AN ILLUSTRATIVE EXAMPLE

of the

ELLIPSOIDAL PENDULUM

by

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1. The Problem Considered.

In a paper to be presented to the Royal Society of Canada in May, 1928, Dr. D. Buchanan has obtained equations which define the periodic orbits of a particle constrained to move, under gravity, on the surface of the ellipsoid of revolution

\[ x^2 + y^2 + z^2(1+\varepsilon) - l^2 = 0, \]

\( \varepsilon \) being a parameter and \( l \) a constant. It is assumed that the surface is smooth and conforms somewhat closely to a sphere. For \( \varepsilon = 0 \) the problem reduces to that of the spherical pendulum which is considered by Moulton in his memoir on "Periodic Orbits." For \( \varepsilon \neq 0 \) Dr. Buchanan has designated the problem as the "ellipsoidal pendulum."

The object of this thesis is to construct the orbit determined by assigning numerical values to the various constants on which the solution depends. It is first necessary to give a brief outline of Dr. Buchanan's paper. This is done in Part II. In Part III several series which enter into the solution of the \( z \)-equation are computed. In Part IV numerical values are assigned to all constants and the resulting orbit eventually constructed.
Part II

Resume of Dr. Buchanan's Paper on the "Ellipsoidal Pendulum"

1. The Differential Equations.

As suggested by the form of equation (1) the origin of co-ordinates is taken at the centre of the ellipsoid with the z-axis vertical and positive upwards. The particle is assumed to be of unit mass and to move on the given surface without friction. The differential equations are then found to be

\[
\begin{align*}
\frac{d^2 x}{dt^2} &= \chi \left[ x_0 + x_1 e + x_2 e^2 + \cdots + x_n e^n \right], \\
\frac{d^2 y}{dt^2} &= \gamma \left[ x_0 + x_1 e + x_2 e^2 + \cdots + x_n e^n \right], \\
\frac{d^2 z}{dt^2} &= z_0 + e z_1 + e^2 z_2 + \cdots + e^n z_n,
\end{align*}
\]

where

\[
\begin{align*}
x_0 &= \frac{g}{\ell} (3z - c), \\
x_1 &= \frac{1}{\ell^2} \left[ 3g z - x_0 z^2 - (\frac{dz}{dt})^2 \right], \\
x_2 &= -\frac{z_1}{\ell^2} [x_0 + x_1], \\
&\vdots \\
z_0 &= \frac{g}{\ell^2} (3 z^2 - c, z - \ell^2), \\
z_1 &= \frac{1}{\ell^2} \left[ 3g z^2 - c, z - z (\frac{dz}{dt})^2 - z^2 z_0 \right], \\
z_2 &= -\frac{z_1}{\ell^2} [z^2 z_0 + z (\frac{dz}{dt})^2 + z^2 z_1], \\
&\vdots
\end{align*}
\]
The constants \( q \) and \( C \) are the acceleration due to gravity and the energy constant respectively.

2. The Spherical Pendulum.

For \( \epsilon = 0 \) equations (2) reduce to those of the spherical pendulum. We shall require here only the \( z \)-equation, viz:

\[
\frac{d^2 z}{d\tau^2} = \frac{q}{\epsilon^2} (3 z^2 - c, z - l^2)
\]

and its solution as obtained by Moulton.

Equation (5) admits the integral

\[
\left( \frac{dz}{d\epsilon} \right)^2 = \frac{q}{\epsilon^2} (z_2 - c_2) z^2 - q (z - c_2),
\]

where \( c_2 \) is the constant of integration. If the right side of (6) has the roots \( \alpha_1, \alpha_2, \alpha_3 \) having \( \alpha_1 \leq \alpha_2 \leq \alpha_3 \), then there are made in (5) the substitutions

\[
\begin{align*}
\tau &= \alpha_3 + (\alpha_2 - \alpha_3) \mu, \\
\mu &= \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}, (0 \leq \mu \leq 1), \\
\tau &= \sqrt{\frac{q(\alpha_2 - \alpha_3)}{2 \epsilon^2(1 + \delta)}} (\tau - t_0).
\end{align*}
\]

On solving the resulting equation for \( \omega \) the following solutions are obtained for \( \tau \) and \( \delta \):

\[
\tau = \psi(\tau) = \alpha_3 + (\alpha_1 - \alpha_3) \left[ \frac{1}{2} (1 - \cos 2\tau) \mu + \frac{1}{16} (1 - \cos 4\tau) \mu^2 + \ldots \right]
\]

\[
\delta = \frac{1}{2} \mu + \frac{11}{16} \mu^2 + \ldots
\]

If the cases of the simple pendulum and of revolution with infinite speed in the \( xy \)-plane are excluded, then \( \alpha_1, \alpha_2, \alpha_3, c_1, c \) satisfy the relations

\[
\begin{align*}
-l < \alpha_3 < 0, & \quad -l < \alpha_2 < l, \quad \alpha_1 > l, \\
c_1 &= 2 \alpha_1 + 4 \alpha_3 + 2 \mu (\alpha_1 - \alpha_3), \\
l^2 &= \mu (\alpha_3^2 - \alpha_2^2) - \alpha_3 (2 \alpha_1 + \alpha_3).
\end{align*}
\]
3. The $z$-equation.

