FINITELY PRESENTED MODULES

AND

STABLE THEORY

by

Ronald Stanley Gentle

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Department of MATHEMATICS

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date Sept 7 /76

ABSTRACT

This thesis is a two pronged affair. Part one is a study of finitely presented modules using the techniques of homological algebra. We establish a theorem involving certain exact sequences, which proves to be highly efficient in dealing with the theory of finitely presented modules. An attempt has been made to unify many of the results found in the literature, (with the inclusion of some original results).

Part two, which can be read independently of part one, is a study of the category of short exact sequences modulo split sequences. Special attention is paid to projectives in this category; an explicit construction of a projective resolution, with its' consequences, for an arbitrary object is given. Part two is related to part one in providing a categorical bedding, thereby enriching the theory of finitely presented modules.

TABLE OF CONTENTS

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Introduction

For a ring R, one has the abelian categories of right and left R-modules. How does the structure of one of these categories affect the other? For instance, how does the structure of the subcategory of finitely generated left modules affect right R-modules. The basic device, in making the passage from left to right modules, is the duality functor * (Hom(-,R)). Unfortunately however the duality functor greatly alters structural properties; for example a finitely generated left module is not necessary carried to a finitely generated right module. A first attempt, to overcome this instability of the duality functor, is to consider for each finitely generated left module N, a projective presentation $P \twoheadrightarrow N$ and let N° be coker : $N^* \to P^*$. N° is then a finitely generated right module. However the corresponding presentation $P^* \twoheadrightarrow N^\circ$, has kernel N^* which is a dual module; so that the set of modules $\{N^\circ\}$ is only a special subset of finitely generated right modules. This assignment is thus deficient, and facts to effectively relate the subcategories of finitely generated right and left modules.

All is not lost, however if one passes to the more restrictive subcategory of finitely presented modules (the importance of finitely presented (f.p.) modules cannot be under estimated, as every module is the direct limit of f.p. modules). This thesis demonstrates how the structure of the category of f.p. left modules determines the structure of f.p. right modules (modulo finitely generated projectives). I will briefly outline the contents of this thesis.

In section two, Thm. 2.1 establishes the connection between f.p. left and f.p. right modules. This theorem sets up the dominoes; all resulting propositions of this section (and indeed for all of part one) fall easily. This section incorporates some results of Jans (13) and Bass (3) (who work with noetherian rings) concerning duality, and McRae (14) (15) on projective dimensions of f.p. modules, but the use of Thm. 2.1 means the proofs are substantially different. A few original results are also present.

In section three, pure theory is introduced following Fielhouse (7). Use of Thm. 2.1 immediately shows this pure theory coincides with that of Cohn's (6) (using tensor product). Absolute purity as defined by Maddox (16) is shown to be equivalent to both the homological concept of f.p. injectivity and copure injectivity of Fieldhouse (7). The basic properties of absolutely purity are established.

Section four covers some results of Stenström (18) and Jain (12) on f.p. injectivity and coherence, and contains the particularly useful Thm. 4.3, for testing flatness and f.p. injectivity. Both of the above authors, rely heavily on the character functor $(-)^\circ = \mathrm{Hom}_Z(-,Q/Z)$, with duality formulas (a) $\mathrm{Ext}^n(F,L^\circ) \cong \mathrm{Tor}_n(F,L)^\circ$, (b) $\mathrm{Ext}^n(F,M)^\circ \cong \mathrm{Tor}_n(F,M^\circ)$, R right coherent, F f.p. and with $\mathrm{Tor}_1(M,F) \cong \mathrm{Hom}(\mathrm{Ext}^1(F,R),M)$, F f.p., M injective, R left coherent. (Cartan and Eilenberg (4)). I feel the introduction of character modules into this theory is somewhat artificial and obscures the results; once again Thm. 2.1 enables alternate proofs.

Flatness can be interpreted in terms of relations (linear equations!) (Chase (5)). Section five is an analysis of this result, (using Thm. 4.3), and shows how f.p. injectivity has a similiar interpretation.

Part two of the thesis is of a more categorical nature, and is a synthesis of results of Auslander (1) (2), Hilton (9) (10) and Freyd (8). Part two originated out of an effort to make Thm. 2.1 (a) (b) functorial (which I subsequentially found to be history, Stable Module Theory (2), hence title of thesis)

Section six contains two major results. Thm. 6.7 (Freyd (8)) gives one a home field (an abelian category $\mathcal{E}/\mathcal{S}$) in which to study the functors of sections eight and ten. Prop. 6.4 is essential for all results of section three concerning purity. It is also of utmost importance because it allows one to determine when morphisms (in $\mathcal{E}/\mathcal{S}$) are zero maps.

Section seven is a study of Hilton's projective homotopy (9) (11). All results are known with possible exception of Prop. 7.5.

Section eight puts some results of Hilton and Ree (10) and Auslander and Bridger (2) into the categorical framework established by Freyd (8).

Section nine contains Thm. 9.3, a result of my own, concerning projective resolutions in the abelian category $\mathcal{E}/\mathcal{S}$; its' many corollaries indicate it is of some value but I wish for a more elegant proof.

In section ten, we return to the subject matter of part one, putting Thm. 2.1 (a) (b) into categorical language.

Finally section eleven is of a miscellaneous nature and poses some problems.

Through somewhat fortunate circumstances (more enjoyment on my part), although most of the results of this thesis are known all the proofs given are original (with exception of parts of Thm. 6.7). Either I felt my own proofs were simplications, or because the given proofs were bound up in too much theory (heavy categorical machinery) and would take the reader too far afield or simply because no proofs were given.

If no reference is given in this thesis, the result is original; unless I overlooked its author, then being only a second creator. A final note : $\rightarrowtail$ denotes monic, $\twoheadrightarrow$ epic. (which it is not. Good reading).

1/ Preliminaries.

Let R be an associative ring with identity, ${}_R\mathcal{M}$ the category of left R-modules. $M^* = \mathrm{Hom}_R(M,R)$, then for M a left module, M^* is a right module by the action
$(fr)(x) = (f(x))r \quad f \in M^*,\ x \in M,\ r \in R.$
The assignment $M \mapsto M^*$ is a contravariant functor. The basic facts concerning this functor are stated in the following theorem.

Theorem 1.0.

(a) There is a natural isomorphism

$\theta : \mathrm{Hom}(N,M^*) = \mathrm{Hom}(M,N^*)$ where $(\theta(f)(m))(n) = (f(n))(m)$.

Hence we have,

(b) Regarding * as a covariant functor ${}_R\mathcal{M} \to \mathcal{M}_R^{op}$ (the opposite category), then * is its' own left adjoint. As a result * preserves colimits, but being contravariant this means * transforms colimits to limits (sums to products, cokernels to kernels, pushouts to pullbacks).

(c) The unit (and counit) of this adjunction is

$n_M : M \to M^{**} \quad (n_M(m))(f) = f(m) \quad f \in M^*$

A module M is called torsionless (reflexive) if n_M is injective (an isomorphism).

(d) The triangular identity for this adjunction is

$$\begin{tikzcd} M^* \arrow[r, "n_{M^*}"] \arrow[dr, equal] & M^{***} \arrow[d, "n_M^*"] \\ & M^* \end{tikzcd}$$

so that $M^* \to M^{***}$ is a split monic.//

The basic example $0 \to L \to R \to R/L \to 0$, L a left ideal, gives $0 \to (R/L)^* \to R^*$ exact, and $(R/L)^* = \mathrm{Ann}(L)$ via $f \mapsto f(\bar{1})$.

A module M is said to be finitely presented if there exists an exact sequence $P' \to P \to M \to 0$ with P, P' finitely generated projective. The full sub-category of finitely presented (f.p.) modules is closed under cokernels but not necessarily closed under kernels. Left Noetherian rings are precisely those rings for which the category of finitely generated (f.g.) modules is abelian; a ring is called left coherent if the category of f.p. modules is abelian. A ring is coherent if and only if finitely generated submodules of f.p. modules are in turn f.p. modules.

2/ Duality and Finitely Presented Modules

For reference the following is a slight extension of the standard Snake-lemma.

Snake Lemma - Ker-Coker Sequence ((4) Lemma 10.1 page 101)

$$\begin{tikzcd}
\cdots \arrow[r] & A_5 \arrow[r] & A_4 \arrow[r] & A_3 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_1 \arrow[r] \arrow[d, "f_3"] & 0 & & \\
& & 0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \arrow[r] & \cdots
\end{tikzcd}$$

commutative with exact rows, $K_i = \ker f_i$, $C_i = \operatorname{coker} f_i$, then we have an exact sequence

$$\cdots \to A_5 \to A_4 \to K_1 \to K_2 \to K_3 \to C_1 \to C_2 \to C_3 \to B_4 \to B_5 \to \cdots //$$

Theorem 2.1.

Given a finite presentation $P' \to P \to A$ of a right module A, there exists a finitely presented left module $\hat{A}$ such that

(a) $0 \to \hat{A}^* \to P' \to P \to A \to 0$ and $0 \to A^* \to P^* \to P'^* \to \hat{A} \to 0$

(b) Any finitely presented left module B, is of the form $\hat{A}$ for some finitely presented right module A (and vice versa).

(c) $0 \to \operatorname{Ext}^1(\hat{A},-) \to A\otimes - \to \operatorname{Hom}(A^*,-) \to \operatorname{Ext}^2(\hat{A},-) \to 0$

$0 \to \operatorname{Tor}_2(-,\hat{A}) \to -\otimes A^* \to \operatorname{Hom}(A,-) \to \operatorname{Tor}_1(-,\hat{A}) \to 0$

If $0 \to M \to N \to Q \to 0$ is exact then there exists u, v such that

(d) $0 \to \operatorname{Hom}(\hat{A},M) \to \operatorname{Hom}(\hat{A},N) \to \operatorname{Hom}(\hat{A},Q) \xrightarrow{u} A\otimes M \to A\otimes N \to A\otimes Q \to 0$

(e) $\cdots \to \operatorname{Tor}_1(A,N) \to \operatorname{Tor}_1(A,Q) \xrightarrow{v} \operatorname{Ext}^1(\hat{A},M) \to \operatorname{Ext}^1(\hat{A},N) \to \cdots$

(f) Following commutes in every possible way

$$\begin{tikzcd}
\operatorname{Hom}(\hat{A},Q) \arrow[r] \arrow[d] \arrow[dr, "u"] & \operatorname{Ext}^1(\hat{A},M) \arrow[d] \\
\operatorname{Tor}_1(A,Q) \arrow[r] \arrow[ur, "v"'] & A\otimes M
\end{tikzcd}$$

(g) all of this is expressed in following commutative diagram of exact sequences

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & & & & \\
\cdots \arrow[r] & \mathrm{Tor}_3(A,Q) \arrow[r] \arrow[d, "\cong"] & \mathrm{Tor}_2(A,M) \arrow[r] \arrow[d] & \mathrm{Tor}_2(A,N) \arrow[r] \arrow[d] & \mathrm{Tor}_2(A,Q) \arrow[r, "w"] \arrow[d] & {} & & & & \\
\cdots \arrow[r] & \mathrm{Tor}_1(A^*,Q) \arrow[r] \arrow[d] & A^*\otimes M \arrow[r] \arrow[d] & A^*\otimes N \arrow[r] \arrow[d] & A^*\otimes Q \arrow[r] \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathrm{Hom}(A,M) \arrow[r] \arrow[d] & \mathrm{Hom}(A,N) \arrow[r] \arrow[d] & \mathrm{Hom}(A,Q) \arrow[r] \arrow[d] \arrow[dr] & \mathrm{Ext}^1(A,M) \arrow[r] \arrow[d] & \mathrm{Ext}^1(A,N) \arrow[r] \arrow[d] & \mathrm{Ext}^1(A,Q) \arrow[r, "w'"] \arrow[d] & {} & \\
 & {} \arrow[r, "w"] & \mathrm{Tor}_1(A,M) \arrow[r] \arrow[d] & \mathrm{Tor}_1(A,N) \arrow[r] \arrow[d] & \mathrm{Tor}_1(A,Q) \arrow[r] \arrow[d] \arrow[ur] & A\otimes M \arrow[r] \arrow[d] & A\otimes N \arrow[r] \arrow[d] & A\otimes Q \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & & 0 & 0 & 0 \arrow[r] & \mathrm{Hom}(A^*,M) \arrow[r] \arrow[d] & \mathrm{Hom}(A^*,N) \arrow[r] \arrow[d] & \mathrm{Hom}(A^*,Q) \arrow[r] \arrow[d] & \mathrm{Ext}^1(A^*,M) \arrow[r] \arrow[d, "\cong"] & \cdots \\
 & & & & {} \arrow[r, "w'"] & \mathrm{Ext}^2(A,M) \arrow[r] \arrow[d] & \mathrm{Ext}^2(A,N) \arrow[r] \arrow[d] & \mathrm{Ext}^2(A,Q) \arrow[r] \arrow[d] & \mathrm{Ext}^3(A,M) \arrow[r] & \cdots \\
 & & & & & 0 & 0 & 0 & &
\end{tikzcd}
$$

<u>Remarks</u>; Part (a) is partially used by everyone, but mostly the $A \mapsto \hat{A}$ correspondence, that (b) $A \leftrightarrow \hat{A}$ is generally neglected. Part (c) can be found in (1): M. Auslander: <u>Coherent Functors</u>, but the approach is different, more categorical. The proof of the theorem will be divided into sections; the only great difficulty is part (f).

<u>Proof</u>: (i) Let $P' \xrightarrow{L} P \longrightarrow A \longrightarrow 0$, P, P' f.g. and projective. Dualize to obtain K and $\hat{A}$, such that $0 \to A^* \to P^* \xrightarrow{K} P'^* \to \hat{A} \to 0$ Dualize again, $0 \to \hat{A}^* \to P'^{**} \to P^{**}$, but $P^{**} \cong P$ and $P'^{**} \cong P'$ so coker of this sequence is again A. This establishes (a) and shows $A = \hat{\hat{A}}$, so also gives (b) by a left-right switch.

