A COMPARATIVE STUDY OF STABILITY CONDITIONS IN DYNAMICAL SYSTEMS,
COBWEB MODELS, AND DIFFERENTIAL DELAY EQUATIONS

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ABSTRACT

In the study of ordinary differential equations (ODE), a wealth of time has been spent studying dynamical systems. Because of this, much is known about the behaviour of such systems and many methods have been developed to further aid in their analysis. Through comparisons and the construction of actual parallels, this thesis attempts to show how this knowledge of dynamical systems can also be useful in the study of cobweb models and differential delay equations (DDE).

Concentration is placed on the development of methods for stability analysis, with general introductions to stability analysis in dynamical systems, cobweb models and differential delay systems. A special parallel is drawn between Hopf-type bifurcation in dynamical systems and Allwright bifurcation in cobweb models; while the construction of equivalent dynamical systems for S-convertible DDE is presented and stability analysis is carried out for several special examples.
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INTRODUCTION

The documentation for autonomous dynamical systems of the general form

\[ y_t = f(y) \]

is quite extensive. Methods for stability analysis in particular, has been well developed. This thesis describes several ways in which our knowledge of dynamical systems can be used in the study of two other models; the cobweb model of the general form

\[ x \rightarrow f(x) \text{ with } f \in C^1; \]

and the differential delay equation of the general form

\[ x_t = f(x(t), \int_0^\infty x(t - s)\mu(ds) \]

where \( \mu \) is a finite Borel measure on \([0, \infty)\).

The parallel between the definition of an equilibrium point in dynamical systems and DDE, and that of a fixed point in cobweb models forms a basis upon which similarities in stability analysis can be investigated.

The definition of a stable or asymptotically stable point or closed orbit in a two dimensional dynamical system can be paralleled to that of a stable or asymptotically stable fixed point or points of period two in a cobweb model. By comparing closed orbits to points of period two, we have a basis for studying similarities between bifurcations to closed orbits in dynamical systems and bifurca-
tions to points of period two in cobweb models. In this way we seek to establish similarities in the Hopf bifurcation and Allwright bifurcation theories of the two models.

The definitions for a stable or asymptotically stable equilibrium solution in dynamical systems and in DDE are equivalent. We show that the existence and uniqueness theories for the two models also follow the same lines of proof. Hence we should not be surprised to find further similarities when we investigate the stability analysis in each of the two models.

In fact, we see that the stability analysis procedures for an equilibrium solution of a discrete lag DDE are basically equivalent to those for a dynamical system. Conditions for stability depend on the real part of the eigenvalues being nonpositive. To study the effect of a discrete time lag on the stability of a system, we consider the stability of the equilibrium solutions in the model with and without the lag term. Conditions for stability give insight into how the size of the lag affects the stability of the system. The general principle is that discrete time lags are in general destabilizing. There are however exceptions to this general trend and examples can be found in [3], [6].

Comparisons can also be drawn between certain discrete lag DDE and cobweb models. Here, a discrete time change of one unit can be attributed to the otherwise time independent cobweb model. By approximating the DDE with a difference
quotient

\[ x_t = \frac{[x(t) - x(t-1)]}{[t - (t-1)]} = x(t) - x(t-1) = x_{n+1} - x_n \]

--using a suitable change of variable to reduce to the case of a unit time lag--we get a cobweb model.

In studying similarities between dynamical systems and DDE with distributed time lag, we are actually able to construct equivalent dynamical systems for a special class of these DDE--those of the form

\[ x_t = f(x(t), \int_0^\infty k(s) ds) \text{ where } k(s) = \sum_{i=1}^M \sum_{j=0}^N a_{ij} s^j / \exp(-\alpha_i s). \]

We shall call these \( \delta \)-convertible DDE. This equivalence allows us to use all the methods for stability analysis of dynamical systems to study the stability of the DDE. Because of this, we are able to see the effects of delays caused by different delay kernels on the system by comparison to the equivalent dynamical system without the delay term. This actually reduces the study of these special DDE to that of their equivalent dynamical systems.

Furthermore, using the concept of the average time lag of a distributed lag system, we are able to compare these systems with the discrete lag systems. In doing so, we discover that the former type of delay tends to be less destabilizing than the latter with the same lag time.

Many of the examples used are biological or ecological models. These three types of mathematical models are used extensively in the study of biosciences and
ecology [19], [20], [21], [23], [24], [25]. The usefulness of the well developed methodology for dynamical systems is therefore quite important in these applications. Economics also claims a large field of applicability for the models and comparisons studied here. In particular, DDE are used in models of the capital investment (industrial) economy and for macroeconomic forecasting [17].
CHAPTER 1 A REVIEW OF DYNAMICAL SYSTEMS

1.1 Introduction to Dynamical Systems

A dynamical system is a mathematical model describing motion in time. Typically such a system in $\mathbb{R}^n$ can be written as a system of time dependent first order differential equations $y_t = f(t, y_0)$ with $y \in \mathbb{R}^n$, $f(t, y_0) \in C^r(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ uniformly Lipschitz in $y$ and where $y_0 = y(t_0)$ is a given initial value. The unique solution $y(t)$ of such a system is a trajectory running through the point $y_0$ with velocity $y_t = f(t, y_0)$ at any time $t \geq 0$.

We shall consider the autonomous system

$$y_t = f(y)$$

(1)

in which $f$ is not explicitly dependent on $t$, and where we let $y_0$ vary over all values in $\mathbb{R}^n$. $f$ is then called a vector field in $\mathbb{R}^n$ and gives the velocity of the flow at time $t$. Any solution $y(t)$ satisfying (1) with initial condition $y_0 = y(t_0)$ for some time $t_0$ is time invariant, i.e., if $y(t)$ is a solution then so is $y(t + T)$ for any $T$.

In studying the flow of (1), we first try to analyse the behaviour near a particular point $y_1 \in \mathbb{R}^n$. If $f(y_1) \neq 0$ then the vector field $f(y_1)$ gives the direction of the flow at $y_1$. Since $f(y)$ is assumed to be continuous, the flow of the system
near $y_1$ is roughly parallel to $f(y_1)$. If $f(y_1) = 0$ then $y_1$ is called a critical or equilibrium point of the system. In order to study the flow near such a point, we try a Taylor expansion of $f(y)$ about $y_1$ to get

$$f(y) = f(y_1) + Df(y_1)(y - y_1) + \text{higher order terms}.$$ 

We would like to approximate $f(y)$ near $y_1$ by $Df(y_1)(y - y_1)$ which is linear in $y - y_1$. We need then only analyse the linear system $z_t = Df(y_1)z = Az$, where $Df(y_1) = A$ is an $n \times n$ constant matrix and $z = y - y_1$.

1.2 Linear Results

In studying the linear system $z_t = Az$ where $A$ is a nonsingular $n \times n$ constant matrix, we have the following results for the critical point $z_c = 0$ :

For a two dimensional system ($z \in \mathbb{R}^2$), if $A$ has

A) two equal real eigenvalues with two linearly independent eigenvectors, $z_c$ is a proper node (see Fig. 1.1a).

B) two real distinct eigenvalues of the same sign, $z_c$ is an improper node (see Fig.1.1b).

C) equal real eigenvalues, only one eigenvector, $z_c$ is an improper node (see Fig.1.1c).

D) real eigenvalues of opposite sign, $z_c$ is a saddle point (see Fig.1.1d).
Figure 1.1a (eigenvalues are negative)

Figure 1.1b (eigenvalues are negative)
Figure 1.1c (eigenvalues are negative)

Figure 1.1d
Figure 1.1e (where $a < 0$)

Note that the orbits need not be symmetric about the $x$- and $y$-axes.

Figure 1.1f ($a = 0$)
E) complex conjugate eigenvalues $a \pm ib$, then for

i) $a \neq 0$, $z_c$ is a **spiral point** (see Fig. 1.1e);

ii) $a = 0$, $z_c$ is a **centre point** (see Fig. 1.1f).

### 1.3 Stability Analysis

**Definition**: For the dynamical system (1) with critical point $y = y_c$

i) $y_c$ is **stable** if given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|y(t) - y_c| < \varepsilon \text{ whenever } |y(t_0) - y_c| < \delta.$$  

ii) $y_c$ is **asymptotically stable** if $y_c$ is stable and

$$\lim_{t \to \infty} |y(t) - y_c| = 0.$$

**Case 1**: In $\mathbb{R}^2$, stability analysis of the critical point $z_c = 0$ of the linear system

$z_t = Az$ can be summarized as follows:

If $\lambda, \mu$ are eigenvalues of $A$,

i) $z_c$ is asymptotically stable if $\text{Re}(\lambda), \text{Re}(\mu) < 0$;

ii) $z_c$ is unstable if $\text{Re}(\lambda), \text{Re}(\mu) > 0$;

iii) $z_c$ is stable if $\text{Re}(\lambda) = \theta = \text{Re}(\mu)$;

iv) $z_c$ is a saddle point if $\lambda, \mu$ are real and of opposite sign.
Case 2: In $\mathbb{R}^n$ for $n \geq 3$, stability analysis of $z_C = 0$ can be summarized as follows:

i) $z_C$ is asymptotically stable if $A$ has all its eigenvalues with negative real part;

ii) $z_C$ is unstable if $A$ has any eigenvalues with positive real part;

iii) $z_C$ is a stable centre point if all eigenvalues have nonpositive real part and there exist pure imaginary eigenvalues.

iv) If $A$ has no pure imaginary eigenvalues, then $\mathbb{R}^n$ splits into subspaces $\mathbb{R}^n = E^s \oplus E^u$ where $E^s$, the stable manifold, is spanned by the generalized eigenspaces corresponding to the eigenvalues with negative real part and $E^u$, the unstable manifold, is spanned by those corresponding to the eigenvalues with positive real part. $z_C$ is then asymptotically stable if $\dim(E^u) = 0$.

Unfortunately, a nonlinear system cannot in general be linearized near a critical point using Taylor's expansion (or what is called the variational equation method) to give a reasonable approximation of the behaviour of the flow near that critical point. Linearization as an approximation is only valid when there is a diffeomorphism, $\Psi$, which links the nonlinear system with its linearized form.
There are two theorems, the Sternberg Linearization Theorem [26] and the Hartman Linearization Theorem [10] which give conditions in which linearization is plausible.

Sternberg's theorem allows for a $\Psi \in C^\infty$, but the algebraic condition for this excludes many examples and in [26] are stated definite counterexamples when the condition is not met. Hartman's theorem allows only for a homeomorphism $\Psi \in C^0$ which can distort the geometry of the system drastically. Also, it must be noted that in neither theorem is the case of pure imaginary eigenvalues included.

1.4 Nonlinear Results

Paralleling the linear case, we have that the critical point $y_c = \theta$ of the nonlinear dynamical system $y_t = f(y)$ is asymptotically stable if all the eigenvalues of $Df(\theta)$ have negative real part but unstable if even one of them has positive real part. In the former case $y_c$ is called a sink, in the latter a source.

Method of Liapounov Functions

The stability of the critical point $y_c = \theta$ can be determined by studying Liapounov functions $V(y)$, where $V$ is a continuous function in a neighborhood $N$ of $y_c = \theta$ and has a minimum at $y = \theta$.

*Theorem:* Let $V(y) \in C^1(N - \{\theta\})$, $V(\theta) = 0$, $V(y) > 0$ for $y \in N - \{\theta\}$. Then
i) if \( DV(y) \cdot f(y) < 0 \) in \( N - \{0\} \), \( y_c \) is asymptotically stable;

ii) if \( DV(y) \cdot f(y) \leq 0 \) in \( N - \{0\} \), \( y_c \) is stable;

iii) if \( DV(y) \cdot f(y) = 0 \) in \( N \), \( y_c \) is stable but not asymptotically stable.

**Example 1.1:** Consider the Lotka-Volterra model of predator-prey dynamics \([19]\).

\[
\begin{align*}
x_t &= x(k - my) \\
y_t &= y(-p + qx) \\
\end{align*}
\]

\( (x_t, y_t) = f(x, y) \) \( (2) \)

where \( x(t) \) is the population of the prey at time \( t \), \( y(t) \) is that of the predator, and \( k, m, p \) and \( q \) are positive constants.

The system (2) has critical points \((0, 0)\) and \((p/q, k/m)\).

\[
Df = \begin{bmatrix} k - my & -mx \\ qy & -p + qx \end{bmatrix}
\]

i) \( Df(0, 0) = \begin{bmatrix} k & 0 \\ 0 & -p \end{bmatrix} \) with eigenvalues \( \lambda_1 = k > 0 \) and \( \lambda_2 = -p < 0 \).

