OSCILLATIONS NEAR THE CIRCULAR ORBITS OF THE HELIUM ATOM.

by

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INTRODUCTION

In 1928, Rawles published a paper entitled "Two Classes of Periodic Orbits with Repelling Forces," in which he considered a system consisting of one body of very great mass which attracts two mutually repellent bodies of very small mass. The forces of attraction and repulsion are assumed to vary inversely as the squares of the distances. Rawles showed that if certain conditions of symmetry are imposed the orbits of the small particles must lie either in two parallel planes or be coplanar. In the former case they are found to be circles with centres on a line drawn perpendicular to the plane of the orbits, and passing through the large body. In the other case the orbits are coplanar "arc orbits" in which the small bodies move back and forth on arcs which are symmetrical with respect to the large body. These orbits were first calculated by Langmuir by numerical integration, and were also discussed by Van Vleck in his work on Quantum Principles and Line Spectra.

By assigning suitable values to the constants in Rawles' solutions, we have the case of the helium atom, which consists of the heavy compact nucleus, and two electrons, attracted by the positive nucleus, but repelling each other.

It is the purpose of this paper to discuss the
oscillations near the circular orbits. The second genus orbits near the "arc orbits" have also been considered by D. Buchanan.
On computing the second derivatives, and substituting in the differential equations it is found that

\[ f'' - 2\frac{\eta'}{\rho} - \frac{f}{\rho^3} = -\frac{f}{\rho^3}, \quad (2) \]

\[ \eta'' + 2\frac{\eta'}{\rho} - \eta = -\frac{\eta}{\rho^3}; \]

The solutions of these equations which give the circular orbit are:

\[ \chi = (1 - e^2)^{\frac{1}{3}} = \left(\frac{k^2}{4}\right)^{\frac{1}{3}} = m, \]

\[ \eta = 0, \]

\[ f = \sqrt{1 - m^2}. \]

Now let

\[ \eta = \eta_0 + \gamma \eta', \]

\[ f = \sqrt{1 - m^2} + \gamma \nu, \quad (3) \]

\[ t - t_0 = \sqrt{1 + \frac{\gamma}{\nu}} \tau. \]

where \( \gamma \) is a parameter; \( \delta \) a function of \( \eta \); \( \eta, \nu \) dependent variables, and \( \tau \) the independent variable. When these substitutions are made in(2) the following differential equations are obtained

1. Rawles
Where

\begin{align*}
R_1 &= \frac{3}{2} q^2 \sqrt{1-m^2} + 3 r^2 \sqrt{1-m^2} \left( \frac{5 m^2}{2} - 1 \right) \\
R_2 &= \lambda^3 \left( 4 - 20 m^2 + \frac{35}{2} m^4 \right) + 3 r q^2 \left( \frac{5}{2} m^2 - 2 \right) \\
Q_1 &= 3 q r \sqrt{1-m^2} \\
Q_2 &= 3 q r \left( \frac{5}{2} m^2 - 2 \right) + \frac{3}{2} q^3 \\
Q_3 &= 5 \lambda^3 q \sqrt{1-m^2} \left( 2 - \frac{7}{2} m^2 \right) - \frac{15}{2} \lambda q^3 \sqrt{1-m^2}.
\end{align*}

We propose to integrate equations (4) as a power series in \( \gamma \). Accordingly we let

\begin{align*}
\gamma &= \gamma_0 + \gamma_1 \gamma + \gamma_2 \gamma^2 + \ldots \\
r &= \lambda_0 + \lambda_1 \gamma + \lambda_2 \gamma^2 + \ldots \\
\delta &= \delta_0 + \delta_1 \gamma + \delta_2 \gamma^2 + \ldots
\end{align*}

It will be shown that \( \gamma \) and certain constants of integration may be chosen so that \( r \) and \( q \) shall be periodic and satisfy certain initial conditions.
2. CONSTRUCTION OF THE SOLUTIONS.

When the values of q, r, and S are substituted in (4) and the terms independent of y considered we get

\[ \dot{r}_0 - 2\dot{q}_o - 3(1-m^2)r_o = 0. \]
\[ \ddot{q}_o + 2\dot{r}_o = 0. \]

Integration of which gives

\[ \dot{q}_o = -2r_o + C_o \]
\[ \dot{r}_o + \mu^2 r_o - 2C_o = 0. \]

where \( C_o \) is a constant of integration and

\[ \mu = \sqrt{1+3m^2}. \]

Solving we get

\[ r_o = A_o \sin \mu t + B_o \cos \mu t + \frac{2C_o}{\mu^2}. \]
\[ q_o = -\frac{2A_o}{\mu} \sin \mu t + \frac{2B_o}{\mu} \cos \mu t + C_o (1-\frac{\mu^2}{\mu^2}) t + D. \]

In order to have a periodic solution we must put \( C_o = 0 \). Three arbitrary constants \( A_o, B_o, D \), are left. The initial conditions are now chosen such that the particle is projected orthogonally from the \( \bar{f} \)-axis i.e.

\[ \dot{r}(0) = \dot{q}(0) = 0, \]
\[ \lambda_j(0) = q_j(0) = 0 \quad (j = 0, 1, \ldots, \infty). \]
These initial conditions are those of orbits known as symmetric orbits. For a third condition we choose

\[ r^{(0)} = 1, \quad r^{(j)}_{j} = 0 \quad (j = 0, 1, \ldots, \infty) \]

Imposing these conditions we get

\[ A_0 = 1, \quad B_0 = 0, \quad D_0 = 0 \]

\[ q_0 = -\frac{\mu}{r} \sin \mu t \]

\[ r_0 = \cos \mu t. \quad (6) \]

In order to show that the various differential equations can be integrated and the constants of integration and \( f_0 \) determined so as to satisfy the periodicity and initial conditions, we proceed to terms of higher order.

**Terms in \( \chi \).**

\[ q_i + 2q_i = -\delta_i \xi_{0} + 3q_i \cdot r_0 \sqrt{1-m^2} = \Phi^{(1)} \]

\[ i_i - 2q_i - 3(1-m^2)\xi_i = \delta_i \left[ q_0 + 3(1-m^2)r_0 \right] \]

\[ + \sqrt{1-m^2} \left[ \frac{3}{2} q_0^2 - 3(1-\frac{5}{2} m^2) r_0^2 \right] = R^{(1)} \]

Substitution for \( r_0 \) and \( q_0 \) yields

\[ \Phi^{(1)} = \delta_i \mu \sin \mu t - 3 \sqrt{1-m^2} \frac{\mu}{m} \sin 2\mu t. \]

\[ R^{(1)} = \delta_i (1-3m^2) \cos \mu t + \sqrt{1-m^2} \left[ \frac{3}{\mu^2} - \frac{3}{2} \right] \]

\[ + \frac{15}{4} m^2 \left( \frac{3}{\mu^2} + \frac{3}{2} - \frac{15}{4} m^2 \right) \cos 2\mu t. \]
Integrating (7a) we get

\[ \dot{y}_1 = -2y_1 - \delta_1 \cos \mu t + \frac{3 \sqrt{1-m^2}}{2 \mu^2} \cos 2 \mu t + C_1 \]  

(8)

Substitution in (7b) yields.

\[ \dot{y}_1 + \mu^2 y_1 = -\delta_1 (1+3m^2) \cos \mu t + \sqrt{1-m^2} \left[ \frac{3}{\mu^2} - \frac{3}{2} + \frac{15}{4} m^2 \left( 1 - \frac{5}{2} m^2 \right) \cos 2 \mu t \right] \]  

(9)

The term in \( \cos \mu t \) in the right member will give rise to a non-periodic term \( \tau \sin \mu t \) in the solution, and hence for a periodic solution we must put

\[ \delta_1 (1+3m^2) = 0 \]

Since \( 1+3m^2 \neq 0 \), we have \( \delta_1 = 0 \).

