SECOND GENUS ORBITS NEAR THE ARC ORBITS
OF THE HELIUM ATOM.

by

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CONTENTS

I. The Problem.

II. The Equations of Motion.

III. The First Genus Periodic Orbits.

IV. The Periodic Orbits of the Second Genus.

V. The Complete Solution:
   The coefficients of $\lambda$.
   The coefficients of $\lambda^2$.

VI. Numerical Example of the Orbit.
    Table I - First Genus Orbits.
    Table II - Second Genus Orbits.

Illustration.
I. THE PROBLEM

The object of the paper is to construct algebraically and geometrically a periodic orbit of the second genus.

The particular problem considered is that of the helium atom in which two electrons of infinitesimal mass and a positive nucleus of finite mass mutually repel and attract according to the conventions of electrostatics. The motion is restricted to a plane.

The arc-orbits, or first genus orbits, were obtained by T.H. Rawles in a paper entitled, "Two classes of Periodic Orbits with Repelling Forces." It is from these orbits that the Second Genus Orbits under consideration will be developed.

Second Genus Orbits are defined by Poincaré. A discussion of these orbits and the theory for their construction are given by Dr. D. Buchanan in a paper entitled "Periodic Orbits of the Second Genus Near the Straight Line Equilibrium Points in the Problem of Three Bodies."

The construction of the periodic orbits under consideration follows the methods suggested by this latter dissertation.

II. THE EQUATIONS OF MOTION.

Let the units of time and space be so chosen that the constant of proportionality is unity. Then if \( K^2 \) is the ratio of the repulsion to the attraction, the force function of the system is

\[
U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{k^2}{\Delta},
\]

where \( r_1 \) and \( r_2 \) represent the distances between the small particles and the large one and \( \Delta \) is the distance between the small bodies.

Let the masses of the small bodies be unity and the mass of the large body be "m."

Referred to a set of rectangular axes \( \xi, \eta, \zeta \), the equations of motion are:

\[
\begin{align*}
\frac{d^2 \xi}{dt^2} &= \frac{\partial U}{\partial \xi}, & \frac{d^2 \eta}{dt^2} &= \frac{\partial U}{\partial \eta}, & \frac{d^2 \zeta}{dt^2} &= \frac{\partial U}{\partial \zeta},
\end{align*}
\]  

\[
\frac{d^2 \xi}{dt^2} = \frac{\partial U}{\partial \xi_i}, \quad \frac{d^2 \eta}{dt^2} = \frac{\partial U}{\partial \eta_i}, \quad \frac{d^2 \zeta}{dt^2} = \frac{\partial U}{\partial \zeta_i}, \quad i = 1, 2,
\]

where the subscripts 0, 1, 2, refer to the coordinates of the large body and small bodies respectively.

If we consider "m" infinite as compared with the other two bodies, equations (2) may be neglected. Now take a set of coordinates \( x_i, y_i, z_i \) (\( i = 1, 2 \)) with the origin at the nucleus. The equations of motion of the infinitesimals become -
\[ \dot{x}_i = \frac{\partial u}{\partial x_i} ; \quad \dot{y}_i = \frac{\partial u}{\partial y_i} ; \quad \dot{z}_i = \frac{\partial u}{\partial z_i} \quad (4) \]

where \[ u = \frac{1}{r_i} + \frac{1}{L^2} - \frac{K^2}{\Delta} \]

and \[ \lambda_i^2 = x_i^2 + y_i^2 + z_i^2 \quad i = 1, 2. \]

\[ \Delta^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2. \]

The orbits with which we are concerned are those for which
\[ x_1 = -x_2 , \quad y_1 = y_2 , \quad z_1 = z_2 . \quad (5) \]

Let \( \frac{K^2}{\Delta^2} = \epsilon^2 \) and consider only the motion of the body whose coordinates are \( (x_1, y_1, z_1) \). On dropping the subscripts the equations of motion for this body become
\[ \ddot{x} = -\frac{x}{x^3} + \frac{1 - \epsilon^2}{x^2} ; \quad \ddot{y} = -\frac{y}{y^3} ; \quad \ddot{z} = -\frac{z}{z^3} . \quad (6) \]

When these equations have been solved the solutions for the motion of the other particle can be obtained from (5).

