AN ILLUSTRATIVE EXAMPLE
of the
ELLIPSOIDAL PENDULUM

by

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AN ILLUSTRATIVE EXAMPLE

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Part I.

Introduction.

1. The Problem Considered.

In a paper to be presented to the Royal Society of Canada in May, 1928, Dr. D. Buchanan has obtained equations which define the periodic orbits of a particle constrained to move, under gravity, on the surface of the ellipsoid of revolution

$$x^2 + y^2 + z^2(1+\epsilon) - l^2 = 0 , \tag{1}$$

ϵ being a parameter and l a constant. It is assumed that the surface is smooth and conforms somewhat closely to a sphere. For $\epsilon = 0$ the problem reduces to that of the spherical pendulum which is considered by Moulton in his memoir on "Periodic Orbits." For $\epsilon \neq 0$ Dr. Buchanan has designated the problem as the "ellipsoidal pendulum."

The object of this thesis is to construct the orbit determined by assigning numerical values to the various constants on which the solution depends. It is first necessary to give a brief outline of Dr. Buchanan's paper. This is done in Part II. In Part III several series which enter into the solution of the z- equation are computed. In Part IV numerical values are assigned to all constants and the resulting orbit eventually constructed.

Part II

Resumé of Dr. Buchanan's Paper on the "Ellipsoidal Pendulum"

1. The Differential Equations.

As suggested by the form of equation (1) the origin of co-ordinates is taken at the centre of the ellipsoid with the z-axis vertical and positive upwards. The particle is assumed to be of unit mass and to move on the given surface without friction. The differential equations are then found to be

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} &= x\left[X_0 + X_1\epsilon + X_2\epsilon^2 + \cdots\cdots + X_n\epsilon^n + \cdots\right], \\
\frac{d^2y}{dt^2} &= y\left[X_0 + X_1\epsilon + X_2\epsilon^2 + \cdots\cdots + X_n\epsilon^n + \cdots\right], \\
\frac{d^2z}{dt^2} &= Z_0 + \epsilon Z_1 + \epsilon^2 Z_2 + \cdots\cdots + \epsilon^n Z_n + \cdots,
\end{aligned}\right\} \qquad (2)$$

where

$$\left.\begin{aligned}
X_0 &= \frac{g}{l^2}(3z - c_1), \\
X_1 &= \frac{1}{l^2}\left[gz - X_0 z^2 - \left(\frac{dz}{dt}\right)^2\right], \\
X_2 &= -\frac{z^2}{l^2}\left[X_0 + X_1\right], \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad ,
\end{aligned}\right\} \qquad (3)$$

$$\left.\begin{aligned}
Z_0 &= \frac{g}{l^2}(3z^2 - c_1 z - l^2), \\
Z_1 &= \frac{1}{l^2}\left[3gz^2 - c_1 z - z\left(\frac{dz}{dt}\right)^2 - z^2 Z_0\right], \\
Z_2 &= -\frac{1}{l^2}\left[z^2 Z_0 + z\left(\frac{dz}{dt}\right)^2 + z^2 Z_1\right], \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad .
\end{aligned}\right\} \qquad (4)$$

The constants g and c_1 are the acceleration due to gravity and the energy constant respectively.

2. The Spherical Pendulum.

For $\epsilon = 0$ equations (2) reduce to those of the spherical pendulum. We shall require here only the z-equation, viz:

$$\frac{d^2 z}{dt^2} = \frac{g}{l^2}\left(3z^2 - c_1 z - l^2\right) \tag{5}$$

and its solution as obtained by Moulton.

Equation (5) admits the integral

$$\left(\frac{dz}{dt}\right)^2 = \frac{g}{l^2}\left(2z - c_1\right) z^2 - g\left(2z - c_2\right), \tag{6}$$

where c_2 is the constant of integration. If the right side of (6) has the roots $\alpha_1, \alpha_2, \alpha_3$, having $\alpha_1 \geqq \alpha_2 \geqq \alpha_3$, then there are made in (5) the substitutions

$$\left.\begin{aligned} z &= \alpha_3 + (\alpha_2 - \alpha_3) u^2, \\ \mu &= \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}, \quad (0 \leqq \mu \leqq 1), \\ \tau &= \sqrt{\frac{g(\alpha_1 - \alpha_3)}{2 l^2 (1+\delta)}}\,(t - t_0). \end{aligned}\right\} \tag{7}$$

On solving the resulting equation for u the following solutions are obtained for z and δ :

$$\left.\begin{aligned} z &= \psi(\tau) = \alpha_3 + (\alpha_1 - \alpha_3)\left[\tfrac{1}{2}(1 - \cos 2\tau)\mu + \tfrac{1}{16}(1 - \cos 4\tau)\mu^2 + \cdots\right] \\ \delta &= \tfrac{1}{2}\mu + \tfrac{11}{16}\mu^2 + \cdots\cdots \end{aligned}\right\} \tag{8}$$

If the cases of the simple pendulum and of revolution with infinite speed in the xy – plane are excluded, then $\alpha_1, \alpha_2, \alpha_3, c_1$, satisfy the relations

$$\left.\begin{aligned} &-l < \alpha_3 < 0, \quad -l < \alpha_2 < l, \quad \alpha_1 > l, \\ &c_1 = 2\alpha_1 + 4\alpha_3 + 2\mu(\alpha_1 - \alpha_3), \\ &l^2 = \mu(\alpha_3^2 - \alpha_1^2) - \alpha_3(2\alpha_1 + \alpha_3). \end{aligned}\right\} \tag{9}$$

3. The z-equation.

The z-equation of (2) is transformed by means of (7c) and by putting

$$z = \psi + w. \tag{10}$$

This equation then becomes

$$w'' + \left[4 + 6\mu\cos 2\tau - \mu^2\left(\tfrac{3}{8} - 3\cos 2\tau - \tfrac{3}{4}\cos 4\tau\right) + \cdots\right] w$$
$$= \epsilon C_{10} + \epsilon w C_{11} + w^2 C_{02} + \epsilon^2 C_{20} + \epsilon w'^2 C_{12}$$
$$+ \text{ terms of higher degree in } \epsilon', w, w', \tag{11}$$

where the primes signify differentiation with respect to τ. The various C's denote series similar to ψ, being power series in μ with cosines of even multiples of τ in the coefficients. In the sequel, as here, such series are denoted by the foundation letter C with subscripts, superscripts or strokes.