The solution of (9) is

\[ y_1 = A_1 \cos \mu t + B_1 \sin \mu t + \frac{\sqrt{1-m^2}}{\mu^2} \left[ \frac{3}{\mu^2} - \frac{3}{2} + \frac{15}{4} m^2 \right] \]

\[ + \frac{3 \sqrt{1-m^2}}{2 \mu^2} \left( 1 - \frac{5}{2} m^2 \right) \cos 2 \mu t + \frac{2 C_1}{\mu^2} \]  

(10)

Substitution of (10) in (8) gives

\[ \dot{y}_1 = C_1 \left( 1 - \frac{4}{\mu^2} \right) - \frac{6 \sqrt{1-m^2}}{\mu^2} \left( \frac{1}{\mu^2} - \frac{1}{2} + \frac{5}{2} m^2 \right) \]

\[ + \frac{\sqrt{1-m^2}}{2 \mu^2} \left( 1 + 5m^2 \right) \cos 2 \mu t \]

\[ - 2A_1 \cos \mu t - 2B_1 \sin \mu t. \]  

(11)
and for periodic solutions the constant part must vanish or

\[
C_1 = \frac{6 \sqrt{1-m^2} \left( \frac{1}{\mu^2} - \frac{1}{2} + \frac{5}{4} m^2 \right)}{\mu^2 \left(1 - \frac{4}{\mu^2}\right)}
\]

Integration of (11) gives

\[
g_1 = -\frac{\sqrt{1-m^2}}{4 \mu^3} \left(1 + 5m^2\right) \sin \mu t - \frac{2A_1}{\mu} \sin \mu t
\]

\[+ \frac{2B_1}{\mu} \cos \mu t + D_1
\]

(12)

and imposing the initial conditions

\[\begin{align*}
\mathcal{R}(0) &= \mathcal{R}_i(0) = g_1(0) = 0 \\
B_1 &= 0, \quad D_1 = 0
\end{align*}\]

\[A_1 = -\frac{\sqrt{1-m^2}}{\mu^2} \left( \frac{3}{\mu^2} - 1 + \frac{5}{4} m^2 \right) - \frac{2C_1}{\mu^2}
\]

Hence

\[g_1 = -\frac{2A_1}{\mu} \sin \mu t - \frac{\sqrt{1-m^2}}{4 \mu^3} \left(1 + 5m^2\right) \sin 2\mu t.
\]

\[\mathcal{R}_1 = A_1 \cos \mu t + \frac{3 \sqrt{1-m^2}}{2 \mu^2} \left( \frac{1}{\mu^2} - \frac{1}{2} - \frac{5}{4} m^2 \right)
\]

\[+ \frac{2C_1}{\mu^2} + \frac{\sqrt{1-m^2}}{2 \mu^2} \left(1 - \frac{5}{2} m^2\right) \cos 2 \mu t
\]

(13)

where \(A_1\) and \(C_1\) have the values given above.

Terms in \(\mathcal{X}^2\).

\[
\dot{\mathcal{X}}_2^2 + 2 \mathcal{X}_2 = \left[-\delta_2 - \dot{\mathcal{R}}_0 + 3 \sqrt{1-m^2} \left(g_0 \mathcal{R}_1 + \mathcal{R}_1 \mathcal{R}_0 \right)
\]

\[+ 3g_0 \mathcal{R}_0^2 \left(\frac{5}{2} m^2 - 2\right) + \frac{3}{2} g_0 \mathcal{X}_0^3 \right] = \mathcal{Q}^{(2)}
\]

(14.a)
\[\hat{\mathbf{r}}_2 - 2 \frac{\dot{q}_2}{q_2} - 3(1-m^2) \hat{\mathbf{r}}_2 = \delta_2 \left[ \frac{\dot{q}_2 + 3(1-m^2)}{q_2} \right] + \sqrt{1-m^2} \left[ \frac{3q_0 q_1}{q_1} - 3(2-5m^2) \hat{\mathbf{r}}_0 \hat{\mathbf{r}}_1 \right. \\
+ r_0^3 \left( 4 - 20m^2 + \frac{35}{2}m^4 \right) + \left. 3q_0^2 \frac{r_0^2}{q_1^2} \left( \frac{5}{2}m^2 - 2 \right) \right] = R^{(2)} \]

Substitution of the values of \(q_0, r_0, q_1, r_1\), yields:

\[
Q^{(2)} = \sin \mu \tau \left[ \delta_2 \left( \frac{H}{\mu} - \frac{2}{\mu} \left( J + \frac{2C_1}{\mu^2} \right) - \frac{K}{2} \right) \\
+ \frac{3}{2} \frac{\mu(2-\frac{5}{2}m^2)}{\mu^3} \right] - \frac{6A_1}{\mu} \sqrt{1-m^2} \sin 2 \mu \tau \]

\[
+ \sin 3\mu \tau \left[ \frac{3}{2\mu} \left( 2 - \frac{5}{2}m^2 \right) - \frac{H}{\mu} - \frac{K}{2} + \frac{3}{\mu^3} \right].
\]

\[
R^{(2)} = \left[ \frac{6A_1}{\mu^2} \sqrt{1-m^2} \right] + \cos \mu \tau \left[ \delta_2 \left( 1-3m^2 \right) \\
+ \frac{3K}{\mu} \sqrt{1-m^2} - 3 \sqrt{1-m^2} \left( 2 - \frac{5m^2}{2} \right) \left( J + \frac{H}{2} + \frac{2C_1}{\mu^2} \right) \\
+ 3 \left( 1 - \frac{5m^2}{8} + \frac{35}{2}m^4 \right) + \frac{3}{\mu^2} \left( 1 - \frac{5m^2}{2} \right) \right] \\
+ \cos 2\mu \tau \left[ \sqrt{1-m^2} \left\{ 3A_1 \left( \frac{5m^2}{2} - 1 - \frac{2}{\mu^2} \right) \right\} \right] \]

\[
+ \cos 3\mu \tau \left[ 3 \sqrt{1-m^2} \left\{ \frac{H}{2} \left( 5m^2 - 2 \right) + \frac{3K}{K} \right\} \right] \\
+ \left( 1 - \frac{5m^2}{8} + \frac{35}{2}m^4 \right) - \frac{3}{\mu^2} \left( \frac{5}{2}m^2 - 2 \right).}
where

\[
\begin{align*}
J &= \frac{3 \sqrt{1-m^2}}{\mu^2} \left( \frac{1}{\mu^2} - \frac{1}{2} + \frac{5}{4} m^2 \right), \\
K &= \frac{\sqrt{1-m^2}}{4 \mu^3} \left( 1 + 5m^3 \right), \\
H &= \frac{\sqrt{1-m^2}}{2 \mu^2} \left( 1 - \frac{5}{2} m^2 \right).
\end{align*}
\]

The expressions for \(Q^{(2)}\) and \(R^{(2)}\) may be written as follows:

\[
Q^{(2)} = S_1^{(2)} \sin \mu \tau + S_2^{(2)} \sin 2\mu \tau + S_3^{(2)} \sin 3\mu \tau
\]

\[
R^{(2)} = \frac{6 A_1}{\mu^2} \sqrt{1-m^2} + C_1^{(2)} \cos \mu \tau + C_2^{(2)} \cos 2\mu \tau + C_3^{(2)} \cos 3\mu \tau.
\]

where \(S_j^{(2)}\), \(C_j^{(2)}\) \((j=1,2,3)\) are the coefficients of the sine and cosine terms in (15) and (16).