So \((0, 0)\) is unstable with stable manifold tangent to the \( y \)-axis; i.e., \((0, 0)\) is a saddle point.
\[ Df(p/q, k/m) = \begin{bmatrix} 0 & -mp/q \\ qk/m & 0 \end{bmatrix} \text{ with eigenvalues } \lambda = \pm i\sqrt{kp}. \]

Since (2) is a nonlinear system we cannot classify the stability of the critical point without further study. In order to classify this point we will look for a Liapounov function \( V(x, y) = g(x) + h(y) \) which satisfies one of i) - iii) in the theorem above.

\[ V_t(x, y) = g'(x)x_t + h'(y)y_t = g'(x)(x(k - my)) + h'(y)(y(-p + qx)). \]

Suppose \( V_t(x, y) = 0 \), then \( g'(x)x/(-p + qx) = h'(y)y/(k - my) = 1 \).

Thus \( g'(x) = q - p/x \) and \( h'(y) = -k/y + m \)

\[ g(x) = qx - plnx \quad h(y) = my - klny. \]

Then \( V(x, y) = qx - plnx + my - klny \) is constant on the orbits of the flow of (2) and it has an absolute minimum at \((p/q, k/m)\). So \( V(x, y) \) is a Liapounov function and \((p/q, k/m)\) is stable but not asymptotically stable.

### 1.5 Closed Orbits

Evidence of closed orbits as stable structures occurs in the case of the Van der Pol oscillator \( x_t = y - x^3 + x, \quad y_t = -x \) in \( \mathbb{R}^2 \) where we get a unique closed orbit \( C \) encircling the origin, and such that every trajectory except \((x, y) = 0\) approaches \( C \) as \( t \to +\infty \) [10].

The Poincaré-Bendixson theorem also establishes the (possible) existence of
stable closed orbits for vector fields in $\mathbb{R}^2$.

**Theorem (Poincaré-Bendixson):** Let $f$ be a vector field in $\mathbb{R}^2$ satisfying a uniform Lipschitz condition, $z^+(t)$ a trajectory of the dynamical system $z_t = f(z)$, and $C^+$ the set of $\omega$-limit points of $z^+(t)$ which is bounded and includes no critical points. Then $C^+$ is the range of a periodic orbit of $f(z)$, i.e., it is a closed orbit.

As corollaries to the Poincaré-Bendixson theorem, we have the following results:

**Corollary 1:** Suppose $C$ is a limit cycle with an orbit approaching $C$ from the inside/outside. Then there exists a neighborhood $U$ of $C$ such that if $z \in U$ and $z$ is inside/outside of $C$, $\phi_t(z)$ which is the flow of $z$, approaches $C$ as $t \to +\infty$.

**Corollary 2:** Suppose $K$ is a compact set in $\mathbb{R}^2$ and $\phi_t(K) \subset K$ for $t \geq 0$. Then $K$ contains either a critical point or a closed orbit.

In fact, a further corollary shows that if $K$ is also simply connected, there must exist a critical point in $K$ since it is shown that for any closed orbit $C$ of the flow $\phi_t$, $f(z)$ has a critical point inside of $C$ [10].
The following example uses the Poincaré-Bendixson theorem and its corollaries to show the existence of either a stable critical point or a stable closed orbit.

**Example 1.2:** Consider the predator-prey dynamics model

\[ x_t = x \cdot M(x, y) \quad x, y > 0 \]

\[ y_t = y \cdot N(x, y) \quad \text{where } M \text{ and } N \text{ are the exponential growth rates for populations } x \text{ and } y \text{ respectively.} \]

Suppose the following conditions hold and Fig. 1.2 is the corresponding diagram.

i) \( M, N \in C^1(R^+ \times R^+, R^2) \);

ii) \( \partial M/\partial y < 0, \partial N/\partial x > 0, \partial N/\partial y < 0; \)

iii) Suppose the curves \( M^{-1}[0] \) and \( N^{-1}[0] \) represented as \( y = m(x) \) and \( x = n(y) \) respectively, are in the first quadrant for precisely the ranges \( 0 < x < K \) and \( L < x \) respectively where \( L < K \).

iv) Suppose \( m \) and \( n \) intersect transversely at the point \( P \) (see Fig. 1.2).
It is easily verified that \(\Theta\) is a saddle point, and \(K\) is a saddle point with a unique trajectory exiting into region A. At the interior critical point \(P\), the eigenvalues satisfy

\[ \lambda^2 - (xM_x + yN_y)\lambda + xy(M_xN_y - M_yN_x) = 0. \]

If \(\lambda_1\) and \(\lambda_2\) are the eigenvalues, then \(\lambda_1 + \lambda_2 = xM_x + yN_y\) and \(\lambda_1\lambda_2 = xy (M_xN_y - M_yN_x).\)

\[ \text{sgn}(\lambda_1\lambda_2) = \text{sgn}(M_xN_y - M_yN_x) = \text{sgn}(-M_xN_y/M_yN_x + 1) = \text{sgn}(1 - \text{slope } m/\text{slope } n). \]

Since \(P\) is unique, slope \(m < \text{slope } n\), so \(\lambda_1\lambda_2 > 0\). Thus \(P\) is a node (if \(\lambda_1, \lambda_2\) are real and of the same sign), spiral (if \(\lambda_1 = a + ib = \lambda_2\)), or centre (if \(\lambda_1 = ib = \lambda_2\)).
Now let $z_0(t)$ be the unique trajectory exiting from $K$, the progression for $z_0(t)$ is $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ and there are two possible cases. 

i) There are only finitely many progressions for $z_0(t)$. Then $z_0(t)$ must eventually become monotone and thus $z_0(t)$ approaches a critical point as $t \rightarrow +\infty$. In fact $z_0(t) \rightarrow P$ since it is the only accessible critical point (see Fig.1.3). In this case every other trajectory in the first quadrant must obviously also approach $P$ as $t \rightarrow +\infty$. Thus $P$ is either a sink or a centre, and in the above case $P$ is an asymptotically stable critical point with its basin the whole of the first quadrant.

![Figure 1.3](image)

ii) There are infinitely many progressions for $z_0(t)$. By the Poincaré-Bendixson theorem, there are two subcases:

ii)' where $z_0(t) \rightarrow$ limit cycle $C$ as $t \rightarrow \infty$ (see Fig. 1.4), in which case what
happens within C is unclear from the information we have;

Figure 1.4

\( \text{ii) } \) where \( P \) is an \( \omega \)-limit point of \( z_0(t) \). In this case \( P \) is either a sink (see Fig. 1.5) and thus asymptotically stable, or a centre in which case it is again asymptotically stable by the Poincaré-Bendixson theorem.

Figure 1.5
Further study shows that in the case of ii)', if $P$ is not stable then there must exist a stable closed orbit. There are four cases to consider:

a) If $C$ is a limit cycle from the inside as well as the outside, then corollary 1 implies that $C$ is asymptotically stable.

b) If there are only a finite number of closed orbits inside of $C$, then it is easy to locate an asymptotically stable one.

c) If the interior of $C$ is all closed orbits, then each one is stable.

d) If there are infinitely many closed orbits and $P$ is not stable, then there exists a $C_0$ which is asymptotically stable.

**In Higher Dimensions**

Let $\Phi_t$ be the flow of the $C^1$ autonomous system $\dot{y}_t = f(y)$ in $\mathbb{R}^n$ ($n \geq 3$). Let $C$ be a closed orbit of the flow with period $T$ and $p$ a point on $C$. Then it can be shown that if the Jacobian $D_y \Phi_t(p)$ (given explicitly by Liouville's trace formula [10]) has $n - 1$ eigenvalues of absolute value less than 1, $C$ is asymptotically stable.

**1.6 Bifurcations**

Having established the existence of critical points and closed orbits as asymptotically stable structures, we now wish to consider systems depending on a parameter. We shall study how changes in the parameter affect these stable structures causing destabilization and bifurcations. We will study two common
modes of destabilization in two dimensions. The first is called the saddle-node effect where an asymptotically stable critical node destabilizes to a saddle point. In this case, one of the two negative eigenvalues becomes positive as the parameter passes a critical value. The second mode is the Hopf bifurcation phenomenon in which a critical point bifurcates to closed orbits (these occurring before, after or precisely when the parameter is at its critical value).

In these two modes of destabilization, we will study conditions and examples in which stable structures can be located after the critical point destabilizes.

**Mode 1: Saddle-Node Bifurcation**

As an asymptotically stable nodal point bifurcates to a saddle point, there is no reason to assume that another stable structure will take its place nearby. In fact, the following example shows that this is indeed true.

**Example 1.3:** Consider the linear system

\[
\begin{align*}
    x_t &= -x \\
    y_t &= \alpha y
\end{align*}
\]

with eigenvalues \( \lambda_1 = -1, \lambda_2 = \alpha \). For \( \alpha < 0 \), \((0, 0)\) is an asymptotically stable node; for \( \alpha > 0 \), we have a saddle point. There are no stable structures near the critical point \((0, 0)\) for \( \alpha > 0 \)--the stable structure of the system is the point at \( \infty \).

There do exist examples where the saddle-node bifurcation occurs and an
asymptotically stable critical point destabilizes to a saddle point while stability shifts to two new asymptotically stable points as the parameter passes its critical value. The following example is an illustration of this phenomenon.

**Example 1.4:**

Consider the problem in which a rigid hoop of radius $R$ hangs from the ceiling, and a smooth ball rests in the bottom of the hoop (see Fig. 1.6). As the hoop rotates with a frequency of $\omega$ about the vertical axis through its centre, by methods of classical mechanics, the following system can be shown to describe the motion of the ball.

$$
\theta_t = \phi
$$

$$
\phi_t = \omega^2 \sin \theta \cos \theta - \frac{(g/R)\sin \theta}{(k/mR^2)\phi} (\theta_t, \phi_t) = G(\theta, \phi)
$$

where $\theta$ is the angle of displacement of the ball from its resting place at the bottom of the hoop, and $\omega$ is out parameter, $m = \text{mass}$, $g = \text{gravity}$, $k > 0$ is the constant due to friction.
The Jacobian of the system is

\[
DG(\theta, \phi) = \begin{bmatrix}
0 & 1 \\
\omega^2 \cos 2\theta - (g/R) \cos \phi & -k/mR^2
\end{bmatrix}
\]

and critical points occur when \( \phi = 0 \) and \( \omega^2 \sin \phi (\cos \phi - g/R) = 0 \). We have the following possibilities:

i) \( \phi = 0 \) and \( \theta = 0 \), then \( DG(0, 0) = \begin{bmatrix} 0 & 1 \\ \omega^2 - g/R & -k/mR^2 \end{bmatrix} \) and eigenvalues satisfy \( \lambda^2 + (k/mR^2) \lambda - (\omega^2 - g/R) = 0 \). Thus if \( \omega^2 < g/R \), \((0, 0)\) is a sink (asymptotically stable), while if \( \omega^2 > g/R \), \((0, 0)\) is a saddle (unstable).

ii) \( \phi = 0 \) and \( \theta = \pi \), then \( DG(\pi, 0) = \begin{bmatrix} 0 & 1 \\ \omega^2 + g/R & -k/mR^2 \end{bmatrix} \) and eigenvalues satisfy \( \lambda^2 + (k/mR^2) \lambda - (\omega^2 + g/R) = 0 \). Thus \((0, 0)\) is a saddle point (unstable).

iii) \( \phi = 0 \) and \( \cos \theta = g/R \omega^2 \) (\( \theta \) in first or fourth quadrant). These points exist only when \( g/R \omega^2 \leq 1 \) or \( \omega^2 \geq g/R \).

\[
DG(\text{arccos}(g/R \omega^2), 0) = \begin{bmatrix} 0 & 1 \\ (g^2/R^2 \omega^2) - \omega^2 & -k/mR^2 \end{bmatrix}
\]

and we have a sink when \( \omega^2 > g/R \).

Thus i) and iii) give us the information about the bifurcation which occurs. We
have \((\theta, 0)\) asymptotically stable for \(\theta \leq \omega < \sqrt{g/R}\). At the critical parameter value \(\omega_0 = \sqrt{g/R}\), \((\theta, 0)\) destabilizes and two new asymptotically stable critical points \((\pm \arccos(g/R\omega^2), 0)\) occur as \(\omega\) increases past \(\omega_0\).

