MAJORANT PROBLEMS IN HARMONIC ANALYSIS

by

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ABSTRACT

In various questions of Harmonic analysis we encounter the problem of deriving a norm inequality between a pair of functions when we know a (point wise) inequality between the transforms of these functions. Such problems are known as majorant problems. In this thesis we consider two related problems. First, in Chapter two, we extend the known results on the upper majorant property on compact abelian groups to noncompact locally compact abelian groups. We show, using various test spaces and two notions of majorant, that a Lebesgue space has the upper majorant property exactly when its index is an even integer or infinity. Furthermore, if a Lebesgue space has the lower majorant property, then the Lebesgue space with conjugate index has the upper majorant property.

In the final chapter we consider the second problem. Here we are concerned with deriving global integrability conditions from local integrability conditions for functions which have nonnegative transforms. Such a property holds only in Lebesgue spaces whose index is an even integer or infinity. For Lebesgue spaces whose index is not an even integer or infinity the proof of the failure of this property is based on the failure of the majorant property in these spaces.

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CHAPTER 1

INTRODUCTION

The first section of this chapter is devoted to providing background for the problems we treat in this thesis. The remaining sections are devoted to notations and various preliminaries.

(1.1) Background

Throughout this section G is an infinite compact abelian group whose Haar measure has total mass one. The spaces $L_p(G)$ are formed with respect to this measure.

(1.1.1) Definition: If $f, g \in L_1(G)$ and $|\hat{f}| \leqq \hat{g}$ we say that g majorizes f (or g is a majorant of f).

Let $1 \leq p \leq \infty$. We say $L_p(G)$ has the upper majorant property if there is a positive constant D such that if $f, g \in L_p(G)$ and g majorizes f then $||f||_p \leqq D||g||_p$.

We say $L_p(G)$ has the lower majorant property if there is a positive constant C such that every $f \in L_p(G)$ has a majorant $g \in L_p(G)$ for which $||g||_p \leqq C||f||_p$.

These properties will be abbreviated to UMP and LMP respectively.

(1.1.2) The majorant problem is to determine for which p the space $L_p(G)$ has the UMP or the LMP. This problem was initiated by Hardy and Littlewood [11] and from there evolved through Civin [5], Boas [3], and recently Bachelis [1] and Fournier [7]. The latter two authors extend the results of Hardy and Littlewood and of Boas for the circle group to all infinite compact abelian groups. Bachelis completed the

the results for the circle. We now summarize their results as follows:

(a) If p is an even integer or ∞, then $L_p(G)$ has the UMP with constant 1 .

(b) If p is not an even integer or ∞, then $L_p(G)$ does not have the UMP.

(c) For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the space $L_p(G)$ has the UMP if and only if $L_q(G)$ has the LMP, with the same constant.

(d) $L_1(G)$ has the LMP with constant 1.

(e) Consequently, $L_p(G)$ has the LMP if and only if

$p = 1, 2, \frac{4}{3}, \ldots, \frac{2k}{2k-1}, \ldots\ldots, k \in N$.

We can add a further statement if we note that a continuous function on G which does not belong to $A(G)$ (see (1.3) of this thesis) has no majorant in $L_\infty(G)$. To prove this use [13, (37.4) and (31.42)]. Hence we obtain

(f) $L_\infty(G)$ does not have the LMP.

These six statements can be summarized in the following way. If G is an infinite, compact abelian group, then:

(A) $L_p(G)$ has the UMP if and only if p is an even integer or ∞ ; and when $L_p(G)$ has the UMP the constant is 1 ;

(B) If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $L_p(G)$ has the UMP if and only if $L_q(G)$ has the LMP with the same constant.

The statements of (B) could be called "duality theorems". That $L_p(G)$ has the UMP implies $L_q(G)$ has the LMP was proved, for the circle group, by Hardy and Littlewood and tidied up by Bachelis [1]. This is the more difficult of the two assertions. The other statement in (B) is due to Boas [3].

Possible generalizations to noncompact locally compact abelian groups are given by Civin [5] and Boas [3]. Civin proved dual versions of the results of Hardy and Littlewood, while Boas extended Civin's work and considered various interpretations of the expression "$|\hat{f}| \leqq \hat{g}$" in the definition of majorant. Recently Fournier [8, proof of theorem 3, p. 272] used the fact that $L_{2k}(G)$, $k \in N$ and G noncompact, has the UMP (see chapter 2).

In this thesis we will be concerned with analogues of these results for noncompact locally compact abelian groups. We obtain analogues of (A) and half of (B). Even for the integer group and the real line, these results are new.

(1.1.3) In this section we are concerned only with the circle group T.

A problem related to that of (1.1.2) was initiated by N. Wiener and taken up recently by S. Wainger [20] and H. Shapiro [18]. These results concern the derivation of global "good" behaviour from local "good" behaviour for functions belonging to $L_1(T)$ which have non-negative Fourier coefficients. By "good" behaviour, we mean that the function belongs to $L_p(-\delta,\delta)$ for some p, $1 < p \leq \infty$, and for some $\delta > 0$.

A first step in this direction is the well known result that a positive definite function which is continuous in a neighbourhood of the identity is continuous everywhere (see [13, (32.1) and (32.4)]). Furthermore, if f is continuous then f is positive definite if and only if $\hat{f} \geqq 0$ (see [13, (34.12)]). Now consider general f belonging to $L_1(T)$ with $\hat{f} \geq 0$. A classical result, which may be found in

[6, p. 144], states that if f is also essentially bounded in a neighbourhood of the identity, then f belongs to $L_\infty(T)$ with no increase in norm. Boas [4, p. 242, Theorem 12.6.12] credits Wiener with an analogue of this result which uses L_2 instead of L_∞. From Wiener's result it is not difficult to derive the obvious analogue for L_p when p is an even integer. Indeed, suppose that p is an even integer or ∞ and that $\delta > 0$. Then there is a positive constant C, dependent on at most δ and p, such that if $f \in L_1(T)$, $\hat{f} \geqq 0$ and $f \in L_p(-\delta,\delta)$ then $f \in L_p(T)$ and $||f||_{L_p(T)} \leqq C||f||_{L_p(-\delta,\delta)}$.

It is natural to ask what happens when p is finite and not an even integer. It has been shown by Wainger [20] and Shapiro [18] that the analogous statements fail in these cases. The first examples were provided by Wainger and were phrased in terms of H_p theory. Thus, if $0 < p < 2$ and if $0 < \delta < \pi$, then there is a power series $F(z) = \sum_{n=1}^{\infty} a_n z^n$, analytic in the open unit disc with $a_n \geq 0$ $(n=1,2,\ldots)$ such that

$$\sup_{r\leq 1} \int_{-\delta}^{\delta} |F(re^{i\theta})|^p \, d\theta < \infty ,$$

while

$$\sup_{r<1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta = \infty .$$

It was remarked by Shapiro [18, p. 10] that Wainger's results for $1 \leqq p < 2$ can be recast in terms of two-sided trigonometric series, as follows: there is a trigonometric series with nonnegative coefficients which is not the Fourier series of any function of $L_p(T)$, yet (in the sense of distributions) coincides with an L_p-function in the neighbourhood $[-\pi + \delta, \pi - \delta]$ of the identity.

For finite $p > 2$ which are not even integers Shapiro proved the following (see [18, p. 16]): if $\delta > 0$, then there exists g belonging to $L_1(T)$ which has nonnegative coefficients and which satisfies

(i) $g \in L_p(-\delta,\delta)$;

and (ii) $g \notin L_p(T)$.

In Chapter 3, we shall obtain generalizations of these results on any infinite compact abelian group.

(1.2) Notations

We denote the group of real numbers by R , the circle group by T , and the integer group by Z . The positive integers will be denoted by N . The cyclic group of order r , $r \geq 1$, is denoted by $Z(r)$ and will later be realized in two possible forms: either as the rth roots of unity, a subgroup of T , or as $\{0,1,\ldots, r-1\}$ with addition mod r . $Z(r)^{\omega*}$ is the direct sum of countably many copies of $Z(r)$.

If X is any set, V a subset of X , and f any function on X , then $f|_V$ denotes the restriction of f to V and 1_V denotes the characteristic function of V .

Denote by sgn the signum function given by

$$\operatorname{sgn}(z) = \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

If $1 \leq p \leq \infty$, then q denotes the conjugate index given by $\frac{1}{p} + \frac{1}{q} = 1$.

(1.3) Groups: functions and measures thereon

General references for the following preliminaries are [12], [13], and [15].

Throughout this thesis, G will denote a locally compact abelian group (abbreviated LCAG) whose dual group we denote by $\hat{G}$, the group of all continuous homomorphisms of G into T . If H is a subgroup of G then the annihilator of H , denoted $H^{\perp}$, is $\{\gamma \varepsilon \hat{G} | \gamma(x) = 1 \text{ , for all } x \varepsilon H\}$.

When H is a closed subgroup of G , we have $(G/H)\hat{} = H^{\perp}$ and $\hat{H} = \hat{G}/H^{\perp}$. See [15, p. 96].

G carries a special measure, known as Haar measure, which is translation invariant (also inversion invariant when G is abelian) and is unique up to a positive multiple. All integrals are taken with respect to Haar measure. For G compact, the Haar measure is usually assumed to have total mass 1; for G discrete, Haar measure is generally counting measure. One exception, however, is the case where H is an open subgroup of a compact group G . In this case, the total mass of H is $[G:H]^{-1}$, where $[G:H]$ is the index of H in G .

We use Haar measure to construct $L_p(G)$ $(1 \leq p < \infty)$, the Banach space of functions whose absolute value is pth power integrable, and $L_\infty(G)$, the space of essentially bounded measurable functions. For $f \varepsilon L_p(G)$ $(1 \leq p < \infty)$, we denote its norm by $||f||_p = \{\int_G |f(x)|^p dx\}^{1/p}$, where dx denotes Haar measure on G . For $L_p(G)$ $(1 \leq p < \infty)$, functions are identified if they are equal a.e. with respect to Haar measure;

functions in $L_\infty(G)$ are identified when they are equal locally a.e. with respect to Haar measure.

For $f \in L_1(G)$ we define Fourier and inverse Fourier transforms of f by the formulae

$$\hat{f}(\gamma) = \int_G f(x)\,\overline{\gamma(x)}\,dx$$

$$f^{\vee}(\gamma) = \int_G f(x)\,\gamma(x)\,dx$$

respectively, where $\gamma \in \hat{G}$. For $W \subset L_1(G)$, let $W^{\wedge} = \{\hat{f} \mid f \in W\}$.

We shall use several other spaces of functions (and measures). $C_0(G)$ denotes the Banach space of continuous functions vanishing at ∞. The dual space of $C_0(G)$ is denoted by $M(G)$ and is to be realized as a space of complex regular measures on G (see [12, section 14]).

The space of continuous, positive definite functions on G is denoted by $P(G)$ (see [13, (32.1)]) and $A(G)$ denotes the range of the Fourier transform; thus

$$A(G) = [L_1(\hat{G})]^{\wedge} = \{\hat{f} \mid f \in L_1(\hat{G})\} .$$

It is well known that $A(G) \subset C_0(G)$. The compactly supported members of $A(G)$ form a space denoted $A_c(G)$. A class of functions we shall use often is $L_1(G) \cap A(G)$, which for brevity we denote by $S(G)$.

If, for $i = 1,2$, G_i is an LCAG, then so is $G_1 \times G_2$ and its dual group is $\hat{G}_1 \times \hat{G}_2$. The product of Haar measures on G_1 and G_2

gives a Haar measure on $G_1 \times G_2$. If, for $i = 1,2$, f_i is a function on G_i, we define a function $f_1 \otimes f_2$ on $G_1 \times G_2$ by

$$f_1 \otimes f_2 \,(x,y) = f_1(x)\, f_2(y)$$

for each $(x,y) \in G_1 \times G_2$.

It is easy to see that if $1 \leq p < \infty$ and $f_i \in L_p(G_i)$ $(i=1,2)$, then $f_1 \otimes f_2 \in L_p(G_1 \times G_2)$ and

$$||f_1 \otimes f_2||_p = ||f_1||_p \; ||f_2||_p \quad .$$

If $f_i \in L_1(G_i)$ $(i=1,2)$, then

$$\widehat{f_1 \otimes f_2}\,(\gamma_1,\gamma_2) = \hat{f}_1(\gamma_1)\, \hat{f}_2(\gamma_2) = \hat{f}_1 \otimes \hat{f}_2 \,(\gamma_1,\gamma_2)$$

for each $(\gamma_1,\gamma_2) \in \hat{G}_1 \times \hat{G}_2$ (see [13, (31.7)(b)]). Consequently if $f_i \in S(G_i)$, $i = 1,2$, then $f_1 \otimes f_2 \in S(G_1 \times G_2)$.

(1.4) Preliminaries relating to subgroups and quotient groups

In this section we let H be a closed subgroup of G, although in most cases we shall only be concerned with a compact and open subgroup. We are concerned with two situations: firstly, functions which are supported by H, and secondly functions constant on the cosets of H. These two notions are dual in the sense of the following result.

(1.4.1) Lemma. Let $\mu \in M(G)$. Then μ is supported by H if and only if $\hat{\mu}$ is constant on the cosets of $H^{\perp}$.

For a proof, see [15, p. 118].

