

$$z = u,\qquad z_k = u_k,\quad k=1,2,3,\dots, \tag{0.2}$$

then the set $w[u]$ consisting of points

$$(x,z,z_1,z_2,\dots) = (x,u,u_1,u_2,\dots),\qquad x\in R \tag{0.3}$$

defines a curve (more generally a manifold) in w (Fig.1). Under the transformation (0.1) the curve $w[u]$ is mapped into $(w[u])'$ whose equation is obtained by introducing (0.2) into (0.1):

$$x'=\bar{x}'(x,u,u_1,\dots,u_p),\quad z'=\bar{z}'(x,u,u_1,\dots,u_q),\quad z'_k=\bar{z}'_k(x,u,u_1,\dots,u_{q_k})\ . \tag{0.4}$$

Solving the first equation of (0.4) for x and introducing it into the rest of (0.4) we obtain $\bar{z}'$, $\bar{z}'_k$ as functions of x':

$$z'=u'(x'),\quad z'_k=v^{(k)}(x').$$

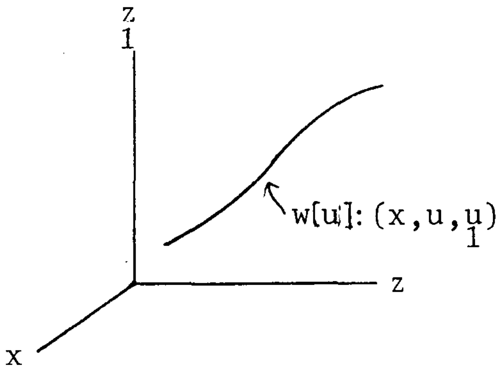

Fig.1

Obviously not all transformations (0.1) have the property

$$v^{(k)}(x') = (d/dx')^k\ u'(x') \equiv u'_k. \tag{0.5}$$

When the equality (0.5) is satisfied for any choice of u(x), i.e.,

$$z' = u',\quad z'_k = u'_k,\quad k=1,2,3,\dots\ , \tag{0.6}$$

we call transformation (0.1) a <u>contact transformation</u>. Namely, a contact transformation maps a curve w[u] defined by (0.3) into a curve w[u'] (Fig.2) consisting of points

$$(x,z,z_1,z_2,\dots) = (x',u',u'_1,u'_2,\dots),\quad x'\in R.$$

It is intuitively clear that in contact transformations

$\bar{z}'_k$ is related to $\bar{x}'$ and $\bar{z}'$ because $\bar{z}'_k$ must behave as the k-th derivative of $\bar{z}'$ with respect to x' when (0.3) is introduced. Obtaining an explicit form of $\bar{z}'_k$ in terms of $\bar{x}'$ and $\bar{z}'$ from the condition that (0.2) yields (0.6) is not only messy but also becomes confusing for vector x and vector z.

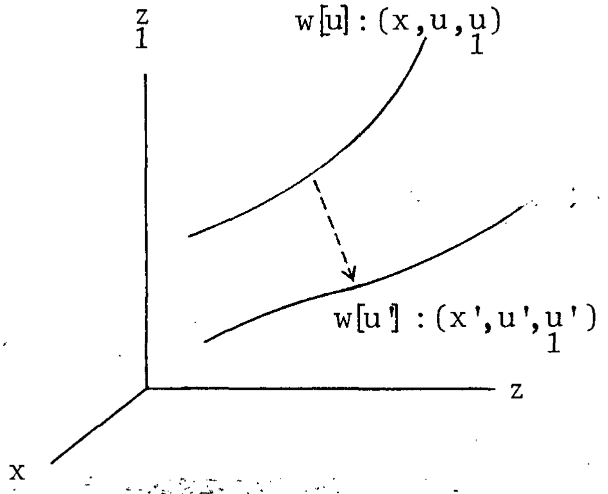

Fig. 2

It is very convenient to replace Eq.(0.2) and Eq.(0.6) by equivalent differential forms. Eq.(0.2) implies

$$dz = u_1 dx, \quad dz_1 = u_2 dx, \quad dz_2 = u_3 dx, \quad \ldots, \tag{0.7}$$

and if we use (0.2) it can be written as

$$dz = z_1 dx, \quad dz_1 = z_2 dx, \quad dz_2 = z_3 dx, \quad \ldots . \tag{0.8}$$

If we let z = u in the first equation of (0.8), we find $z_1=u_1$ and if we let $z_1=u_1$ in the second we find $z_2=u_2$ and so on, recovering (0.2). Thus, we may replace (0.2) by (0.8) which we represent collectively by $\{dz_k - z_{k+1}dx=0\}$. Similarly, we replace (0.6) by $\{dz'_k - z'_{k+1}dx'=0\}$. We redefine a contact transformation as a transformation (0.1) which has the property that:

$$\text{if } \{dz_k - z_{k+1}\,dx=0\},\ \text{then } \{dz'_k - z'_{k+1}dx'=0\}. \tag{0.9}$$

We call (0.8) a tangent (or contact) condition.†

The simplest contact transformation is a point transformation:††

$$x' = \bar{x}'(x,z), \quad z' = \bar{z}'(x,z), \quad z'_k = \bar{z}'_k(x,z,z_1,\dots,z_k), \tag{0.10}$$

and the next simplest one is Lie's contact transformation

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† The first equation of (0.8) written in the form of scalar product $(dz,dx)\cdot(1,-z_1)=0$ implies that the vector (dz,dx) is perpendicular to the vector $(1,-z_1)$ which represents a normal vector to a curve $z-u=0$ when z_1 is replaced by u_1. Consequently, the variation (dz,dx) must always be tangent to the curve $z-u=0$. In order to determine $\bar{z}'_1$ from $\bar{z}'$ and $\bar{x}'$ we introduce $\bar{z}',\bar{x}'$ into $dz'-z'_1dx'=0$, i.e.,

$$0=d\bar{z}'-z'_1d\bar{x}'=(\bar{z}'_xdx+\bar{z}'_zdz+\bar{z}'_{z_1}dz_1+\dots) - z'_1(\bar{x}'_xdx+\bar{x}'_zdz+\bar{x}'_{z_1}dz_1+\dots),$$

where $\bar{z}'_x=\partial_x\bar{z}'$, $\bar{x}'_x=\partial_x\bar{x}'$, ... We eliminate dz, dz_1, dz_2,... using (0.8), and obtain

$$0=(\bar{z}'_x + \bar{z}'_z z_1 + \bar{z}'_{z_1} z_2 + \dots) - z'_1(\bar{x}'_x + \bar{x}'_z z_1 + \bar{x}'_{z_1} z_2 + \dots).$$

Solving this for z'_1, we obtain $\bar{z}'_1$. Using this $\bar{z}_1$ in $dz'_1-z'_2dx'=0$ we find $\bar{z}'_2$, and so on.

†† Usually a set consisting of the first two equations in (0.10) is called a point transformation. In the following we call (0.10) a point transformation.

$$x' = \bar{x}'(x,z,\underset{1}{z}), \quad z' = \bar{z}'(x,z,\underset{1}{z}), \quad \underset{k}{z}' = \underset{k}{\bar{z}}'(x,z,\underset{1}{z},\ldots,\underset{k}{z}).^{\dagger} \quad (0.11)$$

The most general contact transformation of the form (0.1) was considered by Bäcklund [26].

A particularly important class of contact transformations is that of infinitesimal contact transformations. Consider a transformation

$$\begin{aligned} x' &= x + \varepsilon\xi(x,z,\underset{1}{z},\ldots,\underset{p}{z}) \\ z' &= z + \varepsilon\zeta(x,z,\underset{1}{z},\ldots,\underset{q}{z}) \\ \underset{k}{z}' &= \underset{k}{z} + \varepsilon\underset{k}{\zeta}(x,z,\underset{1}{z},\ldots,\underset{q_k}{z}), \quad k=1,2,3,\ldots \end{aligned} \quad (0.12)$$

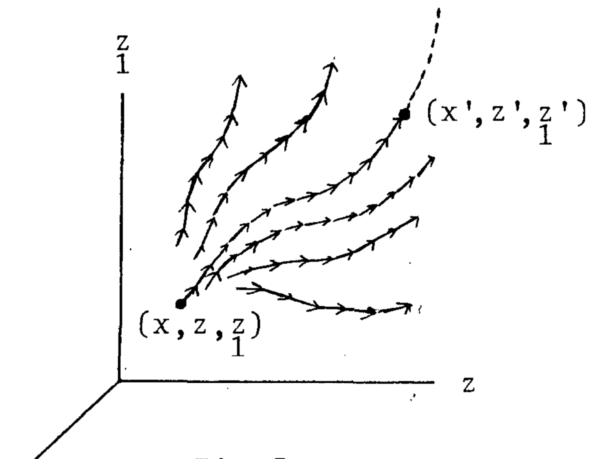
where ξ,ζ and $\underset{k}{\zeta}$ are functions of their arguments and ε is a small parameter. We call (0.12) an <u>infinitesimal contact transformation</u> if it satisfies the condition (0.9) to order $O(\varepsilon)$. When $\xi=\xi(x,z)$, $\zeta=\zeta(x,z)$ the transformation (0.12) is called an <u>infinitesimal point transformation</u> and when ξ,ζ and $\underset{1}{\zeta}$ are functions of $x,z,\underset{1}{z}$, the transformation is called <u>Lie's infinitesimal contact transformation</u>. All other cases

† Usually the term contact transformation refers to Lie's contact transformation. Lie's contact transformation has the property of mapping two curves $z=u(x)$ and $z=v(x)$ which are tangent to each other at x_0 into two curves $z'=u'(x')$ and $z'=v'(x')$ which are also tangent at the transformed point x_0'. Under the definition (0.9), not all contact transformations have this property.

will be called <u>higher order infinitesimal contact transformations</u>. As in the case of finite contact transformations (0.1), once ξ and ζ are given the function ζ_k are determined from the condition (0.9) as described in the footnote on page five.

A succession of infinitesimal transformations (0.12) leads to a finite transformation which is called a <u>group transformation</u>. Its geometrical picture is the following. Eq.(0.12) associates a variation vector $(\Delta x, \Delta z, \Delta z_1, \ldots) = \varepsilon(\xi, \zeta, \zeta_1, \ldots)$ with every point $(x, z, z_1, \ldots)$ and defines a flow in the space w (Fig.3). Let the equation of a flow curve originating at a point $(x, z, z_1, \ldots)$ be

$$\begin{aligned} x' &= \bar{x}'(x, z, z_1, \ldots; a) \\ z' &= \bar{z}'(x, z, z_1, \ldots; a) \qquad (0.13) \\ z_k' &= \bar{z}_k'(x, z, z_1, \ldots; a), \end{aligned}$$



<u>Fig.3</u>

where <u>a</u> is a parameter of the curve. For simplicity, we denote $\omega = (x, z, z_1, \ldots)$ and write (0.13) as

$$\omega' = \bar{\omega}'(\omega; a) = T(a)\omega, \qquad (0.14)$$

where T(a) is generally a nonlinear operator acting on ω.

The transformation (0.14) forms a one-parameter group of transformations. Namely, there exists a parametrization with parameter a such that

$$\bar{\omega}'(\bar{\omega}'(\omega;a);b) = \bar{\omega}'(\omega;a+b), \quad \bar{\omega}'(\omega;0) = \omega, \tag{0.15}$$

or equivalently,

$$T(b)T(a) = T(a+b), \quad T(0) = I, \tag{0.16}$$

where I represents the identity transformation. The explicit form of the operator T(a) is given by

$$T(a) = e^{a\ell} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \ell^n, \tag{0.17}$$

where ℓ is defined by

$$\ell = \xi\partial_x + \zeta\partial_z + \underset{1}{\zeta}\partial_{\underset{1}{z}} + \underset{2}{\zeta}\partial_{\underset{2}{z}} + \ldots , \tag{0.18}$$

The operator ℓ is called a generator of the group T(a). If (0.12) is an infinitesimal contact transformation, then its "integrated form" (0.13) is also a contact transformation which we call a group contact transformation. Depending on the type of infinitesimal contact transformations mentioned above we call the corresponding group contact transformations a point group transformation, Lie's group contact transformation or a higher order group contact transformation.

The first two group transformations were studied extensively by Lie [2,4] in the last century while the higher order group contact transformations were introduced recently by Anderson, Kumei and Wulfman [9,10,11,12]. They are also called Lie-Bäcklund (L-B) group transformations [13] .

Up to now a picture of differential equations has been absent. To find the significance of contact transformations in the study of differential equations we consider an equation

$$f(x,z,\underset{1}{z},\ldots,\underset{n}{z}) = 0, \tag{0.19}$$

where f is an analytic function $R^{n+2} \to R$. Eq.(0.19) defines a hypersurface (or a manifold) in the space $(x,z,\underset{1}{z},\ldots,\underset{n}{z})$. We again consider (0.2). Let $w^{(n)}[u]$ be the set consisting of points

$$(x,z,\underset{1}{z},\ldots,\underset{n}{z}) = (x,u,\underset{1}{u},\ldots,\underset{n}{u}), \quad x \in R. \tag{0.20}$$

The set $w^{(n)}[u]$ defines a curve (generally a manifold) in the space $(x,z,\underset{1}{z},\ldots,\underset{n}{z})$. Let us demand that the curve $w^{(n)}[u]$ be imbedded on the hypersurface f=0 (Fig.4). This will be possible only if the function u happens to be a solution of the differential equation

$$f(x,u,\underset{1}{u},\ldots,\underset{n}{u}) = 0. \tag{0.21}$$

When u solves (0.21), we call $w^{(n)}[u]$ a solution curve (more generally a solution manifold). The projection of the curve onto the x-z plane defines a solution curve $z=u$ in the usual sense (Fig.4).

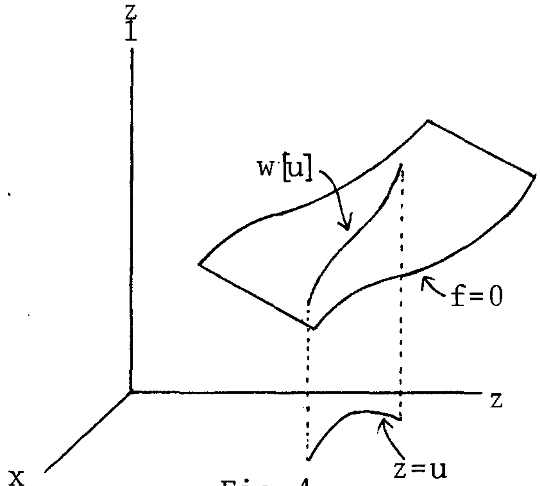


Fig.4

We now suppose that the contact transformation (0.1) maps the hypersurface $f=0$ into a hypersurface

$$f'(x',z',z'_1,\dots,z'_n) = 0. \tag{0.22}$$

Since the transformation is a contact transformation, the curve $w^{(n)}[u]$ is mapped into the curve $w^{(n)}[u']$ (Fig.2) consisting of points

$$(x,z,z_1,\dots,z_n) = (x',u',u'_1,\dots,u'_n), \qquad x'\in R. \tag{0.23}$$

Obviously, if the curve $w^{(n)}[u]$ is on the surface $f=0$, then the curve $w^{(n)}[u']$ must be on the surface $f'=0$ (Fig.5), namely,

$$f'(x',u',u'_1,\dots,u'_n) = 0. \tag{0.24}$$

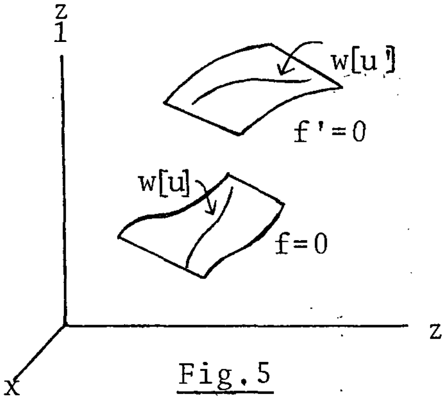


Fig.5

In other words, a contact transformation maps a solution of the differential equation (0.21) into a solution of the differential equation (0.24). When the contact transformation maps the hypersurface f=0 into itself, the transformation is called an <u>invariance contact transformation</u> of f=0. Obviously, such a transformation maps a solution of the differential equation (0.21) into another solution of the same equation (Fig.6). When a solution happens to be mapped into itself under such a transformation, it is called an <u>invariant solution</u> of the transformation. Particularly important invariance contact transformations are those which form groups in the sense of (0.13)-(0.15). We call them invariance group contact transformations, or <u>invariance groups</u> for short.

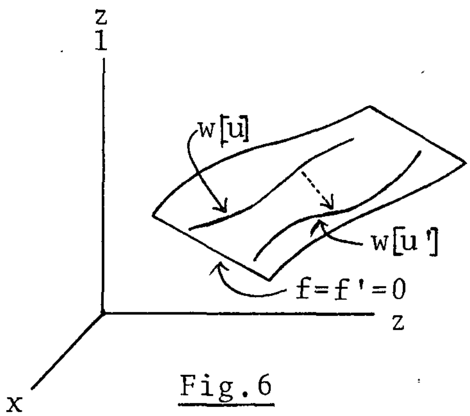


Fig.6

Lie [2-6] studied invariance groups extensively and established the foundation of a group analysis of differential equations.† More recently, Ovsjannikov [6,7] extended Lie's theory and applied it extensively to partial differential

† From a practical point of view it is important that Lie gave the algorithm for finding the invariance groups of any given differential equation. We give one explicit example of the computation of such an invariance group in Appendix 3.

equations. Bluman and Cole [8] used invariance groups to construct solutions to certain types of boundary value problems. All these works just mentioned are concerned with either point groups or Lie's group contact transformations.

It has been found that the Lie-Bäcklund group transformations are also useful, particularly for the study of nonlinear differential equations. It is shown [Appendices 5-7]† that the well known infinite number of conservation laws admitted by the Korteweg-deVries equation $u_{xxx}+uu_x+u_t=0$, the cubic Schroedinger equation $u_{xx}+u^2u^*-iu_t=0$ and the sine-Gordon equation $u_{xt}-\sin u=0$ are all related to invariances of the corresponding equations under L-B groups.†† More interestingly, soliton solutions admitted by these equations are shown to be invariant solutions of these invariance L-B groups [Appendix 6]. These findings lead to a general theorem [Appendix 7] that with any conservation law, admitted by a Hamiltonian system,

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† Appropriate references are given in the appendices.

†† A group theoretical aspect of conservation laws of the Korteweg-deVries and sine-Gordon equations was also studied by Steudel using Noether's theorem [14]. He called groups, leading to conservation laws, Noether transformations [15,16].

$$\frac{\partial p_n}{\partial t} = - \frac{\delta H}{\delta q_n}, \qquad \frac{\partial q_n}{\partial t} = \frac{\delta H}{\delta p_n}, \tag{0.25}$$

is associated an invariance L-B group of the equation. Further studies of invariance L-B groups have been reported [17-24].

Now we return to the question posed at the beginning of this introduction. Suppose that there exists an invertible transformation mapping a given system of nonlinear differential equations to a system of linear differential equations. We should expect that the invariance groups of the two systems have the same structure. Since it is well known that any linear system admits an invariance group related to the superposition principle, it is evident that the given nonlinear system must admit the corresponding invariance group. From such an observation, we establish a theorem which tells one definitively:

1) when a given nonlinear system can be mapped into a linear system;
2) how to construct the mapping when it exists.

It should be emphasized that in applying this theorem one needs only to calculate a system's invariance <u>infinitesimal</u> contact transformations (0.12) by a known algorithm. Moreover, the types of infinitesimal transformations to be considered are simple: in general one may assume $\xi=0$, $\zeta=\zeta(x,z,z_1)$; for a system, i.e., z is a vector, it turns out that ζ is at most linear in z_1 (corresponding to a point group).

In the first chapter, we define and formulate mathematically those basic concepts which were illustrated above and lay out the basis for subsequent developments. In the second chapter we establish the theorem mentioned above. A number of examples are given. In the third chapter, we examine non-invertible mappings which connect nonlinear equations to linear equations. Appendices 5,6,7 consist of already published works on L-B groups and conservation laws.

CHAPTER 1.

CONTACT TRANSFORMATIONS.

In this chapter we discuss various properties of transformations which will be considered in subsequent chapters. All transformations considered in this work are basically of "contact" type. The term "contact transformation" will be used in a context more general than it is usually referred to.

Throughout the work, we adopt the customary summation rule for repeated indices: Roman indices are summed from 1 to M and Greek indices from 1 to N.

1.1 Contact transformations.

Let w be an infinite dimensional vector space with coordinates

$$\omega = (x, z, \underset{1}{z}, \underset{2}{z}, \ldots, \underset{n}{z}, \ldots)$$

where $x = (x_1, x_2, \ldots, x_M) \in R^M$, $z = (z^1, z^2, \ldots, z^N) \in R^N$ and $\underset{n}{z} \in R^{NM^n}$ consists of coordinates $z^\nu_{i_1 i_2 \ldots i_n}$ with $\nu = 1, 2, \ldots, N$ and $i_k = 1, 2, \ldots, M$. For instance,

$$\underset{1}{z} = (z^1_1, z^1_2, \ldots, z^N_{M-1}, z^N_M), \quad \underset{2}{z} = (z^1_{11}, z^1_{12}, z^1_{21}, \ldots, z^N_{MM}).$$

$w^{(n)}$ denotes the space with coordinates $\omega^{(n)} = (x, z, \underset{1}{z}, \ldots, \underset{n}{z})$, where $\omega^{(0)} = (x,z)$. Let γ be a space of functions $w^{(k)} \to R$, $k = 1,2,3,\ldots$, analytic in a given domain $D(w)$ of w.

We consider a transformation $T: w \to w$ defined by

$$\begin{aligned} x' &= \bar{x}'(x, z, \underset{1}{z}, \ldots), \qquad z' = \bar{z}'(x, z, \underset{1}{z}, \ldots), \\ \underset{k}{z}' &= \underset{k}{\bar{z}}'(x, z, \underset{1}{z}, \ldots), \qquad k = 1,2,3,\ldots \end{aligned} \tag{1.1}$$

where $\bar{x}_i' \in \gamma$, $\bar{z}'^{\nu} \in \gamma$, $\bar{z}'^{\nu}_{i_1 i_2 \ldots i_k} \in \gamma$. We write (1.1) as

$$\omega' = \bar{\omega}'(\omega) = T\omega, \tag{1.2}$$

and the first n+2 expressions of (1.1) as

$$\omega'^{(n)} = \bar{\omega}'^{(n)}(\omega) .$$

A set of equations

$$\begin{cases} dz^{\nu} - z^{\nu}_i dx_i = 0 \\ dz^{\nu}_{i_1 i_2 \ldots i_k} - z^{\nu}_{i_1 i_2 \ldots i_k j} dx_j = 0, \quad k = 1,2,3,\ldots \end{cases} \tag{1.3}$$

is called a contact condition. We express the set (1.3) simply by $\{d\underset{k}{z} - \underset{k}{z}_i dx_i = 0\}$.

Definition 1. The transformation (1.1) is a contact transformation only if it preserves the contact condition, i.e.,

$$\{dz_k - z_{ki}dx_i = 0\} \rightarrow \{dz'_k - z'_{ki}dx'_i = 0\}, \qquad (1.4)$$

Let $u(x)$ be a function $R^M \rightarrow R^N$. We consider a space $w[u] \subset w$ consisting of points

$$\omega = (x, u, u_1, u_2, \ldots, u_k, \ldots), \qquad x \in R^M,$$

where, as z_k, u_k is a vector with components

$$u^{\nu}_{i_1 i_2 \ldots i_k} = \partial_{x_{i_1}} \partial_{x_{i_2}} \ldots \partial_{x_{i_k}} u^{\nu}(x) \ .$$

The transformation (1.1) maps the space $w[u]$ into a new space which we denote by $Tw[u]$. If T is a contact transformation, then $Tw[u] = w[u']$ for some function $u'(x)$. Namely, for any function $u(x)$ the transformed space $Tw[u]$ consists of points expressed as $\omega = (x, u', u'_1, u'_2, \ldots, u'_k, \ldots)$ for some function $u'(x)$.

1.2 Invertible contact transformations.

We call the transformation $\omega'^{(n)} = \bar{\omega}'^{(n)}(\omega)$ the n-th extended transformation of $\omega'^{(0)} = \bar{\omega}'^{(0)}$. Given $\bar{\omega}'^{(0)}$, one can determine $\bar{\omega}'^{(n)}$, $n>0$, from the condition (1.4).

A contact transformation is said to be invertible, or 1-1, if there exists a space $w^{(n)}$ in which the n-th extended transformation is a 1-1 mapping $w^{(n)} \rightarrow w^{(n)}$, $n=0,1,2,\ldots$. Under the present difinition (1.4), not all contact transformations are 1-1. Actually only very limited classes of contact transformations are invertible. Two cases, z scalar and z vector, are considered separately.

<u>z scalar.</u> Bäcklund [26] proved for scalar z that the most general 1-1 contact transformation is the extended Lie contact transformation. The following theorem by Meyer [27,28] characterizes Lie's contact transformation:

<u>Theorem [Meyer]</u>. A transformation $T: w^{(1)} \rightarrow w^{(1)}$,

$$x' = \bar{x}'(x,z,\underset{1}{z}), \quad z' = \bar{z}'(x,z,\underset{1}{z}), \quad \underset{1}{z}' = \bar{\underset{1}{z}}'(x,z,\underset{1}{z}), \qquad (1.5)$$

$\bar{x}': w^{(1)} \rightarrow R^M$, $\bar{z}': w^{(1)} \rightarrow R$, $\bar{\underset{1}{z}}': w^{(1)} \rightarrow R^M$, is a Lie contact transformation, i.e.

$$dz' - z_i' dx_i' = \rho(x,z,\underset{1}{z})(dz - z_i dx_i) \qquad (1.6)$$

if and only if

1) $\bar{x}_i'$, $i=1,2,\ldots,M$, and $\bar{z}'$ are M+1 independent functions of $x,z,\underset{1}{z}$ and satisfy $[\bar{x}_i',\bar{x}_j'] = 0$, $[\bar{z}',\bar{x}_i'] = 0$,

2) $\bar{z}_i'$, $i=1,2,\ldots,M$, are determined from

$$\partial_{x_i}\bar{z}' + z_i\partial_z\bar{z}' = z_j'(\partial_{x_i}\bar{x}_j' + z_i\partial_z\bar{x}_j') \tag{1.7}$$

or from

$$\partial_{z_i}\bar{z}' = z_j'\partial_{z_i}\bar{x}_j' \tag{1.8}$$

and $\rho(x,z,\underset{1}{z})$ from

$$\rho = \partial_z\bar{z}' - z_i'\partial_z\bar{x}_i' = [\bar{x}_k', \bar{z}_k'], \quad k=1,2,\ldots,M, \tag{1.9}$$

where the Lagrange bracket, [,] , of two functions $\phi(x,z,\underset{1}{z})$ and $\psi(x,z,\underset{1}{z})$ is defined by

$$[\phi,\psi] = (\partial_{z_i}\phi)(\partial_{x_i}\psi + z_i\partial_z\psi) - (\partial_{z_i}\psi)(\partial_{x_i}\phi + z_i\partial_z\phi). \tag{1.10}$$

The extensions of the transformation (1.5) to higher order coordinates $\underset{n}{z}'$, $n>1$, are found from the contact condition (1.4).

<u>Remark 1</u>. In the literature, the term "contact transformation" usually refers to Lie's contact transformation.

<u>z vector</u>. For a vector $z \in R^N$, $N>1$, the most general 1-1 contact transformation is the extended point transformation [29] of

$$x' = \bar{x}'(x,z), \qquad z' = \bar{z}'(x,z), \tag{1.11}$$

$\bar{x}':w^{(0)} \to R^M$, $\bar{z}':w^{(0)} \to R^N$. The transformations for $\underset{n}{z}$, $n>0$, are determined from (1.4).

Remark 2. In the present context, a canonical transformation of classical mechanics

$$t' = t, \quad q' = \bar{q}'(p,q,t), \quad p' = \bar{p}'(p,q,t), \tag{1.12}$$

t time, q generalized coordinates, p generalized momenta, corresponds to a point transformation with $t = x$, $q = (z^1,z^2,\dots z^n)$, $p = (z^{n+1},z^{n+2},\dots,z^{2n})$.

We write the m-th extensions of (1.5) and (1.11) and their inverses as

$$\omega'^{(m)} = \bar{\omega}'^{(m)}(\omega^{(m)}) \equiv T\omega^{(m)}, \quad \omega^{(m)} = \bar{\omega}^{(m)}(\omega'^{(m)}) \equiv T^{-1}\omega'^{(m)} \tag{1.13}$$

and the infinite extensions as

$$\omega' = \bar{\omega}'(\omega) \equiv T\omega, \quad \omega = \bar{\omega}(\omega') \equiv T^{-1}\omega'. \tag{1.14}$$

1.3 Infinitesimal contact transformations.

We consider a contact transformation which depends analytically on some parameter α and reduces to the identity transformation I at $\alpha=0$:

$$\omega' = \bar{\omega}'(\omega;\alpha), \quad \omega = \bar{\omega}'(\omega;0) \tag{1.15}$$

or

$$\omega' = T(\alpha)\omega, \quad T(0) = I. \tag{1.16}$$

Expanding $\bar{\omega}'$ in a power series in α, we obtain

$$\omega' = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}[(\partial_\alpha)^n \bar{\omega}']_{\alpha=0} \ . \tag{1.17}$$

Defining an operator

$$\ell = \xi^i \partial_{x_i} + \zeta^\nu \partial_{z^\nu} + \zeta_i^\nu \partial_{z_i^\nu} + \dots$$

with $\xi^i = (\partial_\alpha \bar{x}_i^{\prime})_{\alpha=0}$, $\zeta^\nu = (\partial_\alpha \bar{z}'^\nu)_{\alpha=0}$, $\zeta_i^\nu = (\partial_\alpha \bar{z}_i^{\prime\nu})_{\alpha=0}$,....,

we write (1.17) as

$$\omega' = (1 + \alpha\ell)\omega + O(\alpha^2) \tag{1.18}$$

where $O(\alpha^2)$ represents terms of order α^2. For (1.15) to be a contact transformation it is necessary that ζ_i^ν, ζ_{ij}^ν, ... in ℓ satisfy the recursion relation

$$\zeta_{i\dots jk}^\nu = D_{x_k}\zeta_{i\dots j}^\nu - z_{i\dots jm}^\nu D_{x_k}\xi^m \tag{1.19}$$

where

$$D_{x_k} = \partial_{x_k} + z_k^\nu \partial_{z^\nu} + z_{ik}^\nu \partial_{z_i^\nu} + \dots \ .$$

Now instead of starting from a transformation of the form (1.15), we start with an operator

$$\ell = \xi^i \partial_{x_i} + \zeta^\nu \partial_{z^\nu} + \zeta_i^\nu \partial_{z_i^\nu} + \dots , \tag{1.20}$$

where ξ^i, $\zeta^\nu \epsilon \gamma$, and $\zeta^\nu_i, \zeta^\nu_{ij}, \ldots$ are determined recursively from (1.19). We consider a group transformation

$$\omega' = e^{\alpha \ell}\omega = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (\ell)^n \omega. \tag{1.21}$$

Transformation (1.21) always satisfies the contact condition (1.4). It was previously shown [Appendix 5] that a transformation of the type (1.15) can be represented as an infinite composition of group transformations of the form (1.21). Namely, (1.15) can be written as

$$\omega' = (e^{\alpha \ell_1} e^{\frac{\alpha^2}{2!} \ell_2} e^{\frac{\alpha^3}{3!} \ell_3} \ldots)\omega = T(\alpha)\omega, \tag{1.22}$$

where ℓ_i are operators of the form (1.20). ℓ is called a generator of a group contact transformation (1.21). Depending on the forms of ξ and ζ, the one-parameter groups (1.21) are classified as

a) Point groups: $\xi^i = \xi^i(x,z)$, $\zeta^\nu = \zeta^\nu(x,z)$

b) Lie contact transformation groups: z is scalar and there exists a function $G: w^{(1)} \to R$ such that

$$\xi^i = \partial_{z_i} G, \quad \zeta = z_i \partial_{z_i} G - G. \tag{1.23}$$

c) Lie-Bäcklund groups: all other forms of ξ^i and ζ^ν.

The transformation (1.21) corresponding to the first two cases defines a mapping $w^{(n)} \to w^{(n)}$ for any $n>0$. For the third case, however, the transformation (1.21) must be considered in the infinite dimensional space as $w \to w$. In all these cases, group transformations are 1-1 in the domain where $e^{\alpha \ell}$ exists.

1.4 Properties of generators.

The commutator of two generators ℓ and ℓ' is defined by

$$[\ell,\ell'] = (\ell\xi'^i - \ell'\xi^i)\partial_{x_i} + (\ell\zeta'^\nu - \ell'\zeta^\nu)\partial_{z^\nu} + \dots$$
$$+ (\ell\zeta'^\nu_{i\dots j} - \ell'\zeta^\nu_{i\dots j})\partial_{z^\nu_{i\dots j}} + \dots\dots \quad . \tag{1.24}$$

Let λ be the space of all generators ℓ. It is easy to show:

Proposition 1. If $\ell\epsilon\lambda$ and $\ell'\epsilon\lambda$, then $[\ell,\ell']\epsilon\lambda$.

Thus, the form of the commutator is determined from the two leading terms using (1.19). From now on we represent the generators by two leading terms as $\ell = \xi^i\partial_{x_i} + \zeta^\nu\partial_{z^\nu}$.

The following property is very useful [in Appendix 6]:

Proposition 2. Generators of the form $\ell = \zeta^\nu \partial_{z^\nu}$ commute with the total derivative operator D_{x_i}.

Proof. Noting that $D_{x_i} \in \lambda$, we have

$$[\ell, D_{x_i}] = (\ell \cdot 1)\partial_{x_i} + (\ell z_i^\nu - D_{x_i}\zeta^\nu)\partial_{z^\nu} = (D_{x_i}\zeta^\nu - D_{x_i}\zeta^\nu)\partial_{z^\nu} = 0. \quad \square$$

We introduce into the space λ an equivalence relation $\mathcal{R}$ by:

Definition 2. Two generators ℓ and ℓ' are equivalent if and only if $(\ell - \ell')g|_{g=0} = 0$ for any $g \in \gamma$. The equivalence is represented by $\ell \doteq \ell'$.

The symbol $|_{g=0}$ indicates the evaluation of the quantity for those values of $\omega = (x, z, z_1, \ldots)$ which satisfy the equations

$$g = 0, \quad D_{x_i} g = 0, \quad D_{x_i} D_{x_j} g = 0, \quad \ldots \quad . \tag{1.25}$$

The equivalence class N of $0 \in \lambda$ consists of generators

$$\ell = \xi^i D_{x_i}, \qquad \xi^i \in \gamma. \tag{1.26}$$

Proposition 3. A generator $\ell = \xi^i \partial_{x_i} + \zeta^\nu \partial_{z^\nu}$ is equivalent to $\hat{\ell} = (\zeta^\nu - z_i^\nu \xi^i)\partial_{z^\nu}$.

Proof. Obviously, $\ell - \hat{\ell} = \xi^i \partial_{x_i} + z_i^\nu \xi^i \partial_{z^\nu}$. From (1.19) we find its full expression to be

$$\ell - \hat{\ell} = \xi^i (\partial_{x_i} + z_i^\nu \partial_{z^\nu} + z_{ij}^\nu \partial_{z_j^\nu} + \ldots) = \xi^i D_{x_i}. \quad \square$$

As a result, elements of the quotient space λ/k can be represented by generators of the form $\ell = \theta^\nu(\omega^{(m)})\partial_{z^\nu}$, $\theta^\nu \epsilon \gamma$. The generators ℓ and $\hat{\ell}$ satisfy the same commutation relation. Namely,

Proposition 4. $[\ell_1, \ell_2] = \ell_3$ if and only if $[\hat{\ell}_1, \hat{\ell}_2] = \hat{\ell}_3$.

Proof. From (1.24) we have

$$[\ell_1, \ell_2] = (\ell_1 \xi_2^i - \ell_2 \xi_1^i)\partial_{x_i} + (\ell_1 \zeta_2^\nu - \ell_2 \zeta_1^\nu)\partial_{z^\nu} = \ell_3.$$

Therefore, $\hat{\ell}_3 = \{(\ell_1 \zeta_2^\nu - \ell_2 \zeta_1^\nu) - z_i^\nu(\ell_1 \xi_2^i - \ell_2 \xi_1^i)\}\partial_{z^\nu}$.

On the other hand, we have

$$[\hat{\ell}_1, \hat{\ell}_2] = \{\hat{\ell}_1 \zeta_2^\nu - z_i^\nu \hat{\ell}_1 \xi_2^i - \xi_2^i D_{x_i}(\zeta_1^\nu - \xi_1^j z_j^\nu)$$

$$- \hat{\ell}_2 \zeta_1^\nu + z_i^\nu \hat{\ell}_2 \xi_1^i + \xi_1^i D_{x_i}(\zeta_2^\nu - \xi_2^j z_j^\nu)\}\partial_{z^\nu}$$

$$= \{(\hat{\ell}_1 - \xi_1^i D_{x_i})\zeta_2^\nu - (\hat{\ell}_2 - \xi_2^i D_{x_i})\zeta_1^\nu$$

$$- z_i^\nu(\hat{\ell}_1 - \xi_1^j D_{x_j})\xi_2^i + z_i^\nu(\hat{\ell}_2 - \xi_2^j D_{x_j})\xi_1^i\}\partial_{z^\nu}$$

$$= \{\ell_1\zeta_2^\nu - \ell_2\zeta_1^\nu - z_i^\nu(\ell_1\xi_2^i - \ell_2\xi_1^i)\}\partial_{z^\nu} = \hat{\ell}_3.$$

The converse is obvious. □

1.5 Invariance contact transformations.

A set of K C^1 functions $w^{(n)} \to R$,

$$f^\nu(\omega^{(n)}) = f^\nu(x, z, \underset{1}{z}, \ldots \underset{n}{z}), \quad \nu=1,2,\ldots,K, \tag{1.27}$$

are said to be functionally independent in the domain D(w) iff there exist K components of $(\underset{1}{z}, \underset{2}{z}, \ldots, \underset{n}{z})$, denoted by $y^1, y^2, \ldots, y^K$, for which the Jacobian of $(f^1, f^2, \ldots, f^K)$ is nonzero:

$$\frac{D(f^1, f^2, \ldots, f^K)}{D(y^1, y^2, \ldots, y^K)} \neq 0 \text{ in } D(w). \tag{1.28}$$

We denote by D(w;f=0) the set of points $\omega^{(n)}$ satisfying the equations $f^\nu(\omega^{(n)})=0$, $\nu=1,2,\ldots,K$. K equations

$$f^\nu(\omega^{(n)}) = 0, \quad \nu=1,2,\ldots,K, \quad K \le N, \tag{1.29}$$

are said to be <u>independent</u> iff the set of functions f^ν are functionally independent in $D(w)$ and $D(w;f=0)$ is nonempty.