4. The Equation of Variation and its Solution.

If we equate the left member of (11) to zero we obtain the equation of variation,

$$w'' + \left[4 + 6\mu\cos 2\tau - \mu^2\left(\tfrac{3}{8} - 3\cos 2\tau - \tfrac{3}{4}\cos 4\tau\right) + \cdots\right] w = 0 \tag{12}$$

of which the generating solution is $z = \psi$. Applying the method of solution developed by Poincaré for this type of equation, we find that its solution is

$$w = A\varphi + B(\chi + K\tau\varphi), \tag{13}$$

where A and B are constants of integration, and

$$\varphi = \sin 2\tau + \tfrac{1}{4}\mu\sin 4\tau + \tfrac{1}{256}\mu^2(-3\sin 2\tau + 32\sin 4\tau + 9\sin 6\tau) + \cdots\cdots$$
$$\chi = \cos 2\tau + \tfrac{1}{4}\mu(-3 + 2\cos 2\tau + \cos 4\tau) + \tfrac{1}{256}\mu^2(-192 + 119\cos 2\tau + 64\cos 4\tau + 9\cos 6\tau) + \cdots\cdots \tag{14}$$
$$K = \tfrac{15}{16}\mu^2 + \text{terms of higher degree in } \mu.$$

5. The Integration of Equation (11).

Equation (11) will now be integrated as a power series in ϵ. Accordingly the substitution

$$w = \sum_{j=1}^{\infty} w_j \epsilon^j \tag{15}$$

is made in (11). On equating coefficients of like powers of ϵ in the resulting equation, cited as (11'), a sequence of differential equations is determined which give successively the values of w_j. The initial condition $w'(0) = 0$ is imposed. It then follows from (15) that

$$w_j'(0) = 0, \quad (j = 1, \ldots \infty.) \tag{16}$$

<u>Coefficients of ϵ.</u>

The terms in ϵ in (11') yield the equation

$$w_1'' + \left[4 + 6\mu \cos 2\tau - \left(\tfrac{3}{8} - 3\cos 2\tau - \tfrac{3}{4}\cos 4\tau\right)\mu^2 + \cdots\right] w_1 = W_1 \tag{17}$$

where $W_1 = C_{10}$ in (15). The complementary function of (17) is

$$w_1 = a_1 \varphi + b_1 (\chi + K\tau\varphi). \tag{18}$$

The particular integral is obtained by varying the parameters a_1 and b_1. It is found that

$$\begin{aligned} a_1 &= \frac{1}{\Delta}\left[-\int \chi W_1 d\tau - K\tau \int \varphi W_1 d\tau + \iint \varphi W_1 d\tau d\tau\right], \\ b_1 &= \frac{1}{\Delta}\int \varphi W_1 d\tau, \end{aligned} \tag{19}$$

the determinant Δ being constant. It is shown that

$$\Delta = -\varphi'(0)\chi(0). \tag{20}$$

The product χW_1 gives rise to constant terms of the type

$$d_1 = d_1^{(0)} + d_1^{(1)}\mu + d_1^{(2)}\mu^2 + \cdots .$$

When the integrations in (19) are performed and the resulting values of a_1 and b_1 substituted in (18) the complete solution for w_1 becomes

$$w_1 = A_1 \varphi + B_1 (\chi + K\tau\varphi) + d_1 \tau \varphi + \overline{C_1}, \tag{21}$$

where A_1 and B_1 are the constants of integration and $\overline{C}_1$ is a cosine series described in § 3. As w_1 is to be periodic we must have $\quad K B_1 + d_1 = 0,$ or

$$B_1 = -\frac{1}{K} d_1 = \frac{1}{\mu^2} d^{(1)} \tag{22}$$

where $d^{(1)}$ is a power series in μ similar to d_1. From the initial conditions (16) it follows that $A_1 = 0$. Hence the desired solution at this step is

$$w_1 = \frac{1}{\mu^2} C^{(1)}. \tag{23}$$

<u>Coefficients of ϵ^2.</u>

The coefficients of ϵ^2 in (11') give

$$w_2'' + [4 + 6\mu \cos 2\tau + \cdots] w_2 = W_2 = \frac{1}{\mu} + C_2 . \tag{24}$$

The integration of this equation is similar to that of (17) and

$$\left.\begin{aligned} d_2 &= d_2^{(0)} + d_2^{(1)}\mu + \cdots\cdots, \\ B_2 &= \frac{1}{\mu^6} d^{(2)}, \\ w_2 &= \frac{1}{\mu^6} C^{(2)}. \end{aligned}\right\} \tag{25}$$

The succeeding steps are similar to the preceding and the solution at the nth step is

$$w_n = \frac{1}{\mu^{4n-2}} C^{(n)}.$$

This completes the solution of (11). On substituting the resulting value of w in (10) the solution of (10) for the vertical motion becomes

$$z = \psi + \frac{\epsilon}{\mu^2} C^{(1)} + \frac{\epsilon^2}{\mu^6} C^{(2)} + \cdots\cdots + \frac{\epsilon^n}{\mu^{4n-2}} C^{(n)} + \cdots\cdots . \tag{26}$$