(ii) For any right module N and left module M consider $N \otimes M \to \mathrm{Hom}(N^*, M)$ where $(n \otimes m)f = f(n)m$, $f \in N^*$; this is an isomorphism if N is f.g. and projective (since it is true for the ring R). Using the exact sequence $0 \to K \to P'^* \to \hat{A} \to 0$ to obtain

$$\begin{tikzcd}
 & P'\otimes M \arrow[r] \arrow[d] & P\otimes M \arrow[r] \arrow[dd, "\cong"] & A\otimes M \arrow[r] \arrow[dd] & 0 & \\
 & \mathrm{Hom}(P'^*,M) \arrow[d] & & & & \\
0 \arrow[r] & \mathrm{Hom}(K,M) \arrow[r] \arrow[d] & \mathrm{Hom}(P^*,M) \arrow[r] \arrow[d] & \mathrm{Hom}(A^*,M) \arrow[r] & \mathrm{Ext}^1(K,M) \arrow[r] & 0 \\
 & \mathrm{Ext}^1(\hat{A},M) \arrow[r] & 0 & & &
\end{tikzcd} \qquad (\#1)$$

$\mathrm{Ext}^1(K,M) = \mathrm{Ext}^2(\hat{A},M)$ amd everything is natural with respect to M; so apply the snake lemma (w the connecting homomorphism is actually an isomorphism) to obtain

$0 \to \mathrm{Ext}^1(\hat{A},-) \to A\otimes- \to \mathrm{Hom}(A^*,-) \to \mathrm{Ext}^2(\hat{A},-) \to 0$

(iii) Now consider the map $N^* \otimes Q \to \mathrm{Hom}(N,Q)$ given by $(f \otimes q)n = f(n)q$, which is an isomorphism if N is f.g. and projective (again because it is true for R).
(<u>Remark</u>: The maps in (ii) and (iii) are connected by the following commutative diagrams:

$$\begin{tikzcd} N^*\otimes Q \arrow[d] \arrow[dr] & \\ \mathrm{Hom}(N,Q) & \mathrm{Hom}(N^{**},Q) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} M\otimes Q \arrow[r] \arrow[d] & M^{**}\otimes Q \arrow[dl] \\ \mathrm{Hom}(M^*,Q) & \end{tikzcd}$$

induced by $N \to N^{**}$ and $M \to M^{**}$; for f.g. projectives, when P is identified with P^{**}, the maps of (i) and (iii) are the same.) Using $0 \to L \to P \to A \to 0$ to obtain

$$\begin{tikzcd} & & & 0 \arrow[r] \arrow[d] & \mathrm{Tor}_1(A,Q) \arrow[d] & \\ 0 \arrow[r] & \mathrm{Tor}_1(L,Q) \arrow[r] & \hat{A}^*\otimes Q \arrow[r] \arrow[dd] & P'\otimes Q \arrow[r] \arrow[dd, "\cong"'] & L\otimes Q \arrow[r] \arrow[d] & 0 \\ & & & & P\otimes Q \arrow[d] & \\ & 0 \arrow[r] & \mathrm{Hom}(\hat{A},Q) \arrow[r] & \mathrm{Hom}(P'^*,Q) \arrow[r] & \mathrm{Hom}(P^*,Q) & \end{tikzcd} \qquad (\#2)$$

Again the connecting homomorphism is an isomorphism, $\mathrm{Tor}_1(L,Q) = \mathrm{Tor}_2(A,Q)$ and with naturality the snake lemma gives $0 \to \mathrm{Tor}_2(A,-) \to \hat{A}^*\otimes - \to \mathrm{Hom}(\hat{A},-) \to \mathrm{Tor}_1(A,-) \to 0$ thus (c) has been established.

(iv) Now suppose $0 \to M \to N \to Q \to 0$, then

$$\begin{tikzcd} \mathrm{Hom}(\hat{A},Q) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\hat{A},M) \arrow[d] \\ \mathrm{Tor}_1(A,Q) \arrow[r] & A\otimes M \end{tikzcd}$$

horizontal maps from homology, vertical maps from diagrams (#1) and (#2). Commutivity must be verified. To compute $\mathrm{Hom}(\hat{A},Q) \to \mathrm{Ext}^1(\hat{A},M)$, use the presentation $0 \to K \to P' \xrightarrow{\#} \hat{A} \quad 0$ of $\hat{A}$. The required map is the connecting homomorphism of:

$$\begin{tikzcd}
 & & (\mathrm{Hom}(\hat{A},Q)) \arrow[d] & \\
 & \mathrm{Hom}(P'^{*},M) \arrow[r] \arrow[d] & \mathrm{Hom}(P'^{*},N) \arrow[r] \arrow[d] & \mathrm{Hom}(P'^{*},Q) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, dash] & \mathrm{Hom}(K,M) \arrow[r] \arrow[d] & \mathrm{Hom}(K,N) \arrow[r] & \mathrm{Hom}(K,Q) & \\
 & (\mathrm{Ext}^1(\hat{A},M)) & & &
\end{tikzcd} \qquad (\#3)$$

Similiarly $\mathrm{Tor}_1(A,Q) \to A\otimes M$ is the connecting homomorphism of the following diagram, using the presentation $0 \to L \to P \to A \to 0$

$$\begin{tikzcd}
 & & & (\mathrm{Tor}_1(A,Q)) \arrow[d] & \\
 & L\otimes M \arrow[r] \arrow[d] & L\otimes N \arrow[r] \arrow[d] & L\otimes Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, dash] & P\otimes M \arrow[r, dash] \arrow[d] & P\otimes N \arrow[r, dash] & P\otimes Q & \\
 & (A\otimes M) & & &
\end{tikzcd} \qquad (\#4)$$

There is a natural map from (#3) to (#4) which induces maps into the kernels of (#4) out of the cokernels (#3), connecting the two corresponding ker-coker sequences naturally. Explicitly:

$$\begin{tikzcd}
\mathrm{Hom}(P'^{*},-) \arrow[r] \arrow[d] & P'\otimes - \arrow[r] \arrow[d] & L\otimes - \arrow[d] \\
\mathrm{Hom}(K,-) \arrow[r] & \mathrm{Hom}(P^{*},-) \arrow[r] & P\otimes -
\end{tikzcd}$$

fill in M, N, Q for $-$. So we have

$$\begin{tikzcd}
\mathrm{Hom}(\hat{A},Q) \arrow[r, "w"] \arrow[d] & \mathrm{Ext}^1(\hat{A},M) \arrow[d] \\
\mathrm{Tor}_1(A,Q) \arrow[r, "w"] & A\otimes M
\end{tikzcd} \quad \text{commuting}$$

The first vertical map induced into the kernel $\mathrm{Tor}_1(A,Q)$ of the kercoker sequence of (#4) and the second vertical map induced out of the cokernal $\mathrm{Ext}^1(\hat{A},M)$ of the ker-coker sequence of (#3). Finally to show these maps are the same as those arising from (#1) and (#2). Let X be either

M, N or Q, the maps connecting the two ker-coker sequences are then the unique maps making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}(\hat{A},X) \arrow[r] \arrow[dd, dashed] & \mathrm{Hom}(P'^*,X) \arrow[r] \arrow[d] & \mathrm{Hom}(K,X) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\hat{A},X) \arrow[dd, dashed] \\
 & P'\otimes X \arrow[d] & \mathrm{Hom}(P^*,X) \arrow[d, "\cong"] & \\
\mathrm{Tor}_1(A,X) \arrow[r] & L\otimes X \arrow[r] & P\otimes X \arrow[r] & A\otimes X
\end{tikzcd} \qquad (\#5)$$

Examine diagram (#1) where $\mathrm{Ext}^1(A,M) \to A\otimes M$ is defined via w , and the "snaking" shows it is required map making (#5) commute at the right end for X = M. The same applies for (#2) with X = Q looking at the left end of (#5).

(v) Also:

$$\begin{tikzcd}
\mathrm{Hom}(\hat{A},Q) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\hat{A},M) \arrow[d] \\
\mathrm{Tor}_1(A,Q) \arrow[r] \arrow[ur, shift left, "v_1"] \arrow[ur, shift right, "v_2"'] & A\otimes M
\end{tikzcd}$$

v_1 the map induced out of the cokernel $\mathrm{Tor}_1(A,Q)$ and

v_2 the map induced into the kernel $\mathrm{Ext}^1(A,M)$.

But $\downarrow\nearrow\downarrow = \overset{\rightarrow}{}\downarrow = \downarrow\rightarrow$, and epics can be cancelled so $\nearrow\downarrow = \rightarrow$, hence $v_1 = v_2$. This gives (d) using ker-coker lemma.

(vi)

$$\begin{tikzcd}
 & & \mathrm{Tor}_1(A,Q) \arrow[dr] & & \\
\mathrm{Hom}(\hat{A},N) \arrow[r] & \mathrm{Hom}(\hat{A},Q) \arrow[ur, two heads] \arrow[rr, "u"] \arrow[dr] & & A\otimes M \arrow[r] & A\otimes N \\
 & & \mathrm{Ext}^1(\hat{A},M) \arrow[ur, tail] & &
\end{tikzcd}$$

Im u = $\mathrm{Im}(\mathrm{Tor}_1(A,Q) \to A\otimes M)$ because $\mathrm{Hom}(\hat{A},Q) \twoheadrightarrow \mathrm{Tor}_1(A,Q)$

$= \ker(A\otimes M \to A\otimes N)$.

ker u = $\ker(\mathrm{Hom}(\hat{A},Q) \to \mathrm{Ext}^1(\hat{A},M))$ because $\mathrm{Ext}^1(\hat{A},M) \rightarrowtail A\otimes M$

$= \mathrm{Im}(\mathrm{Hom}(\hat{A},N) \to \mathrm{Hom}(\hat{A},Q))$.

This establishes (e) and (iv), (v) and (vi) give (f). and (g) comes from (a) to (f) and the snake lemma again. //

Corollary 2.2 Let A be as in the theorem, then

$0 \to \mathrm{Ext}^1(\hat{A},R) \to A \to A^{**} \to \mathrm{Ext}^2(\hat{A},R) \to 0$ ((11) page 142.)

$0 \to \mathrm{Ext}^1(A,R) \to \hat{A} \to \hat{A}^{**} \to \mathrm{Ext}^2(A,R) \to 0$//

Corollary 2.3 ((13) page 81) All f.p. left modules are torsionless if and only if all f.p. right modules are W-modules (B is W-module if $\mathrm{Ext}^1(B,R) = 0$).//

Corollary 2.4 In the situation of the theorem, where L and K are defined so that:

$$\begin{tikzcd} 0 \arrow[r] & \hat{A}^* \arrow[r] & P' \arrow[rr] \arrow[dr] & & P \arrow[r] & A \arrow[r] & 0 \\ & & & L \arrow[ur] & & & \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & A^* \arrow[r] & P^* \arrow[rr] \arrow[dr] & & P'^* \arrow[r] & \hat{A} \arrow[r] & 0 \\ & & & K \arrow[ur] & & & \end{tikzcd}$$

then (a) $0 \to L \to K^* \to \mathrm{Ext}^1(\hat{A},R) \to 0$

(a') $0 \to K \to L^* \to \mathrm{Ext}^1(A,R) \to 0$

and (b)
$$\begin{tikzcd} P' \arrow[r] \arrow[dr] & K^* \arrow[r] & P \\ & L \arrow[u] \arrow[ur] & \end{tikzcd}$$
(b')
$$\begin{tikzcd} P^* \arrow[r] \arrow[dr] & L^* \arrow[r] & P'^* \\ & K \arrow[u] \arrow[ur] & \end{tikzcd}$$
commute.

Proof
$$\begin{tikzcd} 0 \arrow[r] & A^* \arrow[r] \arrow[d, equal] & P' \arrow[r] \arrow[d, "\cong"'] & L \arrow[r] \arrow[d, "h"] & 0 & \\ 0 \arrow[r] & A^* \arrow[r] & P'^{**} \arrow[r] & K^* \arrow[r] & \mathrm{Ext}^1(\hat{A},R) \arrow[r] & 0 \end{tikzcd}$$

h is induced out of the cokernel and the snake lemma gives (a). The left half side of (b) commutes by definition of L. for right half side

$$\begin{tikzcd} P' \arrow[dr] & K^* \arrow[r] & P \\ & L \arrow[u] & \end{tikzcd} = P' \longrightarrow K^* \longrightarrow P = P' \longrightarrow P = \begin{tikzcd} P' \arrow[dr] & & P \\ & L \arrow[ur] & \end{tikzcd}$$

$P' \twoheadrightarrow L$ can be cancelled, so the right half commutes also. (a') and (b') by left-right symmetry.

Corollary 2.5 ((13) page 71) If A is a torsionless f.p. right module there exists K a finitely generated left submodule of a free module such that:

$0 \to A^* \to P^* \to K \to 0$, $\quad 0 \to K^* \to P \to A \to 0$

$0 \to A \to A^{**} \to \mathrm{Ext}^1(K,R) \to 0$, $\quad 0 \to K \to K^{**} \to \mathrm{Ext}^1(A,R) \to 0$

Conversely such K give rise to A torsionless and f.p.

Proof Apply Corollary 2.2 and Cor. 2.4 to get $L = K^*$

From Cor. 2.2 also; $0 \to A \to A^{**} \to \mathrm{Ext}^2(\hat{A},R) \cong \mathrm{Ext}^1(K,R) \to 0$

and $0 \to K \to L^* \cong K^{**} \to \mathrm{Ext}^1(A,R)$ (from Cor. 2.4).//

Remarks

(a) This establishes a correspondence between A_r the class of f.p. torsionless right modules and B_l the class of f.g. modules isomorphic to submodules of free left modules. Now if R is left coherent $B_l \subseteq A_l$, and also duals of f.p. rights are f.p. left (see ahead Prop. 4.1). So for $A \in A_r$, let $F \twoheadrightarrow A^*$, F f.g. free, then $0 \to A \to A^{**} \to F^*$ embeds A in a free module so $A_r \subseteq B_r$. This is the appropriate generalization of : R left noetherian, a f.g. torsionless right module can be embedded in a f.g. free module ((3) 4.5). For R left and right coherent $A_r = B_r$, $A_l = B_l$.

(b) The restriction that A be f.p. is not necessary here in the following sense:

Proposition 2.6 Let A be f.g., define K and L such that $0 \to L \to P \to A \to 0$, $0 \to A^* \to P^* \to K \to 0$ (P f.g. proj.)

then (i) $0 \to K \to L^* \to \mathrm{Ext}^1(A,R) \to 0$

(ii) $0 \to L \to K^* \to \mathrm{Ker}(A \to A^{**}) \to 0$

(iii) K is f.g. and torsionless.