The implicit function theorem ensures that if $\{f^\nu\}$ is a set of functionally independent functions and $D(w;f=0)$ is nonempty, then in every neighbourhood of a point $\omega\in D(w;f=0)$ there exists a unique set of K C^1 functions $\psi^\nu(\omega^{(n)})$, $\nu=1,2,\ldots,K$, independent of $y^1,y^2,\ldots,y^K$, with the property that the functions f^ν all vanish with the substitutions

$$y^\nu = \psi^\nu(\omega^{(n)}), \quad \nu=1,2,\ldots,K. \tag{1.30}$$

The system (1.30) is called an <u>explicit form</u> of the system (1.29). The system (1.29) is said to be a <u>linear system</u> iff its explicit form is linear in $z, \underset{1}{z}, \ldots, \underset{n}{z}$, namely,

$$y^\nu = A^\nu_\mu z^\mu + \phi^\nu(x), \quad \nu=1,2,\ldots,K, \tag{1.31}$$

where A^ν_μ is a linear operator defined by

$$A^\nu_\mu z^\mu = \{a^\nu_\mu(x) + a^{\nu i}_\mu(x) D_{x_i} + \ldots + a^{\nu i_1 i_2 \ldots i_n}_\mu(x) D_{x_{i_1}} D_{x_{i_2}} \ldots D_{x_{i_n}}\} z^\mu, \tag{1.32}$$

$a^{\nu i\ldots j}_\mu : R^M(x)\to R$, $\phi^\nu : R^M(x)\to R$. Defining an operator matrix $A = |A^\nu_\mu|$, we sometimes write (1.31) in the form

$$y = Az + \phi(x) \tag{1.33}$$

where y,z and ϕ are column vectors.

We consider a system of equations

$$f = 0, \quad D_{x_i} f = 0, \quad D_{x_i} D_{x_j} f = 0, \ \ldots \tag{1.34}$$

where f=0 is the independent system (1.29). We use the same notation D(w;f=0) to represent the set consisting of points $\omega \in D(w)$ satisfying the system (1.34). The set D(w;f=0) defines a manifold in w.

A contact transformation $\omega' = \bar{\omega}'(\omega)$ is called an invariance contact transformation of the equation f = 0 iff it transforms D(w;f=0) into itself, namely,

$$f(\bar{\omega}'(\omega))\big|_{f(\omega)=0} = 0. \tag{1.35}$$

For a group contact transformation (1.21) to be an invariant transformation, it is necessary and sufficient that

$$\ell \cdot f(\omega)\big|_{f(\omega)=0} = 0. \tag{1.36}$$

Eq. (1.36) is called the determining equation. Because of the local nature of the generators of an invariance group, we have:

Proposition 5. A system of independent equations $\{f^\nu(\omega)=0\}$ and its explicit form $\{y^\nu = \phi^\nu(\omega)\}$ admit the same invariance group generators.

Lie gave an algorithm [6,7,23] to determine invariance group generators $\ell=\xi^i(\omega^{(m)})\partial_{x_i}+\zeta^\nu(\omega^{(n)})\partial_{z^\nu}$ satisfying the condition (1.36) for given m and n.

Clearly we have

<u>Proposition</u> 6. If $\ell\doteq\ell'$ and ℓ is an invariance group generator of the system $f=0$, then ℓ' is also an invariance group generator of $f=0$.

Thus, recalling Propositions 3 and 4, we see that for the study of invariance groups, it is sufficient to consider generators of the form $\ell=\theta^\nu(\omega^{(m)})\partial_{z^\nu}$. Often generators of this form are easier to work with than generators of the form (1.20) and in the rest of this work we only deal with such generators.

1.6 Invariance contact transformations of differential equations.

In §1.1 we introduced the notation w[u] to represent a space consisting of ω given by

$$\omega = (x,u,\underset{1}{u},\underset{2}{u},\ldots), \quad x\in R^M, \quad u:R^M(x)\to R^N.$$

When u is a solution of a system of differential equations

$$f(x,u,\underset{1}{u},\underset{2}{u},\ldots,\underset{n}{u}) = 0, \tag{1.37}$$

we call w[u] a solution surface of the differential equations (1.37). It is clear that for w[u] to be a solution surface it is necessary and sufficient that w[u] be imbedded in D(w;f=0). Now let $T:\omega'=\bar{\omega}'(\omega)$ be an invariance contact transformation of $f(\omega^{(n)})=0$. Then from the contact property of T we have, as mentioned in §1.1,

$$Tw[u] = w[u'], \tag{1.38}$$

and from the invariance property of T, we have for a solution surface w[u],

$$Tw[u] \subset D(w;f=0), \tag{1.39}$$

hence w[u'] must be a solution surface of the equations (1.37). Therefore, any invariance contact transformation of $f(\omega^{(n)})=0$ maps a solution surface of the differential equations (1.37) to another solution surface.

1.7 Invariance groups of linear equations.

Generators of invariance group contact transformations of linear equations bear special properties.

We consider two cases separately:

$$\text{i)}\quad \ell = \theta^{\nu}(x)\partial_{z^{\nu}}, \qquad \theta^{\nu}: R^{M}(x)\to R$$

$$\text{ii)}\quad \ell = \theta^{\nu}(\omega^{(n)})\partial_{z^{\nu}}, \qquad \theta^{\nu}: w^{(n)}\to R.$$

To be consistent with notations to be used later, capital letters $X, Z, U, \Theta, \ldots$ will be used in this section. We use the term "linear" in the sense defined in §1.5.

<u>i) The first case</u>. It is known that any linear equation admits a generator of the form $L=\Theta^{\nu}(X)\partial_{Z^{\nu}}$. Namely,

<u>Proposition 7</u>. A system of linear equations $A^{\nu}_{\mu}Z^{\mu} - \Phi^{\nu}(X) = 0$, $\nu=1,2,\ldots,K$, $K\leq N$, $\Phi^{\nu}: R^{M}(X)\to R$, with linear operator A^{ν}_{μ} defined by

$$A^{\nu}_{\mu}Z^{\mu} = \{A^{\nu}_{\mu}(X) + A^{\nu i}_{\mu}(X)D_{X_i} + \ldots + A^{\nu i_1 i_2\ldots i_n}_{\mu}(X)D_{X_{i_1}}D_{X_{i_2}}\ldots D_{X_{i_n}}\}Z^{\mu} \tag{1.40}$$

$A^{\nu i\ldots j}_{\mu}: R^{M}(X)\to R$, admits a generator $L=U^{\mu}(X)\partial_{Z^{\mu}}$ depending upon an arbitrary solution $\{U^{\nu}(X);\nu=1,2,\ldots,N\}$ of the system of linear differential equations $A^{\nu}_{\mu}U^{\mu}(X)=0$, i.e.,

$$A^{\nu}_{\mu}(X)U^{\mu} + A^{\nu i}_{\mu}(X)U^{\mu}_{i} + \ldots + A^{\nu i_1 i_2\ldots i_n}_{\mu}(X)U^{\mu}_{i_1 i_2\ldots i_n} = 0, \tag{1.41}$$

$\nu=1,2,\ldots,K$.

Here and in the following, $U^{\mu}_{ij\ldots k} = \partial_{x_i}\partial_{x_j}\ldots\partial_{x_k}U^{\mu}(X)$.
Let $\bar{Z}=(1+cL)Z$, c constant and $L=U^{\mu}(X)\partial_{Z^{\mu}}$ defined above.
It is easy to see that if $A^{\nu}_{\mu}Z^{\mu}-\Phi^{\nu}(X)=0$, then the $\bar{Z}$, a superposition of $cU(X)$ with Z, satisfies the equation $A^{\nu}_{\mu}\bar{Z}^{\mu}-\Phi^{\nu}(X)=0$.

<u>Definition 3.</u> An operator $L=U^{\mu}(X)\partial_{Z^{\mu}}$ is said to be a superposition generator of the equation $A^{\nu}_{\mu}Z^{\mu}-\Phi^{\nu}(X)=0$, $\nu=1,2,\ldots,K$, if $A^{\nu}_{\mu}U^{\mu}(X) = 0$.

It is clear from Proposition 5 that:

<u>Proposition 8</u>. A linear system $F^{\nu}(\Omega^{(n)})=0$, $\nu=1,2,\ldots,K$, $K\leq N$, admits a superposition generator $L=U^{\mu}(X)\partial_{Z^{\mu}}$ of its explicit form

$$Y^{\nu} = \hat{A}^{\nu}_{\mu}Z^{\mu} + \Phi^{\nu}(X). \tag{1.42}$$

<u>ii) The second case</u>. In this subsection, we restrict ourselves to a linear system $F^{\nu}(\Omega^{(n)})=0$ whose explicit form is linear <u>homogeneous</u>, i.e. $A^{\nu}_{\mu}Z^{\mu}=0$. In this case, two types of generators can arise:

a) Θ^{ν} is linear in $Z, \underset{1}{Z}, \ldots, \underset{m}{Z}$, i.e., $\Theta^{\nu}=B^{\nu}_{\mu}Z^{\mu}+\Psi^{\nu}(X)$ for some linear operator B .

b) Θ^{ν} is nonlinear in $Z, \underset{1}{Z}, \ldots, \underset{m}{Z}$.

The generators will be called linear generators and nonlinear generators respectively. For a linear system, we often assume its invariance group generators to be linear generators as the computation of generators becomes much simpler. Although this assumption has been found to be valid for most of linear systems, there exists no theorem stating the range of validity of this assumption.†) Indeed, there exist exceptions. For instance, the wave equation $Z_{XX} - Z_{YY} = 0$ admits a nonlinear generator $L=\Theta(Z_X+Z_Y,X+Y)\partial_Z$, $\Theta(\cdot,\cdot)$ being an arbitrary function. We now show the completeness of linear generators, namely, that if the system admits nonlinear generators, then they all can be found by examining linear generators admitted by the system.

Theorem 1. A linear homogeneous system $A^{\nu}_{\mu}Z^{\mu} = 0$, A^{ν}_{μ} defined by (1.40), admits a nonlinear generator $\Theta^{\nu}(\Omega^{(m)})\partial_{Z^{\nu}}$ iff the system admits a linear generator of the form

$$L = \{B^{\nu}_{\mu}(X)Z^{\mu} + B^{\nu i}_{\mu}(X)Z^{\mu}_{i} + \ldots + B^{\nu i_1 i_2 \ldots i_m}_{\mu}(X)Z^{\mu}_{i_1 i_2 \ldots i_m}\}\partial_{Z^{\nu}} \tag{1.43}$$

---

†) Only for a limited class of scalar linear equations has this assumption been shown to be valid [6,7].

where

$$B_{\mu}^{\nu i\ldots j}(X) = \{\partial_{Z^{\nu}_{i\ldots j}} \Theta^{\nu}(\Omega^{(m)})\}|_{Z=U, \underset{1}{Z}=\underset{1}{U}, \ldots, \underset{m}{Z}=\underset{m}{U}}, \qquad (1.44)$$

and $U: R^M(X) \to R^N(Z)$ is an arbitrary solution of a system of differential equations $A^{\nu}_{\mu} U^{\mu} = 0$.

This is a new result and the proof is given in Appendix 1. According to this theorem, if a linear homogeneous system admits a nonlinear generator, then the system must admit a linear generator which depends on an arbitrary solution of the corresponding system of differential equations. By virtue of this result, in the study of a linear homogeneous system, we may restrict ourselves only to linear generators of the form

$$L = (B^{\nu}_{\mu} Z^{\mu} + \Psi^{\nu})\partial_{Z^{\nu}} \qquad (1.45)$$

where

$$B^{\nu}_{\mu} = B^{\nu}_{\mu}(X) + B^{\nu i}_{\mu}(X) D_{X_i} + \ldots + B^{\nu i_1 i_2 \ldots i_m}_{\mu}(X) D_{X_{i_1}} D_{X_{i_2}} \ldots D_{X_{i_m}} \qquad (1.46)$$

$B^{\nu i\ldots j}_{\mu} : R^M(X) \to R$, and $\Psi^{\nu} : R^M(X) \to R$.

To illustrate a use of the theorem we consider the following example.

Example. Consider the equation

$$Z_{XX} - Z_{YY} = 0. \tag{1.47}$$

This equation admits a linear generator

$$L = \{g(U_X+U_Y)\cdot(Z_X+Z_Y)\}\partial_Z \tag{1.48}$$

where g is an arbitrary function of U_X+U_Y, U being an arbitrary solution of the differential equation $U_{XX}-U_{YY}=0$. In this example, the equation is scalar, hence we drop all the Greek indices in the theorem. Comparing (1.43) and (1.48), we have

$$B(X,Y) = 0, \quad B^X(X,Y) = g(U_X+U_Y), \quad B^Y(X,Y) = g(U_X+U_Y). \tag{1.49}$$

Comparing (1.49) with (1.44), we look for a function Θ satisfying

$$\Theta_Z = 0, \quad \Theta_{Z_X} = g(Z_X+Z_Y), \quad \Theta_{Z_Y} = g(Z_X+Z_Y). \tag{1.50}$$

Clearly, we have $\Theta=G(Z_X+Z_Y)$, G being an arbitrary function of Z_X+Z_Y, and consequently Eq(1.47) admits a nonlinear generator $L=G(Z_X+Z_Y)\partial_Z$.

We state a few basic properties of the linear generators. Their proofs are given in Appendix 2.

Let B be an operator matrix $B = |B_\mu^\nu|$.

<u>Proposition 9</u>. If $L=(B_\mu^\nu Z^\mu)\partial_{Z^\nu}$, $\bar{L}=(\bar{B}_\mu^\nu Z^\mu)\partial_{Z^\nu}$ and $\tilde{L}=(\tilde{B}_\mu^\nu Z^\mu)\partial_{Z^\nu}$ satisfy the commutation relation $[L,\bar{L}]=\tilde{L}$, then $[B,\bar{B}]=-\tilde{B}$.

<u>Proposition 10</u>. If $L=(B_\mu^\nu Z^\mu)\partial_{Z^\nu}$ and $\bar{L}=(\bar{B}_\mu^\nu Z^\mu)\partial_{Z^\nu}$ are generators of invariance groups of a linear homogeneous system $A_\mu^\nu Z^\mu=0$, then so is $\tilde{L}=(B_\kappa^\nu \bar{B}_\mu^\kappa Z^\mu)\partial_{Z^\nu}$.

Let A be an operator matrix $A = |A_\mu^\nu|$.

<u>Proposition 11</u>. $L=(B_\mu^\nu Z^\mu)\partial_{Z^\nu}$ is an invariance group generator of a linear system $AZ=0$ iff $[A,B]Z=0$ for any Z satisfying $AZ=0$.

<u>Proposition 12</u>. If $L=(B_\mu^\nu Z^\mu)\partial_{Z^\nu}$ is an invariance group generator of the system $AZ=0$, and if U(X) is a solution of a system of differential equations $AU(X)=0$, then $(B)^k U(X)$ also solves the same differential equations, i.e. $A(B)^k U(X)=0$, $k=1,2,3,\ldots$.

Invariance groups of linear equations have been studied extensively in recent years in connection with representation theories of groups [30,31] and with quantum physics [25] In these works, problems were formulated in terms of the linear differential operator B instead of the operator $L = (B_\mu^\nu Z^\mu)\partial_{Z^\nu}$. A systematic method of finding the operator B was first presented by Winternitz et al.[32,33] in their study of symmetries of Schroedinger equations. The relationship between the operators L and B was first established by Anderson, Kumei and Wulfman and Propositions 10-12 have been known to them. Proposition 9 is a new result.

1.8 Contact transformations of the spaces γ and λ.

Functions f and generators ℓ were defined in space w. If the space w is mapped into a new space W by some transformation T, then f and ℓ undergo corresponding transformations. We assume the transformation T to be a 1-1 contact transformation, analytic in D(w) and write it as

$$\Omega = \bar{\Omega}(\omega) \equiv T\omega \quad \text{with} \quad \Omega = (X, Z, Z_1, Z_2, \ldots). \tag{1.51}$$

The inverse is written as $\omega = \bar{\omega}(\Omega) \equiv T^{-1}\Omega$. D(W) denotes the image of D(w), and Γ the space of functions $F: W^{(k)} \to R$, $k = 1, 2, 3, \ldots$, analytic in D(W). Λ represents the space of generators L of group contact transformations in W.

The transformation $T: \gamma \to \Gamma$. The transformation of the function $f \in \gamma$ by T is written as $Tf(\omega^{(n)})$ and defined by

$$Tf(\omega^{(n)}) \equiv f(T^{-1}\Omega^{(n)}). \tag{1.52}$$

The inverse transformation $\Gamma\to\gamma$ is defined by

$$T^{-1}F(\Omega^{(n)}) \equiv F(T\omega^{(n)}). \tag{1.53}$$

The transformation $T:\lambda\to\Lambda$. We write the transformation of ℓ by T as $T\ell T^{-1}$. Regarding (1.51) as a change of variables $\omega\to\Omega$ one finds

$$T\ell T^{-1} = (\ell\bar{X}_i)\partial_{X_i} + (\ell\bar{Z}^\nu)\partial_{Z^\nu}, \tag{1.54}$$

where $\bar{X},\bar{Z}$ are X,Z components of (1.51). Since the transformation (1.51) is a contact transformation, if $\ell\in\lambda$, then $T\ell T^{-1}\in\Lambda$. In view of Poposition 3, we have

$$T\ell T^{-1} \doteq (\ell\bar{Z}^\nu - Z_i^\nu\ell\bar{X}_i)\partial_{Z^\nu}. \tag{1.55}$$

Similarly,

$$T^{-1}LT \doteq (L\bar{z}^\nu - z_i^\nu L\bar{x}_i)\partial_{z^\nu}. \tag{1.56}$$

We let $\ell=\theta^\nu(\omega^{(m)})\partial_{z^\nu}$ and $L = \Theta^\nu(\Omega^{(m)})\partial_{Z^\nu}$ and consider two cases, z scalar and z vector, separately.

z scalar. As mentioned in §1.2, the most general 1-1 contact transformation in this case is the extended Lie contact transformation. It takes the form

$$X_i = \bar{X}_i(x,z,\underset{1}{z}), \quad Z = \bar{Z}(x,z,\underset{1}{z}), \quad Z_i = \bar{Z}_i(x,z,\underset{1}{z}), \; \dots \; . \tag{1.57}$$

Using (1.7)-(1.9) where we let $x'=X$, $z'=Z$, $\underset{1}{z}'=\underset{1}{Z}, \dots,$ we obtain from (1.55) and (1.56) the expressions

$$T\theta(\omega^{(m)})\partial_z T^{-1} \doteq \{\theta(T^{-1}\Omega^{(m)})\rho(T^{-1}\Omega^{(1)})\}\partial_Z, \tag{1.58}$$

and

$$T^{-1}\Theta(\Omega^{(m)})\partial_Z T \doteq \{\Theta(T\omega^{(m)})\sigma(\omega^{(1)})\}\partial_z, \tag{1.59}$$

where

$$\rho(T^{-1}\Omega^{(1)}) = \rho(\omega^{(1)})|_{\omega=\bar{\omega}(\Omega)} \quad \text{and} \quad \sigma(\omega^{(1)}) = \{\rho(\omega^{(1)})\}^{-1} \tag{1.60}$$

with $\rho(\omega^{(1)}) = \partial_z\bar{Z} - \bar{Z}_i\partial_z\bar{X}_i$.

<u>z vector</u>. In this case the most general 1-1 contact transformation is the extended point transformation:

$$X_i = \bar{X}_i(x,z), \quad Z^\nu = \bar{Z}^\nu(x,z), \; \dots \; , \tag{1.61}$$

yielding

$$T\theta^\nu(\omega^{(m)})\partial_{z^\nu}T^{-1} \doteq \{\theta^\mu(T^{-1}\Omega^{(m)})\rho_\mu^\nu(T^{-1}\Omega^{(1)})\}\partial_{Z^\nu}, \tag{1.62}$$

where

$$\rho_\mu^\nu(T^{-1}\Omega^{(1)}) = \{\partial_{z^\mu}\bar{Z}^\nu - \bar{Z}_i^\nu\partial_{z^\mu}\bar{X}_i\}|_{\omega=\bar{\omega}(\Omega)} \tag{1.63}$$

and

$$T^{-1}\Theta^{\nu}(\Omega^{(m)})\partial_{Z^{\nu}}T \doteq \{\Theta^{\mu}(T\omega^{(m)})\sigma_{\mu}^{\nu}(\omega^{(1)})\}\partial_{z^{\nu}}, \tag{1.64}$$

where

$$\sigma_{\mu}^{\nu}(\omega^{(1)}) = \{\partial_{Z^{\mu}}\bar{z}^{\nu} - \bar{z}_{i}^{\nu}\partial_{Z^{\mu}}\bar{x}_{i}\}|_{\Omega=\bar{\Omega}(\omega)} \tag{1.65}$$

From these results follows that:

<u>Proposition 13</u>. An operator $L=\Theta^{\nu}(\Omega^{(m)})\partial_{Z^{\nu}}$ is an invariance group generator of the equation $F(\Omega^{(n)})=0$ iff $T^{-1}F=f(\omega^{(n)})=0$ admits the generator (1.59) (z scalar) or the generator (1.64) (z vector).

CHAPTER 2.

INVERTIBLE MAPPINGS OF NONLINEAR SYSTEMS TO LINEAR SYSTEMS

In this chapter, we study transformations mapping nonlinear differential equations to linear differential equations in a 1-1 manner. Based upon the group analysis of differential equations, we obtain necessary and sufficient conditions for the existence of such transformations. The established theorems not only allow us to determine the existence of the transformations but also enable us to actually construct these transformations from invariance groups of the nonlinear equations.

In the following analysis, two types of transformations are considered:

1) The invariance group transformations of differential equations; and

2) The mappings which transform nonlinear differential equations to linear differential equations.

Theorems will be proved based upon the following observations. Clearly if there exists a 1-1 mapping between any two differential equations it must inject properties of one equation into the other, including their invariance properties.[†)] For this reason the often ignored fact, mentioned in §1.7, that any linear differential equation admits a superposition generator becomes significant. As we have seen, this particular generator depends upon an arbitrary solution of the linear equation. It follows then that any nonlinear equation transformable to a linear equation by a 1-1 mapping must admit a generator which depends upon an arbitrary solution of some linear differential equation.[††)]

2.1 Theorems on the existence of 1-1 mappings.

We consider two cases, z scalar and z vector, separately since each admits a different type of 1-1 contact transformation. First we consider the case z scalar. In the following theorem, the Lie contact transformation discussed in §1.2 will be used with new notations

$X=x'$, $Z=z'$, $Z_1=z_1'$, ...

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†) The idea of comparing invariance groups of differential equations in the search of mappings connecting the equations was first used by Bluman in his study of Burgers' equation [34] and it was applied to the study of mappings of one dimemsional linear parabolic equations to the heat equation [35].

††) The results in this chapter and a part of the results in the next chapter have been reported as Technical Report 81-3 of the Institute of Applied Mathematics and Statistics, University of British Columbia.

Theorem 2. A scalar n-th order nonlinear equation

$$f(\omega^{(n)}) = f(x,z,\underset{1}{z},\underset{2}{z},\ldots,\underset{n}{z}) = 0, \quad x\in R^M, \quad z\in R, \tag{2.1}$$

is transformable by a 1-1 contact transformation to a linear equation if and only if the equation f=0 admits a generator ℓ of the form

$$\ell = \{\sigma(\omega^{(1)})U(\bar{X}(\omega^{(1)}))\}\partial_z, \tag{2.2}$$

where

1) $U(X):R^M\to R$ is an arbitrary solution of some n-th order linear differential equation

$$AU \equiv \{A(X) + A^i(X)D_{X_i} + \ldots + A^{i_1\ldots i_n}(X)D_{X_{i_1}}\ldots D_{X_{i_n}}\}U = 0,$$

and (2.3)

2) $\bar{X}(\omega^{(1)}):w^{(1)}\to R^M$ is a component of a Lie contact transformation

$$X = \bar{X}(\omega^{(1)}), \quad Z = \bar{Z}(\omega^{(1)}), \quad \underset{1}{Z} = \underset{1}{\bar{Z}}(\omega^{(1)}) \tag{2.4}$$

and

$$\sigma(\omega^{(1)}) = \{\rho(\omega^{(1)})\}^{-1} = (\partial_z\bar{Z} - \bar{Z}_i\partial_z\bar{X}_i)^{-1}. \tag{2.5}$$

The transformation (2.4) maps Eq.(2.1) to a linear equation which has an explicit form $AZ-\Phi(X)=0$ with A defined by (2.3).

Proof. Suppose that Eq.(2.1) is transformable to a linear equation by an extended Lie contact transformation $\omega \to \Omega$. By Proposition 8, this linear equation admits the superposition generator $L=U(X)\partial_Z$ of the equation $AZ-\Phi(X)=0$. Hence, according to Proposition 13, Eq.(2.1) must admit (2.2).

Conversely, suppose that Eq.(2.1) admit a generator of the form (2.2) with the properties 1) and 2). The transformed equation of Eq.(2.1) by (2.4) is written as $Tf=F(X,Z,\underset{1}{Z},\ldots,\underset{n}{Z})=0$. In view of Proposition 13, $F=0$ admits the generator $L=U(X)\partial_Z$. Thus, by the invariance condition (1.36), we have

$$LF = F_Z U + F_{Z_i} U_i + \ldots + F_{Z_{i_1 i_2 \ldots i_n}} U_{i_1 i_2 \ldots i_n} = 0 \qquad (2.6)$$

for any $\Omega^{(n)}$ satisfying $F(\Omega^{(n)})=0$ and for any $U(X)$ satisfying the differential equation (2.3). It is easy to show that (2.3) and (2.6) involve the same set of $U, U_i, \ldots$. We assume without a loss of generality that both contain U. Eliminating U between the two equations, we get

$$0 = (AF_{Z_i} - A^i F_Z)U_i + (AF_{Z_{ij}} - A^{ij} F_Z)U_{ij} + \ldots . \qquad (2.7)$$

Since U represents an arbitrary solution of Eq.(2.3), at any point X an arbitrary set of values may be assigned to $U_i, U_{ij}, \ldots$, and thus all the coefficients in (2.7) must vanish. This is possible only if F has the form $G(AZ,X)$,

$G(\cdot,\cdot)$ being an arbitrary function. Therefore, the extended transformation of (2.4) maps Eq.(2.1) to an equation $G(AZ,X)=0$, which is solvable in the explicit form $AZ-\Phi(X)=0$. □

In this theorem, $z,z_i,\ldots,Z,Z_i,\ldots$ are coordinates of the spaces w, W. Because of the contact condition (1.4) imposed upon T this theorem implies:

Corollary 1. A scalar nonlinear differential equation

$$f(x,u,\underset{1}{u},\ldots,\underset{n}{u}) = 0, \qquad x \in R^M, \quad u:R^M(x)\to R \tag{2.8}$$

is transformable to a linear differential equation by a 1-1 mapping if and only if the equation $f(x,z,\underset{1}{z},\ldots,\underset{n}{z})=0$ admits a generator of the form (2.2). The mapping is given by the extension of (2.4) and it transforms Eq.(2.8) into a differential equation which is solvable in an explicit form

$$AU - \Phi(X) = 0. \tag{2.9}$$

We now turn to a system of nonlinear equations. We have:

Theorem 3. A system of K independent n-th order nonlinear equations

$$f^{\nu}(\omega^{(n)}) = f^{\nu}(x, z, \underset{1}{z}, \underset{2}{z}, \ldots, \underset{n}{z}) = 0, \quad \nu=1,2,\ldots,K, \quad K \le N, \tag{2.10}$$

$x \in R^{M}$, $z \in R^{N}$, is transformable by a 1-1 contact transformation to a linear system if and only if the system (2.10) admits a generator of the form

$$\ell = \{U^{\mu}(X(x,z))\,\sigma_{\mu}^{\nu}(x,z,\underset{1}{z})\}\partial_{z^{\nu}} \tag{2.11}$$

where

1) $U^{\nu}(X)$, $\nu=1,2,\ldots,N$, is an arbitrary solution of some system of n-th order linear differential equations

$$A_{\mu}^{\nu}U^{\mu} = \{A_{\mu}^{\nu}(X) + A_{\mu}^{\nu i}(X)D_{X_i} + \ldots + A_{\mu}^{\nu i_1 \ldots i_n}(X)D_{X_{i_1}} \ldots D_{X_{i_n}}\}U^{\mu} = 0, \tag{2.12}$$

$\nu=1,2,\ldots,K$, and

2) $\bar{X}(x,z): w^{(0)} \to R^{M}$ is a component of a point transformation

$$X = \bar{X}(x,z), \qquad Z = \bar{Z}(x,z) \tag{2.13}$$

with inverse transformation $x=\bar{x}(X,Z)$, $z=\bar{z}(X,Z)$ and

$$\sigma_{\mu}^{\nu}(x,z,\underset{1}{z}) = \{\partial_{Z^{\mu}}\bar{z}^{\nu} - \underset{i}{z}^{\nu}\partial_{Z^{\mu}}\bar{x}_i\}\Big|_{X=\bar{X},\,Z=\bar{Z}}. \tag{2.14}$$

The extended point transformation of (2.13) maps Eq.(2.10) to a linear system with an explicit form

$$A^{\nu}_{\mu} Z^{\mu} - \Phi^{\nu}(X) = 0, \quad \nu=1,2,\ldots,K. \tag{2.15}$$

Proof. We recall that the most general 1-1 contact transformation involving vector z is the point transformation. Suppose that there exists a point transformation

$$X = \bar{X}(x,z), \quad Z = \bar{Z}(x,z), \quad \underset{1}{Z} = \underset{1}{\bar{Z}}(x,z,\underset{1}{z}), \ \ldots \tag{2.16}$$

mapping Eq.(2.10) to a linear system (2.15). By Proposition 8 this linear system admits the superposition generator $L=U^{\nu}(X)\partial_{Z^{\nu}}$. In view of Proposition 13 the system (2.10) must admit the generator (2.11) with properties 1) and 2).

Conversely, suppose that Eq.(2.10) admit the generator (2.11). Under transformation (2.13), the generator (2.11) is transformed into

$$L = U^{\nu}(X)\partial_{Z^{\nu}} \tag{2.17}$$

and Eq.(2.10) into, say, $F^{\nu}(X,Z,\underset{1}{Z},\ldots,\underset{n}{Z})=0$, $\nu=1,2,\ldots,K$. The system $\{F^{\nu}=0\}$ is solvable in explicit forms for K components of the $\underset{1}{Z},\underset{2}{Z},\ldots,\underset{n}{Z}$. Without loss of generality, for these we can choose Z_1^ν, $\nu=1,2,\ldots,K$, and write the explicit forms as

$$Z_1^\nu + \phi^\nu(X,Z,\underset{1}{Z},\ldots,\underset{n}{Z}) = 0, \quad \nu=1,2,\ldots,K, \tag{2.18}$$

where ϕ^ν are independent of Z_1^μ, $\mu=1,2,\ldots,K$. According to Proposition 13, the system $\{F^\nu=0\}$ admits the generator (2.17), and hence so does the system (2.18). Thus,

$$U_1^\nu + \phi^\nu_{Z^\mu}U^\mu + \phi^\nu_{Z_i^\mu}U_i^\mu + \ldots + \phi^\nu_{Z^\mu_{i_1\ldots i_n}}U^\mu_{i_1\ldots i_n} = 0, \tag{2.19}$$

where $\nu=1,2,\ldots,K$ and $\phi^\nu_{Z^\mu}=\partial_{Z^\mu}\phi^\nu$, etc.. This holds for any $U(X)$ satisfying the differential equations (2.12). It is easy to show that Eq.(2.19) can involve only those $U^\mu, U_i^\mu,\ldots$ appearing in Eq.(2.12). Eq.(2.12) is solvable for U_1^μ, $\mu=1,2,\ldots,K$. This is seen as follows. Suppose this be not the case, i.e. rank $|A_\mu^{\nu 1}|<K$, $\mu,\nu\leq K$. We fix X at $X=X^0$ and assign to $U^\nu(X^0)$, $U_i^\nu(X^0)$, $U_{ij}^\nu(X^0),\ldots$ a set of values consistent with Eq.(2.12). Here, the indices on $U_i^\nu(X^0)$ are restricted either to $i>1$ or to $\{i=1,\nu>K\}$. For this set of values, there exist non-unique values of $U_1^\nu(X^0)$, $\nu\leq K$, satisfying Eq.(2.12) because of the above rank condition. On the other hand, the introduction of the same set of values into Eq.(2.19) uniquely determines the values of $U_1^\nu(X^0)$. This contradicts the condition that

Eq.(2.19) holds for any solution of Eq.(2.12). Thus, Eq.(2.12) is solvable as

$$U_1^\nu + \hat{A}_\mu^\nu U^\mu + \hat{A}_\mu^{\nu i} U_i^\mu + \ldots + \hat{A}_\mu^{\nu i_1 \ldots i_n} U_{i_1 \ldots i_n}^\mu = 0, \quad \nu=1,2,\ldots,K. \tag{2.20}$$

Eliminating U_1^ν from (2.19) and (2.20), we have

$$0 \equiv (\phi^\nu_{Z^\mu} - \hat{A}_\mu^\nu) U^\mu + (\phi^\nu_{Z_i^\mu} - \hat{A}_\mu^{\nu i}) U_i^\mu + \ldots + (\phi^\nu_{Z^\mu_{i_1 \ldots i_n}} - \hat{A}_\mu^{\nu i_1 \ldots i_n}) U^\mu_{i_1 \ldots i_n}. \tag{2.21}$$

The equality (2.21) is possible only if

$$\phi^\nu = \hat{A}_\mu^\nu Z^\mu + \hat{A}_\mu^{\nu i} Z_i^\mu + \ldots + \hat{A}_\mu^{\nu i_1 \ldots i_n} Z^\mu_{i_1 \ldots i_n} + \Phi^\nu(X)$$

where $\hat{A}_\mu^{\nu 1} \equiv 0$ for $\mu \leq K$, and consequently the explicit form (2.18)of the transformed system $\{F^\nu = 0\}$ is linear. It is also clear that Eq.(2.18) is equivalent to Eq.(2.15). □

As in the case of scalar z, from this theorem follows:

<u>Corollary 2</u>. A system of K independent nonlinear differential equations

$$f^\nu(x, u, \underset{1}{u}, \ldots, \underset{n}{u}) = 0, \quad \nu=1,2,\ldots,K, \quad K \leq N, \tag{2.22}$$

$x \in R^M$, $u: R^M(x) \to R^N$, is transformable by a 1-1 mapping to a system of linear differential equations if and only if the system $f^\nu(x, z, \underset{1}{z}, \ldots, \underset{n}{z}) = 0$ admits a generator of the form (2.11). The mapping is given by (2.13) and it transforms Eq.(2.22) to a system solvable in explicit form as

$$A^\nu_\mu U^\mu - \Phi^\nu(X) = 0, \quad \nu = 1, 2, \ldots, K. \tag{2.23}$$

2.2 Remarks on the use of theorems.

These results just obtained ensure that if a given system of nonlinear equations is transformable to a system of linear equations by a 1-1 mapping, one can always find the mapping by examining the nature of the invariance group of the nonlinear equations. The type of groups to be considered depends on the dimension of the space z.

For a scalar equation we need only to consider a generator ℓ of the form

$$\ell = \theta(x, z, \underset{1}{z})\partial_z \; . \tag{2.24}$$

If the equation is transformable to a linear equation, then it admits a generator of the form (2.2). It should be emphasized that the function $\bar{X}(\omega^{(1)})$, the factor $\sigma(\omega^{(1)})$ and the linear differential equation (2.3) can all be found by examining the generators admitted by Eq.(2.1).

Once $\bar{X}$ is obtained, the function $\bar{Z}(\omega^{(1)})$ is determined from the condition $[\bar{Z},\bar{X}_i]=0$ which represents a system of first order partial differential equations for $\bar{Z}$. At this point, $\bar{Z}$ still admits functional arbitrariness. From $\bar{Z}$ and $\bar{X}$ we determine $\underset{1}{\bar{Z}}$ using conditions (1.7) or (1.8). Next we use (2.5) for the known $\rho=\sigma^{-1}$ to limit the arbitrariness in $\bar{Z}$. The resulting transformation $X=\bar{X}$, $Z=\bar{Z}$, $\underset{1}{Z}=\underset{1}{\bar{Z}}$ maps the non-linear equation to an equation with an explicit form $AZ - \Phi(X) = 0$. The form of $\Phi(X)$ depends upon the remaining arbitrariness in $\bar{Z}$.

For a system of equations, in view of (2.11) and (2.14), we need to consider generators of the form

$$\ell = \{\zeta^{\nu}(x,z) - z_i^{\nu}\xi^i(x,z)\}\partial_{z^{\nu}}. \tag{2.25}$$

By Proposition 3, (2.25) is equivalent to a generator of a point group

$$\ell = \xi^i(x,z)\partial_{x_i} + \zeta^{\nu}(x,z)\partial_{z^{\nu}}.$$

If there exists a mapping to a linear system, we can find the functions $\bar{X}(x,z)$, $\sigma_{\mu}^{\nu}(x,z,\underset{1}{z})$ by comparing the resulting invariance group generators (2.25) with (2.11). The functions $\bar{Z}^{\nu}(x,z)$ are to be determined from these functions using equations (2.14). Eq.(2.12) is found on determining the invariance group.

Remark 1. It is possible for differential equations to admit generators whose forms are more general than those of (2.2) or (2.11) with the forms (2.2) or (2.11) as special cases. The Monge-Ampère equations of a special type are such examples as we see in the following examples. A system of ordinary differential equations also admits such generators.

2.3 Examples.

To illustrate the use of our theorems, we consider some well known equations transformable to linear equations. Since the linearization of differential equations $f^\nu(x,u,\underset{1}{u},\ldots,\underset{n}{u})=0$ is equivalent to that of the equations $f^\nu(x,z,\underset{1}{z}\ldots,\underset{n}{z})=0$ by a contact transformation, we only deal with the latter. In the following examples we let $x_1=x$, $x_2=y$ and, where convenient, adopt the customary notations $z_x=p$, $z_y=q$, $z_{xx}=r$, $z_{xy}=s$, $z_{yy}=t$.

A. The equation $z_{xx} + \frac{1}{2}(z_x)^2 - z_y = 0$.

We consider the equation†

$$f = z_{xx} + \tfrac{1}{2}(z_x)^2 - z_y = 0. \tag{2.26}$$

† This is an integrated form of Burgers' equation $z_{xx}+zz_x-z_y=0$ which will be discussed in 3.1.

