ASYMPTOTIC SOLUTIONS OF STEADILY SPINNING SHALLOW
SHELLS OF REVOLUTION UNDER UNIFORM PRESSURE

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Abstract

The problem of a steadily spinning shallow elastic shell of revolution under a uniform pressure distribution is investigated by perturbation and asymptotic methods. Accurate numerical solutions are also obtained to confirm the adequacy of perturbation and asymptotic solutions. The limiting case of a flat plate is first solved for the entire range of values of the two load parameters. The results provide a different interpretation of the existing nonlinear membrane solution for the same problem. Depending on the particular geometric and load configuration, the boundary value problem for the shell case is reduced either to the solution of a sequence of linear problems or to the solution of a previously solved nonlinear problem (including one for a flat plate) modified by the solutions of a sequence of linear problems. The analysis for the shell problem under a combined centrifugal and pressure loading shows a complex interplay among the load and geometric parameters. It also enables us to consider a number of previously investigated problems as special cases of our problem and offers a unified treatment of these problems.

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Nomenclature

A = extensional flexibility

D = bending rigidity

E = Young's modulus

ν = Poisson's ratio

h = thickness of plate and shell

V, H = axial and radial stress resultant

$\psi = rH$

M_r, M_θ = stress couples

N_r, N_θ = tangential and circumferential stress resultant

ω = steady angular velocity of the rotating shell

p_V = axial load intensity

$p_H = \rho h \omega^2 r$, radial load intensity

$p_0 = (3 + \nu) \rho h \omega^2 r_0$

Q = transverse shear resultant

R_θ = circumferential curvature radius

r, θ = polar coordinates of a point on the midsurface of the shell

r_0 = radial coordinate of outer edge of the midsurface of the shell frustum

r_1 = radial coordinate of the inner edge of the shell frustum

u, w = radial and axial displacement component of the midsurface of the shell

$x = r/r_0$, nondimensional radial coordinate

$x_1 = r_1/r_0$

z = axial coordinate of a point on the middle surface of the shell

ϵ_r = radial component of midsurface strain

ϵ_θ = circumferential component of midsurface strain

κ_r, κ_θ = curvature changes of the midsurface of the shell

ξ = meridional slope of midsurface of the shell ($= dz/dr$)

$\xi_0 = \xi$ at outer edge

ρ = density of the shell material

σ_r, σ_θ = radial and circumferential stress component

ϕ = meridional change of slope of the midsurface ($= dw/dr$)

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I. Introduction

This report investigates the elastostatic behavior of steadily rotating, shallow, elastic shells of revolution under a uniformly distributed pressure.* The shells are in the form of a hollow frustum bounded by two parallel circular edges in planes normal to the axis of revolution. With different types of edge constraints, these structures, including the limiting case of a flat disc, have many applications in space exploration [1,2]. They will be analyzed in this report by way of a shallow, thin, elastic shell theory which takes into account the simplifications associated with the inherent axisymmetry of our class of problems [3,4,5]. To make the report self-contained, the development of this shell model will be sketched out in Chapter II. A two-point boundary value problem (BVP) for two coupled second order nonlinear ordinary differential equations (ODE) appropriate for the analysis of our particular class of shell problems will be formulated in section (4) of that chapter. Except for a narrow range of geometric and load parameter values, an appropriate dimensionless form of this BVP contains one or more small parameters. Perturbation and asymptotic methods [6] are therefore appropriate for the analysis of the BVP. Accurate numerical solutions have also been obtained to confirm the predictions of our asymptotic analyses. To be concrete, we consider shell frusta with an outer edge free of edge tractions and an inner edge clamped to a rigid hub. Shells with other edge conditions can also be similarly investigated.

* Within the framework of a shallow shell theory, there is no distinction between normal and vertical loadings.

The main difficulty in an asymptotic analysis of the BVP central to our shell model lies in a proper scaling of the unknown dependent variables to arrive at a dimensionless form of the BVP. The proper scaling varies with the range of load and geometric parameter values, and the dimensionless quantities which play the role of small parameters also change accordingly. The correct scalings in most cases are deduced from limiting situations for which exact or adequate approximate solutions are known (or can be obtained); in other cases, they are inferred by physical reasoning. Once the correct scaling is known, an approximate solution by perturbation and/or asymptotic methods [6,7,8] is usually straightforward. In cases where an asymptotic solution cannot be obtained in terms of elementary or special functions, the application of perturbation or asymptotic methods still enables us to deduce some qualitative features of the solution. The order of magnitude of the unknowns, the existence of a layer phenomenon, the width of the boundary or internal layer, and other qualitative solution features contribute significantly to an efficient and accurate numerical solution for the same problem [13,14,22,23,26].

Because of its (relative) mathematical simplicity, the Foppl-Hencky nonlinear membrane theory [9,10] has been widely used in engineering analyses of sheetlike elastic bodies. J.G. Simmonds [1] applied this theory to a limiting case of our problem, namely, a steadily spinning disc subject to uniform pressure. The Foppl-Hencky theory seems particularly appropriate for the problem as it gives bounded transverse deflections throughout the flat disc even when the inner hole size shrinks to zero. (In contrast, a linear membrane theory gives an unbounded transverse deflection at the center for this limiting case.) For a sufficiently high pressure load magnitude, the

theory also predicts compressive circumferential stresses occurring in some region(s) of the disc. As an ideal elastic membrane would wrinkle in the presence of compressive stresses, it was concluded in [1] that the axisymmetric nonlinear membrane analysis should not be used for such a high pressure magnitude (relative to the magnitude of the radially directed centrifugal load distribution induced by the steady rotation). Our asymptotic analyses for this problem (in Chapter III) provide a different perspective for the Foppl-Hencky theory. In particular, it will be shown that, for the high pressure range, the nonlinear membrane solution is in fact the leading term outer asymptotic expansion of the exact solution, and it alone provides an accurate approximate solution for the in-plane stretching action throughout the entire plate. This and other related results enable us to give a new interpretation of Simmonds' work [1]. The insight on the nonlinear membrane theory gained from our asymptotic analyses is one of the highlights of this report.

For the general case, the undeformed shell shape introduces an additional degree of complexity. Aided by the results for some special cases obtained in [2,13,14,22,23], the shell behavior in different regions of the load-geometric parameter space will be analyzed in Chapter IV, again by taking advantage of the presence of one or more parameters after proper scalings. In all cases, the nonlinear BVP for shells may be reduced either to a sequence of linear problems or to a nonlinear problem already solved herein or elsewhere, corrected by terms determined by linear problems. As such, the shallow shell problem may be considered completely solved. Accurate numerical solutions of the same BVP confirm the results of our asymptotic analyses.

II. Formulation

1. Axisymmetric Deformations of Shallow Shells of Revolution

In cylindrical coordinates (r, θ, z) , the middle surface of a shallow shell of revolution may be described by

$$z = z(r). \quad (\text{II.1})$$

A shell is shallow if $(dz/dr)^2 \ll 1$ so that $1 + (dz/dr)^2 \approx 1$. For shells subject to axisymmetric external loads which induce only axisymmetric radial and axial deformation, components of displacement, strain and stress are independent of the angular coordinate θ .

Let $u(r)$ and $w(r)$ be the radial and axial displacement components, respectively, of points on the middle surface of a shell of revolution. For an infinitesimally small strain theory, the midsurface strain components in radial and circumferential directions, ϵ_r and ϵ_θ , can be expressed in terms of u and w as follows [3-5]:

$$\epsilon_\theta = \frac{u}{r} \quad (\text{II.2})$$

$$\epsilon_r = u' + z'w' + \frac{1}{2}(w')^2. \quad (\text{II.3})$$

where a prime denotes differentiation with respect to r . Eq. (II.2) and the first term of the righthand side of (II.3) are the same as the linear plane strain case [11]. The second and third terms of the right hand side of (II.3) are the contributions to ϵ_r by w due to the curvature of the deformed shell. They are easy to explain geometrically as shown in Fig. 1.

Arc AB of length ds with

$$ds^2 = dr^2 + (z'dr)^2,$$

is moved to A'B' due to displacement components w and u . A'B' has length ds' with

$$\begin{aligned}(ds')^2 &= (1+u')^2 dr^2 + (w'+z')^2 dr^2 \\ &\cong [1+2u' + (w'+z')^2] dr^2\end{aligned}$$

where, consistent with the small strain approximation (II.3), we have neglected the quadratic term in u' . Hence, the change in length is given to a good approximation by

$$ds' - ds \cong [1+u' + \frac{1}{2}(w'+z')^2 - 1 - \frac{1}{2}z'^2] dr = [u' + \frac{1}{2}(w')^2 + w'z'] dr$$

and the relative change by

$$\frac{ds' - ds}{ds} = \frac{ds' - ds}{\sqrt{dr^2 + z'^2 dr^2}} \cong \frac{ds' - ds}{dr} \cong u' + z'w' + \frac{1}{2}(w')^2. \quad (\text{II.4})$$

The approximation of ds by dr in the denominator is consistent with the shallow shell restriction. That completes the derivation of (II.3).

The midsurface curvature changes, κ_θ and κ_r , are given by [3-5]

$$\kappa_\theta = -\frac{w'}{r}, \quad \kappa_r = -w'. \quad (\text{II.5,6})$$

It is not difficult to see why curvature change in the meridional direction is given by (II.6). The derivation of (II.5) is helped by Fig. 2. As shown in that figure, we have

$$z' \cong \frac{r}{R_\theta} \quad \text{or} \quad \frac{1}{R_\theta} \cong \frac{z'}{r}.$$

After deformation, z' becomes $z' + w'$. Hence, $1/R_\theta$ becomes

$$\frac{1}{R'_\theta} \approx \frac{z' + w'}{r + u} = \frac{z' + w'}{r(1 + \epsilon_\theta)} \approx \frac{z' + w'}{r}$$

(with $1 + \epsilon_\theta \cong 1$ for infinitesimal strains) so that

$$\kappa_\theta = \frac{1}{R'_\theta} - \frac{1}{R_\theta} = \frac{z' + w'}{r} - \frac{z'}{r} = \frac{w'}{r}.$$

The negative sign in (II.5) and (II.6) is consistent with our definition of M_θ and M_r and the relations between M_θ , M_r and κ_θ , κ_r .

2. Stress Resultants, Stress Couples, Distributed Loads and Equilibrium

As in the theory of flat plates, stress resultants N_r , N_θ and stress couples M_r , M_θ for shallow shells are given in terms of stresses σ_r , σ_θ as follows [4,5,12]:

$$\{N_r, N_\theta\} = \int_{-h/2}^{h/2} \{\sigma_r, \sigma_\theta\} d\zeta \quad (II.7)$$

$$\{M_r, M_\theta\} = \int_{-h/2}^{h/2} \{\sigma_r, \sigma_\theta\} \zeta d\zeta.$$

The stress-strain relations are also taken in the same form as for flat plates [4,5,12]:

$$\epsilon_r = A(N_r - \nu N_\theta), \quad \epsilon_\theta = A(N_\theta - \nu N_r) \quad (II.8)$$

$$M_r = D(\kappa_r + \nu \kappa_\theta), \quad M_\theta = D(\kappa_\theta + \nu \kappa_r).$$

For a homogeneous material, we have in terms of Young's modulus E , Poisson's ratio ν , and the shell thickness h ,

$$A = \frac{1}{Eh}, \quad D = \frac{Eh^3}{12(1-\nu^2)}. \quad (II.9)$$

The stress resultants and couples of the shell must be in equilibrium with external surface force intensity (Fig. 3) for any portion of the midsurface. Force and moment equilibrium equations for the class of problems of interest here may be taken in the form [3-5]:

$$(rH)' - N_\theta + rp_H = 0 \quad (II.10)$$

$$(rV)' + rp_V = 0 \quad (II.11)$$

$$(rM_r)' - M_\theta - rQ = 0 \quad (II.12)$$

where p_H , p_V are radial and vertical surface load components. (II.10) is the equilibrium equation in radial direction. As shown in Fig. 3, a differential element with side length $rd\theta$ and ds is considered. The area of the element is $rd\theta ds$ and the radial component of applied load is $p_H rd\theta ds$. The radial components of the force vectors are:

$$Hrd\theta + \frac{d}{ds} (Hrd\theta) ds - Hrd\theta = \frac{d}{ds} (Hrd\theta) ds$$

and

$$- N_\theta dsd\theta.$$

Hence, the equilibrium equation in radial direction is

$$\frac{d}{ds} (Hrd\theta) ds - N_\theta dsd\theta + p_H rdsd\theta = 0.$$

For a shallow shell, $\frac{d}{ds}$ is replaced by $\frac{d}{dr}$, and the equation becomes

$$\frac{d}{dr} (rH) - N_\theta + rp_H = 0. \quad (II.10)$$

(II.11) is the equilibrium equation in vertical direction. As shown in Fig. 4, the equation should be

$$Vrd\theta + \frac{d}{ds} (Vrd\theta) ds + p_V rdsd\theta - Vrd\theta = 0$$

or

$$\frac{d}{dr} (rV) + rp_V = 0. \quad (II.11)$$

(II.12) is a moment equilibrium equation. As shown in Fig. 5, this equilibrium equation should be

$$M_r rd\theta + \frac{d}{ds} (M_r rd\theta) ds - M_r rd\theta - Qrd\theta ds - M_\theta dsd\theta = 0$$

or

$$\frac{d}{dr}(rM_r) - M_\theta - rQ = 0. \quad (\text{II.12})$$

Consistent with the shallow shell approximation, the radial and axial stress resultants, H and V , are related to the meridional stress resultant N_r and the transverse shear resultant Q by the relations (see Fig. 6):

$$\begin{aligned} Q &= V - (z' + w')H \\ N_r &= H + (z' + w')V \approx H. \end{aligned} \quad (\text{II.13})$$

3. Reduction to Two Simultaneous Equations

Equation (II.11) can be integrated to give

$$rV(r) = \frac{F_0}{2\pi} - \int_{r_1}^r p_V(x) x dx \quad (II.14)$$

where F_0 is a constant of integration.

Let

$$\psi = rH \quad \phi = w', \quad \xi = z'. \quad (II.15)$$

Then, (II.10) becomes

$$N_\theta = (rH)' + rp_H = \psi' + rp_H \quad (II.16)$$

while (II.13) and (II.8) become

$$N_r \approx H = \frac{\psi}{r}, \quad Q = V - (\xi + \phi) \frac{\psi}{r} \quad (II.17)$$

and

$$\begin{aligned} \epsilon_r &= A \left(\frac{\psi}{r} - v\psi' - vrp_H \right), & \epsilon_\theta &= A \left(\psi' + rp_H - v \frac{\psi}{r} \right), \\ M_r &= -D \left(\phi' + \frac{v}{r} \phi \right), & M_\theta &= -D \left(v\phi' + \frac{1}{r} \phi \right), \end{aligned} \quad (II.18)$$

respectively. With (II.17) and (II.18), Eq. (II.12) becomes

$$D \left[\phi'' + \frac{1}{r} \phi' - \frac{1}{2} \frac{\phi}{r} \right] - \frac{1}{r} (\xi + \phi) \psi = -V. \quad (II.19)$$

Substituting (II.2) into (II.3) to eliminate u , we get a compatibility equation

$$\epsilon_r = (r\epsilon_\theta)' + (z' + \frac{1}{2}\phi)\phi. \quad (II.20)$$

We now use (II.8), (II.15), (II.16) to write (II.20) as

$$A[\psi'' + \frac{1}{r}\psi' - \frac{1}{r^2}\psi] + \frac{1}{r}(\xi + \frac{1}{2}\phi)\phi = -A(rp_H)' - (1+\nu)Ap_H. \quad (\text{II.21})$$

(II.19) and (II.21) are the two basic differential equations for the class of problems investigated in this thesis.

4. Boundary Value Problems

We consider in this thesis a shell frustum subject to a uniform normal pressure distribution and to a meridionally varying radial load distribution induced by steadily rotating the shell about its axis of revolution. In this case, we have $p_V = \text{constant}$ and $p_H = \rho h \omega^2 r$ so that

$$-A(rp_H)' - (1+\nu)Ap_H = -(3+\nu)A\rho h\omega^2 r \equiv -Ap_0x \quad (\text{II.22})$$

and

$$-V(r) = -\frac{F_0}{2\pi r} + \frac{1}{r} \int_{r_1}^r p_V x dx = -\frac{F_0}{2\pi r} + \frac{p_V}{2r} (r^2 - r_1^2). \quad (\text{II.23})$$

where F_0 is a constant of integration.