(1.4.2) If π is the canonical projection of G onto G/H, then the map $\phi \longmapsto \phi \circ \pi$ is a positivity preserving one - one correspondence between functions on G/H and functions on G which are constant on the cosets of H.

(1.4.3) Suppose that H is an open subgroup of G and let $f_H \in L_1(H)$ be continuous. If we extend f_H to a function f on G by letting $f(x) = 0$ if $x \notin H$, then $f \in L_1(G)$ is continuous and f is supported by H. Thus $\hat{f}$ is constant on the cosets of $H^{\perp}$ and so we can identify $\hat{f}$ with a function on $\hat{G}/H^{\perp} = \hat{H}$. As in [15, p. 122] we identify $\hat{f}$ with $\hat{f}_H$.

(1.4.4) At the root of (1.4.5) and a later calculation is a specialized version of Weil's formula (see [15, p. 70]) which says that if H is a closed subgroup of G, then

$$\int_{G/H}\left\{\int_H f(xy)\,dy\right\}d\dot{x} = \int_G f(x)\,dx$$

for $f \in L_1(G)$; here dy, $d\dot{x}$, dx denote Haar measure on H, G/H, and G respectively.

We want in particular the special case where G is compact and H is open. If f is constant on the cosets of H, let F be the function on G/H for which $F \circ \pi = f$. Then Weil's formula unveiled is

$$\int_{G/H} F(\dot{x})\,d\dot{x} = [G:H]\int_G f(x)\,dx .$$

(1.4.5) By combining (1.4.4) with [15, p. 118], we obtain, for compact open H,

$$A(H) = A(G)|_H = \{f|_H \mid f \varepsilon A(G)\} .$$

(1.5) Some properties of $S(G)$

In this section we list the properties of $S(G) = L_1(G) \cap A(G)$ which we will use later. Unless otherwise stated, all proofs of these statements can be found in [10, chapter 3].

(1.5.1) $S(G)$ can be made into a Banach space via the norm

$$||f||_S = ||f||_1 + ||\hat{f}||_1 .$$

Then the Fourier transform maps $S(G)$ isometrically onto $S(\hat{G})$.

$S(G)$ is an ideal in $M(G)$ and thus also is an ideal in $L_1(G)$. $S(G)$ is dense in $L_p(G)$, $1 \leq p < \infty$.

(1.5.2) The Banach space dual of $S(G)$, denoted by $S^*(G)$, may be regarded as a space of distribtuions on G. If $1 \leq p \leq \infty$, $L_p(G)$ can be embedded in $S^*(G)$, since if $f \varepsilon L_p(G)$ the formula

$$<f,u> = \int_G f(x)\, u(x)dx , \quad u \varepsilon S(G) ,$$

defines a continuous linear functional, denoted by L_f, on $S(G)$. Note that for $1 \leq r \leq \infty$ we have $||u||_r \leq ||u||_S$ when $u \varepsilon S(G)$. Similarly one embeds $M(G)$ in $S^*(G)$ by setting, for $\mu \varepsilon M(G)$,

$$<\mu,u> = \int_G u(x)d\mu(x) , \quad u \varepsilon S(G) .$$

<u>(1.5.3)</u> The Fourier transform for members of $S^*(G)$ is defined in the usual dual manner. If $L \in S^*(G)$ we define $\hat{L} \in S^*(\hat{G})$ by

$$<\hat{L},u> = <L,\hat{u}> , \quad u \in S(\hat{G}) .$$

A similar formula defines inverse transforms.

We now have two ways of defining $\hat{f}$ when $f \in L_p(G)$ and $1 \leq p \leq 2$; one via the Hausdorff-Young theorem (see [13, (31.21)]) and the second via distributions. For $f \in L_p(G)$ and $1 \leq p \leq 2$, these two definitions agree in the sense that $\widehat{L_f} = L_{\hat{f}}$. For $p > 2$ we have, in the noncompact case, only the definition as a distribution. In this case, by combining [10, 3.10] and [9, Chapter 6, Theorem 6.6], we see that there is an $f \in L_p(G)$ for which $\hat{f}$ (that is $\widehat{L_f}$) is not defined by a measure.

<u>(1.5.4)</u> We need to know that $S(G)$ contains functions with nonnegative transforms. This fact is contained in the following result (for a proof, see [13, (33.12)]).

<u>Lemma</u>. $L_1(G)$ contains an approximate identity $\{u_i | i \in I\}$, each member of which belongs to $S(G)$, and satisfies the following:

(a) each u_i belongs to $S(G) \cap P(G)$ and is nonnegative;

(b) for each $i \in I$, $\int_G u_i(x)dx = 1$;

(c) each $\hat{u}_i$ is nonnegative and belongs to $A_c(\hat{G}) \cap P(\hat{G})$;

(d) $\lim \hat{u}_i = 1$ pointwise and uniformly on compact sets.

It follows from [13, (32.33)(b) and (32.48)(a)] that

(e) for $1 \leq p < \infty$ we have $\lim_i ||u_i * f - f||_p = 0$ for $f \in L_p(G)$, and if $f \in C_0(G)$ we have

$$\lim_i ||u_i * f - f||_\infty = 0 .$$

<u>(1.5.5)</u> If $f \in S(G)$, we have $f \in P(G)$ if and only if $\hat{f} \geqq 0$. This is simply [13, (33.3) and (33.10)].

CHAPTER 2

THE MAJORANT PROBLEM

In this chapter we present the main result of the thesis, along with some related topics. The main theorem is stated in section 1 and proved in the succeeding sections. Throughout this chapter, unless otherwise stated, G will be a noncompact LCAG. We are grateful to J. Fournier and M. Cowling for conversations regarding S(G).

(2.1) Definitions and examples.

Since we are dealing with noncompact groups we have problems regarding the Fourier transform on $L_p(G)$ when $p > 2$. Specifically, we want to interpret the inequality

$$|\hat{f}| \leqq \hat{g}$$

when $f, g \in L_p(G)$, $p > 2$. As noted in (1.5.3), $\hat{f}$ need not be a measure and so not a function.

The definitions (1.1.1) were used by Boas and Bachelis, and in [7], Fournier uses a definition involving only trigonometric polynomials. We state this now.

(2.1.1) Definition. Suppose that G is compact. We say $L_p(G)$ has the UMPT if there is a positive constant D such that whenever f, g are trigonometric polynomials (finite linear combinations of characters) and g majorizes f, we have $||f||_p \leqq D||g||_p$.

We say $L_p(G)$ has the LMPT if there is a positive constant C such that every trigonometric polynomial f has a trigonometric polynomial majorant g for which $||g||_p \leqq C||f||_p$.

We show that this definition is equivalent to that given in (1.1.1).

<u>(2.2.2) Lemma</u>. Let G be compact.

a) If $1 \leq p < \infty$, the UMPT as defined in (2.1.1) is equivalent to the UMP as defined in (1.1.1). The constants are the same.

b) If $1 < p < \infty$, the LMPT as defined in (2.1.1) is equivalent to the LMP as defined in (1.1.1).

<u>Proof</u>

a) It is obvious that the definition of (1.1.1) implies that of (2.1.1). Conversely, if $1 \leq p < \infty$ and f , $g \in L_p(G)$ are such that $|\hat{f}| \leqq \hat{g}$, let (u_α) be an approximate identity as in (1.5.4). Then for every α , we have

$$|\widehat{u_\alpha * f}| \leqq \widehat{u_\alpha * g} ;$$

and thus

$$||u_\alpha * f||_p \leqq D||u_\alpha * g||_p ,$$

where D is the constant of the definition. Taking limits we obtain

$$||f||_p \leqq D||g||_p .$$

b) Let (u_α) be as in (1.5.4) and let $f \in L_p(G)$. Then $u_\alpha * f$ is a trigonometric polynomial. Let g_α be a trigonometric polynomial majorizing $u_\alpha * f$ and satisfying

$$||g_\alpha||_p \leqq C||u_\alpha * f||_p \leqq C||f||_p \ .$$

Here we use the fact that $||u_\alpha||_1 = 1$.

Then the net $\{g_\alpha\}$ is norm-bounded in the Banach space $L_p(G)$ and so has a weak * convergent subnet, $\{g_\beta\}$ say, with weak * limit g in $L_p(G)$. Then

$$||g||_p \leqq \underline{\lim_\beta} \, ||g_\beta||_p \leqq C||f||_p \ ,$$

and by weak * convergence we have

$$\hat{g}(\gamma) = \lim_\beta \hat{g}_\beta(\gamma) \geqq \lim_\beta |\hat{u}_\beta(\gamma)\hat{f}(\gamma)| = |\hat{f}(\gamma)|$$

for every $\gamma \in \hat{G}$. Hence $L_p(G)$ has the LMP in the sense of (1.1.1).

Conversely, if f is a trigonometric polynomial, then there is g belonging to $L_p(G)$ which majorizes f and satisfies $||g||_p \leqq C||f||_p$. We show that g can be replaced by a trigonometric polynomial which changes only slightly the constant for the LMP. For any $\varepsilon > 0$ we can (according to [13, (31.37)] find an $h \in L_1(G)$ with the properties

(i) $0 \leqq \hat{h} \leqq 1$;

(ii) $\hat{h}$ is compactly supported and $\hat{h}(\gamma) = 1$ for all γ such that $\hat{f}(\gamma) \neq 0$;

(iii) $||h||_1 < 1 + \varepsilon$.

Then $h * g$ is a trigonometric polynomial majorizing f . Moreover, we have

$$||h*g||_p \leqq ||h||_1 \; ||g||_p < (1+\varepsilon) \; ||g||_p \leqq (1+\varepsilon)C \; ||f||_p \; .$$

This proves the lemma.

It is clear that the LMP implies the LMPT for $L_1(G)$ as well. For the moment, we can not give a direct proof, as in (2.1.2)(a), for the converse of this statement. However, that the converse is true follows from the fact that the LMP and the LMPT hold for $L_1(G)$. The LMP is easy to prove directly, and the LMPT for $L_1(G)$ may be derived from the LMP for $L_1(G)$.

Lemma (2.1.2) suggests that, when G is not compact, we could use a suitable test space rather than all of $L_p(G)$ in our definitions of majorant and majorant properties. When G is compact, the space of trigonometric polynomials coincides with $[A_c(\hat{G})]^{\wedge}$. We could use this test space in our definitions for noncompact G , but we prefer to use $S(G)$ and return to the consideration of other spaces in Section 6. Roughly speaking, any test space contained in $L_1(G)$ and dense in $L_p(G)$ (p finite) will do.

<u>(2.1.3) Definition</u>. If f , $g \in S(G)$ and $|\hat{f}| \leqq \hat{g}$, we shall say that <u>g majorizes f</u> or that <u>g is a majorant of f</u> .

We say $L_p(G)$ has the <u>upper majorant property</u> if there is a positive constant D such that whenever f , $g \in S(G)$ and g majorizes

f , we have $||f||_p \leqq D||g||_p$.

Due to the frequent appearence of this phrase, upper majorant property will be abbreviated to UMP from now on.

(2.1.4) Proposition. If p is an even integer or ∞ then $L_p(G)$ has the UMP with constant 1.

Proof:

First suppose $p = 2$; then the result is obvious by the Parseval formula, since if f , $g \varepsilon S(G)$ and $|\hat{f}| \leqq \hat{g}$ we have

$$||f||_2 = ||\hat{f}||_2 \leqq ||\hat{g}||_2 = ||g||_2 .$$

If $p = 2k$ $(k\varepsilon N)$ then $h^k \varepsilon L_2(G)$ for every $h \varepsilon L_{2k}(G)$. If f , $g \varepsilon S(G)$ and $|\hat{f}| \leqq \hat{g}$, then f^k , $g^k \varepsilon S(G)$ and

$$|\widehat{f^k}| = |\hat{f}*\cdots\cdots*\hat{f}| < \hat{g}*\cdots\cdots*\hat{g} = \widehat{g^k} .$$

There are k $\hat{f}$'s in the first product and k $\hat{g}$'s in the second. From the L_2 case we have

$$||f||_{2k}^{2k} = ||f^k||_2^2 \leqq ||g^k||_2^2 = ||g||_{2k}^{2k} ,$$

that is

$$||f||_{2k} \leqq ||g||_{2k} .$$

If $p = \infty$, we have $S(G) \subset \bigcap_{1 \leq p \leq \infty} L_p(G)$ and the relation

$$||h||_\infty = \lim_{k \to \infty} ||h||_{2k}$$

holds for at least every $h \in L_1 \cap L_\infty$. The result follows from the L_{2k} results.

Our main result is, essentially, that the converse of (2.1.4) is also true. We now state this theorem.

(2.1.5) Main Theorem. Suppose that G is a noncompact LCAG and $1 \leq p < \infty$. If p is not an even integer, then $L_p(G)$ does not have the UMP.

To prove this theorem we must show that for every positive constant D , there exist f , $g \in S(G)$ such that

a) $|\hat{f}| \leqq \hat{g}$

and b) $||f||_p > D||g||_p$.

This will be accomplished in the next four sections.

(2.2) Some Reductions.

Our proof of the main theorem is based on the structure theorem for LCAG's, and in this section we show that it suffices to prove the theorem for certain classes of groups.

(2.2.1) Theorem. Suppose that G_1 and G_2 are LCAG's, and suppose that $L_p(G_1)$ or $L_p(G_2)$ does not have the UMP. Then $L_p(G_1 \times G_2)$ does not have the UMP.