**Mode 2:** Hopf Bifurcation (in \(\mathbb{R}^2\)) [22]

**Hopf Bifurcation Theorem:** Let \(F(x, \mu) \in C^k(\mathbb{R}^2 \times \mathbb{R}^1), k \geq 4\) and \(F(\theta, \mu) = 0\) for every \(\mu\). Suppose \(D_x F(\theta, \mu)\) has distinct complex conjugate eigenvalues \(\lambda(\mu)\) and \(\lambda(\mu)\). If \(\text{Re}(\lambda(\theta)) = 0\) and \(d/d\mu(\text{Re}(\lambda(\mu)))\big|_{\mu=0} > 0\), then for \(x_t = F(x, \mu)\)

a) for some \(\varepsilon > 0\), there exists a \(C^{k-2}\) function \(\mu: \theta < |x| < \varepsilon \rightarrow \mathbb{R}\) such that if \(\theta < |x| < \varepsilon\), the point \((x, \mu(x))\) is on a closed orbit of period \(T(x)\) where

\[
\lim_{x \rightarrow \theta} T(x) = \frac{2\pi}{|\lambda(\theta)|}.
\]

b) there exists a neighborhood \(U\) of \((\theta, 0)\) such that any closed orbit intersecting \(U\) is one of those in a).

**Definition 1:** Let \(R(\mu, t) = \sup_{|x| = r} |x(t)| = \text{maximum radius at time } t \text{ of orbits starting at radius } r; R(\mu) = \limsup_{t \rightarrow \infty} R(\mu, t); r(\mu) = \inf_{r > 0} R(\mu) = \text{smallest radius at which orbits can be confined for all } t > 0 \text{ by choosing initial radius } r\) sufficiently small. Then the Hopf bifurcation is a

i) **soft bifurcation** when \(\lim_{\mu \rightarrow 0^+} r(\mu) = 0;\)

ii) **hard bifurcation** when \(\liminf_{\mu \rightarrow 0^+} r(\mu) > 0.\)
Definition 2: For \( \mu(x) \) = the parameter value at which there occurs a closed orbit through \( x \), we have the following:

i) If \( \mu(x) > 0 \) for all \( |x| < \varepsilon \) (some \( \varepsilon > 0 \)), then we have \textit{supercritical bifurcation} (\( \Rightarrow \) soft bifurcation).

ii) If \( \mu(x) < 0 \) for all \( |x| < \varepsilon \) (some \( \varepsilon > 0 \)), then we have \textit{subcritical bifurcation} (\( \Rightarrow \) hard bifurcation).

In case i) of definition 2 above, the closed orbits occur after the critical destabilizing value of the parameter has been passed (here \( \mu_c = 0 \)). These orbits are stable; and given any fixed radius \( \delta > 0 \) we can find an initial radius \( r \) small enough (say \( r < r_0 \)) and a parameter value \( \mu \) small enough (say \( 0 < \mu < \mu_0 \)) so that all orbits starting at a radius \( r \) from the origin remain within a distance \( \delta \) from the origin for all \( t \geq 0 \) (\( r_0 \) and \( \mu_0 \) depend on \( \delta \)). Thus the destabilization of the critical point \( x = 0 \) is "soft" since trajectories starting near the origin do not escape to large distances as destabilization occurs; instead, they are forced to remain in a neighborhood of the origin for small positive values of \( \mu \).

In case ii) of definition 2, the closed orbits occur before the critical value of the parameter is reached. Thus the critical point \( x = 0 \) is still stable and co-exists with unstable closed orbits. Once \( \mu \) passes its critical value, \( x = 0 \) becomes unstable but there are no closed orbits to confine the trajectories, so trajectories starting near \( x = 0 \) will be able to escape to large distances in finite
time.

Marsden and MacCracken [22] describe a way in which we can determine which of the two cases occur for the system described in the theorem. If we let $\pi_\mu(x)$ be the Poincare map of the system near $x = 0$ and define $V_\mu(x) = \pi_\mu(x) - x$ = the displacement map, then if $\frac{\partial^3 V_\mu(x)}{\partial x^3}\Big|_{(0,0)} < 0$ the closed orbits occur for $\mu > 0$ and are asymptotically stable; if $\frac{\partial^3 V_\mu(x)}{\partial x^3}\Big|_{(0,0)} > 0$ then the closed orbits occur for $\mu < 0$ and are unstable. For the detailed formula of $\frac{\partial^3 V}{\partial x^3}$, see [22].

Hopf theory extends to higher dimensions by considering the two dimensional manifold in which the pair of complex conjugate eigenvalues cross the imaginary axis. The two dimensional analysis can then be applied in this manifold. It must be noted that the manifold may change depending on the parameter $\mu$ [22].

**Example 1.5a: Subcritical Bifurcation**

Consider the system in polar coordinates

$$r_t = (r^2 + \mu)r = \psi(r)$$

$$\theta_t = 1.$$

This describes counterclockwise motion about the origin with a constant angular velocity. The critical point in cartesian coordinates is $(0, 0)$ and the critical parameter value is $\mu_c = 0$. So we have the following:
i) If $\mu > 0$ then $r > 0$ for all $r \neq 0$. This implies that $r(t)$ is increasing in $t$ and $r(t) \to +\infty$ in finite time. Thus there are no closed orbits (see Fig. 1.7).

![Figure 1.7](image1)

We let $\mu = 0.25$; there are no closed orbits and $x = 0$ is unstable.

![Figure 1.8](image2)
We let $\mu = -0.25$; there exists a closed orbit of radius $r = 0.5$ and $x = 0$ is asymptotically stable.

ii) If $\mu < 0$ then the graph of $r$ to $\psi(r)$ is given by Fig. 1.8 above. So $r(t)$ is decreasing in $t$ for all $t \geq 0$ if $r(0) < \sqrt{|\mu|}$, i.e. $r(t) \to 0$ as $t \to +\infty$. But if $r(0) > \sqrt{|\mu|}$, then $r(t)$ is increasing in $t$ for all $t > 0$ and $r(t) \to \infty$ in finite time. For $r(0) = \sqrt{|\mu|}$, $r(t)$ is a closed orbit of radius $\sqrt{|\mu|}$ (see Fig. 1.9). Thus all closed orbits here occur at subcritical values of $\mu$ as in Fig. 1.8.
Example 1.5b: Supercritical Bifurcation

If we consider instead the slightly changed system

\[ r_t = r(\mu - r^2) = \psi(r) \]

\[ \theta_t = 1 \]

then we see that the closed orbits will occur this time when \( r^2 = \mu > 0 \), i.e. for supercritical values of \( \mu \). For \( \mu < 0 \), \( r_t < 0 \) for all \( r \neq 0 \), which implies that \((0, 0)\) is an asymptotically stable critical point and no closed orbits occur (see Fig.1.10).

For \( \mu > 0 \), if \( r(0) > \sqrt{\mu} \), then \( r_t < 0 \) and \( r(t) \) is decreasing in \( t \) for all \( t \geq 0 \); if \( r(0) < \sqrt{\mu} \), then \( r_t > 0 \) and \( r(t) \) is increasing in \( t \) for all \( t \geq 0 \); if \( r(0) = \sqrt{\mu} \), then \( r(t) \) is a closed orbit of radius \( \sqrt{\mu} \). Thus all trajectories \( r(t) \) with \( r(0) > \sqrt{\mu} \) will spiral inward toward the closed orbit \( r(t) = \sqrt{\mu} \), while all trajectories with \( r(0) < \sqrt{\mu} \) will spiral outward and be confined by the closed orbit \( r(t) = \sqrt{\mu} \). So the critical point \((0, 0)\) is now unstable (see Fig.1.11); but for all small fixed \( \mu > 0 \), sufficiently small perturbations from \((0, 0)\) will not result in large scale instability since the closed orbit will confine trajectories to a neighborhood of \((0, 0)\).
We let $\mu = -0.25$; there are no closed orbits and $x = 0$ is asymptotically stable.

We let $\mu = 0.25$; a closed orbit occurs at $r = 0.5$ and $x = 0$ is unstable.
The closed orbits may also all occur precisely when the parameter reaches its critical value. This may be referred to as **critical bifurcation**. An example of this is the linear system

\[
\begin{align*}
    x_t &= y + \mu x \\
    y_t &= -x + \mu y.
\end{align*}
\]

The critical point is \((x, y) = (0, 0)\) with critical parameter value \(\mu_c = 0\). The eigenvalues are \(\lambda = \mu \pm i\) so \((0, 0)\) is asymptotically stable when \(\mu < 0\) (see Fig. 1.12) and unstable when \(\mu > 0\) (see Fig. 1.13). There are no closed orbits in either of these cases. When \(\mu = \mu_c = 0, \lambda = \pm i\), so \((0, 0)\) is a centre point and every orbit is closed (see Fig. 1.14).

![Figure 1.12 (where \(\mu = -0.25\)](image-url)
Figure 1.13 (where $\mu = 0.25$)

Figure 1.14 (where $\mu = 0$)
Remarks:

1) There are always closed orbits associated with Hopf bifurcation. We have shown that sometimes these closed orbits are asymptotically stable structures—example 1.5b.

2) Since the Hopf theorem is a local theorem, the confinement of trajectories near \((0, 0)\) after destabilization in the case of soft bifurcation, may occur when \(\mu(x)\) has variable sign. An example of this is shown by the system

\[
\begin{align*}
\dot{r}_t &= \begin{cases} 
0 & \text{when } r = 0 \\
\{ r(\mu - r^{18}\cos(1/r^2)) \} & \text{when } r \neq 0
\end{cases} \\
\dot{\theta}_t &= 1,
\end{align*}
\]

where \(r = 0\) when \(r = 0\) or \(r^{18}\cos(1/r^2) = \mu\) (note \(|\mu| \leq r^{18}\)). If we consider \(\mu\) as a function of \(r\), we see that \(\mu(r)\) oscillates with an increasing amplitude in \(r\) until \(r = \sqrt{2/\pi}\). Once \(r\) increases past \(\sqrt{2/\pi}\), \(1/r^2 < \pi/2\) and \(\cos(1/r^2) > 0.\) Soft bifurcation occurs here since \(r(\mu)\) can be made as small as we want by picking \(\mu > 0\) sufficiently small and since \(\lim_{r \to 0^+} \mu(r) = 0.\)

In summary, for a one parameter family \(x_t = F(x, \mu)\) of dynamical systems, the destabilization of a critical point can lead to a shifting of the stability typically in two ways.
i) An asymptotically stable critical point destabilizes to a saddle point, while two new asymptotically stable points are created.

ii) An asymptotically stable critical point destabilizes and asymptotically stable closed orbits are created—supercritical Hopf bifurcation.

After the critical value of the parameter $\mu$ has been passed, it is natural to ask if further bifurcations occur for even larger values of $\mu$, and what kind of shifting in stable structures—if any—may occur. There has been study of this in the case of Hopf bifurcation where closed orbits occurring after destabilization themselves destabilize and new stable closed orbits are created or absorbed. See [7], [14], [27], [28] and [29] for further discussion.
2.1 Introduction to Cobweb Models

We would now like to study the stability of fixed points in cobweb models. The term cobweb model was originally used in economics [8], [16]. In mathematics, these models are usually referred to as maps of the unit interval, and no special terminology seems to exist. Thus, we shall adapt the terminology of the economists in our discussions here.

We will observe similarities and differences in stability conditions for a fixed point as compared to those for a critical point in a dynamical system. We will also observe that the destabilization of a fixed point due to a change in the parameter of a parametrically dependent cobweb model seems to have the same kind of properties as in dynamical systems.

Definition 1: Consider $F$ a $C^1$ function of $\mathbb{R} \to \mathbb{R}$ and $\{x_n\}$ a sequence in $\mathbb{R}$ such that $x_{n+1} = F^n(x_n)$ or $x_n = F^n(x_0)$. Then the map $(F, x_0) \to \{x_n\}$ is a cobweb model (see Fig. 2.1).