We limit our discussion to the case of a shell with a traction free outer radial edge $r = r_0$ so that $V(r_0) = H(r_0) = M_r(r_0) = 0$. The condition $V(r_0)$ determines F_0 so that

$$-V(r) = -\frac{p_V}{2r} (r_0^2 - r^2). \quad (\text{II.24})$$

With (I.22) and (I.24), the governing differential equations (II.19) and (II.20) become

$$D(\phi'' + \frac{1}{r}\phi' - \frac{1}{2}\phi) - \frac{1}{r}(\xi + \phi)\psi = -p_V \frac{r_0^2 - r^2}{2r} \quad (\text{II.25})$$

$$A(\psi'' + \frac{1}{r}\psi' - \frac{1}{2}\psi) + \frac{1}{r}(\xi + \frac{\phi}{2})\phi = -Ap_0x. \quad (\text{II.26})$$

The remaining traction free conditions at $r = r_0$ may be expressed in terms of ψ and ϕ by way of (II.15) and (II.18)

$$r = r_0: \quad \frac{\psi}{r} = 0, \quad -D \left(\frac{d\phi}{dr} + \frac{\nu}{r} \phi \right) = 0. \quad (\text{II.27,28})$$

The inner edge of the shell at $r = r_1$ is clamped (to a rigid hub) so that

$$r = r_1: \quad \phi = u = w = 0. \quad (\text{II.29})$$

The condition of no radial displacement $u = 0$ may be expressed in terms of ψ by way of $u = r\epsilon_\theta$. Thus the first two conditions of (I.27) may be written as

$$r = r_1: \quad \phi = 0, \quad A \left(\frac{d\psi}{dr} + r p_H - \nu \frac{\psi}{r} \right) = 0. \quad (\text{II.30,31})$$

The two ODE (II.19) and (II.21), and the four boundary conditions, (II.27), (II.28), (II.30) and (II.31), form a two point boundary value problem (BVP) for the determination of ϕ and ψ . All remaining quantities which describe the shell behavior can be obtained from ϕ and ψ and/or their first derivative. The only exception is the axial displacement w which is given by

$$w = \int_{r_1}^r \phi(t) dt \quad (\text{II.32})$$

where we have made use of the third conditions in (I.29) for a clamped edge. The remaining chapters of this thesis are devoted to a characterization of the solution of the BVP for different load magnitudes p_V and p_0 .

Edge conditions different from (II.27,28) and/or (II.30,31) can also be analyzed. We mention here only the special case of a shell closed at the apex of a dome shaped shell of revolution. For this case, the boundary conditions (II.30,31) are replaced by the regularity conditions:

$$r = r_1: \quad \phi = \psi = 0. \quad (\text{II.33,34})$$

The second condition along with the boundedness of ψ' implies the symmetry condition of no radial displacement at the apex.

III. Flat Plates

1. Dimensionless Formulation

Let

$$x = \frac{r}{r_0}, \quad x_i = \frac{r_i}{r_0}, \quad \hat{f}(x) = \frac{\phi}{\bar{\phi}}, \quad \hat{g}(x) = \frac{\psi}{\bar{\psi}} \quad (\text{III.1})$$

where $\bar{\phi}$ and $\bar{\psi}$ are scale factors to be specified. With the help of these dimensionless quantities, the two point BVP for ϕ and ψ defined by (II.9), (II.21), (II.27), (II.28), (II.30) and (II.31) with $\xi \equiv 0$, become

$$\frac{D\bar{\phi}}{r_0^2} \left(\hat{f}'' + \frac{1}{x} \hat{f}' - \frac{1}{x^2} \hat{f} \right) - \frac{\bar{\phi}\bar{\psi}}{r_0} \frac{1}{x} \hat{f}\hat{g} = -\frac{(1-x^2)}{2x} r_0 p_V \quad (\text{III.2})$$

$$\frac{A\bar{\psi}}{r_0^2} \left(\hat{g}'' + \frac{1}{x} \hat{g}' - \frac{1}{x^2} \hat{g} \right) + \frac{\bar{\phi}^2}{r_0} \frac{1}{2x} \hat{f}^2 = (-x) p_0 A \quad (\text{III.3})$$

where now a prime indicates differentiation with respect to x . To allow for finite deformations, $\bar{\phi}$ and $\bar{\psi}$ must be chosen so that the nonlinear terms are not of higher order. This requirement can be met by (tentatively) setting

$$\frac{D\bar{\phi}}{r_0^2} = \frac{\bar{\phi}\bar{\psi}}{r_0}, \quad \frac{A\bar{\psi}}{r_0^2} = \frac{\bar{\phi}^2}{r_0}, \quad (\text{III.4})$$

or

$$\bar{\phi} = \frac{\sqrt{DA}}{r_0}, \quad \bar{\psi} = \frac{D}{r_0}. \quad (\text{III.5})$$

With (III.5), Eqs. (III.2) and (III.3) become

$$\hat{f}'' + \frac{1}{x} \hat{f}' - \frac{1}{x^2} \hat{f} - \frac{1}{x} \hat{f}\hat{g} = -k_V \frac{(1-x^2)}{2x} \quad (\text{III.6})$$

$$\hat{g}'' + \frac{1}{x} \hat{g}' - \frac{1}{x^2} \hat{g} + \frac{1}{2x} \hat{f}^2 = -k_H x \quad (\text{III.7})$$

where

$$k_V = \frac{r_0^4}{D\sqrt{DA}} P_V, \quad k_H = \frac{r_0^3}{D} P_0 \quad (\text{III.8})$$

while the boundary conditions (II.27), (II.28), (II.30) and (II.31) assume the following forms:

$$\hat{g}(1) = 0, \quad (\text{III.9})$$

$$\hat{f}'(1) + \nu f(1) = 0, \quad (\text{III.10})$$

$$\hat{f}(x_i) = 0, \quad (\text{III.11})$$

$$\hat{g}'(x_i) - \frac{\nu}{x_i} \hat{g}(x_i) + \frac{k_H}{3+\nu} x_i^2 = 0. \quad (\text{III.12})$$

The dimensionless load parameters k_V and k_H will play an important role in the characterization of the solution of our BVP. If $k_V < 1$ and $k_H < 1$ the above dimensionless form of the BVP is appropriate for a perturbation solution. The special case with $k_V \ll 1$ and $k_H \ll 1$ will be referred to as low pressure and slowly spinning case; a linear theory is expected to be an adequate approximation for this case. Other possibilities of scalings have been worked out for $k_V \gg 1$ and/or $k_H \gg 1$. Briefly, the (k_V, k_H) plane is divided into four regions: Region (1.1) with $k_V^2 < k_H < 1$, Region (1.2) $k_H < k_V^2 < 1$, Region (2) with $k_H > 1$ and $k_V^{2/3} < k_H$, and Region (3) with $k_V > 1$ and $k_H < k_V^{2/3}$. The solution of the BVP exhibits different qualitative features in each of these regions. Accordingly, the method of solution will be different in different regions and will be discussed in separate sections of this chapter.

2. Small to Moderate Load Magnitude Range

In this section we consider the moderate load magnitude range with $k_V < 1$ and $k_H < 1$. For this range of k_V and k_H values, a regular perturbation solution of the BVP is appropriate. Without further restrictions on k_H and/or k_V , two parameter perturbation series in k_H and k_V for $\hat{f}$ and $\hat{g}$ would be necessary and are obtained in Appendix A. To have a better understanding of the qualitative behavior of the solution, we consider separately the two cases $k_H/k_V^2 < 1$ and $k_H/k_V^2 > 1$. A separate treatment of the two cases enables us to use a perturbation series in one parameter only.

Region (1.1): $k_V^2 < k_H < 1$.

For the special case $k_V/k_H \ll 1$ (including $k_V = 0$), we expect the solution to approach that of a rotating disk, [11]. Therefore, we set

$$\hat{g}(x) = k_H g(x), \quad \hat{f}(x) = k_V f(x) \quad (\text{III.13})$$

and write (III.6) and (III.7) as

$$f'' + \frac{1}{x} f' - \frac{1}{x^2} f - \frac{k_H}{x} fg = -\frac{1-x^2}{2x} \quad (\text{III.14})$$

$$g'' + \frac{1}{x} g' - \frac{1}{x^2} g + \frac{\mu_m}{2x} f^2 = -x \quad (\text{III.15})$$

where $k_H < 1$ and $\mu_m = k_V^2/k_H < 1$. Correspondingly, the three boundary conditions (III.9)-(III.11) retain their form with $\hat{f}$ and $\hat{g}$ replaced by f and g while (III.12) becomes

$$g'(x_i) - \frac{v}{x_i} g(x_i) + \frac{x_i^2}{3+v} = 0 \quad (\text{III.16})$$

For $\mu_m < 1$ ($k_V^2 < k_H$), we may seek a perturbation series solution in powers of μ_m with k_H as a parameter

$$\{f, g\} = \sum_{n=0}^{\infty} \{f_n(x; k_H), g_n(x; k_H)\} \mu_m^n. \quad (\text{III.17})$$

The leading term solutions, g_0 and f_0 , are determined by two uncoupled linear BVP

$$\left\{ \begin{array}{l} g_0'' + \frac{1}{x} g_0' - \frac{1}{x^2} g_0 = -x \quad (x_i < x < 1) \\ g_0(1) = g_0'(x_i) - \frac{v}{x_i} g_0(x_i) + \frac{x_i^2}{3+v} = 0 \end{array} \right. \quad (\text{III.18})$$

$$\left\{ \begin{array}{l} f_0'' + \frac{1}{x} f_0' - \frac{1}{x^2} f_0 - \frac{k_H}{x} f_0 g_0 = -\frac{1-x^2}{2x} \quad (x_i < x < 1) \\ f_0(x_i) = f_0'(1) + v f_0(1) = 0. \end{array} \right. \quad (\text{III.19})$$

We first solve (III.18) for g_0 and then use the result in (III.19) to get a second order linear ODE for f_0 . The BVP (III.18) is identical to the well known problem of a rotating disk free at the outer edge and clamped to a rigid hub at the inner edge*. If $k_H \ll 1$, the contribution of the $f_0 g_0$ term in (III.19) is negligible. Thus, to leading order, the bending and stretching actions of the plate are completely uncoupled for $\mu_m = k_V^2/k_H \ll 1$ and $k_H \ll 1$.

*The solution of this problem is given later in (III.36)

For moderately small values of μ_m (and k_H not small compared to unity), higher order correction terms in (III.17) will have to be calculated. It is not difficult to see that these higher order terms are also determined (sequentially) by linear boundary value problems. Higher order corrections will be analyzed in more detail by the two-parameter perturbation series in Appendix A.

If $k_H \ll (\mu_m <) 1$, we may take instead of (III.17) a perturbation solution in powers of k_H (keeping μ_m as a parameter):

$$\{f, g\} = \sum_{n=0}^{\infty} \{F_n(x; \mu_m), G_n(x; \mu_m)\} k_H^n. \quad (\text{III.20})$$

The leading term solutions F_0 and G_0 in that case are also determined by two uncoupled linear BVP:

$$\left\{ \begin{array}{l} F_0'' + \frac{1}{x} F_0' - \frac{1}{x^2} F_0 = -\frac{1-x^2}{2x} \\ F_0(x_1) = F_0'(1) + \nu F_0(1) = 0 \end{array} \right. \quad (\text{III.21})$$

$$\left\{ \begin{array}{l} G_0'' + \frac{1}{x} G_0' - \frac{1}{x^2} G_0 + \frac{\mu_m}{2x^2} F_0^2 = -x \\ G_0(1) = G_0'(x_1) - \frac{\nu}{x_1} G_0(x_1) + \frac{x_1^2}{3+\nu} = 0. \end{array} \right. \quad (\text{III.22})$$

We first solve (III.21) for F_0 and then use the result in (III.22) to get a second order linear ODE for G_0 . The BVP (III.21) is just the classical problem of axisymmetric bending by uniform pressure of a flat circular plate.

clamped at the inner edge and free at the outer edge. If, we have (in addition to $k_H \ll 1$) $\mu_m = k_V^2/k_H \ll 1$ also, the contribution of the F_0^2 term in (III.22) is negligible; the bending and stretching plate action uncouple as we pointed out previously.

For moderately small values of k_H (and μ_m not small compared to unity), higher order correction terms in (III.20) will have to be calculated. It is not difficult to see that they are also determined (sequentially) by linear boundary problems. Higher order correction terms are also discussed in Appendix A in the framework of a two-parameter perturbation solution.

Region (1.2): $k_H < k_V^2 < 1$.

For the special $k_H \ll k_V^2 < 1$, (including the limiting case $k_H = 0$ and $k_V \neq 0$), we expect the solution to be approximately that for bending by uniform pressure [12]. Therefore we set

$$\hat{f} = k_V f(x), \quad \hat{g} = k_V^2 g(x) \quad (\text{III.23})$$

and write the ODE (III.6) and (III.7) as

$$f'' + \frac{1}{x} f' - \frac{1}{x^2} f - \frac{k_V^2}{x} fg = - \frac{1-x^2}{2x} \quad (\text{III.24})$$

$$g'' + \frac{1}{x} g' - \frac{1}{x^2} g + \frac{1}{2x} f^2 = - \frac{1}{\mu_m} x \quad (\text{III.25})$$

where $k_V^2 < 1$ and $\mu_m^{-1} = k_H/k_V^2 < 1$. Correspondingly, three boundary conditions (III.9)-(III.11) retain their form with $\hat{f}$ and $\hat{g}$ replaced by f and g while (III.12) becomes

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + \frac{x_i^2}{\mu_m(3+\nu)} = 0. \quad (\text{III.26})$$

With $k_V^2 < 1$, we may seek a perturbation series solution in powers of k_V^2 . The leading term solutions, again denoted by f_0 and g_0 , are determined by two uncoupled linear BVP:

$$\begin{cases} f_0'' + \frac{1}{x} f_0' - \frac{1}{x^2} f_0 = -\frac{1-x^2}{2x}, \\ f_0(x_i) = f_0'(1) + \nu f_0(1) = 0, \end{cases} \quad (\text{III.27})$$

$$\begin{cases} g_0'' + \frac{1}{x} g_0' - \frac{1}{x^2} g_0 = -\frac{1}{2x} f_0^2 - \frac{1}{\mu_m} x \\ g_0(1) = g_0'(x_i) - \frac{\nu}{x_i} g_0(x_i) + \frac{1}{\mu_m} \frac{x_i^2}{(3+\nu)} = 0. \end{cases} \quad (\text{III.28})$$

We first solve (III.27) for f_0 and then use the result in (III.28) to get a linear BVP for g_0 . The BVP (III.27) is just the problem for the linear bending by uniform pressure of an annular plate clamped at the inner edge to a rigid hub and free of traction at the outer edge. Whether or not we have $k_H \ll k_V^2$ ($\mu_m^{-1} \ll 1$), the bending and stretching action of the plate are always weakly coupled as (III.28) gives a nontrivial solution even for the limiting case of a non-rotating plate with $k_H = 0$ (so that $\mu_m^{-1} = 0$). This is in contrast to the plate behavior in region (1.1). Note that higher order terms in the perturbation series are also determined by linear BVP and are discussed in Appendix A. The special case of $k_H = 0$ has been solved in [12] by a power series solution in x with no restriction on k_V .

3. The Relatively Fast Spinning Range.

In region (2) where $k_H > 1$ and $k_V^2 < k_H^3$, the plate is expected to experience significant in-plane extension throughout. This suggests that we let

$$\hat{g}(x) = k_H g(x), \quad \hat{f}(x) = \frac{k_V}{k_H} f(x) \quad (\text{III.29})$$

and write (III.6) and (III.77) as

$$\frac{1}{k_H} \left[f'' + \frac{1}{x} f' - \frac{1}{x^2} f \right] - \frac{1}{x} f g = - \frac{1-x^2}{2x} \quad (\text{III.30})$$

$$g'' + \frac{1}{x} g' - \frac{1}{x^2} g + \frac{\mu_\ell^3}{2x} f^2 = -x \quad (\text{III.31})$$

where

$$\mu_\ell^3 = \frac{k_V^2}{k_H^3}. \quad (\text{III.32})$$

Three of the four boundary conditions (III.9)-(III.11) retain the same form with $\hat{f}$ and $\hat{g}$ replaced by f and g . The fourth (III.12) becomes identical (in form) to (III.16).