Proof.

For definiteness we assume that $L_p(G_1)$ does not have the UMP. Hence, for any positive constant D there exist f_1 , $g_1 \in S(G_1)$ which satisfy

a) $|\hat{f}_1| \leqq \hat{g}_1$ on $\hat{G}_1$

and b) $||f_1||_p > D||g_1||_p$.

Suppose that $h \in S(G_2)$ is any nontrivial function with $\hat{h} \geq 0$.

Set $F = f_1 \otimes h$, $G = g_1 \otimes h$; then $F, G \in S(G_1 \times G_2)$ and we have

$$|\hat{F}| = |\hat{f}_1 \otimes \hat{h}| = |\hat{f}_1| \otimes \hat{h} \leq \hat{g}_1 \otimes \hat{h} = \hat{G} .$$

Furthermore,

$$||F||_p = ||f_1||_p \; ||h||_p > D||g_1||_p \; ||h||_p = D||G||_p .$$

Thus $L_p(G_1 \times G_2)$ does not have the UMP.

We now recall the structure theorem for LCAG's (see[12, (24.30)]). This theorem states that any LCAG is of the form $R^n \times G_0$, where n is a nonnegative integer and G_0 is an LCAG containing a compact open subgroup. We apply Theorem (2.2.1) to the structure theorem to determine the groups we must consider.

If $R^n \times G_0$ is infinite, then one of the following statements is true:

(a) $n > 0$;

(b) $n = 0$ and G_0 has an infinite compact open subgroup;

(c) $n = 0$ and G_0 is an infinite discrete group.

It therefore suffices to consider only the group R for case (a); in case (b) we will be able to construct examples from those known for the infinite, compact open subgroup of G_0 (see (2.3)).

In case (c) the discrete group G_0 may have an element of infinite order, in which case it contains a copy of Z . Otherwise G_0 is a torsion group. In a torsion group, either we have elements of arbitrarily large order or there is a bound on the orders of all elements. In this latter case, we know that G_0 contains a copy of $Z(r)^{\omega*}$, the direct sum of countably many copies of $Z(r)$, and $r \geq 2$ (see [12, p. 449]).

We now give a further reduction for the discrete case.

<u>(2.2.2) Proposition</u>. Let G be an infinite discrete abelian group containing a subgroup isomorphic to a discrete group H for which $\ell_p(H)$ does not have the UMP. Then $\ell_p(G)$ does not have the UMP.

<u>Proof</u>.

We note that $S(G) = \ell_1(G)$ when G is discrete. Let $\phi: H \to G$ be an embedding of H in G . For any positive constant D there exist $f , g \in \ell_1(H)$ which satisfy $|\hat{f}| \leqq \hat{g}$ and $||f||_p > D||g||_p$.

For any function h on H , we define $h^{\dagger}$ on G by

$$h'(x) = \begin{cases} h(\phi^{-1}(x)) & \text{if } x \in \phi(H) , \\ 0 & \text{otherwise} . \end{cases}$$

It is clear that if $h \in \ell_r(H)$, then $h' \in \ell_r(G)$ and $||h'||_r = ||h||_r$ for any $r \geq 1$. Defining f' and g' similarly we immediately obtain

$$||f'||_p > D||g'||_p .$$

We need only show that $|\widehat{f'}| \leqq \widehat{g'}$. If $h \in \ell_1(H)$, then $h' \in \ell_1(G)$ is supported by $\phi(H)$ and so $\widehat{h'}$ is constant on the cosets of $\phi(H)^{\perp}$. We can identify $\widehat{h'}$ with a function on $\widehat{G}/\phi(H)^{\perp} = \widehat{\phi(H)} = \hat{H}$. By (1.4.3) we may identify $\widehat{h'}$ with $\hat{h}$. Hence we have

$$|\widehat{f'}| = |\hat{f}| \leqq \hat{g} = \widehat{g'} ,$$

and this produces the required example.

It thus suffices in dealing with discrete groups that have elements of infinite order, to consider only the group Z ; likewise for infinite discrete groups in which there is a bound on the orders of the elements, it is enough to consider only the groups $Z(r)^{\omega*}$, for $r \geqq 2$.

We now use this result to summarize the groups or classes of groups for which we must prove the main theorem. They are:

a) R ;

b) Z ;

c) $Z(r)^{\omega*}$, for $r \geqq 2$;

d) G a nondiscrete, noncompact LCAG with a compact open subgroup;

e) G a discrete abelian torsion group with elements of arbitrarily large order.

We shall see that c) and d) may be derived from the compact case. The integer group Z is treated separately, and the final two cases may be derived from the case of the integers.

(2.3) Examples derived from the compact case

In this section we prove the main theorem for LCAG's that have an infinite compact open subgroup or those of the form $Z(r)^{\omega*}$ for some integer $r \geq 2$.

(2.3.1) Theorem. Let G be a nondiscrete noncompact LCAG containing a compact open subgroup. If p is finite and not an even integer, then $L_p(G)$ does not have the UMP.

Proof.

We recall that we must show that for any positive constant D there are $f, g \in S(G)$ which satisfy $|\hat{f}| \leq \hat{g}$ and $||f||_p > D||g||_p$.

Let G_0 be an (infinite) compact open subgroup of G. Haar measure on G can be chosen to assign mass 1 to G_0; since G_0 is open, its Haar measure is the restriction to G_0 of the Haar measure on G.

Let D be any positive constant. From the compact case (see [7]), there are trigonometric polynomials f_1, g_1 satisfying $|\hat{f}_1| \leq \hat{g}_1$ on $\hat{G}_0$ and $||f_1||_p > D||g_1||_p$. In particular f_1 and g_1 belong to $A(G_0)$.

By (1.4.5), if $h_1 \varepsilon A(G_0)$ there is an $h \varepsilon A(G)$ whose restriction to G_0 is h_1. Since G_0 is compact and open, 1_{G_0} belongs to $A(G)$ (see [13, (31.7(i))]) and so we can extend h_1 to G by defining it to be zero off G_0. In particular, we can extend a function h_1 in $S(G_0)$ to a function in $S(G)$ by letting its extension h be zero off G_0; moreover, it is clear that $||h_1||_p = ||h||_p$.

Let f and g be such extensions to G of f_1 and g_1 respectively. Then it is immediate that $||f||_p > D||g||_p$ and so we need only show that $|\hat{f}| \leq \hat{g}$ on $\hat{G}$.

Since f and g are supported by G_0, so their transforms are constant on the cosets of $G_0^{\perp}$ and hence can be identified with functions on $\hat{G}/G_0^{\perp} = \hat{G}_0$. But by (1.4.3) these latter functions are just $\hat{f}_1$ and $\hat{g}_1$ respectively. Thus we have

$$|\hat{f}| = |\hat{f}_1 \circ \pi| \leq \hat{g}_1 \circ \pi = \hat{g}$$

where π denotes the canonical projection of $\hat{G}$ on $G_0^{\perp}$. This concludes the proof.

We now turn our attention to the groups $G = Z(r)^{\omega^*}$, $r \geq 2$. In this case we will be able to derive our examples from those of the compact group $X = Z(r)^{\omega}$, the direct product of countably many copies of $Z(r)$. We note that for each positive integer N, each of X and G contains a subgroup isomorphic to $Z(r)^N$ which are denoted by X_N and G_N respectively. Thus,

$$X_N = \{(x_j) \in X \mid x_j = 0 \text{ if } j \geq N+1\}$$

$$G_N = \{(y_j) \in G \mid y_j = 0 \text{ if } j \geq N+1\} .$$

These groups are self-dual.

(2.3.2) Theorem. If $G = Z(r)^{\omega^*}$ $(r \geq 2)$ and p is finite and not an even integer, then $L_p(G)$ does not have the UMP.

Proof.

Let D be a positive constant. From the known results for X we can find trigonometric polynomials f, g on X which satisfy $|\hat{f}| \leqq \hat{g}$ on G and $||f||_p > D||g||_p$. As $\hat{f}$ and $\hat{g}$ are finitely supported, there is a positive integer N for which

$$\mathrm{supp}(\hat{f}) \cup \mathrm{supp}(\hat{g}) \subset G_N ,$$

where $\mathrm{supp}(\hat{f}) = \{y \in G \mid \hat{f}(y) \neq 0\}$. Thus the transforms of $\hat{f}$ and $\hat{g}$, and hence also their reflections f and g respectively, must be constant on the cosets of $G_N^{\perp}$ in X. We can now identify f and g with functions on $X/G_N^{\perp} = \hat{G}_N = G_N$. It is this pair of functions on G_N (that is on G, but supported by G_N) that will provide our example.

Before proceeding, we should summarize the main identifications used above. We start with a pair of functions f, g on X, constant on the cosets of $G_N^{\perp}$. This gives rise to a pair of functions F, G on $X/G_N^{\perp} = \widehat{G_N}$. Finally, via a (topological) isomorphism, we obtain a pair F^{-1}, G^{-1} on G_N. It is F^{-1}, G^{-1} which will provide our example, and so we will need to check that the following two conditions are satisfied:

a) $|\widehat{F'}| \leq \widehat{G'}$ on X ,

and b) $||F'||_p > D||G'||_p$.

Since G_N is compact, F' , G' belong to $\ell_1(G) = S(G)$. To obtain a), note first that the final paragraph of the proof of (2.3.1) shows that $|\hat{F}| \leqq \hat{G}$ since $|\hat{f}| \leq \hat{g}$. Passing to $\widehat{F'}$ and $\widehat{G'}$ involves a positivity preserving linear map and so we also have $|\widehat{F'}| \leqq \widehat{G'}$. Hence a) holds.

To show that b) holds, we trace the behaviour of the $L_p(X)$ norm of a function h on X which is constant on the cosets of $G_N^{\perp}$. In the first stage we obtain an H on $X/G_N^{\perp}$ for which $h = H \circ \pi$, π being the canonical projection of X on $X/G_N^{\perp}$. From (1.4.4) we have

$$\int_X h(x)\,dx = r^{-N} \int_{X/G_N} H(\dot{x})\,d\dot{x} ,$$

since $[X: G_N^{\perp}] = r^N$. But we also have $|h|^p = |H|^p \circ \pi$; thus if $h \in L_p(X)$, we have $H \in L_p(X/G_N^{\perp})$ with

$$||h||_p = r^{-\frac{N}{p}} ||H||_p . \qquad (1)$$

In the second stage, H^{1} is obtained from H via a topological isomorphism of the underlying groups. As the image of a Haar measure under such a map is also a Haar measure, it follows that there is a positive constant c for which

$$||H^{1}||_p = c^{1/p} ||H||_p , \qquad (2)$$

for every $H \in L_p(G_N)$.

Using (1) and (2) we now see that b) holds:

$$\begin{aligned} ||F'||_p^p &= c||F||_p^p \\ &= cr^N||f||_p^p \\ &> cr^N D^p||g||_p^p = D^p||G'||_p^p \ . \end{aligned}$$

Hence $D||G'||_p < ||F'||_p$ and this completes the proof.

(2.4) The case of the integers

As in the previous cases, our goal is to show if p is not an even integer or ∞ , then for any positive D there are functions f , g belonging to $\ell_1(Z)$ which satisfy $|\hat{f}| \leqq \hat{g}$ on T and $D||g||_p < ||f||_p$.

Our method, though different in detail, is essentially the same as that used by Bachelis [1] and Fournier [7]. For a discussion of the origins of this method see Shapiro [18]. First note that the constant D in our definitions of the UMP (see (1.1.1) and (2.1.3)) must be at least 1, since $S(G)$ contains nontrivial functions with nonnegative transforms. For the group T , Bachelis proves (1.1.2)(b) by using a suggestion of Y. Katznelson to show, by an iteration method, that if the UMP fails to hold with $D = 1$, then it fails to hold at all (see [1, p.121]).

We now give the iteration method; it is the dualized version of a special case of Fejér's Lemma (see [1, p.121], and [21, p.49]).

<u>(2.4.1) Proposition</u>. Let $1 \leq p < \infty$ and suppose α is a finitely supported sequence on Z. For a positive integer n we define α_n by

$$\alpha_n(m) = \begin{cases} \alpha(\frac{m}{n}) & \text{if } m \in nZ \\ 0 & \text{otherwise.} \end{cases}$$

Then for large n we have

$$||\alpha * \alpha_n||_p = ||\alpha||_p^2 ,$$

and in particular,

$$\lim_{\ell \to \infty} ||\alpha * \alpha_\ell||_p = ||\alpha||_p^2 .$$

<u>Proof</u>. Note that $\alpha_n \in c_c(Z)$ and thus $\alpha * \alpha_n \in c_c(Z)$. We use induction on the number of elements ℓ in the support of α. For $\ell = 1$, the result is obvious. Suppose now that $\ell \geq 2$ and that $||\theta * \theta_n||_p = ||\theta||_p^2$ for large n whenever $\theta \in c_c(Z)$ and the support of θ has ℓ-1 members. Write δ_z for $1_{\{z\}}$. Then $(\delta_z)_n = \delta_{nz}$ for every n, $z \in Z$.