Proof For (iii) K is a submodule of L* which is torsionless and $P^* \twoheadrightarrow K$.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "h"'] & P \arrow[r] \arrow[d, "\cong"'] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K^* \arrow[r] & P^{**} \arrow[r] & A^{**} & \end{tikzcd}$$

h is induced into the kernel K*, snake lemma then gives (ii).

$$\begin{tikzcd} 0 \arrow[r] & A^* \arrow[r] \arrow[d, equal] & P^* \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d, "k"] & 0 & \\ 0 \arrow[r] & A^* \arrow[r] & P^* \arrow[r] & L^* \arrow[r] & \mathrm{Ext}^1(A,R) \arrow[r] & 0 \end{tikzcd}$$

k is induced out of cokernel snake lemma gives (i).//

Corollary 2.7 If A is f.g. and torsionless, there is a f.g. torsionless K, such that: ((13) page 71)

$0 \to A \to A^{**} \to \mathrm{Ext}^1(K,R) \to 0 \quad 0 \to K \to K^{**} \to \mathrm{Ext}^1(A,R) \to 0$

Proof If A is torsionless, then $L \cong K^*$ ((ii) of Prop.) then (i) gives $0 \to K \to K^{**} \to \mathrm{Ext}^1(A,R) \to 0$

Then:
$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\cong"'] & P \arrow[r] \arrow[d, "\cong"'] & A \arrow[r] \arrow[d] & 0 & \\ 0 \arrow[r] & K^* \arrow[r] & P^{**} \arrow[r] & A^{**} \arrow[r] & \mathrm{Ext}^1(K,R) \arrow[r] & 0 \end{tikzcd}$$

gives $0 \to A \to A^{**} \to \mathrm{Ext}^1(K,R) \to 0$ by snake lemma again.//

Thus we have a correspondence between f.g. torsionless right and f.g. torsionless lefts. Following two results extend results of Jans, ((13) page 73).

Proposition 2.8 A f.p. right, $A^* = 0$, then:

(i) $A \otimes - \cong \mathrm{Ext}^1(\hat{A},-)$ (ii) $\mathrm{Hom}(A,-) \cong \mathrm{Tor}_1(-,\hat{A})$

in particular $A \cong \mathrm{Ext}^1(\hat{A},R)$ and if $A \neq 0$ then p.d.$\hat{A} = 1$ (projective dimension).

Proof apply (c) of Thm. 2.1 to get (i), (ii) and $\mathrm{Ext}^2(\hat{A},-) = 0$, but if A is projective then $A \cong \mathrm{Ext}^1(\hat{A},R) = 0$. Hence p.d.$\hat{A} = 1$.//

Proposition 2.9 A f.p. right, p.d.A = 1 then: $\hat{A} \cong \mathrm{Ext}^1(A,R)$ and $\mathrm{Ext}^1(A,R)^* = 0$.

Proof If $0 \to P' \to P \to A \to 0$ is exact, then $\hat{A}^* = 0$. Hence by last proposition $\hat{A} \cong \mathrm{Ext}^1(A,R)$.//

Proposition 2.10 Following are equivalent:

(i) R is Regular.

(ii) All f.p. left modules are projective.

(iii) All f.p. right modules are projective.

In fact A is projective if and only if $\hat{A}$ is projective.

Proof Clearly (i) $\Leftrightarrow$ (ii) and (iii). (f.p. flats are projective)

(ii) $\Rightarrow$ (i) If L is f.g. left ideal then $0 \to L \to R \to R/L \to 0$ splits by (ii), hence L is a direct summand.

(ii) $\Leftrightarrow$ (iii) by (c) of Thm. 2.1 $A\otimes - \xrightarrow{\cong} \mathrm{Hom}(A^*,-)$ implies $\mathrm{Ext}^1(\hat{A},-) = 0$, hence result follows by (b) of Thm.2.1.//

For the moment we jump a dimension.

Proposition 2.11 Sup $\{$p.d.A: A f.p. right$\} \leq 2 \Leftrightarrow$ duals of f.p. left are projective.

Proof $0 \to \hat{A}^* \to P' \to P \to A \to 0$ gives the result.//

Corollary 2.12 ((3) 5.2) If R is right and left noetherian, left global dimension R $\leq 2 \Leftrightarrow$ duals of f.g. right is projective.

Proof f.p. coincides with f.g. and global dimension can be calculated over f.g.//

More generally:

Proposition 2.13 ((14) 3.1) Following are equivalent:

(i) Sup $\{$p.d.A; A right f.p.$\} \leq n+2$

(ii) Dual of any f.p. left has p.d. $\leq n$.//

Proposition 2.14 Let $P' \to P \to A$ and $Q' \to Q \to A$ be two finite presentations of A. Let B $(=\hat{A})$ be coker $P^* \to P'^*$, and C be coker $Q^* \to Q'^*$. Then $\mathrm{Ext}^n(C,-) \cong \mathrm{Ext}^n(B,-)$.

Proof For n = 1,2, follows because they are ker and coker of $A\otimes - \to \mathrm{Hom}(A^*,-)$ by (c) of Thm. 2.1. For $n > 2$, $0 \to A^* \to P^* \to P'^* \to B \to 0$ gives $\mathrm{Ext}^n(B,-) \cong \mathrm{Ext}^{n-2}(A^*,-)$.//

Proposition 2.15 Following are equivalent:

(i) R is left semihereditary.

(ii) Sup $\{$p.d.B; B f.p. left$\} \leq 1$.

(iii) any f.p. right A, is of the form $A = A'\oplus P$ where P is f.g. proj., A' f.p. and $A'^* = 0$.

Proof (i) $\Leftrightarrow$ (ii) is standard.

(ii) $\Rightarrow$ (iii) for A f.p. right, we have as before

$$0 \to A^* \to P^* \to P'^* \to B \to 0, \quad P^* \searrow K \nearrow P'^*$$

p.d.B $\leq 1 \Rightarrow$ K is projective $\Rightarrow A^*$ is projective so $0 \to A^* \to P^* \to K \to 0$ splits, so remains (split) exact when dualed, (Also A^* is also f.g.).

Then

$$\begin{array}{ccccccccc} 0 & \to & L & \to & P & \to & A & \to & 0 \\ & & \downarrow & & \downarrow\cong & & \downarrow & & \\ 0 & \to & K^* & \to & P^{**} & \to & A^{**} & \to & 0 \end{array}$$

Snake lemma gives $A \twoheadrightarrow A^{**}$, but A^{**} is projective. Hence:

$A \cong A^{**}\oplus \ker(A \to A^{**}) \cong A^{**}\oplus \mathrm{Ext}^1(A,R)$, (Cor. 2.2)

and $\mathrm{Ext}^1(A,R)^* = 0$ by Prop. 2.9.

(iii) $\Rightarrow$ (ii) Given B f.p. left, construct $A = \hat{B}$. Then $A \cong A' \oplus P$ where $A'^* = 0$. Let $Q' \to Q \to A' \to 0$ be exact Q', Q, f.g. proj. then $Q' \longrightarrow Q \oplus P \to A' \oplus P \cong A \to 0$ use this presentation to compute $\hat{A}$, since $A'^* = 0$,
$0 \to P^* \to Q^* \oplus P^* \to Q'^* \to \hat{A} \to 0$ this gives $0 \to Q^* \to Q'^* \to \hat{A} \to 0$ (the P^* just rides along, note $\hat{A} = \widehat{A'}$), so p.d.$\hat{A} \leq 1$ but by prop.2.14, p.d.B = p.d.$\hat{A} \leq 1$.//

<u>Corollary 2.16</u> R left semihereditary. A f.p. right

(i) $A^{**} = P$ is a f.g. projective.

(ii) $\mathrm{Hom}(A,-) \cong \mathrm{Hom}(P,-) \oplus \mathrm{Tor}_1(-,\hat{A})$.

(iii) $A \otimes - \cong P \otimes - \oplus \mathrm{Ext}^1(\hat{A},-)$.

(iv) For left f.p.B, the assignment $B \mapsto \hat{B}$ can be made functorial, by setting $\hat{B} = \mathrm{Ext}^1(B,R)$.

<u>Proof</u> For (i), (ii) and (iii) use (c) of Thm. 2.1, and the fact that if $A = A' \oplus P$ then $\hat{A} = \widehat{A'}$, and proof of last proposition that A^{**} is f.g. projective. For (iv) choose for each B, an exact sequence $0 \to Q' \to Q \to B \to 0$, Q', Q f.g. proj., then $\hat{B} = \mathrm{Ext}^1(B,R)$.//

<u>Proposition 2.17</u> A f.p., A is proj. $\Leftrightarrow \mathrm{Ext}^1(A,L) = 0$ for all f.g. left ideals L.

<u>Proof</u> Using $0 \to L \to R \to R/L \to 0$ in (e) of Thm. 2.1 gives $\mathrm{Tor}_1(\hat{A},R/L) = 0$. Thus $\hat{A}$ is flat, hence projective, and so A is also projective.//

<u>Proposition 2.18</u> R left coherent, A f.p. left. Following are equivalent: (i) A is projective.

(ii) $\mathrm{Ext}^1(A,B) = 0$ for all f.p. B.

(iii) $\mathrm{Ext}^1(A,B) = 0$ for all cyclic f.p. B.

Proof (i) $\Leftrightarrow$ (ii) Left coherence implies f.g. left ideals are f.p., apply last proposition.

(ii) $\Leftrightarrow$ (ii) Induct on minimal number of generators. For B f.p. let x be an element of a minimal generating set. Use $0 \to Rx \to B \to B/Rx \to 0$ and induction in long exact Ext sequence. //

Corollary 2.19 ((15) Prop.2) R left coherent, A f.p. left. Following are equivalent:

(i) p.d. $A \leq n$

(ii) $\mathrm{Tor}_{n+1}(C,A) = 0$, for all f.p. C

(ii') $\mathrm{Tor}_{n+1}(C,A) = 0$, for all f.p. cyclic C

(iii) $\mathrm{Ext}^{n+1}(A,B) = 0$, for all f.p. B

(iii') $\mathrm{Ext}^{n+1}(A,B) = 0$, for all f.p. cyclic B.

Proof (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (ii') gives flat dimension $A \leq n$, but A is f.p. and R is left coherent, so flat dimension = proj. dimension.

(i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iii') By the last proposition and dimension shifting (R is left coherent). //

Proposition 2.20 ((13) page 74) R left coherent, then $\mathrm{Sup}\,\{\text{p.d.}B \,;\, B \text{ f.p. left, p.d. } B < \infty\} = 0 \Leftrightarrow$ for A f.p. right, $A^* = o$ implies $A = 0$.

Proof ($\Rightarrow$) by Proposition 2.8

($\Leftarrow$) by Proposition 2.9. For if B has p.d. ≤ 1 then $\hat{B}^* = 0 \Rightarrow \hat{B} = 0 \Rightarrow B$ is projective. //

<u>Proposition 2.21</u> ((3) 5.3) R left coherent, then $\operatorname{Sup}\{\text{p.d. } B : B \text{ f.p. left, p.d. } B < \infty\} \leq 1 \Leftrightarrow$ for A f.p torsionless, A* proj. implies A proj.

<u>Proof</u> ($\Rightarrow$) Suppose A* is proj., then p.d. $\hat{A} < \infty$, hence p.d. $\hat{A} \leq 1$, hence K (as in Cor. 2.4) is proj. Then by Cor. 2.5, $A \cong A^{**}$ is proj. ($\Leftarrow$) If there is a f.p. module with finite proj. dimension greater than 2, there is a torsionless f.p. A with p.d. equal 1, (left coherence).
By Cor. 2.4

$$\begin{tikzcd}[column sep=small] o \arrow[r] & \hat{A}^* \arrow[r] & P' \arrow[rr] \arrow[dr] & & P \arrow[r] & A \arrow[r] & 0 \\ & & & K^* \arrow[ur] & & & \end{tikzcd}$$

so K* is proj., by assumption K is projective. Then by Cor. 2.5 $A \cong A^{**}$ and $0 \to A^* \to P^* \to K \to 0$, so A* is proj., hence A is proj. //

3/ Absolute Purity (f.p. Injectivity)

In the category $\mathcal{E}$ of short exact sequences consider the set of finite presentations $\mathcal{G}$; that is short exact sequences $\underline{G} : 0 \to G'' \to G \to G' \to 0$, G f.g. proj. G'' f.g.. A sequence $\underline{A}$ is pure if for any $\underline{G} \xrightarrow{f} \underline{A}$, $\underline{G} \in \mathcal{G}$ then f factors over a split short exact sequence, ($f \sim 0$, see ahead Prop. 6.4, so that in $\mathcal{E}/\mathcal{S}$, $f = 0$; $\mathcal{E}/\mathcal{S}(\underline{G},\underline{A}) = 0$ for all $G \in \mathcal{G}$). Let $\mathcal{P}$ be the class of pure sequences. Then $\underline{H}$ is copure if for any $\underline{H} \xrightarrow{f} \underline{A}$, $A \in \mathcal{P} = f\ 0$. A module A (respectively C) is absolutely pure (flat) if whenever $\underline{E} : 0 \to A \to B \to C \to 0$ is exact, then $\underline{E}$ is pure.

Consider the situation
$$\begin{tikzcd} & & & M \arrow[d, "f"] & \\ \underline{E} : 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$
where M is f.p. and $\underline{E}$ pure, embed this in the diagram

$$\begin{tikzcd} \underline{G} \arrow[d, "f"'] : & 0 \arrow[r] & K \arrow[r] \arrow[d, "f''"] & P \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & 0 \\ \underline{E} : & 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

P f.g. proj., f' induced by projectivity of P.

Since $\underline{G}$ is copure and $\underline{E}$ is pure, $\underline{f} \sim 0$ so by Prop. 6.4
$$\begin{tikzcd} & M \arrow[d, "f"] \arrow[dl, dashed] \\ B \arrow[r] & C \end{tikzcd}$$
fills in, that is $\mathrm{Hom}(M,B) \twoheadrightarrow \mathrm{Hom}(M,C)$, for all f.p. M.

Proposition 3.1 $\underline{E} : 0 \to A \to B \to C \to 0$ is pure if and only if $\mathrm{Hom}(M,B) \longrightarrow \mathrm{Hom}(M,C) \to 0$ for all f.p. M.

Proof ($\Rightarrow$) by above.

($\Leftarrow$) by definition of purity and Prop. 6.4. //

Corollary 3.2 A is absolutely pure (C is flat) is and only if whenever $0 \to A \to B \to C \to 0$ is exact,.