### III. First Genus Periodic Orbits.

In the determination of the First Genus orbits two different cases arise. In the first case the motion is in a circle in a plane normal to the x-axis. In the second case the orbit lies entirely in a plane containing the x-axis. To find the orbits of case two Rawles took the plane of motion as the xy-plane, that is \( z = 0 \). The arc-orbits obtained are given by:
\[ x = x_0 = 1 + \left( -\frac{\pi}{8} + \frac{1}{4} \cos 2\tau \right) \epsilon^2 + \cdots \quad (7) \]
\[ y = y_0 = \sqrt{3} \left[ \left( \frac{\pi}{3} \sin \tau \right) \epsilon + \left( \frac{\sqrt{3}}{6} \sin 3\tau \right) \epsilon^3 + \cdots \right]. \]
IV. THE PERIODIC ORBITS OF THE SECOND GENUS.

Following the methods in the "Periodic Orbits of the Second Genus," by D. Buchanan we displace the infinitesimal from the first genus orbits by the substitutions:

\[ \chi = \chi_0 + \phi ; \quad y = y_0 + \psi ; \quad \Theta = \Theta_0 + \Theta_1 + \Theta_2 \epsilon^2. \]

When arranged in a power series in \( \epsilon \) the equations of variation become:

\[ \phi'' + \phi \left[ (2 - 4 \cos 2\tau) \epsilon^2 + \ldots \right] - \frac{\partial}{\partial \tau} \left[ 2 \sqrt{3} \epsilon \sin \tau \ldots \right] = 0, \]

\[ \psi'' + \psi \left[ 1 + \left( -\frac{\tau}{3} + \frac{\Phi}{3} \cos 2\tau \right) \epsilon^2 + \ldots \right] - \frac{\partial}{\partial \tau} \left[ 2 \sqrt{3} \epsilon \sin \tau \ldots \right] = 0, \]

the generating solutions of which are (7).

To obtain a solution let \( \phi = e^{\lambda \tau} u \); \( \psi = e^{\lambda \tau} v \);

where

\[ u = a_0 + a_1 \epsilon + \ldots \]
\[ v = d_0 + d_1 \epsilon + \ldots \]
\[ \lambda = \lambda_0 + \lambda_1 \epsilon + \ldots \]

When \( u, v \) and \( \lambda \) are substituted in (8) the resulting differential equations contain only even powers of \( \epsilon \) and are thus independent of the sign of \( \epsilon \). This fact determines the choice of \( u, v \) and \( \lambda \). By arbitrarily placing \( \lambda_0 = 1 \) and applying the initial conditions \( u(0) = \text{constant}, \quad v'(0) = 1 \) we obtain a solution

\[ \phi = a_0 e^{i \tau} u + a_1 e^{-i \tau} u, \]
\[ \psi = i a_0 e^{i \tau} v - i a_1 e^{-i \tau} v. \]
where \( \alpha = (\sqrt{3} \ e^2 + \ldots) \)
\[
\begin{align*}
&u_1 = e \left\{ 1 - \frac{\sqrt{3}}{4} \ i \sin 2\tau \right\} + \ldots \\
&u_2 = e \left\{ 1 + \frac{\sqrt{3}}{4} \ i \sin 2\tau \right\} + \ldots \\
&v_1 = \cos \tau + e^{2} \left\{ \frac{3}{10} \ \cos \tau - \frac{3}{128} \ \cos 3\tau - \frac{3}{128} \ i \sin 3\tau \right\} + \ldots \\
&v_2 = \cos \tau + e^{2} \left\{ \frac{3}{10} \ \cos \tau - \frac{3}{128} \ \cos 3\tau + \frac{3}{128} \ i \sin 3\tau \right\} + \ldots 
\end{align*}
\]
Poincaré has shown that a solution of (8) can be obtained by differentiating the generating solutions partially with respect to \( t_0 \).