Definition 2: A fixed point $x_*$ of a cobweb model is a point where $F(x_*) = x_*$. (Note the similarity to a critical point in dynamical systems.)
The definitions of stability and asymptotic stability as \( n \to \infty \) of a fixed point, are the same as those for a critical point. We have the following lemma:

**Lemma**: A fixed point \( x_m > 0 \) of a cobweb model is asymptotically stable when

\[
|F'(x_m)| < 1 \text{ and unstable when } |F'(x_m)| > 1. 
\]

**proof**: If \( |F'(x_m)| < 1 \) then given \( \epsilon > 0 \) small, \( |F'(x)| < 1 - \epsilon < 1 \) in some neighborhood \( N = \{x: |x - x_m| < \delta\} \) of \( x_m \). If \( x_n \in N \) we have that

\[
|F(x_{n+1}) - x_m| = |F'(X)| |x_n - x_m| \quad \text{(by the Mean-Value Theorem)}
\]

where \( X \) is between \( x_n \) and \( x_m \), i.e., \( X \in N \). Thus, \( |F'(X)| < 1 - \epsilon \) and \( |x_{n+1} - x_m| < (1 - \epsilon)|x_n - x_m| \).

This implies that the distance between \( x_m \) and any point in \( N \) is shrinking at a
geometric rate. Thus $x_n \to x_\infty$ as $n \to +\infty$, and $x_\infty$ is asymptotically stable.

Similarly, we can show that $x_\infty$ is unstable when $|F'(x_\infty)| > 1$.

One distinctive property of the cobweb model which is due to the discrete time change is characterized by the following corollary.

**Corollary** For the fixed point $x_\infty > 0$, where $|F'(x_\infty)| < 1$.

i) $x_n - x_\infty$ alternates in sign for large $n$ where $F'(x_\infty) < 0$;

ii) $x_n - x_\infty$ has a fixed sign for large $n$ when $F'(x_\infty) > 0$.

### 2.2 Comparison to Dynamical Systems

We now proceed with two modes of comparison, the first relates a cobweb model to a one dimensional dynamical system, while the second relates it to a two dimensional dynamical system. The first mode is the natural one, but it is the second which provides interesting results.

**Mode 1**

We can relate cobweb models to the one dimensional dynamical system $x_{n+1} = f(x)$ as follows:
approximate the dynamical system above by
\[ \Delta x = f(x) \Delta t \]

and let \( \Delta t = 1 \), then we have
\[ x(n+1) = x(n) + f(x(n)) \]
or
\[ x_{n+1} = g(x_n) \]
where
\[ g(x) = f(x) + x. \]

Studying the conditions for stability of the two types of equations, we note the similarities for stability of a critical point in the dynamical system as compared to those of a fixed point in the cobweb model. The critical point \( p \) is asymptotically stable if \( f'(p) < 0 \), which corresponds to the condition that \( g'(p) = f'(p) + 1 < 0 + 1 = 1 \) or \( g'(p) < 1 \). In the case of the cobweb model \( x_{n+1} = g(x_n) \), a fixed point \( p \) is stable \( \iff \) \( |g'(p)| < 1 \). For positive values of \( g'(p) \) the two conditions are exactly the same (see Fig.2.2a). However when \( g'(p) \) is negative, because of the discrete time change, the values of \( g(x) \) near \( p \) will oscillate between values greater than \( p \) and those less than \( p \). So for \( g'(p) < -1 \) or \( f'(p) < -2 \), the fixed point of the cobweb model becomes unstable whereas the critical point of the corresponding dynamical system remains asymptotically stable. The reason for this discrepancy is the problem of the values of \( g(x) \) "overshooting" the fixed point, as described above, for negative values of \( g'(p) \) (see Fig.2.2b). So once \( g'(p) \) decreases past \( -1 \), the distance
between $g(x_n)$ and $p$ will increase on the left side of $p$ and thus $p$ becomes unstable. This problem does not occur in the one dimensional dynamical system or when $g'(p) > 0$ since then the approach to $p$ is monotone and there is no problem of overshooting the critical or fixed point.

**Example 2.1:** Compare the linear dynamical system $x_t = rx = f(x)$ where we have asymptotic stability of the critical point $x_c = 0$ for $f'(0) = r < 0$, with the corresponding cobweb model $g(x) = (r + 1)x$ or $x_{n+1} = g(x_n)$ where we have asymptotic stability of the fixed point $x_m = 0$ for $|g'(0)| = |r + 1| < 1$ or $-2 < r < 0$ (i.e., for $r + 1 > 0$ or $0 > r > -1$; for $r + 1 < 0$ or $-2 < r < -1$).

**Mode 2**

An analogy can also be drawn between cobweb models and two dimensional dynamical systems. Consider the two dimensional dynamical system with a closed
orbit $C$. Let $L$ be a transversal through the point $p$ on $C$. The Poincare map $\pi: L \to L$ mapping successive crossings of $L$ by trajectories of the dynamical system, is (locally) a cobweb model. We note then that if $|\pi'(p)| < 1$, $C$ is an asymptotically stable closed orbit while $|\pi'(p)| > 1$ implies that $C$ is unstable. We must note however that because orbits don't cross in the plane, $\pi$ is monotonic and thus there is very little of the general theory of cobweb models included in this analogy.

Remark:

In higher dimensions, we don't have this restriction. We have stability of the Poincare map $\pi: H \to H$ (where $H$ is the hyperplane in $\mathbb{R}^{n-1}$ for a $n$-dimensional system) if the eigenvalues of $D\pi(p)$ satisfy $|\lambda| < 1$. But then we would be dealing with cobweb models in higher dimensions.

2.3 Allwright's bifurcation vs. Hopf bifurcation

Our analogy was not however made in vain. We note that the two models refer to the same underlying system; the only difference is in the time dependence of the Poincaré map as opposed to the discrete time change of the cobweb model. Because the underlying system is the same, it is difficult to show contrasts in the two models being compared. In fact, we note that the Hopf theorem for dynamical
systems is proved by translating the system to cylindrical coordinates and using
the Poincaré map on the equivalent transversals $\theta = 0, \theta = 2\pi$ and etc. We could
like to compare the Hopf theory of bifurcation in dynamical systems with
Allwright's hypergraphic theory of bifurcation in cobweb models.

Allwright's theory presents us with a way to identify mapping functions $F_r$ de­
 pending on a parameter $r$ for which the bifurcation process is "regular".

**Definition:** A function $f \in C^3(\mathbb{R}, \mathbb{R})$ is **hypergraphic** if

$$\text{Hypergraph}(f) = 3(f''(x))^2 - 2f'(x)f'''(x) > 0 \text{ whenever } f'(x) \neq 0.$$  

It is easily verified that this property is closed under compositions. We note
here that we are really dealing with local theories so that for any fixed point, we
will only be considering $f$ locally hypergraphic. Allwright's theory [1] is stated
as follows:

**Theorem (Allwright):** Suppose that for all $r$

1) $F_r(x) \in C^3(r^2, \mathbb{R})$ and $F_r$ is hypergraphic (locally);

2) $x_0$ is a fixed point of $F_r^{(k)}$;

3) $F_r^{(k)}(x_0)$ decreases continuously through $-1$ as $r$ increases through $r_C$. 
its critical destabilizing value. Then there are two fixed points of $F_r^{(2k)}$ which branch out from $x_0$ as $r$ increases above $r_c$.

In the following example, we see a cobweb model $x \rightarrow F_r(x)$ in which $f$ is not hypergraphic but we still have Hopf-type bifurcation occurring.

**Example 2.2:** Let $x \rightarrow F_r(x) = r(x + x^3)$ by a cobweb model with fixed point $p = 0$ for all $r$.

**Case 1:** $r \geq 0$

For $r$ nonnegative, we have destabilization when $|F_r'(p)| = |r(1 + 3x^2)|_{x=0} = |r| = r = 1$. Since $F_r'(0) = r$, $F_r''(0) = 0$, and $F_r'''(0) = 6r$;

$$\text{Hypergraph}(F_r) = 3(0)^2 - 2(0)6r = -12r^2 < 0$$

Thus $F_r$ is not hypergraphic near $p = 0$ and Allwright’s theory does not apply.

We note however that for $0 < r < 1$, there exist two other fixed points $p_\pm(r) = x$ as follows:

$$r(x + x^3) = x$$
$$rx^3 + (r - 1)x = 0$$
$$x(rx^2 + r - 1) = 0.$$
So \( x = 0 \) or \( x^2 = (1 - r)/r \) (\( x = \pm \sqrt{1/r - 1} = p_{\pm}(r) \)). Now we have

\[
\text{as } r \to 0^+, p_{\pm}(r) \to \pm \infty;
\]

\[
\text{as } r \to 1^-, p_{\pm}(r) \to 0^\pm.
\]

Also (see Fig. 2.3),

\[
\lim_{n \to \infty} x_n = \theta^\pm \quad \text{for } \quad 0 < x_0 < p_+(r) \text{ or } p_-(r) < x_0 < \theta;
\]

\[
\lim_{n \to \infty} x_n = \pm \infty \quad \text{for } \quad x_0 > p_+(r) \text{ or } x_0 < p_-(r).
\]

![Figure 2.3](image)

*Figure 2.3*

**case 2: \( r < 0 \)**

For negative values of \( r \), \( F_r' (\theta) = r < 0 \) and destabilization occurs when \( r \) decreases through the value \( r = -1 \). Since \( F_r' (\theta) \) is negative, we have the problem of overshoot and in fact, values of \( F_r \) oscillate between positive and negative values. Thus, the model have a flip bifurcation at \( r = -1 \), but since again Allwright does not apply, there is no bifurcation to closed orbits of period 2. However as we shall see, we do have a Hopf-type phenomenon where closed orbits of period 2 occur for subcritical parameter values \(-1 < r < 0\) (see Fig.2.4).
Solving $F_r(F_r(x)) = F_r(r(x + x^3)) = r(r(x + x^3) + r^3(x + x^3)^3) = x$ for $x$ when $-1 < r < 0$, we get

$$r^2x + r^2x^3 + r^4x^5 + 3r^4x^7 + r^4x^9 = x$$

or

$$x(r^2 + (r^2 + r^4)x^2 + 3r^4x^4 + 3r^4x^6 + r^4x^8) = x.$$  

Let $H(r, x) = r^2 + (r^2 + r^4)x^2 + 3r^4x^4 + 3r^4x^6 + r^4x^8$, then we want $H(r, x) = 1$.

Now fix $r_0$ such that $-1 < r_0 < 0$, then $H(r_0, x) = r_0^2 < 1$. But

$$\lim_{x \to \pm \infty} H(r_0, x) = \infty$$

and since $H$ is an even function in $x$ strictly increasing for $x > 0$, there exist a unique $x_{r_0} > 0$ such that $H(r_0, x_{r_0}) = 1$.

![Figure 2.4 (where $-1 < r_0 < 0$)](image)

For this particular example we can find $x_{r_0}$ explicitly since

$$|F_r(x)| = |r||x + x^3|.$$  

We want $x_{r_0}$ such that

$$|F_{r_0}(x_{r_0})/x_{r_0}| = 1 = |r_0||1 + x_{r_0}^2|.$$  

This allows us to express $x_{r_0}$ explicitly in terms of $r_0$

$$1 + x_{r_0}^2 = 1/|r_0|.$$
or \[ x_{r_0} = \sqrt{1/|r_0|-1}. \]

Note: The symmetry in this problem is due to the fact that \( F_r(x) \) is an odd function in \( x \).

These orbits of period 2 have radii decreasing toward 0 as the critical value of the parameter \( r_c = -1 \) is approached. We can see similarities between this and the case of subcritical bifurcation in Hopf theory. This comparison is further investigated in the following general setting.

We shall attempt to show the analogy by constructing a cobweb model from a Hopf model and showing the correspondence between supercritical bifurcation in Hopf and the hypergraphic property in Allwright's theory; and between subcritical bifurcation in Hopf and the property of \( \text{Hypergraph}(f) < 0 \) in Allwright's theory.

Construction

We begin with a two dimensional one parameter dynamical system \( \dot{x}_t = F(x, \mu) \).

\( F(x, \mu) \in C^4(R^2 \times R) \) with critical point \( x = 0 \) and critical parameter value \( \mu_c = 0 \).

Suppose that the system experiences Hopf bifurcation as \( \mu \) increases past \( \mu_c \). We know that in a neighborhood of \((0, 0)\) there is for point \( x = (x, 0) \) on the \( x \)-axis, and a map \( Q(x) \) which gives the next intercept on the \( x \)-axis of the trajectory.
starting at \((x, \theta)\). Note that \(x\) and \(Q(x)\) have opposite sign except when \(x = 0\). The map \((Q\circ Q)(x, \mu)\) is then the usual Poincaré map of the Hopf model (see Fig.2.5).

Let us denote it by \(\pi(x, \mu) = (Q\circ Q)(x, \mu)\).