$\mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to 0$ is also exact, for all f.p. M. //

Lemma 3.3 $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ exact, a epic $\Leftrightarrow$ b = 0 $\Leftrightarrow$ c monic. //

Following proposition shows flatness and purity as defined on previous page, coincides with the standard concepts.

<u>Proposition 3.4</u> $\underline{E} : 0 \to A \to B \to C \to 0$

(i) $\underline{E}$ pure $\Leftrightarrow \underline{E} \otimes M$ is exact for any f.p. M (hence, any M),

$$(\underline{E} \otimes M = 0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0)$$

(ii) C flat $\Leftrightarrow \underline{E} \otimes M$ exact for all such $\underline{E}$ involving C & M f.p.

$\Leftrightarrow \mathrm{Tor}_1(C,M) = 0$ for any f.p. M (hence any M)

(iii) A absolutely pure $\Leftrightarrow E \otimes M$ exact for any such $\underline{E}$ involving A and all f.p. M.

$\Leftrightarrow \mathrm{Ext}^1(M,A) = 0$ for all f.p. M.

<u>Proof</u>

(i) Apply part (d) of Thm. 2.1, Lemma 3.3 and Prop. 3.1.

$$\mathrm{Hom}(\hat{M},B) \xrightarrow{a} \mathrm{Hom}(\hat{M},C) \xrightarrow{b} A \otimes M \xrightarrow{c} B \otimes M$$

(ii) First equivalence from (i) above. For second equivalence, take B projective then:

$$0 \to \mathrm{Tor}_1(C,M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

So if $A \to B \to C$ is pure (C flat by definition on previous page) then (i) implies $\mathrm{Tor}_1(C,M) = 0$.

Conversely
$$\begin{tikzcd}
\mathrm{Hom}(\hat{M},B) \arrow[r, "a"] & \mathrm{Hom}(\hat{M},C) \arrow[r, "b"] \arrow[d] & \mathrm{Ext}^1(\hat{M},A) \arrow[d] \\
& \mathrm{Tor}_1(M,C) \arrow[r] & M \otimes A
\end{tikzcd}$$

$\mathrm{Tor}_1(M,C) = 0 \Rightarrow b = 0 \Rightarrow a$ is onto for all M, apply Cor. 3.2.

(iii) First equivalence by (i).

$$\mathrm{Hom}(M,B) \xrightarrow{a} \mathrm{Hom}(M,C) \to \mathrm{Ext}^1(M,A) \to \mathrm{Ext}^1(M,B)$$

If A is absolutely pure a is epic, take B injective,

$A \twoheadrightarrow B$. Conversely $\mathrm{Ext}^1(M,A) = 0$, implies a epic for any M, hence A absolutely pure by Cor. 3.2.//

A is called copure injective if given $\underline{G}$:

$$\begin{tikzcd} \underline{G} : 0 \arrow[r] & G'' \arrow[r] \arrow[dr, "f"'] & \underline{G} \arrow[r] \arrow[d, dashed, "\bar{f}"] & G' \arrow[r] & 0 \text{ copure} \\ & & A & & \end{tikzcd}$$

then f has an extension $\bar{f}$.

<u>Proposition 3.5</u> A copure inj. $\Leftrightarrow$ A is absolutely pure.

<u>Proof</u> ($\Rightarrow$)

$$\begin{tikzcd} \underline{G} : 0 \arrow[r] \arrow[d, "\underline{f}"'] & G'' \arrow[r] \arrow[d, "f''"] & G \arrow[r] \arrow[d, "f"] & G' \arrow[r] \arrow[d, "f'"] & 0 \\ \underline{E} : 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

, $\underline{G}$ any copure.

since A is copure inj. f" extends to G, by Prop.6.4, $\underline{f} \sim 0$, hence $\underline{E}$ is pure, and A is absolutely pure,

($\Leftarrow$) given

$$\begin{tikzcd} \underline{G} : 0 \arrow[r] & G'' \arrow[r] \arrow[d, "f''"'] & G \arrow[r] & G' \arrow[r] & 0 \\ & A & & & \end{tikzcd}$$

Consider a sequence $\underline{E}$: $0 \to A \to B \to C \to 0$ with B inj.

then

$$\begin{tikzcd} \underline{G} : 0 \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, dashed] & G' \arrow[r] \arrow[d, dashed] & 0 \\ \underline{E} : 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

f" induces $\underline{f} : \underline{G} \to \underline{E}$, since B is injective; $\underline{E}$ is pure since A is absolutely pure, hence $\underline{f} \sim 0$, and the required extension of f" exists. //

<u>Remark</u> The condition $\mathrm{Ext}^1(M,A) = 0$ for all f.p. M is called <u>f.p. injectivity</u>.

Thus absolute pure = f.p. injective = copure injective.

<u>Corollary 3.6</u> ((17) Cor.2 ,page 562)

A is absolutely pure $\Leftrightarrow$ given

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "f"] & P \\ A & \end{tikzcd}$$

with N f.g. and P projective, f has an extension to P.

<u>Proof</u> ($\Rightarrow$) embed $P \subseteq F$, F free, then $N \subseteq F' \subseteq F$ for some

f.g. direct summand of F. Then $0 \to N \to F' \to F'/N \to 0$ is a finite presentation hence is copure. f thus has an extension to F' by Prop. 3.5, this can further by extended to F since F' is a direct summand and then restricted to P. (note $0 \to N \to P \to P/N \to 0$ is copure).

($\Leftarrow$)
$$\begin{array}{cccccccccc} \underline{G} & : & 0 & \to & G'' & \to & G & \to & G' & \to 0 \\ \underline{f}\downarrow & & & & f''\downarrow & & \downarrow & & \downarrow & \\ \underline{E} & : & 0 & \to & A & \to & B & \to & C & \to 0 \end{array}$$

$\underline{G}$ a finite presentation, by hypothesis f" extends to G, hence $\underline{f} \sim 0$, so $\underline{E}$ is pure by definition, $\underline{E}$ is arbitrary so A is absolutely pure.

Corollary 3.7 ((16) Thm .2)

Suppose (M_i) is a directed system, such that $M_i \rightarrowtail \varinjlim M_i$ (for instance submodules of some fixed M, ordered by inclusion) Then M_i absolutely pure for all i $\Rightarrow$ M absolutely pure.

Proof $\underline{G} : 0 \to G'' \to G \to G' \to 0$, $\underline{G}$ a finite presentation
$$\begin{array}{c} G'' \\ f\downarrow \\ \varinjlim M_i \end{array}$$

since G" is f.g. and $M_i \rightarrowtail \lim M_i$, f factors over some M_i,
$$\begin{array}{ccccc} 0 & \to & G'' & \to & G \\ & & f\downarrow & \searrow^{\bar f} & \downarrow \\ & & \varinjlim M_i & \leftarrowtail & M_i \end{array}$$
then $\bar{f}$ extends to G, hence f extends to G.//

Proposition 3.8 For any family (A_i), following are equivalent (i) A_i absolute pure all i.

(ii) $\prod A_i$ ab. pure.

(iii) $\oplus A_i$ ab. pure.

Proof (i) $\Leftrightarrow$ (ii) $\mathrm{Ext}^1(M,\prod A_i) \cong \prod \mathrm{Ext}^1(M,A_i)$.

(i) $\Rightarrow$ (iii) by Cor. 3.7

(iii) $\Rightarrow$ (i) direct summands of ab. pure are absolutely pure, $\mathrm{Ext}^1(M,A\oplus A') = \mathrm{Ext}^1(M,A)\oplus\mathrm{Ext}^1(M,A')$. //

Proposition 3.9 ((17) Thm 2.)

R is semi-hereditary $\Leftrightarrow$ The homomorphic image of an absolutely pure module is absolutely pure.

Proof Let A f.p. left and $0 \to L \to M \to N \to 0$ exact, then $\mathrm{Ext}^1(A,M) \to \mathrm{Ext}^1(A,N) \to \mathrm{Ext}^2(A,L) \to \mathrm{Ext}^2(A,M)$. ($\Rightarrow$) let M be f.p. inj., if R is semi-hereditary. p.d. $A \leq 1$. So above sequence implies $\mathrm{Ext}^1(A,N) = 0$. A is arbitrary so N is f.p. injective. ($\Leftarrow$) For any L let M be an inj. module containing L. Then N is f.p. inj., hence the sequence implies $\mathrm{Ext}^2(A,L) = 0$ so p.d. $A \leq 1$. //

4/ Coherence and f.p. Injectivity

In this section, imposing the conditions of coherence and f.p. injectivity on the ring R, are investigated. A ring is left coherent if f.g. submodules of f.p. lefts are f.p.

Proposition 4.1 R is right coherent $\Leftrightarrow$ duals of f.p. left are f.g. (in which case they are also f.p.)

Proof ($\Rightarrow$) A left f.p.

$$\begin{tikzcd} 0 \arrow[r] & A^* \arrow[r] & P^* \arrow[rr] \arrow[dr] & & P'^* \arrow[r] & \hat{A} \arrow[r] & 0 \\ & & & L \arrow[ur] & & & \end{tikzcd}$$

L is f. g. right, hence f.p. $\Rightarrow$ A* is f.g. (and coherence gives A* is f.p.) ($\Leftarrow$) Suppose $K \twoheadrightarrow P$, P f.g. projective, K f.g. form

$$\begin{tikzcd} 0 \arrow[r] & \hat{A}^* \arrow[r] & P' \arrow[rr] \arrow[dr] & & P \arrow[r] & A \arrow[r] & 0 \\ & & & K \arrow[ur] & & & \end{tikzcd}$$

by hypothesis $\hat{A}^*$ is f.g. hence K is f.p.

If now $K \subseteq M$, M f.p. and K f.g., form

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & L'' \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P'' \arrow[r] \arrow[d] & P'\oplus P'' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/K \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

P', P" f.g. proj.

by what has just been proved L, L' are f.p. hence L" is f.g., thus K is f.p.//

Proposition 4.2 If R is right coherent then following are equivalent (i) A is absolutely pure.

(ii) $\mathrm{Ext}^1(M,A) = 0$ for all cyclic f.p. M.

(iii) $\begin{tikzcd} I \arrow[r, tail] \arrow[dr] & R \arrow[d, dashed] \\ & A \end{tikzcd}$ can be filled in for I f.g. left ideal.

<u>Proof</u> (i) ($\Leftrightarrow$) (ii) Induct on number of generators and use $\mathrm{Ext}^1(M/xR,A) \to \mathrm{Ext}^1(M,A) \to \mathrm{Ext}^1(xR,A)$ taking out a generator of an arbitrary f.p. M. (with right coherence "shifting" on the f.p.'s is allowed).

(ii) ($\Leftrightarrow$) (iii) from $\mathrm{Hom}(R,A) \to \mathrm{Hom}(I,A) \to \mathrm{Ext}^1(R/I,A) \to 0$. //

<u>Theorem 4.3</u>

(a) M is flat $\Leftrightarrow \theta_A : A^*\otimes M \twoheadrightarrow \mathrm{Hom}(A,M)$ for all f.p. A.
If θ_A is epic for all A, it is necessarily an isomorphism.

(b) M is absolutely pure $\Leftrightarrow \Psi_A : A\otimes M \rightarrowtail \mathrm{Hom}(A^*,M)$ for all f.p. A.
If R is left coherent, then Ψ_A monic for all A implies Ψ_A is a isomorphism.

<u>Proof</u> (a) $A^*\otimes M \to \mathrm{Hom}(A,M) \to \mathrm{Tor}_1(\hat{A},M) \to 0$ from Thm. 2.1 and $\ker\theta_A = \mathrm{Tor}_2(\hat{A},M)$.

(b) $0 \to \mathrm{Ext}^1(\hat{A},M) \to A\otimes M \to \mathrm{Hom}(A^*,M) \to \mathrm{Ext}^2(\hat{A},M) \to 0$, from Thm. 2.1, gives the result. $\mathrm{Ext}^1(B,M) = 0$ implies $\mathrm{Ext}^2(B,M) = 0$ for all B f.p., only if shifting is possible, so left coherence is needed for an isomorphism. //

<u>Proposition 4.4</u> (Stated without proof by Stenström, page 323 (18)). If A is f.p. then for any direct system (M_i) $\varinjlim \mathrm{Hom}(A,M_i) \cong \mathrm{Hom}(A,\varinjlim M_i)$. Conversely if $\varinjlim \mathrm{Hom}(A,M_i) \twoheadrightarrow \mathrm{Hom}(A,\varinjlim M_i)$ for any directed system then A is f.p.

<u>Proof</u> (<u>Remark</u> The map $\varinjlim \mathrm{Hom}(A,M_i) \longrightarrow \mathrm{Hom}(A,\varinjlim M_i)$ is the unique map out of the direct limit, induced by the

compatible maps $\mathrm{Hom}(A,M_i) \to \mathrm{Hom}(A,\varinjlim M_i)$ which arise from $M_i \to \varinjlim M_i$.) Let $P' \to P \to A \to 0$ be exact with P, P' f.g. proj.

$$\begin{array}{ccccccc} 0 \to & \varinjlim \mathrm{Hom}(A,M_i) & \to & \varinjlim \mathrm{Hom}(P,M_i) & \to & \varinjlim \mathrm{Hom}(P',M_i) \\ & \downarrow h & & \downarrow \cong & & \downarrow \cong \\ 0 \to & \mathrm{Hom}(A,\varinjlim M_i) & \to & \mathrm{Hom}(P,\varinjlim M_i) & \to & \mathrm{Hom}(P',\varinjlim M_i) \end{array}$$

implies h is an isomorphism. Conversely, $A = \varinjlim A_i$, for some directed system (A_i) of f.p. modules.

If $\varinjlim \mathrm{Hom}(A,A_i) \twoheadrightarrow \mathrm{Hom}(A,\varinjlim A_i) = \mathrm{Hom}(A,A)$ then the identity factors over some A_i, $\begin{array}{c} A_i \\ \nearrow \searrow \\ A = A \end{array}$, hence A is a direct summand of a f.p. module A_i and is f.p. //

Following is a result of Watts (21).

Let E be an injective cogenerator,
$$\begin{array}{ccccc} 0 \to & A & \xrightarrow{n_A} & E\mathrm{Hom}(A,E) \\ & f\downarrow & & \downarrow E(f) \\ 0 \to & B & \to & E\mathrm{Hom}(B,E) \end{array}$$

$$E(f)\ (e_g)_{g\in \mathrm{Hom}(A,E)} = (e_{gf})_{g\in \mathrm{Hom}(B,E)}$$
$$n_A(a) = (g(a))_{g\in \mathrm{Hom}(A,E)}$$

E is a functor and n_A a natural transformation from the identity to E. Hence any direct system can be embedded in a directed system of injective modules.