With $\mu_\ell < 1$, a perturbation solution for f and g in powers of μ_ℓ^3 is appropriate. The leading term solutions f_0 and g_0 satisfy the ODE

$$\frac{1}{k_H} \left[f_0'' + \frac{1}{x} f_0' - \frac{1}{x^2} f_0 \right] - \frac{1}{x} f_0 g_0 = - \frac{1-x^2}{2x} \quad (\text{III.33})$$

$$g_0'' + \frac{1}{x} g_0' - \frac{1}{x^2} g_0 = -x \quad (\text{III.34})$$

and the same boundary conditions as the original problem. The simplification of (III.31) for the leading term approximation is significant as g_0 may now be determined separately. With the boundary conditions,

$$g_0(1) = 0, \quad g_0'(x_1) - \frac{\nu}{x_1} g_0(x_1) + \frac{x_1^2}{3+\nu} = 0, \quad (\text{III.35})$$

we have

$$g_0(x) = A(x - \frac{1}{x}) + \frac{1}{8}(\frac{1}{x} - x^3), \quad A = \frac{1+\nu}{8(3+\nu)} \frac{(3+\nu)+(1-\nu)x_1^4}{(1+\nu)+(1-\nu)x_1^2}. \quad (\text{III.36})$$

Upon substituting (III.36) for $g_0(x)$, (III.33) becomes a linear ODE for f_0

$$f_0'' + \frac{1}{x} f_0' - \left[\frac{1-Ak_H}{x^2} + k_H(A + \frac{1}{8}) - \frac{k_H}{8} x^2 \right] f_0 = -k_H \frac{1-x^2}{2x}. \quad (\text{III.37})$$

The two boundary conditions for f_0 are

$$f_0(x_1) = 0, \quad f_0'(1) + \nu f_0(1) = 0. \quad (\text{III.38})$$

The solution for this linear BVP can be obtained by asymptotic or numerical methods.

For moderately small values of μ_{ℓ} , higher order terms in the perturbation series in μ_{ℓ}^3 may be necessary for a more accurate solution. These correction terms are also determined sequentially by linear boundary value problems similar to those for the leading term solution. In particular, we have for the coefficients μ_{ℓ}^{3n}

$$g_n'' + \frac{1}{x} g_n' - \frac{1}{x^2} g_n = -\frac{1}{2x} \sum_{k=0}^{n-1} f_k f_{n-1-k} \quad (\text{III.39})$$

$$g_n(1) = 0, \quad g_n'(x_1) - \frac{\nu}{x_1} g_n(x_1) = 0$$

and

$$f_n'' + \frac{1}{x} f_n' - \left[\frac{1}{x^2} + \frac{k_H}{x} g_0(x) \right] f_n = \frac{k_H}{x} \sum_{k=0}^{n-1} f_k g_{n-k} \quad (\text{III.40})$$

$$f_n(x_1) = 0, \quad f_n'(1) + \nu f_n(1) = 0.$$

Thus, the restriction of fast spinning, $\mu_\ell < 1$ ($k_V^2 < k_H^3$), reduces the nonlinear problem to a sequence of linear problems with the bending and stretching action of the plate weakly coupled through the forcing terms in the governing (linear) differential equations.

For $k_H \gg 1$, an asymptotic solution of the linear BVP for f_0 is given in Appendix A. Except for a particular solution, the same asymptotic solution is also applicable to f_n . Even without an explicit solution, it is clear from the form of the BVP (III.37)-(III.38) that the (dimensionless) layer width near $x = x_1$ is $O(k_H^{-1/2})$ while, because $g_0(1) = 0$, the (dimensionless) layer width near the turning point $x = 1$ is $O(k_H^{-1/3})$. Furthermore, the layer strength is considerably weaker near the outer edge.

4. The Relatively High Pressure Range

In Region (3) where $k_V > 1$ and $k_H^3 < k_V^2$, the plate is expected to experience significant bending throughout. This suggests that we let

$$\hat{g}(x) = k_V^{2/3} g(x), \quad \hat{f}(x) = k_V^{1/3} f(x) \quad (\text{III.42})$$

and write (III.6) and (III.7) as

$$k_V^{-2/3} \left[f'' + \frac{1}{x} f' - \frac{1}{x^2} f \right] - \frac{1}{x} fg = - \frac{1-x^2}{2x} \quad (\text{III.43})$$

$$g'' + \frac{1}{x} g' - \frac{1}{x^2} g + \frac{1}{2x} f^2 = - \frac{x}{\mu_\ell} \quad (\text{III.44})$$

where μ_ℓ is as given in (III.32). The three boundary conditions (III.9)-(III.11) retain the same form with $\hat{f}$ and $\hat{g}$ replaced by f and g . The fourth, (III.12), becomes

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + \frac{x_i^2}{\mu_\ell(3+\nu)} = 0. \quad (\text{III.45})$$

With $\mu_\ell > 1$, a regular perturbation solution in powers of μ_ℓ^{-1} is possible. The leading term f_0 and g_0 of such the perturbation series are determined by the differential equations and boundary conditions for f and g but with terms multiplied by $1/\mu_\ell$ replaced by zero. Evidently the BVP for f_0 and g_0 remains a nonlinear problem and an elementary solution is not possible.

On the other hand, for extremely high pressure loading so that $k_V^{2/3} \gg 1$, a solution by the method of matched asymptotic expansions is appropriate for all $\mu_\ell \geq 1$. The leading term outer (asymptotic expansion) solutions, denoted by $F_0(x)$ and $G_0(x)$, are determined by

$$F_0 = \frac{1-x^2}{2G_0}, \quad G_0'' + \frac{1}{x} G_0' - \frac{1}{x^2} G_0 + \frac{(1-x^2)^2}{8xG_0^2} = - \frac{x}{\mu_\ell} \quad (\text{III.46})$$

and the boundary conditions $G_0(1) = 0$ and (III.45) with g replaced by G_0 . The two equations in (III.46) are just the governing differential equations for the Foppl-Hencky nonlinear membrane theory [9,10] (in dimensionless form) used in [1]. Methods of solution for these equations have been discussed there as well as in [15-21]. Higher order correction terms of the outer solution may also be obtained by a regular perturbation series solution in powers of $k_v^{-2/3}$. Evidently, the outer solution can only satisfy two of the four boundary conditions as the governing differential equations for the n -th order correction terms G_n and F_n constitute only a second order system. It is therefore necessary to obtain an inner (asymptotic expansion) solution valid in the neighborhood of each edge which satisfies all four boundary conditions on that edge. The actual asymptotic solution to the original problem is then obtained by matching the outer and inner solutions in some overlapping (intermediate) region of validity and forming a uniformly valid composite solution.

In general, the outer solution G_0 cannot be obtained in terms of elementary or special functions. The matched asymptotic expansion technique nevertheless provides us with many qualitative features (such as the order of magnitude of $\hat{f}$ and $\hat{g}$ in various regions of the solution domain, the width of the boundary layers adjacent to each edge, etc.) of the exact solution of the original BVP very useful in obtaining numerical solutions for that problem. For the present problem, we can do more. Even without an explicit solution for G_0 and F_0 , it is possible to show (see Appendix A) that G_0 is in fact the dominant term in the exact solution for g of our problem throughout the plate. The nonlinear membrane solution thus provides an accurate approximate solution for the stretching actions of the plate

whenever $k_V^{2/3}$ is large compared to unity. We will return to this point later in Section (7).

5. Direct and Bending Stresses

In structural designs, it is often important to know whether the bending or stretching action of the plate is dominant. Therefore, the relative magnitude of the direct (membrane) and bending stresses, σ_D and σ_B , in the plate for a particular loading is of interest. The order of magnitude of σ_D and σ_B , given by

$$\sigma_D = \frac{N_\theta}{h}$$

$$\sigma_B = \frac{6M_\theta}{h^2},$$

generally varies from region to region in the (k_V, k_H) plane.

In region (1.1) where $k_H < 1$ and $k_V^2/k_H < 1$, we have

$$\phi = \bar{\phi} \hat{f} = \frac{\sqrt{DA}}{r_0} [k_V f(x)]$$

$$\psi = \bar{\psi} \hat{g} = \frac{D}{r_0} [k_H g(x)]$$

and correspondingly

$$N_\theta = \frac{d\psi}{dr} + r p_H = \frac{\bar{\psi}}{r_0} k_H \left(g' + \frac{x^2}{3+\nu} \right)$$

$$M_\theta = -D \left(\nu \frac{d\phi}{dr} + \frac{1}{r} \phi \right) = -\frac{D\bar{\phi}}{r_0} k_V \left(\nu f' + \frac{1}{x} f \right).$$

As differentiation with respect to x does not change order of magnitude, we have

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{6\sqrt{DA}}{h} \frac{k_V}{k_H}\right) = O\left(\frac{k_V}{k_H}\right).$$

Therefore, bending action dominates if $k_V \gg k_H$ and stretching action dominates if $k_V \ll k_H$.

In region (1.2) where $k_V < 1$ and $k_H/k_V^2 \leq 1$, we have

$$\phi = \bar{\phi} \hat{f} = \frac{\sqrt{DA}}{r_0} [k_V f(x)]$$

$$\psi = \bar{\psi} \hat{g} = \frac{D}{r_0} [k_V^2 g(x)]$$

and correspondingly

$$M_\theta = -\frac{D\sqrt{DA}}{2r_0} k_V \left(\nu f' + \frac{1}{x} f \right), \quad N_\theta = \frac{D}{2r_0} k_V^2 \left[g' + \frac{x^2}{\mu_m(3+\nu)} \right].$$

As there are no layer phenomena and $\mu_m \geq 1$, we have

$$\frac{\sigma_B}{\sigma_D} = \frac{6M_\theta/h^2}{N_\theta/h} = O\left(\frac{6\sqrt{DA}}{hk_V}\right) = O\left(\frac{1}{k_V}\right)$$

so that σ_D is $O(\sigma_B)$ at most and is considerably smaller if $k_V \ll 1$.

In region (2) where $k_H \geq 1$ and $k_V^2 < k_H^3$, we have

$$\phi = \bar{\phi} \hat{f} = \frac{\sqrt{DA}}{r_0} \left[\frac{k_V}{k_H} f(x) \right], \quad \psi = \bar{\psi} \hat{g} = \frac{D}{r_0} [k_H g(x)]$$

and correspondingly,

$$M_\theta = -\frac{D\sqrt{DA}}{2r_0} \frac{k_V}{k_H} \left(\nu f' + \frac{1}{x} f \right), \quad N_\theta = \frac{D}{2r_0} k_H \left(g' + \frac{x^2}{3+\nu} \right).$$

In the interior of the plate, we have then

$$\frac{\sigma_B}{\sigma_D} = 0 \left(\frac{\sqrt{DA}}{h} \frac{k_V}{k_H^2} \right) = 0 \left(\frac{k_V}{k_H^2} \right) = 0 \left(\frac{k_V}{k_H^{3/2}} \frac{1}{k_H^{1/2}} \right);$$

therefore, σ_D is at least of the same magnitude as σ_B ; it is considerably larger if either $k_H \gg 1$ or $k_V^2 \ll k_H^3$. Near the edges however, f_0 exhibits a boundary layer phenomenon if $k_H \gg 1$ so that we have $f' = 0(\sqrt{k_H} f)$ at most while g' remains $O(g)$. Therefore, we have near an edge

$$\frac{\sigma_B}{\sigma_D} = 0 \left(\frac{k_V}{k_H^{1/2}} \right) = 0 \left(k_H \left[\frac{k_V^2}{k_H^3} \right] \right) = 0 \left(k_V^{2/3} \left[\frac{k_V^2}{k_H^3} \right]^{1/6} \right);$$

either σ_D or σ_B may be dominant depending on the magnitude of the ratio $k_V/k_H^{1/2}$.

Finally in region (3) where $k_V \geq 1$ and $k_H^3 < k_V^2$, we have

$$\phi = \bar{\phi} \hat{f} = \frac{\sqrt{DA}}{r_0} [k_V^{1/3} f(x)], \quad \psi = \bar{\psi} \hat{g} = \frac{D}{r_0} [k_V^{2/3} g(x)]$$

and correspondingly

$$M_\theta = - \frac{D\sqrt{DA}}{r_0^2} k_V^{1/3} [v f' + \frac{1}{x} f], \quad N_\theta = \frac{D}{r_0^2} k_V^{2/3} [g' + \frac{x^2}{\mu_\ell(3+v)}].$$

With $\mu_\ell > 1$, we get

$$\frac{\sigma_B}{\sigma_D} = 0 \left(\frac{\sqrt{DA}}{h} \frac{1}{k_V^{1/3}} \right) = 0(k_V^{-1/3})$$

so that σ_B is $O(\sigma_D)$ at most and is significantly smaller in the plate

interior if $k_V \gg 1$. Evidently, in this range of parameter values, the plate resists the predominantly transverse loading by its membrane action. Near the edges, f_0 and g_0 both exhibit a boundary layer phenomenon if $k_V \gg 1$. From the results of section (4), we have near the inner edge

$$f' = O(k_V^{1/3}), \quad g' = O(1)$$

so that

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{\sqrt{DA}}{h} k_V^{-1} \frac{f'}{g'}\right) = O(1).$$

Thus, for $k_V \gg 1$, the bending and direct stresses of the plate are of the same order of magnitude near $x = x_1$. Similarly, σ_B is much smaller than σ_D near $x = 1$ as well as in the plate interior.

6. Comparison with Finite Difference Solution

A FORTRAN computer program has been developed to calculate finite difference solutions of the dimensionless two-point BVP defined by (III.6), (III.7), (III.9)-(III.12). For a given set of input values of the parameters

ν = Poisson's ratio

$x_i = r_i/r_0$, the inner-to-outer hole radius ratio

$k_V = p_V r_0^4 / D \sqrt{DA}$, the dimensionless pressure load magnitude

$k_H = p_0 r_0^3 / D \equiv (3+\nu) \rho h \omega^2 r_0^4 / D$, the dimensionless radial load magnitude,

the computer program obtains $\hat{f}(x)$ and $\hat{g}(x)$ to a specified accuracy and then calculates the corresponding distributions of stress resultants, stress couples and displacement components from the auxiliary formulas, (II.15)-(II.18), (II.32) and $u = r \epsilon_\theta$. The actual methods used in the various steps toward a finite difference solution are described in Appendix B.

Numerical solutions for a number of check cases have been generated by the above computer code to validate the code; these include cases examined in [2] and [22]. For a typical case, the interval $[x_i, 1]$ is usually divided into 320 subintervals. (With $x_i = 0.1$, the subinterval length is 0.0028125.) The local truncation error associated with finite difference scheme is $O(h^2) \leq 10^{-5}$. The computation typically begins with a coarse mesh using some limiting solution as an initial guess. After convergence for the given mesh, the program proceeds to a finer mesh (usually obtained by doubling the mesh points) with (the linear interpolation of) the last solution as an initial guess; the process continues until the

mesh contains 320 subintervals*. At every step, solution is substituted back into the corresponding finite difference equations to compute the maximum (residual) error. The process terminates when the maximum-absolute residual error is below the specified tolerance, which is taken to be less than 10^{-8} ; the solution is said to have converged for the given mesh.

The above computer code has been used to generate numerical solutions for a large number of new cases to confirm the theoretical conclusions of sections (2)-(4). A small selection of the numerical solutions will be presented below to bring out the qualitative features of the solutions in different regions of the k_H, k_V -parameter plane. The selection usually consists of (i) $\hat{f}(1) = r_0 \phi(r_0)/\sqrt{DA}$, (ii) $\hat{m}_r(x_1) = \hat{f}'(x_1) + \nu \hat{f}(x_1)/x_1$, (iii) $\hat{g}(x_1) = r_0 \psi(r_1)/D$, (iv) $\hat{n}_\theta^m = \max [\hat{g}' + k_H x^2/(3+\nu)]$. The first two quantities give typical features of the out-of-plane bending actions of the plate while the last two give typical in-plane stretching plate actions. The two stress measures $\hat{n}_\theta^m$ and $\hat{m}_r(x_1)$ give the maximum direct stress and the maximum bending stress, respectively; they are important in design studies. In all cases reported herein, ν is taken to be 0.3. All numerical results given are accurate at least to the number of digits displayed, with n_θ^m being read from computer output for discrete values of x .

Region (1.1): $k_V^2 < k_H < 1$

From the results for $k_H = 10^0, 10^{-1}, 10^{-2}$ and 10^{-4} and for $\mu_m \equiv k_V^2/k_H = 10^0, 10^{-1}, 10^{-2}, 10^{-3}$ and 0 given in Table (1), we make the following observations in support of the asymptotic results of section (2) for this region of the parameter plane:

*The mesh size would be halved again only if the solutions for the last two meshes still differ by more than a prescribed tolerance.