Integration of (25) gives

\[
\dot{q}_2 = -2r_2 - \frac{S_1^{(2)}}{\mu} \cos \mu \tau - \frac{S_2^{(2)}}{2\mu} \cos 2\mu \tau - \frac{S_3^{(2)}}{3\mu} \cos 3\mu \tau
\]

Substitution of (18) in (16) yields,

\[
\begin{align*}
\dot{r}_2 + \mu^2 r_2 &= 2C_2 + \frac{6 A_1}{\mu^2} \sqrt{1-m^2} + (C_1^{(2)} - \frac{2}{\mu} S_1^{(2)}) \cos \mu \tau \\
+ (C_2^{(2)} - \frac{1}{\mu} S_2^{(2)}) \cos 2\mu \tau + (C_3^{(2)} - \frac{2}{3\mu} S_3^{(2)}) \cos 3\mu \tau
\end{align*}
\]

The term in \(\cos \mu \tau\) in the right member upon integration gives rise to a non-periodic term; therefore, for a periodic solution we must put

\[
C_1^{(2)} - \frac{2}{\mu} S_1^{(2)} = 0.
\]
When the values for \( C_1^{(2)} \), \( S_1^{(2)} \) in (15) and (16) are put in (19) and the resulting equation simplified we have

\[
\mu^2 S_2 = 3(1-m^2) \left[ \frac{1}{2 \mu^4} (1 + 5m^2) + \frac{4}{\mu^2} \frac{1}{\mu^2} 
- \frac{1}{2} + \frac{5}{4} m^2 \right] + 3 \left( 1 - 5m^2 \right)
+ \frac{3}{8} m^4 - \frac{2}{\mu^2} + \frac{5}{2} \frac{m^2}{\mu^2} + \frac{6}{\mu^4}
\]

Solving 18(b) we get

\[
\hat{\gamma}_2 = A_2 \cos \mu \tau + B_2 \sin \mu \tau + \frac{2}{\mu^2} \left( C_2 + \frac{3A_1}{\mu^2} \sqrt{1-m^2} \right)
+ \left( \frac{1}{\mu} S_2^{(2)} - C_2^{(2)} \right) \cos 2\mu \tau
+ \frac{1}{8\mu^2} \left( \frac{2}{3\mu} S_3^{(2)} - C_3^{(2)} \right) \cos 3\mu \tau.
\]

The substitution of (21) in 18(a) gives

\[
\hat{\gamma}_2 = - \frac{4}{\mu^2} \left( C_2 + \frac{3A_1}{\mu^2} \sqrt{1-m^2} \right) + C_2 - \left( 2A_2 - \frac{S_2^{(2)}}{\mu} \right) \cos \mu \tau
+ \left\{ \frac{2}{3\mu^2} \left( C_2^{(2)} - \frac{1}{\mu} S_2^{(2)} \right) - \frac{S_2^{(2)}}{2\mu} \right\} \cos 2\mu \tau - 2B_2 \sin \mu \tau
\]

Since \( q_2 \) must be periodic, the constant part of \( q_2 \) must vanish i.e.

\[
C_2 = - \frac{12A_1 \sqrt{1-m^2}}{\mu^2 (4-\mu^2)}.
\]
When (22) is integrated we get

\[ q_2 = -\frac{1}{\mu} \left( 2A_2 - \frac{S_2^{(2)}}{\mu} \right) \sin \mu \tau \]

\[ + \frac{1}{2\mu} \left\{ \frac{2}{3\mu^2} \left( C_2^{(2)} - \frac{1}{\mu} S_2^{(2)} \right) - \frac{S_2^{(2)}}{2\mu} \right\} \sin 2\mu \tau \]

\[ + \frac{1}{3\mu^2} \left\{ \frac{1}{4\mu} \left( C_3^{(2)} - \frac{2}{3\mu} S_3^{(2)} \right) - \frac{S_3^{(2)}}{3} \right\} \sin 3\mu \tau \]

\[ + \frac{2B_2}{\mu} \cos \mu \tau + D_2 \tag{23} \]

Imposing the initial conditions we find

\[ B_2 = 0 , \quad D_2 = 0 , \]

\[ A_2 = \frac{1}{\mu^2} \left[ \frac{C_2^{(2)}}{3} - \frac{1}{3\mu} S_2^{(2)} - 2C_2 - \frac{6A_2}{\mu^2} \sqrt{1-m^2} \right. \]

\[ \left. + \frac{C_3^{(2)}}{3} - \frac{1}{12\mu} S_3^{(2)} \right] \]

The solutions are therefore (21) and (23) with the above values of \( B_2 , D_2 , S_2 , A_2 \).
3. COMPUTATION FOR THE FIRST CASE

For the helium atom $\mu^2 = \frac{1}{2}$ and the parameter $e$ must be given the value $\left(\frac{7}{8}\right)^2$.

Therefore $m = 0.5$; $\gamma$ was put equal to $0.05$.

and

$$\gamma = 0.05\left(1.58 \sin \mu + 0.11 \sin 2\mu + 0.003 \sin 3\mu\right)$$

$$\lambda = 0.05\left(0.043 + 0.955 \cos \mu + 0.005 \cos 2\mu + 0.00004 \cos 3\mu\right)$$

The orbit was drawn, but is not shown as it is similar to the Y-Z plane in the three dimensional case.
PART II

ORBITS OF THREE DIMENSIONS.

4. The Differential Equations.

The problem is one of constructing the three dimensional orbits which are in the vicinity of the circular orbits. The differential equations of motion are the same as before, namely, those given by (1) in Part I.

After transferring to rotating axis as in the first case we have

\[ \chi'' = -\frac{x}{\rho^3} + \frac{h^2}{4x^2}, \quad \frac{h^2}{4} = 1 - e^2. \]

\[ \eta'' + 2\eta' - \omega = -\frac{\eta}{\rho^3}, \quad (24) \]

\[ \xi'' - 2\xi' - \xi = -\frac{\xi}{\rho^3}. \]

In this case we let

\[ \chi = m + \rho \gamma, \]

\[ \eta = o + \eta \gamma, \quad (25) \]

\[ \xi = \sqrt{1-m^2} + \eta \gamma, \]

\[ t - t_0 = \sqrt{1 + \xi} \tau. \]
where \( y \) is a parameter; \( \delta \) is a function of \( y, \varphi, q, r \) dependent variables, and \( t \) the independent variable.