<u>Theorem 4.5</u> ((18) Thm. 3.2)

Following are equivalent

(i) R right coherent.

(ii) The direct limit of absolutely pure modules is absolutely pure.

(iii) $\varinjlim \mathrm{Ext}^1(A,M_i) \xrightarrow{\cong} \mathrm{Ext}^1(A,\varinjlim M_i)$ for every f.p. A and direct system (M_i).

Proof (i) $\Rightarrow$ (ii) For A f.p. left, A^* is f.p. right by Prop. 4.1.

$$\begin{tikzcd} & (\varinjlim M_i)\otimes A \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(A^*, \varinjlim M_i) \arrow[d, "h"'] \\ 0 \arrow[r] & \varinjlim (M_i \otimes A) \arrow[r] & \varinjlim \mathrm{Hom}(A^*, M_i) \end{tikzcd}$$

h is an isomorphism by Prop.4.4; bottom row is injective since each M_i is absolutely pure, hence top row is injective and by Thm. 4.3, $\varinjlim M_i$ is absolutely pure. (ii) $\Rightarrow$ (iii) Let A be f.p. and (M_i) a direct system. Embed (M_i) in a direct system of injective modules (E_i) (see note before theorem). Let $\vec{N_i}$ denote $\varinjlim N_i$.

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim \mathrm{Hom}(A,M_i) \arrow[r] \arrow[d, "\cong"] & \varinjlim \mathrm{Hom}(A,E_i) \arrow[r] \arrow[d, "\cong"] & \varinjlim \mathrm{Hom}(A,E_i/M_i) \arrow[r] \arrow[d, "\cong"] & \varinjlim \mathrm{Ext}^1(A,M_i) \arrow[r] \arrow[d, "\theta"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(A,\vec{M_i}) \arrow[r] & \mathrm{Hom}(A,\vec{E_i}) \arrow[r] & \mathrm{Hom}(A,\overrightarrow{E_i/M_i}) \arrow[r] & \mathrm{Ext}^1(A,\vec{M_i}) \arrow[r] & \mathrm{Ext}^1(A,\vec{E_i} \end{tikzcd}$$

By hypothesis $\varinjlim E_i$ is absolutely pure, hence last term of the bottom row is zero, implying θ is an isomorphism.

(iii) $\Rightarrow$ (i) Let K be a f.g. submodule of a f.g. projective module P, it suffices to prove K is f.p. (by proof of Prop.4.1). If $0 \to K \to P \to A \to 0$, and (M_i) any direct system

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim \mathrm{Hom}(A,M_i) \arrow[r] \arrow[d, "\cong"] & \varinjlim \mathrm{Hom}(P,M_i) \arrow[r] \arrow[d, "\cong"] & \varinjlim \mathrm{Hom}(K,M_i) \arrow[r] \arrow[d, "g"] & \varinjlim \mathrm{Ext}^1(A,M_i) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(A,\vec{M_i}) \arrow[r] & \mathrm{Hom}(P,\vec{M_i}) \arrow[r] & \mathrm{Hom}(K,\vec{M_i}) \arrow[r] & \mathrm{Ext}^1(A,\vec{M_i}) \arrow[r] & 0 \end{tikzcd}$$

By hypothesis the last map is an isomorphism, hence g is also, then by Prop. 4.4, K is f.p. //

R is <u>right self-f.p. injective</u> if it is f.p. injective as a right module.

<u>Proposition 4.6</u> ((12) Thm. 2.3) R is right self-f.p. inj. $\Leftrightarrow$ all left f.p. are torsionless.

<u>Proof</u> $0 \to \mathrm{Ext}^1(\hat{A},R) \to A \to A^{**}$. //

Proposition 4.7 R is right coherent and right self-f.p. inj. $\Leftrightarrow$ all left f.p. are torsionless and their duals are f.g. (and in which case they are reflexive and f.p. respectively).

Proof Combine Prop. 4.1 and 4.6. Reflexive by "shifting." //

Proposition 4.8 R is right coherent and right self-f.p. inj.

$\Leftrightarrow$ (i) $I_1 \cap I_2$ is f.g. and

(ii) $l(I_1 \cap I_2) = l(I_1) + l(I_2)$ for all f.g. right ideals I_1, I_2.

(iii) $r(a)$ is f.g. and

(iv) $lr(a) = Ra$ for every $a \in R$.

Proof (i) and (iii) is equivalent to R being right coherent ((5) Thm. 2.2) (ii) and (iv) is equivalent to property (iii) of Prop 4.2 ((19),Prop.18.4), hence applying Prop.4.2 gives the result. //

Proposition 4.9 R right coherent and right self-f.p. inj. If A is f.p. right, then p.d. A = 0 or ∞.

Proof If B is f.p. left, and $B^* = 0$ then $B = 0$ because B is torsionless by Prop. 4.6, now apply Prop. 2.20. //

Proposition 4.10 ((18) Lemma 4.1) R right coherent and right f.p. inj. then flat right modules are absolutely pure. (f.p. inj.)

Proof For A f.p. left

$$\begin{tikzcd} 0 \arrow[r] & M\otimes A \arrow[r, "f"] \arrow[d, "h"] & M\otimes A^{**} \arrow[dl, "g", "\cong"'] \\ & \mathrm{Hom}(A^*,M) & \end{tikzcd}$$

f is monic because A is torsionless (Prop. 4.6) and M is flat. g is an isomorphism by Thm. 4.3, because A* is f.p. (Prop.4.1). Hence h is monic, and by Thm. 4.3, M is absolutely pure. //

<u>Proposition 4.11</u> ((18) Prop. 4.2) For R right and left coherent, following are equivalent:

(i) R is right self-f.p. inj.

(ii) f.p. inj. left modules are flat.

(iii) inj. left modules are flat.

<u>Proof</u>

$$\begin{tikzcd} A^*\otimes M \arrow[d, "\Psi"'] \arrow[dr, "\theta"] & \\ \mathrm{Hom}(A^{**},M) \arrow[r, "h"] & \mathrm{Hom}(A,M) \end{tikzcd}$$

A f.p. left.

(i) $\Rightarrow$ (ii) $A \cong A^{**}$ by Prop. 4.7., Ψ is an isomorphism since R is left coherent (Thm. 4.3). Thus θ is an isomorphism and M is flat by Thm. 4.3.

(iii) $\Rightarrow$ (i) If M is inj., hence flat, then by Thm.4.3, θ is an isomorphism. Thus h is epic for any inj. M, taking M an inj. cogenerator implies $A \twoheadrightarrow A^{**}$, and by Prop. 4.6, R is right self-f.p. inj. //

Rings for which injective $\Rightarrow$ flat are called IF rings. (Jain(12))

<u>Corollary 4.12</u> ((12) Thm.3.3)

Left IF rings are right self-f.p. inj.

<u>Proof</u> In (iii) $\Rightarrow$ (i) of the proposition, coherence was not used. //

5/ Generators and Relations

A finite presentation of B with n-generators and m-relations is an exact sequence of the form $R^m \to R^n \to B \to 0$. If $\widehat{B}$ is computed using this presentation, the resulting presentation of $\widehat{B}$ has m-generators and n-relations.

Example 5.1 Abelian Groups

For a finitely generated abelian group, $A^* \cong Z^r$, where r is the rank of A. If $A = Z/p^rZ$, choose the presentation $0 \to Z \xrightarrow{f} Z \to Z/p^rZ \to 0$, where f is multiplication by p^r. The dual of f, $f^* : Z \to Z$, is simply f again. Coker $f^* = \widehat{Z/p^rZ} = Z/p^rZ$. It can thus be arranged that $\hat{A} = A$ for any finitely generated abelian group. For any exact sequence $0 \to M \to N \to Q \to 0$ of abelian groups, then by Thm. 2.1 (g),

$$\begin{tikzcd}
0 \arrow[d] & \\
A^*\otimes Q \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}(A,Q) \arrow[r] \arrow[d] & \mathrm{Ext}^1(A,M) \arrow[d] \\
\mathrm{Tor}_1(A,Q) \arrow[r] \arrow[d] & A\otimes M \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(A^*,M) \arrow[d] \\
 & 0
\end{tikzcd}$$

If $A = Z^r \oplus T(A)$, $T(A)$ torsion subgroup, this becomes

$$\begin{tikzcd}
0 \arrow[d] & \\
Q^r \arrow[r] \arrow[d] & 0 \arrow[d] \\
Q^r\oplus \mathrm{Hom}(T(A),Q) \arrow[r] \arrow[d] & \mathrm{Ext}^1(A,M) \arrow[d] \\
\mathrm{Tor}_1(A,Q) \arrow[r] \arrow[d] & M^r\oplus(T(A)\otimes M) \arrow[d] \\
0 \arrow[r] & M^r \arrow[d] \\
 & 0
\end{tikzcd}$$

That is $\mathrm{Tor}_1(A,Q) \cong \mathrm{Hom}(T(A),Q) \cong \mathrm{Hom}(T(A),T(Q))$.

$\mathrm{Ext}^1(A,M) \cong T(A)\otimes M$

Example 5.2 Cyclic Modules

This will be a special case of Example 5.5. Let H be a finitely generated right ideal. Say $x_1, x_2, \ldots x_n$, generates H. Let $X = (x_1, x2, \ldots x_n) \in R^n$

$R \xrightarrow{X_r} R^n$, right multiplication (action) by X, $y \longmapsto (yx_1, yx_2, \ldots yx_n)$

induces

$$
\begin{tikzcd}
0 \arrow[r] & l(H) \arrow[r] & R \arrow[r] \arrow[d] & R^n \arrow[r] & R^n/RX \arrow[r] & 0 \\
& & R/l(H) \arrow[ur] & & &
\end{tikzcd} \qquad (A)
$$

where $l(H)$ is the left annihilator of H. The dual map of X_r is X_l let multiplication $(y_1, y_2 \ldots y_n) \longmapsto \sum^n x_i y_i$

inducing

$$
\begin{tikzcd}
0 \arrow[r] & (R^n/RX)^* \arrow[r] & R^n \arrow[r] \arrow[d] & R \arrow[r] & R/H \arrow[r] & 0 \\
& & H \arrow[ur] & & &
\end{tikzcd} \qquad (B)
$$

Thus $\widehat{R/H} = R^n/RX$ and $(R^n/RX)^* = \{Y = (y_1, \ldots y_n) : \sum x_i y_i = 0\}$ ($\{Y : (X,Y) = 0\}$ inner product notation).

In particular $\widehat{R/aR} = R/Ra$

Proposition 5.3 H f.g. right ideal. R/H is torsionless $\Leftrightarrow rl(H) = H$.

Proof Using Cor. 2.2 and 2.4, (A) and (B) above, and the fact that $(R/H)^* = r(H)$ (page 2), one obtains $0 \to H \to rl(H) \to (\ker : R/H \to (R/H)^{**})$. //

Corollary 5.4 If R is left self-f.p. inj. every finitely generated right ideal is a right annihilator.

Proof R/H is torsionless for all f.g. right ideals H, by Prop. 4.6. //

Example 5.5 General Case

Any f.p. arises as coker : $R^n \xrightarrow{X_l} R^m$

where $X = (x_{ij})$, $i = 1, \dots m$, $j = 1, \dots n$, $x_{ij} \in R$.

So that X_l is the left action by the matrix X on R^n, and the dual of X_l is X_r right action on R^m (the transpose of the matrix X)

$$0 \to \ker X_l \to R^n \to R^m \to R^m/XR^n \to 0$$

$$\text{and} \quad 0 \to \ker X_r \to R^m \to R^n \to R^n/R^mX \to 0 \ .$$

So $\widehat{R^m/XR^n} = R^n/R^mX$ and $(R^n/R^mX)^* \cong \ker X_l$ via $f \longmapsto (f(e_1), \dots f(e_n)$, where $\{e_i\}$ a basis of R^n, and f is regarded as an element of $(R^n)^*$ which annihilates R^mX.

5.6 Flatness By Thm. 4.3, and Ex. 5.5, M is flat if $\theta : \ker X_l \otimes W \longrightarrow \mathrm{Hom}(R^n/R^mX, W) \longrightarrow 0$ is exact for all possible choices of the matrix X.

$\mathrm{Hom}\ (R^n/R^mX, W) \cong \{w : Xw = 0\} = \ker X_l \subseteq W^n$ via $f \longmapsto (f(e_1), \dots f(e_n))$, where X acts on the left on W^n in the natural way.

θ is then the map $y \otimes w \longmapsto yw$, $y \in R^n$

$((y_1, \dots y_n) \otimes w \longmapsto (y_1 w, \dots y_n w))$.

W is thus flat only if given any $w = (w_1, \dots w_n) \in W^n$, such that $Xw = 0$, there is some k and

$Y_i \in \ker X_l \subseteq R^n$; $b_i \in W$, $i = 1 \dots k$

(or $Y \in \ker X_l \subseteq M_{n,k}(R)$ matrices; $b \in W^k$) with $\sum^{k} Y_i \otimes b_i \longmapsto w$.

But under θ, $\sum Y_i \otimes b_i \longmapsto Yb$; recapping : For any $w \in W^n$ such that $Xw = 0$ there exists k, $Y \in M_{n,k}(R)$, $b \in W^k$ with $XY = 0$

and $w = Yb$. Further, flatness need only be tested on cyclic modules, that is when $m = 1$ (Example 5.2), so one has:

Proposition 5.7 (Chase (5)) Following are equivalent:

(i) W is flat

(ii) Given any $w_1, \dots w_n \in W$ and $x_1, \dots x_n \in R$, such that $\sum x_i w_i = 0$, there exists k, $b_j \in W$, $j = 1, \dots k$, $Y_{ij} \in R$, $i = 1, \dots k$; $j = 1, \dots n$, with $\sum_j^n x_j Y_{ji} = 0$ and $\sum_i^k Y_{ji} b_i = w_j$.