Thus we have

1) \(\pi(x) = Q(Q(x))\);

2) \(\pi'(x) = Q'(Q(x))Q'(x)\);

3) \(\pi''(x) = Q''(Q(x))(Q'(x))^2 + Q'(Q(x))Q''(x)\);

4) \(\pi'''(x) = Q'''(Q(x))(Q'(x))^3 + 3Q''(Q(x))Q'(x)Q''(x) + Q'''(x)Q'(Q(x))\).

Evaluating the above at the critical point \(x = 0\), we have

1) \(\pi(0) = Q(0) = 0\);

2) \(\pi'(0) = (Q'(0))^2\);

3) \(\pi''(0) = Q''(0)(Q'(0))^2 + Q'(0)Q''(0)\);

4) \(\pi'''(0) = Q'''(0)(Q'(0))^3 + 3Q''(0)Q'(0) + Q'''(0)Q'(0)\).
Hopf theory however gives us the following properties:

a) $\pi'(0) = 1$ and

b) $\pi''(0) = 0$.

Since $\pi'(0) = 1 = (Q'(0))^2$ and $Q'(0) < 0$ (since $x$ and $Q(x)$ have opposite sign) we deduce that $Q'(0) = -1$ and equation 3) reduces to

$$\pi'''(0) = -2Q''(0) - 3(Q'(0))^2$$

or

$$\pi'''(0) = -[3(Q''(0))^2 - 2Q'''(0)Q'(0)] = -\text{Hypergraph}(Q) = \text{Schwarzian}(Q) [12].$$

According to Hopf theory, we have the following two cases and their corresponding bifurcations (recall $V(x, \mu) = \pi(x, \mu) - x$) [22].

1) $V''''(0) = \pi''''(0) < 0$ implies supercritical bifurcation and have stable closed orbits occurring for $\mu > 0$ (locally).

2) $V''''(0) = \pi''''(0) > 0$ implies subcritical bifurcation and we have unstable closed orbits occurring for $\mu < 0$ (locally).

Note: Case 2) can be deduced from case 1) by reversing the flow and thus changing the Poincare map $\pi$ to its inverse $\pi^{-1}$.

We observe that as $\mu$ increases through the critical value $\mu_c$, $\pi'(0, \mu)$ increases through the value +1. This follows since $\pi'(0, \mu) = \partial V/\partial x(0, \mu) - 1$; so
\[ \frac{dP'(\theta, \mu)}{d\mu}\big|_{\mu=0} = \frac{\partial^2 V}{\partial \mu \partial x(\theta, \theta)} > 0 \text{ from proof in [22].} \]

But since \( Q'(\theta, \mu) < 0 \) near \( \mu = 0 \) and its square is \( \pi'(\theta, \mu) \), \( Q'(\theta, \mu) \) must decrease through the value \(-1\) as \( \mu \) increases through 0. \( d\pi'(\theta, \mu)/d\mu\big|_{\mu=0} = d/d\mu(Q'(\theta, \mu))^2\big|_{\mu=0} \)

\[ = 2Q'(\theta, \mu)dQ/d\mu(\theta, \mu)\big|_{\mu=0} > 0 \]

\[ \Rightarrow dQ/d\mu(\theta, \mu)\big|_{\mu=0} < 0. \]

The map \( x \to Q(x, \mu) \) thus induced from the above Hopf model is a cobweb model near \( (x, \mu) = (0, 0) \). It has a fixed point \( x = 0 \) for all values of \( \mu \). The model undergoes "flip bifurcation" as \( \mu \) increases through 0 since \( Q'(\theta, \mu) \) simultaneously decreases through \(-1\), as shown above. Furthermore, we have the following two cases corresponding to the two previously mentioned for the Hopf theory.

1) \( V''(\theta) < 0 \Rightarrow \text{Hypergraph}(Q) > 0 \), gives Allwright's regular bifurcation in which stable closed orbits of period 2 occur for \( \mu > 0 \).

2) \( V''(\theta) > 0 \Rightarrow \text{Hypergraph}(Q) < 0 \), gives unstable closed orbits of period 2 occurring for \( \mu < 0 \).

We have thus shown that we can recover the conclusions of Allwright's bifurcation theorem for cobweb models with flip bifurcation which can be induced by a Hopf model. The following definition, theorem and corollary summarizes the results of the above comparison.
Definition: The cobweb model \( x \rightarrow f(x, \mu) \) is said to be \textit{Hopf-induced} if there exist a two dimensional dynamical system \( x = F(x, \mu) \) such that the map \( Q(x) \) of the first return onto the \( x \)-axis of the flow, is the same as the cobweb model \( f(x) \), and such that \( F \) undergoes Hopf bifurcation. We then also say that \( F \) induces \( f \).

Theorem: Suppose the cobweb model \( x \rightarrow f(x) \) is Hopf-induced; and suppose the inducing system \( F \) undergoes supercritical Hopf bifurcation. Then \( f \) undergoes Allwright's regular bifurcation.

Corollary: If \( F \) undergoes subcritical Hopf bifurcation, then \( f^{-1} \) (which exists locally since \( f'(0) = -1 \)) undergoes Allwright's regular bifurcation.

In constructing parallels between cobweb models and Hopf models, we are restricted to looking at the case of flip bifurcation in cobweb models, i.e., for \( x \rightarrow f(x, \mu) \) we need \( f'(0, \mu) < 0 \). The reason for this is obvious since our construction would fail otherwise. Hopf bifurcation always involves closed orbits, thus we are restricted to comparing it to cobweb models with flip bifurcation. Note however that we have not shown that all cobweb models with flip bifurcation can be induced by Hopf models. In fact, this is probably not true. We will however show that cobweb models with flip bifurcation can be induced by dynamical systems exhibiting Hopf-like behaviour.

First consider the Cartesian model of the form
\[ x_t = L(x, y)x - Ay = f(x, y) \]
\[ y_t = L(x, y)y + Ax = g(x, y) \quad \text{where } A > 0. \tag{1} \]

Let \( W(r, \theta) = L(r \cos \theta, r \sin \theta) \), then
\[ f(r \cos \theta, r \sin \theta) = W(r, \theta) r \cos \theta - A r \sin \theta, \]
\[ g(r \cos \theta, r \sin \theta) = W(r, \theta) r \sin \theta + A r \cos \theta. \]

Converting to polar form, the model is then
\[ r_t = \cos \theta [W(r, \theta) r \cos \theta - A r \sin \theta] + \sin \theta [W(r, \theta) r \sin \theta + A r \cos \theta] \]
\[ \theta_t = -\sin \theta [W(r, \theta) r \cos \theta - A r \sin \theta]/r + \cos \theta [W(r, \theta) r \sin \theta + A r \cos \theta]/r \]
or \[ r_t = r W(r, \theta) \]
\[ \theta_t = A. \tag{1}' \]

The planar differential equation is \( dr/d\theta = r W(r, \theta)/A \).

For the Cartesian system (1) (assuming \( L \in C^1 \)), the Jacobian at \( x=\theta \) is
\[ DF(\theta) = \begin{bmatrix} L(\theta) & -A \\ A & L(\theta) \end{bmatrix}. \]
The eigenvalues are \( \lambda = L(\theta) \pm iA \).

Suppose that the system were dependent on a parameter \( \mu \), such that \( L = L(x, y, \mu) = L(x, \mu) \). Then \( d\text{Re}(\lambda)/d\mu = \text{Re}L(\theta, \mu)/\partial \mu \); so in order for the conditions of the Hopf bifurcation theorem to hold we need

i) \( L(x, \mu) \in C^q(\mathbb{R}^2; \mathbb{R}) \) and
Now to attack the problem itself, consider the parametrized cobweb model \( x \to P(x, \mu) \) with

a) \( P(0, \mu) = 0 \) for all \( \mu \) (i.e., \( x = 0 \) is a fixed point);

b) \( P \in C^2(\mathbb{R}^n, \mathbb{R}) \) (i.e., smoothness);

c) \( \partial P(0, 0)/\partial x = -1 \) and \( \partial^2 P(0, 0)/\partial x^2 > 0 \) (i.e., bifurcation at \( \mu = 0 \); asymptotic stability for \( \mu < 0 \); unstable for \( \mu > 0 \)).

We seek a Hopf-like model of the form (1) such that its first return map is \( P \).

We begin by constructing a polar model of the form (1)' whose first return map coincides with the cobweb model \( x \to P(x, \mu) \). We must then verify that the constructed model exhibits geometric behaviour similar to Hopf models.

Construction

In constructing the trajectories \( r_x(\theta) \) of the model we seek, we want \( r_x(\theta) \) to satisfy the following:

i) \( r_x(\theta) = x \);

ii) \( r_x(\pi) = P(x) = P(x, \mu) \) and

iii) \( \partial r_x(\theta)/\partial \theta = 0 \) for \( \theta \) near \( 0 \) or \( \pi \).

Condition iii) ensures \( C^1 \) continuity as we complete the first return and begin the second loop.
Let \( H(\theta) > 0, H(\theta) \in C_0^\infty(\mathbb{R}) \) with \( \text{supp}(H) \subset [1, 2] \) (for \( x < 0 \), let \( \text{supp}(H) \subset [4, 5] \)) and \( \int_0^\theta H(\theta) d\theta = 1 \). Let \( r_x(\theta) = x + (P(x) - x) \int_0^\theta H(\omega) d\omega \). Then \( r_x(\theta) \) obviously satisfies the conditions above.

Having constructed the trajectories, we try to recover the vector field of the model we seek. Note that \( \partial r_x(\theta)/\partial x = 1 - (P'(x) - 1) \int_0^\theta H(\omega) d\omega \). But \( \partial P/\partial x(\theta, \theta) = -1 \), so for \( x, \mu \) small, \( \partial r_x(\theta)/\partial x \) is near 1; in particular \( \partial r_x/\partial x(\theta) > 0 \). It follows that the map \((x, \theta) \rightarrow (r_x(\theta), \theta)\) is a diffeomorphism and its inverse defines \( x \) as a function of \( r \) and \( \theta \): \( x = x(r, \theta) \). This function gives the starting point of the trajectory through \((r, \theta)\).

From the definition of \( r_x(\theta) \), \( \partial r_x(\theta)/\partial \theta = (-P(x) - x)H(\theta) \)

\[ = (-P(x(r, \theta)) - x(r, \theta))H(\theta). \]

So the planar form of the dynamical system is

\[ \partial r/\partial \theta = (-P(x(r, \theta)) - x(r, \theta))H(\theta) \]

\[ = rW(r, \theta). \]

That is, \( W(r, \theta) = (-P(x(r, \theta)) - x(r, \theta))H(\theta)/r \).

By the methods used in the proof of Hopf's theorem [22], we can show that \( W \) has \( C^1 \) smoothness. From this we can recover \( C^1 \) smoothness for \( x_L \) and \( y_L \). The following theorem summarizes the result we have just proved.
Theorem: Suppose the cobweb model $x \rightarrow f(x, \mu)$ undergoes Allwright bifurcation. Then there is a two dimensional dynamical system $x_t = F(x, \mu)$ with $F \in C^1$ such that

i) $F$ induces $f$ and

ii) the system $x_t = F(x, \mu)$ bifurcates with eigenvalues $\pm i\lambda$ and there are closed orbits as in Hopf.

As in the case of dynamical systems, the effect of a cascading bifurcation of the periodic points may occur through the doubling of periods [7], [24].
3.1 Introduction to Differential Delay Equations

We have so far investigated the behaviour of critical points in systems of ordinary differential equations (or dynamical systems) and of fixed points in cobweb models. We have been able to draw an interesting—though not thoroughly complete—parallel between the bifurcation theories for the two models mentioned, as the value of a parameter passes some critical value. We would now like to study a model of yet a different kind and consider possible parallels in its stability behaviour with those of the above mentioned models.

A particular solution of a first order ODE $x_t = f(t, x)$, is dependent on a given initial value $x(0) = x_0$; similarly the behaviour of a cobweb model $x \rightarrow g(x)$ depends on an initial value $x_0$. We will now proceed to study differential equations in which the past history of a function must be known in order to determine its behaviour in the future (i.e., there is a delayed effect). Such equations, in which the rate of change of a variable depends on its past history, are called differential delay equations (DDE).

Here, we will be concerned with two different kinds of delays and their effects on the stability of the system, i.e., on the equilibrium structures of the system. We will again restrict ourselves to the case of autonomous systems and work only
in one dimension in our examples.