6. The Horizontal Motion.

As the x- and y-equations in (2) are the same it is necessary only to consider the x-equation. When the

transformations (7c), (26) and

$$\epsilon = \rho\,\mu^4$$

are applied to (2a) it becomes

$$\begin{aligned} x'' + x\Big[& a_{00}^{(0)} + \left(a_{00}^{(1)} + a_{02}^{(1)}\cos 2\tau\right)\mu \\ & + \left(b_{00}^{(2)} + a_{02}^{(2)}\cos 2\tau + a_{04}^{(2)}\cos 4\tau\right)\mu^2 \\ & + \left(b_{00}^{(3)} + b_{02}^{(3)}\cos 2\tau + a_{04}^{(3)}\cos 4\tau + a_{06}^{(3)}\cos 6\tau\right)\mu^3 \\ & + \cdots\cdots \Big] = 0 \end{aligned} \tag{27}$$

where

$$a_{00}^{(0)} = \frac{2(2\alpha_1 + \alpha_3)}{\alpha_1 - \alpha_3}, \quad a_{00}^{(1)} = \frac{3\alpha_1}{\alpha_1 - \alpha_3}, \quad a_{02}^{(1)} = 3\cos 2\tau,$$

$$\vdots$$

$$b_{00}^{(2)} = a_{00}^{(2)} + a_{10}^{(0)}\rho + a_{20}^{(0)}\rho^2 + \cdots + a_{n0}^{(0)}\rho^n + \cdots$$

$$b_{00}^{(3)} = a_{00}^{(3)} + a_{10}^{(1)}\rho + a_{20}^{(1)}\rho^2 + \cdots + a_{n0}^{(1)}\rho^n + \cdots$$

$$b_{02}^{(3)} = a_{02}^{(3)} + a_{12}^{(1)}\rho + a_{22}^{(1)}\rho^2 + \cdots + a_{n2}^{(1)}\rho^n + \cdots$$

$$\vdots$$

The constant term and the terms in μ in (27) are identical with the corresponding terms in the x - equation of the spherical pendulum. The solutions of (27) will be similar in form to those of the spherical pendulum and with the exception of two slight variations noted below there will be no difference in the constant term and the terms in μ in the resulting equation. Making use of the solutions obtained by Moulton, we have

$$x = A_1\left[x_1\cos\beta\tau - x_2\sin\beta\tau\right],$$

$$y = B_2\left[x_1\sin\beta\tau + x_2\cos\beta\tau\right],$$

$$\left.\begin{aligned}
\beta &= \frac{\sqrt{2}}{\sqrt{\alpha_1-\alpha_3}}\left[\sqrt{2\alpha_1+\alpha_3} + \frac{3\alpha_1}{4\sqrt{2\alpha_1+\alpha_3}}\,\mu + \cdots\right],\\
x_1 &= 1 - \frac{\alpha_1-\alpha_3}{4(\alpha_1+\alpha_3)}(\cos 2\tau - 1)\,\mu + \cdots,\\
x_2 &= \frac{\sqrt{2(\alpha_1-\alpha_3)(2\alpha_1+\alpha_3)}}{4(\alpha_1+\alpha_3)}(\sin 2\tau)\,\mu + \cdots,\\
A_1^2 &= l^2 - (1+\epsilon)\,z(0)^2,\\
B_2^2 &= \frac{2l^2(1+\delta)(c_1 - 2z(0))}{[\beta + x_2'(0)]^2(\alpha_1-\alpha_3)}.
\end{aligned}\right\} \qquad (28)$$

The constants A_1^2 and B_2^2 differ from the corresponding constants in the spherical pendulum as the expression for A_1^2 contains $(1+\epsilon)$ and those for both A_1^2 and B_2^2 contain $z(0)$ which in this problem has a slightly different value to that of the spherical pendulum.

This completes the solution.

Part III

The Algebraic Expressions for Certain Series.

1. Introduction.

The object of this part is to compute all coefficients in equation (11) which contribute terms in ϵ and ϵ^2 to (11'). These coefficients are $C_{10}, C_{20}, C_{02}, C_{11}$, that is those the sum of whose subscripts is not greater than two.

It will be remembered that these coefficients resulted from the transformation of (2c) by (7c) and (10). Hence we have

$$\left.\begin{aligned}
C_{10} &= \frac{2\ell^2(1+\delta)}{g(\alpha_1-\alpha_3)} Z_1(\psi), \\
C_{20} &= \frac{2\ell^2(1+\delta)}{g(\alpha_1-\alpha_3)} Z_2(\psi), \\
C_{02} &= \frac{2\ell^2(1+\delta)}{g(\alpha_1-\alpha_3)} \left(\text{coefficients of } w^2 \text{ in } Z_0\right) \\
&= \frac{2\ell^2(1+\delta)}{g(\alpha_1-\alpha_3)} \times 3, \\
C_{11} &= \frac{2\ell^2(1+\delta)}{g(\alpha_1-\alpha_3)} \left(\text{coefficients of } w \text{ in } Z_1\right) \\
&= \frac{2\ell^2(1+\delta)}{g(\alpha_1-\alpha_3)} \cdot \frac{1}{\ell^2}\left[6g\psi - c_1 - \left(\frac{d\psi}{dt}\right)^2 - 2\psi Z_0(\psi) \right. \\
&\qquad \left. - \psi^2 \frac{g}{\ell^2}(6\psi - c_1)\right].
\end{aligned}\right\} \tag{29}$$

These series and all others contributing to the z-equation will be carried out to and including terms in μ^2.