(iii) Given any $w_1, \dots w_n \in W$, and $X_{ij} \in R$, $i = 1, \dots m$; $j = 1, \dots k$ such that $\sum_j^n X_{ij} w_j = 0$, there exists k, $b_j \in W$, $j = 1, \dots k$ and $Y_{ij} \in R$, $i = 1, \dots n$; $j = 1, \dots k$ with $\sum_j^n X_{ij} Y_{jk} = 0$ for all (i,k) and $\sum_j^k Y_{ij} b_j = w_i$ for all i. //

5.8 f.p. Injectivity By Thm. 4.3 and Ex. 5.5, W is f.p. inj. if $\Psi : 0 \longrightarrow (R^m/XR^n) \otimes W \longrightarrow \mathrm{Hom}(\ker X_r, M)$ is exact for all choices of matrices X.

$R^n \otimes W \xrightarrow{X \otimes 1} (R^m/XR^n) \otimes W \longrightarrow 0$ implies $(R^m/XR^n) \otimes W \cong W^m/XW^n$. From Ex.5.5 $(R^m/XR^n)^* \cong \ker X_r$, via $f \longmapsto (f(e_1), \dots f(e_n))$. Hence Ψ is $w \longmapsto (y \longmapsto yw)$ ($= w_r$ right multiplication by w).

$w = (w_1, \dots w_m) \in W^n/XW^m$. In co-ordinates $(y_1, \dots y_n) \mapsto \sum y_i w_i$.

Ψ is injective if $w_r = 0$ implies $w \in XW^n$.

Proposition 5.9 W is f.p. inj. $\Leftrightarrow$ given $w_1, \dots w_m \in W$ ($w \in W^m$) and $X_{ij} \in R$, $i = 1 \dots m$; $j = 1 \dots n$ ($X \in M_{m,n}(R)$) such that $\sum_i^m y_i X_{ij} = 0$, $y_i \in R$ ($yX = 0$, $y \in R^m$) implies $\sum^m y_i w_i = 0$ ($yw = 0$) then $w_i = \sum^n X_{ij} b_j$, for some $b_j \in M$ ($w = Xb$, for some $b \in M^n$). //

Corollary 5.10 If R is left coherent.

W is f.p. inj. $\Leftrightarrow$ Given $w_1, \ldots w_m \in W$, and $x_1, \ldots x_m \in R$ such that $\sum y_i x_i = 0$, $y_i \in R$, implies $\sum y_i w_i = 0$ $(yx = 0 \Rightarrow yw = 0)$ then for all i, $w_i = x_i b$ for some $b \in M$, (then $w = xb$).

Proof For all left coherent rings, one can check for f.p. injectivity on cyclic f.p. by Prop. 4.2. Hence by Ex. 5.2, one can take $n = 1$ in the proposition. //

Corollary 5.11 If W is f.p. injective and H a f.g. right ideal then $r_W l_R(H) = HW$.

Proof Let $w \in r_W l_R(H)$, and $x_1, \ldots x_n$ generate H. If $yx_i = 0$ for all i, then $y \in l_R(H)$, and so $yw = 0$, hence by the proposition $w = \sum x_i b_i$, $b_i \in W$. //

Part Two

6/ The Category of Exact/Split Sequences

The assignment $A \mapsto \hat{A}$ is not in general functorial. One of the obstructions being that for P f.g. proj. one can choose $\hat{P}$ to be 0, P* or any other f.g. projective. From $0 \to \hat{A}^* \to P' \to P \to A \to 0$, and Schanuel's Lemma, $\hat{A}^*$ is uniquely determined up to projective summands; this suggests that $\hat{A}$ is also 'uniquely' determined. This is indeed the case, and we proceed to set up the machinery to demonstrate this fact (and also examine the machinery itself!)

Starting with an Abelian category $\mathcal{A}$, one can kill the projectives $\mathbb{P}$, to form the quotient category $\mathcal{A}/\mathbb{P}$. However $\mathcal{A}/\mathbb{P}$ is not a 'nice' category, and in general will not be Abelian. One can also form category $\mathcal{E}$ of exact sequences; $\mathcal{E}$ is never Abelian ((22)page 375), however killing off the split exact sequences $\mathcal{S}$, results in an Abelian category $\mathcal{E}/\mathcal{S}$ (Thm. 6.7 ahead). If $\mathcal{A}$ has sufficient projectives, one can assign to each object A, a specific projective presentation $0 \to K \to P \to A \to 0$. This assignment determines a full embedding of $\mathcal{A}/\mathbb{P}$ into $\mathcal{E}/\mathcal{S}$, for which the image of $\mathcal{A}/\mathbb{P}$ constitutes a resolving class of projectives for $\mathcal{E}/\mathcal{S}$ (Thm. 8.5 ahead), (kill projectives only to become projectives). Hence it is natural to work in the Abelian category $\mathcal{E}/\mathcal{S}$ rather than $\mathcal{A}/\mathbb{P}$.

The next three lemmas are recorded for reference.

<u>Lemma 6.1</u> ((11)page 83)

Given
$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d, "s"] \\ B \arrow[r, "r"] & D \end{tikzcd}$$

then $(0\to)\ C \xrightarrow{(a,b)} A\oplus B \xrightarrow{(r,-s)} D[\to 0]$ is exact

$\Leftrightarrow$ the square is a (pull-back), [push-out]. //

<u>Lemma 6.2</u> ((11)page 84)
$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commutative and exact rows, then the right hand square is a pull-back and a push-out. //

<u>Lemma 6.3</u> ((23)page 163) Any $\underline{A} \to \underline{B}$ in $\mathcal{E}$, the category of short sequences, has a factorization:

$$\begin{tikzcd} 0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B'' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B'' \arrow[r] & B \arrow[r] & B' \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{A} \arrow[d] \\ \underline{B} \end{tikzcd} \quad //$$

The objects of $\mathcal{E}$ being exact sequences can be thought of as chain complexes so the notion of homotopy naturally arises.

<u>Proposition 6.4</u> Given
$$\begin{tikzcd} 0 \arrow[r] & A'' \arrow[r, "a"] \arrow[d, "f''"'] & A \arrow[r, "a'"] \arrow[d, "f"] \arrow[dl, dashed, "g"'] & A' \arrow[r] \arrow[d, "f'"] \arrow[dl, dashed, "h"'] & 0 \\ 0 \arrow[r] & B'' \arrow[r, "b"'] & B \arrow[r, "b'"'] & B' \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{A} \arrow[d, "\underline{f}"] \\ \underline{B} \end{tikzcd}$$

following are equivalent:

(i) There exists g such that $ga = f''$.

(ii) There exists h such that $b'h = f'$.

(iii) There exists g and h such that $bg + ha' = f$.

(iv) $\underline{f}$ factors through a split exact sequence.

(v) $\underline{f}$ is chain homotopic to zero ($\underline{f}\sim o$).

<u>Note</u> (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) Fieldhouse (7),

(i) $\Leftrightarrow$ (iv) Freyd (8).

<u>Proof</u> We prove (i) $\Leftrightarrow$ (iv), then (ii) $\Leftrightarrow$ (iv) is proved dually. That (i) and (ii) combined $\Leftrightarrow$ (iii) is clear, and (i) (ii) and (iii) constitute (v).

(iv) $\Rightarrow$ (i)

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C\oplus D \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B'' \arrow[r] & B \arrow[r] & B' \arrow[r] & 0
\end{tikzcd}$$

The required g is achieved via the projection $C\oplus D \twoheadrightarrow C$.

(i) $\Rightarrow$ (iv) Consider

$$\begin{tikzcd}
A'' \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, "g"] & \\
B'' \arrow[r] \arrow[drr, equal] & E \arrow[dr, dashed, "k"'] & \\
 & & B''
\end{tikzcd}$$

, where the square is a push-out.

k exists to give a commutative diagram. Hence $B'' \to E$ is split monic, and result now follows by Lemmas 6.2 and 6.3. //

Suppose $\mathbb{C}$ and $\mathbb{D}$ are acyclic (exact) chain complexes, and $\underline{f} : \mathbb{C} \to \mathbb{D}$, a chain map. Breaking the complexes into short exact sequences, $\underline{f}$ induces maps $\underline{f}_n$ between these pieces.

<u>Proposition 6.5</u> If $\mathbb{C}$ and $\mathbb{D}$ are acyclic, and $\underline{f}\sim o$ (chain homotopic), then $\underline{f}_n \sim o$.

<u>Proof</u>

$$\begin{tikzcd}
C_{n+1} \arrow[rr, "c"] \arrow[dd] & & C_n \arrow[rr, "c"] \arrow[dd] \arrow[dr] \arrow[ddll, "\theta_n"'] & & C_{n-1} \arrow[dd] \\
 & & & K_n \arrow[ur] \arrow[dd] & \\
D_{n+1} \arrow[rr, "d"'] & & D_n \arrow[rr] \arrow[dr] & & D_{n-1} \\
 & & & L_n \arrow[ur] &
\end{tikzcd}$$

$\theta_{n-1}c$ factors through coker $(C_{n+1} \to C_n) = K_n$, so there exists $\theta : K_n \to D_n$,

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r, "c'"] \arrow[d, "f_n"'] & K_n \arrow[d, "f_n'"] \arrow[dl, "\theta"'] \\
D_{n+1} \arrow[r] & D_n \arrow[r, "d'"] & L_n
\end{tikzcd}$$

Then $d'\theta c' = d'(\theta_{n-1}c) = d'(f_n - d\theta_n) = d'f_n = f_n'c'$ and c' is epic, so $d'\theta = f_n'$. Hence by Prop. 6.4 the induced map $\underline{f}_n : (K_{n+1} \to C_n \to K_n) \to (L_{n+1} \to D_n \to L_n)$ is homotopic to zero. //

There is a partial converse, whose proof is the same as showing, two chain maps $\mathbb{P} \to \mathbb{B}$, inducing the same maps $H_0(\mathbb{P}) \to H_0(\mathbb{B})$ are then homotopic. ((11)page 127).

<u>Proposition 6.6</u> Let $\mathbb{P}$ be a projective resolution of A, and $\mathbb{B}$ an acyclic complex, suppose $f : A \to B_0$ factors over B_1 :

$$\begin{tikzcd}
\arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & A \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \\
\arrow[r] & B_2 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r] & 0
\end{tikzcd}$$

then the induced chain map $\mathbb{P}_A \to \mathbb{B}$ is homotopic to zero, where $\mathbb{P}_A$ is $\mathbb{P}$ with A adjoined.

<u>Proof</u> By induction, to construct the homotopy

$$\begin{tikzcd}
P_{n+1} \arrow[rr] \arrow[dr] \arrow[dd] & & P_n \arrow[r, "d_n"] \arrow[dd, "f_n"'] & P_{n-1} \arrow[dd, "f_{n-1}"] \arrow[ddl, "\theta_n"'] \\
& K_{n+1} \arrow[ur] \arrow[dd] & & \\
B_{n+1} \arrow[rr] \arrow[dr] & & B_n \arrow[r, "b_n"'] & B_{n-1} \\
& L_{n+1} \arrow[ur] & &
\end{tikzcd}$$

$b_n(f_n - \theta_n d_n) = b_n f_n - (f_{n-1} - \theta_{n-1}d_{n-1})d_n = b_n f_n - f_{n-1}d_n = 0$

First equality by induction; so $f_n - \theta_n d_n$ factors through $\ker b_n = L_{n+1}$, but P_n is projective and $B_{n+1} \twoheadrightarrow L_{n+1}$, so this factorization can be lifted to B_{n+1}, giving θ_{n+1}. //

For each hom set (A,B) in $\mathcal{E}$, those $\underline{f}\sim 0$ form a subgroup, and induce an equivalence relation compatible with the additive structure of $\mathcal{E}$. Consider the quotient category $\mathcal{E}/\mathcal{S}$, whose objects are those of $\mathcal{E}$ but whose hom sets are $(A,B)/\sim$.

Theorem 6.7 (Freyd (8) Thm. 3.3)

$\mathcal{E}/\mathcal{S}$ is Abelian.

Note: Following proof is adapted from Freyd's, however for our purposes we need the explicit calculation of the kernel and cokernel of a morphism and its canonical factorization, for further propositions.

Proof $\mathcal{E}/\mathcal{S}$ is additive because $\mathcal{E}$ is additive. Hence it will suffice to prove that every morphism $\underline{f}$ has a kernel and cokernel; and a factorization $\underline{f} = \underline{gh}$ where $\underline{h}$ is a cokernel and $\underline{g}$ a kernel ((20) page 87).

Given $\underline{f} : \underline{A} \to \underline{B}$, we will show

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] \arrow[d, equal] & B''\oplus A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 & {} \arrow[d, phantom, "\underline{k}"] \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal] & 0 & {} \arrow[d, phantom, "\underline{h}"] \\
0 \arrow[r] & B'' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 & {} \arrow[d, phantom, "\underline{g}"] \\
0 \arrow[r] & B'' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, equal] & 0 & {} \arrow[d, phantom, "\underline{l}"] \\
0 \arrow[r] & E \arrow[r] & B\oplus A' \arrow[r] & B' \arrow[r] & 0 & {}
\end{tikzcd}$$

represents $0 \to \ker \underline{f} \to A \to \operatorname{im} \underline{f} \to \underline{B} \to \operatorname{coker} \underline{f} \to 0$.

By Lemma 6.3. $\underline{f} = \underline{gh}$. The exact sequences at top and bottom result from Lemma 6.1, using Lemma 6.2 and its dual.

We prove (a) $\underline{k} = \ker \underline{f}$ (b) $\underline{g} = \ker \underline{l}$ then dually

(a') $\underline{l} = \operatorname{coker} \underline{f}$ (b') $\underline{h} = \operatorname{coker} \underline{k}$.

(a) (i) $\underline{k}$ is monic :

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[ddl, dashed, "\theta"'] & X' \arrow[d, "\underline{x}"] \\
A'' \arrow[r] \arrow[d, equal] & B''\oplus A \arrow[r] \arrow[d] & E \arrow[d, "\underline{k}"] \\
A'' \arrow[r] & A \arrow[r] & A'
\end{tikzcd}$$

if $\underline{kx} = 0$ then θ exists by Prop. 6.4; the same θ then shows $\underline{x} = 0$.

(ii) $\underline{hk} = 0$:

$$\begin{tikzcd}
A'' \arrow[r] \arrow[d, equal] & B''\oplus A, \arrow[ddl, dashed, "\theta"] \\
A'' \arrow[d] & \\
B'' &
\end{tikzcd}$$

take θ to be the projection.