- (1) For a fixed value of μ_m , numerical solutions for $\hat{g}(x_1)$ and $\hat{n}_\theta^m$ are effectively proportional to k_H .
- (2) For a fixed value of μ_m , numerical solutions for $\hat{f}(1)$ and $\hat{m}_r(x_1)$ are effectively proportional to $\sqrt{k_H}$ ($= k_V/\sqrt{\mu_m}$) and therefore to k_V .
- (3) For a fixed k_H , $\hat{g}(x_1)/k_H$, $\hat{n}_\theta^m/k_H$, $\hat{f}(1)/k_V$ and $\hat{m}_r(x_1)/k_V$ do not change appreciably as μ_m decreases and hardly change at all for $\mu_m \ll 1$.
- (4) As $\mu_m \rightarrow 0$, $\hat{g}(x_1)$, $\hat{n}_\theta^m$, $\hat{f}(1)$ and $\hat{m}_r(x_1)$ tend to the corresponding values for the limiting case of no pressure load (which is the leading term of a perturbation solution in μ_m); in particular $\hat{f}(1)$ and $\hat{m}_r(x_1)$ tend to zero as $\mu_m \rightarrow 0$.

Thus, the accurate numerical solutions confirm the fact that a truncated perturbation series solution is expected to be adequate for $\mu_m < 1$.

Region (1.2): $k_H < k_V^2 < 1$

From the results for $k_V = 10^0, 10^{-1}$ and 10^{-2} and for $\mu_m = 10^0, 10^1, 10^2, 10^3, 10^4$ and 10^6 given in Table (2), we make the following observations in support of the asymptotic results of section (2) for Region (1.2) in the parameter plane:

- (1) For a fixed value of μ_m , numerical solutions for $\hat{f}(1)$ and $\hat{m}_r(x_1)$ are effectively proportional to k_V while $\hat{g}(x_1)$ and $\hat{n}_\theta^m$ are effectively proportional to k_V^2 .

- (2) For a fixed value of k_V , the four quantities do not change appreciably for large values of μ_m and all tend to a limit as $\mu_m \rightarrow \infty$ corresponding to the case of no spinning, i.e. $k_H = 0$.
- (3) For μ_m sufficiently large (with the critical value of μ_m depending on k_V), compressive circumferential stresses appear at and adjacent to the outer edge of the plate. For very large k_V , the critical value of μ_m coincides with the corresponding critical value for the nonlinear membrane solution observed in [1].

It appears that for small values of μ_m^{-1} , the (truncated) perturbation solution proposed in section (2) for Region (1.2) gives an accurate approximate solution for the problem.

Region (2): $k_V^2 < k_H^3$, $k_H \geq 1$

From the numerical results for $k_H = 10^0, 10^1, 10^2, 10^3, 10^4$ and 10^5 and for $\mu_\ell \equiv k_V^{2/3}/k_H = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ and 0 given in Table (3), we make the following observations in support of the asymptotic results of section (3) for this region in the parameter plane:

- (1) For a fixed value of $\mu_\ell < 1$, numerical solutions for $\hat{g}(x_1)$ and $\hat{n}_\theta^m$ are effectively proportional to k_H as predicted by the asymptotic solution.
- (2) For a fixed value of $\mu_\ell < 1$, the numerical solutions for $\hat{f}(1)$ are effectively proportional to $\mu_\ell^{3/2} \sqrt{k_H} = k_V/k_H$ for $k_H > 10^2$ and to $\mu_\ell^{3/2} k_H^{3/2} = k_V$ for $k_H < 10^2$. This is due to the fact that $g(x)$ is numerically small, of order 10^{-2} near x_1 , so that we have $f = 0(k_H)$ for $k_H = 0(10^2)$ or smaller.

- (3) For a fixed value of k_H , $\hat{g}(x_i)$, $\hat{n}_\theta^m$, $\hat{f}(1)/(k_V/k_H)$ and $\hat{m}_r(x_i)/(k_V/k_H)$ do not change appreciably with μ_ℓ as μ_ℓ decreases to zero. In the limit as $\mu_\ell \rightarrow 0$, it is expected that $\hat{f}$ and $\hat{m}_r$ tend to zero and $\hat{g}(x_i)$ and $\hat{n}_\theta^m$ tend to the rotating disc solution and they do.
- (4) For $k_H \gg 1$, the (dimensionless) boundary layer width (near x_i) is seen from Table (4) to be $O(k_H^{-1/2})$ as predicted by the asymptotic analysis. For a measure of the layer width, we use the dimensionless distance $\Delta x = \bar{x} - x_i$ with $\bar{x}$ being the location where $\hat{m}_r$ decreases to one tenth of its value at x_i . Table (4) gives Δx for a range of values of k_H and for $\mu_\ell = 1$ and 10^{-2} taken from our numerical solutions of the BVP for f and g . The layer width near $x = 1$ has also been verified to be $O(k_H^{-1/3})$.
- (5) For $k_H \gg 1$ and $\mu_\ell \ll 1$, the magnitudes of $\hat{f}$ and $\hat{g}$ are approximately those of the linear membrane solution, i.e. the leading term outer solution of our BVP; the layer solution near x_i is needed only to make $\hat{f}(x_i) = 0$. The magnitude of $\hat{m}_r$ in the layer adjacent to x_i is seen from the numerical solutions to be larger than the outer solution by a factor proportional to $k_H^{1/2}$ as predicted by the asymptotic analysis. On the other hand, the strength of the layer solution adjacent to the outer edge of the plate is weak and has very little effect on $\hat{f}$ and $\hat{g}$; its main function is to ensure $\hat{m}_r(1) = 0$. This is again consistent with the behavior of the asymptotic solution in the neighborhood of the turning point obtained in section (2) of Appendix A.

The above observations confirm that a truncated perturbation solution in μ_ℓ is adequate in Region (2). In fact, the numerical solution for $k_V = k_H = 10^{-5}$ agrees with the leading term perturbation solution to within 2%.

Region (3): $k_V^{2/3}/k_H \geq 1, k_V > 1$

From the numerical results for $k_V = 10^0, 10^3, 10^6$ and 10^9 and for $\mu_\ell = 10^0, 10^1, 10^2, 10^3$ and 10^4 given in Table (5), we make the following observations in support of the asymptotic results of section (4) for this region of the parameter plane:

- (1) For a fixed value of μ_ℓ , $\hat{g}(x_1)$ and $\hat{n}_\theta^m$ are effectively proportional to $k_V^{2/3}$ while $\hat{f}(1)$ and $\hat{m}_r(x_1)$ are effectively proportional to $k_V^{1/3}$ when k_V is sufficiently large compared to unity.
- (2) For a fixed value of k_V , the same four quantities (and in fact the solution of the BVP itself) do not change appreciably for $\mu_\ell > 1$. The larger is the value of μ_ℓ , the sooner they reach their respective limiting values.
- (3) From a table similar to Table (4) for Region (2), the width of the boundary layer adjacent to $x = x_1$ is seen to be proportional to $k_V^{-1/3}$ for sufficiently large k_V . (We will not display the table for Δx here.) The layer width near $x = 1$ is seen to be proportional to $k_V^{-1/6}$.
- (4) For $\hat{g}$, the strength of these layers are weak compared to the nonlinear membrane component of the solution. For $k_V = 10^{-5}$, the numerical solution for $g = \hat{g}/k_V^{2/3}$ differs from the

corresponding nonlinear membrane solution by less than 0.2% for $\mu_\ell \geq 1$.

- (5) For $\hat{f}$, the strength of the layer near x_1 is comparable to the nonlinear membrane component of the solution. The strength of the layer near the outer edge is considerably weaker than the nonlinear membrane component. In fact, outside the layer, near $x = x_1$, the numerical solution for $\hat{f}/k_V^{1/3} = f$ differs from the nonlinear membrane solution by less than 0.4% for $k_V = 10^5$ and $\mu_\ell \geq 1$, the layer region adjacent to the outer edge included.

The above observations confirm that a truncated perturbation solution in μ_ℓ^{-1} should be adequate in Region (3).

7. On Nonlinear Membrane Theory

Because of its mathematical simplicity, the Foppl-Hencky nonlinear membrane theory has been widely used in engineering analysis of sheetlike elastic bodies. Of interest here is the use of this theory in the analysis of a steadily spinning disk under a uniform normal pressure distribution [1]. With recent advances in both nonlinear theories of plates and shells and in methods for asymptotic solutions of differential equations, we now see that the Foppl-Hencky theory may be considered as the outer asymptotic expansion of the exact solution of the von Karman plate theory (see section 4). Physically, the Foppl-Hencky theory is an approximation of the von Karman theory neglecting in the latter the effect of the small bending stiffness. In this section, we examine the range of applicability of the Foppl-Hencky theory taking advantage of the asymptotic results obtained in sections (3) and (4) of this chapter.

It is evident from our asymptotic results that even within the framework of a moderately small rotation theory, a nonlinear membrane theory is appropriate only in the relatively high pressure range, i.e., $k_H^3 < k_V^2$, or in the notation of [1]

$$\tilde{k} = 16 k_V^2 / h_H^3 > 16.$$

It is found in [1] that compressive circumferential stresses appear for $\tilde{k} = \tilde{k}_{wp} = 0.3154$ in the limiting case of a vanishing small hole ($x_i \rightarrow 0$) and for smaller $\tilde{k}$ if $x_i > 0$. Now, compressive stresses induce wrinkling in the elastic membrane and are therefore unacceptable for various design purposes. However, these same compressive stresses pose no design problems when the

membrane theory which generates them constitutes only the outer asymptotic expansion of the exact solution for the von Karman plate theory. The sheet does have a finite bending stiffness, however small, which would induce bending stresses whenever necessary to inhibit the occurrence of wrinkles. In particular, significant bending stresses develop in a region adjacent to a clamped edge of the annular plate where the outer solution ceases to be valid. A different type of asymptotic solution, which incorporates the significant effect of the small bending stiffness, is appropriate in the edge zone. Thus, in the framework of a matched asymptotic expansion solution, the results in [1] obtained from Foppl-Hencky nonlinear membrane theory continues to be useful whether or not the resulting membrane stresses are compressive.

For $\tilde{k} = 16k_V^2/k_H^3 < 16$ and $k_H > 1^*$, we know from section (3) that a "quasilinear" plate theory resulting from a regular perturbation solution in powers of (h_V^2/h_H^3) for the von Karman equations is appropriate for our problem. As we found in section (3), this theory generates membrane stresses by a (quasi) linear membrane theory (III.34)-(III.35). The bending action of the plate is then determined by a linear bending theory (III.37)-(III.38) which incorporates the effect of the membrane solution just obtained. It has been argued [1] that a nonlinear membrane theory should be used as it leads to finite transverse deflection throughout the sheet while the (quasi) linear membrane theory gives an unbounded transverse deflection at the center of an annular plate with an infinitesimally small hole. With the

*It is clear that for $k_H < 1$ (and therefore $k_V < 1$ also), a suitable linearized theory suffices as we are in region (1.1).

asymptotic results of section (3), we now see that a (quasi) linear membrane theory is in fact appropriate for the determination of the membrane stresses in this range of parameter values. However, it should be supplemented by a (quasi) linear bending theory for the bending plate action, with $\phi \sim \bar{\phi}(k_V/k_H)f_0$ determined by (III.37)-(III.38), resulting in a bounded transverse deflection everywhere. On the other hand, the use of a nonlinear membrane theory without a matching inner solution also may not give an adequate approximation of actual plate bending behavior, even though it gives a finite transverse deflection and an accurate approximation for the stretching action throughout the sheet.

IV. Shallow Shells

1. Introduction

When $\xi \neq 0$ so that the shell is not a flat plate, we may again use the dimensionless quantities introduced in (III.1) along with

$$s(x) = \frac{\xi(r)}{\xi_0}, \quad k_G = \frac{\xi_0 r_0}{\sqrt{DA}} = \sqrt{12(1-\nu^2)} \frac{\xi_0 r_0}{h}, \quad (\text{IV.1})$$

where ξ_0 is chosen so that $\max |s(x)| = 1$, and tentatively write (II.25) and (II.26) as

$$L[\hat{f}] - \frac{1}{x} [k_G s(x) + \hat{f}] \hat{g} = -k_V \frac{1-x^2}{2x} \quad (\text{IV.2})$$

$$L[\hat{g}] + \frac{1}{x} [k_G s(x) + \frac{1}{2} \hat{f}] \hat{f} = -k_H x$$

with k_V and k_H given by (III.8) and

$$L[] \equiv []'' + \frac{1}{x} []' - \frac{1}{x^2} []. \quad (\text{IV.3})$$

In terms of $\hat{f}$ and $\hat{g}$, the relevant boundary conditions for the shell case have the same form as those for the plate case given by (III.9)-(III.12). With $k_G = 0$, (IV.2) and (IV.3) reduce to the corresponding plate equations (III.6) and (III.7). The appearance of the $k_G s(x)$ term in (IV.2) and (IV.3) introduces an unusual degree of complexity which makes the analysis of the solution of our BVP considerably more involved..

Consider for example the extreme case when the shell is subject only to normal pressure loading. We know from the results in [13,14,22,23] that "polar dimpling" may occur for some range of k_G values. That is, depending on the relative magnitude of k_G , k_V and k_H , the solution of our BVP at a given location may be either $f \cong 0$ or $f \approx -2k_G s(x)$ and that an internal layer phenomenon may exist at a location determined by the load magnitude. The occurrence of polar dimpling depends on the interplay among k_G , k_V and k_H in a complicated way which is not at all obvious from the analyses for the plate case.

To deduce the correct shell behavior corresponding to different combinations of load and geometric parameter values, the (k_H, k_V, k_G) parameter space will be divided into three principal regions (Fig. 8). In the next few sections, we will work out the appropriate re-scalings of the BVP (IV.2), (IV.3), (III.9)-(III.12) in the various subregions of these principal regions and discuss the behavior of the shell implied by these re-scalings.

2. The Light to Moderate Load Magnitude Range

We consider in this section the load magnitude range $0 \leq k_V < 1$ and $0 \leq k_H < 1$, designated as light to moderate load magnitude range in Chapter III. In the load-geometric parameter space, this range spans the unit square column with $0 \leq k_G < \infty$ and will be designated as Region I of the (k_V, k_H, k_G) space. We subdivide this region into three subregions: Region (I.1) with $k_G < k_V$, Region (I.2) $k_V \leq k_G < 1$ and Region (I.3) with $k_G \geq 1$ (see Figure 9).

Region (I.1): $k_G < k_V$ (and $k_V < 1, k_H < 1$)

For $k_G/k_V \ll 1$, we expect the shell to behave very much as a flat plate and the scaling for Region (1.1) or Region (1.2) of the plate problem applies. To be consistent with the flat plate results, we subdivide Region (I.1) for shells into Region (I.1.1) in which $k_V^2 < k_H$ and Region (I.1.2) in which $k_V^2 > k_H$.*

For Region (I.1.1) with $k_V^2 < k_H$, we use the scaling (III.13) and write (IV.2) as

$$L[f] - \frac{k_H}{x} (\mu_G s + f)g = -\frac{1-x^2}{2x} \quad (\text{IV.4})$$

$$L[g] + \frac{\mu_m}{x} (\mu_G s + \frac{1}{2}f)f = -x \quad (\text{IV.5})$$

where $\mu_G = k_G/k_V$ and $\mu_m = k_V^2/k_H$ is as previously defined for plate and $\mu_m < 1$ in Region (I.1.1). The boundary conditions for f and g are exactly

*It is possible to seek a perturbation solution in k_G in Region (I.1) with a leading term identical to the $k_G = 0$ case already treated in section (2) of Chapter III. However, for k_G near (but still less than) unity, a different scaling from the one used in Regions (1.1) and (1.2) for flat plates will lead to a more efficient solution. The new scalings in Region (I.2) for shells also provide a gradual transition into Region (I.3).

as in the plate case. With $\mu_G < 1$, a (regular) perturbation solution in (powers of) μ_G is appropriate. The leading term solution is identical to the flat plate solution in Region (1.1) given in section (2) of Chapter III; it requires no further comments. Higher order correction terms are determined by a sequence of linear inhomogeneous BVP which differ from each other only in the forcing term.

For Region (I.1.2) where $k_H < k_V^2$, we use the scaling for plate for the same load parameter range given in (III.23) (namely, $\hat{f} = k_V f$ and $\hat{g} = K_V^2 g$). Write the ODE for $\hat{f}$ and $\hat{g}$ as

$$L[f] - \frac{k_V^2}{x} (\mu_G s + f)g = - \frac{1-x^2}{2x} \quad (\text{IV.7})$$

$$L[g] + \frac{1}{x} (\mu_G s + \frac{1}{2}f)f = - \frac{1}{\mu_m} x \quad (\text{IV.8})$$

where $\mu_m = k_V^2 / k_H$ is now greater than one. The boundary conditions for f and g in this region are identical in form to those for the flat plate in Region (1.2), namely, (III.9)-(III.11) and (III.26). Again a perturbation solution in μ_G (< 1) is appropriate. The leading term solution is identical to the flat plate solution in Region (1.2) and requires no further comments. Higher order correction terms in the perturbation series are again determined by a sequence of linear BVP.