Let $\alpha \in c_c(Z)$ have ℓ elements in its support and write

$$\alpha = \sum_{j=1}^{\ell} \alpha_j \, \delta_{k_j} .$$

If $\beta = \sum_{j=1}^{\ell-1} \alpha_j \delta_{k_j}$, then we have

$$\alpha * \alpha_n = (\beta + \alpha_\ell \, \delta_{k_\ell}) * (\beta_n + \alpha_\ell \, \delta_{nk_\ell})$$

$$= \beta * \beta_n + \alpha_\ell(\beta_n * \delta_{k_\ell} + \beta * \delta_{nk_\ell}) + \alpha_\ell^2 \, \delta_{(n+1)k_\ell} ,$$

since $\delta_z * \delta_u = \delta_{z+u}$ whenever $z , u \in z$.

The supports of these four terms are pairwise disjoint when n is sufficiently large, and so

$$||\alpha * \alpha_n||_p^p = ||\beta * \beta_n||_p^p + |\alpha_\ell|^p (||\beta_n * \delta_{k_\ell}||_p^p + ||\beta * \delta_{nk_\ell}||_p^p) + |\alpha_\ell|^{2p} , \quad (1)$$

since $||\delta_z||_p = 1$ for every z .

Now

$$\delta_{k_\ell} * \beta_n = \sum_{j=1}^{\ell-1} \alpha_j \, \delta_{k_\ell} * \delta_{nk_j} = \sum_{j=1}^{\ell-1} \alpha_j \, \delta_{nk_j + k_\ell} ;$$

since all the $nk_j + k_\ell (1 \le j \le \ell-1)$ are distinct, we have

$$||\delta_{k_\ell} * \beta_n||_p^p = \sum_{j=1}^{\ell-1} |\alpha_j|^p = ||\beta||_p^p .$$

$$\text{Similarly,} \quad ||\delta_{nk_\ell} * \beta||_p^p = ||\beta||_p^p . \qquad (2)$$

Combining (1) and (2) we obtain

$$||\alpha * \alpha_n||_p^p = ||\beta * \beta_n||_p^p + 2|\alpha_\ell|^p \; ||\beta||_p^p + (|\alpha_\ell|^p)^2 , \quad \text{for large } n$$

$$= (||\beta||_p^p + |\alpha_\ell|^p)^2$$

$$= ||\alpha||_p^{2p}.$$

Taking pth roots we obtain the result.

<u>(2.4.2) Corollary</u>. Let $1 \leq p < \infty$ be an exponent for which there is a pair of finitely supported functions f , g on Z satisfying $|\hat{f}| \leq \hat{g}$ and $||g||_p < ||f||_p$. Then $\ell_p(Z)$ fails to have the UMP.

<u>Proof</u>.

Since $||g||_p < ||f||_p$, there is a constant $C > 1$ for which

$$C||g||_p < ||f||_p \ .$$

Form f_n and g_n as defined in the statement of (2.4.1). We note that $\hat{f}_n(x) = \hat{f}(nx)$ since

$$\hat{f}_n(x) = \sum_{m=-\infty}^{\infty} f_n(m)\, e^{-imx} = \sum_{\ell=-\infty}^{\infty} f(\ell)\, e^{-in\ell x} = \hat{f}(nx) \ .$$

Similarly for g_n . Since $|\hat{f}| \leq \hat{g}$, we have

$$\begin{aligned} |\widehat{f*f_n}(x)| &\leq |\hat{f}(x)|\ |\hat{f}_n(x)| \\ &= |\hat{f}(x)|\ |\hat{f}(nx)| \\ &\leq \hat{g}(x)\ \hat{g}(nx) \\ &= \widehat{g * g_n}(x) \ . \end{aligned}$$

Thus $g*g_n$ majorizes $f * f_n$ when g majorizes f , for all positive integers n . Applying (2.4.1) to each sequence $g*g_n$ and $f * f_n$, and noting that $C^2||g||_p^2 < ||f||_p^2$, it follows that there is an n for which

$$C^2 ||g*g_n||_p < ||f*f_n||_p \ .$$

For any positive constant D , there is a positive integer ℓ for which $D \leqq C^{\ell}$. Iteration of the above procedure shows that there is a pair of finitely supported functions d , e on Z with e majorizing d and $C^{\ell}||e||_p < ||d||_p$. Hence $D||e||_p < ||d||_p$ and thus $\ell_p(Z)$ does not have the UMP .

To complete the proof of the main theorem for the integer group, we must, for a given finite p which is not an even integer, find a pair of finitely supported functions on Z which satisfy the hypotheses of Corollary (2.4.2). We require some preliminaries about the functions which operate on $P_r(Z)$, the real-valued members of $P(Z)$.

Recall that if Δ is a subset of the complex plane C and $F : \Delta \to C$, then we say <u>F operates on $P(Z)$</u> if $F \circ \phi \in P(Z)$ whenever $\phi \in P(Z)$ and range$(\phi) \subseteq \Delta$. If $\Delta = (-1,1)$ and F is real-valued, then Rudin [16] has shown that F must be of the form

$$\left.\begin{aligned} & F(x) = \sum_{n=0}^{\infty} c_n x^n \ , \text{ for } |x| < 1 \\ \text{and} \qquad & c_n \geqq 0 \ \text{ for } n = 0, 1, 2, \ldots . \end{aligned}\right\} \qquad (3)$$

A fore-runner of this result can be found in Schoenberg [17].

Our next result is stated and proved only for the case at hand. It will be clear that a more general formulation is possible. Its proof requires an elementary application of the convergence theorem for sequences of positive definite functions (see [2, p. 17]).

(2.4.3) Lemma. Let $F : (-1,1) \to R$ be continuous and have the property that $F \circ \psi \in P_r(Z)$ whenever ψ is a finitely supported member of $P_r(Z)$ with $\text{range}(\psi) \subseteq (-1,1)$. Then F operates on $P_r(Z)$.

Proof.

Let $\psi \in P_r(Z)$ have $\text{range}(\psi) \subseteq (-1,1)$ and let (K_N) be the Fejér kernel in $L_1(T)$. It is well known that $\hat{K}_N$ belongs to $P_r(Z)$ and is finitely supported for every N . Moreover, we have $0 \leq \hat{K}_N \leq 1$ and so $\text{range}(\hat{K}_N\psi) \subseteq (-1,1)$ if $\text{range}(\psi) \subseteq (-1,1)$. Thus $F \circ (\hat{K}_N\psi)$ belongs to $P_r(Z)$. Since the (K_N) are an approximate identity for $L_1(T)$, (1.5.4) shows that $\psi = \lim_N \hat{K}_N\psi$ pointwise and so $F \circ \psi = \lim_N F \circ (\hat{K}_N\psi)$ belongs to $P_r(Z)$. Thus F operates on $P_r(Z)$.

(2.4.4) Proposition. If p is not an even integer or ∞ , there exists a function ϕ belonging to $P_r(Z)$ which is finitely supported and for which $|\phi|^p$ does not belong to $P(Z)$.

Proof.

Let $H : (-1,1) \to R$ be defined by $H(x) = |x|^p$. Then it is easy to see that H is not of the form (3) (unless p is an even integer) and so H does not operate on $P_r(Z)$. As H is continuous, we can apply (2.4.3) to conclude that there is a finitely supported $\phi \in P_r(Z)$ for which $\text{range}(\phi) \subseteq (-1,1)$ and $H \circ \phi \notin P(Z)$; that is, $|\phi|^p \notin P(Z)$.

We can now present the necessary examples. The germ of this method is to be found, in a disguised form, in [7, p. 163].

(2.4.5) Theorem. Suppose that $p \geq 1$ is not an even integer or ∞. Then $\ell_p(Z)$ does not have the UMP with constant 1.

Proof.

We want to find finitely supported functions f, g on Z which satisfy

a) $|\hat{f}| \leqq \hat{g}$

and b) $||g||_p < ||f||_p$.

Let ϕ be as in (2.4.4). Positive definite functions are self-adjoint and the real ones are symmetric ($\phi(-n) = \phi(n)$ for every $n \in Z$). This implies that $|\phi|^p$ is self-adjoint and so has a real-valued Fourier transform. As $|\phi|^p \notin P_r(Z)$, there is an $x \in T$ for which $(|\phi|^p)^\wedge(x) < 0$, that is,

$$\sum_{\ell=-\infty}^{\infty} |\phi(\ell)|^p e^{-i\ell x} < 0$$

(see (1.5.5)).

Define $\gamma \in \hat{Z}$ by $\gamma(\ell) = e^{i\ell x}$, for every $\ell \in Z$. Then for a real parameter t we define functions f_t on Z by

$$f_t = (1+t\gamma)\,\phi\,,$$

where 1 stands for the constant function with value 1. Note that each f_t is finitely supported and $f_t \in P(Z)$ if $t \geqq 0$. It is clear that if t is nonnegative, we have

$$|\widehat{f_{-t}}| \leqq \widehat{f_t}\,.$$

It thus suffices to find a positive t_0 for which

$$||f_{t_0}||_p < ||f_{-t_0}||_p \ .$$

In other words, for positive t we set $f = f_{-t}$, $g = f_t$ and show that b) holds for some positive t_0 ;

To this end we define a function F on R by

$$F(t) = ||f_t||_p^p$$

$$= \sum_{\ell=-\infty}^{\infty} |\phi(\ell)|^p \ |1+t\gamma(\ell)|^p \ .$$

Notice that, for every ℓ , the function $t \mapsto |1+t\gamma(\ell)|^p$ is differentiable at 0 . Since ϕ is compactly supported, the sum defining $F(t)$ is really finite; hence we have:

$$F'(0) = \sum_{\ell=-\infty}^{\infty} \frac{d}{dt} [|\phi(\ell)|^p \ |1+t\gamma(\ell)|^p]_{t=0}$$

$$= p \sum_{\ell=-\infty}^{\infty} |\phi(\ell)|^p \frac{d}{dt} [|1+t\gamma(\ell)|^p]_{t=0}$$

$$= p \sum_{\ell=-\infty}^{\infty} |\phi(\ell)|^p \ \mathrm{Re}(\overline{\gamma(\ell)})$$

$$= p \ \mathrm{Re}\left(\sum_{\ell=-\infty}^{\infty} |\phi(\ell)|^p \ \overline{\gamma(\ell)}\right)$$

$$= p \ \widehat{|\phi|^p} \ (x) \ .$$

Thus $F'(0) < 0$, and thus there must be a positive t_0 for which

$$F(t_o) < F(-t_0) .$$

Taking pth roots we obtain

$$||f_{t_0}||_p < ||f_{-t_0}||_p ,$$

and b) follows. This completes the proof.

(2.4.6) In this section we sketch an alternative proof of (2.4.5) for p a rational number but not an even integer. This method, except for a minor modification due to Z not being compact, is directly analogous to the example first produced by Hardy and Littlewood for $L_3(T)$ (see [11, p. 305] and [3, p. 255]).

Suppose that $1 \leq p < \infty$ and that p is not an even integer. For each positive integer n, let c_n be the binomial coefficient $\dfrac{\frac{p}{2}(\frac{p}{2}-1)\cdots\cdots(\frac{p}{2}-n+1)}{n!}$. Let k be the least positive integer n for which $c_n < 0$. We need a function ϕ in $\ell_1(Z)$ with the following properties:

(a) $\phi = \hat{g}$, for some nontrivial, nonnegative C^∞ function g on T;

(b) for an integer L, to be specified later, the functions $\{\phi^\ell | 1 \leq \ell \leq L\}$ are mutually orthogonal in $\ell_2(Z)$. Note that these products are defined by pointwise multiplication.

To obtain such a function ϕ we need a nontrivial, nonnegative C^∞ function g on T for which the convolution powers $g^{*\ell} (1 \leq \ell \leq L)$

are mutually orthogonal in $L_2(T)$. Since these convolution powers are nonnegative they are orthogonal if and only if their supports are disjoint. Suppose g is supported by a small interval $[a,b]$ with $0 < a < b < 2\pi$. Then the convolution powers $g^{*\ell}(1 \le \ell \le L)$ will have disjoint supports provided that b is sufficiently small and a is sufficiently close to b.

Now let $\psi = \phi^{\frac{2}{p}}$. Since ϕ is the Fourier transform of a C^∞ function, the same is true of ψ ; in particular, ψ belongs to $\ell_1(Z)$. Let λ be a real number with $|\lambda| \leqq 1$ and let t be a positive parameter. We define a function f_λ by

$$f_\lambda = \psi(1 + t\phi + \lambda t^k \phi^k) ,$$

where 1 is the constant function whose sole value is 1 . Then f_λ belongs to $\ell_1(Z)$.

We first examine the $\ell_p(Z)$-norm of f_λ . We have $||f_\lambda||_p^p = ||f_\lambda^{p/2}||_2^2$ and, it t is small enough, we can apply the binomial theorem to $(1 + t\phi + \lambda t^k \phi^k)^{p/2}$. When we do this we obtain an absolutely convergent double series (in $\ell_2(Z)$) and thus

$$f_\lambda^{p/2} = \psi^{p/2}(1 + t\phi + \lambda t^k \phi^k)^{p/2}$$
$$= \phi \sum_{\ell=0}^{\infty} \beta_\ell (t\phi)^\ell ,$$

where $\beta_\ell = C_\ell$ for $0 \le \ell \le k - 1$ and $\beta_k = C_1\lambda + C_k$. If we write

$$f_\lambda^{p/2} = \phi \{ \sum_{\ell=0}^{2k} \beta_\ell (t\phi)^\ell + o(t^{2k+1})\} \quad ,$$

then using (b), with $L = 2k+1$, we obtain

$$||f_\lambda||_p^p = ||f_\lambda^{p/2}||_2^2 = \sum_{\ell=0}^{k} |\beta_\ell|^2 \, ||\phi^{\ell+1}||_2^2 \, t^{2\ell} + o(t^{2k}) \, .$$

Our choice of k shows that

$$||f_1||_p^p < ||f_{-1}||_p^p$$

if t is sufficiently small. This takes care of the norm inequality.