The generator (2.24) is now $\ell=\theta(x,y,z,z_x,z_y)\partial_z$. Applying Lie's algorithm, we find that Eq.(2.26) admits

$$\ell_1 = (y^2 z_y + yxz_x + \tfrac{1}{2}x^2 + y)\partial_z, \quad \ell_2 = (yz_y + \tfrac{1}{2}xz_x)\partial_z,$$

$$\ell_3 = z_y\partial_z, \quad \ell_4 = z_x\partial_z, \quad \ell_5 = \partial_z, \tag{2.27}$$

$$\ell_6 = (yz_x + x)\partial_z, \quad \ell_7 = U(x,y)e^{-\frac{1}{2}z}\partial_z,$$

where, in ℓ_7, $U(x,y)$ is an arbitrary solution of the heat equation $U_{xx} - U_y = 0$. This indicates that Eq.(2.26) is equivalent to a linear equation. To find the mapping, we compare ℓ_7 with (2.2) to get $\bar{X}=x$, $\bar{Y}=y$ and $\sigma=e^{-\frac{1}{2}z}$. From the conditions $[\bar{X},\bar{Z}]=[\bar{Y},\bar{Z}]=0$, we obtain $\bar{Z} = \bar{Z}(x,y,z)$. Thus, the mapping is a point transformation. From (2.5) and $\rho=(\sigma)^{-1}=e^{\frac{1}{2}z}$, we find that $\partial_z\bar{Z}=e^{\frac{1}{2}z}$. The mapping is then

$$X = x, \quad Y = y, \quad Z = 2e^{\frac{1}{2}z} + h(x,y), \tag{2.28}$$

where $h(x,y)$ is an arbitrary function of x and y. It is easy to check that the extended point transformation of (2.28) maps Eq.(2.26) to the linear equation

$$AZ - \Phi(X,Y) \equiv Z_{XX} - Z_Y - (h_{XX} - h_Y) = 0. \tag{2.29}$$

Setting $h=0$, from Corollary 1 we see that the transformation

$$X = x, \quad Y = y, \quad U = 2e^{\frac{1}{2}u} \tag{2.30}$$

maps the differential equation $u_{xx} + \frac{1}{2}(u_x)^2 - u_y = 0$ to the heat equation $\Delta U = U_{XX} - U_Y = 0$, and moreover the inverse of (2.30),

$$x = X, \quad y = Y, \quad u = 2\ln(\tfrac{1}{2}U) \tag{2.31}$$

defines an implicit solution $u(x,y)$ of this nonlinear differential equation for any solution $U(X,Y)$. In this case the explicit form is

$$u = 2\ln\{\tfrac{1}{2}U(x,y)\}. \tag{2.32}$$

B. Hodograph transformations.

In this example we let $z^1 = w$, $z^2 = v$ and consider a system of quasilinear equations

$$f^i = a^{ix}(w,v)w_x + a^{iy}(w,v)w_y + b^{ix}(w,v)v_x + b^{iy}(w,v)v_y = 0, \quad i=1,2, \tag{2.33}$$

where the coefficients a's and b's are functions of w and v. For the invariance group of this equation we have:

Proposition 14. Provided $J = w_x v_y - v_x w_y \neq 0$, the system (2.33) admits a generator of the form

$$\ell = -\{U^1(w,v)w_x + U^2(w,v)w_y\}\partial_w - \{U^1(w,v)v_x + U^2(w,v)v_y\}\partial_v \tag{2.34}$$

where $\{U^1(w,v), U^2(w,v)\}$ is an arbitrary solution of the system of linear differential equations

$$b^{iy}(w,v)U^1_w - a^{iy}(w,v)U^1_v - b^{ix}(w,v)U^2_w + a^{ix}(w,v)U^2_v = 0, \quad i=1,2, \tag{2.35}$$

where $U^i_w = \partial_w U^i$, $U^i_v = \partial_v U^i$.

<u>Proof</u>. Since $D_x f^i = D_y f^i = 0$, we find that

$$\begin{aligned} -\ell f^i = {} & w_x U^1_w(a^{ix}w_x + a^{iy}w_y) + v_y U^2_v(b^{ix}v_x + b^{iy}v_y) \\ & + w_y U^2_w(a^{ix}w_x + a^{iy}w_y) + v_x U^1_v(b^{ix}v_x + b^{iy}v_y) \\ & + v_x U^1_w(b^{ix}w_x + b^{iy}w_y) + w_y U^2_v(a^{ix}v_x + a^{iy}v_y) \\ & + v_y U^2_w(b^{ix}w_x + b^{iy}w_y) + w_x U^1_v(a^{ix}v_x + a^{iy}v_y). \end{aligned}$$

Using Eq.(2.33) in the first two rows of this expression, we find that

$$\ell f^i \big|_{f_1=f_2=0} = J\cdot(b^{iy}U^1_w - a^{iy}U^1_v - b^{ix}U^2_w + a^{ix}U^2_v)$$

which vanishes by the condition (2.35). □

To construct the mapping to a linear system, we compare (2.34) with (2.11) which in the present case takes the form

$$\ell = \{U^1(\bar{X},\bar{Y})\sigma_1^1 + U^2(\bar{X},\bar{Y})\sigma_2^1\}\partial_W + \{U^1(\bar{X},\bar{Y})\sigma_1^2 + U^2(\bar{X},\bar{Y})\sigma_2^2\}\partial_V .$$

Clearly $\bar{X}=w$, $\bar{Y}=v$, $\sigma_1^1=-w_x$, $\sigma_2^1=-w_y$, $\sigma_1^2=-v_x$, $\sigma_2^2=-v_y$. The definition of σ_μ^ν leads to $\partial_W\bar{x}=1$, $\partial_V\bar{x}=0$, $\partial_W\bar{y}=0$, $\partial_V\bar{y}=1$. Thus we have a solution $\bar{x}=W$, $\bar{y}=V$. Combining these together, we find the hodograph transformation [36]

$$x = W, \quad y = V, \quad w = X, \quad v = Y \tag{2.36}$$

which maps Eq. (2.33) to a linear system

$$A_\mu^i Z^\mu = b^{iy}(X,Y)W_X - a^{iy}(X,Y)W_Y - b^{ix}(X,Y)V_X + a^{ix}(X,Y)V_Y = 0, \tag{2.37}$$

where $Z^1=W$, $Z^2=V$ and $i=1,2$.

C. The Legendre transformation.

We consider a second order quasilinear equation

$$f = a(p,q)r + 2b(p,q)s + c(p,q)t = 0, \tag{2.38}$$

where p,q,r,s and t denote the variables defined at the beginning of this section and a,b and c are functions of p and q. We have

Proposition 15. Eq.(2.38) admits a generator $\ell=U(p,q)\partial_z$ depending on an arbitrary solution U(p,q) of the linear differential equation

$$U = a(p,q)U_{qq} - 2b(p,q)U_{pq} + c(p,q)U_{pp} = 0, \qquad (2.39)$$

where $U_{pp}=(\partial_p)^2U$, $U_{pq}=\partial_p\partial_qU$, $U_{qq}=(\partial_q)^2U$.

Proof. Introducing $w=p$, $v=q$, $w_x=r$, $v_y=t$, $w_y=v_x=s$, we write Eq.(2.38) as a system

$$a(w,v)w_x + b(w,v)(w_y + v_x) + c(w,v)v_y = 0 \qquad (2.40)$$

$$w_y - v_x = 0. \qquad (2.41)$$

According to Proposition 14, this system admits the generator of the form (2.34) where $\{U^1,U^2\}$ is an arbitrary solution of the linear differential equations

$$c(w,v)U^1_w - b(w,v)(U^1_v + U^2_w) + a(w,v)U^2_v = 0 \qquad (2.42)$$

$$U^1_v - U^2_w = 0. \qquad (2.43)$$

Eq.(2.43) allows us to introduce a function U(w,v) with a property

$$U^1 = \partial_w U \equiv U_w, \qquad U^2 = \partial_v U \equiv U_v, \tag{2.44}$$

and then Eq.(2.42) takes the form

$$aU_{vv} - 2bU_{wv} + cU_{ww} = 0. \tag{2.45}$$

Introducing (2.44) into (2.34) and using (2.41), we find that $\ell \doteq (D_x U)\partial_w + (D_y U)\partial_v$. Recalling that $w=p=z_x$ and $v=q=z_y$, we see that this is the first extended part of the operator $\ell = U(p,q)\partial_z$, and hence follows the assertion. □

In order to construct a mapping of Eq.(2.38) to a linear equation we compare the generator $\ell=-U(p,q)\partial_z$ with (2.2). Clearly, $\bar{X}=p$, $\bar{Y}=q$ and $\sigma=1$. To find $\bar{Z}$, we use $[\bar{Z},\bar{X}]=[\bar{Z},\bar{Y}]=0$, i.e.,

$$\bar{Z}_x + p\bar{Z}_z = 0, \qquad \bar{Z}_y + q\bar{Z}_z = 0.$$

The solution is $\bar{Z}=\bar{Z}(p,q,\alpha)$, $\alpha=-z+px+qy$. From (1.8), we get

$$\bar{Z}_p + x\bar{Z}_\alpha = P, \qquad \bar{Z}_q + y\bar{Z}_\alpha = Q$$

with $P=Z_X$ and $Q=Z_Y$. Now from (2.5) with $\sigma=1$, we get $\bar{Z}_\alpha=1$, and consequently $\bar{Z}=-z+px+qy+h(p,q)$ for an arbitrary function h. Setting h=0, we get the Legendre transformation [36]:

$$X = p, \quad Y = q, \quad Z = -z + px + qy, \quad P = x, \quad Q = y. \tag{2.46}$$

This transformation maps Eq.(2.38) to

$$\mathcal{A}Z \equiv a(X,Y)T - 2b(X,Y)S + c(X,Y)R = 0 \tag{2.47}$$

where $T=Z_{YY}$, $S=Z_{XY}$ and $R=Z_{XX}$.

D. Lie's Theorem on the Monge-Ampère equation.

The Monge-Ampère equation takes the form

$$f = A(rt - s^2) + Br + Cs + Dt + E = 0, \tag{2.48}$$

where the coefficients A,B,C,D and E are functions of x,y,z,p and q. In studying this equation, the concept of intermediate integrals plays an important role [37].
An equation

$$I(\alpha(\omega^{(1)}),\beta(\omega^{(1)})) = 0, \tag{2.49}$$

$\alpha:w^{(1)}\to R$, $\beta:w^{(1)}\to R$, $I(\cdot,\cdot)$ an arbitrary function $R^2\to R$, is said to be a general intermediate integral of Eq.(2.48) if α and β satisfy the equality

$$(D_x\alpha)(D_y\beta) - (D_y\alpha)(D_x\beta) = f. \tag{2.50}$$

Lie [38,37] proved a theorem which in our notation reads as

Theorem[Lie] . A Monge-Ampère equation admitting two general intermediate integrals $I^1(\alpha^1,\beta^1)=0$ and $I^2(\alpha^2,\beta^2)=0$ is transformable to the equation $Z_{XY}=0$ by a Lie contact transformation $\Omega^{(1)} = \bar{\Omega}^{(1)}(\omega^{(1)})$ whose four components are given by

$$\bar{X} = \alpha^1, \quad \bar{Y} = \alpha^2, \quad \bar{P} = \beta^1, \quad \bar{Q} = \beta^2. \tag{2.51}$$

Two intermediate integrals in this theorem are related to an invariance group of the corresponding Monge-Ampère equation:

Proposition 16. A Monge-Ampère equation possessing two general intermediate integrals $I^i(\alpha^i,\beta^i)=0$, $i=1,2$, admits two invariance group generators

$$\ell_i = \sigma(\omega^{(1)}) I^i(\alpha^i,\beta^i)\partial_z, \quad i=1,2, \tag{2.52}$$

where $\sigma^{-1} = \rho = [\alpha^1,\beta^1] = [\alpha^2,\beta^2]$.

Proof. It is easy to check that the equation $Z_{XY}=0$ admits generators $L_1=I^1(X,P)\partial_Z$ and $L_2=I^2(Y,Q)\partial_Z$ with arbitrary functions I^1 and I^2. On the other hand, according to Lie's theorem above, there exists a Lie contact transformation mapping the Monge-Ampère equation in question to $Z_{XY}=0$. In view of (2.51) and Proposition 13, the inverse of this transformation maps L_1 and L_2 to (2.52). □

The generators (2.52) appear to have different forms from the generator (2.2). However, they contain (2.2) as a special case. To see this we choose special forms $I^1=I^1(\alpha^1)$, $I^2=I^2(\alpha^2)$ and let $\ell=\ell_1+\ell_2$. Then from (2.52) we obtain

$$\ell = \sigma\{I^1(\alpha^1) + I^2(\alpha^2)\}\partial_z \equiv \sigma\cdot U(\alpha^1,\alpha^2)\partial_z . \qquad (2.53)$$

Observing that $U(X,Y)=I^1(X)+I^2(Y)$ is the general solution of the equation $U_{XY}=0$, we see that indeed the Monge-Ampère equation in question admits a generator of the form (2.2) with $\bar{X}=\alpha^1(\omega^{(1)})$ and $\bar{Y}=\alpha^2(\omega^{(1)})$. From this result it is clear that we can find Lie's linearization mapping of a Monge-Ampère equation to $Z_{XY}=0$, when it exists, by examining the invariance group of the equation .

E. The equation $(z_x)^\alpha z_{xx} - z_{yy} = 0$.

A special Monge-Ampère equation of the form

$$g(z_x)z_{xx} - z_{yy} = 0, \qquad g:R\to R, \qquad (2.54)$$

arises in a variety of physical problems such as nonlinear vibrations ($g(z_x)>0$) [39], and irrotational transonic flows ($g(z_x)=1+az_x$) [40]. In the following, we consider a class of equations of the form

$$f = (z_x)^\alpha z_{xx} - z_{yy} = 0, \ \alpha \text{ real}, \tag{2.55}$$

and apply the foregoing analysis to examine a possible mapping to a linear equation other than the Legendre transformation. The invariance group of Eq.(2.55) depends upon the value of α. Assuming a generator to be of the form (2.24), i.e. $\ell=\theta(x,y,z,p,q)\partial_z$, we find that the following cases occur:

(1) $\alpha \neq 0, -2, -4$:

$$\begin{aligned} \ell_1 &= \{-(\alpha+4)xpq + \alpha zq - 2(\alpha+1)yq^2 - 4y\textstyle\int p^{\alpha+1}dp\}\partial_z \\ \ell_2 &= \{(\alpha+4)xp + (3\alpha+4)yq - \alpha z\}\partial_z, \\ \ell_3 &= (z + \tfrac{1}{2}\alpha yq)\partial_z, \quad \ell_4 = y\partial_z, \quad \ell_5 = U(p,q)\partial_z. \end{aligned} \tag{2.56}$$

(2) $\alpha = -2$: ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5 as above and $\ell_6 = zp\partial_z$. (2.57)

(3) $\alpha = -4$:
$$\begin{aligned} \ell_1 &= (xp - yq)\partial_z, \quad \ell_2 = U(p,q)\partial_z, \\ \ell_3 &= \{pI^1(p^{-1}-q,(p^{-1}-q)y+z)\}\partial_z, \\ \ell_4 &= \{pI^2(p^{-1}+q,(p^{-1}+q)y-z)\}\partial_z. \end{aligned} \tag{2.58}$$

$$(4)\ \alpha = 0: \quad \ell_1 = z\partial_z, \quad \ell_2 = h(x,y)\partial_z, \qquad (2.59)$$
$$\ell_3 = I^1(x+y,p+q)\partial_z, \quad \ell_4 = I^2(x-y,p-q)\partial_z.$$

In all cases, the function $U(p,q)$ represents an arbitrary solution of the differential equation

$$U_{pp} - (p)^{\alpha}U_{qq} = 0 \qquad (2.60)$$

and the corresponding generators are related to the Legendre transformation discussed above. The function $h(x,y)$ in the last case is an arbitrary solution of the differential equation $h_{xx} - h_{yy} = 0$ and the corresponding generator ℓ_2 is the superposition generator of the equation $z_{xx}-z_{yy}=0$. I^1 and I^2 in the last two cases are arbitrary functions of their arguments and the equations $I^1=0$ and $I^2=0$ are the general intermediate integrals of the corresponding equations.

We examine the case $\alpha=-4$ in some detail. The equation is

$$(z_x)^{-4}z_{xx} - z_{yy} = 0. \qquad (2.61)$$

Comparing (2.52) and the generators ℓ_3 and ℓ_4 of (2.58), and using (2.51), relations $[\bar{X},\bar{Z}]=[\bar{Y},\bar{Z}]=0$ and (1.8),(2.5), we find the Lie contact transformation mapping (2.61) to $Z_{XY}=0$ to be

$$X = p^{-1} - q, \quad Y = p^{-1} + q, \quad Z = -2p^{-1}(z - px - qy), \tag{2.62}$$
$$P = -(p^{-1} - q)y - z, \quad Q = (p^{-1} + q)y - z.$$

The inverse transformation is

$$x = \tfrac{1}{2}Z - \tfrac{1}{4}(X + Y)(P + Q), \quad y = -(X + Y)^{-1}(P - Q) \tag{2.63}$$
$$z = -(X+Y)^{-1}(YP+XQ), \quad p = 2(X+Y)^{-1}, \quad q = \tfrac{1}{2}(-X+Y).$$

If we introduce the general solution of $U_{XY}=0$, $U=F(X)+G(Y)$, where F and G are arbitrary functions, into (2.63) by $Z=U$, $P=U_X$ and $Q=U_Y$, then we obtain a parametric representation of the general solution of the differential equation

$$(u_x)^{-4}u_{xx} - u_{yy} = 0. \tag{2.64}$$

Explicitly this is

$$\left\{ \begin{aligned} x &= \tfrac{1}{2}\{F(X)+G(Y)\} - \tfrac{1}{4}(X+Y)\{F'(X)+G'(Y)\} \\ y &= -(X+Y)^{-1}\{F'(X)-G'(Y)\} \\ u &= -(X+Y)^{-1}\{YF'(X)+XG'(Y)\}, \end{aligned} \right. \tag{2.65}$$

where F' and G' denote derivatives.

We note in passing that Eq.(2.61) is also transformable by the Legendre transformation (2.46) to

$$Z_{XX} - (X)^{-4} Z_{YY} = 0, \tag{2.66}$$

and in turn Eq.(2.66) can be mapped into $Z_{XY}=0$ by a composition of the transformations (2.46) and (2.62). Explicitly, the transformation

$$x = 2(X+Y)^{-1}, \quad y = \tfrac{1}{2}(-X+Y), \quad z = (X+Y)^{-1}Z$$

maps the equation $z_{xx} - (x)^{-4} z_{yy} = 0$ to the equation $Z_{XY}=0$.

Remark 2. Let Eq.(2.55) be written as

$$D_x(z_x)^{\alpha+1} - D_y(\alpha+1)z_y = 0. \tag{2.67}$$

If we introduce a potential $\tilde{z}$ by

$$(z_x)^{\alpha+1} = \tilde{z}_y, \qquad (\alpha+1)z_y = \tilde{z}_x, \tag{2.68}$$

$\tilde{z}$ satisfies the equation

$$(\tilde{z}_y)^{\beta}\, \tilde{z}_{yy} - \tilde{z}_{xx} = 0, \qquad \beta = -\alpha(\alpha+1)^{-1}. \tag{2.69}$$

The transformation (2.68) may be viewed as a Bäcklund

transformation[†)] between Eq.(2.67) and Eq.(2.69).
For $\alpha=-2$, the transformation (2.68) becomes an auto-Bäcklund transformation:

$$(z_x)^{-2} z_{xx} - z_{yy} = 0 \xleftrightarrow{(2.68)} (\tilde{z}_y)^{-2} \tilde{z}_{yy} - \tilde{z}_{xx} = 0. \quad (2.70)$$

For $\alpha=-4$, Eq.(2.69)takes the form

$$(\tilde{z}_y)^{-\frac{4}{3}} \tilde{z}_{yy} - \tilde{z}_{xx} = 0 \quad (2.71)$$

and the transformation (2.68) together with the general solution (2.65) yields a general solution for Eq.(2.71). Seymour and Varley [41] obtained the general solution of Eq.(2.54) when $g(z_x)$ satisfies the equation

$$\frac{d}{dz_x} g = \mu g^{\frac{5}{4}} + \nu g^{\frac{7}{4}}, \qquad \mu, \nu \text{ constants.}$$

The case $\mu=0$ yields Eq.(2.61) and the case $\nu=0$ leads to Eq.(2.71).

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†) Consider a system $\{g^\nu(x,z,\ldots,\underset{k}{z},\tilde{z},\ldots,\underset{\ell}{\tilde{z}})=0\}$. If this has the property that for any solution of a system $\{F^\mu(x,z,\ldots,\underset{m}{z})=0\}$ the corresponding $\tilde{z},\underset{1}{\tilde{z}},\underset{2}{\tilde{z}},\ldots$ satisfying $\{g^\nu=0\}$solve a system $\{\tilde{F}^\mu(x,\tilde{z},\ldots,\underset{n}{\tilde{z}})=0\}$, then the system $\{g^\nu=0\}$ is called a Bäcklund transformation between $\{F^\mu=0\}$ and $\{\tilde{F}^\mu=0\}$. In particular, if $F=\tilde{F}$, then it is called an auto-Bäcklund transformation. For the discussion of Bäcklund transformations, see the article by Lamb in Ref.1.

CHAPTER 3.

NON-INVERTIBLE MAPPINGS OF NONLINEAR SYSTEMS TO LINEAR SYSTEMS.

In the preceding chapter we have considered mappings which transform a system of nonlinear equations to a system of linear equations in a 1-1 manner. If we require only that a mapping transform a solution of some linear system to a solution of a given nonlinear system, the class of mappings widens and includes non-invertible (non 1-1) mappings. In the following we investigate these types of mappings and show that such mappings are frequently related to invariance groups.

3.1 Examples of non-invertible mappings.

We first consider Burgers' equation

$$\tilde{\ddot{f}} = \tilde{z}_{xx} + \tilde{z}\tilde{z}_x - \tilde{z}_y = 0. \tag{3.1}$$

This equation admits a five parameter point Lie group [42]. However, none of the generators is of the form (2.2), and hence by Theorem 2 there exists no 1-1 mapping to a linear equation. It is known that the Hopf-Cole transformation [43,44]

$$x = X, \quad y = Y, \quad \tilde{z} = \frac{2Z_X}{Z}, \tag{3.2}$$

relates Eq.(3.1) to the heat equation $Z_{XX}-Z_Y=0$. Introducing the transformation (3.2) into Eq.(3.1), we find that

$$\tilde{f} = 2Z^{-2}(ZZ_{XXX} - Z_X Z_{XX} - ZZ_{XY} + Z_X Z_Y) = 0 \tag{3.3}$$

which factorizes as

$$\tilde{f} = 2Z^{-2}(ZD_X - Z_X)(Z_{XX} - Z_Y) = 0. \tag{3.4}$$

It follows from (3.4) that the transformation (3.2) maps a solution of the heat equation to a solution of Burgers' equation. It is incorrect to say that the Hopf-Cole transformation maps Burgers' equation to the heat equation. It is also clear that (3.2) is not a 1-1 mapping. Although this type of mapping is out of the scope of the discussion in the preceding chapter, this particular transformation is found to be related to a Lie group. One standard argument [45] to rationalize the Hopf-Cole transformation is to introduce z through $\tilde{z}=z_x$ and, after integrating once, one considers the equation

$$z_{xx} + \tfrac{1}{2}(z_x)^2 - z_y = 0. \tag{3.5}$$

One then says "by inspection" that the transformation

$$x = X, \quad y = Y, \quad z = 2\ln(cZ), \quad c \text{ constant}, \tag{3.6}$$

maps Eq.(3.5) to the heat equation $Z_{XX}-Z_Y=0$ and from this follows the transformation (3.2). Eq.(3.5) corresponds to Example A in the previous chapter where we found the transformation (3.6) with $c=\frac{1}{2}$ by applying Theorem 2.

The second example is a nonlinear diffusion equation

$$\tilde{f} = D_x(\tilde{z}^{-2}\tilde{z}_x) - \tilde{z}_y = 0. \tag{3.7}$$

This equation admits a four parameter point Lie group [6] with no generator of the form (2.2). As in the above we let $\tilde{z}=z_x$ and instead of Eq.(3.7) we consider an integrated form

$$f = (z_x)^{-2}z_{xx} - z_y = 0. \tag{3.8}$$

This equation admits [23] seven generators of point transformations including the generator

$$\ell = U(z,y)z_x\partial_z, \tag{3.9}$$

involving an arbitrary solution of $U_{zz}-U_y=0$. Comparing (3.9) with (2.2), we find a point transformation

$$X = z, \quad Y = y, \quad Z = x \tag{3.10}$$

which maps Eq.(3.8) to the heat equation $\Delta Z = Z_{XX} - Z_Y = 0$. From (3.10) it follows that $\tilde{z} = z_x = (Z_X)^{-1}$ and one can verify that the transformation

$$x = Z, \quad y = Y, \quad \tilde{z} = (Z_X)^{-1} \tag{3.11}$$

transforms Eq.(3.7) to

$$\tilde{f} = Z_X^{-3}(Z_X D_X - Z_{XX})(Z_{XX} - Z_Y) = 0. \tag{3.12}$$

Hence the transformation (3.11) maps a solution of the heat equation to a solution of Eq.(3.7).

3.2 A use of potential functions.

The equations we have just considered are of the form

$$D_x g(\tilde{z}_{x\ldots x}, \ldots, \tilde{z}_x, \tilde{z}) - D_y \tilde{z} = 0. \tag{3.13}$$

For such an equation we can always introduce a potential z by

$$\tilde{z} = z_x, \qquad g = z_y. \tag{3.14}$$

The equation governing z is

$$g(z_{x\ldots xx},\ldots,z_{xx},z_x) - z_y = 0. \tag{3.15}$$

As the examples in §3.1 show, Eq.(3.15) can admit a larger invariance group than the original equation (3.13). Although this is not always the case, it is well worth keeping in mind. The possibility of introducing a potential, of course, is not limited only to equations of the form (3.13). To illustrate the importance of considering such a potential in more general circumstances, we take two examples.

A. A nonlinear wave equation. We consider a system

$$\begin{aligned} v_y - w_x &= 0 \\ v_y &= avw + bv + cw, \end{aligned} \tag{3.16}$$

where a,b and c are constants. Equations of this form arise in physical problems. For instance an equation governing a fluid flow through a reacting medium [46,45]

$$\begin{aligned} w_y + v_y + c\, w_x &= 0 \\ v_y &= k_1(a - w)v - k_2w(b - v) \end{aligned} \tag{3.17}$$

and an equation describing a two wave interaction [47-49],

$$\begin{aligned} v_y + c_1v_x &= -avw - bv - cw \\ w_y + c_2w_x &= avw + bv + cw \end{aligned} \tag{3.18}$$

can be put in the form (3.16) by simple changes of variables.†) Rescaling the variables in (3.16) as

$$(x,y,v,w) \longrightarrow \left(\frac{x}{c}, \frac{y}{b}, \frac{cv}{a}, \frac{bw}{a}\right), \tag{3.19}$$

we obtain

$$v_y - w_x = 0 \tag{3.20a}$$

$$v_y = vw + v + w. \tag{3.20b}$$

This equation admits only a trivial invariance point group generated by

$$\ell_1 = \partial_x, \quad \ell_2 = \partial_y, \quad \ell_3 = x\partial_x - y\partial_y - (v+1)\partial_v + (w+1)\partial_w. \tag{3.21}$$

However, if we introduce a potential z by

$$v = z_x, \quad w = z_y, \tag{3.22}$$

then the corresponding equation for z, i.e.

$$z_{xy} - z_x z_y - z_x - z_y = 0 \tag{3.23}$$

admits a larger point group with generators

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†) For Eq.(3.17), $x \to cx$, $y \to x+y$ and for Eq.(3.18), $x \to c_2 x - c_1 y$, $y \to x-y$.

$$\ell_1 = \partial_x, \quad \ell_2 = \partial_y, \quad \ell_3 = x\partial_x - y\partial_y - (x-y)\partial_z,$$
$$\partial_4 = \partial_z, \quad \ell_5 = U(x,y)e^z\partial_z, \tag{3.24}$$

where $U(x,y)$ is an arbitrary solution of the differential equation

$$U_{xy} - U_x - U_y = 0. \tag{3.25}$$

Now applying Theorem 2 to ℓ_5, we find a transformation

$$X = x, \quad Y = y, \quad Z = e^{-z}, \tag{3.26}$$

which maps Eq.(3.23) to†)

$$Z_{XY} - Z_X - Z_Y = 0. \tag{3.27}$$

The transformation connecting Eq.(3.27) to Eq.(3.20) is then

$$x = X, \quad y = Y, \quad v = -\frac{Z_X}{Z}, \quad w = -\frac{Z_Y}{Z}. \tag{3.28}$$

It is known that Eq.(3.17) and Eq.(3.18) admit transformations of the form (3.28) [45,46,48,49].

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†) When bc=0, the equation corresponding to Eq.(3.23) admits an additional generator and the equation can be mapped into $Z_{XY}=0$.

To gain some insight as to why the introduction of the potential, (3.22), enlarges the invariance group let us express the generators (3.24) in terms of variables x,y,v and w. The generators ℓ_1 and ℓ_2 are unchanged. The first extension of ℓ_3 takes the form

$$\ell_3 = x\partial_x - y\partial_y - (x-y)\partial_z - (z_x+1)\partial_{z_x} + (z_y+1)\partial_{z_y}. \tag{3.29}$$

Clearly, this corresponds to ℓ_3 of (3.21). The first extension of ℓ_4 is

$$\ell_4 = \partial_z + 0\cdot\partial_{z_x} + 0\cdot\partial_{z_y}, \tag{3.30}$$

and hence $\ell_4 \equiv 0$ in the (x,y,v,w) system. For the generator ℓ_5 we have

$$\ell_5 = Ue^z\partial_z + (U_x+Uz_x)e^z\partial_{z_x} + (U_y+Uz_y)e^z\partial_{z_y}. \tag{3.31}$$

Because of the appearance of z in the coefficients of $\partial_{z_x} \equiv \partial_v$ and $\partial_{z_y} \equiv \partial_w$, the first extended part of (3.31) can not be expressed in terms of x,y,v,w alone and consequently ℓ_5 is not a point group generator in the (x,y,v,w) system as we should expect. Now suppose that we consider z,v and w as functions of x and y and express z by a line integral as

$$z = \int z_x dx + z_y dy = \int v dx + w dy. \tag{3.32}$$

Then the first extended part of (3.31) can be written as

$$\ell_5 = \{(U_x+Uv)e^{\int vdx+wdy}\}\partial_v + \{(U_y+Uw)e^{\int vdx+wdy}\}\partial_w. \tag{3.33}$$

One can verify that Eq.(3.20) indeed admits the generator (3.33). We note that the generator (3.33) depends not only on x,y,v and w but also on the integral $\int vdx+wdy$. In other words by introducing the potential, we have in effect introduced an "integral dependent" generator which is beyond the framework of the Lie-Bäcklund groups. The same can be said for the preceding two examples. For instance the generator $\ell=U(x,y)e^{-\frac{1}{2}z}\partial_z$ of Eq.(3.5) becomes, via $\tilde{z}=z_x$, an integral dependent generator,

$$\ell = \{(U_x-\tfrac{1}{2}U\tilde{z})e^{-\frac{1}{2}\int\tilde{z}dx}\}\partial_{\tilde{z}} \tag{3.34}$$

of Burgers' equation (3.1). We will discuss some aspects of integral dependent generators in the following chapter.

B. <u>An equation of a fluid flow.</u> Sukharev [50] investigated the invariance point group of the equation

$$\left.\begin{aligned} &w_y + v_x = 0, \\ &w_x - v^\alpha w^{-\beta} = 0, \end{aligned}\right. \qquad \alpha,\beta \text{ real} \tag{3.35}$$

which describes a fluid flow through a long pipe-line.

The system (3.35) was found to admit an extra generator when $\alpha = -1$:

$$\begin{aligned} w_y + v_x &= 0 \\ w_x - v^{-1} w^{-\beta} &= 0. \end{aligned} \tag{3.36}$$

The extra generator is

$$\ell = g(w,y)\partial_x + \{w^{-\beta}\partial_w g(w,y)\}\partial_v, \tag{3.37}$$

where $g(w,y)$ is an arbitrary solution of a linear differential equation

$$\partial_w(w^{-\beta}\partial_w g) - \partial_y g = 0. \tag{3.38}$$

The property of the generator (3.37) was not studied. It is shown here that it is related with a linear equation associated with Eq.(3.36).

According to Proposition 3, we have

$$\ell \doteq (w_x g)\partial_w + (v_x g - w^{-\beta}\partial_w g)\partial_v. \tag{3.39}$$

This, however, is not in the form of (2.11)†) and consequently

†) With $z^1=w$, $z^2=v$, the operator (2.11) takes the form $\ell=(U^1\sigma_1^1+U^2\sigma_2^1)\partial_w+(U^1\sigma_1^2+U^2\sigma_2^2)\partial_v$, where $U^i=U^i(\bar{X}(x,y),\bar{Y}(x,y))$, $U^i(X,Y)$ being an arbitrary solution of some linear system of differential equations. Clearly, (3.39) is not of this form.

there exists no 1-1 mapping of the system (3.36) to a linear system. Now, introducing a potential z by

$$w = z_x, \qquad v = -z_y, \tag{3.40}$$

we write the second equation of (3.36), as

$$(z_x)^{\beta} z_{xx} + (z_y)^{-1} = 0 \tag{3.41}$$

The generator (3.39) can be written in terms of x,y and z as

$$\ell = (D_x U)\partial_w - (D_y U)\partial_v = (D_x U)\partial_{z_x} + (D_y U)\partial_{z_y}, \tag{3.42}$$

where $U=U(z_x,y)$ is defined by

$$U(z_x,y) = \int^{z_x} g(s,y)ds. \tag{3.43}$$

From (3.38) and (3.43), we obtain an equation for U:

$$(z_x)^{-\beta}(\partial_{z_x})^2 U - \partial_y U = 0. \tag{3.44}$$

Noticing that (3.42) is the first extended part of the generator

$$\ell = U(z_x,y)\partial_z, \tag{3.45}$$

we expect that Eq.(3.41) admits the generator (3.45) subject to the condition (3.44). A direct calculation verifies this. We can now identify (3.45) with (2.2) obtaining $\bar{X}=z_x$, $\bar{Y}=y$. Going through the steps illustrated in the example of the Legendre transformation in the preceding chapter, we find a Lie contact transformation

$$x = -Z_X, \quad y = Y, \quad z = Z - XZ_X, \quad z_x = X, \quad z_y = Z_Y, \tag{3.46}$$

which transforms Eq.(3.41) to a linear equation

$$X^{-\beta}Z_{XX} - Z_Y = 0. \tag{3.47}$$

Let U(X,Y) be a solution of the differential equation

$$X^{-\beta}U_{XX} - U_Y = 0. \tag{3.48}$$

Then, from the first three relations in (3.46), we obtain an implicit solution of Eq.(3.41):

$$x = -U_X(X,Y), \quad y = Y, \quad z = U(X,Y) - XU_X(X,Y), \tag{3.49}$$

and the rest of (3.46) leads to

$$w = X, \quad v = -U_Y(X,Y), \tag{3.50}$$

which together with (3.49) defines an implicit solution of Eq.(3.36). To obtain an explicit solution, we solve the first two equations of (3.49) with respect to X and Y, say, $X=f(x,y)$ and $Y=y$, and introduce these into (3.50) to get

$$w = f(x,y), \quad v = -U_Y(f(x,y),y). \tag{3.51}$$

So far in this chapter, we only considered those equations which admit potential functions. The following equation does not admit a potential, but it is related to a linear equation.

3.3 The Liouville equation.

The Liouville equation is defined by

$$z_{xy} - e^z = 0. \tag{3.52}$$

One of the generators admitted by this equation is

$$\ell = \{f(x)z_x + g(y)z_y + f_x(x) + g_y(y)\}\partial_z, \tag{3.53}$$

where $f(x)$ and $g(y)$ are arbitrary functions and f_x and g_y are their derivatives. This is the only generator which depends on arbitrary functions. The generator (3.53) is not of the form (2.2) and hence there exists no 1-1 mapping of Eq.(3.52) to a linear equation. Let us consider the invariant solution $z = u(x,y)$ associated with the generator (3.53). It is a solution of the system

$$\begin{aligned} & f(x)u_x + g(y)u_y + f_x(x) + g_y(y) = 0 \\ & u_{xy} - e^u = 0. \end{aligned} \tag{3.54}$$

The solution is found to be

$$u = \ell n\left|\frac{2\phi_x\psi_y}{(\phi + \psi)^2}\right| \tag{3.55}$$

where $\phi(x) = \int f^{-1}dx$ and $\psi(y) = \int g^{-1}dy$. This is the general solution of the Liouville equation [37]. Introducing $U=\phi+\psi$, we write (3.55) as

$$u = \ell n|2U^{-2}U_xU_y|. \tag{3.56}$$

Recognizing U as the general solution of the equation $U_{xy}=0$, we conclude that the transformation

$$z = \ell n|2Z^{-2}Z_xZ_y| \tag{3.57}$$

maps a solution of $Z_{xy}=0$ to a solution of $z_{xy} - e^z = 0$. One can also see this from the equality

$$z_{xy} - e^{z} = (Z_x^{-1}D_x + Z_y^{-1}D_y - Z_x^{-2}Z_{xx} - Z_y^{-2}Z_{yy} - 2Z^{-1})Z_{xy}, \tag{3.58}$$

hence $Z_{xy}=0 \rightarrow z_{xy}-e^{z}=0$.

3.4 A series of L-B generators and the linearization.

Another sign which indicates a possible connection of a nonlinear equation to a linear equation is the admission of an infinite sequence of Lie-Bäcklund generators by the nonlinear equation. From Proposition 10 it is clear that a linear homogeneous system admitting a generator of the form $L = (B_{\mu}^{\nu}Z^{\mu})\partial_{Z^{\nu}}$ admits an infinite sequence of L-B generators. Consequently, if there exists a mapping connecting this linear system to a given nonlinear system, then the mapping is likely to transform these generators into group generators of the nonlinear system. In Appendix 4, we investigate L-B generators of a nonlinear diffusion equation $D_x\{(z)^{\alpha}z_x\} - z_t = 0$ and it is shown that only for $\alpha=-2$ the equation admits an infinite sequence of L-B generators. The analysis of these generators in turn leads to a transformation similar to (3.11). The Hopf-Cole transformation of Burgers' equation can be obtained in a similar manner.

CHAPTER 4.

A SUMMARY AND FUTURE PROBLEMS.

4.1 A summary of the main reuslts.

In the second chapter we proved that by examining the invariance group of a system of nonlinear differential equations one can determine definitively whether the system is transformable to a linear system by an invertible mapping. Moreover, the mapping can be constructed from a generator of the group. In all cases, we need only to consider group generators of the form (2.24) or (2.25) in which no higher coordinates than $\underset{1}{z}$ appears, i.e. $\theta^{\nu} = \theta^{\nu}(x, z, \underset{1}{z})$.

In the third chapter we investigated the question of the existence of non-invertible mappings relating linear and nonlinear equations. It is a considerably more complex question than that of invertible mappings. The problem of finding such mappings is equivalent to finding a condition under which a given nonlinear equation admits a transformation leading to a factorization such as (3.4),(3.12) and (3.58). No definitive condition has been found yet. However, as it has been demonstrated here, the group analysis supplemented by the introduction of a potential function and higher order Lie-Bäcklund generators are effective means to discover such non-invertible mappings.