2. A Number of Auxiliary Functions of χ.

With the exception of C_{02} the coefficients (29) are

the sums of a number of different functions of χ. Accordingly the computation will be simplified considerably by first computing the auxiliary functions ψ^2, $\left(\frac{d\psi}{dt}\right)^2$, $\psi\left(\frac{d\psi}{dt}\right)^2$, $Z_0(\psi)$, $\psi^2 Z_0(\psi)$, $Z_1(\psi)$, $\psi^2 Z_1(\psi)$, $\psi Z_0(\psi)$, $\psi^2(6\psi - c_1)$

It is found that

$$
\left.
\begin{aligned}
\psi^2 &= a_{11} + a_{12}(1-\cos 2\tau)\mu + (a_{13} + a_{14}\cos 2\tau + a_{15}\cos 4\tau)\mu^2 + \cdots, \\
\left(\frac{d\psi}{dt}\right)^2 &= \frac{g\alpha}{2l^2(1+\delta)}\psi'^2 = \frac{g\alpha^3}{4l^2(1+\delta)}(1-\cos 4\tau)\mu^2 + \cdots\cdots, \\
\psi\left(\frac{d\psi}{dt}\right)^2 &= \frac{g\alpha_3\alpha^3}{4l^2(1+\delta)}(1-\cos 4\tau)\mu^2 + \cdots\cdots, \\
Z_0(\psi) &= a_{21} + a_{22}(1-\cos 2\tau)\mu + (a_{23} + a_{24}\cos 2\tau + a_{25}\cos 4\tau)\mu^2 + \cdots, \\
\psi^2 Z_0(\psi) &= a_{31} + a_{32}(1-\cos 2\tau)\mu + (a_{33} + a_{34}\cos 2\tau + a_{35}\cos 4\tau)\mu^2 + \cdots, \\
Z_1(\psi) &= \frac{1}{l^2}\left[3g\psi^2 - c_1\psi - \psi\left(\frac{d\psi}{dt}\right)^2 - \psi^2 Z_0(\psi)\right] \\
&= \frac{1}{l^2}\left[a_{41} + a_{42}(1-\cos 2\tau)\mu + (a_{43} + a_{44}\cos 2\tau + a_{45}\cos 4\tau)\mu^2\right. \\
&\qquad \left. + \frac{g\alpha^3\alpha_3}{4l^2(1+\delta)}(1-\cos 4\tau)\mu^2 + \cdots\right], \\
\psi^2 Z_1(\psi) &= a_{51} + a_{52}(1-\cos 2\tau)\mu + (a_{53} + a_{54}\cos 2\tau + a_{55}\cos 4\tau)\mu^2 \\
&\qquad + a_{56}\frac{g\alpha^3\alpha_3}{4l^2(1+\delta)}(1-\cos 4\tau)\mu^2 + \cdots, \\
\psi Z_0(\psi) &= a_{61} + a_{62}(1-\cos 2\tau)\mu + (a_{63} + a_{64}\cos 2\tau + a_{65}\cos 4\tau) + \cdots, \\
\psi^2(6\psi - c_1) &= a_{71} + a_{72}(1-\cos 2\tau)\mu + (a_{73} + a_{74}\cos 2\tau + a_{75}\cos 4\tau) + \cdots,
\end{aligned}
\right\} \quad (30)
$$

where

$$
\left.
\begin{aligned}
\alpha &= \alpha_1 - \alpha_3, \\
a_{11} &= \alpha_3^2, & a_{21} &= \frac{g}{l^2}(3\alpha_3^2 - c_1\alpha_3 - l^2), \\
a_{12} &= \alpha\alpha_3, & a_{22} &= \frac{\alpha g}{l^2}\left(3\alpha_3 - \frac{c_1}{2}\right), \\
a_{13} &= \frac{\alpha}{8}(3\alpha + \alpha_3), & a_{23} &= \frac{\alpha g}{8l^2}\left(9\alpha + 3\alpha_3 - \frac{c_1}{2}\right), \\
a_{14} &= -\frac{\alpha^2}{2}, & a_{24} &= -\frac{3\alpha^2 g}{2l^2}, \\
a_{15} &= \frac{\alpha}{8}(\alpha - \alpha_3), & a_{25} &= \left(3\alpha - 3\alpha_3 + \frac{c_1}{2}\right)\frac{\alpha g}{8l^2},
\end{aligned}
\right\}
$$

$$
\left.
\begin{aligned}
a_{31} &= a_{11} a_{21}, \\
a_{32} &= a_{11} a_{22} + a_{12} a_{21}, \\
a_{33} &= a_{11} a_{23} + \tfrac{3}{2} a_{12} a_{22} + a_{13} a_{21}, \\
a_{34} &= a_{11} a_{24} - 2 a_{12} a_{22} + a_{14} a_{21}, \\
a_{35} &= a_{11} a_{25} + \tfrac{1}{2} a_{12} a_{22} + a_{15} a_{21}, \\
\\
a_{41} &= 3g a_{11} - c_1 \alpha_3 - a_{31}, \\
a_{42} &= 3g a_{12} - c_1 \tfrac{\alpha}{2} - a_{32}, \\
a_{43} &= 3g a_{13} - c_1 \tfrac{\alpha}{16} - a_{33}, \\
a_{44} &= 3g a_{14} \qquad - a_{34}, \\
a_{45} &= 3g a_{15} - c_1 \tfrac{\alpha}{16} - a_{35}, \\
\\
a_{51} &= \tfrac{1}{\ell^2} a_{11} a_{41}, \\
a_{52} &= \tfrac{1}{\ell^2} (a_{11} a_{42} + a_{12} a_{41}), \\
a_{53} &= \tfrac{1}{\ell^2} (a_{11} a_{43} + \tfrac{3}{2} a_{12} a_{42} + a_{13} a_{41}), \\
a_{54} &= \tfrac{1}{\ell^2} (a_{11} a_{44} - 2 a_{12} a_{42} + a_{14} a_{41}), \\
a_{55} &= \tfrac{1}{\ell^2} (a_{11} a_{45} + \tfrac{1}{2} a_{12} a_{22} + a_{15} a_{41}), \\
a_{56} &= -a_{11}, \\
\\
a_{61} &= a_{21} \alpha_3, \\
a_{62} &= a_{21} \tfrac{\alpha}{2} + a_{22} \alpha_3, \\
a_{63} &= a_{21} \tfrac{\alpha}{16} + a_{22} \tfrac{3\alpha}{4} + a_{23} \alpha_3, \\
a_{64} &= \qquad - a_{22} \alpha + a_{24} \alpha_3, \\
a_{65} &= - a_{21} \tfrac{\alpha}{16} + a_{22} \tfrac{\alpha}{4} + a_{25} \alpha_3,
\end{aligned}
\right\} \qquad (31)
$$