We will begin by looking at the case of a discrete or sharp time delay of \( T < \infty \). Our DDE then takes the form

\[ x_t = f(x(t), x(t - T)) \text{ for } t \geq 0, \]

with initialization given by \( x(t) = p(t) \) for \(-T \leq t \leq 0\).

The second case we will consider is that of a distributed delay in which the importance or weight of the past behaviour of the variable \( x \) at any particular time is determined by a delay kernel \( \mu(ds) \), where \( \mu \) is a finite Borel measure on \([0, \infty)\) with \( \int_0^\infty \mu(ds) = 0 \) and \( s > 0 \). Our delay term is then in integral form

\[ \int_0^T x(t - s)\mu(ds) \text{ for } t > 0, T \geq 0. \]

The first case mentioned can be thought of as a restriction of the second by letting \( \mu(ds) = \delta(s - T)ds \).

### 3.2 Existence and Uniqueness Theory

Before we begin our analysis of the two cases, we would like to present a sketch of the existence and uniqueness theory for solutions to the above mentioned problems. The general form in which the problem will be written will cover both the cases before mentioned.

**Theorem:** Consider \( f \in C^1(\mathbb{R}, \mathbb{R}) \) with \( K = \sup_x |f'(x)| < \infty \); \( \mu \) a finite Borel mea-
sure on \([0, \infty)\) with total variation \(M < \infty; p(t) \in C \cap L^\infty((-\infty, 0]).\) Then the DDE

\[
x(t) = f(\int_0^\infty x(t - s)\mu(ds)), \quad t > 0
\]

\[
x(t) = p(t), \quad t \leq 0
\]

(1)

\[
x(t) \in C(R, R)
\]

has a unique solution.

Sketch of proof:

To solve the initial value problem above, we construct a sequence of approximating solutions \(x_n(t)\) as in the case of ODE.

For \(t \leq 0,\) \(x_n(t) = p(t).\)

For \(t > 0,\) let \(x_0(t) = p(0)\)

\[
x_1(t) = p(0) + \int_0^t f(\int_0^\infty x_0(v - s)\mu(ds))dv
\]

\[
x_2(t) = p(0) + \int_0^t f(\int_0^\infty x_1(v - s)\mu(ds))dv
\]

\[
\vdots
\]

\[
x_{n+1}(t) = p(0) + \int_0^t f(\int_0^\infty x_n(v - s)\mu(ds))dv.
\]

In this way we have a recursion formula for the sequence \(\{x_n\}^\infty_1\) and

\[
|x_1(t) - x_0(t)| \leq \int_0^t f(\int_0^\infty x_0(v - s)\mu(ds))|dv
\]

but \(x_0(v - s) = p(\theta) =\) constant and so \(\int_0^\infty x_0(v - s)\mu(ds) = \mu([\theta, \infty))p(\theta) =\) constant.

Thus \(|x_1(t) - x_0(t)| \leq Lt\) where \(L =\) constant > 0 and

\[
|x_2(t) - x_1(t)| \leq \int_0^t f(|\int_0^\infty x_1(v - s)\mu(ds)| - f(\int_0^\infty x_0(v - s)\mu(ds))|dv
\]
Similarly,

\[ |x_3(t) - x_2(t)| \leq K \int_0^t L(v - s) \mu(ds)dv \]
\[ \leq K \int_0^t KLM(v - s)^2/2! \mu(ds)dv \]
\[ \leq K \int_0^t KLM(v/2)!Mdv \]
\[ \leq L(KM)^2 t^3/3! \]

and \[ |x_{n+1}(t) - x_n(t)| \leq L(KM)^n t^{n+1}/(n + 1)! \].

So \( \sum_{n=0}^{\infty} |x_{n+1} - x_n| \leq L(e^{KMt} - 1)/KM \), and the sequence \( \{x_n\}_{n=0}^{\infty} \) converges to \( x(t) \in C(R, R) \). Moreover, we get that \( x(t) = p(0) + \int_0^t \int_0^\infty x(v - s)\mu(ds)dv \) by taking the limit as \( n \to \infty \) of the recursion formula for \( x_{n+1}(t) \).

Differentiating the above integral formula for \( x(t) \), we see that it does indeed satisfy the initial value problem (1). Moreover, we have an estimate for \( |x(t)| \),

\[ |x(t)| \leq x_0 + \sum_{n=0}^{\infty} |x_{n+1} - x_n| \leq p(0) + L(e^{KMt} - 1)/KM \]

which gives an exponential bound on the growth rate of \( |x(t)| \).

The uniqueness of the solution is proved as in the case of the ODE, by using the integral equation for two solutions \( x \) and \( x^* \) and showing they must be the same.
3.3 Stability Theory

Before we actually go on to study the stability properties of the solutions, it is appropriate to give a definition of exactly what we mean by a stable or asymptotically stable equilibrium solution of a DDE.

**Definition:** For the initial value problem (1) with equilibrium solution \( x = x_C \), without loss of generality let \( \mu \) be normalized so that \( \int_0^\infty \mu(ds) = 1 \). Then \( f(x_C^*, \mu) = f(x_C) = 0 \), and given any \( \varepsilon > 0 \)

i) \( x = x_C \) is a **stable** equilibrium solution if there exists \( \delta > 0 \) such that

if \( \sup |x(t) - x_C| < \delta \) for \( t < 0 \), then \( |x(t) - x_C| < \varepsilon \) for \( t > 0 \);

ii) \( x = x_C \) is an **asymptotically stable** equilibrium solution if \( x = x_C \) is stable and \( \lim_{t \to +\infty} |x(t) - x_C| = 0 \).

3.4 Case of Discrete Time Lag

The problem of differential delay equations with discrete time lags is one of particular interest to those in electrical engineering and control theory. Consider the general problem, \( x_t = f(x(t), x(t - T)) \) for \( t > 0, 0 < T < \infty \) with initialization \( x(t) = p(t) \) for \( -T \leq t \leq 0 \); where \( f, p \) and \( x \) satisfy the conditions in our existence
and uniqueness theorem. A continuous solution \( x(t) \) can be found for all \( t > 0 \) by piecing together the solutions for \( 0 < t < T, \ T < t < 2T, \) and etc.

Looking at the general linear equation of the form

\[
x_t = ax(t) - bx(t - T). \tag{2}
\]

Stability analysis of the equilibrium solution \( x(t) = 0 \) gives the following results by the method of Laplace transforms [2].

**Theorem:** The equilibrium solution \( x(t) = 0 \) of the linear DDE (2) is stable/asymptotically stable \( \iff \) all roots of the Laplace multiplier \( G(s) = s - a + be^{-sT} \) are in the left half plane/the interior of the left half plane (i.e., \( \text{Re}(s) < 0/\text{Re}(s) < 0 \)).

For the case where \( 0 < a < b, \) this gives a condition on the time lag \( T. \) So we have asymptotic stability of \( x = 0 \) \( \iff T < \arccos(a/b) / \sqrt{b^2 - a^2} = T_C. \) We will often refer to \( T_C \) as the critical value of \( T \) (see Figs. 3.1a and 3.1b).
Figure 3.1a: Here we have set $a = 0.5$, $b = 0.8$ and $T = 1$. Thus $T < \arccos(a/b)/$
Without the delay term, the ODE $x_t = (a - b)x(t)$ has an asymptotically stable equilibrium point at $x = 0$ as long as $a < b$. Thus we note that in this case time delays of $T > T_c$ have a destabilizing effect on the system.

Nonlinear equations are linearized using the method of variational equations. Thus local stability analysis can be carried out for both linear and nonlinear discrete delay equations using the method of Laplace transforms.

**Example 3.1:** Consider the well known logistic model for population dynamics with a discrete time lag $T$ introduced [20], [24], [25].

$$x_t = r x(1 - x(t - T)/K).$$

(3)

$K = \text{constant}$, is the carrying capacity and the parameter $r$ is a measure of the net reproductive rate. Equation (3) has an equilibrium solution at $x = 0$ which is unstable and at $x = K$ which is asymptotically stable in the instantaneous model (without lag).
Figure 3.2a

Here $K = 5$, $r = 1$ and $T = 1$. Thus $T < \pi/2$ and $x = K$ is asymptotically stable. We have used initialization $x(t) = \cos(\pi t/2)$ for $t < 0$.

Figure 3.2b

Here again $K = 5$ and $T = 1$, but $r = 2$. Thus $T > \pi/4$ and $x = K$ is unstable. We have used initialization $x(t) = 4.8$ for $t < 0$ and started plotting at $t = 1$. 
Let $X = K - x$ and linearize to get the linear DDE $X_t = -rX(t - T)$ with equilibrium solution $X = 0$ corresponding to $x = K$. The Laplace multiplier is then $G(s) = s + re^{-sT}$. Roots cross the imaginary axis from the left half plane to the right when $s = \pm i\omega_0$, or when $\omega_0 = r \Rightarrow T = \pi/2r = T_C$. Thus $x = K$ is asymptotically stable for $0 < T < \pi/2r$ (see Fig. 3.2a) and unstable for $T > \pi/2r$ (see Fig. 3.2b).

The discrete time lag of a DDE can be likened to the discrete time change of $\Delta t = 1$ often associated with cobweb models. Consider the linear DDE with discrete time lag $T = 1, x_t = rx(t - 1)$ and replace $x$ by the difference ratio $[x(t) - x(t - 1)]/[t - (t - 1)] = x(t) - x(t - 1)$. Then we have $x(t) = (r + 1)x(t - 1)$ or the cobweb model $x \rightarrow (r + 1)x$. The equilibrium point $x = 0$ is asymptotically stable in the DDE for $T = 1 < -\pi/2r$ or for $-\pi/2 < r < 0$; it is asymptotically stable in the cobweb model for $-2 < r < 0$.

The reason for the lower bounds for $r$ in both cases (as opposed to the condition $r < 0$ for the ODE $x_t = rx(t)$) is due to the problem of overshoot. We have discussed this in the case of the cobweb models, for the DDE the problem arises from the complex roots $s = \alpha \pm i\beta$ of the Laplace multiplier $G(s) = s - re^{-sT}$. The DDE has solutions of the form $x(t) = e^{at} = e^{\alpha t}(\cos\beta t \pm i\sin\beta t)$ giving us two real solutions $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$. These indeed oscillate about the equilibrium
point \( x = 0 \), overshooting it as they approach it for \( \alpha < 0 \) as \( t \to \infty \).

In the preceding comparison, we restricted ourselves to the case of discrete delay \( T = 1 \), but in fact the comparison can be extended for general \( T < \infty \) by the following substitution. For \( x_t = r x(t - T) \), let \( \tau = t/T \) and \( x^\star(\tau) = x(T\tau) \), then

\[
\frac{dx}{d\tau} = T x = T r x(T\tau - T) = T r x(T(\tau - 1)) = T r x^\star(\tau - 1); \text{ but } \frac{dx^\star}{d\tau} = \frac{dx}{d\tau} T, \text{ so } \frac{dx^\star}{d\tau} = T^2 r x^\star(\tau - 1) \text{ is now a DDE with discrete delay of one unit time. The above comparison is then valid.}
\]

### 3.5 Case of Distributed Time Lag

This is the case of concern for the study of elastic and magnetic properties of materials, and often it gives more realistic models for ecological and biological studies [211, 251]. Here we consider the problem where the behaviour of a variable at a particular time in the future will depend not merely on its value at one particular time in the past, but rather on an average over past times. This case allows us to work with infinite time lag \( T = \infty \). We shall see that it is precisely in this special case that we are able to analyse our DDE by way of a trick reduction to a model we can analyse by well developed methods already discussed.

In this case our delay equation is then an integro-differential equation of the following form

\[
x_t = f(\int_0^\infty x(t - s)\mu(ds)) \quad t > 0.
\]

(4)

Again the distribution of the delay is determined by the delay kernel \( \mu(ds) \); we
shall restrict ourselves to the case where \( u(ds) = k(s)ds \) is a continuous density
with \( k(s) \in C([0, \infty)) \) and \( \int_0^\infty |k(s)|ds < \infty \). (Note that with this restriction we can
no longer include the case of the discrete delay in our problem.) The new form is

\[
x_t = f(x(t), \int_0^\infty x(t - s)k(s)ds) \quad t > 0.
\]

In studying the stability of an equilibrium solution of (4), we can deal with
the linear case \( f(x, y) = ax + by \), by the method of Laplace transforms. This is
strictly for the linear case however, and in order to study the problem form non-
linear \( f \) we will first restrict our delay kernel \( k(s) \) to a special class of continu-
ous functions which will allow us to reduce our DDE to an ordinary dynamical
system. I shall refer to this class of distributed delay integro-differential equa-
tions as \( \delta \)-convertible DDE (\( \delta \) for dynamical systems).