This gives rise to a solution of the form
\[
\begin{align*}
&\phi = A_{\phi} \ u_{\phi} \\
&\psi = A_{\psi} \ v_{\psi} \\
&\theta = A_{\theta} \ u_{\theta} \\
&\psi = A_{\psi} \ v_{\psi}
\end{align*}
\]
where
\[
\begin{align*}
&u_{\phi} = e \left[ -\frac{1}{2} \ \sin 2\tau \right] + \ldots \\
&v_{\phi} = \sqrt{3} \left[ \frac{3}{2} \ \cos \tau + \left( \frac{1}{32} \ \cos 3\tau \right) e^2 + \ldots \right] \\
&u_{\theta} = e \left[ -\frac{1}{2} \ \sin 2\tau \right] + \ldots \\
&v_{\theta} = \sqrt{3} \left[ \frac{3}{2} \ \cos \tau + \left( \frac{1}{32} \ \cos 3\tau \right) e^2 + \ldots \right]
\end{align*}
\]
To obtain all the solutions put
\[
\begin{align*}
&\phi = \Theta + \kappa T \phi \\
&\psi = \Psi + \kappa T \psi
\end{align*}
\]
where
\[
\begin{align*}
&\kappa = \kappa_1 \ e + \kappa_2 \ e^2 + \ldots \\
&\Theta = \Theta_0 + \Theta_1 \ e + \Theta_2 \ e^2 + \ldots \\
&\Psi = \Psi_0 + \Psi_1 \ e + \Psi_2 \ e^2 + \ldots
\end{align*}
\]
The variables \( \Theta \) and \( \Psi \) are then determined by applying the initial conditions
\[
\begin{align*}
&\Theta(0) = \text{constant} \quad , \quad \Theta(0) = 0 \\
&\Psi(0) = 0 \quad , \quad \Psi(0) = \text{constant}
\end{align*}
\]
The new solutions obtained are:
\[
\begin{align*}
&\phi = A_{\phi} \left\{ u_{\phi} + \kappa T \ u_{\phi} \right\} \\
&\psi = A_{\psi} \left\{ v_{\psi} + \kappa T \ v_{\psi} \right\}
\end{align*}
\]
where

\[ K = -\frac{3}{2} \lambda + \cdots \cdots \]

\[ u_\lambda = 1 + \left( \frac{1}{4} \cos 2\tau - 1 \right) \lambda^2 + \cdots \cdots \]

\[ v_\lambda = \lambda \left( \frac{3}{2} \sin \tau \right) + \cdots \cdots \]

Combining the above results we obtain

\[ f = a, \lambda \left( u_\lambda + a^2 \lambda \right) + a \lambda \left( u_\lambda + a \lambda \right) \]

\[ g = i a, \lambda \left( v_\lambda - a^2 \lambda \right) - a \lambda \left( v_\lambda - a \lambda \right) \]

where \( a \) and \( \lambda \) are constants of integration, and \( \lambda \) and \( \lambda \) are power series in \( \lambda \) having constant coefficients. The \( u_j \), \( v_j \), \( (j = 1, 2, \ldots) \) are periodic functions and have the values given in the previous section as far as the computation was carried out.

V. THE COMPLETE SOLUTION.

Besides making the substitutions \( \chi = \chi_0 + f \)
and \( \chi = \chi_0 + g \), we make the further substitutions \( \tau = (\lambda + \gamma) \tau \)
and \( \lambda = \lambda_0 + \gamma \), where \( \gamma \) is a function of \( \lambda \). We now proceed to show that the differential equations in \( p \) and \( q \) can be integrated as power series in \( \lambda \) and therefore \( \gamma \) must be determined likewise as a power series in \( \lambda \).

To do this we put

\[ f = \sum_{j=0}^{\infty} f_j \lambda^j \]

\[ g = \sum_{j=0}^{\infty} g_j \lambda^j \]

\[ \gamma = \sum_{j=0}^{\infty} \gamma_j \lambda^j \]
These values are now substituted in the differential equations (8). The coefficients of $\lambda$ in the power series thus obtained are found by equating like powers of $\lambda$.

Since $\lambda$ can be made arbitrarily small the construction is made only for the first power of $\lambda$ and the third power of $\epsilon$. Certain terms, however, in the equations for $\lambda^2$ are considered in the determination of $\gamma_i$. There will be no terms independent of $p$, $q$, $\gamma$, and $\lambda$ since $x_0$ and $y_0$ are solutions.

Make these substitutions in equations (6) and let the notation $\ddot{x}$ denote $\frac{d^2x}{dt^2}$.