(iii) Suppose $\underline{hx} = 0$:

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[ddl, "\theta"'] & X', \arrow[d] \\
A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[d] \\
B'' \arrow[r] & B \arrow[r] & B'
\end{tikzcd}$$

so that θ exists with the properties of Prop. 6.4.

then

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X' \arrow[d] \\
A'' \arrow[r] \arrow[d, equal] & B''\oplus A \arrow[r] \arrow[d] & E \arrow[d] \\
A'' \arrow[r] & A \arrow[r] & A'
\end{tikzcd}$$

gives a factorization of $\underline{x}$ thru $\underline{k}$.

(b) (i) $\underline{g}$ is monic, proof same as for $\underline{k}$.

(ii) $\underline{lg} = 0$:

$$\begin{tikzcd}
B'' \arrow[r] \arrow[d, equal] & E, \arrow[ddl, dashed, "\theta"] \\
B'' \arrow[d] & \\
E &
\end{tikzcd}$$

take θ to be identity.

(iii) Suppose $\underline{lx} = 0$:

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[ddl, "\theta"'] & X' \arrow[d] \\
B'' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B' \arrow[d, equal] \\
E \arrow[r] & B\oplus A' \arrow[r] & B'
\end{tikzcd}$$

then θ exists as in Prop. 6.4.

let $\underline{x}^\circ$ be :

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\theta"'] & X' \arrow[d, dotted] \\
B'' \arrow[r] \arrow[d, equal, "g"'] & E \arrow[r] \arrow[d] & A' \arrow[d] \\
B'' \arrow[r] & B \arrow[r] & B'
\end{tikzcd}$$

, $\underline{x} - \underline{x}^\circ \sim 0$ because left leg of $\underline{x} - \underline{x}^\circ$ is the zero map.

Hence in $\mathcal{E}/\mathcal{S}$, $\underline{x} = \underline{x}^\circ$, and $\underline{x}$ can be factored through $\underline{g}$. //

7/ Projective Homotopy

Now assume $\mathcal{A}$ has enough projectives; let $\mathcal{P}$ be the full subcategory of projectives.

Proposition 7.1 For $f : A \to B$, the following are equivalent:

(i) f can be factored through some projective.

(ii) f can be factored through any projective Q, such that $Q \twoheadrightarrow B$.

(iii) f can be factored through any $C \twoheadrightarrow B$.

Proof (9)page 131. (proof is straightforward). //

f is projectively homotopic to g if f - g factors through a projective. Let P(A,B) be the subgroup of (A,B) consisting of those maps which can be factored through a projective. Let $\pi(A,B) = (A,B)/P(A,B)$.

Proposition 7.2 If $e : Q \twoheadrightarrow B$, Q projective then $\pi(A,B) = \operatorname{coker} e^* : (A,Q) \to (A,B)$.

Proof Im e^* is the set of f which can be factored through $Q \twoheadrightarrow B$, hence equals P(A,B) by Prop. 7.1. //

Corollary 7.3 ((11)page 135).

The functor $\pi(A,-)$ is additive.

Proof To evaluate $\pi(A, B\oplus B')$, take $Q\oplus Q' \twoheadrightarrow B\oplus B'$. //

Let
$$\begin{tikzcd} Q_n \arrow[rr] \arrow[dr] & & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \\ & S_n \arrow[ur] & & & \end{tikzcd}$$
, be a projective resolution of B, with n^{th} syzygy S_n.

Define $\pi_n(A,B) = \pi(A,S_n)$; since $\pi(A,-)$ kills projectives, Schanuel's Lemma gives $\pi_n(A,-)$ is independent of choices of presentations.

Proposition 7.4 ((11)page 142)

Given $0 \to B'' \to B \to B' \to 0$ there is an exact sequence

$$\ldots \to \tilde{\pi}_n(A,B) \to \tilde{\pi}_n(A,B') \to \tilde{\pi}_{n-1}(A,B'') \to \tilde{\pi}_{n-1}(A,B) \to \ldots$$

$$\ldots \to \tilde{\pi}(A,B) \to \tilde{\pi}(A,B') \to \mathrm{Ext}^1(A,B'') \to \mathrm{Ext}^1(A,B) \to \ldots$$

In particular $\tilde{\pi}_n(A,-)$ is half-exact.

Proof Construct projective resolutions $\mathbb{Q}'$, $\mathbb{Q}''$, of B' and B'', let $\mathbb{Q} = \mathbb{Q}' \oplus \mathbb{Q}''$, then

$$\begin{tikzcd} 0 \arrow[r] & (A,Q_0'') \arrow[r] \arrow[d] & (A,Q_0) \arrow[r] \arrow[d] & (A,Q_0') \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (A,B'') \arrow[r] & (A,B) \arrow[r] & (A,B'') \arrow[r] & \mathrm{Ext}^1(A,B'') \arrow[r] & \ldots \end{tikzcd}$$

apply snake lemma and Prop. 7.2 to get

$$0 \to (A,S_1'') \to (A,S_1) \to (A,S_1') \to \tilde{\pi}(A,B'') \to \tilde{\pi}(A,B) \to$$
$$\to \tilde{\pi}(A,B') \to \mathrm{Ext}^1(A,B'') \to \mathrm{Ext}^1(A,B) \to \ldots$$

$S_1 = \ker : Q_0 \to B$ (first syzygy).

Going back another step, using Prop. 7.2

$$\begin{tikzcd} 0 \arrow[r] & (A,Q_1'') \arrow[r] \arrow[d] & (A,Q_1) \arrow[r] \arrow[d] & (A,Q_1') \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (A,S_1'') \arrow[r] \arrow[d] & (A,S_1) \arrow[r] \arrow[d] & (A,S_1') \arrow[r] \arrow[d] & \tilde{\pi}(A,B'') \arrow[r] & \tilde{\pi}(A,B) \arrow[r] & \ldots \\ & \tilde{\pi}(A,S_1'') \arrow[r] & \tilde{\pi}(A,S_1) \arrow[r] & \tilde{\pi}(A,S_1') \end{tikzcd}$$

apply snake lemma, noting $\tilde{\pi}(A,S_1) = \tilde{\pi}_1(A,B)$

then induction completes the sequence. //

Proposition 7.5 Let

$$\begin{tikzcd} P_n \arrow[r] \arrow[dr] & P_{n-1} \arrow[r] & \ldots \arrow[r] & P_0 \\ & Z_n(A) \arrow[u] \end{tikzcd}$$

be a projective resolution of A, with n^{th} syzygy $Z_n(A)$.

If $Q \twoheadrightarrow B$, Q projective then for $n \geq 0$

$\mathrm{Ext}^n(A,Q) \to \mathrm{Ext}^n(A,B) \to \tilde{\pi}(Z_n(A),B) \to 0.$

Proof $n = 0$ is Prop. 7.2 ($Z_0(A) = A$)

For $n>1$ replace A by $Z_{n-1}(A)$ to reduce to the case $n = 1$.

Let $P = P_0$ and $Z = Z_1(A)$, if $0 \to S \to Q \to B \to 0$, apply snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & (P,S) \arrow[r] \arrow[d] & (P,Q) \arrow[r] \arrow[d] & (P,B) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & (Z,S) \arrow[r] \arrow[d] & (Z,Q) \arrow[r] \arrow[d] & (Z,B) \arrow[r] \arrow[d] & \tilde{\pi}(Z,B) \arrow[r] & 0 \\
& \mathrm{Ext}^1(A,S) \arrow[r] & \mathrm{Ext}^1(A,Q) \arrow[r] & \mathrm{Ext}^1(A,B) & &
\end{tikzcd}$$ //

Proposition 7.6 ((11)page 142) $L_n(A,-) \cong \tilde{\pi}_{n+1}(A,-)$, $n \geq 1$.

(L_n the n^{th} left derived functor)

Proof From

$$\begin{tikzcd}
\arrow[r] & Q_2 \arrow[rr, "b"] \arrow[dr] & & Q_1 \arrow[rr, "a"] \arrow[dr] & & Q_0 \\
& & S_2 \arrow[ur] & & S_1 \arrow[ur] &
\end{tikzcd}$$

one obtains

$$\begin{tikzcd}
(A,Q_2) \arrow[rr, "b^*"] \arrow[dr] & & (A,Q_1) \arrow[rr, "a^*"] \arrow[dr] & & (A,Q_0) \\
& (A,S_2) \arrow[ur] & & (A,S_1) \arrow[ur] &
\end{tikzcd}$$

$\ker a^* \cong (A,S_2)$, regarding $(A,S_2) \subseteq (A,Q_1)$ then $\mathrm{im}\, b^*$ are those $f \in (A,S_2)$ which factor through $Q_2 \to Q_1$, so $\mathrm{im}\, b^* = P(A,S_2)$ by Prop. 7.1. Then the first left derived functor of $(A,-)$:

$L_1(A,B) = H_1((A,Q)) = \ker a^*/\mathrm{im}\, b^* = \tilde{\pi}(A,S_2) = \tilde{\pi}_2(A,B).$

For $n \geq 1$, $L_n(A,B) \cong L_1(A,S_{n-1}) = \tilde{\pi}_2(A,S_{n-1}) = \tilde{\pi}_{n+1}(A,B)$. //

Any $f : A \to B$ induces a natural transformation $\mathrm{Ext}^1(B,-) \xrightarrow{f^*} \mathrm{Ext}^1(A,-)$, which can be computed in terms of extensions by using pull-backs:

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, dashed] \arrow[d, equal] & E' \arrow[r, dashed] & A \arrow[r] \arrow[d] & 0 & \mathcal{E}' \\ 0 \arrow[r] & C \arrow[r] & E \arrow[r] \arrow[u, dashed, leftarrow] & B \arrow[r] & 0 & \mathcal{E} \arrow[u] \end{tikzcd}$$

Proposition 7.7 ((10)Cor. to Thm. 1.3)

Every natural transformation $\theta : \mathrm{Ext}^1(B,-) \to \mathrm{Ext}^1(A,-)$, is induced by a map $f : A \to B$.

Proof Let $0 \to K \to P \to A \to 0$ and $0 \to L \to Q \to B \to 0$ be projective presentations.

$$\begin{tikzcd} 0 \arrow[r] & (B,-) \arrow[r] \arrow[d, dashed, "F_1"'] & (Q,-) \arrow[r] \arrow[d, dashed, "F_2"'] & (L,-) \arrow[r] \arrow[d, dashed, "F_3"'] & \mathrm{Ext}^1(B,-) \arrow[r] \arrow[d, "\theta"] & 0 \\ 0 \arrow[r] & (A,-) \arrow[r] & (P,-) \arrow[r] & (K,-) \arrow[r] & \mathrm{Ext}^1(A,-) \arrow[r] & 0 \end{tikzcd} \qquad (*)$$

F_i exist because representables $(M,-)$ are projective, (by Yoneda's Lemma) in the functor category $Ab^{\mathcal{C}}$. Also by Yoneda's L. each F_i is induced from some f_i such that:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "f_3"'] & P \arrow[r] \arrow[d, "f_2"'] & A \arrow[r] \arrow[d, "f_1"'] & 0 \\ 0 \arrow[r] & L \arrow[r] & Q \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Then f_1 induces a natural transformation $\mathrm{Ext}^1(B,-) \to \mathrm{Ext}^1(A,-)$ which also makes (*) commutative, by the uniqueness of maps induced out of the cokernel, this map must be θ. //

Proposition 7.8 $f : A \to B$ induces the zero map $f^\circ : \mathrm{Ext}^1(B,-) \to \mathrm{Ext}^1(A,-) \Leftrightarrow f$ factors through a projective.

Proof
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & C \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

$f^\circ = 0 \Leftrightarrow$ top row splits for all extensions of B

$\Leftrightarrow$ there exists g such that
$$\begin{tikzcd} C \arrow[r, dash] \arrow[d, equal] & E' \arrow[dl, "g"] \\ C & \end{tikzcd}$$

$\Leftrightarrow$ there exists h such that
$$\begin{tikzcd} & A \arrow[dl, "h"'] \arrow[d, "f"] \\ E \arrow[r] & B \end{tikzcd}$$
, by Prop. 6.4.

$\Leftrightarrow$ f factors over a projective, by Prop. 7.1. //

Proposition 7.9 ((2) Thm. 1.40)

There is an exact sequence

$0 \to P(A,B) \to (A,B) \to [\mathrm{Ext}^1(B,-), \mathrm{Ext}^1(A,-)] \to 0.$

Hence $\pi(A,B) \cong [\mathrm{Ext}^1(B,-), \mathrm{Ext}^1(A,-)]$.

Proof The last map is onto by Prop. 7.7, and exactness at the middle by Prop. 7.8. //

8/ Killing Projectives

8.1 Consider the quotient category $\mathfrak{A}/\mathbb{P}$, whose objects are those of $\mathfrak{A}$, but with hom sets $\widetilde{\pi}(A,B)$. For each A of $\mathfrak{A}$ chose a projective presentation $0 \to K \to P \to A \to 0$, then

$$\begin{array}{c} A \\ f\downarrow \\ B \end{array} \mapsto \begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \to & L & \to & Q & \to & B & \to & 0 \end{array} = \begin{array}{c} F(A) \\ \downarrow F(f) \\ F(B) \end{array}$$

constitutes a functor $F : \mathfrak{A} \to \mathcal{E}/\mathcal{S}$. In fact, if f induces two maps $\underline{f}'$, $\underline{f}''$: $F(A) \to F(B)$, then $\underline{f}' - \underline{f}''\sim 0$, hence $\underline{f}' = \underline{f}''$ in $\mathcal{E}/\mathcal{S}$, so F(f) is well-defined. If F' were defined using different presentations, then

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & A & \to & 0 \\ & & \uparrow\downarrow & & \uparrow\downarrow & & \| & & \\ 0 & \to & K' & \to & P' & \to & A & \to & 0 \end{array} = \Psi_A \uparrow \downarrow \theta_A \begin{array}{c} F(A) \\ \\ F'(A) \end{array}$$

$1 - \Psi_A\theta_A \sim 0$ since right leg of $1 - \Psi_A\theta_A$ is the zero map $A \to A$, hence in $\mathcal{E}/\mathcal{S}$, Ψ_A and θ_A are inverses of each other, and determine a natural equivalence between the functors F and F'.

To determine ker F, suppose $f : A \to B$,
$F(f) = 0 \Leftrightarrow F(f)\sim 0 \Leftrightarrow f$ factors over $Q \to B \Leftrightarrow f = 0$ in $\mathfrak{A}/\mathbb{P}$
Hence F factors $\mathfrak{A} \xrightarrow{F} \mathcal{E}/\mathcal{S}$ through $\mathfrak{A} \to \mathfrak{A}/\mathbb{P} \to \mathcal{E}/\mathcal{S}$ which embeds $\mathfrak{A}/\mathbb{P}$ as a full subcategory of the abelian category $\mathcal{E}/\mathcal{S}$.