When the substitutions (25) are made in (24) and the factor \( y \) taken out, the following differential equations are obtained

\[
\ddot{p} + 3(1 + \delta)(1 - m^2) p - 3(1 + \delta) m \sqrt{1 - m^2} \dot{r} = (1 + \delta)(y P_2 + y^2 P_3 + \cdots + y^j P_{j+1} + \cdots)
\]

\[
\ddot{r} + 2 \sqrt{1 + \delta} \dot{r} \dot{t} = (1 + \delta)(y Q_2 + y^2 Q_3 + \cdots + y^j Q_{j+1} + \cdots) \quad (26)
\]

\[
\dot{r} - 2 \sqrt{1 + \delta} \dot{q} - 3(1 + \delta)(1 - m^2) \dot{r} + 3(1 + \delta) m \sqrt{1 - m^2} \dot{p} = (1 + \delta)(y R_2 + y^2 R_3 + \cdots + y^j R_{j+1} + \cdots)
\]

where

\[
P_2 = 3 \left( \frac{1}{m} + \frac{3}{2} \left[ m - \frac{5}{2} m^3 \right] \right) \dot{r}^2 + \frac{3}{2} m q^2\]

\[
-3 \left( 2 m - \frac{5}{2} m^3 \right) \dot{r}^2 + 3 \sqrt{1 - m^2} (1 - 5 m^2) \dot{p} \dot{r} + \frac{3}{2} \left( 1 - 5 m^2 \right) \dot{q}^2 - 3 \left( 2 - \frac{15}{2} m^2 + \frac{35}{2} m^4 \right) \dot{p} \dot{r}^2
\]

\[
P_3 = \left( \frac{3}{2} - 15 m^2 + \frac{35}{2} m^4 \right) \dot{p}^3 + \frac{15}{2} m \sqrt{1 - m^2} (7 m^2 - 3) \dot{p}^2 \dot{r}
\]

\[
+ \frac{3}{2} \left( 1 - 5 m^2 \right) \dot{q}^2 - 3 \left( 2 - \frac{15}{2} m^2 + \frac{35}{2} m^4 \right) \dot{p} \dot{r}^2
\]

\[- \frac{15}{2} m \sqrt{1 - m^2} q^2 \dot{r} + 5 \left( 2 - \frac{7}{2} m^2 \right) \dot{r}^3.
\]
The variable $\dot{q}$ enters the $R_j$ and $Q_j$ to even degrees and the $Q_j$ to odd degrees.

On integrating 26(b) we have

$$\dot{q} = -2 \sqrt{1-\delta} \ r + (1+\delta) \sum \gamma Q_2 + \gamma^2 Q_3 + \ldots \ d\tau + C_1^{(e)} \quad (27)$$

where $C_1^{(e)}$ is a constant of integration.

Substitution of (27) in 26(c) yields

$$\ddot{r} + (1+\delta)(1+3m^2) \ r - 3(1+\delta) \ m \sqrt{1-m^2} \ \rho \ 
= (1+\delta)(\gamma R_2 + \gamma R_3 + \ldots ) + 2(1+\delta) \sum \gamma Q_2 + \gamma Q_3 + \gamma Q_4 + \ldots \ d\tau + C_1^{(e)} \quad (28)$$
We shall take 26(a), 27, 28, as the three equations defining $p$, $q$, $r$. As in Part I, we shall integrate these equations as power series in $\gamma$ and shall show that and other constants of integration can be chosen so that $p$, $q$, $r$ will be periodic and satisfy certain initial conditions. We first let

$$p = \sum_{j=0}^{\infty} p_j \gamma^j,$$
$$q = \sum_{j=0}^{\infty} q_j \gamma^j,$$
$$r = \sum_{j=0}^{\infty} r_j \gamma^j,$$

and then substitute these series in 26(a), 27, 28. The resulting equations are denoted by 26(a), 27, 28, respectively.

5. THE GENERATING SOLUTIONS.

We consider first only the terms of 26(a), 27, 28, which are independent of $\gamma$ and thus obtain

$$\ddot{p} + 3(1-m^2)p - 3m\sqrt{1-m^2}r = 0$$

$$\dot{q} = -2r + C^{(o)}_i$$

$$\dot{\dot{r}} + (1+3m^2)r - 3m\sqrt{1-m^2}\dot{p} = 2C^{(o)}_i$$

(30)
These are the equations of variation. Equations 30(a) and 30(c) are independent of 30(b) and will be considered first. Neglecting the right member of 30(c) and denoting $\frac{d}{dt}$ by $D$ we obtain

$$\left[D^2 + 3(1-m^2)\right] \rho_0 - 3m \sqrt{1-m^2} \lambda_0 = 0,$$

$$-3m \sqrt{1-m^2} \rho_0 + \left[D^2 + 1 + 3m^2\right] \lambda_0 = 0.$$

The functional determinant is

$$\Delta = D^4 + 4D^2 + 3(1-m^2),$$

and equating this functional determinant to zero we get

$$D^2 = -2 + \sqrt{1 + 3m^2}, \quad -2 - \sqrt{1 + 3m^2}.$$

The second root of $D^2$ is always negative, and the first root is negative so long as $m^2 < 1$. But for the circular solutions $m^2$ must be less than 1, if they are to be real. That is, both roots are negative. We may therefore let

$$-2 + \sqrt{1 + 3m^2} = -\sigma_1^2; \quad -2 - \sqrt{1 + 3m^2} = -\sigma_2^2.$$

Then the solutions of (31) are

$$\rho_0 = A_1^{(s)} e^{i \sigma_1 \tau} + A_2^{(s)} e^{-i \sigma_1 \tau} + A_3^{(s)} e^{i \sigma_2 \tau} + A_4^{(s)} e^{-i \sigma_2 \tau},$$

$$\lambda_0 = B_1^{(s)} e^{i \sigma_1 \tau} + B_2^{(s)} e^{-i \sigma_1 \tau} + B_3^{(s)} e^{i \sigma_2 \tau} + B_4^{(s)} e^{-i \sigma_2 \tau},$$

where $A_j^{(s)}$, $B_j^{(s)}$ ($j = 1, 2, 3, 4$) are constants of integration. Of the eight constants only four are independent.
Upon substituting (33) in (31) we get

\[ A_j^{(0)} = \omega_j, \quad B_j^{(0)} \quad (j = 1, 2, 3, 4) \]

\[ \omega_1 = \frac{3m \sqrt{1 - m^2}}{1 - 3m^2 + \sqrt{1 + 3m^2}} \]

\[ \omega_2 = \frac{3m \sqrt{1 - m^2}}{1 - 3m^2 - \sqrt{1 + 3m^2}} \quad (34). \]

We have therefore, three sets of generating solutions,

**I**

\[ \rho_0 = \omega_1 \left( B_1^{(0)} e^{i\sigma_1 \tau} + B_2^{(0)} e^{-i\sigma_1 \tau} \right) \]

\[ \mathcal{R}_0 = B_1^{(0)} e^{i\sigma_1 \tau} + B_2^{(0)} e^{-i\sigma_1 \tau} \]

\[ \text{Period} = \frac{2\pi}{\sigma_1} = \mathcal{P}_1 \]