We would like to know if f_1 majorizes f_{-1} . This would certainly be the case if ψ were positive definite. For general p the author has had no luck in determining if we can assume that $\psi = \phi^{2/p}$ is positive definite in addition to conditions (a) and (b). However, it is quite easily done when p is rational. For suppose that $p = \frac{m}{\ell}$; then $\frac{2}{p} = \frac{2\ell}{m}$. If we let $\psi = \hat{g}^{2\ell}$ and $\phi = \hat{g}^m$, then $\psi^{p/2} = \phi$, ψ is positive definite, and ψ belongs to $\ell_1(Z)$. Hence f_1 is a majorant of f_{-1} . The only change required for the norm computation is to replace L by mL in condition (b).

(2.5) Examples derived from the integer group

This section gives the proof of the main theorem for the remaining two classes of groups, namely R and discrete torsion groups with elements of arbitrarily large orders. In both cases we derive these results from the corresponding results for the integers.

We first deal with R . A slightly modified form of a device due to de Leeuw (see [14, p. 375]) is required. The proof is the same as his and so is omitted. Before stating this lemma, we remark that finitely supported functions on Z can be identified with finite sums of point masses on R via $\Sigma\, \alpha_\ell\, 1_{\{\ell\}} \leftrightarrow \Sigma\, \alpha_\ell\, \delta_\ell$, where δ_ℓ is the unit point mass at ℓ on R .

<u>(2.5.1) Lemma</u>. Let $1 \leq p < \infty$. Let $\phi \in A_c(R)$ have the following properties:

(1) $\phi \geq 0$;

(2) $\hat{\phi} \geq 0$;

(3) $\int \phi^p(x)dx = 1$;

(4) $\mathrm{supp}(\phi) = \overline{\{x \in R \mid \phi(x) \neq 0\}} \subset [a, a+1]$ for some real number a .

Then for any finitely supported $\hat{\sigma}$ on Z (identified with the corresponding discrete measure on R) , we have $\sigma * \phi \in S(R)$ and $||\sigma*\phi||_p = ||\sigma||_p$.

Since any interval of length one will work in (4), an example of such a ϕ is, except for the normalizing factor required for (3),

$$\phi(x) = \begin{cases} 1 - 2|x| & |x| \leq 1/2 \\ 0 & |x| \geq 1/2 . \end{cases}$$

<u>(2.5.2) Theorem.</u> Suppose that $p \geq 1$ is not an even integer or ∞. Then $L_p(R)$ does not have the UMP.

<u>Proof.</u>

Let D be any positive constant. In the proof of (2.4.5) we found finitely supported λ, μ on Z which satisfy

(a) $|\hat{\lambda}| \leqq \hat{\mu}$ on T, hence on R

and (b) $D||\mu||_p < ||\lambda||_p$.

Let ϕ be as in (2.5.1) and set $f = \lambda*\phi$, $g = \mu*\phi$. Then f and g belong to $S(R)$ since $S(R)$ is an ideal in $M(R)$ (see (1.5.1)). By (a) we obtain

$$|\hat{f}| = |\hat{\lambda}|\hat{\phi} \leqq \hat{\mu}\hat{\phi} = \hat{g} ,$$

and from (2.5.1) and (b) we have

$$||f||_p = ||\lambda*\phi||_p = ||\lambda||_p > D||\mu||_p = D||\mu*\phi||_p = D||g||_p .$$

This completes the proof.

<u>(2.5.3) Theorem.</u> Let G be a discrete abelian group containing elements of arbitrarily large order. Suppose that $p \geq 1$ is not an even integer or ∞. Then $\ell_p(G)$ does not have the UMP.

<u>Proof.</u>

Let D denote an arbitrary positive constant. We know that there are finitely supported functions f' and g' on Z which satisfy $|\widehat{f'}| \leqq \widehat{g'}$ on T and $D||g'||_p < ||f'||_p$.

For a positive integer n, let $S_n = \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$. Then there is a positive integer n for which

$$\mathrm{supp}(f') \cup \mathrm{supp}(g') \subset S_n .$$

S_n has $2n+1$ members and by hypothesis G contains a cyclic group H with at least $2n+1$ members. Suppose that H has r elements, $r \geq 2n+1$, and identify H with the subgroup $\{1, \xi, \xi^2, \ldots, \xi^{r-1}\}$ of T, where $\xi = \exp(\frac{2\pi i}{r})$.

Define a map $\rho: Z \to G$ by $\rho(k) = \xi^k$; then ρ is a continuous homomorphism with image contained in H. It is easy to see that $\rho|_{S_n}$ is injective. We define functions on G as follows:

$$\text{set} \quad f(x) = \begin{cases} f'(j) & \text{if } x = \rho(j) \text{ where } j \in S_n \\ 0 & \text{if } x \notin \rho(S_n) \, ; \end{cases}$$

define similarly g in terms of g'.

As f, g are finitely supported, they belong to $S(G)$; since $\rho|_{S_n}$ is injective we have

$$||f'||_p = ||f||_p \quad \text{and} \quad ||g'||_p = ||g||_p .$$

Thus $D||g||_p < ||f||_p$.

Since f, g are supported by H, we can identify $\hat{f}$ and $\hat{g}$ with functions on $\hat{G}/H^{\perp} = \hat{H} = H$ regarded as a subgroup of T. As in previous cases, we are really identifying $\hat{f}$ with $\widehat{f'}|_H$, since if $\gamma \in \hat{H}$

and γ corresponds to ξ^k $(0\le k\le r-1)$, then

$$\hat{f}(\gamma) = \sum_{x\varepsilon\rho(S_n)} f(x)\,\overline{\gamma(x)} = \sum_{|j|\le n} f'(j)\exp(-2\pi i\tfrac{jk}{r})$$

$$= \widehat{f'}\,(\xi^k)\ .$$

A similar relation holds for $\hat{g}$ and $\widehat{g'}$, so we have

$$|\hat{f}| = |\widehat{f'}|_{\hat H}| \leqq \widehat{g'}|_{\hat H} = \hat{g}\ .$$

This completes the proof.

We note now that by combining (2.3.1), (2.3.2), (2.4.5), (2.5.2) and (2.5.3), we obtain the main theorem (2.1.5).

(2.6) Miscellany

As mentioned in (2.1), other dense subspaces of $L_p(G)$ (p finite) have equal claims to consideration in the definition of the UMP. Obvious candidates are $L_1(G) \cap L_\infty(G)$, $L_1(G) \cap L_p(G)$, $A_c(G)$, and $[A_c(\hat{G})]^\wedge = \{f\varepsilon L_1(G)/\hat{f}$ has compact support$\}$. The latter space will be denoted by $S_c(G)$.

Our first aim is to show that neither the positive results (2.1.4) nor the negative results (2.1.5) are affected by using any of the above four test spaces instead of $S(G)$ in the definition of the UMP. Before doing this, we should note that $L_1 \cap L_\infty$ cannot give us anything new. For if f , $g \varepsilon L_1 \cap L_\infty$ with $|\hat{f}| \leqq \hat{g}$, we have $\hat{g} \geq 0$ and so $\hat{g} \varepsilon L_1(\hat{G})$, by [13, (31.42)]. Hence also $\hat{f}$ belongs to $L_1(\hat{G})$. The inversion theorem now says that f and g are equal a.e. to functions in $S(G)$.

<u>(2.6.1) Lemma.</u> Let $1 \leq p \leq \infty$ and suppose that $L_p(G)$ has the UMP as defined in (2.1.3). Then $L_p(G)$ also has the UMP when either $S_c(G)$, $A_c(G)$, $L_1 \cap L_\infty$ or $L_1 \cap L_p$ is used instead of $S(G)$ in the definition of the UMP.

<u>Proof</u>.

This is obvious for $S_c(G)$ and $A_c(G)$, since each is contained in $S(G)$. Suppose that $L_p(G)$ has the UMP as defined in (2.1.3) and let f , $g \in L_1 \cap L_p$. Let $(\tilde{u}_\alpha)$ be an approximate identity for $L_1(G)$ as in (1.5.4). Then $u_\alpha * f$, $u_\alpha * g \in S(G)$ for every α and we have

$$|\hat{u}_\alpha \hat{f}| = \hat{u}_\alpha |\hat{f}| \leqq \hat{u}_\alpha \ \hat{g} \ ;$$

thus

$$||u_\alpha * f||_p \leqq D||u_\alpha * g||_p \quad ,$$

D being the constant of the definition. Taking limits, we obtain

$$||f||_p \leqq D||g||_p \ .$$

When $p < \infty$ a similar argument shows that we can also replace $S(G)$ by $L_1 \cap L_\infty$.

(2.6.2) Lemma: Suppose that $1 \leq p < \infty$ and that $L_p(G)$ fails to have the UMP as defined in (2.1.3). Then it also fails to have the UMP when $S(G)$ is replaced by either $L_1 \cap L_\infty$, $L_1 \cap L_p$, $A_c(G)$, or $S_c(G)$ in the definition.

Proof.

Since $L_1 \cap L_\infty$ and $L_1 \cap L_p$ contain $S(G)$, the statement for these two cases follows immediately from (2.1.5). In the other two cases it is enough to show that our examples can always be assumed either to be compactly supported or to have compactly supported transforms.

We first deal with $S_c(G)$. Let (u_α) be an approximate identity for $L_1(G)$ as in (1.5.4). Each $\hat{u}_\alpha$ is compactly supported. Now suppose that D is an arbitrary positive constant and let f , $g \in S(G)$ satisfy $|\hat{f}| \leqq \hat{g}$ and $D||g||_p < ||f||_p$. We obviously have $|\hat{u}_\alpha \hat{f}| \leqq \hat{u}_\alpha \hat{g}$ for every α and there is an index β for which $D||u_\beta * g||_p \leqq ||u_\beta * f||_p$, by (1.5.4)(e). Replacing f and g by $u_\beta * f$ and $u_\beta * g$ proves the lemma for $S_c(G)$.

We now consider the case of $A_c(G)$. As is to be expected, we simply dualize the argument just given for $S_c(G)$. Let (v_i) be an approximate identity for $L_1(\hat{G})$, with the properties listed in (1.5.4). Now $\hat{v}_i \to 1$ uniformly on compact sets so that $||\hat{v}_i h - h||_p \to 0$ $(1 \leqq p < \infty)$ for every $h \in L_p(G)$. In particular we have $||\hat{v}_i h||_p \to ||h||_p$. Now let D be an arbitrary positive constant and let f , $g \in S(G)$ satisfy $|\hat{f}| \leqq \hat{g}$ and $D||g||_p < ||f||_p$. Then there is an index j for which

$$D||\hat{v}_j g||_p < ||\hat{v}_j f||_p .$$

We also have

$$|(\hat{v}_j f)^\wedge| = |v_j * \hat{f}| \leqq v_j * |\hat{f}| \leqq v_j * \hat{g} = (\hat{v}_j g)^\wedge \quad .$$

Thus if we replace f and g by $\hat{v}_j f$ and $\hat{v}_j g$, respectively, we have examples arising from $A_c(G)$. This completes the proof.

<u>(2.6.3) Remark</u>. When $G = R$, we can use $C_c^\infty(R)$, the space of compactly supported infinitely differentiable functions on R , as a test space in our definition of the UMP. First, note that $L_1(R)$ has a bounded approximate identity of nonnegative functions belonging to $C_c^\infty(R)$ with nonnegative transforms. For let $\phi \in C_c^\infty(R)$ be nonnegative and satisfy $\int_{-\infty}^{\infty} \phi(x)dx = 1$. If, for $\eta > 0$, we set $\phi_\eta(x) = \eta^{-1}\phi(\frac{x}{\eta})$, then $\{\phi_\eta\}$ is an approximate identity for $L_1(R)$ (see [18, p. 10]). Let $\psi_\eta = \phi_\eta * (\phi_\eta)^\sim$, where $h^\sim(x) = \overline{h(-x)}$. Then $\{\psi_\eta\}$ is an approximate identity for $L_1(R)$ consisting of nonnegative functions in $C_c^\infty(R)$ and $0 \leqq \widehat{\psi_\eta} \leqq 1$ for every η . The proof now follows those of (2.6.1) and (2.6.2).

The proof now follows those of (2.6.1) and (2.6.2).

It is possible to give a definition of majorant which is valid for every pair of functions belonging to $L_p(G)$. To do this we require the notion of distribution as described in (1.5).

<u>(2.6.4) Definition</u>. Let G be any LCAG and let $L, M \in S^*(G)$. By $|L| \leqq M$ we mean $|\langle L, u\rangle| \leqq \langle M, u\rangle$ for every $u \in S_+(G)$, the set of nonnegative real-valued members of $S(G)$.

If $L^{\vee}$ denotes the inverse transform of an $L \in S^*(G)$, we say that $\underline{M^{\vee} \text{ majorizes } L^{\vee}}$ when $|L| \leqq M$.