These equations become:

$$\ddot{x} = -(1+2\gamma+\gamma^2)(x_0+p)\left[1 + \frac{1}{6}(1+2\cos \pi \tau)(\epsilon_0^2 + 2\epsilon_0 \lambda + \lambda^2)\right]$$

$$+ \rho \left[-3 + \frac{1}{6} (-3 + 2 \cos \pi \tau)(\epsilon_0^2 + 2\epsilon_0 \lambda + \lambda^2)\right]$$

$$- 2 \sqrt{3} \sin \pi \tau (\epsilon_0 + \lambda)$$

$$+ \text{terms in } \rho^2, \rho^3, \rho^4.$$ 

$$\ddot{y} = -(1+2\gamma+\gamma^2)(y_0+q)\left[1 + \frac{1}{6}(1+3 \cos \pi \tau)(\epsilon_0^2 + 2\epsilon_0 \lambda + \lambda^2)\right]$$

$$+ \rho \left[-3 + \frac{1}{6} (-3 + 2 \cos \pi \tau)(\epsilon_0^2 + 2\epsilon_0 \lambda + \lambda^2)\right]$$

$$- 2 \sqrt{3} \sin \pi \tau (\epsilon_0 + \lambda)\right].$$

In the subsequent work the subscript $o$ in $\epsilon_0$ will be dropped and restored after the solutions have been obtained.

(10)
The Coefficients of \( \lambda \).

Let the equations in \( \lambda \) be denoted by

\[
\begin{align*}
\ddot{P} + P, (2l^2 - 4e^2 \cos 2\tau + \ldots) - \ddot{Q}, (2iv \sin \tau + \ldots) &= 0 \\
\ddot{Q} + Q, (1 - \frac{1}{8} l^2 + \frac{1}{8} e^2 \cos 2\tau) + P, (2iv \sin \tau + \ldots) &= 0.
\end{align*}
\]

Then

\[
P'' = -\frac{3}{4} l - \frac{3}{8} \dot{l}^2 - \frac{3}{8} l \cos 2\tau + \frac{17}{8} l^3 \cos 2\tau
\]

and

\[
Q'' = -\frac{\sqrt{3}}{6} l^2 \sin 3\tau - \gamma \left\{ \frac{4\sqrt{3}}{3} \sin 3\tau + \frac{4\sqrt{3}}{3} l \sin 3\tau \right\}.
\]

The constant \( \gamma \) enters only in the \( Q^{(1)} \) and \( P^{(1)} \) equations as indicated. The complementary functions were obtained above in equations (9). In order to obtain the particular integrals we employ the method of the variation of parameters and consider \( a_1^{(1)} \ldots a_4^{(1)} \) not as constants but as functions of \( \tau \).

Thus we have

\[
\begin{align*}
\Delta \dot{a}_1^{(1)} &= -l \sin \tau (P^{(1)} M_{12} + Q^{(1)} M_{14}) \\
\Delta \dot{a}_2^{(1)} &= l \sin \tau (P^{(1)} M_{22} + Q^{(1)} M_{24}) \\
\Delta \dot{a}_3^{(1)} &= -l (P^{(1)} M_{32} + Q^{(1)} M_{34}) \\
\Delta \dot{a}_4^{(1)} &= (P^{(1)} M_{42} + Q^{(1)} M_{44}),
\end{align*}
\]

where \( \Delta \) is the determinant

\[
\begin{vmatrix}
l \sin \tau & l \sin \tau & l \sin \tau & l \sin \tau \\
\sin \tau (i u_1 + i u_2) & -i \sin \tau (i u_2 + i u_1) & -i \sin \tau (i u_2 + i u_1) & -i \sin \tau (i u_2 + i u_1) \\
i \sin \tau v_1 & -i \sin \tau v_2 & -i \sin \tau v_3 & -i \sin \tau v_4 \\
i \sin \tau (i v_1 + i v_2) & -i \sin \tau (i v_2 + i v_1) & -i \sin \tau (i v_2 + i v_1) & -i \sin \tau (i v_2 + i v_1)
\end{vmatrix}.
\]
This determinant is a constant and can be evaluated with the least difficulty by putting $\tau = 0$.

The value of $\Delta$ thus obtained is

$$-4.03 i e^5 + i e^7 (\quad) + i e^9 (\quad) + \ldots \ldots \ldots (13)$$

The $M_{jk} (f=1, \ldots 4; k = 2, 4, \ldots)$ are the co-factors of the elements $j, k$ in the determinant $\Delta$, where $j$ and $k$ refer to the column and row respectively.