**II**

\[ \rho_0 = \omega_2 \left( B_3^{(0)} e^{i\sigma_2 \tau} + B_4^{(0)} e^{-i\sigma_2 \tau} \right) \]

\[ \mathcal{R}_0 = B_3^{(0)} e^{i\sigma_2 \tau} + B_4^{(0)} e^{-i\sigma_2 \tau} \]

\[ \text{Period} = \frac{2\pi}{\sigma_2} = \mathcal{P}_2 \]

**III**

\[ \rho_0 = \omega_1 \left( B_1^{(0)} e^{i\sigma_1 \tau} + B_2^{(0)} e^{-i\sigma_1 \tau} \right) + \omega_2 \left( B_3^{(0)} e^{i\sigma_2 \tau} + B_4^{(0)} e^{-i\sigma_2 \tau} \right) \]

\[ \mathcal{R}_0 = \left( B_1^{(0)} e^{i\sigma_1 \tau} + B_2^{(0)} e^{-i\sigma_1 \tau} \right) + \left( B_3^{(0)} e^{i\sigma_2 \tau} + B_4^{(0)} e^{-i\sigma_2 \tau} \right) \]

\[ \text{Period} = \mathcal{P}_3 = \mathcal{N}_2 \mathcal{P}_1 = \mathcal{N}_1 \mathcal{P}_2. \]

**WHERE** \( \frac{\mathcal{N}_1}{\mathcal{N}_2} = \frac{\sigma_1}{\sigma_2} \) **AND** \( \mathcal{N}_1, \mathcal{N}_2 \) **ARE INTEGERS RELATIVELY PRIME.**
The constructions for generating solutions I and II only will be considered.

6. CONSTRUCTION OF THE SOLUTIONS.

Since the constructions are the same for generating solutions I and II, we shall drop the subscripts in the following work. We shall now consider the differential equations obtained by equating the coefficients of the various powers of $\gamma$ in $26^1(a)$, $27^1$, $28^1$.

Terms independent of $\gamma$.

The complete solutions of $30(a)$, (c) having the period $P Are$

$$\beta_0 = \omega \left( B_{1}^{(o)} e^{i\sigma t} + B_{2}^{(o)} e^{-i\sigma t} \right) + \frac{2m}{\sqrt{r - m^2}} C^{(o)}$$

(35)

$$\beta_0 = B_{1}^{(o)} e^{i\sigma t} + B_{2}^{(o)} e^{-i\sigma t} + 2 C^{(o)}$$

Substitution of 35(b) in 30(b) yields

$$q_0 = \frac{2i}{\sigma} \left( B_{1}^{(o)} e^{i\sigma t} - B_{2}^{(o)} e^{-i\sigma t} \right) - 3 C^{(o)} \tau + C_2^{(o)}$$

(36)

where $C_2^{(o)}$ is a constant of integration.
Since \( q_0 \) is to be periodic we put \( C_1^{(0)} = 0 \).

Three constants \( B_1^{(0)}, B_2^{(0)}, C_2^{(0)} \) remain and we may therefore impose three initial conditions upon the solutions. As in Part I we take the case where the orbits are symmetric viz.,

\[
\dot{p}_j^{(0)} = \dot{q}_j^{(0)} = q_j^{(0)} = 0 \quad (j = 0, \ldots, \infty)
\]

(37)

This gives \( B_1^{(0)} = B_2^{(0)} \) and \( C_2^{(0)} = 0 \). That is, we have \( B_1^{(0)} \) or \( B_2^{(0)} \) arbitrary, and may take

\[
\lambda_j^{(0)} = 1, \quad \gamma_j^{(0)} = 0 \quad (j = 1, \ldots, \infty)
\]

(38).

without loss of generality, since \( r \) carries \( \gamma \) as a factor, and \( \gamma \) is arbitrary.

Therefore \( B_1^{(0)} = B_2^{(0)} = \frac{1}{2} \).

and

\[
\lambda_0 = \frac{e^{i \sigma \tau} + e^{-i \sigma \tau}}{2} = \cos \sigma \tau
\]

\[
\gamma_0 = -\frac{2}{\sigma} \sin \sigma \tau
\]

\[
\beta_0 = \omega \cos \sigma \tau
\]

(39).

Terms in \( \gamma \).

\[
\left[ D^2 + 3(1-m^2) \right] p_1 - 3m \sqrt{1-m^2} \lambda_1 = \dot{p}^{(i)} =
\]

\[
= -3 \delta_i \left[ (1-m^2) \omega - m \sqrt{1-m^2} \right] \cos \sigma \tau +
\]
From 28(b) we have

\[ \dot{\varphi} = -2 \lambda + C_i^{(0)} \]

and if we expand we get

\[ \dot{q}_1 = -2 \lambda_1 + C_i^{(0)} \]

\[ \left[ D^2 + \frac{1}{1 + 3 m^2} \right] \dot{r}_1 - 3 m \sqrt{1 - m^2} \beta_1 = R^{(i)} \]

\[ = \delta_i \left[ 3 m \sqrt{1 - m^2} \varpi + 1 - 3 m^2 \right] \cos \sigma \tau \]

\[ + 2 C_i^{(0)} + \frac{3}{4} \left( 1 - 5 m^2 \right) \sqrt{1 - m^2} \varpi^2 - 3 m \left( 2 - \frac{5}{2} m^2 \right) \]

\[ - 3 \sqrt{1 - m^2} \left( \frac{1}{2} - \frac{1}{\sigma^2} - \frac{5}{4} m^2 \right) \]

\[ + \left[ \frac{3}{4} \left( 1 - 5 m^2 \right) \sqrt{1 - m^2} \varpi^2 - 3 m \left( 2 - \frac{5}{2} m^2 \right) \right. \]

\[ - \frac{3}{2} \sqrt{1 - m^2} \left( 1 + \frac{1}{\sigma^2} - \frac{5}{2} m^2 \right) \cos 2 \sigma \tau \]

\[ = \delta_i C_i^{(0)} \cos \sigma \tau + 2 C_i^{(0)} \cos \sigma \tau + C_i^{(0)} \cos 2 \sigma \tau. \]
and
\[ \dot{\beta}_1 = -2 \lambda_1 - \delta_1 \cos \sigma \tau + \frac{3}{2} \left( m \omega + \sqrt{1 - m^2} \right) \cos 2 \sigma \tau + C_i^{(i)} \]  
\hspace{1cm} (43)

Neglecting all terms of the right side of (40) and (42) and solving we find
\begin{align*}
\beta_1 &= \omega \left( B_1^{(i)} e^{i \sigma \tau} + B_2^{(i)} e^{-i \sigma \tau} \right) + \frac{2m}{\sqrt{1 - m^2}} C_i^{(i)} \\
\lambda_1 &= B_1^{(ii)} e^{i \sigma \tau} + B_2^{(ii)} e^{-i \sigma \tau} + Z C_i^{(ii)}
\end{align*}
\hspace{1cm} (44)

where B's and C's are constants of integration.