The $z$-equation of (2) is transformed by means of (7c)
and by putting

$$z = \psi + \omega.$$  \hfill (10)

This equation then becomes

$$\omega'' + \left[ 4 + \mu \cos 2\tau - \mu \left( \frac{3}{8} - 3 \cos 2\tau - \frac{3}{4} \cos 4\tau \right) + \ldots \right] \omega = \psi C_0 + \psi C_1 + \psi^2 C_2 + \psi^3 C_3 + \psi^4 C_4 + \ldots$$

where the primes signify differentiation with respect to $\tau$.

The various $C$'s denote series similar to $\psi$, being power series in $\mu$ with cosines of even multiples of $\tau$ in the coefficients.

In the sequel, as here, such series are denoted by the foundation letter $C$ with subscripts, superscripts or strokes.

4. The Equation of Variation and its Solution.

If we equate the left member of (11) to zero we obtain

the equation of variation,

$$\omega'' + \left[ 4 + \mu \cos 2\tau - \mu \left( \frac{3}{8} - 3 \cos 2\tau - \frac{3}{4} \cos 4\tau \right) + \ldots \right] \omega = 0$$

of which the generating solution is $z = \psi$. Applying the method of solution developed by Poincaré for this type of equation, we find that its solution is

$$\omega = A \varphi + B (\chi + \tau \varphi),$$

where $A$ and $B$ are constants of integration, and

$$\varphi = \sin 2\tau + \frac{1}{2\mu} \sin 4\tau + \frac{1}{256} \mu^2 (-3 \sin 2\tau + 32 \sin 4\tau + 9 \sin 6\tau) + \ldots$$

$$\chi = \cos 2\tau + \frac{1}{4} \mu \left( -3 + 2 \cos 2\tau + \cos 4\tau \right) + \frac{1}{256} \mu^2 (-19 \tau + 119 \cos 2\tau + 6 \cos 4\tau + 9 \cos 6\tau) + \ldots$$

$$K = \frac{15}{16} \mu^2 + \text{terms of higher degree in } \mu.$$
5. The Integration of Equation (11).

Equation (11) will now be integrated as a power series in $e$. Accordingly, the substitution

$$\omega = \sum_{j=1}^{\infty} \omega_j e^j$$

is made in (11). On equating coefficients of like powers of $e$ in the resulting equation, cited as (11'), a sequence of differential equations is determined which give successively the values of $\omega_j$. The initial condition $\omega_j(0) = 0$ is imposed. It then follows from (15) that

$$\omega_j'(0) = 0, \quad (j = 1, \ldots, \infty).$$

The terms in $e$ in (11') yield the equation

$$\omega_{,\nu} + \left[4 + 6 \mu \sigma a_\tau \tau - \left(3 \frac{3}{2} e_\nu a_\tau \tau - \frac{3}{4} \sigma^2 a_\tau \tau \right) \mu^2 + \ldots \right] \omega_{,\nu} = \omega_{,\nu}.$$  

where $W_1 = C_{10}$ in (15). The complementary function of (17) is

$$\omega_{,\nu} = a_j (\chi + k \tau \phi).$$

The particular integral is obtained by varying the parameters $a_j$ and $b_j$. It is found that

$$a_j = \frac{1}{\Delta} \left[ - \int \phi \omega, a \tau \tau - k \tau \int \phi \omega, a \tau \tau + \int \phi \omega, a \tau \tau - \int \phi \omega, a \tau \tau \right],$$

$$b_j = \frac{1}{\Delta} \int \phi \omega, a \tau \tau,$$

the determinant $\Delta$ being constant. It is shown that

$$\Delta = - \phi' (0) \chi (0).$$

The product $\chi \omega_1$ gives rise to constant terms of the type

$$a_j = a_{j(0)} a_{j(1)} a_{j(2)} a_{j(3)} \ldots.$$  

When the integrations in (19) are performed and the resulting values of $a_j$ and $b_j$ substituted in (18) the complete solution for $\omega$ becomes