These co-factors are found to be

$$M_{12} = e^{-i\tau} \left[ \left( \frac{4}{3} + \frac{3\sqrt{3}}{6} \right) \left[ i \left( 2 \sin 2\tau + \sin 4\tau \right) - (2 \cos 4\tau + \cos 2\tau) \right] \right]$$

$$M_{14} = e^{-i\tau} \left[ \left( \frac{3}{6} \right) \left[ i \left( \frac{9}{25} \cos 5\tau - \frac{3}{32} \cos 3\tau - \frac{527}{256} \cos \tau \right) + \left( \frac{3}{256} \sin 5\tau - \frac{13}{6} \sin 3\tau + 128/3 + 9 \sin \tau \right) \right] \right]$$

$$M_{22} = e^{-i\tau} \left[ \left( \frac{3}{6} \right) \left[ i \left( 2 \sin 2\tau + \sin 4\tau \right) + (2 \cos 4\tau + \cos 2\tau) \right] \right]$$

$$M_{24} = e^{-i\tau} \left[ \left( \frac{3}{6} \right) \left[ i \left( -\frac{9}{25} \cos 5\tau + \frac{3}{32} \cos 3\tau + \frac{527}{256} \cos \tau \right) + \left( \frac{3}{256} \sin 5\tau - \frac{13}{6} \sin 3\tau + 128/3 + 9 \sin \tau \right) \right] \right]$$

$$M_{32} = e^{-i\tau} \left[ \left( \frac{9}{6} \right) \left[ i \left( -\frac{4\sqrt{3}}{3} + \frac{9}{6} \left( 2 \cos 2\tau - \cos 4\tau \right) \right) \right] + i \tau \left[ \frac{9}{6} \left( 5 \sin 4\tau + 7 \sin 2\tau \right) \right] \right]$$

$$M_{34} = e^{i\tau} \left[ \left( \frac{9}{25} \right) \left( \frac{3\sqrt{3}}{2} \sin 5\tau + \frac{1}{2} \sin 3\tau - \frac{527}{256} \sin \tau \right) + i \tau \left[ \frac{9}{6} \left( \frac{2}{25} \cos 5\tau - \frac{3}{12} \cos 3\tau - \frac{527}{128} \cos \tau \right) \right] \right]$$

---

1. Moulton - 'Periodic Orbits,' § 18
When the above values are substituted in (12) the
values of \( \dot{a}^{(0)} \), \( \dot{a}^{(1)} \) are found. Upon integration we have
the constants \( A^{(0)}, \ldots, A^{(n)} \) arising, which are constants of
integration. Also it will be seen that constant terms arise
in \( \dot{a}^{(3)} \) which upon integration give rise to terms in \( \tau \).
Substitution of these terms in the complete solution produces
terms which are non-periodic, consequently all terms in \( \tau \)
must be equated to zero. Hence

\[
\left[ A^{(n)} \kappa + \left\{ \left( \frac{27}{128} - \sqrt{3} \right) \epsilon^3 + \left( \frac{27}{128} - \sqrt{3} \right) \epsilon^2 \gamma \right\} \right] i \tau = 0,
\]

where

\[ \kappa = - \frac{3}{2} \epsilon + \ldots \ldots \ldots \]

On reduction we obtain the equation: (14)

\[
- \frac{3}{2} A^{(n)} + \left( \frac{27}{128} - \sqrt{3} \right) \epsilon^2 + \left( \frac{27}{128} - \sqrt{3} \right) \epsilon^3 \gamma = 0.
\]

Substitution of the values of \( a^{(0)}, \ldots, a^{(n)} \)
obtained in the complete solution gives

\[
\phi = A^{(n)} i \epsilon^\tau \varphi + A^{(n)} i \epsilon^\tau \varphi + \ldots + A^{(n)} i \epsilon^\tau \varphi + \text{cosines},
\]

\[
\psi = i (A^{(n)} i \epsilon^\tau \psi - A^{(n)} i \epsilon^\tau \psi) + A^{(n)} i \epsilon^\tau \psi + \text{sines}.
\]

When the symmetric conditions are imposed upon these
solutions, that is

\[ F^{(0)} = \psi^{(0)} = 0 \]

and therefore \( F^{(0)} = \psi^{(0)} = 0 \quad j = 1 \ldots \ldots \ldots \)
it is found that $A_1^{(r)} = A_2^{(r)}$ and $A_3^{(r)} = 0$.

The constant $A_1^{(1)}$ remains arbitrary, but since $p_1$ carries the factor $\lambda$ we may put $A_1^{(1)} = 1$ without loss of generality.