$$a_{71} = a_{11}(6\alpha_3 - c_1),$$
$$a_{72} = a_{12}(6\alpha_3 - c_1) + 3\alpha a_{11},$$
$$a_{73} = a_{13}(6\alpha_3 - c_1) + \frac{3}{8}\alpha a_{11} + \frac{9}{2}\alpha a_{12},$$
$$a_{74} = a_{14}(6\alpha_3 - c_1) \qquad - 6\alpha a_{12},$$
$$a_{75} = a_{15}(6\alpha_3 - c_1) - \frac{3}{8}\alpha a_{11} + \frac{3}{2}\alpha a_{12}.$$

3. The Series $C_{10}, C_{20}, C_{02}, C_{11}$.

We now find from (29) and (30) that

$$\left.\begin{aligned}
C_{10} &= \frac{2}{\alpha g}\Big\{a_{41} + \left(\tfrac{1}{2}a_{41} + a_{42} - a_{42}\cos 2\tau\right)\mu \\
&\quad + \Big[\left(\tfrac{11}{32}a_{41} + \tfrac{1}{2}a_{42} + a_{43} - \frac{\alpha^3\alpha_3 g}{4l^2}\right) + \left(a_{44} - \tfrac{1}{2}a_{42}\right)\cos 2\tau \\
&\quad + \left(a_{45} + \frac{\alpha^3\alpha_3 g}{4l^2}\right)\cos 4\tau\Big]\mu^2 + \cdots\cdots\Big\}, \\
C_{20} &= -\frac{2}{\alpha g}\Big\{a_{81} + \left(\tfrac{1}{2}a_{81} + a_{82} - a_{82}\cos 2\tau\right)\mu \\
&\quad + \Big[\left(\tfrac{11}{32}a_{81} + \tfrac{1}{2}a_{82} + a_{83} - \frac{\alpha g a_{86}}{2}\right) + \left(a_{84} - \tfrac{1}{2}a_{82}\right)\cos 2\tau \\
&\quad + \left(a_{85} + \frac{\alpha g a_{86}}{2}\right)\cos 4\tau\Big]\mu^2 + \cdots\cdots\Big\}, \\
C_{11} &= \frac{2}{\alpha g}\Big\{a_{91} + \left(\tfrac{1}{2}a_{91} + a_{92} - a_{92}\cos 2\tau\right)\mu \\
&\quad + \Big[\left(\tfrac{11}{32}a_{91} + \tfrac{1}{2}a_{92} + a_{93} - \frac{\alpha^3 g}{4l^2}\right) + \left(a_{94} - \tfrac{1}{2}a_{92}\right)\cos 2\tau \\
&\quad + \left(a_{95} + \frac{\alpha^3 g}{4l^2}\right)\cos 4\tau\Big]\mu^2 + \cdots\cdots\Big\}, \\
C_{02} &= \frac{6}{\alpha_1 - \alpha_3}\left(1 + \tfrac{1}{2}\mu + \tfrac{11}{32}\mu^2 + \cdots\right),
\end{aligned}\right\} \qquad (32)$$

where

$$a_{81} = a_{31} + a_{51},$$
$$a_{82} = a_{32} + a_{52},$$
$$a_{83} = a_{33} + a_{53},$$
$$a_{84} = a_{34} + a_{54},$$
$$a_{85} = a_{35} + a_{55},$$
$$a_{86} = -(1 + a_{56})\frac{\alpha^2 \alpha_3}{2\ell^2},$$

$$a_{91} = -2a_{61} - \frac{g}{\ell^2} a_{71} + 6g\alpha_3 - c_1,$$
$$a_{92} = -2a_{62} - \frac{g}{\ell^2} a_{72} + 3g\alpha,$$
$$a_{93} = -2a_{63} - \frac{g}{\ell^2} a_{73} + \frac{3}{8} g\alpha,$$
$$a_{94} = -2a_{64} - \frac{g}{\ell^2} a_{74},$$
$$a_{95} = -2a_{65} - \frac{g}{\ell^2} a_{75} - \frac{3}{8} g\alpha.$$

Part IV

The Construction of the Orbit.

1. The Values assigned the constants.

When $\epsilon = 0$, ℓ is the length of the spherical pendulum. It is natural, therefore, to put $\ell = 1$. The expression $\alpha_1 - \alpha_3$ occurs frequently in the computation so it might be well to assign values to α_1 and α_3 which will give this expression a simple value. Suitable values are found to be 1.4 for α_1 and -.6 for α_3. These give $\alpha_1 - \alpha_3 = 2$. Then we have from (9b), (9c) and (7b) that $c_1 = 1{\cdot}2$, $\mu = {\cdot}2$, $\alpha_2 = -{\cdot}2$. The resulting value for μ is one of the most suitable we could choose. It is not too small and yet is small enough to make the various series in μ converge with sufficient rapidity.

We must choose ϵ so that (26) will converge. For this to be true $\frac{\epsilon}{\mu^4}$ must be less than unity, that is $\epsilon < {\cdot}0016$ Hence we may take $\epsilon = {\cdot}001$.