Region (I.2): $0 \leq k_V < k_G < 1$ ($k_H < 1$)

With $k_G/k_V \equiv \mu_G > 1$, the scalings used in Region (I.1) are no longer appropriate. As expected, the correct scaling in the new parameter range depends on the relative magnitude of k_V and k_H . Again, we separate two subregions: Region (I.2.1) where $k_G k_H > k_V$ and Region (I.2.2) where $k_G k_H < k_V$.

In Region (I.2.1), the radial load is relatively large compared to the pressure load; the shell response to the combined loading should have a significant component of rotating disc type behavior, especially when $k_G k_H \gg k_V$. Therefore, we set

$$\hat{g} = k_H g, \quad \hat{f} = k_G k_H f. \quad (\text{IV.9})$$

Note that for $k_G \leq k_V$, we have $\hat{g} = 0(k_H)$ and $\hat{f} = 0(k_G k_H)$ also in Region (I.1.1). Thus, there is a gradual transition in the magnitude of the solution from the interior of Region (I.1.1) to its border adjacent to Region (I.2.1). Our choice of scaling (IV.9) for Region (I.2.1) is consistent with this transition. With (IV.9), the governing ODE become

$$\begin{aligned} L[f] - \frac{1}{x} (s + k_H f) g &= - \frac{k_V}{k_H k_G} \frac{1-x^2}{2x} \\ L[g] + \frac{k_V^2}{x} (s + \frac{1}{2} k_H f) f &= -x. \end{aligned} \quad (\text{IV.10})$$

The boundary conditions become

$$f(x_1) = 0, \quad f'(1) + \nu f(1) = 0, \quad (\text{IV.11,12})$$

$$g(1) = 0, \quad g'(x_1) - \frac{\nu}{x_1} g(x_1) + \frac{x_1^2}{3+\nu} = 0, \quad (\text{IV.13,14})$$

With $k_V^2 \leq k_G^2 k_H^2 < k_H < 1$ and $k_V^2 \leq k_V/k_H k_G$, a perturbation solution in k_V^2 is appropriate for the new BVP. The leading term solution $\{f_0, g_0\}$ is determined by

$$L[g_0] = -x \quad (\text{IV.15})$$

$$L[f_0] - \frac{k_H}{x} g_0 f_0 = \frac{s}{x} g_0 - \frac{k_V}{k_G k_H} \frac{1-x^2}{2x}. \quad (\text{IV.16})$$

Thus, to a first approximation, the problem is uncoupled into a rotating disc problem and a problem of linear bending of a "pre-stressed" flat plate by a (nonuniform) distribution of pressure load.

In Region (I.2.2), we have $k_V \geq k_H k_G$ and the pressure load is mainly responsible for the shell behavior. Therefore, we set

$$\hat{f} = k_V f, \quad \hat{g} = \frac{k_V}{k_G} g \quad (\text{IV.17})$$

and write the governing ODE as

$$L[f] - \frac{1}{x} \left(s + \frac{k_V}{k_G} f \right) g = - \frac{1-x^2}{2x} \quad (\text{IV.18})$$

$$L[g] + \frac{k_G^2}{x} \left(s + \frac{1}{2} \frac{k_V}{k_G} f \right) f = - \frac{k_H k_G}{k_V} x. \quad (\text{IV.19})$$

Three of the boundary conditions for this range are also given by (IV.11)-(IV.13); the fourth becomes

$$g'(x_1) - \frac{\nu}{x_1} g(x_1) + \frac{k_H k_G}{k_V} \frac{x_1^2}{3+\nu} = 0. \quad (\text{IV.20})$$

The form of the new BVP suggests that a perturbation solution in k_G^2 would reduce the nonlinear problem to a sequence of linear boundary value problems. The leading term solution again consists of a rotating disc solution and the solution of a problem in linear bending of a "pre-stressed" flat plate by a (nonuniform) distribution of pressure load.

On the other hand, if $k_G^2 \gg k_V/k_G$, then it would be more efficient to work with a perturbation solution in k_V/k_G . The leading term solution in

that case would be a linear bending of a shallow shell of revolution by the combined radial and axial loads which appear on the right side of (IV.18) and (IV.19).

Region (I.3): $k_G \geq 1$ ($k_V, k_H < 1$)

Let $k_L = \max [k_V, k_H]$ and

$$\{\hat{f}, \hat{g}\} = k_L \{f, g\}. \quad (\text{IV.21})$$

Then the ODE (IV.2) may then be written as

$$\frac{1}{k_G} L[f] - \frac{1}{x} \left(s + \frac{k_L}{k_G} f \right) g = - \frac{k_V}{k_L k_G} \frac{1-x^2}{2x} \quad (\text{IV.22})$$

$$\frac{1}{k_G} L[g] - \frac{1}{x} \left(s + \frac{1}{2} \frac{k_L}{k_G} f \right) f = - \frac{k_H}{k_L k_G} x. \quad (\text{IV.23})$$

The boundary conditions now consist of (IV.11)-(IV.13) and

$$g'(x_1) - \frac{\nu}{x_1} g(x_1) + \frac{k_H}{k_L} \frac{x_1^2}{3+\nu} = 0. \quad (\text{IV.24})$$

With $k_L/k_G < 1$ and $\{k_V/k_L, k_H/k_L\} \leq 1$, the new BVP may be linearized by a perturbation solution in k_L/k_G giving us as the leading term solution the conventional linear bending problem for a shallow shell of revolution under the combined radial and axial loads.

3. Curved Plate Behavior at High Load Intensities

Having completed the analysis of the light to moderate load magnitude range characterized by $k_V < 1$ and $k_H < 1$, we now turn to the high load intensity range $k_V \geq 1$ or $k_H \geq 1$. As in the flat plate case, we expect the shell to respond in a qualitatively different way for $k_H^3 < k_V^2$ and for $k_H^3 > k_V^2$. We consider in this section the load magnitude ranges, $\{k_V > k_G^3, k_V^2 > k_H^3\}$ and $\{k_H > k_G^2, k_H^3 > k_V^2\}$. In the load-geometric parameter space $\{k_V, k_H, k_G\}$ with the k_G axis pointing upward (Figure 10), the high load magnitude range spans the positive octant of the space outside the unit square column (anchored at the origin) and below the two intersecting surfaces $k_G = k_V^{1/3}$ and $k_G = k_H^{1/2}$. The two subregions $k_H^3 < k_V^2$ and $k_H^3 > k_V^2$ are designated as Region (II.1) and Region (II.2), respectively.

Region (II.1): $k_H^3 < k_V^2$ and $k_G^3 < k_V$

For $k_G \ll 1$, we expect the shell behavior is platelike; therefore we use the scaling (III.42) (with $\hat{f} = k_V^{1/3} f$ and $\hat{g} = k_V^{2/3} g$) for the plate case and write the two governing ODE as

$$\frac{1}{k_V^{2/3}} L[f] - \frac{1}{x} \left(\frac{k_G}{k_V^{1/3}} s + f \right) g = - \frac{1-x^2}{2x} \quad (\text{IV.25})$$

$$L[g] + \frac{1}{x} \left(\frac{k_G}{k_V^{1/3}} s + \frac{1}{2} f \right) f = - \frac{x}{\mu_\ell} \quad (\text{IV.26})$$

where $\mu_\ell^3 = k_V^2 / k_H^3$ is as defined in (III.32). The boundary conditions take the form (IV.11)-(IV.13) and (III.45). It is clear from (IV.25) and (IV.26) that the same scaling may be used for all $k_G < k_V^{1/3}$, and a perturbation solution in $k_G/k_V^{1/3}$ is appropriate. The leading term solution provides an accurate approximation of the exact solution if $k_G \ll k_V^{1/3}$. This leading

term solution is just the solution for the flat plate problem already analyzed in section (4) of Chapter III.

Region (II.2): $k_H \geq k_V^{2/3}$ and $k_G < k_H^{1/2}$

The proper scaling in this region depends on whether we have $k_G < k_V/k_H$ or $k_G > k_V/k_H$. We designate these two subregions as Region (II.2.1) and Region (II.2.2), respectively.

For the subregion (II.2.1) with $k_G k_H < k_V$, we expect the shell to behave nearly as a flat plate, especially if $k_G \ll 1$. Therefore, the scaling (III.29) (with $\hat{f} = (k_V/k_H)f$ and $\hat{g} = k_H g$) for a relatively fast spinning plate should apply for $k_G \ll 1$. With this scaling, the governing ODE may then be written as

$$\frac{1}{k_H} L[f] - \frac{1}{x} \left(\frac{k_G k_H}{k_V} s + f \right) g = - \frac{1-x^2}{2x} \quad (\text{IV.27})$$

$$L[g] + \frac{1}{x} \mu_\ell^3 \left(\frac{k_G k_H}{k_V} s + \frac{1}{2} f \right) f = -x \quad (\text{IV.28})$$

while the boundary conditions are as given by (IV.11)-(IV.14). Both $k_G k_H/k_V$ and μ_ℓ^3 are smaller than unity. We may therefore consider a perturbation in $k_G k_H/k_V$. The leading term solution in this range of the parameter values has already been worked out in section (3) of Chapter III.

For the complementary subregion (II.2.2) with $k_G k_H > k_V$, the pressure load is too small to prevent the flattening effect of the radial load. For the shell to be nearly flattened out, we have $\phi + \xi = \bar{\phi}(k_G s + \hat{f}) \cong 0$. Therefore, we take in this Region (II.2.2)

$$\hat{f} = k_G f, \quad \hat{g} = k_H g \quad (\text{IV.29})$$

and write the governing ODE as

$$\frac{1}{k_H} L[f] - \frac{1}{x} (s+f) g = - \frac{k_V}{k_G k_H} \frac{1-x^2}{2x} \quad (\text{IV.30})$$

$$L[g] + \frac{k_G^2}{k_H} \frac{1}{x} (s + \frac{1}{2}f) f = -x. \quad (\text{IV.31})$$

The boundary conditions now take the form (IV.11)-(IV.14).

With $k_G^2/k_H < 1$, we seek a perturbation solution in k_G^2/k_H to reduce the nonlinear problem to a sequence of linear problems. The leading term for g is the solution of a rotating disc problem while the leading term solution for f is the solution for the linear bending of a "pre-stressed" flat plate by a nonuniform axial load distribution.

When $k_H \gg 1$ and $k_G k_H \gg k_V$, we have to a first approximation, $s+f = 0$ except for layer phenomena. The shell is more or less flattened out whenever the radial load is sufficiently high, i.e. when the shell is spinned at a sufficiently high constant angular velocity.

4. Thin Shell Behavior at High Load Intensities

The only range of the load and geometric parameter values not analyzed up to this point is the region in the parameter space outside the unit square column, $0 \leq k_H < 1$ and $0 \leq k_V < 1$, and above the two intersecting surface $k_G = k_V^{1/3}$ and $k_G = k_H^{1/2}$, i.e., $k_G^3 > k_V$ and $k_G^2 > k_H$. The projection of the intersection curve on the k_H, k_V -plane is the base curve $k_H^3 = k_V^2$. We designate this remaining region as Region III and consider separately the two subregions (III.1) where $k_H^3 < k_V^2$ and (III.2) where $k_H^3 > k_V^2$,

Region (III.1): $k_H^3 < k_V^2$ (and $k_G^3 > k_V > 1$)

In this region, we take

$$\hat{f} = k_G f, \quad \hat{g} = k_V^{2/3} g \quad (\text{IV.32})$$

and write the ODE (IV.2) as

$$\frac{1}{k_V^{2/3}} L[f] - \frac{1}{x} (s+f) g = - \left(\frac{k_V}{k_G^3} \right)^{1/3} \frac{1-x^2}{2x} \quad (\text{IV.33})$$

$$\left(\frac{k_V}{k_G^3} \right)^{1/3} L[g] + \frac{1}{x} (s + \frac{1}{2} f) f = - \frac{k_H}{k_G^2} x. \quad (\text{IV.34})$$

Correspondingly, the boundary conditions take the form (IV.11)-(IV.13) and

$$g'(x_i) - \frac{\nu}{x_i} g(x_i) + \left(\frac{k_H^3}{k_V^2} \right)^{1/3} \frac{x_i^2}{3+\nu} = 0 \quad (\text{IV.35})$$

with $k_H/k_G^2 < (k_V/k_G^3)^{2/3} \leq 1$, we seek a perturbation solution in k_H/k_G^2 . The leading term solution is identical to the case of no spinning which has already been analyzed in [13,14,22,23,26].

Region (III.2): $k_H^3 > k_V^2$ (and $k_G^2 > k_H > 1$)

The shell is likely to be flattened out in this region. Therefore, we take

$$\hat{f} = k_G f, \quad \hat{g} = k_G^2 g \quad (\text{IV.36})$$

and write the two governing ODE as

$$\frac{1}{k_G^2} L[f] - \frac{1}{x} (s + f) g = - \frac{k_V}{k_G^3} \frac{1 - x^2}{2x} \quad (\text{IV.37})$$

$$L[g] + \frac{1}{x} (s + \frac{1}{2} f) f = - \frac{k_H}{k_G^2} x. \quad (\text{IV.38})$$

Correspondingly, the boundary conditions take the form (IV.11)-(IV.13)

and

$$g'(x_1) - \frac{\nu}{x_1} g(x_1) + \frac{k_H}{k_G^2} \frac{x_1^2}{3+\nu} = 0. \quad (\text{IV.39})$$

With $k_V/k_G^3 < (k_H/k_G^2)^{3/2} \leq 1$ (which follows from $k_V^2 < k_H^3$), a perturbation solution in k_V/k_G^3 is appropriate. The leading term solution is identical to the solution for no pressure load already analyzed in [2].

5. Summary of Re-scalings and Asymptotic Solutions

To summarize the results of this chapter, we write the re-scaling of $\hat{f}$ and $\hat{g}$ in the form

$$\hat{f} = c_f f, \quad \hat{g} = c_g g \quad (\text{IV.40})$$

with the choice of c_f and c_g depending on the value of the parameters k_H , k_V and k_G . The re-scaled form of the two differential equations may be written as

$$c_V L[f] - \frac{1}{x} (c_{Vs} s + c_{fg} f) g = -c_{Vk} \frac{1-x^2}{2x} \quad (\text{IV.41})$$

$$c_H L[g] + \frac{1}{x} (c_{Hs} s + \frac{1}{2} c_{ff} f) f = -c_{Hk} x. \quad (\text{IV.42})$$

Of the four boundary conditions, the form of three of them remains invariant upon re-scaling; the fourth may be written as

$$g'(x_1) - \frac{v}{x_1} g(x_1) + c_{HB} \frac{x_1^2}{3+v} = 0 \quad (\text{IV.43})$$

The eleven multiplicative factors

$$[c_f, c_g], \quad \begin{bmatrix} c_V & c_{Vs} & c_{fg} & c_{Vk} \\ c_H & c_{Hs} & c_{ff} & c_{Hk} \end{bmatrix}, \quad [c_{HB}]$$

completely describe the re-scaled BVP. The re-scaling of the BVP in the various regions of the parameter space will now be summarized in the Table (6) with the help of these multiplicative factors.

The asymptotic solutions in different regions of the parameter space are summarized below:

Region (I.1): A perturbation solution in $\mu_G < 1$ reduces the problem to solving the corresponding plate problems (done in Chapter III) and a sequence of linear problems which give corrections to the plate solutions.

Region (I.2): A perturbation solution in $k_V < 1$ reduces the problem (in each subregion) to a sequence of linear problems. For the subregion (I.2.1), a perturbation solution in $k_H < 1$ also leads to the same result.

Region (I.3): Small load amplitudes naturally lead to a perturbation solution in $k_L/k_G < 1$ (where $k_L = \max [k_H, k_V] < 1$) resulting in a linear shell problem (of the singular perturbation type if $k_G \gg 1$) which has been extensively studied in the literatures [24,25].

Region (II.1): A perturbation solution in $k_G/k_V^{1/3} < 1$ reduces the problem to solving the corresponding flat plate problem and determining corrections to the plate solution from a sequence of linear problems.

Region (II.2.1): A perturbation solution in $k_H k_G/k_V < 1$ reduces the problem to solving the corresponding plate problem and determining corrections to the flat plate solution from a sequence of linear problems. Alternatively, a perturbation solution in $\mu_\ell^3 < 1$ has as its leading term the well known rotating disc solution and the solution for the linear bending by a modified axial load of a flat plate pre-stressed by a nonuniform distribution of in-plane (radial) stress resultant induced by the stretching action of the

shell. Corrections in the second perturbation solution are also determined by linear problems.

Region (II.2.2): A perturbation solution in $k_G^2 k_H^{-1}$ gives results similar to the perturbation solution in μ_ℓ^3 for Region (II.2.1).