Note from the definition that M must be a positive linear functional on $S(G)$, that is, $\langle M,u\rangle \geq 0$ for every $u \in S_+(G)$. This leads to an extension of a result from the Schwartz theory which was pointed out to the author by M. Cowling. Although this result must be well known, we could find no reference, so a proof is included here.

(2.6.5) Lemma. Let $M \in S^*(G)$ be positive. Then M is a measure (though not necessarily a bounded measure when G is noncompact).

Proof.

By the density of $A_c(G)$ in $C_0(G)$ (see [13, (33.13)]), it suffices to prove that the restriction of M to $A(G) \cap C(G;K)$ is continuous on $C(G;K)$, where $C(G;K)$ denotes the space of compactly supported members of $C_0(G)$ whose support is contained in the compact set K . We have to find a positive constant B , which may be K-dependent, such that $|\langle M,u\rangle| \leqq B||u||_\infty$ for every $u \in A(G) \cap C(G;K)$.

Let $u \in A(G) \cap C(G;K)$ and let $v \in A_c(G)$ be such that $v(K) = \{1\}$ and $v(G) \subset [0,1]$. Such functions are guaranteed by [13, (31.37)]. If u is real-valued we have

$$-v||u||_\infty \leqq u \leqq v||u||_\infty ,$$

and by the positivity of M we obtain

$$|\langle M,u\rangle| \leqq \langle M,v\rangle \; ||u||_\infty .$$

It is easy to show now that, if u is complex-valued and belongs to $C(G;K)$, then $|\langle M,u\rangle| \leqq \langle M,v\rangle \, ||u||_\infty$ (use the method of [12, proof of (11.5)]). Hence M is a measure.

If G is not compact, then Haar measure is an unbounded measure which defines a positive member of $S^*(G)$.

In the compact case it was quite clear how to produce a majorant for a trigonometric polynomial. For examples of majorants in the non-compact case, see [8, pp. 272, 273]. We now give an example to show that elements of $S^*(G)$ need not have any majorant at all.

(2.6.6) Example. We will show how to regard the Hilbert transform as a member of $S^*(R)$ and then show that it's inverse transform has no majorant in $S^*(R)$. All that we use here concerning the Hilbert transform may be found in [19, chapter 6]. In what follows, $S(R)$ denotes $L_1(R) \cap A(R)$ and is not to be confused with the Schwartz class of C^∞ functions which, along with all their derivatives, are of rapid decrease.

For $u \in S(R)$ we define a linear functional H on $S(R)$ by

$$\langle H,u\rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{u(x)}{x} \, dx .$$

We now show that for $u \in S(R)$ this expression is sensible.

Let $\tilde{u}$ denote the usual Hilbert transform. By the L_2 theory of this transform, we have

$$(\tilde{u})^\wedge(\xi) = - i \operatorname{sgn}(\xi) \, \hat{u}(\xi) ,$$

since $u \in L_2(R)$. But $\hat{u} \in L_1(R) \cap L_2(R)$ and so we also have $(\tilde{u})^{\wedge} \in L_1(R) \cap L_2(R)$. By the inversion formula, we may regard $\tilde{u}$ as a continuous function on R for which

$$\tilde{u}(x) = -i \int_R \operatorname{sgn}(\xi)\, \hat{u}(\xi)\, e^{i\xi x}\, d\xi \quad ,$$

and in particular,

$$\tilde{u}(0) = \int_R [-i\operatorname{sgn}(\xi)]\, \hat{u}(\xi)\, d\xi \quad . \qquad (1)$$

But $\tilde{u}(0)$ is just $\langle H,u\rangle$. From (1) we obtain

$$|\langle H,u\rangle| \leqq \|\hat{u}\|_1 \leqq \|u\|_S \ ,$$

and so $H \in S^*(R)$. Furthermore, H is the Fourier transform of the L_∞ function $H^\vee(\xi) = -i\operatorname{sgn}(\xi)$.

We now show that $H^\vee$ has no majorant in $S^*(R)$. By (2.6.5) it is enough to show that there is no positive measure μ satisfying

$$|\langle H,u\rangle| \leqq \langle \mu,u\rangle$$

for every $u \in S_+(R)$.

$$\text{Let } u_n(x) = \begin{cases} nx & 0 \leq x \leq \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq 1 \\ -nx+n+1 & 1 \leq x \leq 1 + \frac{1}{n} \\ 0 & \text{otherwise} . \end{cases}$$

Then each u_n is a trapezoidal function and it is well known that $u_n \in S_+(R)$. Since $||u_n||_\infty = 1$ for every n and the support of every u_n is contained in $[0,2]$ it follows that $\{u_n\}$ is a bounded subset of $C(R; [0,2])$. If μ is any positive measure we then have

$$\sup_n \langle\mu,u_n\rangle \leqq \mu([0,2]) < \infty .$$

However, simple computations show that

$$\langle H,u_n\rangle = \log n + (n+1) \log(1 + \tfrac{1}{n}) ,$$

and this sequence is unbounded. Thus $H^\vee$ has no majorant in $S^*(R)$.

We consider next a definition of upper majorant property based on (2.6.4).

<u>(2.6.7) Definition</u>. We say that $L_p(G)$ has the UMPD if there is a positive constant D such that whenever $f, g \in L_p(G)$ and $|\hat{f}| \leqq \hat{g}$ in the distributional sense of (2.6.4), then $||f||_p \leqq D||g||_p$.

It is immediate that if $L_p(G)$ has the UMPD, then $L_p(G)$ has the UMP, since for $f, g \in S(G)$, $|\hat{f}| \leqq \hat{g}$ pointwise implies $|\hat{f}| \leqq \hat{g}$

distributionally. Hence $L_p(G)$ can have the UMPD only when p is an even integer or ∞. We now prove the converse is also true.

(2.6.8) Lemma. If $f, g \in L_1(G)$ satisfy $|\langle\hat{f},u\rangle| \leqq \langle\hat{g},u\rangle$ for every $u \in S_+(\hat{G})$, then $|\hat{f}| \leqq \hat{g}$ pointwise. If $f, g \in S(G)$ satisfy $|\langle\hat{f},u\rangle| \leqq \langle\hat{g},u\rangle$ for every $u \in S_+(\hat{G})$ with $\hat{u}$ lying in some sphere in $L_q(G)$, then $|\hat{f}| \leqq \hat{g}$ pointwise.

Proof.

We prove the first statement only, since the proof of the second is essentially the same. It is easy to see that $\hat{g} \geqq 0$ pointwise. Suppose that the conclusion is false and let $\gamma_0 \in \hat{G}$ be such that $|\hat{f}(\gamma_0)| > \hat{g}(\gamma_0)$. We first consider the case where $\hat{f}$ is real-valued. Then either $\hat{f}(\gamma_0) > \hat{g}(\gamma_0)$ or $\hat{f}(\gamma_0) < -\hat{g}(\gamma_0)$. Suppose, for definiteness, that the first case occurs. By continuity there is an open neighbourhood V of γ_0 such that $\hat{g}(\gamma) < \hat{f}(\gamma)$ for every $\gamma \in V$ and by [13, (31.34)] there exists $w \in S_c(G)$ such that $0 \leqq \hat{w} \leqq 1$ and the support of $\hat{w}$ is contained in V. In particular, $\hat{w} \in S_+(\hat{G})$. Then

$$\langle\hat{g},\hat{w}\rangle = \int_V \hat{g}(\gamma)\hat{w}(\gamma)d\gamma < \int_V \hat{f}(\gamma)\hat{w}(\gamma)d\gamma = \langle\hat{f},\hat{w}\rangle ,$$

and $\langle\hat{g},\hat{w}\rangle < |\langle\hat{f},\hat{w}\rangle|$, which is a contradiction.

Suppose that $\hat{f}$ is complex-valued and $|\hat{f}(\gamma_0)| > \hat{g}(\gamma_0)$. After multiplication by a complex number of absolute value one we can assume that $\mathrm{Re}\,\hat{f}(\gamma_0) > \hat{g}(\gamma_0)$. Now $\mathrm{Re}\,\hat{f}$ is the transform of an L_1-function, and $|\langle\mathrm{Re}\,\hat{f},u\rangle| \leqq \langle\hat{g},u\rangle$ for all u in $S_+(\hat{G})$. By the previous analysis

Re $\hat{f}(\gamma_0) \leqq \hat{g}(\gamma_0)$. This contradiction completes the proof of the lemma.

<u>(2.6.9) Proposition</u>. Let $1 \leq p < \infty$. If $L_p(G)$ has the UMP (as defined in (2.1.3)) then $L_p(G)$ has the UMPD with at most the same constant.

<u>Proof</u>.

Let $f, g \in L_p(G)$ satisfy $|\hat{f}| \leqq \hat{g}$ distributionally and let D be the constant of the definition of the UMP. Let (u_n) , (v_m) be sequences in $S(G)$ such that $u_n \to f$ and $v_m \to g$ in $L_p(G)$. Then we also have $|\langle \hat{u}_n, u\rangle| \to |\langle \hat{f}, u\rangle|$ and $\langle \hat{v}_m, u\rangle \to \langle \hat{g}, u\rangle$ for every $u \in S(\hat{G})$. We can assume that $\hat{v}_m$ is real-valued, and thus that $\langle \hat{v}_m, u\rangle$ is real-valued for every $u \in S_+(G)$. For if the $\hat{v}_m$ are not real-valued, replace v_m by $w_m = \frac{1}{2}(v_m + \tilde{v}_m)$. As $\langle \hat{g}, u\rangle \geqq 0$ when $u \in S_+(\hat{G})$ it is easy to show that $\langle (\tilde{g})^\wedge, u\rangle = \langle \hat{g}, u\rangle$ whenever u belongs to $S_+(\hat{G})$. Hence $\langle \hat{w}_m, u\rangle \to \langle \hat{g}, u\rangle$ for every $u \in S_+(\hat{G})$. Furthermore, $||w_m||_p \to ||\frac{1}{2}(g+\tilde{g})||_p \leqq ||g||_p$. We now assume that the $\hat{v}_m$ are real-valued.

Let $\delta > 0$ and, for $r > 0$, let $S_r = \{u \in S_+(\hat{G}) \mid ||\hat{u}||_q \leq r\}$. Then there is a positive integer m_0 such that for every $m \geq m_0$, we have $|\langle \hat{f}, u\rangle| \leqq \frac{1}{1-\delta} \langle \hat{v}_m, u\rangle$ for all $u \in S_r$ and $||v_m||_p < ||g||_p + \delta$. Similarly, there is a positive integer n_0 such that for every $n \geqq n_0$, $|\langle \hat{u}_{n_0}, u\rangle| \leqq (\frac{1+\delta}{1-\delta}) \langle \hat{v}_{m_0}, u\rangle$ for all $u \in S_r$ and $||f||_p - \delta < ||u_n||_p$.

In particular, we have $|\langle \hat{u}_{n_0}, u\rangle| \leqq (\frac{1+\delta}{1-\delta}) \langle \hat{v}_{m_0}, u\rangle$ for every $u \in S_r$. By Lemma (2.6.8), $|\hat{u}_{n_0}| \leq (^{1+\delta}$

Lemma (2.6.8), $|\hat{u}_{n_0}| \leqq (\frac{1+\delta}{1-\delta}) \hat{v}_{m_0}$ pointwise; since $L_p(G)$ has the UMP,

we also have $||u_{n_0}||_p \leqq (\frac{1+\delta}{1-\delta}) D ||v_{m_0}||_p$. Hence $||f||_p - \delta < (\frac{1+\delta}{1-\delta})$

(inequality continues over)

$D[||g||_p + \delta]$ and, as $\delta > 0$ is arbitrary, $||f||_p \leqq D||g||_p$. This completes the proof.

(2.7) The Lower Majorant Property

In this section we discuss analogues for noncompact LCAG's of the LMP for the compact case.

(2.7.1) Proposition. Let $1 \leq p \leq \infty$ and consider the following statements:

(1) There is a positive constant C_1 such that if $f \in S(G)$, there exists $g \in S(G)$ such that $|\hat{f}| \leqq \hat{g}$ pointwise and $||g||_p \leqq C_1||f||_p$.

(2) There is a positive constant C_2 such that if $f \in S(G)$, there exists $g \in L_p(G)$ such that $|\hat{f}| \leqq \hat{g}$ distributionally and $||g||_p \leqq C_2||f||_p$.

(3) There is a positive constant C_3 such that if $f \in L_p(G)$, there exists $g \in L_p(G)$ satisfying $|\hat{f}| \leqq \hat{g}$ distributionally and $||g||_p \leqq C_3||f||_p$.

Then (1) implies (2), and if $1 < p < \infty$, (2) implies (3) .

Proof.

(1) => (2) is obvious, since $S(G) \subset L_p(G)$ and since $|\hat{f}| \leqq \hat{g}$ pointwise implies $|\hat{f}| \leqq \hat{g}$ distributionally. The constant C_2 is at most C_1 . (2) => (3). Let $f \in L_p(G)$ and let (u_n) be a sequence in $S(G)$ such that $u_n \to f$ in $L_p(G)$. By hypothesis, for each n , there exists $g_n \in L_p(G)$ such that $|\hat{u}_n| \leqq \hat{g}_n$ distributionally and $||g_n||_p \leqq C_2||u_n||_p$. Thus $\{g_n\}$ is a norm-bounded subset of the reflexive Banach space $L_p(G)$ and so there is a weak * convergent s

subsequence, still denoted by (g_n), with weak limit g, say, in $L_p(G)$. Then

$$||g||_p \leqq \varliminf_{n\to\infty} ||g_n||_p \leqq C_2 \lim_{n\to\infty} ||u_n||_p = C_2||f||_p ,$$

and by weak convergence (and (1.5.3)), if $u \in S_+(\hat{G})$,

$$|\langle\hat{f},u\rangle| = \lim_n |\langle\hat{u}_n,u\rangle| \leqq \lim_n \langle\hat{g}_n,u\rangle = \langle\hat{g},u\rangle .$$

Hence (3) follows, with C_3 being at most C_2.