$$\omega_j = \Lambda_j \phi + \beta_j (\chi + k \tau \phi) + \psi_j \tau \phi + \frac{1}{\Delta} \int \phi \omega, a \tau \tau - \int \phi \omega, a \tau \tau \right],$$

$$b_j = \frac{1}{\Delta} \int \phi \omega, a \tau \tau,$$
where \( A_i \) and \( B_i \) are the constants of integration and \( \overline{C_i} \) is a cosine series described in \( \S \) 3. As \( \omega_i \) is to be periodic we must have

\[
- \frac{1}{k} \alpha_i \pm \beta_i = \frac{1}{\mu} \alpha_i \tag{22}
\]

where \( \alpha_i \) is a power series in \( \mu \) similar to \( \alpha_i \). From the initial conditions (16) it follows that \( A_i = 0 \). Hence the desired solution at this step is

\[
\omega_i = \frac{1}{\mu} C_i. \tag{23}
\]

**Coefficients of \( \varepsilon^2 \)**

The coefficients of \( \varepsilon^2 \) in \( (11') \) give

\[
\omega_{2''} + \left[ 4 + 6 \mu^2 \alpha_2 + \cdots \right] \omega_{2'} = \omega_{2} = \frac{1}{\mu} + C_2. \tag{24}
\]

The integration of this equation is similar to that of (17) and

\[
\begin{align*}
\omega_2 &= \alpha_2 (\nu) + \beta_2 (\nu) + \cdots, \\
\omega_2 &= \frac{1}{\mu} C_2.
\end{align*}
\tag{25}
\]

The succeeding steps are similar to the preceding and the solution at the \( n \)th step is

\[
\omega_n = \frac{1}{\mu + \mathcal{N} - 2} C(\nu). \tag{26}
\]

This completes the solution of (11). On substituting the resulting value of \( \omega \) in (10) the solution of (10) for the vertical motion becomes

\[
\mathcal{Z} = \nu + \frac{\varepsilon}{\mu^2} C(\nu) + \frac{\varepsilon^2}{\mu^3} C(\nu) + \cdots + \frac{\varepsilon^\mathcal{N}}{\mu^{\mathcal{N} - 2}} C(\nu) + \cdots. \tag{26}
\]

6. The Horizontal Motion.

As the \( \kappa \) and \( \eta \)-equations in (2) are the same it is necessary only to consider the \( \chi \)-equation. When the
transformation (7c), (26) and
\[ \zeta = \rho \omega^4 \]
are applied to (2a) it becomes
\[ x'' + x \left[ a_{00}^{(0)} + (a_{00}^{(0)} + a_{02}^{(0)} \cos 2 t) \right] \omega^2 + \left( b_{00}^{(2)} + a_{02}^{(2)} \cos 2 t + a_{04}^{(2)} \cos 2 t \right) \mu^2 + \left( b_{00}^{(3)} + b_{02}^{(3)} \cos 2 t + a_{04}^{(3)} \cos 4 t + a_{06}^{(3)} \cos 6 t \right) \mu^3 \]
\[ (27) \]
where
\[ a_{00}^{(0)} = \frac{2 (2 \alpha + \alpha_3^2)}{\alpha_1 - \alpha_3}, \quad a_{00}^{(1)} = \frac{3 \alpha_1}{\alpha_1 - \alpha_3}, \quad a_{02}^{(0)} = 3 \cos 2 t, \]
\[ b_{00}^{(2)} = a_{00}^{(2)}, \quad b_{02}^{(2)} = a_{02}^{(2)}, \quad b_{04}^{(2)} = a_{04}^{(2)}, \quad \]
\[ b_{00}^{(3)} = a_{00}^{(3)} + a_{02}^{(3)}, \quad b_{02}^{(3)} = a_{02}^{(3)}, \quad b_{04}^{(3)} = a_{04}^{(3)}, \quad \]
\[ b_{06}^{(3)} = a_{06}^{(3)}, \quad \]

The constant term and the terms in \( \mu \) in (27) are identical with the corresponding terms in the \( x \) - equation of the spherical pendulum. The solutions of (27) will be similar in form to those of the spherical pendulum and with the exception of two slight variations noted below there will be no difference in the constant term and the terms in \( \mu \) in the resulting equation. Making use of the solutions obtained by Moulton, we have
\[ x = A_1 \left[ x, \cos \left( \frac{t}{\beta} - x_2 \sin \left( \frac{t}{\beta} \right) \right) \right], \]
\[ y = B_2 \left[ x_1 \sin \left( \frac{t}{\beta} \right) + x_2 \cos \left( \frac{t}{\beta} \right) \right], \]
The constants $\alpha_1$ and $\alpha_3$, differ from the corresponding constants in the spherical pendulum as the expression for $\beta^2$ contains $(1 + \varepsilon)$ and those for both $\alpha_1^2$ and $\beta_2^2$ contain $\mathcal{Z}(0)$ which in this problem has a slightly different value to that of the spherical pendulum.