The particular integrals for \( p_1 \) and \( r_1 \) are given by
\begin{align*}
p_1 &= \frac{\left( D^2 + 1 + 3m^2 \right) P^{(i)} + 3m \sqrt{1 - m^2} R^{(i)}}{D^4 + 4D^2 + 3(1 - m^2)} \\
\lambda_1 &= \frac{3m \sqrt{1 - m^2} P^{(ii)} + \left\{ D^2 + 3(1 - m^2) \right\} R^{(ii)}}{D^4 + 4D^2 + 3(1 - m^2)}
\end{align*}
\hspace{1cm} (45)

The coefficient of \( \cos \sigma \tau \) in the numerators must vanish since \(-\sigma^2\) is a root of the denominator.

These coefficients are
\begin{align*}
\delta_1 \left[ a_i^{(i)} \left\{ -\sigma^2 + 1 + 3m^2 \right\} + C_i^{(ii)} \left\{ 3m \sqrt{1 - m^2} \right\} \right] \\
\delta_1 \left[ a_i^{(ii)} \left\{ 3m \sqrt{1 - m^2} \right\} + C_i^{(iii)} \left\{ -\sigma^2 + 3(1 - m^2) \right\} \right]
\end{align*}
respectively and these expressions vanish for $\delta_1 = 0$

$$a^{(i)} = c^{(i)} = 0.$$  

The latter are trivial solutions and shall be neglected.

Complete solutions are, therefore

$$b_1 = \omega \left( B_1^{(i)} e^{i \sigma r} + B_2^{(i)} e^{-i \sigma r} \right) + \frac{2m}{\sqrt{1-m^2}} c^{(i)} + \alpha_0^{(i)} + \alpha_2^{(i)} \cos 2\sigma r, \quad (46)$$

$$r_1 = B_1^{(i)} e^{i \sigma r} + B_2^{(i)} e^{-i \sigma r} + 2 c^{(i)} + \gamma_0^{(i)} + \gamma_2^{(i)} \cos 2\sigma r, \quad (47)$$

Where

$$\alpha_0^{(i)} = \frac{1+3m^2}{3(1-m^2)} a_0^{(i)} + \frac{m}{\sqrt{1-m^2}} c_0^{(i)},$$

$$\alpha_2^{(i)} = \frac{-4\sigma^2 + 1 + 3m^2}{16 \sigma^4 - 16 \sigma^2 + 3(1-m^2)} a_2^{(i)} + 3m \sqrt{1-m^2} c_2^{(i)},$$

$$\gamma_0^{(i)} = \frac{m}{\sqrt{1-m^2}} a_0^{(i)} + c_0^{(i)},$$

$$\gamma_2^{(i)} = \frac{3m \sqrt{1-m^2} a_2^{(i)} + \left\{ -4 \sigma^2 + 3(1-m^2) \right\} c_2^{(i)}}{16 \sigma^4 - 16 \sigma^2 + 3(1-m^2)},$$
where \( a^{(i)}_0, c^{(i)}_0, a^{(i)}_2, c^{(i)}_2 \) are given in (42) and (44).

If we substitute for \( r_1 \) in (41) and solve we get

\[
q_1 = \frac{2i}{\sigma} \left( B_1^{(i)} e^{\sigma \tau} - B_2^{(i)} e^{-i \sigma \tau} \right) - \left( 3 c^{(i)}_1 + 2 c^{(i)}_2 \right) \tau \\
+ \beta^{(i)}_2 \sin \sigma \tau + c^{(i)}_2
\]  

(48)

where

\[
\beta^{(i)}_2 = \frac{1}{2 \sigma} \left\{ \frac{3}{2 \sigma^2} \left( m \omega + \sqrt{1 - m^2} - 2 \gamma^{(i)}_x \right) \right\}
\]

Periodicity conditions require the coefficient of \( t \) to be zero, hence

\[ c^{(i)}_1 = -\frac{2}{3} \gamma^{(i)}_0. \]

and from the initial conditions \( c^{(i)}_2 = 0 \)

\[ B_1^{(i)} = B_2^{(i)} = \frac{1}{6} \gamma^{(i)}_0 - \frac{1}{2} \gamma^{(i)}_2. \]

The desired solutions at this step are,

\[
q_1 = q_1^{(i)} = F_0^{(i)} + F_1^{(i)} \cos \sigma \tau + F_2^{(i)} \cos 2 \sigma \tau
\]

\[ q_1 = G_1^{(i)} \sin \sigma \tau + G_2^{(i)} \sin 2 \sigma \tau
\]

\[ h_1 = h_1^{(i)} \cos \sigma \tau + h_2^{(i)} \cos 2 \sigma \tau
\]

WHERE

\[ F_0^{(i)} = \alpha_0^{(i)} - \frac{4}{3} \frac{m}{\sqrt{1 - m^2}} \gamma^{(i)}_0
\]

\[ F_1^{(i)} = 2 \beta^{(i)}_1 \omega
\]

\[ F_2^{(i)} = \alpha_2^{(i)}
\]

\[ G_1^{(i)} = -\frac{4}{\sigma} B_1^{(i)}
\]

\[ G_2^{(i)} = \beta^{(i)}_2
\]

\[ h_0^{(i)} = -\frac{1}{3} \gamma^{(i)}_0
\]

\[ h_1^{(i)} = 2 \beta^{(i)}_1
\]

\[ h_2^{(i)} = \gamma^{(i)}_2. \]
\[ \begin{aligned}
\left[ D^2 + 3(1-m^2) \right] \beta_2 - 3m \sqrt{1-m^2} \beta_2 &= P^{(2)} \\
\dot{\beta}_2 + 2 \dot{\beta}_2 + \delta_2 \beta_2 &= \left\{ Q_1^{(2)} \sin \omega t + Q_2^{(2)} \sin 2\omega t + Q_3^{(2)} \sin 3\omega t \right\} \\
\left[ D^2 + (1+3m^2) \right] \beta_2 - 3m \sqrt{1-m^2} \beta_2 &= R^{(2)}
\end{aligned} \]

**WHERE**

\[ \begin{aligned}
P^{(2)} &= \left[ \delta_2 \left\{ 3m \sqrt{1-m^2} - 3(1-m^2) \omega \right\} + a_1^{(2)} \right] \cos \omega t \\
&\quad + a_0^{(2)} + a_2^{(2)} \cos 2\omega t + a_3^{(2)} \cos 3\omega t \\
R^{(2)} &= \left[ \delta_2 \left\{ 1 - 3m^2 + 3m \omega \sqrt{1-m^2} \right\} + c_1^{(2)} \right] \cos \omega t \\
&\quad + c_0^{(2)} + c_2^{(2)} \cos 2\omega t + c_3^{(2)} \cos 3\omega t + 2c_1^{(2)} \\
a_1^{(2)} &= K_1^{(i)} + \frac{c}{\sigma^2} \left\{ \omega (1 - 5m^2) - 5m \sqrt{1-m^2} \right\} \\
&\quad + \frac{1}{2} K_2^{(i)} - \frac{3m}{\sigma} G_2^{(i)} + \frac{5}{4} L^{(i)} \\
a_0^{(2)} &= \frac{1}{2} K_1^{(i)} + \frac{3m}{\sigma} G_1^{(i)} \\
a_2^{(2)} &= \frac{1}{2} K_1^{(i)} - \frac{3m}{\sigma} G_1^{(i)} \\
a_3^{(2)} &= \frac{1}{2} K_2^{(i)} + \frac{3m}{\sigma} G_2^{(i)} + \frac{L^{(i)}}{4} \\
Q_1^{(2)} &= -\frac{c}{\sigma} \left( m F_0^{(i)} + \sqrt{1-m^2} H_0^{(i)} \right) + \frac{3}{2} G_2^{(i)} (m \omega + \sqrt{1-m^2}) \\
&\quad + \frac{3}{8} \left( m F_2^{(i)} + \sqrt{1-m^2} H_2^{(i)} \right) + \frac{3}{8} \frac{\omega^2}{\sigma} \left( 1 - 5m^2 \right) + \frac{15}{2} \frac{\omega m \sqrt{1-m^2}}{\sigma} \\
&\quad + \frac{3}{2\sigma} \left( 2 - \frac{5}{2} m^2 \right) - \frac{9}{16}
\end{aligned} \]
\[ Q_2^{(2)} = \frac{3}{2} \left[ m \left( \omega \ C_1^{(1)} - \frac{3}{5} F_1^{(1)} \right) + \sqrt{1-m^2} \left( G_1^{(1)} - \frac{3}{5} H_1^{(1)} \right) \right] \]