However we shall keep $A_1^{(1)}$ unassigned until the second step has been completed. The complete solutions at the first step are therefore: 

$$
\begin{align*}
\bar{f} &= A_1^{(r)} \left( e^{i\tau} u_1 + e^{-i\tau} u_2 \right) + A_2^{(r)} \lambda u_1 + \cos \theta
\end{align*}
$$

$$
\begin{align*}
\bar{g} &= i A_1^{(r)} \left( e^{i\tau} v_1 - e^{-i\tau} v_2 \right) + A_2^{(r)} \lambda v_1 + \sin \theta
\end{align*}
$$

The Coefficients of $\lambda$

At the second step of the integration another relation between $A_4^{(1)}$ and $\gamma$ will arise from the conditions of periodicity, similar to that found in (14) and from these two equations the constants may be determined.

The differential equations at this step are in the independent variables $p_2$ and $q_2$ and the right members are $P^{(2)}$ and $Q^{(2)}$.

The complementary functions are the same, as at the previous step, except that the parameters are denoted by $A_1^{(s)}$, $A_2^{(s)}$, $A_3^{(s)}$, and $A_4^{(s)}$.

Considering first only the terms in $P^{(2)}$ and $Q^{(2)}$ which enter in the determination of $A_4^{(1)}$ and $\gamma$, we deal only with those terms carrying the factors $e^{i\tau}$ and $e^{-i\tau}$. When $A_4^{(1)}$ and $\gamma$ are found the solutions $p_1$ and $q_1$ can be simplified. Picking out the terms in $f_1$, $g_1$, $f_2$, $g_2$, and $p_2$, we obtain...
we obtain -

\[ x^{(2)} = \mathcal{F}_{ij} \left\{ -\frac{7}{4} l^2 + \frac{7n}{8} \cos^2 2\tau - \frac{9}{4} \gamma^2 \cos 2\tau + \frac{13}{2} l \cos 2\tau - \frac{3}{4} l^3 \cos 2\tau + \frac{8}{3} \gamma \cos 2\tau + \frac{1}{2} l^3 \cos 4\tau \right\} \]

\[ + \phi, \left\{ 2\sqrt{3} \sin 2\tau - \frac{13}{6} \gamma \sin 2\tau + \frac{1}{2} l^2 \cos 2\tau + \frac{1}{2} l^3 \cos 4\tau \right\} \]

\[ + \phi, \left\{ -13 \gamma \sin 2\tau + \frac{1}{2} l^2 \cos 2\tau \right\} \]

\[ + \phi, \left\{ -\frac{2}{5} l^2 + \frac{2}{5} l^3 \cos 2\tau \right\} \]

\[ + \phi, \left\{ -8 \sqrt{3} \sin 2\tau + \frac{19}{8} l^2 \sin 2\tau + \frac{19}{8} l^3 \sin 3\tau \right\}, \]

and

\[ y^{(2)} = -\left[ \mathcal{F}_{ij} \right] \left\{ 2\sqrt{3} \gamma \sin 2\tau - \frac{4}{3} l^2 \sin 3\tau \right\} \]

\[ + \phi, \left\{ -\frac{7}{4} l^2 + 2\gamma - \frac{13}{4} \gamma \sin 2\tau + \frac{7}{8} l \cos 2\tau + \frac{13}{8} l^2 \cos 2\tau \right\} \]

\[ + \phi, \left\{ 4\sqrt{3} \gamma \sin 2\tau - \frac{8}{3} l^2 \sin 3\tau + \frac{26}{3} l^2 \cos 3\tau \right\} \]

\[ + \phi, \left\{ -8\sqrt{3} \gamma \sin 2\tau + \frac{3}{8} l^2 \cos 3\tau \right\} \]

\[ + \phi, \left\{ -3 + \frac{21}{8} l^2 - 12 \gamma \cos 2\tau \right\} \]

where

\[ \mathcal{F}_{ij} = A_{ij} \left( i \omega, -i \omega \right) + A_{i}^{(2)} \gamma \sin \omega + \text{cosines}. \]

\[ q_1 = i A_{i}^{(2)} \left( i \omega, -i \omega \right) + A_{i}^{(2)} \gamma \sin \omega + \text{sines}. \]