This completes the solution.
Part III

The Algebraic Expressions for Certain Series.

1. Introduction.

The object of this part is to compute all coefficients in equation (11) which contribute terms in $\zeta$ and $\zeta^2$ to (11'). These coefficients are $c_{10}, c_{20}, c_{02}, c_{11}$, that is those the sum of whose subscripts is not greater than two.

It will be remembered that these coefficients resulted from the transformation of (2c) by (7c) and (10). Hence we have

$$
c_{10} = \frac{2l^2(1+\delta)}{g(\alpha_1 - \alpha_3)} \zeta_1(\psi),
$$

$$
c_{20} = \frac{2l^2(1+\delta)}{g(\alpha_1 - \alpha_3)} \zeta_2(\psi),
$$

$$
c_{02} = \frac{2l^2(1+\delta)}{g(\alpha_1 - \alpha_3)} (\text{coefficients of } \omega^2 \text{ in } Z_0)
$$

$$
= \frac{2l^2(1+\delta)}{g(\alpha_1 - \alpha_3)} \times 3,
$$

$$
c_{11} = \frac{2l^2(1+\delta)}{g(\alpha_1 - \alpha_3)} (\text{coefficients of } \omega \text{ in } Z_1),
$$

$$
= \frac{2l^2(1+\delta)}{g(\alpha_1 - \alpha_3)} \cdot \frac{1}{\ell^2} \left[ 6g \psi - c_1 - (\frac{d\psi}{dt})^2 - 2\psi \zeta_0(\psi) - 4 \frac{g}{\ell^2} (\psi - c_1) \right].
$$

These series and all others contributing to the $\zeta$ - equation will be carried out to and including terms in $\mu^4$.

2. A Number of Auxiliary Functions of $\chi$.

With the exception of $c_{02}$, the coefficients (29) are
the sums of a number of different functions of $X$. Accordingly the computation will be simplified considerably by first computing the auxiliary functions

$$\psi^2, \left(\frac{d\psi}{dt}\right)^2, \psi^2 Z_0(\psi), Z_0(\psi), Z_1(\psi), \psi^2 Z_1(\psi), \psi Z_0(\psi), \psi^2 (\psi - c_1)$$

It is found that

$$\psi^2 = a_{11} + a_{12} (1 - \cos 2\tau) + \left(\frac{d\psi}{dt}\right)^2, \psi^2 Z_0(\psi), Z_0(\psi), Z_1(\psi), \psi^2 Z_1(\psi), \psi Z_0(\psi), \psi^2 (\psi - c_1)$$

$$\psi^2 = a_{11} + a_{12} (1 - \cos 2\tau) + \left(\frac{d\psi}{dt}\right)^2, \psi^2 Z_0(\psi), Z_0(\psi), Z_1(\psi), \psi^2 Z_1(\psi), \psi Z_0(\psi), \psi^2 (\psi - c_1)$$

$$\psi^2 = a_{11} + a_{12} (1 - \cos 2\tau) + \left(\frac{d\psi}{dt}\right)^2, \psi^2 Z_0(\psi), Z_0(\psi), Z_1(\psi), \psi^2 Z_1(\psi), \psi Z_0(\psi), \psi^2 (\psi - c_1)$$

$$\psi^2 = a_{11} + a_{12} (1 - \cos 2\tau) + \left(\frac{d\psi}{dt}\right)^2, \psi^2 Z_0(\psi), Z_0(\psi), Z_1(\psi), \psi^2 Z_1(\psi), \psi Z_0(\psi), \psi^2 (\psi - c_1)$$