If length is measured in feet and time in seconds g will be equal to 32.2.

Tabulating these values, we have

$$\left.\begin{array}{l} \alpha_1 = 1{\cdot}4,\ \alpha_2 = -{\cdot}2,\ \alpha_3 = -{\cdot}6,\ c_1 = 1{\cdot}2,\ \ell = 1, \\ \mu = {\cdot}2,\ \epsilon = {\cdot}001,\ g = 32{\cdot}2. \end{array}\right\} \qquad (33)$$

2. The Numerical Coefficients of Certain Series.

From (11') it is found in (24) that

$$C_2 = C^{(1)2} C_{02} + \mu^2 C^{(1)} C_{11} + \mu^4 C_{20}. \qquad (34)$$

As we are not taking the series further than terms in μ^2 we will not need the series C_{20} and only the first term of the

series C_{11} . By noting this fact now quite a bit of needless computation will be saved.

We therefore will only need the auxiliary constants in (31) and (33) which contribute to C_{10} and to the first term of C_{11} . The expansion for C_{02} contains none of those constants. The required auxiliary constants are α, a_{1j}, a_{2j}, a_{3j}, a_{4j}, a_{61}, a_{71}, a_{91}, $(j = 1, \ldots, 5)$.

It is found that

$$\alpha = 2,$$
$$a_{11} = -\cdot 2160,\ a_{12} = -1\cdot 200,\ a_{13} = 1\cdot 350,\ a_{14} = -2\cdot 000,\ a_{15} = \cdot 6500,$$
$$a_{21} = 25\cdot 76,\ a_{22} = -154\cdot 6,\ a_{23} = 125\cdot 6,\ a_{24} = -193\cdot 2,\ a_{25} = 67\cdot 62,$$
$$a_{31} = -5\cdot 564,\ a_{32} = 2\cdot 480,\ a_{33} = 285\cdot 9,\ a_{34} = -380\cdot 8,\ a_{35} = 94\cdot 89,$$
$$a_{41} = -14\cdot 59,\ a_{42} = -119\cdot 6,\ a_{43} = -170\cdot 7,\ a_{44} = 187\cdot 6,\ a_{45} = -32\cdot 25,$$
$$a_{61} = -15\cdot 46, \qquad a_{71} = 1\cdot 037, \qquad a_{91} = -119\cdot 6.$$

From these we find that

$$\begin{aligned} C_{10} &= -\cdot 4531 + (-3\cdot 941 + 3\cdot 714 \cos 2\tau)\mu \\ &\quad + (-6\cdot 115 + 7\cdot 683 \cos 2\tau - 2\cdot 202 \cos 4\tau)\mu^2 + \cdots, \\ C_{02} &= 3\left(1 + \tfrac{1}{2}\mu + \tfrac{11}{32}\mu^2 + \cdots\right) \\ &= 3\cdot 341, \\ \psi &= -\cdot 6 + (1 - \cos 2\tau)\mu + \tfrac{1}{8}(1 - \cos 4\tau)\mu^2 + \cdots. \end{aligned} \tag{35}$$

3. The z-equation.

We now proceed to find the value of the z-equation. It will be necessary to go through most of the steps which are done in terms of general symbols in § 5 of Part II.

First it is found from (20) and (14) that

$$\left.\begin{aligned}\Delta &= -2.228,\\ \frac{1}{\Delta} &= -.4488.\end{aligned}\right\} \tag{36}$$

and

Next we have from (18), (19) and (21) that

$$w_1 = A_1\varphi + B_1(\chi + K\tau\varphi) + \frac{1}{\Delta}\Big\{-\varphi\int\chi c_{10}\,d\tau + K\varphi\int\int\varphi c_{10}\,d\tau\,d\tau + \chi\int\varphi c_{10}\,d\tau\Big\}, \tag{37}$$

where the three integrals do not contain constants of integration, as these constants are expressed in A_1 and B_1. As these integrals are later multiplied by μ^2 we are only interested in the term independent of μ in the third integral. In the first we must also consider the terms contributing to d_1. As K is proportional to μ^2 the second will drop out altogether.

Considering the product χc_{10} where χ is defined in (14) and c_{10} in (35), and using (36) it is found that

$$\begin{aligned} d_1 &= -\frac{1}{\Delta}\left(2.197\mu + 8.066\mu^2 + \cdots\right)\\ &= .342. \end{aligned}$$

Next, from (22) and (14)

$$B_1 = -\frac{1}{K}d_1 = -.3648\frac{1}{\mu^2}. \tag{38}$$

We now wish the first terms of the series $-\frac{1}{\Delta}\varphi\int\chi c_{10}\,d\tau$ and $-\frac{1}{\Delta}\chi\int\varphi c_{10}\,d\tau$, the sum of which gives the first term of $\overline{c}_1$ in (21). Using (14), (35) and (36) we find that

$$\begin{aligned}\overline{c}_1 &= -\frac{1}{\Delta}\varphi\int(-.4531\cos 2\tau + \cdots)\,d\tau + \frac{1}{\Delta}\chi\int(-.4531\sin 2\tau + \cdots)\,d\tau\\ &= .0508(1-\cos 4\tau) + \cdots\cdots - .0508(1+\cos 4\tau) + \cdots\\ &= -.1016 + \cdots\cdots. \end{aligned} \tag{39}$$