\[ Q_3^{(2)} = 3 \left[ \frac{G_2^{(1)}}{2} \left( m \omega + \sqrt{1-m^2} \right) - \frac{1}{\sigma} \left( m \ F_2^{(1)} + \sqrt{1-m^2} \ H_2^{(1)} \right) \right. \]

\[ \left. - \frac{\omega^2}{4\sigma} \left( 1 - 5m^2 \right) + \frac{5m \omega \sqrt{1-m^2}}{2\sigma} + \frac{1}{2\sigma} \left( 2 - \frac{5}{2}m^2 \right) + \frac{1}{3} \right]. \]

\[ C_1^{(2)} = 3 \sqrt{1-m^2} \left\{ \left( 1 - 5m^2 \right) \omega \left( F_0^{(1)} + \frac{F_2^{(1)}}{2} \right) - 2 \left( 1 - \frac{5}{2}m^2 \right) \left( H_0^{(1)} + \frac{H_2^{(1)}}{2} \right) \right\} \]

\[ - 3m \left( 4 - 5m^2 \right) \left\{ \left( F_0^{(1)} + \omega H_0^{(1)} \right) + \frac{1}{2} \left( F_2^{(1)} + \omega H_2^{(1)} \right) \right\} \]

\[ - \frac{3}{\sigma} \sqrt{1-m^2} \ G_2^{(1)} + \frac{3}{4} \sqrt{\frac{5\omega m}{2}} m \sqrt{1-m^2} \left( 7m^2 - 3 \right) \]

\[ + 3 \frac{\omega^2}{\sigma} \left( \frac{1}{2} + 15m^2 - 35m^4 - \frac{5}{2}m \sqrt{1-m^2} \right) \]

\[ - \frac{10 \omega m}{\sigma} \sqrt{1-m^2} + 15 \omega m \sqrt{1-m^2} \left( 2 - \frac{5}{2}m^2 \right) \]

\[ + 14 \left( \frac{1}{2} - 5m \sqrt{1-m^2} \right) + \left\{ - \frac{27}{2} + 15m^2 + 35 \left( 1 - m^2 \right)^{\frac{5}{2}} \right\} \].
\[
+ \frac{1}{4} \left[ \frac{5}{2} \omega^3 m \sqrt{1-m^2} \left( 7m^2 - 3 \right) + 3 \omega^2 \left( \frac{1}{2} + 15m^2 \right. \right. \\
- 35m^4 - \frac{5}{2} m \sqrt{1-m^2} \left. \right) + 15 \omega m \sqrt{1-m^2} \left( \frac{2}{5} + 2 - \frac{7}{2} m^2 \right) \\
- \frac{12}{5} \left( \frac{1}{2} - 5 \sqrt{1-m^2} \right) + \left\{ -\frac{3}{2} + 15m^2 + \frac{35}{2} \left( 1-m^2 \right)^{\frac{5}{2}} \right\} \right] \\

k_j^{(i)} = 3 \left[ 2 \omega F_j^{(a)} \left( \frac{1}{m} + \frac{3}{2} m - \frac{5}{2} m^3 \right) - 2 m H_j^{(a)} \left( 2 - \frac{5}{2} m^2 \right) \\
+ \sqrt{1-m^2} \left( 1 - 5m^2 \right) \left( F_j^{(a)} + \omega H_j^{(a)} \right) \right] \\

l_j^{(i)} = \omega^3 \left( \frac{3}{2} - 15m^2 + \frac{35}{2} m^4 \right) - \frac{15}{2} \omega^3 m \sqrt{1-m^2} \left( 3 - 7m^2 \right) \\
- 3 \omega \left\{ 2 \left( 1 + \frac{1}{\sigma^2} \right) - 5 \left( \frac{1}{2} + \frac{2}{\sigma^2} \right) m^2 - \frac{35}{2} m^4 \right\} \\
+ 5 \left( 2 - \frac{7}{2} m^2 + \frac{6}{\sigma^2} m \sqrt{1-m^2} \right).
\]
Neglecting all terms of the right side of (50) (A), (C) except $2 C_1^{(2)}$ and solving, we get

$$p_2 = \omega \left( B_1^{(2)} e^{i \sigma \tau} + B_2^{(2)} e^{-i \sigma \tau} \right) + \frac{2m}{\sqrt{1-m^2}} C_1^{(2)}$$

$$\Lambda_2 = B_1^{(2)} e^{i \sigma \tau} + B_2^{(2)} e^{-i \sigma \tau} + 2 C_1^{(2)}$$

where $B_j^{(2)}$, $C_j^{(2)}$ ($j = 1, 2$) are constants of integration.

The particular integrals are given by

$$p_2 = \frac{(D^2 + 1 + 3m^2) P^{(2)} + 3m \sqrt{1-m^2} R^{(2)}}{D^4 + 4D^2 + 3(1-m^2)},$$

$$\Lambda_2 = \frac{3m \sqrt{1-m^2} P^{(2)} + \left\{ D^2 + 3(1-m^2) \right\} R^{(2)}}{D^4 + 4D^2 + 3(1-m^2)}.$$

The coefficient of $\cos \sigma \tau$ must vanish at the previous step and therefore

$$\delta_2 = -a_i^{(2)} \left( -\sigma^2 + 1 + 3m^2 \right) - 3C_1^{(2)} m \sqrt{1-m^2}$$

$$\left\{ 3m \sqrt{1-m^2} - 3(1-m^2) \omega \right\} \left\{ -\sigma^2 + 1 + 3m^2 \right\} + \left\{ -3m^2 + 3m \omega \sqrt{1-m^2} \right\} 3m \sqrt{1-m^2}$$

The complete solutions for $r_2$ and $p_2$ are

$$p_2 = \omega \left( B_1^{(2)} e^{i \sigma \tau} + B_2^{(2)} e^{-i \sigma \tau} \right) + \frac{2m}{\sqrt{1-m^2}} C_1^{(2)}$$

$$+ \alpha_0^{(2)} + \alpha_2^{(2)} \cos 2 \sigma \tau + \alpha_3^{(2)} \cos 3 \sigma \tau$$
\[ \Lambda_2 = B_1 e^{i \sigma T} + B_2 e^{-i \sigma T} + 2 C_1^{(2)} + \int_0^{\sigma} + \int_2^{(2)} \cos 2 \sigma T + \int_3^{(2)} \cos 3 \sigma T. \]