where

$$\alpha = \alpha_1 - \alpha_3$$

$$\begin{align*}
a_{11} &= \alpha_3^2 \frac{a_1}{2}, \\
a_{12} &= \alpha_3 \frac{a_1}{2}, \\
a_{13} &= \alpha_3 (3\alpha + \alpha_3), \\
a_{14} &= \alpha_3 \left(\frac{\alpha_1}{2} - \frac{\alpha_3}{2}\right), \\
a_{15} &= \frac{\alpha_3}{2} (\alpha - \alpha_3), \\
a_{21} &= \frac{a_1}{2} (3\alpha_3 - c_3, \alpha_3 - \alpha_3), \\
a_{22} &= \frac{a_1}{2} (3\alpha_3 - c_3, \alpha_3 - \alpha_3), \\
a_{23} &= \frac{a_1}{2} (3\alpha_3 - c_3, \alpha_3 - \alpha_3), \\
a_{24} &= \frac{a_1}{2} (3\alpha_3 - c_3, \alpha_3 - \alpha_3), \\
a_{25} &= \frac{a_1}{2} (3\alpha_3 - c_3, \alpha_3 - \alpha_3)
\end{align*}$$
\[ a_{31} = a_{11} a_{21} \]
\[ a_{32} = a_{11} a_{22} + a_{12} a_{21} \]
\[ a_{33} = a_{11} a_{23} + \frac{3}{2} a_{12} a_{22} + a_{13} a_{21} \]
\[ a_{34} = a_{11} a_{24} - 2a_{12} a_{22} + a_{14} a_{21} \]
\[ a_{35} = a_{11} a_{25} + \frac{1}{2} a_{12} a_{22} + a_{15} a_{21} \]

\[ a_{41} = 3g a_{11} - c_1 a_3 - a_{31} \]
\[ a_{42} = 3g a_{12} - c_1 a_3 \]
\[ a_{43} = 3g a_{13} - c_1 a_3 \]
\[ a_{44} = 3g a_{14} - a_{34} \]
\[ a_{45} = 3g a_{15} - c_1 a_3 \]

\[ a_{51} = \frac{1}{\lambda} a_{11} a_{41} \]
\[ a_{52} = \frac{1}{\lambda} (a_{11} a_{42} + a_{12} a_{41}) \]
\[ a_{53} = \frac{1}{\lambda} (a_{11} a_{43} + \frac{3}{2} a_{12} a_{42} + a_{13} a_{41}) \]
\[ a_{54} = \frac{1}{\lambda} (a_{11} a_{44} - 2a_{12} a_{42} + a_{14} a_{41}) \]
\[ a_{55} = \frac{1}{\lambda} (a_{11} a_{45} + \frac{1}{2} a_{12} a_{42} + a_{15} a_{41}) \]
\[ a_{56} = a_{11} \]

\[ a_{61} = a_{21} a_3 \]
\[ a_{62} = a_{21} \frac{a_3}{2} + a_{22} a_3 \]
\[ a_{63} = a_{21} a_3 + a_{22} \frac{3a_3}{4} + a_{23} a_3 \]
\[ a_{64} = -a_{22} a_3 + a_{24} a_3 \]
\[ a_{65} = -a_{21} \frac{a_3}{2} + a_{22} \frac{a_3}{4} + a_{25} a_3 \]
3. The Series \( C_0, C_2, C_{02}, C_{11} \).  

We now find from (29) and (30) that

\[
C_0 = \frac{2a_0}{x_3^2} \left[ a_{q_1} + \left( \frac{1}{2} a_{q_1} + a_{q_2} - a_{q_2} \cos 2\tau \right) \mu \right.
\]
\[
+ \left[ \left( \frac{1}{32} a_{q_1} + \frac{1}{2} a_{q_2} + a_{q_3} - \frac{\alpha q_3}{2} \right) \cos 2\tau \right.
\]
\[
+ \left( a_{q_5} + \frac{\alpha q_3}{2} \right) \cos 4\tau \left] \mu^2 + \ldots \right. \right)
\]

\[
C_{20} = -\frac{2a_0}{x_3^2} \left[ a_{q_1} + \left( \frac{1}{2} a_{q_1} + a_{q_2} - a_{q_2} \cos 2\tau \right) \mu \right.
\]
\[
+ \left[ \left( \frac{1}{32} a_{q_1} + \frac{1}{2} a_{q_2} + a_{q_3} - \frac{\alpha q_3}{2} \right) \cos 2\tau \right.
\]
\[
+ \left( a_{q_5} + \frac{\alpha q_3}{2} \right) \cos 4\tau \left] \mu^2 + \ldots \right. \right)
\]

\[
C_{11} = \frac{2a_0}{x_3^2} \left[ a_{q_1} + \left( \frac{1}{2} a_{q_1} + a_{q_2} - a_{q_2} \cos 2\tau \right) \mu \right.
\]
\[
+ \left[ \left( \frac{1}{32} a_{q_1} + \frac{1}{2} a_{q_2} + a_{q_3} - \frac{\alpha q_3}{2} \right) \cos 2\tau \right.
\]
\[
+ \left( a_{q_5} + \frac{\alpha q_3}{2} \right) \cos 4\tau \left] \mu^2 + \ldots \right. \right)
\]

\[
C_{02} = \frac{6}{x_3^3} \left( 1 + \frac{1}{2} \mu + \frac{\mu^2}{32} \right)
\]
where