3.6 \( \delta \)-Convertible DDE

The study of reducing certain types of DDE to equivalent dynamical systems
dates back to the early 1960's and was emphasized by Vogel [30]. Once the equi-
valence of the DDE to the dynamical system has been established, stability ana-
lysis follows by the methods for the latter.

The class to which we will restrict \( k(s) \), may seem very limiting at first, but
it has been shown that even working within this restricted class many DDE pro-
blems can be reasonably approximated.

It can be shown that if \( k(s) \in \{ f \in C([0, \infty)) : f(s) = \sum_{i=1}^N p_i(s)e^{-\alpha_i s}; \alpha_i > 0 \} \) for
\begin{align*}
i = 1, 2, 3, \ldots, N; \ p_i(s) &= \text{polynomial in } s \text{ for all } i, \text{ then (4) with initialization } x(t) = p(t) \text{ continuous for } t \leq 0, \text{ has a solution. Furthermore, if we let } k(s) \\
&\text{have the special form (without loss of generality) } k(s) = \sum_{i=1}^{M} \sum_{j=0}^{N_i} a_{ij} (s^{j}/j!) e^{-\alpha_i s} \\
&\text{and let } w_{ij} = x * k_{ij} \text{ where } k_{ij} = (s^{j}/j!) e^{-\alpha_i s}, \text{ then } w(t) \text{ is a vector with components } w_{ij}. \text{ There is a } p \times p+1 \ (p = \max(j+1)) \text{ constant matrix } E \text{ such that} \\
x(t) \text{ is the solution of (4) with } x(t) = p(t) \text{ on } (-\infty, 0] \leftrightarrow (x(t), w(t)) \text{ is a solution of the ordinary dynamical system} \\
x_t = f(x, \sum_{i,j} a_{ij} w_{ij}(t)) \\
w_t = E [ x(t) ] \\
\{ w(t) \} \text{ with suitable initial values } w(0) = w_0. \quad (5) \\
\end{align*}

Note that the first equation is just the original DDE and the added system is a linear dynamical system. The entries of \( E \) are given by the coefficients of the equations

\[
dw_{ij}(t)/dt = x(t)\delta(j, 0) + w_{ij-1}(t)(1 - \delta(j, 0)) - \alpha_i w_{ij}(t). 
\]

Thus we have converted our \( \delta \)-convertible DDE to its equivalent dynamical system. We have claimed that we can now proceed to use the methods of dynamical systems to study the stability of solutions. Before we study some examples of this application, we will first show that stability and asymptotic stability of a
solution in a DDE is equivalent to that in the corresponding dynamical system.

Without loss of generality, we will work on the equilibrium solution $x = \theta$ and normalize $k(s)$ so $\int_0^\infty k(s)ds = 1$. It is obvious that if $x = (x, w) = \theta$ is a stable or asymptotically stable equilibrium solution of the dynamical system, then $x = \theta$ is respectively so for the DDE. The converse can be shown to hold also.

Let $x = \theta$ be a stable equilibrium solution for the DDE, then by the definition before stated, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|x(t)| < \varepsilon$ for $t > 0$ if $\sup_{t \leq 0} |x(t)| < \delta$. To show that $x = \theta$ is then stable in (5), we must show that given any $\varepsilon' > 0$ there exists for each $w_{ij}$, $\delta' > 0$ such that $|w_{ij}(t)| < \varepsilon'$ for $t > 0$ if $|w_{ij}(0)| < \delta'$. This is obvious since

$$|w_{ij}(t)| = |x - k_{ij}| \leq \int_0^\infty |x(t - s)||k_{ij}(s)|ds \leq \varepsilon < \varepsilon_1$$

and

$$|w_{ij}(0)| \leq \int_0^\infty |x(-s)||k_{ij}(s)|ds < \delta.$$  

Thus given $\varepsilon'$, pick $\delta' < \delta$ and pick $\delta$ so that $\varepsilon < \varepsilon_1$. Similarly we can see that if $x = \theta$ is asymptotically stable for the DDE, then $x = \theta$ must be asymptotically stable in (5) since $\lim_{t \to \infty} |x(t)| = 0$ implies $\lim_{t \to \infty} |w_{ij}(t)| = 0$.

$$\lim_{t \to \infty} |w_{ij}(t)| \leq \lim_{t \to \infty} \int_t^\infty |x(t - s)||k_{ij}(s)|ds + \int_0^t |x(t - s)||k_{ij}(s)|ds = 0.$$  

since $k_{ij}(s)$ is continuous and thus bounded in the first integral, while $\int_0^\infty |k_{ij}(s)|ds \to 0$ as $t \to T < \infty$ for some $T$. 


To study the effect on stability due to delays in $\delta$-convertible DDE, we will first look at some simple examples.

**Example 3.2a:** Consider the integro-differential equation

$$x_t = f\left(\int_0^\infty x(t - s)k(s)ds\right)$$  \hspace{1cm} (6)

with $k(s) = ae^{-\alpha s}$; $a, \alpha > 0$ (i.e., $p = 1$). Then by our conversion method, (6) is equivalent to the dynamical system

$$x_t = f(aw)$$  \hspace{1cm} (6a)

$$w_t = x - \alpha w$$

For this simple case we can get global stability analysis by using a Liapounov function. Let $V(x, w) = -\int_0^w f(au)du + 1/2(x - \alpha w)^2$. It is easily verified that $V(x, w)$ is a suitable Liapounov function for (6a) and that $(0, 0)$ is an asymptotically stable equilibrium point. The basin of this asymptotically stable point is the full $x,w$-plane. This delay kernel then, is not destabilizing for any function $f \in C^1(R, R)$ in this one dimensional integro-differential equation (see Fig.3.3). Note again that this gives global results whereas the Laplace transform and variational equation methods give only local results.
Figure 3.3a: This shows the asymptotically stable behaviour of the $\delta$-convertible DDE (6) with $f(x) = x\cos x/\sqrt{1 + x^2}$ and $k(s) = 2 e^{-s}$—i.e., $a = 2$ and $\alpha = 1$. 
Figure 3.3b: Here we let $a = 1$ and $\alpha = 1$ with $f(x)$ as in Fig. 3.3a. Note that $x = 0$ is again asymptotically stable.

**Example 3.2b:** If we increase in the delay kernel $k(s)$, the order of the polynomial multiple by 1, we then have $k(s) = (a_0 + a_1s)e^{-\alpha s}$ ($a_0 > 0$, $p = 2$). Our matrix $E$ is then given by

$$E = \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \end{bmatrix}$$

and the equivalent dynamical system is

$$x_t = f(a_0w_0 + a_1w_1)$$

$$w_{0t} = x - \alpha w_0$$

$$w_{1t} = w_0 - \alpha w_1$$

Using a linear approximation for $f$ near 0 we can determine the eigenvalues of the corresponding matrix and thus study the conditions needed for stability. Suppose $f'(0) = -1$, then $x_t = -a_0w_0 - a_1w_1$ would give a linearization of $f$. Thus our system becomes

$$X_t = \begin{bmatrix} 0 & -a_0 & -a_1 \\ 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \end{bmatrix}X$$

and the eigenvalues satisfy $\lambda^3 + 2\alpha \lambda^2 + (a_0^2 + a_0)\lambda + a_0\alpha + a_1 = 0$. For $a_1 = 0$, we have the case of example 3.2a and all three roots have negative real part.
giving asymptotic stability. As $a_1$ increases from zero, stability persists until a pure imaginary root occurs, i.e., when $\lambda = \pm i \beta$ (see Fig. 3.4). Calculations show that this occurs when $a_1 = \alpha(2\alpha^2 + a_0)$; so $X = 0$ is asymptotically stable (locally) for $0 < a_1 < \alpha(2\alpha^2 + a_0)$ (see Fig. 3.4a). This condition can also be determined by the Routh-Hurwitz criterion. In other words, for $a_1$ large enough, destabilization occurs (see Fig. 3.4b).

\begin{figure}[h]
\centering
\includegraphics{figure3.4a.png}
\caption{Figure 3.4a}
\end{figure}

This graph shows the asymptotically stable behaviour of the 5=convertible DDE (6) where $f(x) = x\cos x/\sqrt{1 + x^2}$, $k(s) = (1 + 2s)e^{-s}$—i.e., $a_0 = 1$, $a_1 = 2$ and $\alpha = 1$ thus $a_1 < \alpha(2\alpha^2 + a_0) = 3$. 
Figure 3.4b

This graph shows the unstable behaviour of the δ-convertible DDE (6) where
\[ f(x) = x \cos x / \sqrt{1 + x^2}, \quad k(s) = (1 + 4s)e^{-s} \text{--i.e., } a_0 = 1, a_1 = 4 \text{ and } \alpha = 1 \text{ thus } a_1 > \alpha(2\alpha^2 + a_0) = 3. \]

**Example 3.2c:** Suppose there are two exponential terms in the delay kernel, that is, \( k(s) = a(e^{-\alpha_1 s} - e^{-\alpha_2 s}) \). If we assume \( 0 < \alpha_1 < \alpha_2 \) and \( f'(0) = -1 \), the linearized equivalent dynamical system is

\[
\begin{bmatrix}
    1 & -\alpha_1 & 0 \\
    1 & 0 & -\alpha_2 \\
    0 & a & a
\end{bmatrix}
\]

and eigenvalues satisfy \( \lambda^3 + \lambda^2(\alpha_1 + \alpha_2) + \lambda\alpha_1\alpha_2 + a(\alpha_2 - \alpha_1) = 0. \) The
Routh-Hurwitz criterion gives that the roots lie in the left half plane for
\[ 0 < a < \alpha_1 \alpha_2 (\alpha_2 + \alpha_1) / (\alpha_2 - \alpha_1). \]

So we see that once the delay kernel becomes more complex than a constant
multiplied by an exponential, destabilization becomes a possibility. In the
following example, we will use MacDonald's (21) "linear chain trick" to reduce to
an equivalent dynamical system. This method allows us, for a special form of
\( k(s) \), to write a general formula for the interval of asymptotic stability and re­
late it back to the concept of the size of the time lag by looking at the average
time lag \( T = \int_0^\infty k(s)ds \) (\( k(s) \) normalized).

**Example 3.3:** MacDonald considers the delay kernel (or what he refers to as the
"memory function") \( k(s) = a^{p+1} s^p e^{-as} / p! \) \( (a > 0 \) and \( p > 0 \) an integer) for the
logistic equation with distributed lag
\[
\dot{x}_t = r x (1 - \frac{1}{K} \{ \int_0^\infty x(t - s)k(s)ds \}).
\]

The linearized equivalent dynamical system about \( x_c = K \) has eigenvalues which
satisfy
\[
(a + \lambda)^{p+1} \lambda + ra^{p+1} = 0. \quad (7)
\]

**case 1:** For \( p = 0 \) asymptotic stability persists for all values of \( a > 0 \), as we
have already seen to be true in example 3.2a.

**case 2:** For \( p \geq 1 \) destabilization occurs for sufficiently small \( a \).

2i) For \( p = 1 \) (7) becomes \( \lambda^3 + 2a\lambda^2 + a^2 \lambda + a^2 r = 0 \), and the Routh-
Hurwitz criterion gives that \(2a^3 - ra^2 > 0\) or \(a > r/2\) for asymptotic stability. This corresponds to the average lag time \(T = \int a^2s^2e^{-as}ds = 2/a < 4/r\). So destabilization occurs for \(T > 4/r\).

2ii) For \(p = 2\), we have critical average lag value \(T = 3/a = 8/3r\) which is a shorter time lag than in 2i).

2iii) As \(p\) increases past 2, we note that the pure imaginary roots occur when \(a = r(\cos(\pi/2(p+1)))^{p+2}/\sin(\pi/2(p+1))\). Then if we consider \(T\) a constant, the limiting case (as \(p \to +\infty\)) is then that for a discrete time lag of \(T\) (as discussed earlier in example 3.1); that is, asymptotic stability persists for \(0 \leq T < \pi/2r\).