The examples investigated in this work cover all linearizable equations known to the author and two new equations, i.e., Eq.(3.7) and Eq.(3.36). With our method it should be particularly emphasized that even if one is unable to linearize given nonlinear differential equations, one is always left with their invariance groups. In turn these can be used for the construction of invariant solutions, conservation laws and other invariance properties of equations [6,7, 8]. Some such examples are given in Appendices 5-7.

4.2 A generalization of the concept of invariance.

In the group analysis of differential equations, it is very important to find the largest invariance group associated with the equations. During the course of the present work a question concerning the possibility to enlarge an invariance group has arisen. Obviously, the meaning of "largest" changes according to the type of groups we consider. The group can be enlarged by considering higher order L-B groups or by introducing more general types of invariance. In the following we discuss some aspects of integral dependent invariance. We use notations

$u, u_x, u_{xx}, \ldots$ in place of $z, z_x, z_{xx}, \ldots$.

a. A hierarchic structure in L-B sequences. To present the basic idea clearly, we take a specific example, namely, Burgers' equation,

$$u_{xx} + uu_x - u_t = 0. \tag{4.1}$$

A generator of the L-B invariance group of Eq.(4.1),

$$\ell = \theta(x,t,u,u_x,\ldots,u_{x\ldots x})\partial_u , \tag{4.2}$$

must satisfy the determining equation

$$(D_x)^2\theta + uD_x\theta + u_x\theta - D_t\theta = 0 \tag{4.3}$$

for any u satisfying Eq.(4.1). Now we let $u \to u + \varepsilon v$, $|\varepsilon| \ll 1$, in Eq.(4.1). Then v satisfies the linearized equation†)

$$v_{xx} + uv_x + u_x v - v_t = 0. \tag{4.4}$$

In view of Eq.(4.3) and Eq.(4.4), it is clear that θ is

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†) Here, the term "linearization" is used in a different sense than in the preceding chapters.

a very special solution of the linearized equation (4.4): A solution expressed in terms of a solution of the original equation (4.1). This observation leads us to examine the invariance group of the linearized equation (4.4) for from such an invariance group we may be able to construct those particular solutions which satisfy Eq.(4.3). So, we consider the following problem:

Find invariance groups of the differential equation $v_{xx} + uv_x + u_x v - v_t = 0$ with unknown v and an arbitrary solution u(x,t) of the equation $u_{xx} + uu_x - u_t = 0$.

Since Eq.(4.4) is linear in v, it is sufficient, according to Theorem 1, to consider a linear generator

$$\hat{\ell} = (\mathcal{B}v)\partial_v = \hat{\theta}\partial_v, \tag{4.5}$$

where $\mathcal{B}$ is some linear operator. Once $\mathcal{B}$ is found, we can find, using Proposition 12 in §1.7, an infinite sequence of solutions of Eq.(4.4) in the form

$$v^{(n)}(x,t) = (\mathcal{B})^n v(x,t), \quad n=1,2,3,\dots , \tag{4.6}$$

where v(x,t) is any solution of Eq.(4.4). In particular, if we choose as v one of θ satisfying Eq.(4.3), we obtain a sequence of functions

$$\theta^{(n)}(x,t,u,u_x,u_{xx},\ldots) = (\mathcal{B})^n\theta(x,t,u,u_x,\ldots), \qquad (4.7)$$

which solve Eq.(4.4), and hence satisfy Eq.(4.3). In other words, once we find a generator $\hat{\ell} = (\mathcal{B}v)\partial_v$ of the linearized equation, we can construct a sequence of L-B generators

$$\ell = \{(\mathcal{B})^n\theta\}\partial_u, \quad n=1,2,3,\ldots, \qquad (4.8)$$

for the <u>original</u> nonlinear equation from any known generator $\ell=\theta\partial_u$. Obviously, this statement holds for any system of nonlinear equations. We apply this result to analyze L-B invariance groups of Eq.(4.1).

Some years ago I found that Burgers' equation admitted hierarchies of L-B generators. The question is whether these have the structure of the form (4.8). A simple calculation shows that the only point group generator, i.e.

$$\hat{\ell} = (\mathcal{B}v)\partial_v = \{b^x(x,t)v_x + b^t(x,t)v_t + b(x,t)v\}\partial_v \qquad (4.9)$$

admitted by Eq.(4.4) is $\hat{\ell}=v\partial_v$, i.e. $\mathcal{B}=1$. Thus, as long as $\hat{\ell}$ is restricted to the form (4.9), there is no sequence of L-B generators of the form (4.8). Olver found in his study [18] of symmetries of time evolution equations that Burgers' equation admits an infinite sequence of L-B generators of the form

$$\ell = \{(\mathcal{D})^n u_x\}\partial_u, \quad n=1,2,3,... \tag{4.10}$$

with

$$\mathcal{D} = D_x + \tfrac{1}{2}u + \tfrac{1}{2}u_x D_x^{-1}, \tag{4.11}$$

where D_x^{-1} denotes an integral operator with the property $D_x D_x^{-1} = D_x^{-1} D_x = 1$. Comparing (4.10) with (4.8), we let $\mathcal{B} = \mathcal{D}$ and consider an operator corresponding to (4.5):

$$\hat{\theta}\partial_v = (\mathcal{D}\, v)\partial_v = (v_x + \tfrac{1}{2}uv + \tfrac{1}{2}u_x D_x^{-1} v)\partial_v \ . \tag{4.12}$$

It is easy to check that Eq.(4.4) indeed admits (4.12). The sequence (4.10) is one of the hierarchies of generators I found previously. This suggests the existence of other linear operators $\mathcal{B}$. An important aspect of the generator (4.12) is that $\hat{\theta}$ depends not only on x,t (through u), v and v_x but also on the integral $D_x^{-1}v$. This leads us to consider a generalization of (4.9) by including a $D_x^{-1}v$ term:

$$\hat{\ell} = (\mathcal{B}v)\partial_v = \{b^x(x,t)v_x + b^t(x,t)v_t + b(x,t)v + b^{-x}(x,t)D_x^{-1}v\}\partial_v \tag{4.13}$$

The b's are functions of x and t. For this type of integral dependent generator, we can adapt Lie's algorithm for finding generators. With a straightforward calculation we find that Eq.(4.4) admits, in addition to (4.12) and the trivial $\hat{\ell} = v\partial_v$, the generators with following $\mathcal{B}$'s:

$$\begin{aligned}
\mathcal{B}' &= D_t + (\tfrac{1}{2}u_x + \tfrac{1}{4}u^2) + \tfrac{1}{2}u_t D_x^{-1} \\
\mathcal{B}'' &= t\mathcal{B}' + \tfrac{1}{2}xD_x + \tfrac{1}{4}xu + \tfrac{1}{4}(u+xu_x)D_x^{-1} \qquad (4.14a\text{-}c) \\
\mathcal{B}''' &= t\mathcal{D} + \tfrac{1}{2}x + \tfrac{1}{2}D_x^{-1}
\end{aligned}$$

where in $\mathcal{B}'''$, $\mathcal{D}$ is the operator (4.11). Using any of (4.14) we can produce sequences of L-B generators of the form (4.10). More generally, recalling Proposition 10, we see that operators of the form

$$\ell = \{f(\mathcal{D}, \mathcal{B}', \mathcal{B}'', \mathcal{B}''')\theta\}\partial_u \qquad (4.15)$$

where $f(\cdot,\cdot,\cdot,\cdot)$ is an arbitrary polynomial of its arguments are all invariance group generators of Burgers' equation for any θ satisfying the determining equation (4.3), for example $\theta = u_x$. They include all hierarchies mentioned above.

<u>b. On integral dependent generators</u>. The example we have just seen and the example in §3.2 lead us to generalize the concept of invariance by allowing generators to depend not only on derivatives but also on integrated quantities. A fundamental difficulty in considering integral dependent generators is that there are too many possibilities. Consider a time evolution equation

$$G(u,u_x,u_{xx},\ldots,u_{x\ldots x}) - u_t = 0 \tag{4.16}$$

with invariance group generator $\ell=\theta\partial_u$. We want to allow θ to depend on integrals such as

$$D_x^{-1}u = \int u\ dx, \qquad (D_x^{-1})^2 u = \iint u\ dxdx, \ \ldots\ . \tag{4.17}$$

A problem is that these are not the only possible integrals involving u. There is no a priori reason not to include integrals such as

$$D_x^{-1}(u)^2, \qquad D_x^{-1}\left(u_x(D_x^{-1}u)\right), \quad \text{etc.} \quad . \tag{4.18}$$

One way to get around this problem is again to study the linearized equation of Eq.(4.16):

$$G_u v + G_{u_x} v_x + \ldots + G_{u_{x\ldots x}} v_{x\ldots x} - v_t = 0. \tag{4.19}$$

Let $\hat{\ell}=\hat{\theta}\partial_v$ be an invariance group generator of (4.19). The merit of considering Eq.(4.19) instead of Eq.(4.16) is that since Eq.(4.19) is a linear equation in v, we can expect that $\hat{\theta}$ of $\hat{\ell}$ depends linearly on v, i.e. $\hat{\theta}=\mathcal{B}v$, for some integro-differential linear operator $\mathcal{B}$. The form of $\mathcal{B}$, however, is still quite arbitrary. For instance, $\mathcal{B}v$ could contain terms such as

$$D_x^{-1}(fv), \qquad D_x^{-1}\left(gD_x^{-1}(fv)\right), \ \ldots \tag{4.20}$$

where f and g are functions of x,t. We can avoid considering such complex expressions if we observe that quantities such as (4.20) can be formally represented by a series of the form

$$\left\{ \sum_{k=0}^{m} b^{(k)}(D_x)^k + \sum_{k=1}^{\infty} b^{(-k)}(D_x^{-1})^k \right\} v \equiv \mathcal{B}v, \tag{4.21}$$

where $b^{(k)}$ and $b^{(-k)}$ are functions of x,t. For example, by repeated integration by parts, we have for the first expression of (4.20),

$$D_x^{-1}(fv) = \int fv dx = fD_x^{-1}v - (D_x f)(D_x^{-1})^2 v + \left((D_x)^2 f\right)(D_x^{-1})^3 v + \ldots . \tag{4.22}$$

For the second expression of (4.20), we apply this procedure to each integration operator D_x^{-1}. Thus, we should start with a generator $\hat{\ell} = (\mathcal{B}v)\partial_v$ with $\mathcal{B}v$ of the form (4.21) rather than quantities such as (4.20). Once such an operator $\mathcal{B}$ is found, we can obtain a sequence of invariance group generators for the original equation (4.16) using the formula (4.8). Although the approach described here is of great generality, it will be of little practical value unless a closed form of the infinite sum in (4.21) is found. Such a closed expression may be found by examining the first few terms in the sum.

In applying this analysis to the KdV equation $u_{xxx} + uu_x + u_t = 0$, a variety of $\mathcal{B}$'s of the form (4.21) has been found. The only B with a closed form is

$$B = (D_X)^2 + \frac{1}{3} u + \frac{2}{3} u_x D_x^{-1}. \qquad (4.23)$$

This particular operator arose in the study of the inverse scattering method [51] to solve nonlinear time evolution equations. It was also obtained by Olver in connection with invariances of the KdV equation. We use (4.21) with (4.23) to obtain invariance group generators of the KdV equation. The equation admits point group generators $\ell=\theta\partial_u$ with the following θ:

$$\theta^{(a)} = u_x, \quad \theta^{(b)} = -u_t, \quad \theta^{(c)} = tu_x - 1, \quad \theta^{(d)} = -tu_t - \frac{1}{3}xu_x - \frac{2}{3}u. \qquad (4.24)$$

Using these θ's in (4.21) we obtain sequences of invariance group generators. It turns out that there are only two sequences instead of four. With $\theta^{(n)}=(B)^n\theta^{(0)}$ and notations $u_x=u_1$, $u_{xx}=u_2, \ldots$, one sequence is:

$$\theta^{(0)} = \theta^{(a)} = u_1, \quad \theta^{(1)} = u_3 + uu_1 = \theta^{(b)},$$

$$\theta^{(3)} = u_5 + \frac{5}{3} uu_3 + \frac{10}{3} u_1u_2 + \frac{5}{6} u^2u_x, \ \ldots\ ,$$

and the other is

$$\theta^{(0)} = \theta^{(c)} = tu_1-1, \quad \theta^{(1)} = t(u_3+uu_1) - \frac{1}{3}xu_1 - \frac{2}{3}u = \theta^{(d)},$$

$$\theta^{(2)} = tu_5 + \frac{4}{3}tuu_3 - \frac{1}{3}xu_3 + \frac{11}{3}tu_1u_2 - \frac{4}{3}u_2 + \frac{2}{3}tu^2u_1 - \frac{1}{3}xuu_1 - \frac{2}{9}u^2 - \frac{2}{9}u_1D_x^{-1}u, \tag{4.26}$$

Since the first sequence involves no integral quantity, it corresponds to L-B invariance groups and was first found in connection with conservation laws associated with the KdV equation (Appendix 5). The second sequence is new and involves integrals.

Olver was interested in the operator D with the property that $\ell=(D^n\theta)\partial_u$ is an L-B invariance group generator of a nonlinear time evolution equation. The present analysis shows that D is related with a generalized invariance of the corresponding linearized equation. Clearly, our formulation is not restricted to time evolution equations and can lead to more general invariance groups than L-B groups. There should be further investigations of these generalized symmetries and their uses.

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APPENDIX 1. PROOF OF THEOREM 1.

Proof. For brevity we write $\Theta^{\nu}(X,Z,\underset{1}{Z},\ldots,\underset{m}{Z}) = \Theta^{\nu}(X,Z)$. Suppose the system $A^{\nu}_{\mu}Z^{\mu}=0$ admits $L = \Theta^{\nu}(X,Z)\partial_{Z^{\nu}}$. Then, since the system is linear homogeneous, $L=\Theta^{\nu}(X,\varepsilon Z+U)\partial_{Z^{\nu}}$ is also a generator for an arbitrary solution of $A^{\nu}_{\mu}U^{\mu}=0$. We write its power series expansion in ε as

$$\Theta^{\nu}(X,\varepsilon Z+U)\partial_{Z^{\nu}} = h^{\nu(0)}\partial_{Z^{\nu}} + \varepsilon h^{\nu(1)}\partial_{Z^{\nu}} + \varepsilon^{2}h^{\nu(2)}\partial_{Z^{\nu}} + \ldots . \tag{A1}$$

Since ε is arbitrary, each term must be a generator of an invariance group. In particular, the $O(\varepsilon)$ term has the form (1.43).

Conversely, suppose that the system $A^{\nu}_{\mu}Z^{\mu}=0$ admit (1.43). The function $\Theta^{\nu}(X,Z)$ yielding (1.44) is not unique, but if we restrict to $\Theta^{\nu}(X,Z)$ which depends on X and only on those Z's which appear in the coefficients of $\partial_{Z^{\nu}}$ in (1.43), then Θ^{ν} is determined within an arbitrary additive function $\phi^{\nu}(X)$. We let $Z\to\varepsilon Z$ in such $\Theta^{\nu}(X,Z)$ and expand $\Theta^{\nu}(X,\varepsilon Z)$ in the series in ε:

$$\Theta^{\nu}(X,\varepsilon Z) = \bar{h}^{\nu(0)} + \varepsilon\bar{h}^{\nu(1)} + \varepsilon^{2}\bar{h}^{\nu(2)} + \ldots , \tag{A2}$$

where $\bar{h}^{(0)} = \Theta^{\nu}(X,0)$ which is a function of X alone. We may set $\bar{h}^{\nu(0)}=0$ since we have an arbitrary function $\phi^{\nu}(X)$ at our disposal. With such a choice of $\phi^{\nu}(X)$, the operator $\Theta^{\nu}(X,Z)\partial_{Z^{\nu}}$ becomes an invariance group generator. To see this we let $U\to Z$ in (1.43), then let $Z\to\varepsilon Z$. The resulting operator is still an invariance group generator of the same system. We write its power series expansion in ε as

$$L = \varepsilon g^{\nu(1)}\,\partial_{Z^{\nu}} + \varepsilon^2 g^{\nu(2)}\partial_{Z^{\nu}} + \ \dots\ . \tag{A3}$$

By the construction of $\Theta^{\nu}(X,Z)$ above, we have $g^{\nu(k)} \propto \bar{h}^{\nu(k)}$. Since every $g^{\nu(k)}\partial_{Z^{\nu}}$ in (A3) is an invariance group generator of $A^{\nu}_{\mu}Z^{\mu}=0$, we see that the operator

$$\Theta^{\nu}(X,Z)\partial_{Z^{\nu}} = \bar{h}^{\nu(1)}\partial_{Z^{\nu}} + \bar{h}^{\nu(2)}\partial_{Z^{\nu}} + \dots$$

is an invariance group generator of $A^{\nu}_{\mu}Z^{\mu}=0$.

APPENDIX 2. PROOFS OF PROPOSITIONS 9-12.

1. Proof of Proposition 9. From Proposition 2, we have $L\bar{B}^{\nu}_{\mu}=\bar{B}^{\nu}_{\mu}L$ and $\bar{L}B^{\nu}_{\mu}=B^{\nu}_{\mu}\bar{L}$. Then,

$$[L,\bar{L}] = (L\bar{B}^{\nu}_{\mu}Z^{\mu} - \bar{L}B^{\nu}_{\mu}Z^{\mu})\partial_{Z^{\nu}} = (\bar{B}^{\nu}_{\mu}LZ^{\mu} - B^{\nu}_{\mu}\bar{L}Z^{\mu})\partial_{Z^{\nu}}$$

$$= (\bar{B}^{\nu}_{\kappa}B^{\kappa}_{\mu}Z^{\mu} - B^{\nu}_{\kappa}\bar{B}^{\kappa}_{\mu}Z^{\mu})\partial_{Z^{\nu}} = ([\bar{B},B]^{\nu}_{\mu}Z^{\mu})\partial_{Z^{\nu}}.$$

By hypothesis, we have $[L,\bar{L}]=\tilde{L}=(\tilde{B}^{\nu}_{\mu}Z^{\mu})\partial_{Z^{\nu}}$. Comparing these, we obtain $[B,\bar{B}]=-\tilde{B}$. □

2. Proof of Proposition 10. It is sufficient to prove for the explicit form $A^{\nu}_{\mu}Z^{\mu}=0$ of $f=0$. By Proposition 2 and by the invariance condition (1.36), we have

$$A^{\nu}_{\mu}Z^{\mu}=0 \rightarrow LA^{\nu}_{\mu}Z^{\mu} = A^{\nu}_{\kappa}B^{\kappa}_{\mu}Z^{\mu} \equiv A^{\nu}_{\mu}Z'^{\mu} = 0,$$

$$A^{\nu}_{\mu}Z'^{\mu}=0 \rightarrow \bar{L}A^{\nu}_{\mu}Z'^{\mu} = \bar{L}LA^{\nu}_{\mu}Z^{\mu} = 0,$$

i.e., $A^{\nu}_{\mu}Z^{\mu}=0 \rightarrow \bar{L}LA^{\nu}_{\mu}Z^{\mu}=0$. □

3. <u>Proof of Proposition 11</u>. As in the proof of Proposition 10, we have $LA^{\nu}_{\mu}Z^{\mu}=A^{\nu}_{\mu}B^{\mu}_{\kappa}Z^{\kappa}=0$. We also have $B^{\nu}_{\mu}A^{\mu}_{\kappa}Z^{\kappa}=0$ since $A^{\mu}_{\kappa}Z^{\kappa}=0$. From these two we obtain $(A^{\nu}_{\mu}B^{\mu}_{\kappa} - B^{\nu}_{\mu}A^{\mu}_{\kappa})Z^{\kappa}=0$ provided $A^{\mu}_{\kappa}Z^{\kappa}=0$, i.e. $[A,B]Z=0$ provided $AZ=0$. □

4. <u>Proof of Proposition 12.</u> We have $LAZ=ALZ=A^{\nu}_{\mu}B^{\mu}_{\kappa}Z^{\kappa}=0$ for any values of $X, Z, \underset{1}{Z}, \ldots$ satisfying the equation $AZ=0$. Obviously, $Z=U(X)$, $\underset{1}{Z}=\underset{1}{U}(X)$, ..., satisfy this equation and consequently, $A^{\nu}_{\mu}B^{\mu}_{\kappa}U^{\kappa}(X)=0$, i.e. $ABU(X)=0$. Repeating the same argument, we find $A(B)^{m}U(X)=0$. □

APPENDIX 3. A DETERMINATION OF AN INVARIANCE GROUP OF THE EQUATION $(z_{xy})^2 - 4z_x z_y = 0$.

To illustrate the process of determining an invariance group of a differential equation, we consider the equation†

$$f = (z_{xy})^2 - 4z_x z_y = 0. \tag{1}$$

To simplify notations in the following computation, we let $z_x=p$, $z_y=q$, $z_{xx}=r$, $z_{xy}=s$ and $z_{yy}=t$. Then, Eq.(1) becomes

$$s^2 - 4pq = 0. \tag{2}$$

We consider a generator of the form

$$\ell = g(x,y,z,p,q)\,\partial_z. \tag{3}$$

By operating its second extended form on (2), we get

$$\ell(s^2-4pq) = 2sD_xD_yg - 4(D_xg)q - 4p(D_yg)$$

† This equation is taken from Forsyth [37] page 198 ,

$$
\begin{aligned}
= 2\{ & g_{xy}s + g_{xz}qs + g_{xp}s^2 + g_{xq}st \\
& + g_{zy}ps + g_{zz}pqs + g_{zp}ps^2 + g_{zq}pst + g_z s^2 \\
& + g_{py}rs + g_{pz}qrs + g_{pp}rs^2 + g_{pq}rst + g_p ss_x \\
& + g_{qy}s^2 + g_{qz}qs^2 + g_{qp}s^3 + g_{qq}s^2 t + g_q ss_y \\
& - 2g_y p - 2g_z pq - 2g_p ps - 2g_q pt \\
& - 2g_x q - 2g_z pq - 2g_p qr - 2g_q qs\}. \qquad (4)
\end{aligned}
$$

The invariance condition (1.36) demands that (5) vanish under the condition (1.34) which in this case takes the form

$$
\begin{aligned}
f &= s^2 - 4pq = 0, \\
D_x f &= 2ss_x - 4rq - 4ps = 0, \\
D_y f &= 2ss_y - 4sq - 4pt = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad .
\end{aligned}
$$

We only need the first three equations since (4) involves coordinates only upto the third order. We use these three equations to eliminate the coordinates s, s_x, s_y from (4). The resulting quantity must vanish irrespective of values of x,y,z,p,q,r and t: rearranging terms,

$$
\begin{aligned}
0 = {} & rtg_{pq} + r(g_{py} + g_{pz}q + 2g_{pp}p^{\frac{1}{2}}q^{\frac{1}{2}}) + t(g_{qx} + g_{qz}p + 2g_{qq}p^{\frac{1}{2}}q^{\frac{1}{2}}) \\
& + 2g_{xy}p^{\frac{1}{2}}q^{\frac{1}{2}} + 2g_{xz}q(pq)^{\frac{1}{2}} + 4g_{xp}pq \\
& + 2g_{zy}p(pq)^{\frac{1}{2}} + 2g_{zz}(pq)^{\frac{3}{2}} + 4g_{zp}p^2 q \\
& + 4g_{qy}pq + 4g_{qz}pq^2 + 8g_{qp}(pq)^{\frac{3}{2}} - 2g_y p - 2g_x q. \qquad (5)
\end{aligned}
$$

We note that r and t appear only in the first three terms of (5). Thus their coefficients must vanish:

$$g_{pq} = 0, \quad g_{py} + g_{pz}q + 2g_{pp}(pq)^{\frac{1}{2}} = 0,$$

$$g_{qx} + g_{qz}p + 2g_{qq}(pq)^{\frac{1}{2}} = 0 .$$

It is easy to find that the most general solution to these equations is

$$g = a(x)p + b(y)q + e(x,y,z), \tag{6}$$

where a,b and e are arbitrary functions of their arguments. Introducting (6) into the rest of (5), we get

$$\begin{aligned} 0 = {} & e_{xy}(pq)^{\frac{1}{2}} + e_{xz}q(pq)^{\frac{1}{2}} + 2a_x pq \\ & + e_{yz}p(pq)^{\frac{1}{2}} + e_{zz}(pq)^{\frac{3}{2}} + 2b_y pq \\ & - (b_y q + e_y)p - (a_x p + e_x)q . \end{aligned}$$

Since a,b and e do not involve p and q, coefficients of different powers of $p^m q^n$ must vanish. The resulting equations yield

$$e = \alpha z + \beta, \quad \alpha,\beta \text{ arbitrary constants.}$$

Therefore,

$$g = a(x)p + b(y)q + \alpha z + \beta.$$

The final form of the generator is

$$\ell = \{a(x)p + b(y)q + \alpha z + \beta\}\partial_z, \tag{7}$$

where a(x) and b(y) are arbitrary functions and α and β are arbitrary constants. We note the resemblance between the generator (7) and the generator (3.35) of the Liouville equation. As in the case of the Liouville equation, the invariant solution associated with (7) will lead to the general solution of Eq.(1).

On the remarkable nonlinear diffusion equation $(\partial/\partial x)[a(u+b)^{-2}(\partial u/\partial x)] - (\partial u/\partial t) = 0$

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We study the invariance properties (in the sense of Lie–Bäcklund groups) of the nonlinear diffusion equation $(\partial/\partial x)[C(u)(\partial u/\partial x)] - (\partial u/\partial t) = 0$. We show that an infinite number of one-parameter Lie–Bäcklund groups are admitted if and only if the conductivity $C(u) = a(u+b)^{-2}$. In this special case a one-to-one transformation maps such an equation into the linear diffusion equation with constant conductivity, $(\partial^2\bar{u}/\partial\bar{x}^2) - (\partial\bar{u}/\partial\bar{t}) = 0$. We show some interesting properties of this mapping for the solution of boundary value problems.

1. INTRODUCTION

In recent years nonlinear diffusion processes described by the partial differential equation (p.d.e)

$$\frac{\partial}{\partial x}\left[C(u)\frac{\partial u}{\partial x}\right] - \frac{\partial u}{\partial t} = 0, \qquad (1)$$

with a variable conductivity $C(u)$, have appeared in problems related to plasma and solid state physics.[1,2] Interest in such processes has long occurred in other fields such as metallurgy and polymer science.[3–5]

Some exact solutions are well known for such equations.[6] These can be shown to be included in the class of all similarity solutions to such equations obtained from invariance under a Lie group of point transformations.[7,8]

Recently, it has been shown that differential equations can be invariant under continuous group transformations beyond point or contact transformation Lie groups which act on a finite dimensional space.[9] These new continuous group transformations act on an infinite dimensional space. Such infinite dimensional contact transformations have been called Noether transformations[10] or Lie–Bäcklund (LB) transformations[11] (Noether mentioned the possibility of such transformations in her celebrated paper on conservation laws[12]). Well known nonlinear partial differential equations admitting LB transformations include the Korteweg-deVries,[13,14] sine-Gordon,[10,15] cubic Schrödinger,[14] and Burgers' equations.[16] All of these known examples admit an infinite number of one-parameter LB transformations. Moreover, many of their important properties (existence of an infinite number of conservation laws,[13,14] existence of solitons,[14] and existence[17] of Bäcklund transformations[18]) are related to their invariance under LB transformations.

Any linear differential equation which admits a nontrivial one-parameter point Lie group is invariant under an infinite number of one-parameter LB transformations through superposition. Moreover, every known nonlinear p.d.e., invariant under LB transformations, can be associated with some corresponding linear p.d.e.

With the above views in mind we study the invariance properties of Eq. (1). Previously,[7,8,19] it had been shown that Eq. (1) is invariant under

a) a three-parameter point Lie group for arbitrary $C(u)$,

b) a four-parameter point Lie group if $C(u) = a\cdot(u+b)^{\nu}$,

c) a five-parameter point Lie group if $\nu = -\frac{4}{3}$.

[It is well known that a six-parameter point Lie group leaves invariant Eq. (1) in the case $C(u) = \text{const.}$[20]]

In the present work, we show that *Eq. (1) is invariant under LB transformations if and only if the conductivity is of the form*

$$C(u) = a\cdot(u+b)^{-2}, \qquad (2)$$

i.e., *if Eq. (1) is of the form*

$$\frac{\partial}{\partial x}\left[a\cdot(u+b)^{-2}\frac{\partial u}{\partial x}\right] - \frac{\partial u}{\partial t} = 0. \qquad (3)$$

Furthermore, this equation admits an infinite number of LB transformations.

In this special case, there exists a one-to-one transformation which maps Eq. (3) into the linear diffusion equation with constant conductivity, namely, the heat equation

$$\frac{\partial^2\bar{u}}{\partial\bar{x}^2} - \frac{\partial\bar{u}}{\partial\bar{t}} = 0. \qquad (4)$$

In the course of this paper, we find an operator connecting two infinitesimal LB transformations leaving Eq. (3) invariant. We prove that this operator is a recursion operator which generates an infinite sequence of one-parameter infinitesimal LB transformations leaving Eq. (3) invariant. Moreover, we show that no other LB transformation leaves Eq. (3) invariant.

By examining the linearization of Eq. (3), we are led to construct the transformation mapping Eq. (3) into Eq. (4). It is shown that this transformation maps the recursion operator of Eq. (3) into the spatial translation operator of Eq. (4), giving a simple interpretation of the transformation relating Eq. (3) to Eq. (4). We use this transformation to connect boundary value problems of Eq. (3) to those of Eq. (4).

We construct a new similarity solution of Eq. (3) corresponding to invariance under LB transformations.

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2. DERIVATION OF THE CLASS OF NONLINEAR DIFFUSION EQUATIONS INVARIANT UNDER LB TRANSFORMATIONS

LB transformations include Lie groups of point transformations and finite dimensional contact transformations.[11] The algorithm for calculating infinitesimal LB transformations leaving differential equations invariant is essentially the same as Lie's method[8] for calculating infinitesimal point groups.

Consider the most general one-parameter infinitesimal LB transformation that can leave invariant a time-evolution equation,[21] namely;

$$\begin{aligned} u^* &= u + \epsilon U(x,t,u,u_1,\ldots,u_n) + O(\epsilon^2),\\ x^* &= x, \\ t^* &= t, \end{aligned} \tag{5}$$

where $u_i = \partial^i u/\partial x^i$, $i = 1,2,\cdots$. Let $\partial u/\partial t = u_t$, $\partial u_i/\partial t = u_{it}$, $\partial U/\partial u = U_0$, $\partial U/\partial u_i = U_i$, $\partial^2 U/\partial u_i \partial u_j = U_{i,j}$, $C' = dC/du$, and $C'' = d^2C/du^2$.

In the above notation Eq. (1) becomes

$$u_t = C'(u_1)^2 + Cu_2. \tag{6}$$

Under Eqs. (5) the derivatives of u appearing in Eq. (6) transform as follows:

$$\begin{aligned} (u_t)^* &= u_t + \epsilon U' + O(\epsilon^2),\\ (u_1)^* &= u_1 + \epsilon U^x + O(\epsilon^2),\\ (u_2)^* &= u_2 + \epsilon U^{xx} + O(\epsilon^2), \end{aligned}$$

where

$$\begin{aligned} U' &= D_t U = \frac{\partial U}{\partial t} + U_0 u_t + \sum_{i=1}^{n} U_i u_{it}, \qquad (7)\\ U^x &= D_x U = \frac{\partial U}{\partial x} + \sum_{i=0}^{n} U_i u_{i+1},\\ U^{xx} &= (D_x)^2 U = \frac{\partial^2 U}{\partial x^2} + 2\sum_{i=0}^{n} \frac{\partial U_i}{\partial x} u_{i+1} \\ &\quad + \sum_{i,j=0}^{n} U_{i,j} u_{i+1} u_{j+1} + \sum_{i=0}^{n} U_i u_{i+2}. \end{aligned}$$

D_t and D_x are total derivative operators with respect to t and x, respectively.

The transformation (5) is said to leave Eq. (6) invariant if and only if for every solution $u = \theta(x,t)$ of Eq. (6)

$$U' = C''U(u_1)^2 + 2C'U^x u_1 + C'Uu_2 + CU^{xx}. \tag{8}$$

The fact that U must satisfy Eq. (8) for any solution of Eq. (6) imposes severe restrictions on U. Using Eq. (6) the derivatives of u_t with respect to t, i.e., u_{it}, can be eliminated in Eq. (8). Since the invariance condition (8) must hold for every solution of Eq. (6), Eq. (8) becomes a polynomial form in u_{n+1} and u_{n+2}. As a result the coefficients of each term in this form must vanish. This leads us to the determining equations for the infinitesimal LB transformations (5).

If in Eq. (5), $n \leqslant 2$, we obtain the Lie group of point transformations leaving Eq. (6) invariant. Without loss of generality we assume $n \geqslant 3$ in Eq. (5). It turns out that for $n \geqslant 3$, U is independent of x and t.

In our polynomial form, the coefficient of u_{n+2} vanishes and the coefficients of $(u_{n+1})^2$ and u_{n+1}, respectively, lead to determining equations

$$CU_{n,n} = 0, \tag{9}$$

$$nC'U_n u_1 = 2C \sum_{i=0}^{n-1} U_{n,i} u_{i+1}. \tag{10}$$

Solving Eqs. (9) and (10) we find that

$$U = \alpha (C)^{(1/2)n} u_n + E(u,u_1,\ldots,u_{n-1}), \tag{11}$$

where E is undetermined, and $\alpha =$ arbitrary constant.

The substitution of Eq. (11) into the remaining terms of Eq. (8) leads to a polynomial form in u_n whose coefficients of $(u_n)^2$ and u_n, respectively, lead to determining equations

$$CE_{n-1,n-1} = 0, \tag{12}$$

$$\begin{aligned} &2C\left[\sum_{i=0}^{n-2} E_{n-1,i} u_{i+1}\right] \\ &+ (1-n)C'E_{n-1} u_1 - \frac{\alpha}{4} n(n+3)C'(C)^{(1/2)n} u_2 \\ &+ \alpha\left[\tfrac{1}{4}n^2(C')^2(C)^{(1/2)n-1} - \tfrac{1}{2}n(n+2)C''(C)^{(1/2)n}\right](u_1)^2 = 0. \end{aligned} \tag{13}$$

Solving Eqs. (12) and (13) we find that

$$\begin{aligned} U = \alpha\, &\left[(C)^{(1/2)n} u_n + \tfrac{1}{4}n(n+3)C'(C)^{(1/2)n-1} u_1 u_{n-1}\right] \\ &+ F(u)u_{n-1} + G(u,u_1,\ldots,u_{n-2}), \end{aligned} \tag{14}$$

where F and G are undetermined and, more importantly, for $\alpha \neq 0$ it is necessary that the conductivity $C(u)$ satisfy the differential equation

$$2CC'' = 3(C')^2. \tag{15}$$

Hence, it is necessary that

$$C(u) = a\cdot(u+b)^{-2}, \tag{16}$$

where a and b are arbitrary constants for the invariance of Eq. (1) under LB transformations. Without loss of generality we can set $a = 1$, $b = 0$, i.e., from now on we consider the equivalent p.d.e.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) \equiv B. \tag{17}$$

This particular equation has been considered as a model equation of diffusion in high-polymeric systems.[4,5]

3. CONSTRUCTION OF A RECURSION OPERATOR; AN INFINITE SEQUENCE OF INVARIANT LB TRANSFORMATIONS OF EQ. (17)

For $n = 3$ it is easy to solve the rest of the determining equations and show that the only LB transformation leaving Eq. (17) invariant is

$$U = U^{(1)} = u^{-3}u_3 - 9u^{-4}u_1u_2 + 12u^{-5}(u_1)^3. \tag{18}$$

For $n = 4$ we obtain two linearly independent LB transformations $U^{(1)}$ and

$$\begin{aligned} U^{(2)} = u^{-4}u_4 &- 14u^{-5}u_1u_3 - 10u^{-5}(u_2)^2 \\ &+ 95u^{-6}(u_1)^2u_2 - 90u^{-7}(u_1)^4. \end{aligned} \tag{19}$$

The existence of $U^{(1)}$ and $U^{(2)}$, combined with the work of Olver,[16] motivates us to seek a linear recursion operator $\mathcal{G}$ leading to infinitesimal LB transformations $U^{(k)}$ defined as follows:

$$(\mathscr{D})^k B = U^{(k)}, \quad k = 1,2,\cdots. \tag{20}$$

The character of $\{B, U^{(1)}, U^{(2)}\}$ leads one to consider for $\mathscr{D}$ the form

$$\mathscr{D} = pD_x + q + r(D_x)^{-1}, \tag{21}$$

where D_x is a total derivative operator, $(D_x)\cdot(D_x)^{-1}$ is the identity operator, and $\{p,q,r\}$ are functions of $\{u,u_1,u_2\}$. Then one can show that $\mathscr{D}B = U^{(1)}$ if and only if

$$p = u^{-1}, \tag{22}$$

and

$$q[u^{-2}u_2 - 2u^{-3}(u_1)^2] + ru^{-2}u_1 = -3u^{-4}u_1u_2 + 6u^{-5}(u_1)^3. \tag{23}$$

Furthermore, $(\mathscr{D})^2 B = U^{(2)}$ if and only if

$$q = -2u^{-2}u_1, \tag{24}$$

and

$$r = -u^{-2}u_2 + 2u^{-3}(u_1)^2. \tag{25}$$

A more concise expression for the operator is

$$\mathscr{D} = (D_x)^2\cdot(u^{-1})\cdot(D_x)^{-1}. \tag{26}$$

We now show that the constructed operator $\mathscr{D}$ is indeed a recursion operator. Let the operator

$$\begin{aligned} A &= \sum_{i=0}^{2} B_i (D_x)^i \\ &= u^{-2}(D_x)^2 - 4u^{-3}u_1 D_x + 6u^{-4}(u_1)^2 - 2u^{-3}u_2 \\ &= (D_x)^2\cdot u^{-2}, \end{aligned} \tag{27}$$

where $B_i = (\partial/\partial u_i)B$. Olver's work[16] shows that $\mathscr{D}$ is a recursion operator for Eq. (17) if and only if the commutator

$$[A - D_t, \mathscr{D}] = 0, \tag{28}$$

for any solution $u = \theta(x,t)$ of Eq. (17). Moreover, if $\mathscr{D}$ is a recursion operator, then the sequence $\{U^{(1)}, U^{(2)}, \cdots\}$ given by Eq. (20) is an infinite sequence of LB transformations leaving Eq. (17) invariant. It is straightforward to show that A and $\mathscr{D}$ satisfy Eq. (28).

The nature of $U^{(1)}$ and the form of a general U given by Eq. (11) show that for $n = l+2$, there are at most $k < l$ linearly independent LB transformations leaving Eq. (17) invariant since U must depend uniquely on u_{l+2}.

The proof that $\mathscr{D}$ is a recursion operator demonstrates that $k = l$ and hence we have found all possible LB transformations leaving Eq. (17) invariant, namely, $\{U^{(k)}\}$, $k = 1,2,\cdots$.