\begin{align*}
\alpha_{g1} &= \alpha_{31} + \alpha_{51}, \\
\alpha_{g2} &= \alpha_{32} + \alpha_{52}, \\
\alpha_{g3} &= \alpha_{33} + \alpha_{53}, \\
\alpha_{g4} &= \alpha_{34} + \alpha_{54}, \\
\alpha_{g5} &= \alpha_{35} + \alpha_{55}, \\
\alpha_{g6} &= -(1 + \alpha_{56}) \frac{x^2 d^2}{2 \ell^2}, \\
\alpha_{q1} &= -2 \alpha_{61} - \frac{3}{2 \ell^2} \alpha_{71} + 6 \frac{g}{\ell} \alpha_3 - \alpha_1, \\
\alpha_{q2} &= -2 \alpha_{62} - \frac{3}{2 \ell^2} \alpha_{72} + 3 \frac{g}{\ell} \alpha_1, \\
\alpha_{q3} &= -2 \alpha_{63} - \frac{3}{2 \ell^2} \alpha_{73} + \frac{3}{8} \frac{g}{\ell} \alpha_1, \\
\alpha_{q4} &= -2 \alpha_{64} - \frac{3}{2 \ell^2} \alpha_{74}, \\
\alpha_{q5} &= -2 \alpha_{65} - \frac{3}{2 \ell^2} \alpha_{75} - \frac{3}{8} \frac{g}{\ell} \alpha_1.
\end{align*}
Part IV
The Construction of the Orbit.

1. The Values assigned the constants.

When $\zeta = 0$, $\zeta$ is the length of the spherical pendulum. It is natural, therefore, to put $\zeta = 1$. The expression $\alpha_1 - \alpha_3$ occurs frequently in the computation so it might be well to assign values to $\alpha_1$ and $\alpha_3$ which will give this expression a simple value. Suitable values are found to be 1.4 for $\alpha_1$ and -.6 for $\alpha_3$. These give $\alpha_1 - \alpha_3 = 2$. Then we have from (9b), (9c) and (7b) that $c_1 = 1.2, \mu = 2, \alpha_2 = -2$. The resulting value for $\mu$ is one of the most suitable we could choose. It is not too small and yet is small enough to make the various series in $\mu$ converge with sufficient rapidity.

We must choose $\zeta$ so that (26) will converge. For this to be true $\frac{\zeta}{\mu}$ must be less than unity, that is $\zeta < 0.0016$. Hence we may take $\zeta = 0.001$.

If length is measured in feet and time in seconds $g$ will be equal to 32.2.

Tabulating these values, we have

\[
\begin{align*}
\alpha_1 &= 1.4, \quad \alpha_2 = -2, \quad \alpha_3 = -0.6, \quad c_1 = 1.2, \quad \zeta = 1, \\
\mu &= 2, \quad \zeta = 0.001, \quad g = 32.2.
\end{align*}
\]

(33)

2. The Numerical Coefficients of Certain Series.

From (11') it is found in (24) that

\[
C_2 = C^{(-2)}C_{02} + \mu^2 C^{(-1)}C_{12} + \mu^4 C_{22}. \tag{34}
\]

As we are not taking the series further than terms in $\mu^2$, we will not need the series $C_{22}$ and only the first term of the
series $C_{11}$. By noting this fact now quite a bit of needless computation will be saved.

We therefore will only need the auxiliary constants in (31) and (33) which contribute to $C_{10}$ and to the first term of $C_{11}$. The expansion for $C_{12}$ contains none of those constants. The required auxiliary constants are $a_i, a_{ij}, a_{2j}, a_{3j}, a_{4j}, a_{6j}, a_{7j}, a_{9j}$, $(j = 1, \ldots, 3)$.

It is found that

\[ a = 2, \]
\[ a_{11} = -2.160, a_{12} = -1.200, a_{13} = 1.350, a_{14} = -2.000, a_{15} = 1.6500, \]
\[ a_{21} = 2.876, a_{22} = -1.546, a_{23} = 12.6, a_{24} = -19.32, a_{25} = 67.62, \]
\[ a_{31} = -5.556, a_{32} = 2.450, a_{33} = 2.859, a_{34} = -380.8, a_{35} = 94.89, \]
\[ a_{41} = -1.459, a_{42} = 1.196, a_{43} = -170.7, a_{44} = 1876, a_{45} = -32.25, \]
\[ a_{61} = -15.46, \quad a_{71} = 1.037, \quad a_{91} = -119.6. \]