WHERE

\[ \alpha_0^{(2)} = \frac{1 + 3 m^2}{3(1 - m^2)} \alpha_0^{(2)} + \frac{m}{\sqrt{1 - m^2}} C_0^{(2)}, \]

\[ \alpha_2^{(2)} = \frac{(-4 \sigma^2 + 1 + 3 m^2) \alpha_2^{(2)} + 3 m \sqrt{1 - m^2} C_2^{(2)}}{16 \sigma^4 - 16 \sigma^2 + 3(1 - m^2)}, \]

\[ \alpha_3^{(2)} = \frac{(-9 \sigma^2 + 1 + 3 m^2) \alpha_3^{(2)} + 3 m \sqrt{1 - m^2} C_3^{(2)}}{8 \sigma^4 - 36 \sigma^2 + 3(1 - m^2)}, \]

\[ \xi_0^{(2)} = \frac{m}{\sqrt{1 - m^2}} \alpha_0^{(2)} + C_0^{(2)}, \]

\[ \xi_2^{(2)} = \frac{3 m \sqrt{1 - m^2} \alpha_2^{(2)} + \{-4 \sigma^2 + 3(1 - m^2)\} C_2^{(2)}}{16 \sigma^4 - 16 \sigma^2 + 3(1 - m^2)}, \]

\[ \xi_3^{(2)} = \frac{3 m \sqrt{1 - m^2} \alpha_3^{(2)} + \{-9 \sigma^2 + 3(1 - m^2)\} C_3^{(2)}}{8 \sigma^4 - 36 \sigma^2 + 3(1 - m^2)}. \]
On substituting for \( r_2 \) in 50(b) and integrating and equating the coefficient of \( t \) to zero we get

\[
C_i^{(2)} = -\frac{2}{3} \gamma_0^{(2)} j
\]

\[
q_2 = \beta_0^{(2)} \sin \sigma t \sin \sigma t + \beta_1^{(2)} \sin 2\sigma t + \beta_2^{(2)} \sin 3\sigma t,
\]

where

\[
\beta_0^{(2)} = \frac{2}{\sigma} \left( B_1^{(2)} e^{-\sigma t} - B_2^{(2)} e^{-\sigma t} \right),
\]

\[
\beta_1^{(2)} = -\frac{1}{\sigma} \left( \delta_2^{(2)} + \frac{\gamma_1^{(2)}}{\gamma} \right),
\]

\[
\beta_2^{(2)} = -\frac{1}{2\sigma} \left( 2\gamma_2^{(2)} + \frac{\gamma_2^{(2)}}{2\sigma} \right),
\]

\[
\beta_3^{(2)} = -\frac{1}{3\sigma} \left( 2\gamma_3^{(2)} + \frac{\gamma_3^{(2)}}{3\sigma} \right).
\]

From the initial conditions we get

\[
B_2^{(2)} = B_1^{(2)} = \frac{1}{6} \gamma_0^{(2)} - \gamma_2^{(2)} + \gamma_3^{(2)},
\]

\[
C_2^{(2)} = 0.
\]

The solutions at the second step are therefore

\[
p_2 = F_0^{(2)} + F_1^{(2)} \cos \sigma t + F_2^{(2)} \cos 2\sigma t + F_3^{(2)} \cos 3\sigma t
\]

\[
p_2 = G_1^{(2)} \sin \sigma t + G_2^{(2)} \sin 2\sigma t + G_3^{(2)} \sin 3\sigma t
\]

\[
\mathbf{A}_2 = H_0^{(2)} + H_1^{(2)} \cos \sigma t + H_2^{(2)} \cos 2\sigma t + H_3^{(2)} \cos 3\sigma t.
\]

\[
F_0^{(2)} = \frac{2m}{1-m^2} C_i^{(2)} + \alpha_0^{(2)}
\]

where

\[
F_1^{(2)} = 2\omega B_1^{(2)}
\]

\[
F_2^{(2)} = \alpha_2^{(2)}
\]

\[
F_3^{(2)} = \alpha_3^{(2)}
\]

\[
C_i^{(2)} = \beta_0^{(2)} (i = 2, 3)
\]

\[
G_1^{(2)} = -4 \beta_1^{(2)} + \beta_1^{(2)}
\]

\[
H_0^{(2)} = 2 C_i^{(2)} + \gamma_0^{(2)}
\]

\[
H_1^{(2)} = 2 \beta_1^{(2)}
\]

\[
H_j^{(2)} = \gamma_j^{(2)} (j = 2, 3)
\]
7. Computation.

We have

\[ x = m + y \left( p_0 + p_1 y + p_2 y^2 + \ldots \right) \]

and similarly for \( y \), we let

\[ y = \frac{x}{\omega} \left( F_0 + F_1 \cos \sigma \tau + F_2 \cos 2\sigma \tau \right) \]

\[ + y^2 \left( F_0 + F_1 \cos \sigma \tau + F_2 \cos 2\sigma \tau + F_3 \cos 3\sigma \tau \right) \]

and similarly for \( \eta \) and \( \xi \).

We let \( \omega = \frac{1}{2}, m = 0.5, y = 0.05 \) and get

\[ p = -0.0025 + 0.064 \cos \sigma \tau + 0.025 \cos 2\sigma \tau - 0.002 \cos 3\sigma \tau, \]

\[ q = -0.19 \sin \sigma \tau + 0.03 \sin 2\sigma \tau - 0.0063 \sin 3\sigma \tau, \]

\[ \eta = -0.043 + 0.077 \cos \sigma \tau - 0.017 \cos 2\sigma \tau - 0.0027 \cos 3\sigma \tau. \]

In the above case the subscripts 1 and 2 on \( \sigma \) and \( \omega \) may be restored, and the two sets of orbits found which have the periods \( \frac{2\pi}{\sigma_1} \) and \( \frac{2\pi}{\sigma_2} \) respectively.

The above equations were used in plotting the orbits for the period \( \frac{2\pi}{\sigma_1} \)

\[ \sigma_1 = 0.825 \quad \text{therefore} \quad p_1 = \frac{12}{5} \pi \text{ nearly} \]

\[ t \tau \]
Values of $t$ were taken from $t=0^\circ$ to $t=2160^\circ$ at $30^\circ$
intervals and the values of $x$, $y$, $z$ computed. (SEE TABLE)

8. The orbits were then plotted in three planes.*

A check was also made on the work, the vis viva
integral being used. THE EXPRESSION FOR THIS CASE

is

\[
\sqrt{V}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \frac{2}{\left(x^2 + \eta^2 + \xi^2\right)^{1/2}} - \frac{1}{4x} - C
\]

WHERE

\[
x = m + \eta \rho
\]
\[
\eta = o + \xi \varphi
\]
\[
\xi = \sqrt{1-m^2} + \xi \rho.
\]

* SEE GRAPHS AFTER §9.
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Various values were taken for $t, x, y, z$ from the Table I and upon substitution in (51) yielded values for the constant of integration $C$ varying between $C = 2.17$ and $C = 2.31$. 
9. REFERENCES.


5. D. Buchanan - Presented at the Royal Society in Ottawa, May 1929.