4. A MAPPING TO THE LINEAR DIFFUSION EQUATION

As far as we know all p.d.e.'s invariant under LB transformations have a recursion operator and, moreover, can be related to linear p.d.e.'s. This suggests the possibility of seeking a transformation relating Eq. (17) to a linear equation. This leads us to consider the linearization of Eq. (17), namely,

$$(A - \partial/\partial t) f = 0, \tag{29}$$

where A is given by Eq. (27) for any solution $u = \theta(x,t)$ of Eq. (17). Introducing a new variable $\bar{u}$ by

$$f = \frac{\partial}{\partial x}(u\bar{u}), \tag{30}$$

we obtain from Eq. (29) the equation

$$\left[\left(u^{-1}\frac{\partial}{\partial x}\right)^2 + u^{-3}u_1\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right]\bar{u} = 0 \tag{31}$$

and if we set

$$\begin{aligned} \frac{\partial}{\partial \bar{x}} &= u^{-1}\frac{\partial}{\partial x}, \\ \frac{\partial}{\partial \bar{t}} &= \frac{\partial}{\partial t} - u^{-3}u_1\frac{\partial}{\partial x}, \end{aligned} \tag{32}$$

Eq. (31) becomes

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \frac{\partial \bar{u}}{\partial \bar{t}} = 0. \tag{33}$$

Since $f = 0$ is always a solution of Eq. (29), the relation (30) suggests that we set $u\bar{u} =$ *constant*. This and Eqs. (32) lead us to the transformation

$$\begin{aligned} d\bar{x} &= u\,dx + u^{-2}u_1\,dt, \\ d\bar{t} &= dt, \\ \bar{u} &= u^{-1}, \end{aligned} \tag{34}$$

relating solutions $u = \theta(x,t)$ of Eq. (17) to solutions $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ of Eq. (33). Choosing a fixed point (x_0,t_0), we have the following integrated form of Eqs. (34):

$$\begin{aligned} \bar{x} &= \int_{x_0}^{x} u\,dx' - \int_{t_0}^{t}\left(\frac{\partial}{\partial x}u^{-1}\right)_{x=x_0} dt', \\ \bar{t} &= t - t_0, \\ \bar{u} &= u^{-1}. \end{aligned} \tag{35}$$

It is easy to check that Eqs. (35) indeed transform Eq. (17) to Eq. (33), and define a map relating the solutions of Eqs. (17) and (33). Moreover, if $u > 0$ ($\bar{u} > 0$), Eqs. (35) define a one-to-one map since $\partial\bar{x}/\partial x > 0$ for each fixed t.[22]

We now show that under the transformation (34) the recursion operator $\mathscr{D}$ of Eq. (17) is transformed into the recursion operator

$$\bar{\mathscr{D}} = D_{\bar{x}}, \tag{36}$$

leading to an infinite sequence of LB transformations of the heat equation (33). The proof is as follows:

An LB transformation of the form (5) induces an LB transformation on the variables $\{\bar{x},\bar{t},\bar{u}\}$ through Eqs. (34), namely,

$$\begin{aligned} \bar{x}^* &= \bar{x} + \epsilon\bar{\xi} + O(\epsilon^2), \\ \bar{t}^* &= \bar{t}, \\ \bar{u}^* &= \bar{u} + \epsilon\bar{\eta} + O(\epsilon^2), \end{aligned} \tag{37}$$

where $\bar{\xi}$ and $\bar{\eta}$ are defined by

$$\begin{aligned} d\bar{\xi} &= \mathscr{A}\,d\bar{x} + \mathscr{B}\,d\bar{t}, \\ \mathscr{A} &= \bar{u}U, \quad \mathscr{B} = \bar{u}_{\bar{x}}U + (\bar{u})^2(U^x - 2U), \\ \bar{\eta} &= -(\bar{u})^2 U. \end{aligned} \tag{38}$$

It turns out that for any solution $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ of Eq. (33), $\mathscr{A}$ and $\mathscr{B}$ satisfy the integrability condition $D_{\bar{t}}\mathscr{A} = D_{\bar{x}}\mathscr{B}$, so that $d\bar{\xi}$ is an exact differential. The integrated form of $\bar{\xi}$ is

$$\bar{\xi} = -(D_{\bar{x}})^{-1}[\bar{\eta}\cdot(\bar{u})^{-1}] + c, \quad (39)$$

where c is an arbitrary constant. Since $U^{(i+1)} = \mathscr{D} U^{(i)}$, where $\mathscr{D}$ is given by Eq. (26), for $c = 0$ we get a corresponding infinite sequence of invariant infinitesimal LB transformations $\{\bar{U}^{(i)}\}$ for Eq. (33), namely,

$$\bar{U}^{(i)} = \bar{\eta}^i - \bar{u}_{\bar{1}}\bar{\xi}^i,$$

where

$$\bar{\eta}^i = -(\bar{u})^2 U^{(i)}, \qquad \bar{\xi}^i = -(D_{\bar{x}})^{-1}[\bar{\eta}^i\cdot(\bar{u})^{-1}], \quad (40)$$

and $\bar{u}_{\bar{i}} = (\partial/\partial\bar{x})^i\bar{u}$. From Eqs. (40) it is simple to show that

$$\bar{U}^{(i+1)} = D_{\bar{x}}\bar{U}^{(i)}, \quad (41)$$

leading to Eq. (36). Moreover,

$$\bar{U}^{(i)} = D_{\bar{x}}(\bar{u}\bar{\xi}^i) = (D_{\bar{x}})^i\bar{u}_{\bar{1}}, \quad i = 1,2,\cdots. \quad (42)$$

$D_{\bar{x}}$ corrsponds to the obvious invariance of Eq. (33) under translations in $\bar{x}$.

It is interesting to note that the recursion operator for the invariant LB transformations of Burgers' equation is also mapped into the space translation operator under the Hopf–Cole transformation relating Burgers' equation to the heat equation. Moreover, we can obtain the Hopf–Cole transformation by examining the linearization equation (29) corresponding to Burgers' equation.

5. PROPERTIES OF SOLUTIONS OF EQ. (17) FROM THE MAPPING

We now consider the use of Eqs. (34) in constructing solutions to Eq. (17). It is easy to show that Eqs. (34) are equivalent to

$$dx = \bar{u}\,d\bar{x} + \bar{u}_{\bar{1}}\,d\bar{t}, \qquad dt = d\bar{t}, \qquad u = (\bar{u})^{-1}, \quad (43)$$

with an integrated form

$$x = \int_{\bar{x}_0}^{\bar{x}} \bar{u}\,d\bar{x}' + \int_{\bar{t}_0}^{\bar{t}} (\bar{u}_{\bar{1}})_{\bar{x}=\bar{x}_0}\,d\bar{t}', \qquad t = \bar{t} - \bar{t}_0, \qquad u = (\bar{u})^{-1}, \quad (44)$$

for some fixed point $(\bar{x}_0,\bar{t}_0)$. In the following, we assume $u > 0$ ($\bar{u} > 0$). Without loss of generality, we set $\bar{x}_0 = \bar{t}_0 = 0$.

A. Explicit formula connecting solutions; examples

First we consider the problem of giving a more explicit formula for relating solutions of Eq. (33) to those of Eq. (17). Let $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ be a solution of Eq. (33) on the domain $\bar{t} > 0$, $\bar{x}\in(\bar{x}_1,\bar{x}_2)$. By Eqs. (43),

$$x = X(\bar{x},\bar{t}) = \int_0^{\bar{x}} \bar{\theta}(\bar{x}',\bar{t})\,d\bar{x}' + \int_0^{\bar{t}} \left(\frac{\partial\bar{\theta}(\bar{x},\bar{t}')}{\partial\bar{x}}\right)_{\bar{x}=0} d\bar{t}'. \quad (45)$$

This uniquely determines the function X^{-1}, $\bar{x} = X^{-1}(x,t)$, where $\bar{t} = t$. Now Eqs. (44) lead to the following solution of Eq. (17):

$$u = \theta(x,t) = \frac{1}{\bar{\theta}(X^{-1}(x,t),t)},$$

on the domain $x\in(x_1(t),\ x_2(t))$, $t > 0$, where (46)

$$x_1(t) = X(\bar{x}_1,t), x_2(t) = X(\bar{x}_2,t).$$

In a similar manner, Eqs. (35) map a solution $u = \theta(x,t)$ of Eq. (17) to

$$\bar{u} = \bar{\theta}(\bar{x},\bar{t}) = \frac{1}{\theta(\bar{X}^{-1}(\bar{x},\bar{t}),\bar{t})}$$

on the domain $\bar{x}\in(\bar{x}_1(\bar{t}),\bar{x}_2(\bar{t}))$, $\bar{t} > 0$ where (47)

$$\bar{x}_1(\bar{t}) = \bar{X}(x_1,\bar{t}), \quad \bar{x}_2(\bar{t}) = \bar{X}(x_2,\bar{t}),$$

$$\bar{x} = \bar{X}(x,t) = \int_0^x \theta(x',t)\,dx' - \int_0^t \left[\frac{\partial}{\partial x}(\theta(x,t'))^{-1}\right]_{x=0} dt', \quad (48)$$

with the corresponding definition of the function $\bar{X}^{-1}(\bar{x},\bar{t}) = x$.

Example 1: The source solution of Eq. (33), i.e., $\bar{u} = \bar{\theta}(\bar{x},\bar{t}) = a(4\pi\bar{t})^{-1/2}e^{-(\bar{x}^2/4\bar{t})}$ on the domain $-\infty < \bar{x} < \infty, \bar{t} > 0$, is mapped by Eqs. (45) and (46) into the following separable solution of Eq. (17):

$$u = \theta(x,t) = a^{-1}(4\pi t)^{1/2}e^{v^2},$$

on the domain $-\frac{1}{2}a < x < \frac{1}{2}a$, $t > 0$, where $v(x)$ is defined by (49)

$$x = \frac{a}{\sqrt{\pi}}\int_0^v e^{-y^2}\,dy.$$

Note that $\lim_{x\to\pm\frac{1}{2}a}\theta(x,t) = +\infty$.

Example 2. The dipole solution of Eq. (33), i.e.,

$$\bar{u} = \bar{\theta}(\bar{x},\bar{t}) = -\frac{\partial}{\partial\bar{x}}[a(4\pi\bar{t})^{-1/2}e^{-(\bar{x}^2/4\bar{t})}],$$

on the domain $0 < \bar{x} < \infty$, $\bar{t} > 0$, is mapped by Eqs. (45) and (46) into the following self-similar solution of Eq. (17):

$$u = \theta(x,t) = x^{-1}(2t)^{1/2}\left[\ln\left(\frac{a^2}{4\pi t x^2}\right)\right]^{-1/2} \quad (50)$$

on the shrinking domain $0 < x < a(4\pi t)^{-1/2}$, $t > 0$.

B. Connection between initial conditions; connection between boundary conditions

The mapping formulas (34) and (43) demonstrate a one-to-one correspondence (within translation of x,t) between initial conditions for Eq. (17) and those for Eq. (33). As for the connection between boundary conditions, from the same formula it is easy to see that $x = s(t)$ is an insulating boundary of Eq. (17), i.e., $[\partial\theta(x,t)/\partial x]_{x=s(t)} = 0$, if and only if the corresponding boundary $\bar{x} = \bar{s}(\bar{t})$ is an insulating boundary of Eq. (33), i.e., the corresponding solution $\bar{u} = \bar{\theta}(\bar{x},\bar{t})$ satisfies $[\partial\bar{\theta}(\bar{x},\bar{t})/\partial\bar{x}]_{\bar{x}=\bar{s}(\bar{t})} = 0$. Moreover, $s(t) = \text{const}$ if and only if $\bar{s}(\bar{t}) = \text{const}$, i.e., there is a one-to-one correspondence between fixed insulating boundaries of Eqs. (17) and (33).

In general, a noninsulating boundary condition for Eq. (17), on a fixed boundary $x = \text{const} = c$, is mapped into a noninsulating boundary condition of Eq. (33) with a corresponding moving boundary $\bar{x} = \bar{s}(\bar{t}) \neq \text{const}$ with speed

$$\frac{d\bar{s}}{d\bar{t}} = \left[[\theta(x,t)]^{-2} \frac{\partial \theta(x,t)}{\partial x} \right]_{\substack{x=c \\ t=\bar{t}}}, \tag{51}$$

where, as previously mentioned, $u = \theta(x,t) > 0$.

6. CONCLUDING REMARKS

(a) From invariance under the LB transformations $\{U^{(i)}\}$, $i = 1,2,\cdots$, there exist similarity solutions of Eq. (17), i.e., $u = \theta(x,t;n)$, whose similarity forms satisfy

$$U^{(n)} + \sum_{k=1}^{n-1} c_k U^{(k)} = 0, \tag{52}$$

where $\{c_1, c_2, \ldots, c_{n-1}\}$ are arbitrary constants, $n = 1,2,\cdots$. For example, for $n = 1$, Eq. (52) leads to the similarity form

$$u = \theta(x,t;1) = [a(t)\cdot(x + b(t))^2 + c(t)]^{-1/2}, \tag{53}$$

where $\{a(t), b(t), c(t)\}$ are arbitrary. Substituting Eq. (53) into Eq. (17) we find that Eq. (53) solves Eq. (17) if and only if $a = \alpha$, $b = \beta$, and $c = \gamma e^{2\alpha t}$, where $\{\alpha, \beta, \gamma\}$ are arbitrary constants. This solution is not contained in the class of similarity solutions of Eq. (17) obtained from invariance under a four-parameter point Lie group.[7,8]

(b) The infinitesimal transformations (5) of the four-parameter point group of Eq. (17) are given by

$$U^a = u + xu_1, \quad U^b = xu_1 + 2tu_t,$$
$$U^c = u_1, \quad U^d = B. \tag{54}$$

Under the mapping (34), these are transformed, respectively, to corresponding infinitesimals of invariant point group transformations of Eq. (33):

$$\bar{U}^a = \bar{u}, \quad \bar{U}^b = \bar{x}\bar{u}_{\bar{1}} + 2\bar{t}\bar{u}_{\bar{t}},$$
$$\bar{U}^c = 0, \quad \bar{U}^d = \bar{B} = \bar{u}_{\bar{2}}. \tag{55}$$

Conversely, the mapping (34) transforms the six-parameter point Lie group of Eq. (33) as follows: The three-parameter subgroup of infinitesimals given by Eq. (55) transforms to $\{U^a, U^b, U^d\}$ given by Eqs. (54) and $\bar{U} = \bar{u}_{\bar{1}}$ transforms to $U = 0$; the remaining infinitesimal point group transformations $\bar{U}^e = \bar{x}\bar{u} + 2\bar{t}\bar{u}_{\bar{t}}$ and $\bar{U}^f = (\frac{1}{4}\bar{x}^2 + \frac{1}{2}\bar{t})\bar{u} + \bar{x}\bar{t}\bar{u}_{\bar{1}} + \bar{t}^2\bar{u}_{\bar{t}}$ are mapped, respectively, into infinitesimals which depend on $\{x,t,u,u_1\}$ and integrals of u.

(c) Generally speaking, the action of a recursion operator $\mathscr{G}$ on any infinitesimal invariance transformation U of the form (5) (whether of point group or LB type) yields a new infinitesimal transformation $U' = \mathscr{G}U$ if $\mathscr{G}U \neq 0$. For Eq. (17), we can show that $\mathscr{G}U^a = \mathscr{G}U^b = \mathscr{G}U^c = 0$.

(d) The heat equation is a special limiting case of Eq. (3) obtained by setting $a = b^2$ and then observing $\lim_{b\to\infty} b^2(u+b)^{-2} = 1$. As one might expect if $a = b^2$, for the corresponding recursion operator $\mathscr{G}$, $\lim_{b\to\infty} \mathscr{G} = \partial/\partial x$, and the mapping formulas reduce to identity mappings.

(e) Since Eq. (1) admits an infinite sequence of LB transformations if and only if $C(u)$ satisfies Eq. (15) with associated mapping (34) whereas Eq. (4) admits an infinite sequence of LB transformations, there is no point transformation of the form

$$\bar{x} = K(x,t,u),$$
$$\bar{t} = L(x,t,u),$$
$$\bar{u} = M(x,t,u),$$

relating solutions of Eq. (1) and those of Eq. (4).

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[21]An infinitesimal LB transformation

$$x^* = x + \epsilon\xi + O(\epsilon^2),$$
$$t^* = t + \epsilon\tau + O(\epsilon^2),$$
$$u^* = u + \epsilon\mu + O(\epsilon^2),$$

acts on a surface $F(x,t,u) = 0$ in the same manner as

$$x^* = x,$$
$$t^* = t,$$
$$u^* = u + \epsilon U + O(\epsilon^2),$$

where $U = \mu - \xi u_x - \tau u_t$.

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Invariance transformations, invariance group transformations, and invariance groups of the sine-Gordon equations*

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We investigate a structure of continuous invariance transformations connected to the identity transformation. The transformations considered do not necessarily form a group. We clarify the relationship between the infinitesimal invariance transformation and the finite invariance transformation by showing explicitly how the infinitesimal transformations are woven into the finite one. The analysis leads to a new method of finding generators of the invariance group transformation. The results are useful in the study of symmetry properties, or group theoretic structure, of differential equations. We use the results in studying the group properties of the sine-Gordon equation $u_{xt} = \sin u$, and indicate that the equation is invariant under an infinite number of one-parameter groups; the groups obtained are of a more general type than that dealt with by Lie. These findings are used to prove the group theoretic origin of the well-known conservation laws associated with the sine-Gordon equation.

INTRODUCTION

The discovery of the puzzling behavior of nonlinear wave "solitons" in various fields of applied science has triggered extensive study of nonlinear dispersive waves.[1] One of the basic properties of the system which admits a soliton appears to be the possession of an infinite number of conserved quantities. As has been shown by Lax,[2] the existence of such conserved quantities is closely related to the soliton behavior of the waves. In spite of their importance in elucidating the nature of nonlinear waves, it seems that no one as yet has obtained a clear understanding of the origin of such conserved quantities.[3]

It is well known that both in classical and quantum mechanics the conservation law reflects the existence of symmetry in the system. In classical mechanics, Noether's theorem associates one conserved quantity with each invariance group of the action integral. In quantum mechanics, we can associate one conserved quantity Q, which satisfies the equation $[Q,H]+i\partial Q/\partial t = 0$, with each invariance group of the time-dependent Schröedinger equation.[4] From these experiences, it is natural to wonder whether there exists an invariance group associated with each conservation law of nonlinear waves.

In the present and in future communications, we will investigate this question by applying Lie's infinitesimal analysis[5] and its generalization[5d] to the differential equations governing the waves. In this paper, we approach the question by studying the group theoretic aspect of continuous invariance transformations, which has been proved useful in systematically deriving a series of conservation laws.

In Sec. II, we analyze continuous invariance transformations (not necessarily a group transformation) connected to the identity transformation, to clarify the relationship between local and global invariance transformations. The results will be used in Sec. III to elucidate the group theoretic aspect of continuous invariance transformations of differential equations. In Sec. IV, we apply the generalization of Lie's theory to find some invariance groups of the sine-Gordon equation, $u_{xt} = \sin u$. In Sec. V, by using the result of Sec. III, we develop a new method of finding generators of an invariance group of differential equations. The method will be used, with the aid of the Bäcklund transformation, to show that the sine-Gordon equation is invariant under an infinite number of one-parameter groups. In Sec. VI, we investigate a relation between these groups and a series of conservation laws of the sine-Gordon equation.

I. PRELIMINARY

We consider a partial differential equation of the form

$$F(z^i, u, u_j, u_{kl}, \cdots) = 0, \tag{1}$$

where $z^i = (z^1, z^2, \ldots, z^n)$, $u_j = (\partial_1 u, \ldots, \partial_n u)$, etc. Let's suppose that there exists a solution $u = u(z^i, \alpha)$, which depends on a parameter α continuously. Assuming that it is analytic near $\alpha = 0$, we expand the solution in a Taylor series in α,

$$u = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} u^k(z^i), \quad u^k = \{(\partial_\alpha)^k u\}_{\alpha=0}. \tag{2}$$

Putting this solution into equation (1), we obtain a sequence of partial differential equations which will successively determine a possible form of the u^k. In particular, the first term u^0 must be a solution of equation (1). If the equation is linear, all the u^k's must also satisfy the same equation. In the case of nonlinear differential equations, however, all the equations are different. First, the differential equation for u^1 becomes homogeneous linear and involves the first solution u^0; we then obtain a nonhomogeneous linear equation for the u^k, $k > 1$, which has the same homogeneous part as the u^1; the nonhomogeneous term depends upon the u^0, $u^1, \ldots, u^{k-1}$ and their derivatives. By a deductive argument, we expect that if only the nonhomogeneous solution is taken for $u^2, u^3, \ldots, u^{k-1}$, the nonhomogeneous solution for u^k will have a strong functional dependence on the u^0. We consider the sine—Gordon equation $u_{xt} - \sin u = 0$ as an example. The equation for u^0, u^1, and u^2 is found to be

$$u^0_{xt} - \sin u^0 = 0, \quad u^1_{xt} - u^1 \cos u^0 = 0,$$
$$u^2_{xt} - u^2 \cos u^0 = -(u^1)^2 \sin u^0.$$

It is surprising that we can find many solutions for u^1 and u^2 which can be expressed as simple functions of u^0 and its derivatives; a few examples are

$$u^1 = u^0_x \quad \text{and} \quad u^2 = u^0_{xx},$$

or (3a–d)

$$u^1 = u^0_{xxx} + \tfrac{1}{2}(u^0_x)^3 \quad \text{and}$$
$$u^2 = u^0_{xxxxxx} + 3(u^0_x)^2 u^0_{xxxx} + \tfrac{9}{4}(u^0_x)^4 u^0_{xx} + 9u^0_x u^0_{xx} u^0_{xxx} + 3(u^0_{xx})^3.$$

The existence of such solutions is directly connected to the origin of the infinite number of conservation laws, and the study of the origin of such solutions will provide a key in understanding the origin of the conservation laws. We ask how a nonhomogeneous solution for u^k will depend on the u^0 if we take only the nonhomogeneous solutions for $u^2, \ldots, u^{k-1}$; this problem requires a careful analysis of invariance transformations.

II. RELATION BETWEEN AN INFINITESIMAL AND A FINITE INVARIANCE TRANSFORMATION

We have considered an example in which one solution u is continuously connected to another solution u^0 through a parameter α. This may be considered as a continuous transformation of u^0 to u; it is a special case of the continuous invariance transformation which is connected to the identity transformation.

We consider a set of transformations of the coordinates of the n-dimensional vector space $R^n(x^1, \ldots, x^n)$ which analytically depends on the parameter α, and becomes the identity transformation for $\alpha = 0$:

$$x^i \to \bar{x}^i = X^i(x^j, \alpha), \quad x^i = X^i(x^j, 0). \tag{4}$$

We also consier an equation $F(x^i) = F(x^1, \ldots, x^m) = 0$ which is defined in the subspace $R^m(x^1, \ldots, x^m)$ of R^n. The equation $F(x^i) = 0$ defines a manifold S, or hypersurface in R^m. We define the invariance transformation in the following way:

> Transformation (4) is a continuous invariance transformation of the equation $F(x^i) = 0$, if the condition $F(X^i(x^j, \alpha)) = 0$ is satisfied for the continuous values of α on the manifold S defined by $F(x^i) = 0$.

Geometrically, this implies that an invariance transformation carries a point on S into another point on S. We first investigate this invariance condition in detail, and will come back to the invariance transformation of the differential equation in the next section.

Under the condition we have imposed on the transformation, we can expand $X^i(x^j, \alpha)$ in a Taylor series in α by:

$$\bar{x}^i = X^i(x^j, \alpha) = x^i + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \xi^i_k, \quad \xi^i_k = \{(\partial_\alpha)^k X^i(x^j, \alpha)\}_{\alpha=0}. \tag{5}$$

Defining the differential operator U_k by

$$U_k = \sum_{j=1}^{n} \xi^j_k \partial_j \;, \tag{6}$$

we can write (5) as

$$\bar{x}^i = \left(1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} U_k\right) x^i. \tag{7}$$

We analyze the effect of this transformation on an analytic function $f(x^i)$ defined in R^n. Expanding $f(\bar{x}^i)$ in a Taylor series in α, we obtain

$$f(\bar{x}^i) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \phi_k(x^i), \quad \phi_k = \{(\partial_\alpha)^k f(\bar{x}^i)\}_{\alpha=0} = A_k f(x^i),$$

where

$$A_0 = 1 \quad \text{and}$$

$$A_k = k! \sum \left\{ \left[\prod_{j=1}^{s} \left([(p_j!)^{q_j}(q_j!)]^{-1} \prod_{l=1}^{q_j} \xi^{r^l_j}_{p_j} \right) \right] \times \prod_{j=1}^{s} \prod_{l=1}^{q_j} \partial_{r^l_j} \right\}, \tag{9}$$

where p_j and q_j are the integers satisfying the conditions

$$\sum_{j=1}^{s} p_j q_j = k, \quad 1 \leq p_i < p_j \leq k \quad \text{for} \quad i < j, \; 1 \leq q_j.$$

Here, we apply the summation convention with respect to the indices r^l_j. The choice of the sets $(p_1, \ldots, p_s)$ and $(q_1, \ldots, q_s)$ satisfying the conditions is not unique, and the sum in (9) is to be taken with respect to each of such sets. Using the differential operator A_k, we can write the effect of the coordinate transformation on the function $f(x^i)$ as

$$f(\bar{x}^i) = \left(1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} A_k\right) f(x^i) = T(\alpha) f(x^i). \tag{10}$$

We note that $T(\alpha)x^i = x^i + \alpha\xi^i_1 + \frac{1}{2}\alpha^2\xi^i_2 + \cdots = \bar{x}^i$ recovers the definition we started with.

Now we suppose that the continuous transformation $T(\alpha)$ leaves the equation $F(x^i) = 0$ defined in R^m invariant in the sense defined above. Then, the following statement will be obvious:

> The transformation $T(\alpha)$ is a continuous invariance transformation of the equation $F(x^i) = 0$, if and only if $A_k F(x^i) = 0$ on the manifold S defined by $F(x^i) = 0$. (A)

Although this provides the condition for a transformation to be an invariance transformation, it is very difficult to get any clear view of the structure of the transformation unless a considerable simplification of expression (9) is made; it is crucial to oberve that we can re-express (9) as

$$A_k = k! \sum \frac{(\bar{U}_1)^{q_1}}{(p_1!)^{q_1}(q_1!)} \cdot \cdots \cdot \frac{(\bar{U}_s)^{q_s}}{(p_s!)^{q_s}(q_s!)}, \tag{11}$$

where we take the same rule of summation as for (9). The remarkable feature of this expression is the fact that all the $\bar{U}_k$'s are first-order differential operators. We write down the first four generators in this form:

$$A_1 = \bar{U}_1, \quad A_2 = (\bar{U}_1)^2 + \bar{U}_2, \quad A_3 = (\bar{U}_1)^3 + 3\bar{U}_1\bar{U}_2 + \bar{U}_3,$$
$$A_4 = (\bar{U}_1)^4 + 6(\bar{U}_1)^2\bar{U}_2 + 3(\bar{U}_2)^2 + 4\bar{U}_1\bar{U}_3 + \bar{U}_4, \qquad \text{(12a)–(12d)}$$

where

$$\bar{U}_1 = U_1 = \xi^i_1 \partial_i, \quad \bar{U}_2 = U_2 - \xi^i_1 \xi^j_{1,i} \partial_j$$
$$\bar{U}_3 = U_3 + (-3\xi^i_1 \xi^j_{2,i} + 2\xi^i_1 \xi^k_{1,i} \xi^j_{1,k} + 2\xi^i_1 \xi^k_1 \xi^j_{1,ik}) \partial_j,$$
$$\bar{U}_4 = U_4 + (-4\xi^i_1 \xi^j_{3,i} - 3\xi^i_2 \xi^j_{2,i} + 9\xi^i_1 \xi^k_{1,i} \xi^j_{2,k} + 6\xi^i_1 \xi^k_1 \xi^j_{2,ik}$$

$$+3\xi_2^i\xi_{1,i}^k\xi_{1,k}^j+3\xi_2^i\xi_1^k\xi_{1,ik}^j-6\xi_1^i\xi_{1,i}^l\xi_{1,l}^k\xi_{1,k}^j$$
$$-3\xi_1^i\xi_1^l\xi_{1,il}^k\xi_{1,k}^j$$
$$-12\xi_1^i\xi_1^l\xi_{1,l}^k\xi_{1,ik}^j-3\xi_1^i\xi_1^l\xi_1^k\xi_{1,ikl}^j)\partial_j. \qquad \text{(13a)–(13d)}$$

The importance of the decomposition into this form will be recognized if we remember the basic lemma used in the theory of continuous group transformations:

> If two first-order differential operators $U_a=\xi_a^i\partial_i$ and $U_b=\xi_b^i\partial_i$, $i=1,2,\ldots,n$, satisfy the conditions $U_jf(x^i)=0$, $j=a,b$, on the manifold defined by $f(x^i)=0$, then we have $U_aU_bf(x^i)=0$ on the same manifold.

Successive applications of the lemma to the invariance condition (A), lead to the conclusion that all the operators $\bar{U}_k$ in the expression (11) must satisfy the condition:

$$\bar{U}_kF(x^i)=0 \text{ on the manifold } S \text{ defined by } F(x^i)=0. \qquad \text{(B)}$$

This allows us to draw the following conclusion:

> *All the A_k's are constructed from first-order differential operators Q, which satisfy the condition $QF(x^i)=0$ on $F(x^i)=0$.* (C)

In Lie's theory of group transformations, the operator which satisfies condition (B) is called *a generator of the invariance group*. We suppose that the largest invariance group of the equation $F(x^i)=0$ is an r-parameter group with generators Q_i. Then, all the operators which satisfy condition (B) can be written as

$$\bar{U}_k=\sum_{i=1}^{r}a_k^iQ_i\,. \qquad (14)$$

In particular, if we let $a_k^i=0$ for $k\geq 2$, we obtain $A_k=(\bar{U}_1)^k$ from (11), and the operator $T(\alpha)$ in (10) reduces to an exponential operator,

$$T(\alpha)=\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}(\bar{U}_1)^k=e^{\alpha\bar{U}_1}. \qquad (15)$$

Result (C) is significant in studying the structure of invariance transformations because it clarifies the constraints on and arbitrariness of an invariance transformation. The vital fact is that if we have a complete set of generators of the invariance group of the equation $F(x^i)=0$, then any continuous invariance transformation connected to the identity transformation can be constructed from these generators.

Now, the problem is how to find such generators for a given equation $F(x^i)=0$. The basic idea of deriving the generators was established by Lie, and we will illustrate it briefly after the discussion of differential equations.

III. INVARIANCE TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS

We have considered a set of coordinate transformations in R^n which leave the equation $F(x^i)=0$ defined in R^m invariant. We now introduce some functional relations among the coordinates, which are compatible with the equation $F(x^i)=0$; such relations will restrict further the domain of manifold S.

We consider a function $u(x^i)$ defined in the $(k-1)$-dimensional space $R^{k-1}(x^1,\ldots,x^{k-1})$ with $k<m$, and assume the following:

> The coordinate x^k is determined by the relation $x^k=u(x^i)$, and the coordinates $x^{k+1},\ldots,x^n$ are determined as the derivatives of $u(x^i)$ with respect to the coordinates $x^1,\ldots,x^{k-1}$. For instance, $x^{k+1}=\partial_1u,\ \ldots\ ,x^{2k-1}=\partial_{k-1}u,\ x^{2k}=\partial_1\partial_1u,\ x^{2k+1}=\partial_1\partial_2u,\ \cdots$. (16)

We suppose that R^n is chosen in such a way that if it contains a coordinate corresponding to a jth derivative, then coordinates for all the other jth derivatives also appear in R^n. The condition we have imposed are compatible with the equation $F(x^i)=0$ only if the function $u(x^i)$ is a solution of the equation $F(x^i)=0$, interpreted as a partial differential equation by considering x^i's as derivatives defined by (16). Each solution will define a submanifold $\bar{S}$ of the manifold S, called the solution surface.

Now, we consider a continuous coordinate transformation (4) under which a manifold satisfying condition (16) is always mapped onto a manifold which also satisfies condition (16), with $x^k=u(x^i,\alpha)$. In analyzing such transformations, it is convenient to introduce the following definitions:

Basic coordinates and *j*th order coordinates

> We call the coordinates $x^1,\ldots,x^k$ basic coordinates; and the coordinates corresponding to the jth order derivatives, jth order coordinates. For instance, in (16), $x^{k+1},\ldots,x^{2k-1}$ are first-order, and x^{2k}, x^{2k+1} are second-order.

Basic space and *j*th extended space

> We call the vector space $(x^1,\ldots,x^l)$, jth extended space if it consists of all and only the basic coordinates and a complete set of the first through the jth order coordinates. In particular, we call the 0th extended space $(x^1,\ldots,x^k)$, the basic space.

Basic transformation

> We call the transformation of a set of basic coordinates, the basic transformation.

Basic operator and *j*th extended operator

> We call the operator $\overset{j}{Q}=\sum_{i=1}^{l}\xi^i\partial_i$ of the transformation in the jth extended space the jth extended operator. The 0th extended operator $\overset{0}{Q}=\sum_{i=1}^{k}\xi^i\partial_i$, will be called the basic operator.

It is clear that under condition (16), the transformation of the basic coordinates will determine the transformation of the rest of the coordinates. In particular, if a basic operator is given, we can determine all the extended operators. Now, we require that such transformation leaves the equation $F(x^i)=0$ invariant. The geometrical meaning of the invariance transformation is more important; the invariance transformation maps one solution surface $\bar{S}$ to another solution surface $\bar{S}'$ (or onto itself), both of which are on S. A discovery of such a transformation will lead to a new solution of the differential equation. The transformation studied most extensively is the group transformation. Lie considered an invariance group transformation of the form

$$x^i \to \bar{x}^i = e^{\alpha Q} x^i = x^i + \alpha \xi^i(x) + \cdots, \quad i=1,\ldots,k,$$
$$x = (x^1,\ldots,x^k), \tag{17}$$

in which infinitesimal terms of the basic transformation depend only on the basic coordinates. It is important to note that, under such assumptions, a finite transformation of the coordinate x^p does not involve any coordinate whose order is higher than the order of x^p. This guarantees that the jth extended space is closed under the transformation. The existence of such a closed space enables us to elegantly construct a finite group transformation, via the method of characteristics, from a generator of the group.

Anderson, Kumei, and Wulfman, however, found that there exist invariance groups of time-dependent Schrödinger equations which are not of Lie's form.[4] They generalized Lie's theory by allowing infinitesimal terms ξ^i of the basic coordinates to depend on the coordinates of higher order:

$$x^i \to \bar{x}^i = e^{\alpha Q} x^i = x^i + \alpha \xi^i(x) + \cdots, \quad i=1,\ldots,k,$$
$$x = (x^1,\ldots,x^l), \quad l \geq k. \tag{18}$$

Here, the order of the coordinates in ξ^i is not restricted, and coordinates of any order may appear.[6] We note, however, that we no longer have any closed finite-dimensional space under such a group transformation.[7] Although this does not cause any problem in finding generators of invariance transformations, we can no longer apply the method of characteristics in finding a finite transformation. This generalization, however, is absolutely necessary to uncover all the invariance groups inherent in the differential equations.[8] Before we show that the sine–Gordon equation admits such invariance groups, we answer the question raised at the end of Sec. I. To put the problem into our present language, we rewrite (1) as

$$F(x^i)=0 \text{ with } x^i = z^i, \quad i=1,\ldots,n,$$
$$x^{n+1} = u, \quad x^{n+2} = u_1, \cdots, \tag{1'}$$

and (2) as

$$\bar{x}^i = x^i, \quad i=1,\ldots,n,$$
$$\bar{x}^{n+1} = X^{n+1}(x^i,\alpha) = x^{n+1} + \alpha\xi_1^{n+1} + \tfrac{1}{2}\alpha^2\xi_2^{n+1} + \cdots. \tag{2'}$$

From a transformational viewpoint, the statement that $\bar{u}$ is a solution of equation (1) is equivalent to saying that transformation (2′) leaves equation (1′) invariant. For such a transformation, as we have found, we can write $\xi_k^{n+1} = A_k x^{n+1}$ (or $u^k = A_k u^0$ in the old notation). This leads to the conclusion,

> If a differential equation $F(z^i, u, u_j, u_{kl}, \cdots)=0$ admits a solution $u(z^i,\alpha) = \sum_{k=0}^{\infty} (\alpha^k/k!) u^k(z^i)$ which depends analytically on α near $\alpha=0$, then u^k is always written as $u^k(z^i) = A_k u^0(z^i)$, where u^0 is a solution of the same equation and the operator A_k is constructed by (11) from the generators Q of invariance group transformations of $F=0$ by which the independent variables z^i are unchanged. In particular, $u^1 = Qu^0$. Furthermore, if only the inhomogeneous solution is taken for every u^k, $k > 1$, then $u^k = (Q)^k u^0$, and a resulting solution is expressed as $u(z^i) = e^{\alpha Q} u^0(z^i)$. (D)

IV. SOME INVARIANCE GROUP TRANSFORMATIONS OF SINE-GORDON EQUATION

We now go back to the analysis of the solutions (3a)–(3d). According to result (D), these solutions clearly indicate the existence of invariance groups, or symmetries, of the sine–Gordon equation. We will systematically determine generators of the invariance group transformations and will reveal new symmetries of the equation.