Substitution of \( p_1 \) and \( q_1 \) in \( x^{(2)} \) and \( y^{(2)} \) gives

for \( p^{(2)} \), so far as the terms in \( l^{\omega} \) are concerned,

\[ A_{ij} \left[ \left( \frac{13}{2} + \frac{9\sqrt{3}}{6} \right) l^2 + \left( \frac{8}{3} + \frac{9\sqrt{3}}{6} \right) \gamma \right] \cos 2\tau \]

\[ + \left\{ \frac{13}{8} l^2 - \frac{9\sqrt{3}}{6} \gamma \right\} \cos 2\tau \]

\[ + \left\{ 13 \gamma + \frac{21\sqrt{3}}{12} \gamma \sin 2\tau + \frac{21}{12} \gamma \cos 2\tau - 2\sqrt{3} A_{i}^{(2)} \sin 2\tau \right\} \sin 2\tau \]

\[ + \left\{ -\frac{9\sqrt{3}}{12} \gamma \right\} \sin 2\tau \]
and for \( Q^{(2)} \) the terms in \( e^{i\alpha t} \) are

\[
A_{(1)}^{(2)} \left[ (\frac{1}{2} i e - 2 V i + \frac{7}{8} i e^2 V + 6 A_{n}^{(n)} i e^3) \cos t + \left(-\frac{5}{4} i e - \frac{1}{12} e^2 V - 6 A_{n}^{(n)} i e^3 \right) \cos 3t \right. 
+ \left. - \frac{8}{64} A_{n}^{(n)} e^2 \sin 2t - \frac{9}{64} e^2 V \sin 3t. \right]
\]

The terms containing \( e^{-i\alpha t} \) are the conjugates of those in \( e^{i\alpha t} \) and will yield the same relation in \( A_{n}^{(n)} \) and \( V_{r} \). On varying the parameters \( a_{1}^{(2)}, \ldots, a_{\infty}^{(2)} \) we find -

\[
\Delta \dot{a}_{1}^{(2)} = - \dot{e} e^{i\alpha t} \left( P^{(2)} M_{12} + Q^{(2)} M_{14} \right)
\]

\[
\Delta \dot{a}_{2}^{(2)} = e^{i\alpha t} \left( P^{(2)} M_{22} + Q^{(2)} M_{24} \right)
\]

\[
\Delta \dot{a}_{3}^{(2)} = - \left( P^{(2)} M_{32} + Q^{(2)} M_{34} \right)
\]

\[
\Delta \dot{a}_{4}^{(2)} = \left( P^{(2)} M_{42} + Q^{(2)} M_{44} \right). \tag{15}
\]

When the terms in \( P^{(2)} \) and \( Q^{(2)} \) carrying the factor \( e^{i\alpha t} \) are substituted in the \( \dot{a}_{1}^{(2)} \) equation of (15), the exponentials cancel off and constant terms arise. The integration of these terms will render terms in \( \tau \) which in turn will render \( f_{2}^{(2)} \) and \( g_{2}^{(2)} \) non-periodic. Hence these terms must be made to vanish, that is

\[
\frac{1185}{96} A_{n}^{(1)} e + \frac{399}{3584} e V + \left( \frac{399}{3584} + \frac{98563}{3 \times 128^2} \right) e^3 \gamma_{1} + \left( \frac{455}{3584} + \frac{11 \times 18 \times 25}{6 \times 64^2} \right) e^3 \gamma_{2} = - \frac{2}{64}. \tag{16}
\]

The co-factors \( M_{12} \) and \( M_{22} \); and \( M_{14} \) and \( M_{24} \) are conjugates. Similarly it was seen that the terms in \( P^{(2)}, Q^{(2)} \) carrying the factor \( e^{-i\alpha t} \) are conjugates. Therefore it is obvious that an exactly similar relation to (16) in \( A_{n}^{(1)} \) and \( V_{r} \) is obtained from the equation \( \dot{a}_{2}^{(2)} \).

The second relation in \( A_{4}^{(1)} \) and \( V_{r} \) was obtained in equation (14). The solution of these equations then
determines \( A_4^{(l)} \) and \( \gamma \), uniquely.

The solution gives:

\[
\gamma = \frac{1}{\lambda} \left\{ -0.06 - 1.49 l^2 \right\}
\]

and

\[
A_\nu^{(l)} = -0.958 l^2.
\]