Using this result with (14) and (38) we now have

$$\omega_1 = B_1 x + \overline{C_1}$$
$$= \frac{1}{\mu^2}\left(-.3648\, x + \mu^2 \overline{C_1}\right)$$
$$= \frac{1}{\mu^2}\Big\{-.3648 \cos 2\tau + \mu(.2736 - .1824 \cos 2\tau - .0912 \cos 4\tau) + \mu^2(.1720 - .1696 \cos 2\tau - .0912 \cos 4\tau - .0128 \cos 6\tau) + \cdots\cdots\Big\}, \tag{40}$$

and
$$C^{(1)} = \mu^2 \omega_1 . \tag{41}$$

On completing this part of the solution it is found from the above result and from (33) that

$$\frac{\epsilon}{\mu^2} C^{(1)} = .025\, C^{(1)}$$
$$= -.009 \cos 2\tau + \mu(.007 - .005 \cos 2\tau - .002 \cos 4\tau) + \mu^2(.004 - .004 \cos 2\tau - .002 \cos 4\tau - .0003 \cos 6\tau) + \cdots\cdots . \tag{42}$$

Although the computation is carried to three terms it cannot be relied on to give accuracy beyond two significant figures. It will be found from (26), (35) and (8) that these two figures are the first two places of decimals. Hence only the first term of (42) will enter into the final solution. This result is to be expected from the very small value assigned to ϵ . From the small value obtained for $\frac{\epsilon}{\mu^2} C^{(1)}$ it might safely be assumed that all terms contributed to the solution by $C^{(2)}$ in (26) are too small to be counted. However, to make the solution complete, we will compute $C^{(2)}$.

From (34) we have in (24) that

$$C_2 = C^{(1)2} C_{02} + \mu^2 C^{(1)} C_{11} + \mu^4 C_{20} .$$

Using (40), (41) and (32) it is found that

$$C^{(1)2} C_{02} = .2222(1 + \cos 4\tau) + \mu(.2222 - .5556 \cos 2\tau$$

$$+\cdot 2222\cos 4\tau + \cdot 1126\cos 6\tau) + \mu^2(\cdot 8459 - \cdot 586\cos 2\tau$$
$$+\cdot 1126\cos 6\tau + \cdot 1667\cos 8\tau) + \cdots\cdots,$$
$$\mu^2 C^{(1)} C_{11} = 1\cdot 355\cos 2\tau\,\mu^2 + \cdots\cdots.$$

So that $C_2 = \cdot 2222(1+\cos 4\tau) + \mu(\cdot 2222 - \cdot 5556\cos 2\tau$
$+\cdot 2222\cos 4\tau + \cdot 1126\cos 6\tau) + \mu^2(\cdot 8459 + \cdot 769\cos 2\tau$
$+\cdot 1126\cos 6\tau + \cdot 1667\cos 8\tau) + \cdots\cdots.$

The remainder of the computation of $C^{(2)}$ is similar to that of $C^{(1)}$. The following results are obtained:

$$\alpha_2 = -\cdot 0431\frac{1}{\mu^4}$$
$$\beta_2 = \cdot 046\frac{1}{\mu^6}$$
$$C^{(2)} = \cdot 046\cos 2\tau + \mu(-\cdot 0345 + \cdot 023\cos 2\tau + \cdot 012\cos 4\tau)$$
$$+\mu^2(\cdot 0154 + \cdot 0214\cos 2\tau - \cdot 0045\cos 4\tau + \cdot 0162\cos 6\tau)$$
$$+\cdots\cdots,$$

and $\frac{\epsilon^2}{\mu^6}C^{(2)} = \cdot 016\,C^{(2)}$
$$= \cdot 0008\cos 2\tau + \mu(-\cdot 0005 + \cdot 0003\cos 2\tau$$
$$+\cdot 0002\cos 4\tau) + \mu^2(\cdot 0003 + \cdot 0003\cos 2\tau$$
$$+\cdot 0003\cos 6\tau) + \cdots\cdots. \qquad (43)$$

This completes the solution of the z-equation. Taking all numbers to two places of decimals we have from (35), (42), (43), and (8) that

$$z = \psi + \frac{\epsilon}{\mu^4}C^{(1)} + \frac{\epsilon^2}{\mu^6}C^{(2)} + \cdots\cdots$$
$$= -\cdot 6 + (1-\cos 2\tau)\mu + \frac{1}{8}(1-\cos 4\tau)\mu^2 + \cdots$$
$$-\cdot 01\cos 2\tau + \cdots\cdots$$
$$= -\cdot 4 - \cdot 21\cos 2\tau + \cdots. \qquad (44)$$

4. The x- and y-equations.

The solutions for the x- and y-equations will not be

carried beyond terms in μ. Thus they may be obtained directly by substituting for the known constants in (28).

It is found successively that

$$\left.\begin{aligned}
&\beta = 1\cdot625,\\
&x_1 = \tfrac{1}{8}(9-\cos 2\tau)+\cdots,\\
&x_2 = \cdot1854\sin 2\tau+\cdots,\\
&A_1 = \cdot792,\\
&B_2 = \cdot678,\\
&x = -\cdot122\cos\cdot4\tau+\cdot891\cos 1\cdot6\tau+\cdot024\cos 3\cdot6\tau+\cdots,\\
&y = \cdot105\sin\cdot4\tau+\cdot763\sin 1\cdot6\tau+\cdot02\sin 3\cdot6\tau+\cdots.
\end{aligned}\right\}\quad(45)$$

To find the period of x and y we must find the period of $x_1\cos\beta\tau$ in (28).

If P_1 is the period of x_1 and P_2 the period of $\cos\beta\tau$ then $P_1 = \pi$, $P_2 = \frac{2\pi}{\beta} = \frac{2\pi}{1\cdot6} = \frac{5\pi}{4}$.

Hence $P = 5P_1 = 4P_2 = 5\pi$. (46)

It may be noted that P_2 is not exactly commensurable with P_1, but is only so because we take β to a finite number of places of decimals.