We first specialize the general formulation given above to a case in which we have three basic coordinates x^1, x^2, and x^3, and $F(x^i)$ is chosen as

$$F(x^1,x^2,x^3,x^4,x^5,x^6,x^7,x^8) = x^7 - \sin x^3.$$

As stated in (16), we establish the following constraints on the coordinates:

$$x^3 = u(x^1,x^2), \quad x^4 = u_1, \quad x^5 = u_2, \quad x^6 = u_{11},$$
$$x^7 = u_{12}, \quad x^8 = u_{22} \quad x^9 = u_{111}, \quad x^{10} = u_{112},$$
$$x^{11} = u_{122}, \quad x^{12} = u_{222}, \quad x^{13} = u_{1111}, \quad x^{14} = u_{1112},$$
$$x^{15} = u_{1122}, \quad x^{16} = u_{1222}, \quad x^{17} = u_{2222}, \quad x^{18} = u_{11111},$$
$$x^{19} = u_{11112}, \quad x^{20} = u_{11122}, \quad x^{21} = u_{11222}, \quad x^{22} = u_{12222},$$
$$x^{23} = u_{22222}, \tag{19}$$

where subscripts 1 and 2 indicate the derivatives with respect to x^1 and x^2. We now consider a transformation, of the generalized form (18), in which x^1 and x^2 are unchanged and the infinitesimal transformation of x^3 depends on x^3 and the first through the third-order coordinates:

$$\bar{x}^1 = x^1, \quad \bar{x}^2 = x^2,$$
$$\bar{x}^3 = x^3 + \alpha\xi^3(x^3,x^4,x^5,x^6,x^8,x^9,x^{12}). \tag{20}$$

We note that the inclusion of the coordinates x^7, x^{10}, and x^{11} is redundant because we can replace them by the coordinates in ξ^3 after we have introduced the condition $F=0$. The infinitesimal transformation, induced by (20), of the coordinate corresponding to the derivative $(\partial_1)^m(\partial_2)^n u$ is calculated as[9]

$$\bar{x}^i = x^i + \alpha(\partial_1)^m(\partial_2)^n\xi^3 = x^i + \alpha\xi^i. \tag{21}$$

Here, the partial derivative should be interpreted as

$$(\partial_1)^m(\partial_2)^n\xi^3 = (\partial_1)^m(\partial_2)^n \times \xi^3(u(x^1,x^2), u_1(x^1,x^2), \ldots, u_{222}(x^1,x^2)) \tag{22}$$

For instance

$$\xi^4 = \xi_3^3 x^4 + \xi_4^3 x^6 + \xi_5^3 x^7 + \xi_6^3 x^9 + \xi_8^3 x^{11} + \xi_9^3 x^{13} + \xi_{12}^3 x^{16}, \tag{23}$$

where ξ_i^3 is the derivative of ξ^3 with respect to the coordinate x^i contained in ξ^3, and should not be confused with the same notation used in Sec. II. As in Sec. II, we write the infinitesimal transformation in the jth extended space as

$$\bar{x}^i = (1+\alpha \overset{j}{Q})x^i, \quad \overset{j}{Q} = \sum_{i=1}^{j} \xi^i\partial_i \text{ with } \xi^1 = \xi^2 = 0. \tag{24}$$

Now, we assume that the equation $F=0$ is invariant under the group transformation whose infinitesimal form is given by (20); the condition is

$$\overset{2}{Q}F = \xi^7 - \xi^3 \cos x^3 = 0 \text{ on } x^7 - \sin x^3 = 0, \tag{25}$$

which is the partial differential equation for ξ^3. The equation $\overset{2}{Q}F=0$ will split into a set of partial differential equation because some of the coordinates which appear in the equation are independent from the coordinates in ξ^3 (Appendix). By solving these equations, we find four independent solutions:

$$\xi_a^3=x^4, \quad \xi_b^3=x^9+\tfrac{1}{2}(x^4)^3, \tag{26a}$$

$$\xi_c^3=x^5, \quad \xi_d^3=x^{12}+\tfrac{1}{2}(x^5)^3. \tag{26b}$$

Obviously, the last two solutions can be obtained from the first two, by interchanging the roles of x^1 and x^2. The second extended operators associated with ξ_a^3 and ξ_b^3 are calculated from (21) and (24) as

$$\overset{2}{Q}_a=x^4\partial_3+x^6\partial_4+x^7\partial_5+x^9\partial_6+x^{10}\partial_7+x^{11}\partial_8, \tag{27a}$$

$$\begin{aligned}\overset{2}{Q}_b=&\{x^9+\tfrac{1}{2}(x^4)^3\}\partial_3+\{x^{13}+\tfrac{3}{2}(x^4)^2x^6\}\partial_4+\{x^{14}\\&+\tfrac{3}{2}(x^4)^2x^7\}\partial_5+\{x^{16}+3x^4(x^6)^2+\tfrac{3}{2}(x^4)^2x^9\}\partial_6\\&+\{x^{19}+3x^4x^6x^7+\tfrac{3}{2}(x^4)^2x^{10}\}\partial_7+\{x^{20}+3x^4(x^7)^2\\&+\tfrac{3}{2}(x^4)^2x^{11}\}\partial_8.\end{aligned} \tag{27b}$$

We can easily check that they satisfy condition (25); thus, the sine-Gordon equation is invariant under the group transformations generated by these operators.

This result explains the origin of the solutions (3a)—(3d); they are obtained from result (D):

$$u_a^1=\overset{0}{Q}_a x^3=x^4 \tag{28a}$$

and

$$u_a^2=(\overset{1}{Q}_a)^2x^3=x^6, \tag{28b}$$

$$u_b^1=\overset{0}{Q}_b x^3=x^9+\tfrac{1}{2}(x^4)^2 \tag{28c}$$

and

$$\begin{aligned}u_b^2=(\overset{3}{Q}_b)^2x^3=&x^{24}+3(x^4)^2x^{13}+\tfrac{9}{4}(x^4)^4x^6\\&+9x^4x^6x^9+3(x^6)^3.\end{aligned} \tag{28d}$$

We note that we need the extended operators $\overset{1}{Q}_a$ and $\overset{3}{Q}_b$ for calculating the second term u^2. As we stated in Sec. III, this is the general character of the generalized transformation (18) and we need the $k(n-1)$th extended operator to calculate the nth term u^n if the basic transformation contains the k-th order coordinate.

V. GENERATING FUNCTION FOR GENERATORS

We have obtained four generators of the invariance group of the sine-Gordon equation by considering a generalized Lie transformation (18). However, if we had assumed a more general form for ξ^3, we might have been able to produce more generators. It is unfortunate that we have no theory which tells us which coordinates we need in ξ^i of (18) to obtain a complete set of generators, hence we must make some assumptions on the form of ξ^i. In practice, it is not possible to retain too many coordinates in ξ^i because the determining equations for ξ^i become too huge to solve. Therefore, it is highly desirable to have another method for producing the generators, which does not require either such assumptions or the construction of the solutions of determining equations. Here, we provide one such method although the completeness of the set of generators obtained is still not assured.

The idea of the method is to reverse the result in Sec. II. We found that the operator Q which satisfies a condition $QF=0$ on $F=0$ is the building block of any invariance transformation connected to the identity transformation. By reversing this, we argue that if we have an invariance transformation connected to the identity transformation, then we can find at least one such operator. More precisely, we proceed in the following way.

Suppose we have an invariance transformation of the equation $F(x^i)=0$,

$$\bar{x}^i=x^i+\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}A_k x^i=x^i+\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\xi_k^i, \quad i=1,\ldots,n, \tag{29}$$

in which all the ξ_k^i are known. Using the result (11), we can write,

$$\xi_1^i=A_1x^i=\bar{U}_1x^i, \quad \xi_2^i=A_2x^i=\{(\bar{U}_1)^2+\bar{U}_2\}x^i,$$

$$\xi_3^i=A_3x^i=\{(\bar{U}_1)^3+3\bar{U}_1\bar{U}_2+\bar{U}_3\}x^i, \cdots,$$

where all the $\bar{U}_k$ are first-order differential operators. From the first equation, we obtain

$$\bar{U}_1=\sum_{i=1}^{n}\xi_1^i\partial_i\ . \tag{30}$$

Feeding this into the second, we get $\bar{U}_2x^i=\xi_2^i-(\bar{U}_1)^2x^i$, which provides

$$\bar{U}_2=\sum_{i=1}^{n}[\xi_2^i-\{(\bar{U}_1)^2x^i\}]\partial_i\ . \tag{31}$$

Next we substitute these for the $\bar{U}_1$ and $\bar{U}_2$ in the third equation to determine $\bar{U}_3$. Continuing this process we can obtain a series of operators $\bar{U}_k$, all of which satisfy the invariance condition "$QF=0$ on $F=0$." We note that if the starting transformation (29) happens to form a group, then we only get $\bar{U}_1$ and all the others are equivalently zero for the reason discussed in Sec. II. We may consider the starting transformation (29) as a generating function for generators of an invariance group. The upshot of the method is that only algebraic computations are involved in the process and a computer can be used, whereas the construction of the solutions of the determining equations by computer is very difficult. Obviously, this method can be used to find generators of an invariance group of a differential equation if the constraints (16) are taken into account. We apply the method to the sine-Gordon equation to find additional generators.

We start with the well-known Bäcklund transformation of the sine-Gordon equation,[1]

$$\bar{x}^4-x^4=2\alpha\sin\tfrac{1}{2}(\bar{x}^3+x^3), \quad \alpha(\bar{x}^5+x^5)=2\sin\tfrac{1}{2}(\bar{x}^3-x^3), \tag{32}$$

with the convention established in (19). This transformation guarantees that if x^3 is a solution of the sine-Gordon equation then so is $\bar{x}^3$ for a continuous value of α. A principal use of the Bäcklund transformation is to construct a new solution $\bar{x}^3$ from a known solution x^3 by solving a set of first-order differential equations (32). We assume that the new solution $\bar{x}^3$, is an analytic function of α in the neighborhood of $\alpha=0$, and so are its derivatives. Then, it is clear from (32) that the transformation is connected to the identity transformation; $\bar{x}^3 \to x^3$ as $\alpha \to 0$. The analyticity assumption allows us to expand the solution $\bar{x}^3$ in the Taylor series in α near $\alpha=0$. Such an expansion is found in the paper by Scott *et al.*,[1] and we rewrite their result:

$$\bar{x}^3 = x^3 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!}\xi_k^3, \tag{33}$$

with:

$$\begin{aligned}
&\xi_1^3 = 2x^4, \quad \xi_2^3 = 4x^6, \quad \xi_3^3 = 12x^9 + 2(x^4)^3,\\
&\xi_4^3 = 48x^{13} + 48(x^4)^2x^6,\\
&\xi_5^3 = 240x^{18} + 360(x^4)^2x^9 + 600x^4(x^6)^2 + 18(x^4)^5,\\
&\xi_6^3 = 1440x^{24} + 2880(x^4)^2x^{13}\\
&\qquad + 12\,960x^4x^6x^9 + 3840(x^6)^3 + 1440(x^4)^4x^6\\
&\xi_7^3 = 10\,080x^{31} + 176\,400(x^6)^2x^9 + 95\,760x^4(x^9)^2\\
&\qquad + 141\,120x^4x^6x^{13} + 25\,200(x^4)^2x^{18}\\
&\qquad + 63\,000(x^4)^3(x^6)^2 + 18\,900(x^4)^4x^9 + 450(x^4)^7, \cdots,
\end{aligned} \tag{34a—g}$$

where we have adopted the convention (19) and $x^{31} = u_{1111111}$. In this specific Bäcklund transformation, the coordinates x^1 and x^2 are unchanged, i.e.,

$$\bar{x}^1 = x^1, \quad \bar{x}^2 = x^2 \quad \text{or} \quad \xi_1^i = \xi_2^i = 0 \text{ for } i \geq 1. \tag{35}$$

The transformations (33) and (35) form the basic transformations, and they provide all the necessary information to follow the above prescription to find $\bar{U}_k$. We list the results up to $\bar{U}_7$:

$$\begin{aligned}
&\bar{U}_2 = \bar{U}_4 = \bar{U}_6 = 0, \quad \bar{U}_1 = 2x^4\partial_3, \quad \bar{U}_3 = \{4x^9 + 2(x^4)^3\}\partial_3,\\
&\bar{U}_5 = \{48x^{18} + 120(x^4)^2x^9 + 120x^4(x^6)^2 + 18(x^4)^5\}\partial_3,\\
&\bar{U}_7 = \{1440x^{31} + 25\,200(x^6)^2x^9 + 15\,120x^4(x^9)^2\\
&\qquad + 20\,160x^4x^6x^{13} + 5040(x^4)^2x^{18} + 12\,600(x^4)^3(x^6)^2\\
&\qquad + 6300(x^4)^4x^9 + 450(x^4)^7\}\partial_3.
\end{aligned} \tag{36a—e}$$

Here, we have given the operators in the basic form; the operators in the extended form can be obtained from (21) and (24). By continuing this process, we will be able to find an infinite number of operators which satisfy the invariance condition (25). We can associate one invariance group transformation of the sine-Gordon equation with each of these operators.

VI. A SERIES OF CONSERVATION LAWS AND INVARIANCE GROUPS

In this section, we use notation (16), hence x^k represents a solution of the differential equation $F(x^i)=0$.

We consider an equation $F(x^i)=0$ which can be put into a conservation form:

$$\sum_{i=1}^{k-1}\partial_i f^i = 0, \quad f^i = f^i(x^1,\ldots,x^{k-1},x^k,\ldots,x^l), \tag{37}$$

where the derivatives are to be taken by considering $x^k,\ldots,x^l$ as functions of $x^1,\ldots,x^{k-1}$. The vector $\mathbf{f} = (f^1, f^2, \ldots, f^{k-1})$ establishes a divergent free flux in the space $R^{k-1}(x^1,\ldots,x^{k-1})$ for each solution of the equation. Now, we assume that the equation $F(x^i)=0$ is invariant under an r-parameter group with the property:

$$\begin{aligned}
&\bar{x}^i = T_\epsilon(\alpha)x^i = x^i \quad \text{for} \quad i=1,\ldots,k-1 \quad \text{with}\\
&T_\epsilon(\alpha) = \exp\Big(\sum_{i=1}^{r}\alpha^i Q_i\Big).
\end{aligned} \tag{38}$$

We suppose that transformation (38) exists if $|\alpha| < \delta$. Here, δ is a positive number. Under such assumptions, $\bar{x}^k$ represents a new solution of the equation and a corresponding flux, $\bar{\mathbf{f}} = (\bar{f}^1, \bar{f}^2, \ldots, \bar{f}^{2k-1})$ is written as

$$\begin{aligned}
\bar{f}^i &= f^i(x^1,\ldots,x^{k-1},\bar{x}^k,\ldots,\bar{x}^l)\\
&= T_\epsilon(\alpha)f^i(x^1,\ldots,x^{k-1},x^k,\ldots,x^l).
\end{aligned} \tag{39}$$

The implication of the new flux is the same as the old one, except that it is now for the new solution. However, its power series expansion in α tells us something new about the starting solution x^k; because we have assumed that the transformation $T_\epsilon(\alpha)$ exists at least for some range of $|\alpha|$, it acts as a generating function of fluxes; each term of the expansion of (39) in $\alpha^1,\ldots,\alpha^r$ also forms a divergent free flux. We state this as follows:

> If a differential equation $F(x^i)=0$ admits an invariance group with property (38), and if a flux $\mathbf{f}$ of the form (37) exists, then, for any polynomial function $G(Q_1,\ldots,Q_r)$ of the generators of the group, the vector $G\mathbf{f}$ forms a divergent free flux. (E)

Here, we see two basic patterns for a series of divergent free fluxes to arise: one associated with a series $Q_i\mathbf{f}$, $i=1,\ldots,r$ and one associated with a series $(Q_i)^n\mathbf{f}$, $n=1,2,\cdots$. It will be reasonable to say in general, that the former is more fundamental than the latter because the series of the second type can be mechanically constructed if Q_i is known, although the reverse is not possible. One, however, should not think that the fluxes of the second type are trivial.[10]

We now apply this analysis to the sine-Gordon equation, $F = x^7 - \sin x^3$. The equation can be put into the conservation form by multiplying by x^5;

$$\partial_1 f^1 + \partial_2 f^2 = 0 \quad \text{with} \quad \mathbf{f} = (f^1, f^2) = (\tfrac{1}{2}(x^5)^2, \ \cos x^3),$$

and the generators (36b)—(36e) can be used to derive new fluxes. We list a few of them, (using the notation $\mathbf{f}_i = \bar{U}_i\mathbf{f}$):

$$\begin{aligned}
&\mathbf{f}_1 : f_1^1 = 2x^5x^7, \quad f_1^2 = -2x^4\sin x^3,\\
&\mathbf{f}_3 : f_3^1 = \{4x^{14} + 6(x^4)^2x^7\}x^5, \quad f_3^2 = -\{4x^9 + 2(x^4)^3\}\sin x^3,\\
&\mathbf{f}_5 : f_5^1 = 24\{2x^{25} + 10x^4x^7x^9 + 5(x^4)^2x^{14} + 5x^7(x^6)^2\\
&\qquad + 10x^4x^6x^{10} + \tfrac{15}{4}(x^4)^4x^7\}x^5,\\
&\qquad f_5^a = -24\{2x^{10} + 5(x^4)^2x^9 + 5x^4(x^6)^2 + \tfrac{3}{4}(x^4)^5\}\sin x^3,\\
&\mathbf{f}_{3,3} : f_{3,3}^1 = (\overset{4}{\bar{U}}_3)^2 f^1 = 16[x^5x^{32} + 3(x^4)^2x^5x^{19} + (x^{14})^2\\
&\qquad + \{3(x^4)^2x^7 + 9x^4x^5x^6\}x^{14} + 6x^4x^5x^7x^{13}\\
&\qquad + \{9x^5(x^6)^2 + 9x^4x^5x^9 + \tfrac{9}{4}(x^4)^4x^5\}x^{10}\\
&\qquad + 9x^5x^6x^7x^9 + 9(x^4)^3x^5x^6x^7 + \tfrac{9}{4}(x^4)^4(x^7)^2].
\end{aligned} \tag{40a—d}$$

Here, we have listed only the first component for $\mathbf{f}_{3,3}$. Among these fluxes, the first flux, $\mathbf{f}_1$ is trivial because it is the derivative of $\mathbf{f}$ with respect to x^1.[12] We analyze the known results from our viewpoint. Our results are clearly different from the fluxes given in the paper by Scott *et al.*[11] Their results, however, can be obtained by taking a linear combination of fluxes with the form $(\bar{U}_1)^a(\bar{U}_2)^b\cdots(\bar{U}_r)^p\mathfrak{f}$. In fact, by using (11) and (36a)–(36e), we find that $A_3\mathfrak{f}$ and $A_5\mathfrak{f}$ recover their results. For instance,

$$A_3 f^1 = \{(\bar{U}_1)^3 + 3\bar{U}_1\bar{U}_2 + \bar{U}_3\} f^1 = 6\{2x^5x^{14} + 4x^4x^{10} + (x^4)^2x^5x^7\}$$
$$= 6\{2u_2u_{1112} + 4u_{12}u_{112} + (u_1)^2u_2u_{12}\},$$

where we interpret $\bar{U}_i$ as a generator extended to a necessary order. Now, we ask which fluxes are most basic among these.

Although this question is very important in analyzing the nature of conservation laws in general, the answer depends on the measure one uses. However, as we have indicated above, the hierarchy becomes quite clear within the framework of group theory; we classify fluxes into two categories:

(1) Basic fluxes: $\mathfrak{f},\ Q_i\mathfrak{f},\ \ i=1,\ldots,r,$

and

(2) Associated fluxes: $(Q_{i_1})^{n_1}(Q_{i_2})^{n_2}\cdots(Q_{i_p})^{n}\mathfrak{f}$

with $i_k=1,\ldots,r$, and $n_1+n_2+\cdots+n_p>1$,

and we use the basic fluxes to characterize the conservation law associated with a solution. The remarkable feature of the sine-Gordon equation is that it possesses a series of basic fluxes.

SUMMARY

To conclude this paper, we briefly summarize the results obtained in the present study. In Sec. II, we studied a structural aspect of continuous invariance transformations connected to the identity transformation, and we stated the explicit relation between a continuous invariance transformation and a continuous invariance group transformation [(A), (11), (C)]. In Sec. III, we used the result of Sec. II to analyze invariance properties of differential equations and we uncovered the group theoretic structure, inherent in any solution which depends on a continuous parameter [(D)]. In Sec. V, a new method was given for obtaining generator of an invariance group and it was used to find a series of new generators of an invariance group of the sine-Gordon Eq. [(36b)–(36e)]. In Sec. VI, we gave a group theoretic criteria for the existence of a series of conservation laws associated with solutions of a differential equation [(E)], and this was used to provide a group theoretic explanation of a series of conservation laws of the sine-Gordon equation. The results (40a)–(40d) explicitly indicate that there exist conservation laws whose existence is inexplicable within the Lie's framework of group theory, but still can be explained by group theory if the generalized theory (Ref. 5d) is used. In the next papers, we will show that the conservation laws of the Korteweg–deVries equation and the cubic Schröedinger equation are also related to invariance groups of the generalized Lie type.[13]

Note added in proof: The transformation (10) with A_k defined by (11), (12a–d) has been found to be the power series expansion, in α, of the expression

$$T(\alpha) = e^{\alpha\bar{U}_1}\, e^{(2!)^{-1}\alpha^2\bar{U}_2}\, e^{(3!)^{-1}\alpha^3\bar{U}_3}\, e^{(4!)^{-1}\alpha^4\bar{U}_4}\cdots .$$

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APPENDIX: DETERMINING EQUATIONS OF GENERATORS

Although our transformation is more general than that of Lie, the basic idea for obtaining the differential equations (determining equations) for ξ^i is the same as Lie's, and for a detailed discussion of the Lie method we refer the reader to the book by Ovsjannikov[5b] or the book by Bluman and Cole.[5c] Using f for ξ^3, the deteter-mining equations for our problem are the following:

$$f_{9,12}=0,$$
$$f_{3,12}x^4 + f_{4,12}x^6 + f_{5,12}x^7 + f_{6,12}x^9 + f_{8,12}x^{11} + f_{12,12}x^{16} = 0,$$
$$f_{3,9}x^5 + f_{4,9}x^7 + f_{5,9}x^8 + f_{6,9}x^{10} + f_{8,9}x^{12} + f_{9,9}x^{14} = 0,$$
$$f_3x^7 + f_4x^{10} + f_5x^{11} + f_6x^{14} + f_8x^{16} + f_9x^{19} + f_{12}x^{22}$$
$$+ \sum_{\substack{i=3,4,5\\6,8,12}} (f_{3,i}x^5 + f_{4,i}x^7 + f_{5,i}x^8 + f_{6,i}x^{10} + f_{8,i}x^{12}$$
$$+ f_{9,i}x^{14}) = 0,$$

with supplementary conditions:

$$x^7 = \sin x^3,\quad x^{10} = x^4\cos x^3,\quad x^{11} = x^5\cos x^3,$$
$$x^{14} = x^6\cos x^3 - (x^4)^2\sin x^3,$$
$$x^{16} = x^8\cos x^3 - (x^5)^2\sin x^3,$$
$$x^{19} = x^9\cos x^3 - 3x^4x^6\sin x^3 - (x^4)^3\cos x^3,$$
$$x^{22} = x^{12}\cos x^3 - 3x^5x^8\sin x^3 - (x^5)^3\cos x^3,$$

where $f_j = \partial_j f$ and $f_{j,k} = \partial_j\partial_k f$.

*This work was supported by a Research Cooperation Grant.

[1]A.C. Scott, F.Y.F. Chu, and D.W. McLaughlin, Proc. IEEE 61, 1444 (1973) (review article).

[2]P.D. Lax, Comm. Pure Appl. Math. 21, 467 (1968).

[3]It has been suspected that some transformation property of the differential equation governing the wave motion is responsible for the existence of a series of conservation laws. In fact, the restricted Bäcklund transformations (R.B.T.) have provided a systematic way of deriving a series of conservation laws. However, the derivation involves a process of power series expansion of a solution with respect to some parameter. Such a method only exemplifies the existence of a series, but does not explain the origin of individual conservation law. On the discussion of R.B.T. in the theory of solitons, we refer to (a) G.L. Lamb, Rev. Mod. Phys. 43, 99 (1971); (b) D.W. McLaughlin and A.C. Scott, J. Math. Phys. 14, 1817 (1973); (c) H.D. Wahlquist and F.B. Estabrook, Phys. Rev. Lett. 31, 1386 (1973). *We add in proof the following papers on the Bäcklund transformations*: G.L. Lamb, Jr., J. Math. Phys. 15, 2157 (1974); M. Wadati, H. Sanuki, and K. Konno, Progr. Theor. Phys. (Kyoto) 53, 419 (1975).

[4]R.L. Anderson, S. Kumei, and C.E. Wulfman, Rev. Mex. Fis. 21, 1,35 (1972); J. Math. Phys. 14, 1527 (1973).

[5]For Lie's work and its later development, we refer the reader to (a) S. Lie, *Transformationgruppen* (Chelsea, New York, 1970), 3 Vols., Reprints of 1888, 1890, and 1893 eds., S. Lie, *Differentialgreichungen* (Chelsea, New York, 1967), reprint of 1891 ed., S. Lie, *Continuierliche Gruppen* (Chelsea, New York, 1967), reprint of 1893 ed. (b) L.V. Ovsjannikov, *Group theory of differential equations* (Siberian Sec. Acad. of Sci., Novosibirsk, USSR, 1962). This book has been translated into English by G.W. Bluman, Department of Mathematics, University of British Columbia (unpublished). L.V. Ovsjannikov, *Some problems arising in group analysis of differential equations* (Proceeding Conference on Symmetry, Similarity and Group Theoretic Methods in Mechanics, edited by P.G. Glockner and M.C. Singh (University of Calgary Press, Canada, 1974). (c) G.W. Bluman and J.D. Cole, J. Math. Mech. 18, 1025 (1969). G.W. Bluman and J.D. Cole, *Similarity Methods for Differential Equation* (Springer, New York, 1974). (d) R.L. Anderson, S. Kumei and C.E. Wulfman, Phys. Rev. Lett. 28, 988 (1972).

[6]We note that the well-known contact transformations of ordinary differential equations, which were extensively studied by Lie, are a realization of the derivative-dependent transformations in which only the first-order derivative appears.

[7]If the equation is an ordinary differential equation, it is always possible to find a closed space.

[8]Several years ago, Professor G.M. Lamb kindly raised the question of the relation between this generalization and the Bäcklund transformation, which depends on first-order derivatives. The basic difference is the fact that the Bäcklund transformation is not a group transformation in general, whereas our generalization allows us to construct a group transformation. We should consider that a Lie type transformation and the Bäcklund transformation are complementary in the sense that neither of them subsumes the other. Lie's infinitesimal approach, however, will be superior in the structural analysis of continuous invariance transformations.

[9]The general formula of the expression of the extended operator will be found in the paper by R.L. Anderson and S. Davison, J. Math. Anal. Appl. 48, 301 (1974).

[10]We define "trivial" flux in the following way. We consider a set $S\{f_1, f_2, \ldots, f_l\}$ *which consists of divergence free fluxes*, $f_1, \ldots, f_l$ and their derivatives of any order. We note that the derivatives are also divergence free. Now, a flux f is said to be trivial with respect to the set S, if f can be expressed as a linear combination of the members of the set S. In this sense, the flux $Q^n f$ is, in general, nontrivial with respect to the set $S\{f, Qf, Q^2 f, \ldots, Q^{n-1} f\}$. For instance, the flux $f_{3,3}$ of (40) is nontrivial with respect to the set $S\{f, \bar{U}_3^1 f\}$.

[11]Eq. VI. B.7 in Ref. 1.

[12]This is due to the special character of the operator $\bar{U}_1$; the operation of $\bar{U}_1$ on the variable x^i, $i > 2$, is equivalent to the differentiation of the function u with respect to x^1. For instance, $\bar{U}_1 x^3 = x^4$, $(\bar{U}_1^1)^2 x^3 = x^6$ and $\bar{U}_1^1 x^5 = x^7$ are the transformations $u \to u_1$, $u \to u_{11}$ and $u_2 \to u_{12}$. Because of this property, the fluxes obtained from $(\bar{U}_1)^n f$ are all trivial.

[13]S. Kumei, "Group theoretic studies of conservation laws of nonlinear dispersive waves" (II, III, IV) (submitted for publication).

Group theoretic aspects of conservation laws of nonlinear dispersive waves: KdV type equations and nonlinear Schrödinger equations*

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Group theoretic properties of nonlinear time evolution equations have been studied from the standpoint of a generalized Lie transformation. It has been found that with each constant of motion of the KdV type equation $f_{xxx} + a(f)f_x + f_t = 0$ and of the coupled nonlinear Schrödinger equation $f_{xx} + a(f,g) + if_t = 0$, $g_{xx} + a(g,f) - ig_t = 0$ one invariance group of the equations is always associated. The well-known series of constants of motion of the KdV equation and the cubic Schrödinger equation will be recovered from the invariance groups of the equations. The doublet solution of the KdV equation will be characterized as the invariant solution of one of the groups. In a more general context, it will be shown that the well-known equation of quantum mechanics $(d/dt)\langle U\rangle = \langle[iH, U] + \partial U/\partial t\rangle$ can be generalized to a class of nonlinear time evolution equations and that if U is a generator of an invariance group of the equation then $(d/dt)\langle U\rangle = 0$. The class includes equations such as the KdV, the cubic Schrödinger, and the Hirota equations.

INTRODUCTION

In this paper, we study group theoretic aspects of time evolution equations of nonlinear waves, particularly of the Korteweg–de Vries (KdV) equation $f_{xxx} + ff_x + f_t = 0$ and of the cubic Schrödinger equation $f_{xx} + f^2 f^* + if_t = 0$.

Some time ago, Anderson, Kumei, and Wulfman proposed a generalization[1] of the Lie–Ovsjannikov[2–4] theory of invariance groups of differential equations, and applied it to a number of quantum mechanical systems to systematically study dynamical groups.[5] Recently it has been shown by Ibragimov and Anderson[6] that this generalized transformation is an infinite dimensional contact transformation.

It has been shown in the preceding paper[7] that the sine-Gordon equation $f_{xt} - \sin f = 0$ admits an infinite number of one-parameter invariance groups of this new type, with each of which one can associate a series of conservation laws. Although the generalization appears to broaden the usefulness of group theoretic analysis of differential equations, particularly of nonlinear ones, the physical implications of the new type of symmetry are still unclear in many respects.

The aim of the present paper is to investigate some of the well studied equations of nonlinear waves[8] from the standpoint of the generalized theory, and to gain a clearer insight into the physical significance of the presence of the new kind of symmetry. It will be shown that some of the fundamental properties of the KdV and the cubic Schrödinger equations are the direct results of the existence of new groups.

In Sec. I, we briefly review a few basic ideas of infinitesimal invariance transformations to fix notations.

In Sec. II, we investigate group theoretic properties of the KdV equation and the related equations. The main results are: (1) With each constant of motion of the KdV type equation $f_{xxx} + a(f)f_x + f_t = 0$, one invariance group is associated, hence the KdV equation admits an infinite number of invariance groups. (2) The doublet solution, as well as the singlet solution, of the KdV equation is the invariant solution (or generalized similarity solution) of one of the groups.

In Sec. III, we prove that with each conservation law of the coupled nonlinear Schrödinger equation $f_{xx} + a(f,g) + if_t = 0$, $g_{xx} + a(g,f) - ig_t = 0$ one can associate one invariance group. The constants of motion of the cubic Schrödinger equation due to Zakharov and Shabat[9] will be recovered from the invariance group of the equation.

In Sec. IV, we investigate some general properties of generators of invariance groups of time evolution equations $H(t, x^i, f, f_i, f_{ij}, \cdots) + f_t = 0$. It will be shown that (1) A generator U of an invariance group of $H + f_t = 0$ always satisfies the relation $[\mathcal{H}, U] + \partial U/\partial t = 0$, where $\mathcal{H}$ is a Lie operator associated with H; (2) For a class of nonlinear time evolution equations, the equation $(d/dt)\langle U\rangle = \langle[\mathcal{H}, U] + \partial U/\partial t\rangle$ can be generalized; in particular, if U is a generator, then $(d/dt)\langle U\rangle = 0$.

I. INFINITESIMAL INVARIANCE TRANSFORMATIONS

We denote m-dimensional real and complex vector space by R^m and C^m, respectively and we consider the following infinite direct sum of the spaces; by denoting $C^{(N+1)^k}$ by $\underset{k}{C}$,

$$V = R^N \oplus \underset{0}{C} \oplus \underset{0}{C'} \oplus \underset{1}{C} \oplus \underset{1}{C'} \oplus \cdots \oplus \underset{k}{C} \oplus \underset{k}{C'} \oplus \cdots. \tag{1}$$

The prime is to distinguish two spaces of the same dimensions. We denote the elements of $\underset{k}{C}$ and $\underset{k}{C'}$ by $\underset{k}{u}$ and $\underset{k}{v}$, thus the elements of V are

$$z = (x, u, v, \underset{1}{u}, \underset{1}{v}, \ldots, \underset{k}{u}, \underset{k}{v}, \cdots), \quad x \in R^{N+1}. \tag{2}$$

The components of $\underset{k}{u}$, $\underset{k}{v}$ are written as $u_{p_1 p_2 \cdots p_k}$, $v_{p_1 p_2 \cdots p_k}$ where each index runs from 0 through N[10]:

$$u = (u), \quad v = (v), \quad \underset{1}{u} = (u_0, u_1, \ldots, u_N), \quad \underset{1}{v} = (v_0, v_1, \ldots, v_N),$$

$$\underset{2}{u} = (u_{00}, u_{01}, \ldots, u_{0N}, \ldots, u_{N0}, u_{N1}, \ldots, u_{NN}),$$

$$\underset{2}{v} = (v_{00}, v_{01}, \ldots, v_{0N}, \ldots, v_{N0}, v_{N1}, \ldots, v_{NN}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{3}$$

Now we consider an infinitesimal transformation in V

$$\bar{z}=z+\epsilon Z, \quad Z=(0,\eta,\zeta,\underset{1}{\eta},\underset{1}{\zeta},\ldots,\underset{k}{\eta},\underset{k}{\zeta},\cdots), \tag{4}$$

where

$$\eta=\eta(z;c), \quad \zeta=\zeta(z;c), \tag{5}$$

and the components of $\underset{k}{\eta}$ and $\underset{k}{\zeta}$ are to be determined by the formula

$$\eta_{p_1p_2\cdots p_k}=D_{p_1p_2\cdots p_k}\eta, \quad \zeta_{p_1p_2\cdots p_k}=D_{p_1p_2\cdots p_k}\zeta, \tag{6}$$

where $D_{ij\cdots m}=D_iD_j\cdots D_m$ with

$$D_i=\partial_{x^i}+(u_i\partial_u+v_i\partial_v)+(u_{ij}\partial_{u_j}+v_{ij}\partial_{v_j})+\cdots$$
$$+(u_{ij\cdots m}\partial_{u_{j\cdots m}}+v_{ij\cdots m}\partial_{v_{j\cdots m}})+\cdots. \tag{7}$$

In this paper, the summation rule will be assumed for repeated indices. In (5), c denotes a collection of all the real and complex numbers appearing in the expression of η or ζ. We write (4) compactly in the usual way as

$$\bar{z}=(1+\epsilon U)z, \tag{8}$$

with

$$U=(\eta\partial_u+\zeta\partial_v)+(\eta_i\partial_{u_i}+\zeta_i\partial_{v_i})+\cdots+(\eta_{i\cdots k}\partial_{u_{i\cdots k}}$$
$$+\zeta_{i\cdots k}\partial_{v_{i\cdots k}})+\cdots. \tag{9}$$

The operator U has the following property (see Appendix A for the proof):

Lemma 1: If a function $A(z)$ is twice differentiable with respect to all the variables, then $(D_iU-UD_i)A(z)=0$ for $i=0,1,\ldots,N$.

We consider a set of differential equations for functions $f(x)$ and $g(x)$,

$$F^i(z;c)=0, \quad i=1,2, \tag{10a}$$

$$u=f(x), \quad v=g(x), \quad \underset{k}{u}=\underset{k}{f}(x), \quad \underset{k}{v}=\underset{k}{g}(x), \quad k=1,2,\ldots,\infty, \tag{10b}$$

where $\underset{k}{f}(x)$ and $\underset{k}{g}(x)$ are functions of the $(N+1)^k$-tuple

$$\underset{1}{f}(x)=(f_0,f_1,\ldots,f_N), \quad \underset{1}{g}(x)=(g_0,g_1,\ldots,g_N),$$
$$\underset{2}{f}(x)=(f_{00},\ldots,f_{NN}), \quad \underset{2}{g}(x)=(g_{00},\ldots,g_{NN}), \tag{11}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

with $f_{i\cdots j}=\partial_{x^i}\cdots\partial_{x^j}f(x)$, $g_{i\cdots j}=\partial_{x^i}\cdots\partial_{x^j}g(x)$. C in (10a) represents a set of parameters (real or complex) appearing in the differential equation. Each solution of Eq. (10) defines a manifold in V which we call a solution manifold.

It is well known[2–4] that a group transformation e^{aU} maps a solution manifold of (10) into another (or the same) solution manifold if and only if

$$UF^i(z;c)\big|_{\mathcal{J}}=0, \quad i=1,2, \tag{12}$$

where $(\cdot\cdot)|_{\mathcal{J}}$ indicates to evaluate the quantity under the conditions

$$F^i=0, \quad D_{p_1\cdots p_k}F^i=0, \quad i=1,2, \quad k=1,2,\ldots,\infty. \tag{13}$$

The operator U is then a generator of an invariance group.

We define #-conjugation of a quantity $A(z;c)=A(x,u,v,\ldots,\underset{k}{u},\underset{k}{v};c)$ by

$$A(z;c)^{\#}=A(x,v,u,\ldots,\underset{k}{v},\underset{k}{u};c^*), \tag{14}$$

where the asterisk represents a complex conjugation. An important subclass of Eq. (10) is

$$F^1(z;c)=0 \quad \text{with } F^2=(F^1)^{\#},$$
$$u=f(x), \quad v=f(x)^*,\cdots. \tag{15}$$

For this equation, the generator U takes the form

$$U=\eta\partial_u+\eta^{\#}\partial_v+\eta_i\partial_u+\eta_i{}^{\#}\partial_v+\cdots$$
$$+\eta_{i\cdots k}\partial_{u_{i\cdots k}}+(\eta_{i\cdots k})^{\#}\partial v_{i\cdots k}+\cdots. \tag{16}$$

In this paper, we consider the infinitesimal transformations of the type (4) which involves no transformation in x. This transformation, however, is not as special as it might look. Let us consider an infinitesimal transformation of a more general type[2–6]

$$\bar{z}=z+\epsilon Z, \quad Z=(\hat{\xi},\hat{\eta},\hat{\zeta},\underset{1}{\hat{\eta}},\underset{1}{\hat{\zeta}},\cdots), \quad \hat{\xi}=(\hat{\xi}^0,\hat{\xi}^1,\ldots,\hat{\xi}^N), \tag{17}$$

where

$$\hat{\eta}_{p_1\cdots p_{k-1}p_k}=D_{p_k}\hat{\eta}_{p_1\cdots p_{k-1}}-u_{p_1\cdots p_{k-1}q}D_{p_k}\hat{\xi}^q,$$
$$\hat{\zeta}_{p_1\cdots p_{k-1}p_k}=D_{p_k}\hat{\zeta}_{p_1\cdots p_{k-1}}-v_{p_1\cdots p_{k-1}q}D_{p_k}\xi^q. \tag{18}$$

It can be proved[11] that if we know the transformations of type (4), then we can also obtain the more general type (17):

Lemma 2: If (4) is an infinitesimal invariance transformation of (10), then for an arbitrary choice of $\hat{\xi}$, $\hat{\eta}$, and $\hat{\zeta}$ subjected to the conditions $\hat{\eta}-\hat{\xi}^iu_i=\eta$, $\hat{\zeta}-\hat{\xi}^iv_i=\zeta$, the transformation (17) is also an invariance transformation of Eq. (10). Conversely, if (17) is an invariance transformation of (10), then so is (4) for $\eta=\hat{\eta}-\hat{\xi}^iu_i$, $\zeta=\hat{\zeta}-\hat{\xi}^iv_i$.

In the following sections, we write the operators (9) and (16) as

$$U=\eta\partial_u+\zeta\partial_v, \quad U=\eta\partial_u+\eta^{\#}\partial_v. \tag{19}$$

They, however, must be always interpreted as their infinite prolongation. Also, we use the following abbreviation:

$$[A(z)]_{\substack{u=f(x)\\v=g(x)}}=[A(u,v)]_{f,g}$$

and

$$\int[A(u,v)]_{f,g}\,dx=\int A(u,v)\,dx.$$

II. A GROUP THEORETIC ANALYSIS OF THE KdV EQUATION

The equation of our interest is $f_{111}+ff_1+f_0=0$.[12–16] The equation is a particular case of (10) for which $F^2=0$, $g=0$. In this section, we use t, x for x^0, x^1, and write coordinates such as u_0, u_{10}, $\cdots$ as u_t, u_{xt}, $\cdots$. Similarly, we write η_0, η_{10}, $\cdots$ as η_t, η_{xt}, $\cdots$. Thus, by definition $\eta_t = D_t\eta$, $\eta_{xt} = D_x D_t \eta$, etc. Also, because the equation involves a single real function, all the v's in the first section are to be ignored.