Region (III.1): A perturbation solution in the smaller radial load amplitude factor $k_H/k_G^2 < 1$ has as its leading term the solution for the same problem but without the radial load. This difficult nonlinear BVP has already been analyzed in [13,14,22,23,26]*. Corrections to the leading term solution are again determined by a sequence of linear problems.

Region (III.2): A perturbation solution in the smaller amplitude factor $k_V/k_G^3 < 1$ has as its leading term the solution of the same problem for a shell with no pressure loading. This difficult nonlinear BVP has already been analyzed in [2,5,22]*. Corrections to the leading term solution are again determined by a sequence of linear problems.

*For the case $k_H/k_G^2 \ll 1$ and $k_V/k_G^3 \ll 1$, the BVP can be linearized.

V. Concluding Remarks

In the preceding pages, the problem of a steadily spinning shallow elastic shell of revolution under a uniform pressure distribution has been analyzed and completely solved by perturbation and asymptotic methods. The results have been confirmed by accurate numerical solutions.

For the limiting case of a flat plate, our asymptotic analysis shows that the solution under the combined radial and axial loading may be one of the following four qualitatively different types depending on the particular load combination. For "light-to-moderate" load magnitude, the plate may either behave effectively as a disc pre-stressed by spinning with (secondary) out of plane bending by the uniform pressure, or as a transversely bent plate by the uniform pressure with the deformed slope producing additional (secondary) in-plane extension beyond the effect of the steady rotation. At high rotating speed, the plate again behaves effectively as a disk pre-stressed by the centrifugal force and bent out of plane by the uniform pressure, with the (secondary) bending action confined to a narrow layer adjacent to the plate edges. At high pressure loading, the plate behaves effectively as a membrane in finite deformation except for some linear bending action in a boundary layer adjacent to the plate edges. For this case, it has been shown in Chapter III, that bending stresses are negligible in the plate interior and are at most of the same order of magnitude as the direct (membrane) stresses near an edge. For design purposes, a solution by the Foppl-Hencky type (or the Reissner type [27]) nonlinear membrane theory should provide a good approximation to the direct stresses and therefore also a good estimate of the bending stress level. Near the borders between load-parameter regions where one of these basic modes of deformations dominates, the expected mode would be modified considerably by the competing neighboring mode(s).

For shallow shells, our asymptotic analysis shows that the structure may exhibit one of three typical mode of deformation, depending on the relative magnitude of the load and geometric parameters. Loosely speaking, the shell is predominantly in a linear bending mode for sufficiently light load magnitude; "very shallow" shells (defined more precisely in chapter IV) are nearly plate-like while "very thin" shells at high load magnitude are usually in a nonlinear membrane or inextensional bending mode. For the first two modes, either bending or stretching action may be dominant in the shell interior, analogous to the flat plate case.

The fact that a shell under the combined radial and axial loading may exhibit one of several modes of behavior is not unexpected. However, our analysis makes precise the ranges of the load and geometrical parameter values for which each of these modes would occur; it also shows how the solution procedure may simplify in each case leading to a complete solution of the problem for the entire parameter space. It is important to note that the same analysis, possibly with some straightforward modifications, applies to shells subject to more general loading and to nonshallow shells. We have focussed on a specific class of problems in order to give a unified treatment for these problems important in applications and to put known results for special cases (such as those obtained in references [1,2,12-23,26]) in a proper context. Our contributions, however, go well beyond the results for this class of problems; our analysis actually constitutes a general approach to the solution of (small strain) axisymmetric finite deformation problems for shells of revolution.

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APPENDIX A. Solutions for Various Classes of Plate Problems.

1. Perturbation Solutions for Small Pressure and Slowly Spinning Case.

The equations and boundary conditions for this case are given by (III.6)-(III.7), (III.9)-(III.12). With $k_H \ll 1$ and $k_V \ll 1$, a perturbation solution of this BVP is appropriate. We take the two parameter perturbation series solutions in the form

$$\hat{f}(x) = k_V f_1(x) + k_H f_2(x) + k_V^2 f_3(x) + k_V k_H f_4(x) + k_H^2 f_5(x) + \dots \quad (\text{A.1})$$

$$\hat{g}(x) = k_V g_1(x) + k_H g_2(x) + k_V^2 g_3(x) + k_V k_H g_4(x) + k_H^2 g_5(x) + \dots \quad (\text{A.2})$$

These kinds of perturbation series are widely used in elastostatics [6-8].

For our problem, we expect leading terms to be proportional to load parameters k_V , k_H , $f(x)$ to be an odd function of k_V and g an even function of k_V . Hence (A.1) and (A.2) may be simplified to

$$\hat{f}(x) = k_V f_1(x) + k_V k_H f_2(x) + \dots \quad (\text{A.3})$$

$$\hat{g}(x) = k_H g_1(x) + k_H^2 g_2(x) + k_V^2 g_3(x) + \dots \quad (\text{A.4})$$

where subscripts of functions have been changed.

Substitution of (A.3), (A.4) into Eq. (III-6)-(III.7) yields

$$f_1'' + \frac{1}{x} f_1' - \frac{1}{2} f_1 = -\frac{1-x^2}{2x} \quad (\text{A.5})$$

$$f_2'' + \frac{1}{x} f_2' - \frac{1}{2} f_2 = \frac{1}{x} f_1 g_1 \quad (\text{A.6})$$

.....

$$g_1'' + \frac{1}{x} g_1' - \frac{1}{2} g_1 = -x \quad (\text{A.7})$$

$$g_2'' + \frac{1}{x} g_2' - \frac{1}{2} g_2 = 0 \quad (\text{A.8})$$

$$g_3'' + \frac{1}{x} g_3' - \frac{1}{2} g_3 = -\frac{1}{2x} f_1^2 \quad (\text{A.9})$$

.....

Substitution of (A.3) and (A.4) into B.C. (III.9)-(III.12) yields

$$f_1(x_i) = 0$$

$$f_2(x_i) = 0$$

$$f_1'(1) + \nu f_1(1) = 0$$

$$f_2'(1) + \nu f_2(1) = 0$$

$$g_1'(x_i) - \frac{\nu}{x_i} g_1(x_i) + \frac{x_i^2}{3+\nu} = 0$$

$$g_2'(x_i) - \frac{\nu}{x_i} g_2(x_i) = 0 \quad (\text{A.10, a-j})$$

$$g_3'(x_i) - \frac{\nu}{x_i} g_3(x_i) = 0$$

$$g_1(1) = 0$$

$$g_2(1) = 0$$

$$g_3(1) = 0$$

etc. We solve (A.5) with (A.10 a,c) to get f_1 , then solve (A.7) with (A.10 e,h) to g_1 , etc. For simplicity, we give here only the solutions for the case $x_i = 0$:

$$f_1 = \frac{1}{16} \left(\frac{1-\nu}{1+\nu} x + x^3 - 4x \ln x \right)$$

$$g_1 = \frac{1}{8} x(1-x^2)$$

$$g_2 = 0$$

$$g_3 = \frac{1}{512} \left\{ \left[\frac{1}{8} \left(\frac{1-\nu}{1+\nu} \right)^2 + \frac{5}{6} \frac{(1-\nu)}{(1+\nu)} + \frac{275}{144} \right] x - \left[\frac{1}{8} \left(\frac{1-\nu}{1+\nu} \right)^2 + \frac{3(1-\nu)}{4(1+\nu)} + \frac{7}{4} - \frac{2(2+\nu)}{1+\nu} \ln x + 2 \ln^2 x \right] x^3 + \left[\frac{1}{3} \ln x - \frac{5}{36} - \frac{1-\nu}{12(1+\nu)} \right] x^5 - \frac{1}{48} x^7 \right\} \quad (\text{A.11 a-d})$$

For the $k_H \ll k_V^2 \ll 1$ case, we have

$$\hat{f}(x) \sim k_V f_1(x)$$

$$\hat{g}(x) \sim k_V^2 g_3(x)$$

or

$$\phi \sim \frac{\sqrt{DA}}{r_0} k_V f_1(x) \quad (A.12)$$

$$\psi \sim \frac{D}{r_0} k_V^2 g_3(x) . \quad (A.13)$$

For the $k_V^2 \ll k_H \ll 1$ case, we get instead

$$\hat{f}(x) \sim k_V f_1(x)$$

$$\hat{g}(x) \sim k_H g_1(x)$$

or

$$\phi \sim \frac{\sqrt{DA}}{r_0} k_V f_2(x) \quad (A.14)$$

$$\psi \sim \frac{D}{r_0} k_H g_1(x) . \quad (A.15)$$

2. Asymptotic Solutions for the Fast Spinning Case

From the view point of an asymptotic analysis, the BVP for $k_H \gg 1$ (see (III.30) and (III.31)) is a singular perturbation problem. When $k_H \gg k_V^{2/3}$ (in addition to $k_H \gg 1$), then the leading term perturbation solution in $\mu_\ell^3 \equiv k_V^2/k_H^3 \ll 1$ for g is (see (III.34)-(III.35))

$$g_0 = A\left(x - \frac{1}{x}\right) + \frac{1}{8}\left(\frac{1}{x} - x^3\right) \quad (\text{A.16})$$

where

$$A = \frac{1+v}{8(3+v)} \frac{(3+v)+(1-v)x_i^4}{(1+v)+(1-v)x_i^2} \quad (\text{A.17})$$

with $A \rightarrow 1/8$ as $x_i \rightarrow 0$.

The leading term solution for f is then determined by

$$f_0'' + \frac{1}{x} f_0' - \left[\frac{k_H}{x} g_0(x) + \frac{1}{x^2} \right] f_0 = -k_H \frac{1-x^2}{2x} \quad (\text{A.18})$$

$$f_0(x_i) = 0, \quad f_0'(1) + v f_0(1) = 0 \quad (\text{A.19})$$

The leading term outer (asymptotic expansion) solution of (A.18) is evidently

$$f_0^{(0)} = \frac{1-x^2}{2g_0} = \frac{4x}{x^2+1-8A} \quad ; \quad (\text{A.20})$$

it does not satisfy either of the boundary conditions in (A.19) and requires boundary layer corrections.

For the boundary layer correction adjacent to the inner edge, we write the leading term solution which is uniformly valid away from $x = 1$ as

$$f_0 \sim f_0^{(0)}(x) + \bar{f}_0(y), \quad y = \sqrt{k_H}(x-x_i) \quad (\text{A.21})$$

Then $\bar{f}_0$ is determined by the BVP

$$\bar{f}_0'' - \alpha^2 \bar{f}_0 = 0 \quad (\text{A.22})$$

$$\bar{f}_0(0) + \frac{1 - x_1^2}{2g_0(x_1)} = 0, \quad \bar{f}_0(y) \rightarrow 0 \text{ as } y \rightarrow \infty$$

where $\alpha^2 = g(x_1)/x_1$. The solution of (A.22) is

$$\bar{f}_0 = -\frac{1 - x_1^2}{2g_0(x_1)} e^{-\alpha\sqrt{k_H}(x-x_1)}. \quad (\text{A.23})$$

Similarly, we write the solution which is uniformly valid away from $x = x_1$ as

$$f_0 \sim f_0^{(0)}(x) + \tilde{f}_0(t), \quad t = k_H^{1/3}(x-1). \quad (\text{A.24})$$

With

$$g_0(x) = B(1-x) + O([1-x]^2), \quad B = \frac{1}{2}(1-4A) \quad (\text{A.25})$$

near $x = 1$, we find that $\tilde{f}_0(t) = k_H^{1/3}\tilde{f}(t)$ with $\tilde{f}$ determined by the BVP

$$\begin{aligned} \tilde{f}'' + Bt\tilde{f} &= 0, \\ \tilde{f}'(0) &= C \equiv \frac{8A-4}{(1-4A)^2}, \quad \tilde{f}(t) \rightarrow 0 \text{ as } t \rightarrow -\infty. \end{aligned} \quad (\text{A.26})$$

The solution of (A.26) is

$$\tilde{f}(t) = C\Gamma\left(\frac{1}{3}\right)\left(\frac{3}{B}\right)^{1/3} A_1([k_H B(1-x)^3]^{1/3}) \quad (\text{A.27})$$

where $A_1(z)$ is the Airy function.

As $\bar{f}_0$ is exponentially small for $x > x_1$ and $\tilde{f}_0$ is exponentially small for $x < 1$, the uniformly valid solution for f_0 in the whole interval $0 \leq x \leq 1$ is given by

$$\begin{aligned}
 f_0 &= f_0^{(0)}(x) + \bar{f}_0(y) + k_H^{1/3} \tilde{f}(t) \\
 &= \frac{1-x^2}{2g_0(x)} - \frac{1-x_1^2}{2g_0(x_1)} e^{-\alpha\sqrt{k_H}(x-x_1)} \\
 &\quad + k_H^{1/3} C \Gamma\left(\frac{1}{3}\right) \left(\frac{3}{B}\right)^{1/3} A_1 ([k_H B(1-x)^3]^{1/3})
 \end{aligned} \tag{A.28}$$

3. Asymptotic Solution for the High Pressure Range

Suppose we write the solution of our BVP in region (3) as

$$\begin{pmatrix} f \\ g \end{pmatrix} \sim \begin{pmatrix} F_0 \\ G_0 \end{pmatrix} + \begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix} \quad (\text{A.29})$$

where $G_0(x)$ and $F_0(x)$ are the nonlinear membrane solutions for f and g , respectively, as determined by (III.46) and the boundary conditions

$$G_0(1) = 0, \quad G'_0(x_1) - \frac{\nu}{x_1} G_0(x_1) + \frac{x_1^2}{\mu_\ell(3+\nu)} = 0. \quad (\text{A.30})$$

Upon substituting (A.26) into the ODE (III.43,44) and the four relevant boundary conditions and observing (III.46) and (A.30), we get

$$\epsilon^2 \left[\bar{f}'' + \frac{1}{x} \bar{f}' - \frac{1}{x^2} \bar{f} \right] - \frac{1}{x} \bar{f} \bar{g} - \frac{1}{x} G_0(x) \bar{f} - \frac{1}{x} F_0(x) \bar{g} = -\epsilon^2 \left[F_0'' + \frac{1}{x} F_0' - \frac{1}{x^2} F_0 \right] \quad (\text{A.31})$$

$$\bar{g}'' + \frac{1}{x} \bar{g}' - \frac{1}{x^2} \bar{g} + \frac{1}{2x} \bar{f}^2 + \frac{1}{x} F_0(x) \bar{f} = 0 \quad (\text{A.32})$$

with

$$\bar{g}(1) = 0, \quad \bar{f}'(1) + \nu \bar{f}(1) + F_0'(1) + \nu F_0(1) = 0 \quad (\text{A.33-36})$$

$$\bar{g}'(x_1) - \frac{\nu}{x_1} \bar{g}(x_1) = 0, \quad \bar{f}(x_1) + F_0(x_1) = 0$$

where $\epsilon = k_V^{-1/3}$. The new problem for $\bar{f}$ and $\bar{g}$ is again a singular perturbation problem for $k_V^{1/3} \gg 1$. We will not be concerned with the

outer solution of this new problem as it is of higher order in ϵ compared to G_0 and F_0 .

For an inner solution near x_1 , we note that a distinguished limit of the problem (A.31-36) as $\epsilon \rightarrow 0$ consists of

$$\bar{f} \sim \bar{f}_0(y), \quad \bar{g} \sim \epsilon^2 \bar{g}_0(y), \quad y = \frac{x - x_1}{\epsilon} = k_V^{1/3} (x - x_1) \quad (\text{A.37})$$

with $\bar{f}_0$ and $\bar{g}_0$ being the solution of the BVP

$$\bar{f}_0'' - \alpha^2 \bar{f}_0 = 0, \quad \bar{g}_0'' + \frac{1}{2x_1} \bar{f}_0^2 + \frac{1}{x_1} F_0(x_1) \bar{f}_0 = 0, \quad (\text{A.38})$$

$$\bar{f}_0(0) + F_0(x_1) = 0, \quad \bar{g}_0'(0) = 0 \quad (\text{A.39})$$

$$\bar{g}_0(y), \quad \bar{f}_0(y) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty \quad (\text{A.40})$$

where $()' = d()/dy$ and $\alpha^2 = G_0(x_1)/x_1$ (known to be positive from the nonlinear membrane solution), and where the limit of $y \rightarrow \infty$ is understood to mean $\epsilon \rightarrow 0$, $\eta(\epsilon) \rightarrow 0$, $\eta(\epsilon)/\epsilon \rightarrow \infty$ (for some suitable $\eta(\epsilon)$) and (the intermediate variable) $z \equiv \epsilon y/\eta(\epsilon) = O(1)$. The solution of this problem is

$$\bar{f}_0(y) = -F_0(x_1) e^{-\alpha y} = -F_0(x_1) e^{-\alpha(x-x_1)/\epsilon} \quad (\text{A.41})$$

$$\bar{g}_0(y) = \frac{3\beta}{4\alpha} \frac{x-x_1}{\epsilon} + C_0 + \frac{\beta}{\alpha^2} \left[e^{-\alpha(x-x_1)/\epsilon} - \frac{1}{8} e^{-2\alpha(x-x_1)/\epsilon} \right] \quad (\text{A.42})$$

where $\beta = F_0^2(x_1)/x_1$ and where C_0 is a constant to be determined in conjunction with the next order term in the asymptotic expansion (and may be set equal to zero for a leading term solution).