![Graph](image)

**Figure 3.5**

The graphs above show the distribution of the delay kernel \(k(s) = a^{p+1}s^pe^{-as}/p!\) for the cases \(p = 1, p = 2, p = 3\) and \(p = 4\) where \(a = 1/2, a = 9/8, a \approx 1.7589\) and \(a \approx 2.3947\) for each of the \(p\)'s respectively. This compares the shapes of the delay kernel as \(p\) increases and for critical values of \(a\). Note how the distribution becomes more concentrated as \(p\) increases from 1 to 4.
We note that for fixed parameter $r$, the critical value of the average lag $T$ decreases to $\pi/2r$ as $p$ increases to $\infty$. This shows that this particular case of distributed delay is less destabilizing than the discrete delay case and that as the order $p$ of the delay kernel increases, so does the instability of the equilibrium solution (see Fig. 3.5).

This result can be generalized to the case of any $\delta$-convertible DDE. By taking linear combinations of these special memory functions due to MacDonald, we are able to get any delay kernel which gives a $\delta$-convertible DDE. Thus it seems that the closer the shape of the distribution curve of the delay kernel is to a spike—as in the discrete delay—so the instability of the equilibrium solution seems to worsen (see Fig. 3.5).

Looking back at example 3.2b and 3.2c, we see that the concept of an average time lag can be applied. Normalizing the $k(s)$ we get that

$$T = \left\{ \int_0^\infty s k(s) ds \right\} / \left\{ \int_0^\infty k(s) ds \right\}.$$

In example 3.2b, we have

$$T = (a_0/\alpha^2 + 2a_1/\alpha^3)/(a_0/\alpha + \alpha_1/\alpha^2) = (1/\alpha)(a_0\alpha + 2a_1)/(a_0\alpha + a_1).$$

Since destabilization occurs when $a_1 = \alpha(2\alpha^2 + a_0)$, stability persists for

$$T < (1/2\alpha)(3a_0 + 4\alpha^2)/(a_0 + \alpha^2).$$

Note that $1/2\alpha$ is small when $\alpha$ is large and large when $\alpha$ is small, while $(3a_0 + 4\alpha^2)/(a_0 + \alpha^2)$ remains between the values 3 and 4. If we fix $\alpha$, say $\alpha = \ldots$
1, then \((3a_0 + 4\alpha^2)/(a_0 + \alpha^2)\) decreases from 4 to 3 as \(a_0\) increases from 0 to \(\infty\).

Studying the shape of \(k(s)/[\int k(s)ds]\) for varying values of \(a_0\) with \(\alpha\) fixed (see Fig.3.6a) and for varying values of \(\alpha\) with \(a_0\) fixed (see Fig.3.6b), we see that the above generalization about the shape of the delay kernel function is indeed supported. Note that \(\alpha_1\) is here still a free parameter such that when chosen so that \(0 < \alpha_1 < \alpha(2\alpha^2 + a_0)\), stability persists. That is, the actual size of \(T\) still varies according to \(\alpha_1\).

The graphs above show the distribution of the normalized delay kernel \(k(s) = (a_0 + a_1 s)e^{-\alpha s}/(a_0/\alpha + a_1/\alpha^2)\) for \(\alpha = 1\) and where \(a_0 = 0, a_0 = 1, a_0 = 10\) and \(a_0 = 1000\). The corresponding critical values of \(\alpha_1\) are used to compare the graphs of the critical destabilizing case. As \(a_0\) increases for fixed \(\alpha = 1\) we see that the distribution of the delay kernel does indeed become more concentrated and stability worsens. As \(a_0\) increases from 0 to 1000, \(T_C\) decreases from 2.0 to 1.5.
The graphs show the distribution of the delay kernel $k(s)$ as described in Fig. 3.6a but this time we fix $a_0 = 1$ and graph for $\alpha = 1, \alpha = 2, \alpha = 3$ and $\alpha = 4$. Again the corresponding critical values of $a_1$ are used so we may compare the critical destabilizing case. Note that as $\alpha$ increases from 1 to 4 for fixed $a_0 = 1$ the graph becomes sharp much more rapidly than in the case of Fig.3.6a. In fact, as $\alpha$ increases from 1 to 4, $T_C$ decreases from 1.75 to 0.5.

In example 3.2, we want $\int_0^a k(s)ds = a/\alpha_1 - a/\alpha_2 = 1 \Rightarrow a = \alpha_1\alpha_2/(\alpha_2 - \alpha_1).

So $T = a/\alpha_1^2 - a/\alpha_2^2 = [\alpha_1\alpha_2/(\alpha_2 - \alpha_1)][1/\alpha_1^2 - 1/\alpha_2^2]$ or $T = (\alpha_2 + \alpha_1)/\alpha_1\alpha_2$.

For asymptotic stability, we want

\[ a = \alpha_1\alpha_2/(\alpha_2 - \alpha_1) < \alpha_1\alpha_2(\alpha_2 + \alpha_1)/(\alpha_2 - \alpha_1) \]

or

\[ 1 < (\alpha_2 + \alpha_1). \]

Thus the critical value at which destabilization occurs when $\alpha_2 + \alpha_1 = 1$ or

\[ T = 1/\alpha_1\alpha_2. \]

Since $0 < \alpha_1 < \alpha_2$, $1/2 < \alpha_2 < 1$ and $0 < \alpha_1 < 1/2$ for asymptotic stability.

For such restriction on the values of the $\alpha$'s, $T = 1/\alpha_1\alpha_2$ is maximum when $\alpha_2 = 1$ and $\alpha_1 = 0$; $T = 1/\alpha_1\alpha_2$ is minimum when $\alpha_2 = \alpha_1 = 1/2$. Thus stability im-
proves as \( \alpha_1 \) decreases from \( 1/2 \) to 0 and correspondingly as \( \alpha_2 \) increases from \( 1/2 \) to 1. The following figure, Fig.3.7, shows that again the conclusion we stated earlier is supported.

\[
\begin{align*}
\text{Figure 3.7} \\
\end{align*}
\]

Considering the critical case when \( \alpha_1 + \alpha_2 = 1 \), the graphs above show the distribution of the normalized delay kernel \( k(s) = a(e^{-\alpha_1 s} - e^{-\alpha_2 s}) \) where \( a = \frac{\alpha_1 \alpha_2}{(\alpha_2 - \alpha_1)} \). We let \( \alpha_1 \) increase from 0.1 to 0.48 while \( \alpha_2 \) correspondingly decreases from 0.9 to 0.52. The stability of the system worsens since \( T_c \) accordingly decreases from 11.11 to 4.01.

So far we have only dealt with single delay terms in our equations, but multiple delay equations and equations in which both discrete and distributed delay terms occur, have also been studied [2], [9]. Also, although we have only worked with scalar DDE, the methods extend to systems of DDE and the results are similar except in the case of the most simple delay kernel \( k(s) = ae^{-\alpha s} \). Stable equilibrium solutions do not necessarily remain stable for all \( a \) as in the case of a single scalar equation— an example of this is presented in [20]. Furthermore,
studies have shown that not all delays have destabilizing effects. We have already seen this to be true in examples 3.2a and 3.3(case 1); other examples are studied in [3], [6].
CHAPTER IV DISCUSSION

Extensive material has been written about dynamical systems and much methodology has been developed for them. Through the parallels drawn in this thesis, we see that insight from dynamical systems can be used to better understand the behaviour of cobweb models as well as differential delay systems.

The similarities in conditions for stability of equilibrium solutions of dynamical systems and DDE as compared to those for fixed points in cobweb models suggest similarities in more general stability behaviour. In fact we have seen this to be the case in at least two ways.

When comparing a cobweb model to a one dimensional dynamical system, we run into a major geometric problem, that of the overshooting of a fixed point in cobweb models which cannot be reproduced in the dynamical system. Hence we move on to compare it to a two dimensional dynamical system.

In particular, we are able to produce corresponding models which actually reproduce special bifurcation behaviour. The closed orbits in dynamical systems are successfully compared with the points of period two in cobweb models. This allows us to compare Hopf bifurcation in dynamical systems to Allwright's regular bifurcation in cobweb models. Through formal construction procedures, by way of first return maps, we are able to construct a cobweb model experiencing
Allwright's regular bifurcation given a dynamical system experiencing Hopf's bifurcation. The correspondence is a very convenient one in which supercritical bifurcation is translated to the property hypergraph $> 0$ and subcritical bifurcation to hypergraph $< 0$.

Conversely, when given a cobweb model with flip bifurcation about a fixed point which experiences either hypergraph $> 0$ or hypergraph $< 0$, we are able to construct a parallel two-dimensional dynamical system which experiences geometric bifurcation behaviour like that described in Hopf's theorem.

All these parallels are of course only valid locally. For any two distinct fixed points of a cobweb model, there will most likely be two distinct dynamical systems to parallel the bifurcation behaviour near the two points. Conversely, for two distinct equilibrium solutions of a dynamical system, each experiencing Hopf bifurcation, there will most likely be two distinct cobweb models to parallel the bifurcation behaviour near the two points. In this way the parallels are limited to the study of local behaviour about equilibrium or fixed points.

This comparison though thus limited, can be very useful in creating dynamical models from data collected at unit time intervals. For example, suppose the population of a certain species were counted every year for 20 years and a cobweb model is found which describes the population fluctuation from year to year. If in addition this data seems to indicate the existence of an equilibrium popu-
lation level nearby, then a corresponding dynamical model in one or two dimensions—depending on the behaviour of the cobweb model—can be found which may indeed model the behaviour of the population at intermediate times in those 20 years. Again, once a dynamical model is found all the methods developed for such models are at disposal for further study.

Truly, the above process may produce a dynamical model which does not at all model the behaviour of the population at any other times than those specified by the original cobweb model. However, when it does, the simple process of constructing a parallel dynamical model can prove quite rewarding.

Although we have dealt only with the case in which one single bifurcation occurs, as we have stated, further bifurcations do occur and such behaviour has been studied [28]. The paralleling of singular bifurcation behaviour may indeed carry over to further bifurcations the phenomena of successive bifurcation in dynamical systems may indeed be reproduced in cobweb models which experience cascading bifurcation.

In regard to differential delay systems, we have seen that they must be considered in two separate cases. The case of the discrete lag of time $0 < T < \infty$ in one dimension, we have shown can be paralleled to a cobweb model. Then by way of the methods applied to cobweb models, we can compare the behaviour near a fixed point to that of a dynamical system near an equilibrium point. We can also
compare the stability conditions of an equilibrium solution of a DDE with lag $T$ to those for the same equation minus the lag (i.e., with $T = 0$).

We have seen in our examples that discrete time lags are in general destabilizing and the larger the time $T$, the more likely it seems that an equilibrium solution stable without the lag, would become unstable with the lag.

For the case of the distributed time lag, we have seen that a restriction on the delay kernel to that of a particular form giving us a $\delta$-convertible DDE, will allow us through the methods described in Chapter III to directly convert the DDE to an equivalent dynamical system of a higher dimension. The equivalence of the two systems allows us then to use the methods for dynamical systems to analyse the stability behaviour of the DDE without worry.

We have seen in our examples that $\delta$-convertible DDE with average time lag $T$ tend to be less destabilizing than the same DDE with discrete time lag $T$. In particular, MacDonald's special memory functions in the logistic model show that as the order $p$ of the memory function increases to $\infty$, the critical average time lag $T_s$ decreases to $T_s = \pi/2r$, which is precisely that for the case of a discrete time lag in the same logistic model. Because $T_s$ increases, stability persists for a smaller range of $T$ and thus we say that stability "worsens".

When we look at the graphs of the delay kernels precisely when $T = T_s$, we see
that as \( p \) increases (or as \( T_c \) decreases) the shape of the graph is such that the distribution of the weight of the delay becomes concentrated, i.e., the graph actually tends to look more and more like the spike of a \( \delta \) function. When we go back to look at the normalized graphs of the delay kernels in examples 3.2b and 3.2c, we see that this phenomenon is again observed. We conclude thus that as \( T_c \) increases or as the stability of the system worsens, so does the distribution of the lag become more like that of the discrete case.

We have thus after converting to the equivalent dynamical system, been able to use the methods for dynamical systems to analyse the behaviour of our original \( \delta \)-convertible DDE. In this thesis, we have only discussed the general stability analysis which is carried over when we convert from the DDE to the dynamical system, but the study of bifurcations may also be investigated to see if their analysis also carries over to similar behaviour in the DDE. Also, the restriction placed on the delay kernel in order to get a \( \delta \)-convertible DDE is to a class of functions which can certainly be used to approximate many other functions not of the same form. Thus parallel dynamical systems can perhaps be found which would approximate the behaviour of a more general class of DDE (at least locally).
BIBLIOGRAPHY