A. A Lie algebra of an invariance group of the KdV equation

We write the equation as

$$F = u_{xxx} + uu_x + u_t = 0, \qquad (20)$$
$$u = f(x,t), \quad u_x = f_x(x,t), \quad u_t = f_t(x,t), \cdots .$$

We look for an operator $U = \eta\partial_u$ which satisfies condition (12) for this equation. We assume the transformation to be a generalized Lie type[1] with $\eta = \eta(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx})$. The absence in η of coordinates corresponding to t derivatives may be justified for time evolution type equations in which the only t derivative contained is f_t.

The application of Lie's algorithm[3,4] for finding generators leads to the following results:

$$\begin{aligned} &U^1 = (tu_x - 1)\,\partial_u, \\ &U^2 = \tfrac{1}{3}\,\{xu_x - 3t(u_{xxx} + uu_x) + 2u\}\,\partial_u, \\ &U^3 = u_x\partial_u, \quad U^4 = (u_{xxx} + uu_x)\,\partial_u, \\ &U^5 = (\tfrac{3}{5}\,u_{xxxxx} + uu_{xxx} + 2u_x u_{xx} + \tfrac{1}{2}u^2 u_x)\,\partial_u. \end{aligned} \qquad (21)$$

The generators form a nonsemisimple algebra (see Appendix B for the definition of a commutator)

$$\begin{aligned} &[U^1, U^2] = \tfrac{2}{3}U^1, \quad [U^1, U^3] = 0, \quad [U^1, U^4] = U^3, \\ &[U^1, U^5] = U^4, \quad [U^2, U^3] = \tfrac{1}{3}U^3, \quad [U^2, U^4] = U^4, \\ &[U^2, U^5] = \tfrac{5}{3}U^5, \quad [U^3, U^4] = 0, \\ &[U^3, U^5] = 0, \quad [U^4, U^5] = 0. \end{aligned} \qquad (22)$$

By making use of Eq. (20), and by applying Lemma 2, one can cast the first four generators into "genuine" Lie generators: They are equivalent to

$$\begin{aligned} &\bar{U}^1 = -t\partial_x - \partial_u, \quad \bar{U}^2 = \tfrac{1}{3}(-x\partial_x - 3t\partial_t + 2u\partial_u), \\ &\bar{U}^3 = -\partial_x, \quad \bar{U}^4 = \partial_t. \end{aligned} \qquad (23)$$

This set of generators is well known.[14,17] The generator U^5, however, is new and its properties will be analyzed later.

Let us consider operators $\partial U/\partial t = (\partial_t\eta)\,\partial_u$ and $H = (u_{xxx} + uu_x)\,\partial_u = U^4$. It is remarkable that all the U^i of (21) satisfy the relation $[H, U^i] + \partial U^i/\partial t = 0$. In Sec. IV, it will be shown that a generator of an invariance group of time evolution equations always satisfies such a relation.

It is well known[13] that the KdV equation admits an infinite number of conservation laws. To study a possible connection between the present groups and the conservation laws, we need to know effects of infinitesimal invariance transformations on constants of motion.

In his analysis of constants of motion of the time evolution equation $H(x, t, u, u_x, u_{xx}, \ldots, u^{(n)}) + u_t = 0$, $u^{(n)} = u_{\underbrace{xx\cdots x}_{n}}$, $u = f(x,t)$, Lax[15] considered an infinitesimal transformation of a solution $f(x,t)$ into a solution $u = f(x,t) + \epsilon\phi(x,t)$. The function ϕ must satisfy the linear equation

$$H_u(f)\,\phi + H_{u_x}(f)\,\phi_x + \cdots + H_{u^{(n)}}(f)\,\phi^{(n)} + \phi_t = 0, \qquad (24)$$

where $H_u(f) = (\partial_u H)_{u=f}$, etc. and $\phi^{(n)} = (\partial_x)^n\phi(x,t)$. We note that $\eta(f)$ of a generator of an invariance group of the equation $H(f) + f_t = 0$ is a special realization of ϕ. The effect of the transformation on the constant of motion $I(f)$ is

$$\begin{aligned} &I(f + \epsilon\phi) = I(f) + \epsilon(\Gamma(f), \phi), \\ &(\Gamma(f), \phi) = \partial_\epsilon I(f + \epsilon\phi)\big|_{\epsilon=0}. \end{aligned} \qquad (25)$$

The function $\Gamma(f)$ is a gradient of the functional $I(f)$.[15] For the constant of motion of integral type, i.e., $I(f) = \int \rho(f)\,dx$, the gradient has a simple expression: Assuming $\rho(u) = \rho(x, t, u, \ldots, u^{(k)})$,

$$\Gamma(u) = \rho_u - D_x\rho_{u_x} + D_x^2\rho_{u_{xx}} + \cdots + (-D_x)^k\rho_{u^{(k)}} \qquad (26)$$

In this case, we have

$$(\Gamma(f), \phi) = \int \Gamma(f)\,\phi\,dx. \qquad (27)$$

Lax observed

$$(\Gamma(f), \phi) \text{ is a constant of the motion.} \qquad (28)$$

B. Constants of motion of $f_{xxx} + a(f)f_x + f_t = 0$ and its groups

Now we prove a theorem which establishes a relationship between a constant of motion of the KdV type equation and its invariance group. We consider an equation

$$f_{xxx} + a(f)f_x + f_t = 0, \qquad (29)$$

where $a(f)$ is a function of f. We assume that an initial value problem for this equation is well posed for a periodic boundary condition $f(x,t) = f(x + x_0, t)$ or for a condition $f(-\infty, t) = f(\infty, t) = 0$. Let us suppose that the system has a constant of motion of integral type $I(f) = \int \rho(f)\,dx$. The limits of the integration are either over the period or from $-\infty$ to ∞. We prove:

Theorem 1. If $\Gamma(u)$ is the gradient of a constant of motion $I(f) = \int\rho(f)\,dx$ associated with the equation $f_{xxx} + a(f)f_x + f_t = 0$, then the operator $U = \eta\partial_u$ which has $\eta(u) = D_x\Gamma(u)$ is a generator of an invariance group of the equation.

Proof: It is sufficient if we prove $\{U(u_{xxx} + a(u)u_x + u_t)\}_f = 0$. We consider a transformation of a solution f to a solution $f + \epsilon\phi$. Then, by (24), $\phi_{xxx} + a(f)\phi_x + a_u(f)f_x\phi + \phi_t = 0$. Thus, $0 = \int\Gamma(f)(\phi_{xxx} + a\phi_x + a_u f_x\phi + \phi_t)\,dx$. Integrating this by parts and assuming null contribution from the boundary terms, we obtain $0 = \int\{-D_x^3\Gamma - D_x(a\Gamma) + \Gamma a_u u_x - D_t\Gamma\}_f\,\phi\,dx + (d/dt)\int\Gamma\,\phi\,dx$. The second term vanishes because of (28). Because we can prescribe an arbitrary admissible function for ϕ at initial time t_0, this equation implies $\{D_x^3\Gamma + D_x(a\Gamma) - \Gamma a_u u_x + D_t\Gamma\}_f = 0$. Differentiating this with respect to x, and defining $\eta = D_x\Gamma$, we find $\{D_x^3\eta + \eta a_u u_x + (D_x\eta)\,a + D_t\eta\}_f = \{U(u_{xxx} + au_x + u_t)\}_f = 0$.

This theorem establishes a relationship between constants of motion and invariance groups of Eq. (29),

$$I(f) = \int\rho(f)\,dx \longrightarrow \Gamma(u) \longrightarrow \{U = \eta\partial_u,\ \eta = D_x\Gamma\}. \qquad (30)$$

The process from U to I involves an integration process and not all the generators are integrable to I. In Sec. IV, we provide another scheme to connect a group to a constant of motion which can supplement such a nonintegrable case.

The application of the theorem to the generators (21) leads to (within constant factors),

$$I^1=\int(\tfrac{1}{2}tu^2-xu)\,dx,\quad I^3=\int\tfrac{1}{2}u^2\,dx,$$
$$I^4=\int(\tfrac{1}{3}u^3-u_x^2)\,dx, \tag{31}$$
$$I^5=\int(\tfrac{1}{4}u^4-3uu_x^2+\tfrac{9}{5}u_{xx}^2)\,dx.$$

The generator U^2 is not integrable. The constants (31) coincide with members of the set of constants of motion due to Miura, Gardner, and Kruskal.[13] The simplest constant $I=\int u\,dx$ is missing; the reason is that it gives $\Gamma=1$, hence $U=0$. In the last section, however, we show that one can associate this with the generator U^2. Thus, we write $I^2=\int u\,dx$.

The fact that there exist an infinite number of constants of motion for the KdV equation means that the equation is invariant under an infinite number of groups; the situation is similar to the case of the sine-Gordon equation $f_{xt}-\sin f=0$.[7]

Now, we study properties of the groups associated with constants of motion of the KdV equation. First we review a few important properties of the gradient found by Lax[15] and Gardner.[16]

C. Properties of gradients (Lax and Gardner)

Lax has proved that the gradients associated with the constants of motion of the KdV equation has the following unique properties:

(1) If $\Gamma^i(u)$ is a gradient of $I^i=\int\rho^i(f)\,dx$, $i,j\geqslant 2$, then $\Gamma^i(u)D_x\Gamma^j(u)=J^{ij}$ with $J^{ij}=$ polynomial in u, u_x, u_{xx}, $\cdots$.

(2) Every solitary wave solution

$$u=3c\,\mathrm{sech}^2\tfrac{1}{2}\sqrt{c}\,(x-ct)\equiv s(x-ct) \tag{32}$$

is an eigenfunction of the gradients

$$\Gamma(s)=\gamma(c)s,\quad \gamma(c)=\text{eigenvalue}. \tag{33}$$

In the study of doublet solutions of the KdV equation, Lax, as well as Kruskal and Zabusky,[12] focused his attention on three constants I^3, I^4, and I^5. For these constants, the gradients are

$$\Gamma^3=u,\quad \Gamma^4=u^2+2u_{xx},$$
$$\Gamma^5=u^3+3u_x^2+6uu_{xx}+\tfrac{18}{5}u_{xxxx}, \tag{34}$$

and correspondingly,

$$\Gamma^3(s)=s,\quad \Gamma^4(s)=2cs,\quad \Gamma^5(s)=\tfrac{18}{5}c^2s. \tag{35}$$

Another remarkable property of $\Gamma(u)$ of the KdV equation is due to Gardner,

(3) If we define an operator W^i associated with $\Gamma^i(u)$ of I^i, $i>2$, by

$$W^i=(D_x\Gamma^i)\,\partial_u+(D_x^2\Gamma^i)\,\partial_{u_x}+(D_x^3\Gamma^i)\,\partial_{u_{xx}}+\cdots, \tag{36}$$

then $[W^i,W^j]=0$.

D. Properties of U^i, $i>2$

We note the similarity between the generator $U^i=(D_x\Gamma^i)\,\partial_u$ and Gardner's operator W^i. They, however, are different in that the prolonged U^i involves terms such as $(\cdot)\,\partial_{u_t}$, $(\cdot)\,\partial_{u_{xt}}$ whereas W^i does not. Nevertheless Gardner's result implies that two generators U^i and U^j associated with I^i and I^j commute,

$$[U^i,U^j]=0,\quad i,j>2. \tag{37}$$

This is obviously the reflection of the fact that the KdV equation is a completely integrable Hamiltonian system.[18,19]

Incidentally, it is often useful to note that: If $I(f)=\int\rho(f)\,dx$ is a constant of motion associated with the differential equation $F(x,t,f,f_x,f_t,f_{xx},f_{xt},f_{tt},\cdots)=0$, and if U is a generator of an invariance group of $F=0$, then the quantity $I'=\int\{U\rho(u)\}_f\,dx$ is also a constant of motion of the same equation. The application of this scheme to the KdV equation, however, fails to generate a constant; indeed, by making use of Eq. (26), Lax's result (1), and Lemma 1, we find

$$\int U^i\rho^j\,dx=\int\sum_k(D_x^k\eta^i)\,\rho^j_{u_{(k)}}\,dx=\int\sum_k\eta^i(-D_x)^k\rho^j_{u_{(k)}}\,dx$$
$$=\int\eta^i\Gamma^j\,dx=\int(D_x\Gamma^i)\,\Gamma^j\,dx=\int D_xJ^{ij}\,dx=0. \tag{38}$$

Although the method fails to generate a string of constants of motion, it has been found that U^4 gives rise to the following recursive relation:

$$0=\int U^4\rho^i\,dx=c\frac{d}{dt}I^{i+1}. \tag{39}$$

This relation has been checked up to $i=4$.

E. Properties of e^{aU^i}

If $u=f(x,t)$ is a solution of the KdV equation, then, by construction, a function $u=\bar f(x,t;a)=\{e^{aU^i}u\}_f$ is also a solution provided a series $\sum_{k=0}^{\infty}\{a^k/k^*)(U^i)^k\}_f$ exists. First, we show that this group transformation does not alter the values of the constants of motion I^j,

$$\int\{\rho^j(u)\}_f\,dx=\int\{\rho^j(u)\}_{\bar f}\,dx,\quad j\geqslant 2. \tag{40}$$

Proof: First, by (38), $\int\{U^i\rho^j\}_f\,dx=0$. This must hold at initial time for it is a constant of motion: $\int\{U^i\rho^j\}_{\mu(x)}\,dx=0$ for any admissible initial condition $f(x,0)=\mu(x)$. It can be proved that this is possible only if $U^i\rho^j=D_xh^{ij}(u)$, $h^{ij}=$ polynomial in $u,u_x,u_{xx},\cdots$. Then, by using Lemma 1, $(U^i)^k\rho^j=(U^i)^{k-1}D_xh^{ij}=D_x(U^i)^{k-1}h^{ij}$. Thus, $\int\{\rho^j\}_{\bar f}\,dx=\int\{\rho^j+D_x\sum_{k=1}^{\infty}(a^k/k!)(U^i)^{k-1}h^{ij}\}_f\,dx=\int\{\rho^j\}_f\,dx$.

This result reminds us of quantum mechanics where group operations e^{iaA}, e^{ibB} do not alter the values of observables $\langle A\rangle$ and $\langle B\rangle$ provided $[A,B]=0$. Here operators U^i and observables I^i are related by (30) and in fact the U^i's commute by (37).

The relation (40) indicates that both solutions $f(x,t)$ and $\bar f(x,t;a)$ will break up into the same set of solitons. To prove this we start from Lax's result (2). We suppose Γ to be a linear combination of Γ^i associated with the constants of motion I^i of integral type. Differentiating Eq. (33) by x and using the relationship between Γ and U, we obtain

$$\{Uu\}_s = \gamma(c)s_x, \quad s = s(x-ct). \tag{41}$$

This and Lemma 1 give rise to

$$\{(U)^n u\}_s = (\gamma\partial_x)^n s = \{(\gamma D_x)^n u\}_s. \tag{42}$$

This relation implies that: For the solitary wave solution (32), we have the operator identity $U = \gamma D_x$. Consequently, the group operation e^{aU} has the effect of translation in x when it is operated on the solitary wave solution,

$$\{e^{aU}u\}_{s(x-ct)} = s(x - ct + a\gamma(c)). \tag{43}$$

Now let us assume that the solution $f(x,t)$ splits into N well separated solitons as $t \to \infty$,

$$f(x,t) \sim \sum_{i=1}^{N} s_i(x - c_i t + \delta_i) \quad \text{as } t \to \infty. \tag{44}$$

For such a wave profile, interactions between solitons are small, hence at least for small a we may assume

$$\{e^{aU}u\}_{f(x,t)} \sim \sum_{i=1}^{N} \{e^{aU}u\}_{s_i(x-c_it+\delta_i)} \quad \text{as } t \to \infty. \tag{45}$$

In view of (43), we can write this as

$$\{e^{aU}u\}_{f(x,t)} \sim \sum_{i=1}^{N} s_i(x - c_i t + \delta_i + a\gamma(c_i)) \quad \text{as } t \to \infty. \tag{46}$$

Thus, two solutions $f(x,t)$ and $\bar f(x,t;a) = \{e^{aU}u\}_{f(x,t)}$ of the KdV equation have the same asymptotic profile as $t \to \infty$ except that the phase of each soliton is shifted by the amount $a\gamma(c_i)$.

F. Invariant solutions of the KdV equation

One curious question would be whether there exists a solution which is mapped onto itself under the transformation e^{aU}. Speaking in a more general context, a solution, of a differential equation $F=0$, which is mapped onto itself by the invariance group of the equation is called an *invariant solution* (or *generalized similarity solution*).[4] The necessary and sufficient condition for f to be the invariant solution of e^{aU} is obviously

$$\{Uu\}_f = 0. \tag{47}$$

One of the best known invariant solutions will be the Green's function of the heat equation $f_{xx} - f_t = 0$, $f = (4\pi t)^{-1/2}\exp(-x^2/4t)$. Here the group involved is the dilation group generated by $U = (xu_x + 2tu_t + u)\partial_u$ (or equivalently $U' = -x\partial_x - 2t\partial_t + u\partial_u$).

It is well known that the singlet solution of the KdV equation (32) is the invariant solution for $U = U^4 - c^{-1}U^3$ $(= \partial_t + c^{-1}\partial_x)$. The simplest generalization of this is to consider a group generated by $U = U^5 + pU^4 + qU^3$, p,q constants. Then the condition (47) yields

$$\tfrac{3}{5}f_{xxxxx} + ff_{xxx} + 2f_xf_{xx} + \tfrac{1}{2}f^2f_x + p(f_{xxx} + ff_x) + qf_x = 0.$$

An integration of this equation with respect to x, assuming $f(\pm\infty,t) = 0$, leads to the fourth order equation obtained by Kruskal and Zabusky,[12] and Lax.[15] The nature of the solution was carefully studied by Lax, and the solution was shown to be the doublet solution. From a group theoretic viewpoint, therefore, *the doublet solution of the KdV equation is the invariant solution of the group* $e^{a(U^5+pU^4+qU^3)}$.

The idea here is precisely parallel to Lax's; Lax uses a condition $\Gamma(f) = 0$ to characterize the doublet solution whereas we use $\{Uu\}_f = 0$; but they are related by (30).

III. INVARIANCE GROUPS AND CONSERVATION LAWS OF NONLINEAR SCHRODINGER EQUATIONS

The cubic Schrödinger equation $-if_{xx} - if^2f^* + f_t = 0$ is another well studied nonlinear equation. It is known to share many common properties with the KdV equation.[9,18,19] In this section we study group theoretic aspects of conservation laws associated with a class of nonlinear Schrödinger equations.

A. Conservation laws of nonlinear Schrödinger equations

We consider a coupled nonlinear Schrödinger equation

$$\begin{aligned} &u_{xx} + a(u,v;c) + iu_t = 0, \quad v_{xx} + a(u,v;c)^{\#} - iv_t = 0,\\ &u = f(x,t), \quad u_x = f_x(x,t), \quad u_t = f_t(x,t), \cdots,\\ &v = g(x,t), \quad v_x = g_x(x,t), \quad v_t = g_t(x,t), \cdots, \end{aligned} \tag{48}$$

where a function a is subject to the condition

$$a_u(f,g;c) = [a_u(f,g;c)]^{\#}, \quad a_u = \partial_u a. \tag{49}$$

[See (14) for the notation #.] Condition (50) amounts to requiring that the equation can be written as a Hamiltonian system,

$$\frac{\delta\hat H}{\delta g} = -if_t, \quad \frac{\delta\hat H}{\delta f} = ig_t, \tag{50}$$

where $\delta\hat H/\delta g$ and $\delta\hat H/\delta f$ are Frechet derivatives of $\hat H = \int E(f,g)\,dx$, E = energy density. Equation (48) reduces to the cubic Schrödinger equation for the special case of $a = u^2v$ and $g = f^*$.

We assume that an initial value problem is well posed either for a periodic condition $f(x,t) = f(x+x_0,t)$, $g(x,t) = g(x+x_0,t)$ or for a boundary condition $f(\pm\infty,t) = 0$, $g(\pm\infty,t) = 0$. Let us suppose that the system described by (48) has a constant of motion $I(f,g) = \int \rho(f,g)\,dx$ where the integration is over the period or from $-\infty$ to $+\infty$. The following theorem establishes the relationship between the I and an invariance group of the equation. In the following, quantities $\delta I/\delta u$ and $\delta I/\delta v$ represent $\{\delta I/\delta f\}_{f=u,g=v}$ and $\{\delta I/\delta g\}_{f=u,g=v}$.

Theorem 2: If $\delta I/\delta f$ and $\delta I/\delta g$ are Frechet derivatives of a constant of motion $I(f,g) = \int\rho(f,g)\,dx$ associated with Eq. (48), *then the operator* $U = i(\delta I/\delta v)\partial_u - i(\delta I/\delta u)\partial_v$ is a generator of an invariance group of the equation.

Proof: We consider infinitesimal transformations of solutions f,g into solutions $f+\epsilon\phi$, $g+\epsilon\psi$. ϕ and ψ must satisfy the equations $A = \phi_{xx} + a_u(f,g;c)\phi + a_v(f,g;c)\psi + i\phi_t = 0$, $B = \psi_{xx} + a_u(g,f;c^*)\psi + a_v(g,f;c^*)\phi - i\psi_t = 0$. The effect of this transformation on I can be found easily; by integration by parts, we arrive at

$$I(f+\epsilon\phi, g+\epsilon\psi) = I(f,g) + \epsilon\int\left(\frac{\delta I}{\delta f}\phi + \frac{\delta I}{\delta g}\psi\right)dx$$
$$\equiv I(f,g) + \epsilon\,\delta I.$$

Thus, $d/dt\,\delta I = 0$. Next obviously,

$$\int \left(i \frac{\delta I}{\delta f} A - i \frac{\delta I}{\delta g} B \right) dx = 0.$$

On integrating by parts this yields $0 = \int (P\phi + Q\psi)\, dx + (d/dt)\,\delta I$ where

$$P = -\{U[v_{xx} + a(u, v;c)^{\#} - iv_t]\}_{f,g} + i[a_u(f,g;c) - a_u(g,f;c^*)] \frac{\delta I}{\delta f},$$

$$Q = -\{U[u_{xx} + a(u, v;c) + iu_t]\}_{f,g} + i[a_u(f,g;c) - a_u(g,f;c^*)] \frac{\delta I}{\delta g}.$$

Because $(d/dt)\,\delta I = 0$, we obtain

$$\int (P\phi + Q\psi)\, dx = 0. \tag{*}$$

One can prescribe arbitrary admissible functions for ϕ and ψ at an initial time. Thus, the Eq. (*) implies that P and Q are identically zero. Furthermore, the second terms of P and Q are zero because of condition (49), hence, $P=0$ and $Q=0$ yield the equations to be proved.

This theorem enables us to find constants of motion if we know the invariance groups of Eq. (48); the process involves a straightforward integration process $(\delta I/\delta f, \delta I/\delta g) \rightarrow I$. However, we note that there may be a generator which is not integrable to a constant of motion. This theorem can be extended to a general Hamiltonian system.[20]

B. Invariance groups of the cubic Schrödinger equation and its conservation laws

We look for the operator of the form (16) which satisfies the invariance condition (12) for $F^1 = f_{xx} + f^2 f^* + if_t = 0$ and $F^2 = (F^1)^{\#} = 0$. Assuming the transformation to be the generalized type with $\eta = \eta(x, t, u, v, u_x, v_x, \ldots, u_{xxxxx}, v_{xxxxx})$, and carrying out Lie's algorithm, we arrive at the following eight generators [writing only the first term of (16)]:

$$\begin{aligned} &U_1 = (-\tfrac{1}{2}ixu + tu_x)\,\partial_u, \\ &U_2 = (itu_{xx} + itu^2 v + \tfrac{1}{2}xu_x + \tfrac{1}{2}u)\,\partial_u, \quad U_3 = iu\partial_u, \\ &U_4 = u_x \partial_u, \quad U_5 = i(u_{xx} + u^2 v)\,\partial_u, \\ &U_6 = (u_{xxx} + 3uvu_x)\,\partial_u, \\ &U_7 = i(u_{xxxx} + u^2 v_{xx} + 4uvu_{xx} + 2uu_x v_x + 3vu_x^2 + \tfrac{3}{2}u^3 v^2)\,\partial_u, \\ &U_8 = [u_{xxxxx} + 5(uvu_{xxx} + uu_x v_{xx} + 2vu_x u_{xx} \\ &\qquad + uv_x u_{xx} + u_x^2 v_x) + \tfrac{15}{2}u^2 v^2 u_x]\,\partial_u. \end{aligned} \tag{51}$$

The first five generators can be cast into "genuine" Lie type operators by Lemma 2:

$$\bar{U}^1 = -t\partial_x - \tfrac{1}{2}ixu\partial_u, \quad \bar{U}^2 = -x\partial_x - 2t\partial_t + u\partial_u,$$
$$\bar{U}^3 = iu\partial_u, \quad \bar{U}^4 = i\partial_x, \quad \bar{U}^5 = -\partial_t.$$

The effects of the group transformation $e^{a\bar{U}^i}$, a real, on a solution $f(x,t)$ can be found easily for $i < 6$,

$$\begin{aligned} &f^1 = \exp[-i(ax + a^2 t/2)/2]\, f(x + at, t), \\ &f^2 = af(ax, a^2 t), \quad f^3 = \exp(ia) f(x,t), \\ &f^4 = f(x + a, t), \quad f^5 = f(x, t + a). \end{aligned} \tag{52}$$

The remaining three generators are of the generalized type, and there exists, at present, no analytic method of finding corresponding global transformations.

The constants of motion associated with the generators (51) can be found by the simple integration process; they are $I^i = \int \rho^i\, dx$ where

$$\begin{aligned} &\rho^1 = xuv - it(u_x v - uv_x), \quad \rho^3 = uv, \quad \rho^4 = iu_x v, \\ &\rho^5 = \tfrac{1}{2}(u_x v_x - \tfrac{1}{2}u^2 v^2), \quad \rho^6 = i(u_{xxx} v + \tfrac{3}{2} uu_x v^2), \\ &\rho^7 = u_{xx} v_{xx} + \tfrac{1}{2}u^3 v^3 - 2(u_x v + uv_x)^2 - 3u_x v_x uv, \\ &\rho^8 = u_{xxxxx} v + 5(uu_{xxx} v + uu_x v_{xx} + 2u_x u_{xx} v + uu_{xx} v_x + u_x^2 v_x) \\ &\qquad + \tfrac{1}{3} u^2 u_x v^3. \end{aligned} \tag{53}$$

The operator U^2 is not integrable. These constants of motion, except the first one, agree with the ones obtained by Zakharov and Shabat.[9] The phase shift operator U^3, the x translation operator U^4, and the t translation operator U^5 have given rise to the probability density ρ^3, the momentum density ρ^4, and the energy density ρ^5. The first constant I^1 also has a simple meaning if we consider the cubic Schrödinger equation as the Schrödinger equation for a particle with negative mass $-\frac{1}{2}$: The I^1 represents the initial position of the particle, $\langle x_0 \rangle = \langle x - tV \rangle = \int f^* \cdot \left(x - t\frac{p}{m} \right) f\, dx = I^1$, $V =$ velocity.

Let us define the Lie Hamiltonian by

$$H = \left(i\frac{\delta \hat{H}}{\delta v} \right) \partial_u - \left(i\frac{\delta \hat{H}}{\delta u} \right) \partial_v = U^5, \quad \hat{H} = \text{energy} = I^5. \tag{54}$$

Then, we find that the operator U^i of (51) satisfies the relation $[H, U^i] + \partial U^i/\partial t = 0$ with

$$\frac{\partial U}{\partial t} = i\left(\partial_t \frac{\delta I^i}{\delta v} \right) \partial_u - i\left(\partial_t \frac{\delta I^i}{\delta_u} \right) \partial_v.$$

We note that the second generator U^2 which is not related to a constant of motion also satisfies the relation. A general analysis of this property of the generators will be given in the next section. Some of the other commutation relations among U^i are $[U^i, U^j] = 0$ for $3 \leq i, j \leq 8$.

IV. GENERAL PROPERTIES OF GENERATORS OF INVARIANCE GROUPS OF TIME EVOLUTION EQUATIONS

Let us assume that Eq. (15) is a time evolution type: $x^0 =$ time coordinate,

$$\begin{aligned} &F^1(z;c) = H(z;c) + u_0 = 0, \\ &F^2(z;c) = [H(z;c)]^{\#} + v_0 = 0. \end{aligned} \tag{55}$$

To carry out a consistent analysis, we must take into account the relation (13),

$$D_{p_1 p_2 \cdots p_k}(H + u_0) = 0, \quad k = 1, 2, \ldots, \infty. \tag{56}$$

We define two operators associated with H^{21} and U by

$$H = H\partial_u + H^{\#}\partial_v, \tag{57}$$

$$\frac{\partial U}{\partial x^0} = (\partial_{x^0}\eta)\,\partial_u + (\partial_{x^0}\eta)^{\#}\partial_v. \tag{58}$$

As was mentioned in the first section, they must be interpreted as their infinite prolongation.

By the definition of a time evolution equation, H is not a function of the coordinates corresponding to x^0-derivatives such as u_{01}, v_{120}. In such a case, we can always express any coordinate of x^0-derivatives in terms of other coordinates by making use of the relations (55) and (56). Thus, we assume, without a loss of generality, that η is free of these coordinates.

A key in the present analysis is to write Eq. (55) as

$$(H+D_0)u=0. \tag{59}$$

We first prove:

Lemma 3: If U is a generator of an invariance group of the equation $H+u_0=0$, then under condition (56) we have $[U,H]+\partial U/\partial x^0=0$.

Proof: We have $[U,H]+\partial U/\partial x^0=a\partial_u+a^{\#}\partial_v$ with $a=UH-H\eta+\partial_{x^0}\eta$. It is sufficient if we prove that a vanishes under (56). Indeed, $0=U(H+u_0)=UH+D_0\eta=UH+\partial_{x^0}\eta+u_0\eta_u+v_0\eta_v+\cdots=UH+\partial_{x^0}\eta-H\partial_u-H^{\#}\partial_v-\cdots=UH+\partial_{x^0}-H\eta$.

Now, we define the following quantity:

$$\langle U\rangle=\mathrm{Re}\int(vUu)_{\substack{u=f(x)\\ v=f^*(x)}}\,dx^1dx^2\cdots dx^N,\quad \mathrm{Re}=\text{real part}, \tag{60}$$

where the integration should be taken over the whole space of interest. Obviously $\langle U\rangle$ is a function of x^0 only. The following lemma describes how it develops in time for a class of nonlinear systems:

Lemma 4: If H of the equation $H+u_0=0$ satisfies the equation

$$H^{\#}+vH_u+u(H_v)^{\#}-D_i[vH_{u_i}+u(H_{v_i})^{\#}]+\cdots+(-1)^rD_{p_1\cdots p_r}[vH_{u_{p_1\cdots p_r}}+u(H_{v_{p_1\cdots p_r}})^{\#}]=0 \tag{61}$$

and if all the boundary integrals

$$\int_S[u\eta_{j\cdots k}H_{u_{ij\cdots k}}]_{\substack{u=f(x)\\ v=f^*(x)}}\,\nu^i\,d\Omega$$

and

$$\int_S[v\eta^{\#}_{j\cdots k}H_{u_{ij\cdots k}}]_{\substack{u=f(x)\\ v=f^*(x)}}\,\nu^i\,d\Omega$$

vanish for $\nu=(\nu^1,\ldots,\nu^N)=$ normal vector on the boundary surface, then

$$\frac{d}{dx^0}\langle U\rangle=\left\langle[U,H]+\frac{\partial U}{\partial x^0}\right\rangle. \tag{62}$$

Proof: For brevity, we write (60) as $\langle U\rangle=\mathrm{Re}\int vUu\,d\mathbf{x}$. Then, we have $d/dx^0\langle U\rangle=\mathrm{Re}\int(v_0Uu+vD_0Uu)\,d\mathbf{x}=\mathrm{Re}\int[-H^{\#}\eta+v(\partial U/\partial x^0-HU)u]\,d\mathbf{x}$. Here, we have used the relations (55) and (56). On the other hand, we have $\langle UH\rangle=\mathrm{Re}\int vUHu\,d\mathbf{x}=\mathrm{Re}\int v\eta_{i\cdots j}H_{u_{i\cdots j}}\,d\mathbf{x}$. Applying Green's theorem repeatedly, and using the hypotheses, we find $\langle UH\rangle=\mathrm{Re}\int(-H^{\#}\eta)\,d\mathbf{x}$. Putting these two together, we obtain $(d/dx^0)\langle U\rangle=\langle[U,H]+\partial U/\partial x^0\rangle$.

The combination of Lemma 3 and 4 leads to a method to associate a conserved quantity with an invariance group of the equation:

Theorem 3: If the operator U defined by (16) is a generator of an invariance group of the equation $H+u_0=0$, and if H satisfies all the conditions in Lemma 4, then the quantity $\langle U\rangle$ defined by (60) is a constant of motion, i.e., $d/dx^0\langle U\rangle=0$.

We note that in proving this we did not assume the quantity $\int f^*(x)f(x)\,dx^1\cdots dx^N$ to be independent of time.

Lemma 3 can be generalized to a set of nonlinear time evolution equations of the form

$$H^i+u_0^i=0,\quad H^i=H^i(x,\mathbf{u},\underset{1}{\mathbf{u}},\underset{2}{\mathbf{u}},\ldots,\underset{r}{\mathbf{u}}),\quad i=1,2,\ldots,M, \tag{63}$$

where $\mathbf{u}=(u^1,u^2,\ldots,u^M)$, $\underset{k}{\mathbf{u}}=(\underset{k}{\mathbf{u}}^1,\underset{k}{\mathbf{u}}^2,\ldots,\underset{k}{\mathbf{u}}^M)$ and $u^i=f^i(x)$, etc. In this case we have

Lemma 3': If $U=\eta^i\partial_{u^i}$ is a generator of an invariance group of Eq. (63), then we have $[U,H]+\partial U/\partial x^0=0$ where $H=H^i\partial_{u^i}$ and $\partial U/\partial x^0=(\partial_{x^0}\eta^i)\partial_{u^i}$.

For Hamilton's equations of a field $[u^1=P=p(x),\ u^2=Q=q(x)]$

$$\frac{\delta\hat H}{\delta Q}+P_0=0,\quad -\frac{\delta\hat H}{\delta P}+Q_0=0,$$

we obtain the familiar expression

$$[U,H]+\frac{\partial U}{\partial x^0}=0,\quad\text{with}\quad H=\frac{\partial\hat H}{\partial Q}\partial_p-\frac{\partial H}{\partial p}\partial_Q.$$

The theorem above can be specialized to a real differential equation: If $H(x,u,\underset{1}{u},\underset{2}{u},\ldots,\underset{r}{u})$, in the equation $H+u_0=0$, satisfies an equation

$$H+uH_u-D_i(uH_{u_i})+\cdots+(-1)^rD_{p_1\cdots p_r}(uH_{u_{p_1\cdots p_r}})=0 \tag{64}$$

and if all the surface integrals $\int_S[u\eta_{j\cdots k}H_{u_{ij\cdots k}}]_{u=f(x)}\,\nu^i\,d\Omega$ vanish for $S=$ boundary, then the quantity $\langle U\rangle=\int_v[uUu]_{u=f}\,dx^1\cdots dx^N$ is a constant of motion. Here, $v=$ the whole space inside S.

The following equations which have been attracting considerable attention in the study of propagation of nonlinear waves satisfy the condition (61) or (64):

generalized Korteweg–de Vries equation

$$(\partial_x)^{2n+1}f+f^m\partial_xf+\partial_tf=0,$$

cubic Schrödinger equation in n dimensions

$$-i\left[\sum_{k=1}^{n}(\partial_{x^k})^2f+f^2f^*\right]+\partial_tf=0,$$

Hirota equation[8]

$$a(\partial_x)^3f+ib(\partial_x)^2f+cff^*\partial_xf+idf^2f^*+\partial_tf=0.$$

However, the heat equation $f_{xx}-f_t=0$ and Burgers equation $f_{xx}+ff_x-f_t=0$, both of which represent a dissipative system, do not satisfy Eq. (64).

The application of Theorem 3 to the KdV equation and to the cubic Schrödinger equation has turned out to produce only a few constants of motion,

KdV equation:

$$\langle U^1\rangle = -\int u\,dx,\quad \langle U^2\rangle = \int \tfrac{1}{2}u^2\,dx,$$
$$\langle U^i\rangle = 0,\quad \text{for } i>2.$$

cubic Schrödinger equation:

$$\langle U^2\rangle = \int \tfrac{1}{2}uu^*\,dx,\quad \langle U^i\rangle = 0 \quad \text{for } i>2.$$

V. CONCLUDING REMARKS

We have shown that provided one considers the group transformation which is more general than the one considered by Lie, one can associate one invariance group with each constant of motion of a class of physical systems. Thus for such a system one can derive the constants of motion by finding the invariance groups of the equation. One of the best known methods of finding conservation laws is to use Noether's theorem. The difference between the two is that the groups in the present approach leave the differential equation invariant whereas the groups in Noether's theorem leave an action integral invariant.

In the following communication, a generalization of Theorems 1 and 2 will be discussed.