Note that each of statements (1), (2), and (3) in this proposition makes sense for $1 \leqq p \leqq \infty$.

<u>(2.7.2) Definition</u>. Let $1 \leq p \leq \infty$. We shall say that $L_p(G)$ has the LMP(j) (j=1,2,3) if statement (j) of (2.7.1) holds for $L_p(G)$.

The content of (2.7.1) is that if $1 \leq p \leq \infty$ and $L_p(G)$ has LMP(1), then it has LMP(2); similarly, if $L_p(G)$ has LMP(2), then it has LMP(3) when $1 < p < \infty$.

Note that in statements (1) and (2) of (2.7.1), the condition "$f \in S(G)$" could equally be replaced by "$f \in S_c(G)$" or "$f \in A_c(G)$" and the conclusions would still hold. This leads to a new definition of LMP(j) when $j = 1,2$.

<u>(2.7.3) Lemma</u>. (a) $L_1(G)$ has LMP(3), with constant 1.

(b) $L_2(G)$ has LMP(2), with constant 1.

Proof. (a) By Lemma (2.6.7), if $f, g \in L_1(G)$, then $|\hat{f}| \leqq \hat{g}$ distributionally if and only if $|\hat{f}| \leqq \hat{g}$ pointwise. We proceed exactly as in Hardy and Littlewood [11, p. 305]. For $f \in L_1(G)$, write $f = f_1 f_2$, where $f_i \in L_2(G)$ and $||f_i||_2^2 = ||f||_1$, for $i = 1,2$. Let $g_i \in L_2(G)$ be such that $\widehat{g_i} = |\widehat{f_i}|$, $i = 1,2$. If $g = g_1 g_2$, then $g \in L_1(G)$ and $|\hat{f}| \leqq \hat{g}$ pointwise, since

$$|\hat{f}| = |\hat{f}_1 * \hat{f}_2| \leq |\hat{f}_1| * |\hat{f}_2| = \hat{g}_1 * \hat{g}_2 = \hat{g}$$

([13, (31.29)]).

Furthermore,

$$\begin{aligned} ||g||_1^2 \leqq ||g_1||_2^2\ ||g_2||_2^2 &= ||\hat{g}_1||_2^2\ ||\hat{g}_2||_2^2 \\ &= ||\hat{f}_1||_2^2\ ||\hat{f}_2||_2^2 \\ &= ||f_1||_2^2\ ||f_2||_2^2 = ||f||_1^2 . \end{aligned}$$

This proves (a) .

(b) If $f \in S(G)$, let $g \in L_2(G)$ be such that $\hat{g} = |\hat{f}|$ a.e. Then $|\hat{f}| \leqq \hat{g}$ distributionally and $||g||_2 = ||f||_2$. Hence (b) follows.

<u>(2.7.4) Remark</u>. There are other senses, not covered by (2.7.2), in which $L_1(G)$ and $L_2(G)$ have a LMP. Two examples are:

(i) $L_1(G)$ has the LMP in the sense that there is a positive constant C_4 such that if $f \in S_c(G)$, there exists $g \in S_c(G)$ for which $|\hat{f}| \leqq \hat{g}$ pointwise and $||g||_1 \leqq C_4 ||f||_1$.

(ii) $L_2(G)$ has the LMP in the sense that there is a positive constant C_5 such that if $f \in L_2(G)$, there exists $g \in L_2(G)$ satisfying $|\hat{f}| \leqq \hat{g}$ a.e. and $||g||_2 \leq C_5||f||_2$.

Each is easy to prove; statement (ii) is essentially (2.7.3)(b) and (i) can be derived from (2.7.1)(a) by using the method of (2.1.2)(b).

We come now to the duality theorem. In the compact case, one proves that a space $L_p(G)$ has the LMP by using the Hardy-Littlewood duality theorem (see(1.1)). For $p = \frac{2k}{2k-1}$ $(k \in N)$, a direct proof that $L_p(G)$ has the LMP is not known, even for the circle group. Unfortunately, we have found neither a generalization of the Hardy-Littlewood duality theorem nor a direct proof that $L_p(G)$ has LMP(j), for some j, when $p = \frac{2k}{2k-1}$ $(k \in N)$. However, it is an easy matter to generalize the Boas duality theorem (see (1.1)), and from this we can at least conclude that $L_p(G)$ can only have LMP(j) for some j (j = 1,2, or 3) if p is 1, 2, or of the form $\frac{2k}{2k-1}$ for $k \in N$.

<u>(2.7.5) Lemma</u>. Suppose that $L, M \in S^*(G)$ and $|L| \leqq M$. If $\phi, \psi \in S(G)$ satisfy $|\phi| \leqq \psi$ pointwise, then $|\langle L,\phi\rangle| \leqq 2\langle M,\psi\rangle$.

<u>Proof.</u>

First suppose that L is real-linear and that ϕ is real-valued. Then we have $-\psi \leqq \phi \leqq \psi$, so that $0 \leq \phi + \psi$ and $0 \leq \psi - \phi$. Since $|L| \leqq M$, we have $|\langle L,\phi+\psi\rangle| \leqq \langle M,\phi+\psi\rangle$ and $|\langle L,\psi-\phi\rangle| \leqq \langle M,\psi-\phi\rangle$. This is equivalent to

$$-\langle M,\psi\rangle - \langle M,\phi\rangle \leqq -\langle L,\phi\rangle - \langle L,\psi\rangle \leqq \langle M,\phi\rangle + \langle M,\psi\rangle$$

and $$-\langle M,\psi\rangle + \langle M,\phi\rangle \leqq -\langle L,\phi\rangle + \langle L,\psi\rangle \leqq \langle M,\psi\rangle + \langle M,\phi\rangle ,$$

Adding, we obtain

$$-2\langle M,\psi\rangle \leqq -2\ \langle L,\phi\rangle \leqq 2\langle M,\psi\rangle\ ,$$

hence $$|\langle L,\phi\rangle| \leqq \langle M,\psi\rangle\ .$$

Suppose now that L is general and ϕ is real-valued. Then $\langle L,\phi\rangle = \langle L_1,\phi\rangle - i\langle L_1, i\phi\rangle$ where L_1 is a real-linear functional on $S(G)$. If α is a complex number of absolute value one such that $\alpha\langle L,\phi\rangle = |\langle L,\phi\rangle|$, then

$$|\langle L,\phi\rangle| = \langle L,\alpha\phi\rangle = \langle L_1,\alpha\phi\rangle\ .$$

But it is easy to check that $|L_1| \leqq M$ and $|\alpha\phi| \leqq \psi$, so that $|\langle L_1, \alpha\phi\rangle| \leqq \langle M,\psi\rangle$. Thus $|\langle L,\phi\rangle| \leqq \langle M,\psi\rangle$ in this case.

For general L and complex-valued ϕ , we have, writing $\phi = \phi_1 + i\phi_2$ with ϕ_1 and ϕ_2 real-valued,

$$|\langle L,\phi\rangle| = |\langle L,\phi_1+i\phi_2\rangle| \leqq |\langle L,\phi_1\rangle| + |\langle L,\phi_2\rangle| \leqq 2\langle M,\psi\rangle\ .$$

<u>(2.7.6) Theorem</u>. Let $1 \leqq p \leqq \infty$. (a) Suppose that $L_p(G)$ has LMP(j) (j = 2 or 3) with constant C . Then $L_q(G)$ has the UMP with constant at most $2C$.

(b) If $L_p(G)$ has LMP(1) with constant C , then $L_q(G)$ has the UMP with constant at most C .

Proof.

See Boas [3, p. 256].

(a) The case $j = 2$ is similar to the case $j = 3$, so we prove only the latter. Let $f, F \in S(G)$ satisfy $|\hat{f}| \leqq \hat{F}$ pointwise. If $g \in L_p(G)$, there exists $h \in L_p(G)$ satisfying $|\hat{g}| \leqq \hat{h}$ distributionally and $||h||_p \leqq C||g||_p$. From (1.5.3) we have

$$\left|\int_G f(x)\, g(-x)dx\right| = |\langle\hat{g},\hat{f}\rangle|$$

$$\leqq 2\, \langle\hat{h},\hat{F}\rangle \qquad \text{(by (2.7.5))}$$

$$= 2\int_G h(x)\, F(-x)dx$$

$$\leqq 2||h||_p\, ||F||_q \leqq (2C||F||_q)\, ||g||_p \,.$$

Thus the map $g \longmapsto \int_G f(x)\, g(-x)dx$ defines a continuous linear functional on $L_p(G)$, with norm at most $2C||F||_q$. However, this functional is defined by the $L_q(G)$ - function f and so has norm $||f||_q$. Consequently we have $||f||_q \leqq 2C||F||_q$, and this proves (a).

(b) First suppose $1 \leqq p < \infty$. We proceed as in (a), using the Parseval formula instead of (2.7.5), and using the density of $S(G)$ in $L_p(G)$. When $p = \infty$, we again proceed as in (a) to obtain

$$\left|\int_G f(x)\, g(-x)dx\right| \leqq ||F||_1\, ||g||_\infty \,. \qquad (1)$$

It is an easy exercise to show that $S(G)$ is weak-*-dense in $L_\infty(G)$, and consequently

$$||f||_1 = \sup\{|\int_G f(x)g(x)dx| \;\; | g\varepsilon S(G) \text{ and } ||g||_\infty \leqq 1 \}$$

Combined with (1), this shows that $||f||_1 \leqq C||F||_1$. This completes the proof.

<u>(2.7.7) Corollary</u>. Suppose p is not 1, 2, or of the form $\frac{2k}{2k-1}$ for some $k \varepsilon N$. Then $L_p(G)$ does not have LMP(j) for $j = 1, 2,$ or 3.

<u>Proof</u>. If p is not of the given form, then q, the index conjugate to p, is not an even integer or ∞ and so $L_q(G)$ does not have the UMP (see (2.1.5)). The result follows from (2.7.6).

CHAPTER 3

MAJORANTS AND FUNCTIONS WITH NONNEGATIVE TRANSFORMS

This chapter is concerned with obtaining generalizations of the results outlined in (1.1.3) to all infinite compact abelian groups.

(3.1) Even integers again

Throughout this section, G is a compact abelian group. We establish analogues of the positive results for the circle group described in (1.1.3).

(3.1.1) Theorem. Let A be a symmetric neighbourhood of the identity in G . Suppose that $f \in L_1(G)$ and $\hat{f} \geqq 0$. If $f \in L_2(A)$, then $f \in L_2(G)$ and there is a positive constant C , dependent on A but independent of f , for which

$$||f||_{L_2(G)} \leqq C||f||_{L_2(A)} .$$

Proof.

Let m be normalized Haar measure on G . Let K be a compact symmetric neighbourhood of the identity in G with $K + K \subset A$. Set $h = \frac{1}{m(K)} 1_K$ and define $g = h * \tilde{h}$, where $\tilde{h}(x) = \overline{h(-x)}$. Then we have $\hat{g} = |\hat{h}|^2 \geqq 0$ and $\hat{g}(0) = 1$. Moreover, $|g| \leqq \frac{1}{m(K)} 1_A$, and so $gf \in L_2(G)$, since $1_A f \in L_2(G)$.

If $\beta \in \hat{G}$, we have

$$\widehat{gf}(\beta) = \hat{g} * \hat{f}\ (\beta) = \sum_{\gamma \in \hat{G}} \hat{g}(\gamma)\ \hat{f}(\beta-\gamma) \geqq \hat{g}(0)\ \hat{f}(\beta) = \hat{f}(\beta) .$$

Applying the Plancherel theorem, we obtain

$$||f||_{L_2(G)} = ||\hat{f}||_{L_2(\hat{G})} \leqq ||\widehat{gf}||_{L_2(\hat{G})} = ||gf||_{L_2(G)} \leqq \frac{1}{m(K)} ||f||_{L_2(A)} .$$

This proves the theorem, with $C = \frac{1}{\check{m}(K)}$.

It is now an easy matter to derive the corresponding result for $L_p(G)$ when p is an even integer or ∞ .

<u>(3.1.2) Corollary</u>. Let A be a symmetric neighbourhood of the identity, and suppose $f \in L_1(G)$ with $\hat{f} \geqq 0$. If p is an even integer or ∞ and if $f \in L_p(A)$, then $f \in L_p(G)$. Moreover, there is a positive constrant C , dependent on A and p but not on f , for which

$$||f||_{L_p(G)} \leqq C||f||_{L_p(A)} .$$

<u>Proof</u>.