Since \( A_\nu^{(l)} \) and \( \gamma \), have been determined the solutions \( p_1 \) and \( q_1 \) can be simplified, as follows:

\[
p_1 = \frac{c}{\Delta} \left[ A_\nu^{(l)} \left\{ 2 \, l \, \cos \sqrt{3} l^2 \, \tau + 0.333 \, l \, \cos (2 - \sqrt{3} l^2) \, \tau 
            - 0.333 \, l \, \cos (2 + \sqrt{3} l^2) \, \tau \right\}
            + 0.0698 \, l^2 + 0.419 \, l^4
            - 0.101 \, l^4 \, \cos 2 \tau + 0.0778 \, l^4 \, \cos 4 \tau 
            + 0.0148 \, l^4 \, \cos 6 \tau - 0.012 \, l^4 \, \cos 8 \tau \right],
\]

\[
q_1 = \frac{c}{\Delta} \left[ A_\nu^{(l)} \left\{ -\left(1 + \frac{3}{16} \, l^2 \right) \sin (1 + \sqrt{3} l^2) \, \tau
            - \left(1 + \frac{3}{16} \, l^2 \right) \sin (\sqrt{3} l^2 - 1) \, \tau
            + \frac{3}{32} \, l^2 \sin (\sqrt{3} l^2 + 3) \, \tau
            - \frac{3}{64} \, l^2 \sin (\sqrt{3} l^2 - 3) \, \tau \right\}
            - 1.135 l^3 + 0.383 l^5 \sin \tau
            + (-0.06 l^3 + 0.403 l^5) \sin 3 \tau
            + (-0.069 l^3 - 0.066 l^5) \sin 5 \tau
            + (-0.027 l^3 - 0.0031 l^5) \sin 7 \tau
            + 0.0078 l^5 \sin 9 \tau \right].
\]
This completes the integration as far as the terms in \( \lambda \) are concerned.

The Second Genus orbits are then found by the expressions \( x = x_0 + \eta_1 \lambda \) and \( y = y_0 + \eta_2 \lambda \) where the calculation has been taken to the first power of \( \lambda \). These give a very close approximation of the orbits since \( \lambda \) is arbitrary and can be made very small.

The solutions of the second genus orbits involve two periods, a multiple of \( \frac{2\pi}{\alpha} \) and a multiple of \( 2\pi \).

In order to obtain this period approximately let the period

\[
P = \eta_1 \left( \frac{2\pi}{\alpha} \right) = \eta_2 (2\pi).
\]

Then \( \frac{\eta_1}{\eta_2} = \alpha = \sqrt{3} e^3 = \frac{17}{10} e^2 \) approximately.

Take \( \eta_1 = 17e^2 \) and \( \eta_2 = 10 \).

Therefore \( P = 20\pi \) or \( 17e^2 \left( \frac{2\pi}{\alpha} \right) \).

The period of the Second Genus orbits is then seen to be in the neighborhood of \( 20\pi \).

VI. A NUMERICAL EXAMPLE OF THE ORBIT.

The following is a numerical example of the second genus orbit. The orbit of one particle will be found but it must be remembered that due to the relations given in (5) the orbit of the other particle is symmetrical to that of the first one with respect to the \( y \)-axis.
Table I gives the solutions of the first genus orbits:

\[ x_0 = 0.625 + 0.25 \cos 2\tau \]
\[ y_0 = 1.15 \sin \tau + 0.018 \sin 3\tau , \]

where \( \ell \) was assigned the value 1.

<table>
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<th>( \tau )</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( \tau )</th>
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<th>( y_0 )</th>
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<td>0</td>
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<td>-0.84</td>
</tr>
<tr>
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<tr>
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<td>( \frac{\tau}{4} )</td>
<td>0.62</td>
<td>0.84</td>
</tr>
<tr>
<td>( \tau )</td>
<td>0.88</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table II gives the solutions of the Second Genus orbits. In obtaining these results \( A'' \), \( \ell \) and \( \lambda \) are assigned the values \( A'' = 1 \); \( \ell = 1 \); \( \lambda = 1 \).

\[ f, \lambda = -0.029 - 0.011 \cos 2\tau - 0.05 \cos 1.73 \tau + 0.011 \cos 3.37 \tau + 0.025 \cos 2\tau - 0.019 \cos 4\tau - 0.04037 \cos 6\tau + 0.003 \cos 7\tau. \]

\[ g, \lambda = 0.03 \sin 1.73 \tau + 0.03 \sin 2.78 \tau - 0.023 \sin 4.73 \tau - 0.011 \sin 1.27 \tau + 0.027 \sin 2\tau - 0.086 \sin 3\tau + 0.031 \sin 4\tau + 0.0073 \sin 7\tau - 0.002 \sin 9\tau. \]
In the illustration the first genus orbit is represented by a red line. The first two oscillations of the second genus orbit are represented by the full line and the final or twentieth oscillation is shown by the dotted line.
REFERENCES


2. Moulton "Periodic Orbits".