From these we find that

\[ C_{10} = -4.4131 + (-3.941 + 3.714 \cos 2\tau) \mu \]
\[ + (-6.115 + 7.683 \cos 2\tau - 2.202 \cos 4\tau - \cdots) \mu^2 + \cdots \]
\[ C_{12} = 3 \left( 1 + \frac{1}{2} \mu + \frac{11}{32} \mu^2 + \cdots \right), \]
\[ \psi = -6 + (1 - \cos 2\tau) \mu + \frac{1}{8} \left( 1 - \cos 4\tau \right) \mu^2 + \cdots. \]

3. The $\alpha$-equation.

We now proceed to find the value of the $\alpha$-equation. It will be necessary to go through most of the steps which are done in terms of general symbols in §5 of Part II. First it is found from (20) and (14) that
\[ \Delta = -2.228 \]
\[ \frac{1}{\Delta} = -4.488 \]

Next we have from (18), (19) and (21) that

\[ \omega_1 = A_0 \phi + B_1 (x + \kappa \tau \phi) + \frac{1}{\Delta} \left\{ - \phi \int x \phi C_{10} \, d\tau \right\} \]
\[ + \frac{k}{\Delta} \int \phi C_{10} \, d\tau \, d\tau + \chi \int \phi C_{10} \, d\tau \]

where the three integrals do not contain constants of integration, as these constants are expressed in \( A_0 \) and \( B_1 \). As these integrals are later multiplied by \( \mu^2 \) we are only interested in the term independent of \( \mu \) in the third integral. In the first we must also consider the terms contributing to \( \omega_1 \). As \( K \) is proportional to \( \mu^2 \) the second will drop out altogether.

Considering the product \( x C_{10} \) where \( x \) is defined in (14) and \( C_{10} \) in (35), and using (36) it is found that

\[ \omega_1 = -\frac{1}{\Delta} \left( 2.947 \mu + 8.066 \mu^2 + \cdots \right) \]
\[ = 3.42 \]

Next, from (22) and (14)

\[ \beta_0 = -\frac{1}{K} \omega_1 = -\frac{3.648}{\mu^2} \]

We now wish the first terms of the series \(-\frac{1}{\Delta} \phi \int x C_{10} \, d\tau\) and \(-\frac{1}{\Delta} x \int \phi C_{10} \, d\tau\), the sum of which gives the first term of \( C \), in (21). Using (14), (35) and (36) we find that

\[ C = -\frac{1}{\Delta} \phi \int \left( 1 - \cos 2 \tau + \cdots \right) \, d\tau + \frac{1}{\Delta} \chi \int \left( -\cos 2 \tau + \cdots \right) \, d\tau \]
\[ = 0.0508 \left( 1 - \cos 4 \tau \right) + \cdots + 0.0508 \left( 1 + \cos 4 \tau \right) + \cdots \]
\[ = -1.016 + \cdots \]

Using this result with (14) and (38) we now have
On completing this part of the solution it is found from the above result and from (33) that
\[
\frac{\varepsilon}{\mu} C^{(1)} = 0.25 C^{(1)}
\]
\[
= -0.009 \cos 2 \alpha + \mu (0.007 - 0.008 \cos 2 \alpha
- 0.002 \cos 4 \alpha) + \mu^2 (0.004 - 0.004 \cos 2 \alpha
- 0.002 \cos 4 \alpha
- 0.003 \cos 6 \alpha) + \ldots
\]
(42)

Although the computation is carried to three terms it cannot be relied on to give accuracy beyond two significant figures. It will be found from (26), (35) and (8) that these two figures are the first two places of decimals. Hence only the first term of (42) will enter into the final solution. This result is to be expected from the very small value assigned to \( \varepsilon \). From the small value obtained for \( \frac{\varepsilon}{\mu} C^{(1)} \) it might safely be assumed that all terms contributed to the solution by \( C^{(2)} \) in (26) are too small to be counted. However, to make the solution complete, we will compute \( C^{(2)} \).

From (34) we have in (24) that
\[
C_2 = C^{(1)} C_0 + \mu C^{(1)} C_1 + \mu^2 C_2.
\]

Using (40), (41) and (32) it is found that
\[
C^{(1)} C_0 = 2.222 (1 + \cos 4 \alpha) + \mu (2.222 - 0.555 \cos 2 \alpha)
\]
\[ +2222 \cos 4 \tau + 1126 \cos 6 \tau \begin{array}{c} -18 \\ +2222 \cos 4 \tau + 1126 \cos 6 \tau + 1667 \cos 8 \tau \end{array} + \mu^2 \left( 8459 + 769 \cos 2 \tau + 1126 \cos 6 \tau + 1667 \cos 8 \tau \right) + \ldots \ldots \]

\[ \mu^2 C''(\tau) = 1.355 \cos 2 \tau \mu^2 + \ldots \ldots \]