For an inner solution near $x = 1$, we note that a distinguished limit of the BVP in this case consists of

$$\begin{aligned} \bar{f} \sim \sqrt{\epsilon} \tilde{f}_0(t) &= k_V^{-1/6} f_0(t), & \bar{g} \sim \epsilon^{3/2} \tilde{g}_0(t) &= k_V^{-1/2} \tilde{g}_0(t) \\ t &= \frac{x-1}{\sqrt{\epsilon}} = k_V^{1/6} (x-1) \end{aligned} \quad (\text{A.43})$$

with $\tilde{f}_0$ and $\tilde{g}_0$ determined by the BVP

$$\tilde{f}_0'' - F_0(1) \tilde{g}_0 = 0, \quad \tilde{g}_0'' + F_0(1) \tilde{f}_0 = 0 \quad (\text{A.44})$$

$$\tilde{g}_0(0) = 0, \quad \tilde{f}_0'(0) + F_0'(1) + \nu F_0(1) = 0 \quad (\text{A.45})$$

$$\tilde{g}_0, \tilde{f}_0 \rightarrow 0 \quad \text{as} \quad t \rightarrow -\infty \quad (\text{A.46})$$

where $()' \equiv d()/dt$ and where the limit $t \rightarrow -\infty$ is understood to mean $\sqrt{\epsilon} \rightarrow 0$, $\zeta(\epsilon) \rightarrow 0$, $\zeta(\epsilon)/\sqrt{\epsilon} \rightarrow \infty$ (for some suitable $\zeta(\epsilon)$) and (the intermediate variable) $s \equiv \sqrt{\epsilon} t / \zeta(\epsilon) = O(1)$. The solution of this problem is

$$\tilde{f}_0 = -\frac{\sqrt{2}}{\gamma} [F_0'(1) + \nu F_0(1)] e^{\gamma t / \sqrt{2}} \cos(\gamma t / \sqrt{2}). \quad (\text{A.47})$$

$$\tilde{g}_0 = \frac{\sqrt{2}}{\gamma} [F_0'(1) + \nu F_0(1)] e^{\gamma t / \sqrt{2}} \sin(\gamma t / \sqrt{2}). \quad (\text{A.48})$$

where $\gamma^2 = F_0(1)$.

With $\{\tilde{f}_0, \tilde{g}_0\}$ and $\{\tilde{f}_0, \tilde{g}_0\}$ exponentially small away from $x = x_1$ and $x = 1$, respectively, we may write the leading term composite solution as

$$f \sim F_0(x) - F_0(x_1)e^{-\alpha(x-x_1)/\epsilon} - \frac{\sqrt{2\epsilon}}{\gamma} [F_0'(1) + \nu F_0(1)] e^{\gamma(x-1)/\sqrt{2\epsilon}} \cos(\gamma[x-1]/\sqrt{2\epsilon}) \quad (\text{A.49})$$

$$g \sim G_0(x) + \epsilon^2 \left[\frac{3\beta}{4\alpha} \frac{x-x_1}{\epsilon} + \frac{\beta}{\alpha^2} \left\{ e^{-\alpha(x-x_1)/\epsilon} - \frac{1}{8} e^{-2\alpha(x-x_1)/\epsilon} \right\} \right] + \frac{\sqrt{2\epsilon}^3}{\gamma} [F_0'(1) + \nu F_0(1)] e^{\gamma(x-1)/\sqrt{2\epsilon}} \sin(\gamma[x-1]/\sqrt{2\epsilon}) \quad (\text{A.50})$$

For $k_V^{1/6} \gg 1$, the nonlinear membrane solution $F_0(x)$ alone is an adequate approximation for $f(x)$ away from the edge. However, it is clear that $F_0(x)$ is not dominant near the inner edge and the edge zone solution $\tilde{f}_0(y)$ must also be included. Furthermore, we have $\tilde{f}_0'(t) = \frac{1}{\sqrt{\epsilon}} O(\tilde{f}_0)$ near $x=1$, so that the outer solution alone does not give an accurate approximation of the bending stresses of the plate near the outer edge.

In contrast, the nonlinear membrane solution $G_0(x)$ is the dominant part of $g(x)$ throughout the entire plate, the edge zones included, for $k_V^{1/6} \gg 1$. Furthermore, we have $d\bar{g}/dx = O(\sqrt{\epsilon})G_0'(x)$ at most for $x_1 \leq x \leq 1$. It follows that the stretching action of the entire plate in region (3) of the k_H, k_V -plane is accurately described by the Foppl-Hencky nonlinear membrane solution alone whenever $k_V^{1/6} \gg 1$.

APPENDIX B. FINITE DIFFERENCE SOLUTIONS

A FORTRAN program has been developed to solve the systems of nonlinear ordinary differential equations numerically. The equations are taken in the form

$$f'' + \frac{1}{x} f' - \frac{1}{x^2} f - \frac{\bar{B}}{x} s g - \frac{\bar{C}}{x} f g = -\bar{D} \frac{1-x^2}{2x} \quad (\text{B.1})$$

$$g'' + \frac{1}{x} g' - \frac{1}{x^2} g + \frac{\bar{F}}{x} s f + \frac{\bar{G}}{2x} f^2 = -\bar{H} x \quad (\text{B.2})$$

where $\bar{B}, \dots, \bar{H}$ are constants depending on the load and geometric parameters of a specific problem. These two equations are supplemented by four boundary conditions:

$$x = x_1: \quad f = 0, \quad g' + \bar{A} x^2 - \frac{\nu}{x} g = 0 \quad (\text{B.3,4})$$

$$x = 1: \quad g = 0, \quad f' + \nu f = 0 \quad (\text{B.5,6})$$

where

$$\bar{A} = \frac{r_0^2 p_0}{(3+\nu)\bar{\psi}}.$$

Upon using central finite difference method, the ODE's are replaced by system of nonlinear algebraic equations with the discretization errors $O(h^2)$, where h is the length of step:

$$f_1 = 0$$

$$-g_3 + 4g_2 - \left[3 + \frac{\nu}{x_i} (2h)\right] g_1 + 2h\bar{A}x_i^2 = 0$$

$$\left(1 - \frac{h}{2x_i}\right) f_{i-1} - \left(2 + \frac{h^2}{x_i^2}\right) f_i + \left(1 + \frac{h}{2x_i}\right) f_{i+1} - \frac{\bar{B}}{x_i} s_i h^2 g_i - \frac{\bar{C}h^2}{x_i} f_i g_i$$

$$+ \frac{1-x_i^2}{2x_i} \bar{D} h^2 = 0 \quad (i = 2, \dots, n-1)$$

$$\left(1 - \frac{h}{2x_i}\right) g_{i-1} - \left(2 + \frac{h^2}{x_i^2}\right) g_i + \left(1 + \frac{h}{2x_i}\right) g_{i+1} + \frac{\bar{F}}{x_i} s_i h^2 f_i + \bar{G} \frac{h^2}{2x_i} f_i^2$$

$$+ x_i \bar{H} h^2 = 0 \quad (i = 2, \dots, n-1)$$

$$(3 + 2vh)f_n - 4f_{n-1} + f_{n-2} = 0$$

$$g_n = 0.$$

To get $O(h^2)$ accuracy, second order interpolations for boundary points have been used. For convenience, we denote (B.7) as

$$F_i(f_1, g_1, f_2, g_2, \dots, f_n, g_n) = 0 \quad (i = 1, 2, \dots, 2n) \quad (\text{B.8})$$

where F_i are nonlinear functions of $f_1, g_1, \dots, f_n, g_n$. The nonlinear system is solved by Newton's method.

The solution procedure is as follows:

1. Take an initial guess of the solution $(f_1^0, g_1^0, \dots, f_n^0, g_n^0)$. For our problem the leading terms of outer solutions are used whenever available.
2. Calculate (residual) errors of the solution, i.e., substitute the solution into the equations, compute the residuals $\epsilon_{10}, \epsilon_{20}, \dots, \epsilon_{2n0}$ and then form

$$\epsilon_0 = (\epsilon_{10}^2 + \epsilon_{20}^2 + \dots + \epsilon_{2n0}^2)^{1/2}.$$

If ϵ_0 is small enough (depending on the accuracy requirements), computation is finished. Because of the finite difference, accuracy requirements should not be higher than $O(h^2)$.

3. If the tolerance is not met in step (2), find a direction and magnitude such that new solution

$$\begin{pmatrix} f_1 \\ g_1 \\ \vdots \\ f_n \\ g_n \end{pmatrix}_1 = \begin{pmatrix} f_1 \\ g_1 \\ \vdots \\ f_n \\ g_n \end{pmatrix}_0 + \begin{pmatrix} \Delta f_1 \\ \Delta g_1 \\ \vdots \\ \Delta f_n \\ \Delta g_n \end{pmatrix} \quad (\text{B.9})$$

yields a smaller error $\epsilon_1 < \epsilon_0$.

4. Using the new solution as a starting point, search for the new "optimal" direction and magnitude for corrections to get a solution which makes the error smaller than ϵ_1 etc. The process repeats until the accuracy requirement is satisfied.

Some features of the computer program are recorded here for reference:

1. Because we minimize the residual errors of solutions by substituting back to the equations, the procedure guarantees that the finite difference equations are satisfied up to some required accuracy. With the errors between ODE's and Finite difference equations being of $O(h^2)$, the total accuracy is under control.

2. The optimal direction in Step 4 of the procedure is known to be the one given by Newton's method if the initial guess is good enough. Hence, we first calculate Newton direction in our program. The principle is as follows. Form the linear Taylor approximations of (B.8):

$$F_i^0 + \frac{\partial F_i^0}{\partial f_1} \Delta f_1 + \frac{\partial F_i^0}{\partial g_1} \Delta g_1 + \dots + \frac{\partial F_i^0}{\partial f_n} \Delta f_n + \frac{\partial F_i^0}{\partial g_n} \Delta g_n + \dots = 0. \quad (i=1,2,\dots,n) \quad (\text{B.10})$$

where

$$F_i^0 = F_i(f_1^0, g_1^0, \dots, f_n^0, g_n^0)$$

$$\frac{\partial F_1^0}{\partial f_1} = \frac{\partial F_1}{\partial f_1} (f_1^0, g_1^0, \dots, f_n^0, g_n^0)$$

If $f_1^0, g_1^0, \dots, f_n^0, g_n^0$ are close to the solution, we have reason to neglect higher order of $\Delta f_1, \dots, \Delta g_n$ in (B.10); then (B.10) can be written as

$$[DF(y_k)]\{\Delta y_k\} = -\{F(y_k)\} \quad (B.11)$$

where we denote

$$\{y_k\} = \{f_1^k, g_1^k, \dots, f_n^k, g_n^k\}$$

$$\{\Delta y_k\} = \{f_1^{k+1} - f_1^k, g_1^{k+1} - g_1^k, \dots, f_n^{k+1} - f_n^k, g_n^{k+1} - g_n^k\}$$

$$\{F(y_k)\} = \{F_1(y_k), F_2(y_k), \dots, F_n(y_k)\}.$$

$$[DF(y_k)] = \begin{vmatrix} \frac{\partial F_1(y_k)}{\partial y_1} & \frac{F_1(y_k)}{\partial y_2} & \dots & \frac{\partial F_1(y_k)}{\partial y_{2n}} \\ \frac{\partial F_2(y_k)}{\partial y_1} & \frac{F_2(y_k)}{\partial y_2} & \dots & \frac{\partial F_2(y_k)}{\partial y_{2n}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial F_{2n}(y_k)}{\partial y_1} & \frac{\partial F_{2n}(y_k)}{\partial y_2} & \dots & \frac{\partial F_{2n}(y_k)}{\partial y_{2n}} \end{vmatrix} \quad (B.12)$$

In our problem $[DF(y_k)]$ is a band matrix, both the length of the lower band and the upper band are 4 (excluding the diagonal). A UBC library routine

DGBAND is used to solve Eq. (B.11). The band of coefficients matrix are stored column by column with some zero elements forming a one dimension array. Upon using the band behavior, calculations are much simplified. The main feature is Gaussian elimination with partial pivoting and back substitution.

3. As mentioned in (2), our program first calculates the Newton direction to reach y_{k+1} from y_k . If $\epsilon_{k+1} < \epsilon_k$, the calculation terminates at the k th step. If $\epsilon_{k+1} > \epsilon_k$, we calculate the negative gradient direction of ϵ_k^2 which guarantees that ϵ_k would descend most rapidly in the local region at the price of slow convergence speed.

$$\begin{aligned}\epsilon_k^2 &= F_1^2(y_k) + F_2^2(y_k) + \dots + F_{2n}^2(y_k) \\ -\frac{1}{2} \nabla(\epsilon_k^2) &= -F_1(y_k) \cdot \nabla F_1(y_k) - F_2(y_k) \cdot \nabla F_2(y_k) \dots - F_{2n}(y_k) \cdot \nabla F_{2n}(y_k) \\ &= -[DF(y_k)]^T \{F(y_k)\}\end{aligned}\tag{B.13}$$

where $[DF(y_k)]$, $\{F(y_k)\}$ have already been obtained in (2).

4. One dimension search is used in both Newton direction and fast descent direction to get a smaller error.

5. The coarse subintervals are automatically equally divided to produce new finer subintervals for the improvement of accuracy. The new initial guess is linear interpolation of results of coarse subintervals.

Table (1): Numerical Solutions for Region (1.1) of the Flat Plate Case
($\nu = 0.3$)

k_H \ $\mu_m = \frac{k_V^2}{k_H}$		10^0	10^{-1}	10^{-2}	10^{-3}	0
10^0	$\hat{g}(x_i)$	$1.933-2^*$	$1.902-2$	$1.899-2$	$1.899-2$	$1.898-2$
	$\hat{n}_\theta^m$	$1.1 -1$	$1.1 -1$	$1.1 -1$	$1.1 -1$	$1.105-1$
	$\hat{f}(1)$	$8.567-2$	$2.709-2$	$8.568-3$	$2.709-3$	0
	$\hat{m}_r(x_i)$	$9.568-1$	$3.026-2$	$9.569-2$	$3.029-2$	0
10^{-1}	$\hat{g}(x_i)$	$1.933-3$	$1.902-3$	$1.899-3$	$1.899-3$	$1.898-3$
	$\hat{n}_\theta^m$	$1.1 -2$	$1.1 -2$	$1.1 -2$	$1.1 -2$	$1.105-2$
	$\hat{f}(1)$	$2.742-3$	$8.671-3$	$2.742-3$	$8.671-4$	0
	$\hat{m}_r(x_i)$	$3.045-1$	$9.628-2$	$3.045-2$	$9.628-3$	0
10^{-2}	$\hat{g}(x_i)$	$1.934-4$	$1.902-4$	$1.899-4$	$1.899-4$	$1.898-4$
	$\hat{n}_\theta^m$	$1.1 -3$	$1.1 -3$	$1.1 -3$	$1.1 -3$	$1.105-3$
	$\hat{f}(1)$	$8.681-3$	$2.745-3$	$8.681-4$	$2.745-4$	0
	$\hat{m}_r(x_i)$	$9.634-2$	$3.047-2$	$9.634-3$	$3.047-3$	0
10^{-4}	$\hat{g}(x_i)$	$1.934-6$	$1.902-6$	$1.899-6$	$1.899-6$	$1.898-6$
	$\hat{n}_\theta^m$	$1.1 -5$	$1.1 -5$	$1.1 -5$	$1.1 -5$	$1.105-3$
	$\hat{f}(1)$	$8.682-4$	$2.746-4$	$8.682-5$	$2.746-5$	0
	$\hat{m}_r(x_i)$	$9.635-3$	$3.047-3$	$9.635-4$	$3.047-4$	0