5. The Orbit.

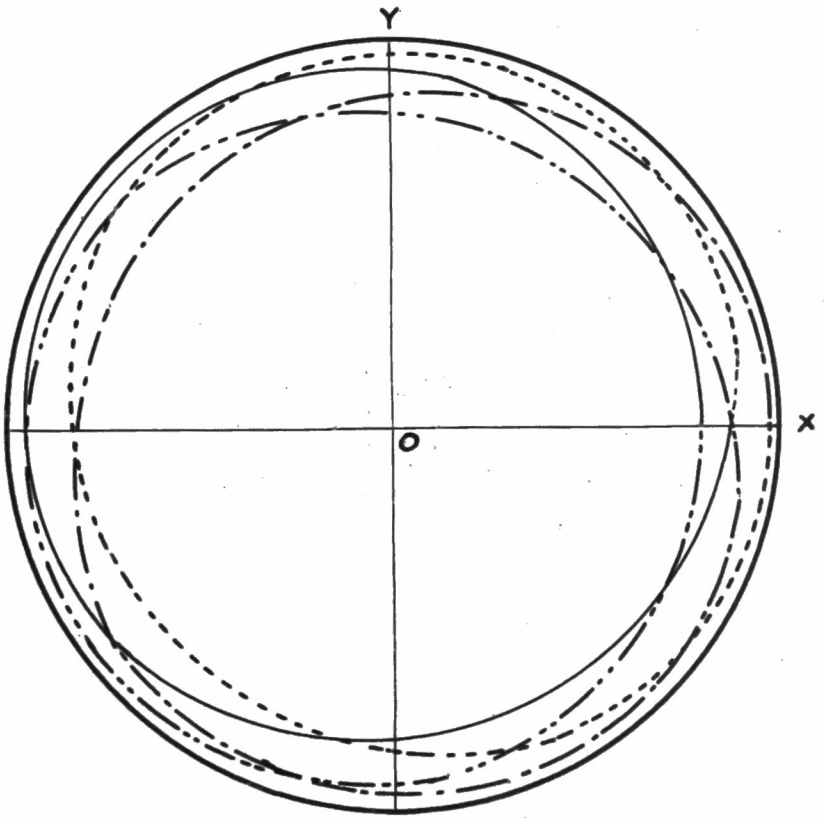
Equations (45) and (44) give x, y, z for any time τ.

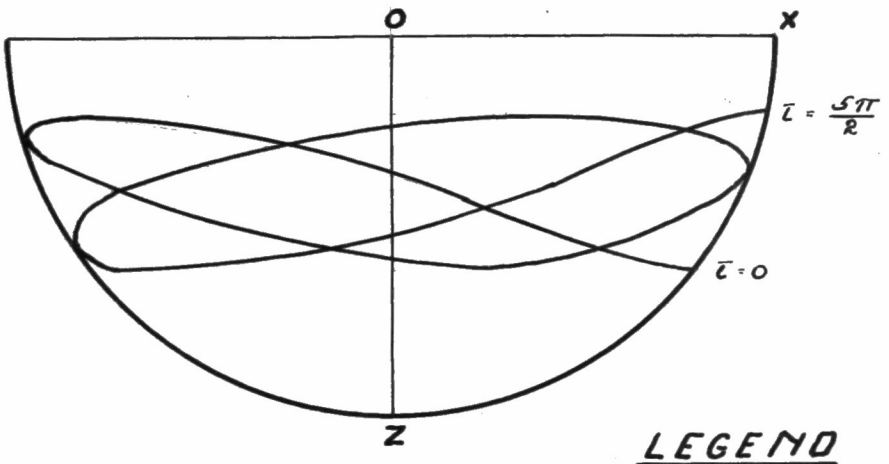
The appended table gives x, y, z for values of τ differing by $\frac{\pi}{4}$ from 0 to 5π, a complete period. A column is added giving the value of $x^2+y^2+z^2(1+\epsilon)$ which gives a check as to the accuracy of the computation as far as it is taken. From (1) the value of this expression

should be $\ell^2 = 1$.

The diagram is self-explanatory. In making the x-y-diagram x was plotted against z, that is for $z = -\cdot 61, -\cdot 4$ $-\cdot 19$, y was so taken that $\sqrt{x^2+y^2} = \cdot 79, \cdot 92, \cdot 98$, respectively. This makes no essential difference in the orbit, it merely serves to lessen the slight error introduced by neglecting the coefficients of μ^2 in the x- and y-equations.

τ	x	y	z	$x^2+y^2+z^2(1+\epsilon)$
0	·79	0	−·61	1·00
$\frac{\pi}{4}$	·14	·73	−·4	·71
$\frac{\pi}{2}$	−·80	·50	−·19	·94
$\frac{3\pi}{4}$	−·78	−·35	−·4	·89
π	·25	−·71	−·61	·94
$\frac{5\pi}{4}$	·89	·12	−·4	·97
$\frac{3\pi}{2}$	·31	·81	−·19	·79
$\frac{7\pi}{4}$	−·63	·56	−·4	·87
2π	−·63	−·40	−·61	·93
$\frac{9\pi}{4}$	·42	−·69	−·4	·81
$\frac{5\pi}{2}$	·99	0	−·19	1·02
$\frac{11\pi}{4}$	·42	·69	−·4	·81
3π	−·63	·40	−·61	·93
$\frac{13\pi}{4}$	−·63	−·56	−·4	·87
$\frac{7\pi}{2}$	·31	−·81	−·19	·79
$\frac{15\pi}{4}$	·89	−·12	−·4	·97
4π	·25	·71	−·61	·94
$\frac{17\pi}{4}$	−·78	·35	−·4	·89
$\frac{9\pi}{2}$	−·80	−·50	−·19	·94
$\frac{19\pi}{4}$	·14	−·73	−·4	·71
5π	·79	0	−·61	1·00





LEGEND

First Circuit ———

Second Circuit - - - - - - - -

Third Circuit —·—·

Fourth Circuit —··—··