First suppose that $p = 2\ell$, where ℓ is a positive integer. We have $f^\ell \in L_2(A)$ and $\widehat{f^\ell} \geqq 0$, since $\widehat{f^\ell} = \hat{f} * \hat{f} * \cdots\cdots * \hat{f} \geqq 0$ (there are ℓ $\hat{f}$'s here). By (3.1.1), $f^\ell \in L_2(G)$ and $||f^\ell||_{L_2(G)} \leqq \frac{1}{\check{m}(K)} ||f^\ell||_{L_2(A)}$, where K is as in the proof of (3.1.1). Thus f belongs to $L_p(G)$ and we have

$$||f||_{L_p(G)} = [||f^\ell||_{L_2(G)}]^{1/\ell} \leqq [\check{m}(K)]^{-1/\ell} ||f^\ell||^{1/\ell}_{L_2(A)}$$

$$= [m(K)]^{-1/\ell} ||f||_{L_p(A)} \qquad (1)$$

Hence, $C = \check{m}(K)^{-1/\ell}$.

Suppose now that $p = \infty$ and $f \in L_\infty(A)$. Then $f \in L_{2\ell}(A)$ for every positive integer ℓ . Since

$$\lim_{p\to\infty} ||h||_p = ||h||_\infty \quad \text{if} \quad h \in L_r \quad \text{for some} \quad r ,$$

it follows from (1) that

$$||f||_{L_\infty(G)} \leqq ||f||_{L_\infty(A)} .$$

In particular, $f \in L_\infty(G)$ and $C = 1$ in this case.

<u>(3.1.3) Remark</u>. If $f \in L_\infty(G)$ and $\hat{f} \geqq 0$, then $\hat{f} \in \ell_1(\hat{G})$ and thus is equal a.e. to a function in $P(G)$, the space of continuous positive definite functions of G (see [13, (34.12)] or use the UMP as in [3, p. 256]). The norm inequality of (3.1.2) tells us nothing new, for if we assume $f \in P(G)$, we have

$$f(0) = ||f||_{L_\infty(G)} \leqq ||f||_{L_\infty(A)} \leqq ||f||_{L_\infty(G)} = f(0) .$$

However, when $p = \infty$, (3.1.2) is an alternative proof of a classical result (see [6, p. 144]).

(3.2) Failure of "good" behaviour when p is not an even integer

In this section we present extensions of the results of Wainger [20] and Shapiro [18]. Throughout, G will be an infinite compact abelian group. One characteristic of the positive results is that we obtain an inequality of the form

$$||f||_{L_p(G)} \leqq C||f||_{L_p(A)} .$$

We refer to this inequality as (W) .

We shall show that if $p \geq 1$ is finite and not an even integer, then no inequality of the form (W) can hold. Of course, we assume that $\hat{f} \geq 0$ in (W) .

<u>(3.2.1) Lemma</u>. Let B a nonempty symmetric open subset of G . Then we can find $u \in A(G)$ with the following properties:

(1) u and $\hat{u}$ are real-valued;

(2) u is supported in B ;

(3) u is bounded away from zero on some symmetric open set contained in B .

Before proceeding with the proof, we need some more notation. For $f \in L_1(G)$, we let $\tilde{f}(x) = \overline{f(-x)}$; for $y \in G$, we let $({}_y f)(x) = f(y+x)$. Note that $({}_x f)^{\sim} = {}_{-x}(\tilde{f})$.

<u>Proof of (3.2.1)</u>. Let $x_0 \in B$ and suppose that V is a symmetric neighbourhood of x_0 contained in B . There is a compact symmetric neighbourhood K of the identity such that $K + K \subset - x_0 + V$.

Set $w = 1_K * \tilde{1}_K$. Then w is supported by $K + K$, $w \in A(G)$ and $w(0) = m(K) > 0$. Thus there is a $\delta > 0$ and a symmetric neighbourhood Y of the identity such that $Y \subset K + K$ and $w(x) \geq \delta$ for every x in Y .

Let $u_1 = {}_{-x_0}w$; then $\operatorname{supp}(u_1) = x_0 + \operatorname{supp}(w) \subset V \subset B$. Since $\hat{u}_1$ may not be real-valued, consider $u = \frac{1}{2}(u_1 + \tilde{u}_1)$. Then both u and $\hat{u}$ are real-valued and $u \in A(G)$. Furthermore,

$$\begin{aligned} \operatorname{supp}(u) &\subset \operatorname{supp}(u_1) \cup \operatorname{supp}(\tilde{u}_1) \\ &= \operatorname{supp}(u_1) \cup \{- \operatorname{supp}(u_1)\} \subset B . \end{aligned}$$

Finally, we note that u is bounded away from zero on $x_0 + Y$, since if $y \in Y$,

$$\begin{aligned} u(x_0+y) &= \frac{1}{2}(w(-x_0+x_0+y) + w(x_0-x_0-y)), \text{ as } ({}_{-x_0}W)^{\sim} = {}_{x_0}(\tilde{w}) . \\ &= \frac{1}{2}(w(y) + w(-y)) \\ &\geqq \delta . \end{aligned}$$

This completes the proof.

Lemma (3.2.1) and the failure of the UMP when p is not an even integer provide the key to the next two theorems. Theorem (3.2.3) is a direct generalization of Shapiro's result [18, p. 16], and the method is the same. Theorem (3.2.3) is a slight variant, the difference being that in (3.2.3) we have $p > 2$ and we can conclude that our example belongs to $L_2(G)$ and hence also belongs to $L_1(G)$. We can not do this when $1 \leq p < 2$.

(3.2.2) Theorem Suppose $p \geq 1$ is finite and not an even integer. Let A be a closed symmetric neighbourhood of the identity which is not of full measure. Then no inequality of the form (W) can hold.

Proof.

We know that the UMP fails in $L_p(G)$ for these p ; see (1.1.2)(b). Thus, for each positive integer n we can find trigonometric polynomials f_n , F_n with the following properties: F_n majorizes f_n , $||F_n||_p^p \leqq 1$ and $||f_n||_p^p \geqq 2^n$.

Apply Lemma (3.2.1) to the set $B = A'$, the complement of A , to obtain u belonging to $A(G)$, supported by A' , and bounded below by $\delta > 0$ on a nonempty open set I contained in A' . Let $u^* = \sum_{\gamma \varepsilon \hat{G}} |\hat{u}(\gamma)|\gamma$. Then u^* belongs to $A(G)$ and u^* majorizes u .

We first show that (1) f_n can be assumed to have real coefficients and still be majorized by F_n , and (2) after a possible replacement of f_n as in (1), we can arrange that the sequence $\{||f_n||^p_{L_p(I)}\}$ is unbounded.

Let ℓ denote the least number of translates of I required to cover G , and suppose that $G = \bigcup_{j=1}^{\ell} (y_j + I)$. Then

$$2^n \leqq ||f_n||^p_{L_p(G)} \leqq \sum_{j=1}^{\ell} \int_{y_j+I} |f_n(x)|^p \, dx \leqq \ell ||{}_z f_n||^p_{L_p(I)} ,$$

where z is that $-y_j$ $(1 \leq j \leq \ell)$ for which $||{}_z f_n||^p_{L_p(I)} = \max_{1 \leq j \leq \ell} ||{}_{-y_j} f_n||^p_{L_p(I)}$.

Thus (2) will still hold when we replace f_n by ${}_z f_n$; note that z may vary with n. We now show how to make the necessary adjustments for (1) and still preserve (2).

If $h_1 = \frac{1}{2}({}_z f_n + ({}_z f_n)^{\approx})$ and $h_2 = \frac{-i}{2}({}_z f_n - ({}_z f_n)^{\sim})$, then both h_1 and h_2 have real-valued Fourier transforms and at least one of h_1 and h_2 has the pth power of its $L_p(I)$-norm $\geqq 2^{n-1}/\ell$. We thus replace ${}_z f_n$ by whichever of h_1 and h_2 has $||h_j||^p_{L_p(I)} \geqq 2^{n-1}/\ell$. Hence we can assume (1) and (2) hold. Denote the replacement by f_n once again.

We now show that (W) cannot hold. Let $g_n = F_n u^* + f_n u$; then $\hat{g}_n \geqq 0$ since F_n majorizes f_n, u^* majorizes u, and the coefficients of u and f_n are real-valued. Now $\sup||g_n||^p_{L_p(A)} < \infty$, since

$$\int_A |g_n(x)|^p dx = \int_A |F_n(x)|^p \ |u^*(x)|^p dx \qquad \text{(u is supported by A')}$$

$$\leqq ||u^*||^p_\infty \ ||F_n||^p_{L_p(G)}$$

$$\leqq ||u^*||^p_\infty \ .$$

However, $\sup\limits_n ||g_n||^p_{L_p(G)} = \infty$; for if this sequence were bounded, the sequence $\{||f_n u||^p_{L_p(G)}\}$ would also be bounded since $f_n u = g_n - F_n u^*$. But

$$||f_n u||^p_{L_p(G)} \geqq \int_I |f_n(x)|^p \ |u(x)|^p \ dx \geqq \delta^p \ 2^{n-1}/\ell$$

by (2). This completes the proof.

We can be much more direct when $p > 2$.

<u>(3.2.3) Theorem</u>. Let $p > 2$ not be an even integer or ∞ . Suppose that A is a closed symmetric neighbourhood of the identity which is not of full measure. Then there exists g belonging to $L_1(G)$ with the following properties:

(a) $\hat{g} \geq 0$;

(b) $g \in L_p(A)$;

(c) $g \notin L_p(G)$.

<u>Proof</u>.

By [7, p. 165], there is a complex-valued function c on $\hat{G}$ with $|c| \in [L_p(G)]^\wedge$ and $c \notin [L_p(G)]^\wedge$. Let $F \in L_p(G)$ be such that $\hat{F} = |c|$; since $p > 2$, we can find f_1 belonging to $L_2(G)$ such that $\hat{f}_1 = c$. Note that f_1 does not belong to $L_p(G)$.

Let u be obtained from Lemma (3.2.1), using $B = A'$. Let I be a symmetric neighbourhood contained in A' on which u is bounded below by a positive number, δ say.

Since $f_1 \notin L_p(G)$ and since only a finite number of translates of I are required to cover G , there must be a $y \in G$ for which ${}_yf_1 \notin L_p(I)$. Since I is symmetric, $({}_yf_1)^\sim$ does not belong to $L_p(I)$, and so not both of $h_1 = \frac{1}{2}({}_yf_1 + ({}_yf_1)^\sim)$ and $h_2 = \frac{-i}{2}({}_yf_1 - ({}_yf_1)^\sim)$ can belong to $L_p(I)$. As both h_1 and h_2 have real coefficients, we replace ${}_yf_1$ with whichever of h_1 and h_2 does not belong to $L_p(I)$. Call this replacement f ; then f is majorized by F .

Let $u^* = \sum_{\gamma \varepsilon \hat{G}} |\hat{u}(\gamma)|\gamma$, and let $g = Fu^* + fu$. Then g belongs to $L_2(G)$, and $\hat{g} \geqq 0$. Furthermore, $g \,\varepsilon\, L_p(A)$ since u is supported by A' , and so

$$||g||^p_{L_p(A)} = \int_A |F(x)|^p \; |u^*(x)|^p \, dx \leqq ||u^*||^p_\infty \;\; ||F||^p_{L_p(G)} \; .$$

However, g does not belong to $L_p(G)$: for otherwise we would have $fu = g - Fu^* \,\varepsilon\, L_p(G)$, and yet

$$||fu||^p_{L_p(G)} \geqq \int_I |f(x)|^p \; |u(x)|^p dx \geqq \delta^p \int_I |f(x)|^p dx = \infty \; .$$

This completes the proof.

For the case $1 \leq p < 2$, we can give a modified version of (3.2.3). This result should be regarded as an extension of Wainger's result [20].

<u>(3.2.4) Proposition</u>. Suppose $1 \leq p < 2$ and let A be a closed symmetric neighbourhood of the identity which is not of full measure. Then there is a trigonometric series with nonnegative coefficients which converges to a function in $L_p(A)$, but whose coefficients are not the Fourier coefficients of any function belonging to $L_p(G)$.

<u>Proof</u>.

Since the UMP fails in $L_p(G)$, for each positive integer n there exist trigonometric polynomials F_n , f_n with F_n majorizing f_n ,

$||F_n||_p \leqq 2^{-n}$, and $||f_n||_p \geqq 2^n$. As in the proof of Theorem (3.2.3) we can assume that f_n has real coefficients and that $||f_n||_{L_p(I)} \geqq c2^n$ for some positive constant c , where I is as in the proof of Theorem (3.2.3). Let u be obtained from Lemma (3.2.1) with $B = A'$, and let $u^* = \sum_{\gamma \varepsilon \hat{G}} |\hat{u}(\gamma)|\gamma$.

If $g_n = F_n u^* + f_n u$, then $\hat{g}_n \geqq 0$ and g_n belongs to $A(G)$ for every n . Let $g = \sum_{n=1}^{\infty} g_n$. Then g is a trigonometric series with nonnegative coefficients. Note that $g_n \varepsilon L_p(A)$ and $||g_n||_{L_p(A)} \leqq 2^{-n} ||u^*||_\infty$. Hence the series defining g converges in $L_p(A)$. Since the sequence $\{||g_n||_{L_p(G)}\}$ is unbounded, g cannot belong to $L_p(G)$.

<u>(3.2.5) Remark</u>. We should note that the series defining g can be assumed to converge a.e. (with respect to Haar measure) by using a subsequence if necessary. For this subsequence, g still cannot belong to $L_p(G)$ (for the same reason as for the original g) .

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