*The notation $x \pm y$ means $(x) \times 10^{\pm y}$

Table (2): Numerical Solutions for Region (1.2) of the Flat Plate Case
 $(\nu = 0.3)$

k_V \ $\mu_m = \frac{k_V^2}{k_H}$		10^0	10^1	10^2	10^3	10^4
10^0	$\hat{g}(x_i)$	$1.933-2^*$	$2.247-3$	$5.393-4$	$3.685-4$	$3.5 -4$
	$\hat{n}_\theta^m$	$1.1 -1$	$1.2 -2$	$\begin{bmatrix} 2.7 -3 \\ -1.56-3 \end{bmatrix}$	$\begin{bmatrix} 1.8 -3 \\ -2.03-3 \end{bmatrix}$	$\begin{bmatrix} 1.7-3 \\ 2.1-3 \end{bmatrix}$
	$\hat{f}(1)$	$8.567-2$	$8.669-2$	$8.680-2$	$8.681-2$	$8.682-2$
	$\hat{m}_r(x_i)$	$9.568-1$	$9.627-1$	$9.633-1$	$9.634-1$	$9.635-1$
10^{-1}	$\hat{g}(x_i)$	$1.934-4$	$2.248-5$	$5.394-6$	$3.686-6$	$3.5 -6$
	$\hat{n}_\theta^m$	$1.1 -3$	$1.2 -4$	$\begin{bmatrix} 2.7 -5 \\ -1.56-5 \end{bmatrix}$	$\begin{bmatrix} 1.8 -5 \\ -2.03-5 \end{bmatrix}$	$\begin{bmatrix} 1.7-5 \\ -2.1-5 \end{bmatrix}$
	$\hat{f}(1)$	$8.681-3$	$8.682-3$	$8.682-3$	$8.682-3$	$8.682-3$
	$\hat{m}_r(x_i)$	$9.634-2$	$9.635-2$	$9.635-2$	$9.635-2$	$9.635-2$
10^{-2}	$\hat{g}(x_i)$	$1.934-6$	$2.248-7$	$5.394-8$	$3.686-8$	$3.5 -8$
	$\hat{n}_\theta^m$	$1.1 -5$	$1.2 -6$	$\begin{bmatrix} 2.7 -7 \\ -1.65-7 \end{bmatrix}$	$\begin{bmatrix} 1.8 -7 \\ -2.03-7 \end{bmatrix}$	$\begin{bmatrix} 1.7-7 \\ -2.1-7 \end{bmatrix}$
	$\hat{f}(1)$	$8.682-4$	$8.682-4$	$8.682-4$	$8.682-4$	$8.682-4$
	$\hat{m}_r(x_i)$	$9.635-3$	$9.635-3$	$9.635-3$	$9.635-3$	$9.635-3$

* The notation $x \pm y$ means $(x) \times 10^{\pm y}$

Table (3): Numerical Solutions for Region (2) of the Flat Plate Case ($\nu = 0.3$)

k_H	$\mu_\ell = \frac{k_V^{2/3}}{k_H}$	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	0
10^0	$\hat{g}(x_i)$	$1.933-2^*$	$1.899-2$	$1.899-2$	$1.899-2$	$1.899-2$	$1.898-2$
	$\hat{n}_\theta^m$	$1.1 -1$	$1.1 -1$	$1.1 -1$	$1.1 -1$	$1.1 -1$	$1.1 -1$
	$\hat{f}(1)$	$8.567-2$	$2.709-3$	$8.568-5$	$2.709-6$	$8.568-8$	0
	$\hat{m}_r(x_i)$	$9.568-1$	$3.029-2$	$9.579-4$	$3.029-5$	$9.579-7$	0
10^1	$\hat{g}(x_i)$	$4.299-1$	$1.901-1$	$1.899-1$	$1.899-1$	$1.899-1$	$1.898-1$
	$\hat{n}_\theta^m$	$\begin{bmatrix} 2.2 +0 \\ -8.45-1 \end{bmatrix}$	$1.1 +0$	$1.1 +0$	$1.1 +0$	$1.1 +0$	$1.1 +0$
	$\hat{f}(1)$	$2.198+0$	$7.653-2$	$2.420-3$	$7.654-5$	$2.420-6$	0
	$\hat{m}_r(x_i)$	$2.700+1$	$9.043-1$	$2.860-2$	$9.044-4$	$2.860-5$	0
10^2	$\hat{g}(x_i)$	$1.386+1$	$1.983+0$	$1.899+0$	$1.899+0$	$1.899+0$	$1.898+0$
	$\hat{n}_\theta^m$	$\begin{bmatrix} 6.0 +1 \\ -4.16+1 \end{bmatrix}$	$1.1 +1$	$1.1 +1$	$1.1 +1$	$1.1 +1$	$1.1 +1$
	$\hat{f}(1)$	$1.210+1$	$1.109+0$	$3.562-2$	$1.126-3$	$3.562-5$	0
	$\hat{m}_r(x_i)$	$3.499+2$	$2.011+1$	$6.418-1$	$2.030-2$	$6.418-4$	0
10^4	$\hat{g}(x_i)$	$2.156+3$		$1.900+2$		$1.899+2$	$1.898+2$
	$\hat{n}_\theta^m$	$\begin{bmatrix} 7.2 +3 \\ -4.67+3 \end{bmatrix}$		$1.1 +3$		$1.1 +3$	$1.1 +3$
	$\hat{f}(1)$	$1.283+2$		$4.261-1$		$4.261-4$	0
	$\hat{m}_r(x_i)$	$3.362+4$		$1.094+2$		$1.094-1$	0
10^6	$\hat{g}(x_i)$	$2.254+5$		$1.899+4$		$1.899+4$	$1.898+4$
	$\hat{n}_\theta^m$	$\begin{bmatrix} 6.8 +5 \\ -4.68+5 \end{bmatrix}$		$1.1 +5$		$1.1 +5$	$1.1 +5$
	$\hat{f}(1)$	$1.294+3$		$4.039+0$		$4.039-3$	0
	$\hat{m}_r(x_i)$	$3.297+6$		$1.136+4$		$1.136+1$	0
10^8	$\hat{g}(x_i)$	$2.265+7$		$1.900+6$		$1.899+6$	$1.898+6$
	$\hat{n}_\theta^m$	$\begin{bmatrix} 6.8 +7 \\ -4.69+7 \end{bmatrix}$		$1.1 +7$		$1.1 +7$	$1.1 +7$
	$\hat{f}(1)$	$1.296+4$		$3.991+1$		$3.992-2$	0
	$\hat{m}_r(x_i)$	$3.290+8$		$1.136+6$		$1.136+3$	0

* The notation $x \pm y$ means $(x) \times 10^{\pm y}$

Table (4): Dimensionless Boundary Layer Width Δx Near the Inner Edge of the Flat Plate in Region (2) of the Parameter Space with $\nu = 0.3$.

$\begin{matrix} k_H \\ \mu_\ell \end{matrix}$	10^1	10^2	10^3	10^4	10^5	10^6
10^0	0.37	0.14	0.0475	0.0160	0.006	0.0016
10^{-2}	0.37	0.24	0.125	0.044	0.017	0.0060

Table (5): Numerical Solutions for Region (3) of the Flat Plate Case ($\nu = 0.3$)

k_V	$\mu_\ell = \frac{k_V^{2/3}}{k_H}$	10^0	10^1	10^2	10^3	10^4	∞
10^0	$\hat{g}(x_i)$	1.933-2	2.247-3	5.393-4	3.685-4	3.512-4	3.5 -4
	$\hat{n}_\theta^m$	1.1 -1	1.2 -2	$\begin{bmatrix} 2.7 -3 \\ -1.56-3 \end{bmatrix}$	$\begin{bmatrix} 1.8 -3 \\ -2.03-3 \end{bmatrix}$	$\begin{bmatrix} 1.7 -3 \\ -2.07-3 \end{bmatrix}$	$\begin{bmatrix} 1.7-3 \\ -2.1-3 \end{bmatrix}$
	$\hat{f}(1)$	8.567	8.669-2	8.680-2	8.681-2	8.681-2	8.681-2
	$\hat{m}_r(x_i)$	9.579-1	9.627-1	9.633-1	9.634-1	9.634-1	9.635-1
10^1	$\hat{g}(x_i)$	1.190-1	4.243-2	3.480-2	3.404-2	3.397-2	3.4 -2
	$\hat{n}_\theta^m$	6.3 -1	2.1 -1	$\begin{bmatrix} 1.7 -1 \\ -1.99-1 \end{bmatrix}$	$\begin{bmatrix} 1.6 -1 \\ 2.01-1 \end{bmatrix}$	$\begin{bmatrix} 1.6 -1 \\ -2.01-1 \end{bmatrix}$	$\begin{bmatrix} 1.6 -1 \\ -2.0 -1 \end{bmatrix}$
	$\hat{f}(1)$	8.062-1	8.492-1	8.537-1	8.542-1	8.542-1	8.542-1
	$\hat{m}_r(x_i)$	9.262+0	9.512+0	9.538+0	9.540+0	9.541+0	9.541+0
10^2	$\hat{g}(x_i)$	1.630+0	1.447+0	1.430+0	1.428+0	1.428+0	1.4 +0
	$\hat{n}_\theta^m$	$\begin{bmatrix} 7.8 +0 \\ -5.26+0 \end{bmatrix}$	$\begin{bmatrix} 6.8 +0 \\ -7.50+0 \end{bmatrix}$	$\begin{bmatrix} 6.7 +0 \\ -7.74+0 \end{bmatrix}$	$\begin{bmatrix} 6.7 +0 \\ -7.76+0 \end{bmatrix}$	$\begin{bmatrix} 6.7 +0 \\ -7.77+0 \end{bmatrix}$	$\begin{bmatrix} 6.7+0 \\ -7.8+0 \end{bmatrix}$
	$\hat{f}(1)$	4.674+0	5.165+0	5.216+0	5.222+0	5.222+0	5.222+0
	$\hat{m}_r(x_i)$	6.895+1	7.155+1	7.182+1	7.185+1	7.185+1	7.185+1
10^3	$\hat{g}(x_i)$	1.387+1	1.342+1	1.338+1	1.337+1	1.337+1	1.3 +1
	$\hat{n}_\theta^m$	$\begin{bmatrix} 6.1 +1 \\ -4.16+1 \end{bmatrix}$	$\begin{bmatrix} 5.8 +1 \\ -5.50+1 \end{bmatrix}$	$\begin{bmatrix} 5.8 +1 \\ -5.65+1 \end{bmatrix}$	$\begin{bmatrix} 5.8 +1 \\ -5.66+1 \end{bmatrix}$	$\begin{bmatrix} 5.8 +1 \\ -5.66+1 \end{bmatrix}$	$\begin{bmatrix} 5.8+1 \\ -5.7+1 \end{bmatrix}$
	$\hat{f}(1)$	1.210+1	1.353+1	1.368+1	1.369+1	1.370+1	1.370+1
	$\hat{m}_r(x_i)$	3.493+2	3.551+2	3.556+2	3.557+2	3.557+2	3.557+2
10^6	$\hat{g}(x_i)$	2.154+3		2.120+3		2.120+3	2.120+3
	$\hat{n}_\theta^m$	$\begin{bmatrix} 7.2 +3 \\ -4.67+3 \end{bmatrix}$		$\begin{bmatrix} 7.0 +3 \\ -6.36+3 \end{bmatrix}$		$\begin{bmatrix} 7.0 +3 \\ -6.38+3 \end{bmatrix}$	$\begin{bmatrix} 7.0 +3 \\ -6.4 +3 \end{bmatrix}$
	$\hat{f}(1)$	1.283+2		1.516+2		1.519+2	1.519+2
	$\hat{m}_r(x_i)$	3.362+4		3.388+4		3.388+4	3.388+4
10^9	$\hat{g}(x_i)$	2.254+5		2.222+5		2.222+5	2.2 +5
	$\hat{n}_\theta^m$	$\begin{bmatrix} 6.9 +5 \\ -4.69+5 \end{bmatrix}$		$\begin{bmatrix} 6.8 +5 \\ -6.39+5 \end{bmatrix}$		$\begin{bmatrix} 6.8 +5 \\ -6.41+5 \end{bmatrix}$	$\begin{bmatrix} 6.8+5 \\ -6.4+5 \end{bmatrix}$
	$\hat{f}(1)$	1.294+3		1.548+3		1.551+3	1.551+3
	$\hat{m}_r(x_i)$	3.297+6		3.321+6		3.321+6	3.321+6

Table (6): Scalings in Parameter Space Regions

Region			Multiplicative Factors	
(I): $k_V < 1$ $k_H < 1$	(I.1): $k_G < 1$	(I.1.1): $\mu_m \equiv \frac{k_V^2}{k_H} < 1$	$[k_V, k_H],$	$\begin{bmatrix} 1 & k_H^{\mu_G} & k_H & 1 \\ 1 & \mu_m^{\mu_G} & \mu_m & 1 \end{bmatrix}, \quad [1]$
	$\mu_G \equiv \frac{k_G}{k_V} < 1$	(I.1.2): $\mu_m > 1$	$[k_V, k_V^2],$	$\begin{bmatrix} 1 & k_V^2 \mu_G & k_V^2 & 1 \\ 1 & \mu_G & 1 & \mu_m^{-1} \end{bmatrix}, \quad [\mu_m^{-1}]$
	(I.2): $k_G < 1$ $\mu_G > 1$	(I.2.1) $k_H^{\mu_G} > 1$	$[k_H, k_G k_H],$	$\begin{bmatrix} 1 & 1 & k_H & (\mu_G k_H)^{-1} \\ 1 & k_V^2 & k_V^2 k_H & 1 \end{bmatrix}, \quad [1]$
		(I.2.2): $k_H^{\mu_G} < 1$	$[k_V, \mu_G^{-1}],$	$\begin{bmatrix} 1 & 1 & \mu_G^{-1} & 1 \\ 1 & k_G^2 & k_V k_G & k_H^{\mu_G} \end{bmatrix}, \quad [k_H^{\mu_G}]$
	(I.3): $k_G > 1$ $(k_L \equiv \max[k_V, k_H])$		$[k_L, k_L],$	$\begin{bmatrix} k_G^{-1} & 1 & k_L k_G^{-1} & \frac{k_V}{k_L} k_G^{-1} \\ k_G^{-1} & 1 & k_L k_G^{-1} & \frac{k_H}{k_L} k_G^{-1} \end{bmatrix}, \quad [\frac{k_H}{k_L}]$

continued ...

Table (6): Scalings in Parameter Space Regions (continued)

Region			Multiplicative Factors			
(II): $k_G < k_V^{1/3}$ $k_G < k_H^{1/2}$	(II.1): $\mu_\ell^3 \equiv \frac{k_V^2}{k_H^3} > 1, \quad k_V > 1$ $(\mu_G < 1)$		$[k_V^{1/3}, k_V^{2/3}]$,	$\begin{bmatrix} k_V^{-2/3} & k_G k_V^{-1/3} & 1 & 1 \\ 1 & k_G k_V^{-1/3} & 1 & \mu_\ell^{-1} \end{bmatrix}$,	$[\mu_\ell^{-1}]$	
	(II.2): $\mu_\ell^3 < 1$	(II.2.1): $k_H \mu_G < 1$	$[\frac{k_V}{k_H}, k_H]$,	$\begin{bmatrix} k_H^{-1} & k_H \mu_G & 1 & 1 \\ 1 & k_H \mu_G \mu_\ell^3 & \mu_\ell^3 & 1 \end{bmatrix}$,	$[1]$	
	$k_H > 1$	(II.2.2): $k_H \mu_G > 1$	$[k_G, k_H]$,	$\begin{bmatrix} k_H^{-1} & 1 & 1 & (k_H \mu_G)^{-1} \\ 1 & k_G^2 k_H^{-1} & k_G^2 k_H^{-1} & 1 \end{bmatrix}$,	$[1]$	
(III): $k_V^{1/3} < k_G$ $k_H^{1/2} < k_G$	(III.1): $\mu_\ell^3 > 1,$ $k_V > 1$		$[k_G, k_V^{2/3}]$,	$\begin{bmatrix} k_V^{-2/3} & 1 & 1 & k_V^{1/3} k_G^{-1} \\ k_V^{2/3} k_G^{-2} & 1 & 1 & k_H k_G^{-2} \end{bmatrix}$,	$[k_H k_V^{-2/3}]$	
	(III.2): $\mu_\ell^3 < 1$ $k_H > 1$		$[k_G, k_G^2]$,	$\begin{bmatrix} k_G^{-2} & 1 & 1 & k_V k_G^{-3} \\ 1 & 1 & 1 & k_H k_G^{-2} \end{bmatrix}$,	$[k_H k_G^{-2}]$

Fig.3 Horizontal force equilibrium

Fig.4 Vertical force equilibrium

(a)

(b)

(c)

Fig.5 Moment equilibrium

(a) Hub-shell configuration

(b) A segment of shell

Fig.6 A normally loaded, spinning shallow shell

Fig.7 k_H - k_V parameter plane of flat plates

Fig.8 k_G - k_V - k_H parameter space of shells

Fig.9 Region(I) of the parameter space for shells

Fig.10 Region(II) of the parameter space for shells