So that \( C_2 = 2222 \left( 1 + \cos 4 \tau \right) + \mu \left( 2222 - 5556 \cos 2 \tau + 2222 \cos 4 \tau + 1126 \cos 6 \tau \right) + \mu^2 \left( 8459 + 769 \cos 2 \tau + 1126 \cos 6 \tau + 1667 \cos 8 \tau \right) + \ldots \ldots \]

The remainder of the computation of \( C^{(2)} \) is similar to that of \( C^{(1)} \). The following results are obtained:

\[ \alpha_2 = -0.0431 \mu \]

\[ \beta_2 = 0.046 \frac{1}{\mu^6} \]

\[ C^{(2)} = 0.046 \cos 2 \tau + \mu \left( -0.025 + 0.023 \cos 2 \tau + 0.012 \cos 4 \tau \right) + \mu^2 \left( 0.0154 + 0.0214 \cos 2 \tau - 0.0045 \cos 4 \tau + 0.0162 \cos 6 \tau \right) + \ldots \ldots \]

and

\[ \frac{\varepsilon_2^2 C^{(2)}}{\mu^6} = 0.016 C^{(2)} \]

\[ = 0.0008 \cos 2 \tau + \mu \left( -0.005 + 0.003 \cos 2 \tau + 0.0002 \cos 4 \tau \right) + \mu^2 \left( 0.0003 + 0.0003 \cos 2 \tau + 0.0003 \cos 6 \tau \right) + \ldots \ldots \]

(43)

This completes the solution of the \( \tau \)-equation. Taking all numbers to two places of decimals we have from (35), (42), (43), and (8) that

\[ \tau = \psi + \frac{\varepsilon}{\mu} C^{(1)} + \frac{\varepsilon^2}{\mu} C^{(2)} + \ldots \ldots \]

\[ = -6 + \left( 1 - \cos 4 \tau \right) \omega + \frac{1}{8} \left( 1 - \cos 2 \tau \right) \mu^2 + \ldots \ldots \]

\[ = -6 - 0.01 \cos 2 \tau + \ldots \ldots \]

\[ = -6 - 0.01 \cos 2 \tau + \ldots \ldots \]

(44)

4. The \( x \)- and \( y \)-equations.

The solutions for the \( x \)- and \( y \)-equations will not be
carried beyond terms in $\mu$. Thus they may be obtained directly by substituting for the known constants in (28).

It is found successively that

$$\beta = 1.625,$$

$$x_1 = \frac{1}{8}(q - c \omega_2 \tau) + \cdots,$$

$$x_2 = 1.854 \sin 2\tau + \cdots,$$

$$\gamma = 1.792,$$

$$\delta = 1.678,$$

$$\chi = -122 \cos 4\tau + 991 \sin 1.6\tau + 0.24 \cos 3.6\tau + \cdots,$$

$$\eta = -10.8 \sin 4\tau + 763 \sin 1.6\tau + 0.2 \cos 3.6\tau + \cdots.$$ (45)

To find the period of $x$ and $\eta$ we must find the period of $x, C\sqrt{3} \tau$ in (28).

If $P_1$ is the period of $x$, and $P_2$ the period of $C\sqrt{3} \tau$
then $P_1 = \pi$, $P_2 = \frac{2\pi}{3} = \frac{2\pi}{1.6} = \frac{5\pi}{4}$.

Hence $P = 5P_1 = 4P_2 = 5.\pi$ . (46)

It may be noted that $P_2$ is not exactly commensurable with $P_1$, but is only so because we take $\beta$ to a finite number of places of decimals.

5. The Orbit.

Equations (45) and (44) give $x, \eta, z$ for any time $\tau$.

The appended table gives $x, \eta, z$ for values of $\tau$
differing by $\frac{\pi}{4}$ from 0 to $5\pi$, a complete period. A column is added giving the value of $x^2 + \eta^2 + z^2 (1 + \epsilon)$
which gives a check as to the accuracy of the computation as far as it is taken. From (1) the value of this expression
should be \( \ell^2 = 1 \).

The diagram is self-explanatory. In making the \( x-y \) diagram, \( x \) was plotted against \( z \), that is for \( z = -61.4 \), \( y \) was so taken that \( \sqrt{x^2 + y^2} = 79, 92, 95 \), respectively. This makes no essential difference in the orbit, it merely serves to lessen the slight error introduced by neglecting the coefficients of \( u \) in the \( x \)- and \( y \)-equations.
<table>
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<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$x^2 + y^2 + z^2(1 + e)$</th>
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