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APPENDIX A: PROOF OF LEMMA 1

It is sufficient if we prove $D_0UA = UD_0A$. To avoid complex indices, we represent a set of indices $i\cdots k$ appearing in the expressions (7) and (9) of D_0 and U by a circle $\circ$ or by a dot $\bullet$, and write D_0 and U as

$$D_0 = \partial_{x0} + \sum_\circ (u_{0\circ}\partial_{u_\circ} + v_{0\circ}\partial_{v_\circ}),$$
$$U = \sum_\circ (\eta_\circ\partial_{u_\circ} + \zeta_\circ\partial_{v_\circ})$$

where the sign $\sum_\circ$ indicates a summation over all the parenthesized quantities in (7) and (9). Then, by the definitions of D_0 and U,

$$D_0UA = D_0\sum_\circ (\eta_\circ A_{u_\circ} + \zeta_\circ A_{v_\circ})$$
$$= \sum_\circ (\eta_\circ D_0A_{u_\circ} + \zeta_\circ D_0A_{v_\circ}) + \sum_\circ [(D_0\eta_\circ)A_{u_\circ} + (D_0\zeta_\circ)A_{v_\circ}].$$

Using $D_0\eta_\circ = \eta_{0\circ} = Uu_{0\circ}$, $D_0\zeta_0 = \zeta_{0\circ} = Uv_{0\circ}$,

$$D_0UA = \sum_\circ (\eta_\circ D_0A_{u_\circ} + \zeta_\circ D_0A_{v_\circ}) + \sum_\circ [(Uu_{0\circ})A_{u_\circ} + (Uv_{0\circ})A_{v_\circ}]. \tag{*}$$

The first term is

$$\sum_\circ (\eta_\circ D_0A_{u_\circ} + \zeta_\circ D_0A_{v_\circ})$$
$$= \sum_\circ \{\eta_\circ[A_{0u_\circ} + \sum_\bullet (u_{0\bullet}A_{u_\circ u_\bullet} + v_{0\bullet}A_{u_\circ v_\bullet})]$$
$$+ \zeta_\circ[A_{0v_\circ} + \sum_\bullet (u_{0\bullet}A_{v_\circ u_\bullet} + v_{0\bullet}A_{v_\circ v_\bullet})]\}$$
$$= \sum_\circ (\eta_\circ A_{0u_\circ} + \zeta_\circ A_{0v_\circ}) + \sum_\bullet u_{0\bullet}\sum_\circ (\eta_\circ A_{u_\circ u_\bullet}$$
$$+ \zeta_\circ A_{u_\bullet v_\circ}) + \sum_\bullet v_{0\bullet}\sum_\circ (\eta_\circ A_{v_\bullet u_\circ} + \zeta_\circ A_{v_\bullet v_\circ})$$
$$= UA_0 + \sum_\bullet u_{0\bullet}UA_{u_\bullet} + \sum_\bullet v_{0\bullet}UA_{v_\bullet}.$$

Hence, (*) gives $D_0UA = U[A_0 + \sum_\circ (u_{0\circ}A_{u\circ} + v_{0\circ}A_{v\circ})] = UD_0A$.

APPENDIX B: A COMMUTATOR OF GENERALIZED LIE TYPE OPERATORS

We consider two operators of the form (19),

$$U^1 = \eta^1\partial_u + \zeta^1\partial_v,\quad U^2 = \eta^2\partial_u + \zeta^2\partial_v.$$

We must interpret these as simplified representations of (9). The commutator of the two is defined as

$$[U^1, U^2] = [(U^1\eta^2) - (U^2\eta^1)]\partial_u + [(U^1\zeta^2) - (U^2\zeta^1)]\partial_v + \cdots$$
$$+ [(U^1\eta^2_{i\cdots k}) - (U^2\eta^1_{i\cdots k})]\partial_{u_{i\cdots k}}$$
$$+ [(U^1\zeta^2_{i\cdots k} - (U^2\zeta^1_{i\cdots k})]\partial_{v_{i\cdots k}} + \cdots.$$

We write this as

$$U = [U^1, U^2] = \eta\partial_u + \zeta\partial_v + \cdots + \eta_{i\cdots k}\partial_{u_{i\cdots k}}$$
$$+ \zeta_{i\cdots k}\partial_{v_{i\cdots k}} + \cdots.$$

We prove that this satisfies the condition imposed on (9), i.e., the condition (6). In fact, by applying Lemma 1,

$$\eta_{i\cdots k} = U^1\eta^2_{i\cdots k} - U^2\eta^1_{i\cdots k} = U^1D_{i\cdots k}\eta^2 - U^2D_{i\cdots k}\eta^1$$
$$= D_{i\cdots k}(U^1\eta^2 - U^2\eta^1) = D_{i\cdots k}\eta.$$

Similarly $\zeta_{i\cdots k} = D_{i\cdots k}\zeta$. Therefore, the operator obtained from the commutator of two operators of the form (9) also assumes the same form.

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On the relationship between conservation laws and invariance groups of nonlinear field equations in Hamilton's canonical form [a)]

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It is shown that whenever fields governed by the equations $\partial/\partial t p_\alpha = -\delta H/\delta q_\alpha$, $\partial/\partial t q_\alpha = \delta H/\delta p_\alpha$ allow a conservation law of the form $\partial\rho/\partial t + \operatorname{div}\mathbf{J} = 0$, there exists a corresponding Lie–Bäcklund infinitesimal contact transformation which leaves the Hamiltonian equations invariant. A condition that an invariant Lie–Bäcklund infinitesimal contact transformation gives rise to a conservation law is established. Each such transformation, which may involve derivatives of arbitrary order, yields a one-parameter local Lie group of invariance transformations. The results are established with the aid of a Lie bracket formalism for Hamiltonian fields. They account for a number of recently discovered conservation laws associated with nonlinear time evolution equations.

INTRODUCTION

In previous papers,[1,2] we have studied invariance properties of various nonlinear time evolution equations by applying the theory of groups of Lie–Bäcklund tangent transformations[3] (not to be confused with the Bäcklund transformations of recent literature[4]) and we have shown that each of the well-known series of conservation laws assoicated with the sine-Gordon equation, the Korteweg–de Vries equation, and the nonlinear Schrödinger equation is related to a different one-parameter group which leaves the corresponding differential equation invariant.

The group generators obtained in these papers depend upon derivatives of arbitrary order, so that they are not of the type considered in Lie's general theory of continuous groups of transformations. The question naturally arises: To what extent can the previous results be generalized?

In the present paper, we study invariance properties of Hamilton's equations governing the time evolution of multicomponent fields $p_\alpha(x)$, $q_\alpha(x)$,

$$\dot{p}_\alpha = -\delta H/\delta q_\alpha, \quad \dot{q}_\alpha = \delta H/\delta p_\alpha, \quad \alpha = 1, 2, \ldots, N, \tag{1}$$

where $x = (x^0, x^1, x^2, x^3)$ and $\dot{p}_\alpha = \partial_{x^0} p_\alpha$, $\dot{q}_\alpha = \partial_{x^0} q_\alpha$. We assume that an energy density $\mathcal{H}$ associated with H can depend on coordinates x (including x^0), p_α, and q_α, and their spatial derivatives of arbitrary order.[5] The main interest of the study is: to examine the relationship between invariance groups admitted by Eq. (1) and conservation laws obeyed by the fields. We will prove that: *The existence of N independent conservation laws associated with the fields of Eq.* (1) *necessarily requires the existence of N one-parameter groups which leave Eq.* (1) *invariant*. The precise result will be stated here as a theorem. The notations in the theorem are the following: $\mathcal{A}$ and $\mathcal{J}^i$ are quantities associated with the fields and are functions of x, p_α, and q_α, and of their spatial derivatives of arbitrary order; D_i represents a differentiation with respect to x^i, and the quantity $\delta A/\delta f$ ($f = q_\alpha$ or p_α) is defined by

$$\frac{\delta A}{\delta f} = \mathcal{A}_f - D_i \mathcal{A}_{f,i} + D_i D_j \mathcal{A}_{f,ij} + \cdots + (-D_i)\cdots(-D_j)\mathcal{A}_{f,i\ldots j} + \cdots,$$

with

$$\mathcal{A}_{f,i\ldots j} = \partial_{f,i\ldots j}\mathcal{A} \quad \text{and} \quad f_{,i\ldots j} = \partial_{x^i}\cdots\partial_{x^j} f.$$

Theorem: If, when p_α and q_α are solutions of the Hamiltonian equations (1), the functions $\mathcal{A}(x, p_\alpha, q_\alpha, \cdots)$ and $\mathcal{J}^i(x, p_\alpha, q_\alpha, \cdots)$ obey the conservation law $D_0\mathcal{A} + \sum_{i=1}^3 D_i \mathcal{J}^i = 0$, then the prolongation of the operator $\mathbf{A} = (\delta A/\delta p_\alpha)\partial_{q_\alpha} - (\delta A/\delta q_\alpha)\partial_{p_\alpha}$ is a generator of an invariance group of the Hamiltonian equations. Conversely, for any operator of the form $\mathbf{A}' = (\delta A'/\delta p_\alpha)\partial_{q_\alpha} - (\delta A'/\delta q_\alpha)\partial_{p_\alpha}$ whose prolongation becomes a generator of an invariance group of the Hamiltonian equations, there exists a flux $\mathcal{J}^i$ which together with a density $\mathcal{A}'$ forms a conservation law $D_0\mathcal{A}' + \sum_{i=1}^3 D_i \mathcal{J}^i = 0$.

The corresponding result for Hamiltonian systems with finite degrees of freedom governed by the equations $\dot{q}_\alpha = \partial H/\partial p_\alpha$, $\dot{p}_\alpha = -\partial H/\partial q_\alpha$ has been obtained by Peterson.[6]

We will prove the theorem by using a Lie bracket formalism, instead of a Poisson bracket formalism, for Eq. (1). To establish the Lie bracket formulation, one needs to associate appropriate operators with physical quantities of the system. Such a formalism is known for Hamiltonian systems with finite degrees of freedom.[7] In the following we will develop a similar formalism for the field equations (1) by applying the theory of Lie–Bäcklund tangent transformations. The formalism turns out to be very appropriate in studying the connection of invariance groups of Eq. (1) to conservation laws. In this approach no reference is made to invariance properties of an action integral $\int L\, dx$: We deal directly with invariance properties of differential equations.

All the results in the following sections remain valid for a general case of n spatial variables.

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I. LIE BRACKET FORMALISM

We consider groups of Lie–Bäcklund tangent transformations generated by the operators[8]

$$U=F_\alpha\partial_{q_\alpha}-G_\alpha\partial_{p_\alpha}+(D_iF_\alpha)\partial_{q_{\alpha,i}}-(D_iG_\alpha)\partial_{p_{\alpha,i}}+\cdots+\{(-D_i)\cdots(-D_j)F_\alpha\}\partial_{q_{\alpha,i\cdots j}}-\{(-D_i)\cdots(-D_j)G_\alpha\}\partial_{p_{\alpha,i\cdots j}}+\cdots, \quad (2)$$

where D_i represents a total derivative operator

$$D_i=\partial_{x^i}+p_{\alpha,i}\partial_{p_\alpha}+q_{\alpha,i}\partial_{q_\alpha}+p_{\alpha,ij}\partial_{p_{\alpha,j}}+q_{\alpha,ij}\partial_{q_{\alpha,j}}+\cdots, \quad (3)$$

and $p_{\alpha,i\cdots j}$, $q_{\alpha,i\cdots j}$ represent coordinates associated with derivatives $\partial_{x^i}\cdots\partial_{x^j}p_\alpha(x)$, $\partial_{x^i}\cdots\partial_{x^j}q_\alpha(x)$. Throughout the paper we adopt a summation convention for repeated indices: a greek index runs from 1 to N and a Roman from 0 to 3. In contrast to conventional contact transformations, we allow F and G to be functions of x and $p_\alpha(x)$, $q_\alpha(x)$, and any of their derivatives of *arbitrary* order. In the study of Eq. (1) which is a time evolution type we can assume without a loss of generality that the F_α and G_α are not functions of time derivatives of $p_\alpha(x)$ and $q_\alpha(x)$. This will be assumed in the following for all the operators of the form (2). To avoid a complex expression we write the operator (2), which we call a Lie–Bäcklund operator, as

$$U=F_\alpha\partial_{q_\alpha}-G_\alpha\partial_{p_\alpha}. \quad (4)$$

We must always consider this to be the infinite series given by (2). We denote a set of operators of the form (2) by Λ. It is known that the U have the properties

(a) If U^1, $U^2\in\Lambda$, then $U^3=[U^1,U^2]\in\Lambda$ with $F^3_\alpha=U^1F^2_\alpha-U^2F^1_\alpha$ and $G^3_\alpha=U^1G^2_\alpha-U^2G^1_\alpha$.[2,9]

(b) If U^1, U^2, $U^3\in\Lambda$, the Jacobi identity holds[9]: $[[U^1,U^2],U^3]+[[U^2,U^3],U^1]+[[U^3,U^1],U^2]=0$.

(c) Members of Λ commute with the total derivative operator D_i: $[U,D_i]=0$.[2,9,10]

This last property will be used frequently in the following without comment. We define the time derivative of the U, which we denote by U_{x^0}, by

$$U_{x^0}=(\partial_{x^0}F_\alpha)\partial_{q_\alpha}-(\partial_{x^0}G_\alpha)\partial_{p_\alpha}. \quad (5)$$

Again, this is a simplified expression; the full expression is obtained by replacing F_α and G_α in (2) by $\partial_{x^0}F_\alpha$ and $\partial_{x^0}G_\alpha$.

Now, let us consider a variational problem of a functional

$$J[p,q,x^0]=\int_G \mathcal{J}(x,p,q)\,dx',\quad dx'=dx^1dx^2dx^3. \quad (6)$$

The density $\mathcal{J}(x,p,q)$ depends on x and p_α, q_α and their derivatives $p_{\alpha,i\cdots j}$, $q_{\alpha,i\cdots j}$ of arbitrary order except ones involving time derivatives. For the variation $p_\alpha(x)\to p_\alpha(x)+\epsilon\psi_\alpha(x)$ we have

$$\delta J=\epsilon\int_G\left(\frac{\delta J}{\delta p_\alpha}\right)\psi_\alpha\,dx'+\text{surface integral} \quad (7)$$

with

$$\frac{\delta J}{\delta p_\alpha}=\mathcal{J}_{p_\alpha}-D_i\mathcal{J}_{p_{\alpha,i}}+D_iD_j\mathcal{J}_{p_{\alpha,ij}}+(-D_i)\cdots(-D_j)\mathcal{J}_{p_{\alpha,i\cdots j}}+\cdots. \quad (8)$$

Similarly, for a variation $q_\alpha(x)\to q_\alpha(x)+\epsilon\phi_\alpha(x)$, we have

$$\frac{\delta J}{\delta q_\alpha}=\mathcal{J}_{q_\alpha}-D_i\mathcal{J}_{q_{\alpha,i}}+D_iD_j\mathcal{J}_{q_{\alpha,ij}}+(-D_i)\cdots(-D_j)\mathcal{J}_{q_{\alpha,i\cdots j}}+\cdots. \quad (9)$$

We adopt (8) and (9) as the defining equations of $\delta J/\delta p_\alpha$ and $\delta J/\delta q_\alpha$. We call $\mathcal{J}$ a density of J. With the functional J we associate an operator $\mathbf{J}$ which is obtained from (2) by substituting $\delta J/\delta p_\alpha$ and $\delta J/\delta q_\alpha$ for F_α and G_α: In simplified notation

$$\mathbf{J}=\left(\frac{\delta J}{\delta p_\alpha}\right)\partial_{q_\alpha}-\left(\frac{\delta J}{\delta q_\alpha}\right)\partial_{p_\alpha}. \quad (10)$$

We designate the operators of this particular form by boldface letters. Then, with the energy functional H, the following Lie–Bäcklund Hamiltonian operator will be associated:

$$\mathbf{H}=\left(\frac{\delta H}{\delta p_\alpha}\right)\partial_{q_\alpha}-\left(\frac{\delta H}{\delta q_\alpha}\right)\partial_{p_\alpha}. \quad (11)$$

The operator corresponding to a functional $\int\partial_{x^0}\mathcal{J}\,dx'$ is found to be equivalent to

$$\mathbf{J}_{x^0}=\left[\partial_{x^0}\left(\frac{\delta J}{\delta p_\alpha}\right)\right]\partial_{q_\alpha}-\left[\partial_{x^0}\left(\frac{\delta J}{\delta q_\alpha}\right)\right]\partial_{p_\alpha}. \quad (12)$$

Let us denote the set of all the operators of the form (10) by Ω. We can prove that Ω closes under the commutation operation defined in (a) above:

Proposition: If two operators $\mathbf{A}$ and $\mathbf{B}$ belong to Ω, the commutator $\mathbf{C}=[\mathbf{B},\mathbf{A}]$ also belongs to Ω, and its density $\mathcal{C}$ is given by any one of the following:

$$\mathcal{C}_1=\left(\frac{\delta B}{\delta p_\alpha}\right)\left(\frac{\delta A}{\delta q_\alpha}\right)-\left(\frac{\delta A}{\delta p_\alpha}\right)\left(\frac{\delta B}{\delta q_\alpha}\right),\quad \mathcal{C}_2=\mathbf{B}\mathcal{A},\quad \mathcal{C}_3=-\mathbf{A}\mathcal{B}. \quad (13)$$

The proof will be given in the Appendix. Following the usual definition of a Poisson bracket for fields, we have $C=\int\mathcal{C}_1\,dx'=\{B,A\}$. Thus, we might state this as: The commutator of the operators associated with the functionals A and B is equal to the operator assoicated with the functional $\{A,B\}$. We note that the canonical commutation relations among p_α and q_α are not carried over to the operator formalism: The operators corresponding to p_α and q_α are $\mathbf{P}_\alpha=\partial_{q_\alpha}$ and $\mathbf{Q}_\alpha=-\partial_{p_\alpha}$ and they all commute.

II. INVARIANCE GROUPS OF HAMILTON'S EQUATIONS AND CONSERVATION LAWS

We now turn our attention to the theorem stated earlier. The well-known equation which describes the time evolution of a functional $A=\int\mathcal{A}\,dx'$ is

$$\frac{d}{dx^0}A=\{H,A\}+\int\partial_{x^0}\mathcal{A}\,dx'. \quad (14)$$

We associate an operator $\mathbf{K}=[\mathbf{H},\mathbf{A}]+\mathbf{A}_{x^0}$ with the quantity on the right-hand side. In view of (12) and (13), it is obvious that:

The density $\mathcal{K}$ corresponding to the operator $\mathbf{K}=[\mathbf{H},\mathbf{A}]+\mathbf{A}_{x^0}$ is any of the following:

$$\mathcal{K}_1=\left(\frac{\delta H}{\delta p_\alpha}\right)\left(\frac{\delta A}{\delta q_\alpha}\right)-\left(\frac{\delta A}{\delta p_\alpha}\right)\left(\frac{\delta H}{\delta q_\alpha}\right)+\partial_{x^0}\mathcal{A},$$

or (**)

$$K_2 = \mathrm{H}\mathcal{A} + \partial_{x^0}\mathcal{A}, \quad K_3 = -\mathbf{A}\mathcal{H} + \partial_{x^0}\mathcal{A}.$$

In the following, we prove the theorem by showing basically the following equivalences:

A is a generator of an invariance group of Eq. (1).
$\updownarrow$
A satisfies $[\mathrm{H}, \mathbf{A}] + \mathbf{A}_{x^0} = 0$
$\updownarrow$
$\mathcal{A}$ satisfies $D_0\mathcal{A} + \sum_{i=1}^{3} D_i\mathcal{J}^i = 0.$

According to the theory of groups of differential equations,[11] the operator U of (2) becomes a generator of an invariance group of Eqs. (1) if and only if U satisfies the equations

$$U\left(\dot{p}_\alpha + \frac{\delta H}{\delta q_\alpha}\right)\bigg|_w = 0, \quad U\left(\dot{q}_\alpha - \frac{\delta H}{\delta p_\alpha}\right)\bigg|_w = 0. \tag{15}$$

Here, the symbol $(\cdots)|_w$ means: Evaluate the quantities under conditions (1) and the conditions implied by them. We note that there exist generators which do not take the special form given by (10). We start from the following properties of a generator of an invariance group of Eq. (1):

Lemma 1: The Lie-Bäcklund operator U defined by (2) satisfies the equations

$$([\mathrm{H}, U] + U_{x^0})p_\alpha|_w = 0, \quad ([\mathrm{H}, U] + U_{x^0})q_\alpha|_w = 0, \tag{16}$$

if and only if U is a generator of an invariance group of Hamilton's equation (1).

Proof: In view of the definition of H, under the condition $(\cdots)|_w$ we have an identity $D_0 = \partial_{x^0} + \mathrm{H}$. Using this relation, we obtain $([\mathrm{H}, U] + U_{x^0})p_\alpha|_w = \{-\mathrm{H}G_\alpha + U(\delta H/\delta q_\alpha) - \partial_{x^0}G_\alpha\}|_w = \{U(\delta H/\delta q_\alpha) - D_0 G_\alpha\}|_w = U(\delta H/\delta q_\alpha + \dot{p}_\alpha)|_w$. Similarly, $([\mathrm{H}, U] + U_{x^0})q_\alpha|_w = U(-\delta H/\delta p_\alpha + \dot{q}_\alpha)|_w$. These relations obviously prove the statement.

In the following analysis, it is often helpful to consider an initial value problem of Eq. (1). We say that functions $f_\alpha(x')$ and $g_\alpha(x')$, $x' = (x^1, x^2, x^3)$, are admissible if the initial value problem $p_\alpha|_{x^0=t} = f_\alpha$, $q_\alpha|_{x^0=t} = g_\alpha$ has a solution. A set of all such admissible functions will be denoted by I. The following lemma states that Eq. (16) holds without the condition $|_w$.

Lemma 2: If U is a generator of an invariance group of Hamilton's equations (1), the operator $[\mathrm{H}, U] + U_{x^0}$ vanishes identically for arbitrary functions $f_\alpha(x')$ and $g_\alpha(x')$ which belong to I.

Proof: We have, by definition, $[\mathrm{H}, U] + U_{x^0} = M_\alpha\partial_q - N_\alpha\partial_p$, where $M_\alpha(x,p,q) = \mathrm{H}F_\alpha - U(\delta H/\delta p_\alpha) + \partial_{x^0}F_\alpha$, $N_\alpha(x,p,q) = \mathrm{H}G_\alpha - U(\delta H/\delta q_\alpha) + \partial_{x^0}G$. By Lemma 1, if p_α, q_α are solutions of Hamilton's equation, then $M_\alpha = N_\alpha = 0$. We let $x^0 \to t$ = initial time. At t, both M_α and N_α are well defined (note that F_α and G_α do not depend on any x^0 derivatives of p_α and q_α), hence, $M_\alpha = N_\alpha = 0$ at t. Suppose that inital conditions were $p_\alpha = f_\alpha(x')$, $q_\alpha = g_\alpha(x')$. Then, $M_\alpha(x,f,g) = N_\alpha(x,f,g) = 0$ with $x = (t, x^1, x^2, x^3)$. Because t is a parameter of arbitrary value, we may replace x by (x^0, x^1, x^2, x^3) to obtain the desired result.

Remark: If f_α, g_α or any of their derivatives were not defined at some point, the function M_α and N_α, hence the operator $[\mathrm{H}, U] + U_{x^0}$, would not be defined at the point. We note that the relation $[\mathrm{H}, U] + U_{x^0} \equiv 0$ holds even pointwise: For any given values of x, p_α, q_α, $p_{\alpha,i}$, $q_{\alpha,i}$, $\cdots$, the operator vanishes. This should be true as long as there exists a solution which takes the designated values at the given point x. It is also clear that we can allow f_α and g_α to be functions of x instead of x' because x^0, if it appears in f_α and g_α, acts simply as a parameter and has no consequence for the proof given here.

Now, we combine the results obtained above to prove the theorem stated at the beginning:

Proof of theorem: In the following, we assume, that the index i runs from 1 to 3. First we show $D_0\mathcal{A} + D_i\mathcal{J}^i = 0 \to \mathbf{A}$ is a generator. Under condition (1), we have $\mathrm{H}\mathcal{A} + \partial_{x^0}\mathcal{A} = D_0\mathcal{A}$, hence, by the hypothesis

$$\mathrm{H}\mathcal{A} + \partial_{x^0}\mathcal{A} = -D_i\mathcal{J}^i. \tag{†}$$

Because we may assume that neither $\mathcal{A}$ nor $\mathcal{J}^i$ contains time derivatives of p_α and q_α, this relation must hold at initial time $x^0 = t$ where arbitrary initial values may be imposed on p_α and q_α. Consequently, the equation (†) holds not only for solutions p_α and q_α but also for arbitrary functions $f_\alpha(x')$ and $g_\alpha(x')$. Thus, noticing that the left-hand side of the equation (†) is the K_2 of (**), we have $K_2 = -D_i\mathcal{J}^i$. This implies that $\delta K/\delta p_\alpha$ and $\delta K/\delta q_\alpha$ vanish identically, and, as a result, $[\mathrm{H}, \mathbf{A}] + \mathbf{A}_{x^0} \equiv 0$ by (**). In view of Lemmas 1 and 2, we see that this is the necessary and sufficient condition for $\mathbf{A}$ to be a generator of an invariance group of Eq. (1). Conversely, if $\mathbf{A}$ is a generator of an invariance group, in view of Lemma 1 we obtain two equations $\delta K/\delta p_\alpha = 0$ and $\delta K/\delta q_\alpha = 0$. According to Lemma 2, these vanish identically. This implies that the density K in (**) must have a divergent form; for instance, $K_2 = (\mathrm{H} + \partial_{x^0})\mathcal{A} = -D_i\mathcal{J}^i$ with $\mathcal{J}^i = \mathcal{J}^i(x,p,q)$. Now if we let p_α and q_α be solutions of Eq. (1), the quantity in the middle of this equation becomes equal to $D_0\mathcal{A}$, and the equation leads to the desired result $D_0\mathcal{A} + D_i\mathcal{J}^i = 0$.

III. INTEGRABILITY OF GENERATORS TO CONSERVED DENSITIES

We have proved that with every conservation law obeyed by the Hamiltonian fields one invariance group is always associated. In the present formulation, the converse of this is true only if the coefficients of the generator U take the special form $F_\alpha = \delta A/\delta p_\alpha$, $G_\alpha = \delta A/\delta q_\alpha$. Because there exists a systematic algorithm for finding generators of invariance groups, it is important to know whether the generators found are integrable to conserved densities. For simplicity, we adopt the following notation:

$$f_\alpha(x) = q_\alpha(x), \quad f_{\alpha+N}(x) = p_\alpha(x) \quad \text{with } \alpha = 1, 2, \ldots, N,$$
$$S_\alpha(f_\lambda) = F_\alpha, \quad S_{\alpha+N}(f_\lambda) = G_\alpha \quad \text{with } \alpha = 1, 2, \ldots, N. \tag{17}$$

In this notation, our problem is to tell whether a given set of S_α have the property $S_\alpha = \delta A/\delta f_\alpha$ for some functional $A[f_\lambda] = \int \mathcal{A}(x, f_\lambda)\,dx'$. As a general property of a functional, we have

$$\left\{\frac{d}{d\epsilon_\alpha}\frac{d}{d\epsilon_\beta}A[f_\lambda + \epsilon_\lambda\phi_\lambda]\right\}_{\epsilon=0} = \left\{\frac{d}{d\epsilon_\beta}\frac{d}{d\epsilon_\alpha}A[f_\lambda + \epsilon_\lambda\phi_\lambda]\right\}_{\epsilon=0},$$

where $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{2N})$. If S_α has the desired property, then, because of the definition $\{(d/d\epsilon_\nu)\mathcal{A}[f_\lambda + \epsilon_\lambda \phi_\lambda]\}_{\epsilon=0} = \int (\delta \mathcal{A}/\delta f_\nu)\phi_\nu \, dx'$ (ν fixed), this relation is written for fixed α and β as,

$$\left\{\frac{d}{d\epsilon_\alpha}\int S_\beta(f_\lambda + \epsilon_\lambda \phi_\lambda)\phi_\beta \, dx'\right\}_{\epsilon=0} = \left\{\frac{d}{d\epsilon_\beta}\int S_\alpha(f_\lambda + \epsilon_\lambda \phi_\lambda)\phi_\alpha \, dx'\right\}_{\epsilon=0}. \tag{18}$$

This is the integrability condition of the set S_α to a conserved density $\mathcal{A}$. It is not difficult to obtain from this a condition which does not involve integration: Using the fact that the functions ϕ_α and ϕ_β are arbitrary, we can reduce (18) to

$$\sum_{(ij)} D^-_{i\cdots j}(\phi \partial_{f_{\beta,i\cdots j}} S_\alpha) = \sum_{(ij)} \phi_{,i\cdots j} \partial_{f_{\alpha,i\cdots j}} S_\beta, \tag{19}$$

$$\phi(x) = \text{arbitrary function}$$

where $D^-_{i\cdots j} = (-D_i)\cdots(-D_j)$, $\phi_{,i\cdots j} = \partial_{x^i}\cdots\partial_{x^j}\,\varphi(x)$, and the notation $\sum_{(ij)} a_{i\cdots j} = a + a_i + a_{ij} + \cdots$. All the Roman indices run from 1 to 3. Because ϕ is arbitrary, the coefficients of each $\phi_{,i\cdots j}$ on both sides of (20) must match.

Example: sine-Gordon equation

To illustrate the results obtained above, we study the sine-Gordon equation, using $t = x^0$, $x = x^1$,

$$u_{tt} - u_{xx} + \sin u = 0. \tag{20}$$

A canonical form $q_t = \delta H/\delta p$, $p_t = -\delta H/\delta q$ for this equation is obtained by letting $q = u$, $H = \frac{1}{2}p^2 + \frac{1}{2}q_x^2 - \cos q$:

$$q_t = p, \quad p_t = -q_{xx} + \sin q. \tag{21}$$

In the previous paper,[1] we have shown the equation $u_{xt} = \sin u$ admits an infinite number of invariance groups, and it is straightforward to adapt these results to Eq. (21); four of the generators of invariance groups of Eq. (21) are

$$U_1 = q_x \partial_q + p_x \partial_p, \quad U_2 = p\partial_q - (-q_{xx} + \sin q)\partial_p.$$

$$U_3 = (4q_{xxx} - 3q_x \cos q + \tfrac{1}{2}q_x^3 + \tfrac{3}{2}q_x p^2)\partial_q - (-4p_{xxx} + 3p_x \cos q - \tfrac{3}{2}q_x^2 p_x - \tfrac{3}{2}p_x p^2 - 3q_x q_{xx} p)\partial_p,$$

$$U_4 = (4p_{xx} - p\cos q + \tfrac{3}{2}q_x^2 p + \tfrac{1}{2}p^3)\partial_q - (-4q_{xxx} + 5q_{xx}\cos q - \tfrac{5}{2}q_x^2 \sin q - \sin q \cos q - \tfrac{3}{2}q_x^2 q_{xx} + \tfrac{1}{2}p^2 \sin q - 3q_x p_x p - \tfrac{3}{2}p^2 q_{xx})\partial_p.$$

Using a theorem given in the previous paper,[8] we see that U_1 and U_2 are equivalent to the space and time translation operators ∂_x and ∂_t. To find conserved densities from these operators we must check condition (19). All of them satisfy the equation, and the conserved density $\mathcal{A}$ associated with each of the generators is found to be:

$$\mathcal{A}_1 = pq_x = \text{momentum density},$$

$$\mathcal{A}_2 = \tfrac{1}{2}p^2 + \tfrac{1}{2}q_x^2 - \cos q = \text{energy density},$$

$$\mathcal{A}_3 = 4pq_{xxx} - 3pq_x \cos q + \tfrac{1}{2}q_x^3 p + \tfrac{1}{2}q_x p^3,$$

$$\mathcal{A}_4 = -2p_x^2 - 2q_{xx}^2 - \tfrac{1}{2}p^2\cos q + \tfrac{3}{4}q_x^2 p^2 + \tfrac{1}{8}p^4 - \tfrac{5}{2}q_x^2 \cos q + \tfrac{1}{2}\cos^2 q + \tfrac{1}{8}q_x^4.$$

These conserved densities are related to those obtained by Lamb,[12] and their group theoretic aspects have been studied by the author[1] and by Steudel.[13]

In the previous paper,[2] we also have shown that a series of conservation laws admitted by the nonlinear Shrödinger equation are related to invariance groups of the equation, where we have made use of a special property of the equation. The present results provide a unified view to the previous results.

CONCLUSION

In this paper, we have developed a new group theoretic way of looking at conservation laws associated with field equations in Hamilton's canonical form, and we have proved that the existence of N independent conservation laws necessarily implies the existence of at least N local one-parameter Lie groups which leave the field equations invariant. The condition that a given invariance group is integrable to a conserved density also has been given. Because there exists a well established algorithm for finding generators of invariance groups of differential equations, and because many Euler–Lagrange equations can be put into Hamilton's canonical form, the present results should be useful in finding conservation laws for a variety of systems.[14]

Clearly, the present approach to conservation laws via a Lie bracket formalism is quite different from conventional approachs which make use of Noether's theorem: Noether's theorem as originally derived is too restrictive to give rise to conservation laws such as those dealt with here. However, as this work was being completed, the author learned in a personal communication from N.H. Ibragimov that he has been able to generalized Noether's theorem and with the aid of Lie–Bäcklund contact transformations he has obtained results similar in part to those obtained here.[10]

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APPENDIX: PROOF OF THE PROPOSITION

To simplify expressions, we use notations $D_i \ldots D_j = D_{i-j}$ and $(-D_i)\cdots(-D_j) = D^-_{i-j}$; $p_{\alpha,i\cdots j} = p_{\alpha i-j}$ and $q_{\alpha,i\cdots j} = q_{\alpha i-j}$. We represent a sum of the form $f + f_i + f_{ij} + \cdots$ by one term f_{i-j}. For instance, Eq. (2) and Eq. (8) become $U = (D_{i-j}F_\alpha)\partial_{q_{\alpha i-j}} - (D_{i-j}G_\alpha)\partial_{p_{\alpha i-j}}$ and $\delta J/\delta p_\alpha = D^-_{i-j}\mathcal{J}_{p_{\alpha i-j}}$. We first prove the following relations:

$$U\left(\frac{\delta M}{\delta q_\beta}\right) = D^-_{i-j}\left[F_\alpha\left(\frac{\delta M}{\delta q_\alpha}\right)_{q_{\beta i-j}} - G_\alpha\left(\frac{\delta M}{\delta p_\alpha}\right)_{q_{\beta i-j}}\right] \tag{A1}$$

$$U\left(\frac{\delta M}{\delta p_\beta}\right) = D^-_{i-j}\left[F_\alpha\left(\frac{\delta M}{\delta q_\alpha}\right)_{p_{\beta i-j}} - G_\alpha\left(\frac{\delta M}{\delta p_\alpha}\right)_{p_{\beta i-j}}\right]. \tag{A2}$$

This relationship is entirely independent from Eq. (1).

We prove the first relation: the second follows similarly. To prove (A1), we assume that functions p_α and q_α decay sufficiently fast as $[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}\to\infty$ so that all the surface integrals which appear in the process vanish. Let us consider an integral $\int v(x')U(\delta M/\delta q_\beta)\,dx'$ where v is some arbitrary function except that it does not diverge at infinity. If we write $\partial_{x^i}\cdots\partial_{x^j}v=v_{i-j}$, then integrating over the whole space

$$\begin{aligned}
&\int vU\left(\frac{\delta M}{\delta q_\beta}\right)dx' \\
&\quad=\int vUD^-_{i-j}\mathcal{M}_{q_{\beta i-j}}\,dx'=\int vD^-_{i-j}U\mathcal{M}_{q_{\beta i-j}}\,dx' \\
&\quad=\int v_{i-j}U\mathcal{M}_{q_{\beta i-j}}\,dx' \\
&\quad=\int v_{i-j}[(D_{i'-j'}F_\alpha)\mathcal{M}_{q_{\beta i-j}q_{\alpha i'-j'}}-(D_{i'-j'}G_\alpha)\mathcal{M}_{q_{\beta i-j}p_{\alpha i'-j'}}]\,dx' \\
&\quad=\int[F_\alpha D^-_{i'-j'}(v_{i-j}\mathcal{M}_{q_{\beta i-j}q_{\alpha i'-j'}}) \\
&\qquad-G_\alpha D^-_{i'-j'}(v_{i-j}\mathcal{M}_{q_{\beta i-j}p_{\alpha i'-j'}})]\,dx' \\
&\quad=\int(F_\alpha v_{i-j}\partial_{q_{\beta i-j}}D^-_{i'-j'}\mathcal{M}_{q_{\alpha i'-j'}} \\
&\qquad-G_\alpha v_{i-j}\partial_{q_{\beta i-j}}D^-_{i'-j'}\mathcal{M}_{p_{\alpha i'-j'}})\,dx' \\
&\qquad\qquad\qquad\qquad(\text{using }[D_i,\,v_{i-j}\partial_{q_{\beta i-j}}]=0) \\
&\quad=\int\left[F_\alpha v_{i-j}\partial_{q_{\beta i-j}}\left(\frac{\delta M}{\delta q_\alpha}\right)-G_\alpha v_{i-j}\partial_{q_{\beta i-j}}\left(\frac{\delta M}{\delta p_\alpha}\right)\right]dx' \\
&\quad=\int vD^-_{i-j}\left[F_\alpha\left(\frac{\delta M}{\delta q_\alpha}\right)_{q_{\beta i-j}}-G_\alpha\left(\frac{\delta M}{\delta p_\alpha}\right)_{q_{\beta i-j}}\right]dx'.
\end{aligned}$$

Now, we have the right hand quantity of Eq. (A1) in the last integrand. Both in the starting and in this final form of the integral the function v appears as a factor. Because v is arbitrary, this equation necessarily implies (A1). Next, by definition, $[\mathbf{B},\mathbf{A}]=C^{q_\alpha}\partial_{q_\alpha}-C^{p_\alpha}\partial_{p_\alpha}$ with

$$C^{q_\alpha}=\mathbf{B}\left(\frac{\delta A}{\delta p_\alpha}\right)-\mathbf{A}\left(\frac{\delta B}{\delta p_\alpha}\right),$$

$$C^{p_\alpha}=\mathbf{B}\left(\frac{\delta A}{\delta q_\alpha}\right)-\mathbf{A}\left(\frac{\delta B}{\delta q_\alpha}\right).$$

In view of the equalities (A2), we obtain

$$\begin{aligned}
C^{q_\alpha}&=D^-_{i-j}\left[-\left(\frac{\delta B}{\delta q_\beta}\right)\left(\frac{\delta A}{\delta p_\beta}\right)_{p_{\alpha i-j}}+\left(\frac{\delta B}{\delta p_\beta}\right)\left(\frac{\delta A}{\delta q_\beta}\right)_{p_{\alpha i-j}}\right. \\
&\qquad\left.+\left(\frac{\delta A}{\delta q_\beta}\right)\left(\frac{\delta B}{\delta p_\beta}\right)_{p_{\alpha i-j}}-\left(\frac{\delta A}{\delta p_\beta}\right)\left(\frac{\delta B}{\delta q_\beta}\right)_{p_{\alpha i-j}}\right] \\
&=D^-_{i-j}\left[\left(\frac{\delta B}{\delta p_\beta}\right)\left(\frac{\delta A}{\delta q_\beta}\right)-\left(\frac{\delta A}{\delta p_\beta}\right)\left(\frac{\delta B}{\delta q_\beta}\right)\right]_{p_{\alpha i-j}} \\
&\equiv D^-_{i-j}[\mathcal{C}_1]_{p_{\alpha i-j}}.
\end{aligned}$$

Similarly, we obtain $C^{p_\alpha}=D^-_{i-j}[\mathcal{C}_1]_{q_{\alpha i-j}}$. Thus, we have proved the assertion for $\mathcal{C}_1$. To prove it for $\mathcal{C}_2$ and $\mathcal{C}_3$, we simply note that they are related to $\mathcal{C}_1$ by $\mathcal{C}_2=\mathcal{C}_1+D_if^i$ and $\mathcal{C}_3=\mathcal{C}_1+D_ig^i$ where f^i and g^i are functions of x, p_α, q_α, and of their derivatives and the index i runs from 1 to 3. The fact that functional derivatives of the functional $\int \mathrm{div}\,h(x,p,q)\,dx'$ always vanish leads to the desired results.

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