Following extends a result of Hilton and Ree ((10) Thm.2.1).

Theorem 8.2 $f : A \to B$, following are equivalent

(i) $Ext^1(B,-) \rightarrowtail\!\!\!\to Ext^1(A,-)$.

(ii) there is B' such that $Ext^1(B,-) \oplus Ext^1(B',-) \cong Ext^1(A,-)$.

(iii) Given $Q \twoheadrightarrow B$, Q projective, then B is a direct summand of $A \oplus Q$.

(iv) B is a direct factor of A in $\mathcal{A}/\mathbb{P}$, that is f is split epic in $\mathcal{A}/\mathbb{P}$.

(v) $F(B)$ is a direct factor of $F(A)$ in $\mathcal{E}/\mathcal{S}$, that is $F(f)$ is split epic in $\mathcal{E}/\mathcal{S}$.

(vi) $F(f)$ is epic.

<u>Proof</u> (iii) $\Rightarrow$ (ii) take B' to be complement of B in $A \oplus Q$.

(ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (iii) If $g : Q \twoheadrightarrow B$ form

$$\begin{tikzcd} K \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dl, dashed, "\theta"'] & A \arrow[d, "f"] \arrow[dl, "v"] \\ K \arrow[r] & Q\oplus A \arrow[r, "{\langle f,-g\rangle}"'] & B \end{tikzcd} \qquad \begin{tikzcd} E_1 \in \mathrm{Ext}^1(A,K) \\ E_2 \in \mathrm{Ext}^1(B,K) \arrow[u, "f^{\circ}_K"'] \end{tikzcd}$$

by taking the pull-back, $v : A \rightarrowtail Q \oplus A$ the inclusion.

By Prop. 6.4, θ exists (because v does), hence the top row splits, and (i) implies the bottom must also split since f°_K is injective.

(iii) $\Rightarrow$ (iv) $\begin{tikzcd} B \arrow[r, "r"] & A\oplus Q \arrow[r, "{\langle f,-g\rangle}"] \arrow[d, shift right, "p"'] & B \\ & A \arrow[u, shift right, "v"'] \arrow[ur, "f"'] & \end{tikzcd}$ where $\langle f,-g\rangle r$ is identity map of B.

In $\mathcal{A}/\mathbb{P}$, v and p are isomorphisms, hence $B \xrightarrow{pr} A \xrightarrow{\langle f,-g\rangle v} B$, gives B as a direct factor of A in $\mathcal{A}/\mathbb{P}$.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) clear.

(vi) $\Rightarrow$ (iii) From proof of Thm. 6.7, coker $F(f)$ is

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d] & Q \arrow[r, "g"] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r] & A\oplus Q \arrow[r, "{\langle f,-g\rangle}"'] & B \arrow[r] & 0 \end{tikzcd}$$

so if $F(f)$ is epic then bottom splits. //

This theorem does not 'dualize' satisfactorily, for example the dual of (vi), F(f) being monic: from Thm. 6.7, ker F(f) is

$$\begin{array}{ccccccccc} 0 & \to & K & \to & L\oplus P & \to & E & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & P & \to & A & \to & 0 \end{array}$$

If F(f) is monic then top row splits, so $\mathrm{Ext}^2(B,-) \cong \mathrm{Ext}^1(L,-) \cong \mathrm{Ext}^1(L\oplus P,-) \cong \mathrm{Ext}^1(K,-)\oplus\mathrm{Ext}^1(E,-) \cong \mathrm{Ext}^2(A,-)\oplus\mathrm{Ext}^1(E,-)$ and $\mathrm{Ext}^2(A,-)$ is a direct summand of $\mathrm{Ext}^2(B,-)$, this is not the corresponding dual of (ii) of the theorem. As for the dual of (i), the following:

<u>Proposition 8.3</u> ((10)Thm. 2.2) $\mathrm{Ext}^1(B,-) \twoheadrightarrow \mathrm{Ext}^1(A,-)$ $\Leftrightarrow$ there exists E, and a projective P, and an exact sequence $0 \to P \to A\oplus E \to B \to 0$. Further P can be chosen to be any projective such that $P \twoheadrightarrow A$.

<u>Proof</u> ($\Rightarrow$) From long exact Ext sequence.

($\Leftarrow$) Given $0 \to K \to P \to A \to 0$ in $\mathrm{Ext}^1(A,K)$, by surjectivity, there is $0 \to K \to E \to B \to 0$ in $\mathrm{Ext}^1(B,K)$ such that

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & A & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & E & \to & B & \to & 0 \end{array}$$

then by Lemma 6.2, $0 \to P \to A\oplus E \to B \to 0$ is exact. //

<u>Theorem 8.4</u> ((12)Thm. 1.44)

Following are equivalent

(i) $\mathrm{Ext}^1(B,-) \cong \mathrm{Ext}^1(A,-)$

(ii) There exists projectives P and Q such that $A\oplus Q \cong B\oplus P$

(iii) $A \cong B$ in $\mathcal{A}/\mathcal{P}$

(iv) $F(A) \cong F(B)$ in $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (ii) if $g : Q \twoheadrightarrow B$;

$$\begin{tikzcd} & & & A \arrow[dll, "v"'] \arrow[d, "f"] & \\ 0 \arrow[r] & P \arrow[r] & Q\oplus A \arrow[r, "\langle f,-g\rangle"] & B \arrow[r] & 0 \end{tikzcd}$$

by Thm. 8.2 this sequence splits, further P is projective since $\mathrm{Ext}^1(B\oplus P,-) \cong \mathrm{Ext}(Q\oplus A,-) \cong \mathrm{Ext}(A,-)$ implying $\mathrm{Ext}^1(P,-) = 0$.

(iii) $\Leftrightarrow$ (iv) because $\mathcal{O}/\mathbb{P} \hookrightarrow \mathcal{E}/\mathcal{S}$ is a full embedding.

(i) $\Rightarrow$ (iii) $f : A \to B$ factors as $A \xrightarrow{v} A\oplus Q \xrightarrow{\cong} B\oplus P \xrightarrow{p} B$ as in the proof of (i) $\Rightarrow$ (ii). But v and p are isomorphisms in $\mathcal{O}/\mathbb{P}$, hence $A \cong B$.

(iii) $\Rightarrow$ (i) $A \underset{h}{\overset{f}{\rightleftarrows}} B$, such that $(1 - fh)\sim 0$ and $(1 - hf)\sim 0$. Then $1 - fh$ induces the zero map $\mathrm{Ext}^1(A,-) \to \mathrm{Ext}^1(A,-)$ by Prop. 7.8. Hence

$$\begin{tikzcd} \mathrm{Ext}^1(A,-) \arrow[rr, equal] \arrow[dr, "h^\circ"'] & & \mathrm{Ext}^1(A,-) \\ & \mathrm{Ext}^1(B,-) \arrow[ur, "f^\circ"'] & \end{tikzcd}$$

. //

<u>Theorem 8.5</u> ((8)Cor. 2.9)

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "F"] \arrow[dr] & & \mathcal{E}/\mathcal{S} \\ & \mathcal{O}/\mathbb{P} \arrow[ur] & \end{tikzcd}$$

, is a full embedding of $\mathcal{O}/\mathbb{P}$ in $\mathcal{E}/\mathcal{S}$, and

Im F $\cong \mathcal{O}/\mathbb{P}$ is a full subcategory of resolving projectives of $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> First statement is contained in section 8.1.

(a) $\mathcal{O}/\mathbb{P}$ resolves for

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P \arrow[r] & A' \arrow[r] \arrow[d, equal] & 0 & = F(A') \\ 0 \arrow[r] & A'' \arrow[r] & A \arrow[r] & A' \arrow[r] & 0 & = \underline{A} \end{tikzcd}$$

can be filled in, and the resulting map $F(A') \to \underline{A}$ is epic in $\mathcal{E}/\mathcal{S}$ (by proof of Thm. 6.7)

(b) F(A) is projective for any A. For if $N \twoheadrightarrow F(A)$; by (a), $F(B) \twoheadrightarrow N$ some B ; to split $N \twoheadrightarrow F(A)$ it suffices to split $F(B) \twoheadrightarrow F(A)$, but this is a split epic by Thm. 8.2. //

Remark 8.6 Given $f : A \to B$, if $g : Q \twoheadrightarrow B$, Q projective then

$$\begin{tikzcd} A \arrow[r, tail, "v"] \arrow[dr, "f"'] & A\oplus Q \arrow[d, "\langle f,-g\rangle"] \\ & B \end{tikzcd}$$

, and in $\mathfrak{A}/\mathbb{P}$, v is an isomorphism, hence replacing f by $\langle f,-g\rangle$, one can assume f is epic in $\mathfrak{A}$.

Proposition 8.7 (Freyd(8))

$\mathfrak{A}/\mathbb{P}$ has weak kernels (no uniqueness property required).

Proof Let $f : A \to B$, by above remark assume f is epic in $\mathfrak{A}$. If $0 \to K \to A \to B \to 0$ is exact in $\mathfrak{A}$, then $K \to A$ is a weak kernel of f in $\mathfrak{A}/\mathbb{P}$. $fg = 0$ in $\mathfrak{A}/\mathbb{P}$, means fg factors over a projective P in :

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "b"'] & A \arrow[r, "f"] & B, \\ & P \arrow[u, dashed, "a"] \arrow[ur] & \end{tikzcd}$$

a is induced since P is projective. Then $f(g - ab) = 0$, so $g - ab$ factors through $K = \ker f$ in $\mathfrak{A}$, but $g - ab = g$ in $\mathfrak{A}/\mathbb{P}$, so g factors through K in $\mathfrak{A}/\mathbb{P}$.

Proof 2 (Freyd) Let $Q \twoheadrightarrow B$, take pull-back

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Q \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

, then $E \to A$ is a weak kernel of f. (use Prop. 7.1). //

Example 8.8 $\mathfrak{A}/\mathbb{P}$ will not in general be Abelian. For example if $\mathfrak{A} = Ab$, $\mathfrak{F}$ = projectives = frees, then in $Ab/\mathfrak{F}$ the canonical map $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/Z$ is both epic and monic but is not an isomorphism. Proof : $Hom(\mathbb{Q},Z) = 0$ implies $Hom(\mathbb{Q},F) = 0$ for F free. Suppose $\mathbb{Q} \twoheadrightarrow X$ in Ab, this map remains epic in $Ab/\mathfrak{F}$, for if

$$\begin{tikzcd} \mathbb{Q} \arrow[r] & X \arrow[r] \arrow[d] & A \\ & F \arrow[ur] & \end{tikzcd}$$

is a factorization

over a free F, then necessarily $\mathbb{Q} \to X \to A = 0$, hence $X \to A$ is zero. Thus in particular $\mathbb{Q} \to \mathbb{Q}/Z$ is epic. By Prop. 8.7 $Z \rightarrowtail \mathbb{Q}$ is a weak kernel of this map, but this is the zero map $Ab/\mathfrak{F}$, hence it is actually the kernel and so $\mathbb{Q} \to \mathbb{Q}/Z$ is monic. This map could not be an isomorphism because $\mathrm{Hom}(\mathbb{Q}/Z,\mathbb{Q}) = 0$.

<u>Proposition 8.9</u> Let $\mathbb{B}$ be a full subcategory of resolving projectives of an abelian category $\mathbb{C}$, then the inclusion $\mathbb{B} \hookrightarrow \mathbb{C}$ preserves kernels.

<u>Proof</u> Suppose $K \to C$ is the kernel of $C \to D$ in $\mathbb{B}$. $K \to C$ is monic in $\mathbb{C}$. For if $N \to K \to C$ is zero let $B \twoheadrightarrow N$, B in $\mathbb{B}$, then $B \twoheadrightarrow N \to K \to C = 0$ implies $B \twoheadrightarrow N \to K = 0$ implies $N \to K = 0$.

Let $L \to C$ be the kernel of $C \to D$ in $\mathbb{C}$ and let $B \twoheadrightarrow L$, B in $\mathbb{B}$.

$$\begin{tikzcd} & K \arrow[r] \arrow[d, dashed, "g"] & C \arrow[r] & D \\ B \arrow[ur, dashed, "h"] \arrow[r, two heads] & L \arrow[ur] & & \end{tikzcd}$$

, g exists since $L = \ker C \to D$ in $\mathbb{C}$, and $K \to C$ monic implies g is monic.

h exists since $K = \ker C \to D$ in $\mathbb{B}$.

$$\begin{tikzcd} & K \arrow[d] & C \\ B \arrow[ur] & L \arrow[ur] & \end{tikzcd} = \begin{tikzcd} & K \arrow[r] & C \\ B \arrow[ur] & & \end{tikzcd} = \begin{tikzcd} & C \\ B \arrow[r] & L \arrow[u] \end{tikzcd}$$

and the monic $L \to C$ can be cancelled hence

$$\begin{tikzcd} & K \arrow[d] \\ B \arrow[ur] \arrow[r] & L \end{tikzcd}$$

is commutative, implying that g is also epic, g is then an isomorphism and $K \to C$ is also the kernel of $C \to D$ in $\mathbb{C}$. //

<u>Corollary 8.10</u> $F : \mathfrak{A}/\mathbb{P} \twoheadrightarrow \mathcal{E}/\mathfrak{S}$ preserves kernels. //

<u>Corollary 8.11</u> (a remark of Freyd (8)page 99)

If $\mathfrak{A}/\mathbb{P}$ has kernels then p.d. $\mathcal{E}/\mathcal{S} \leq 2$.

<u>Proof</u> For N in $\mathcal{E}/\mathcal{S}$, choose $F(B) \to F(C) \to N \to 0$, B, C in $\mathfrak{A}$, by the proposition $0 \to F(A) \to F(B) \to F(C) \to N \to 0$ for some A. Since F(A) is projective by Thm. 8.5, p.d. $N \leq 2$. //

9/ Syzygy Functor

9.1 General reference Auslander and Bridger ((2)pages 48-51).

For each A, chose $P \twoheadrightarrow A$, P projective (for convenience if A is projective chose $A \xrightarrow{=} A$, and if A is f.g. chose P f.g.). Let $Z(A) = \ker(P \to A)$. Z is not a functor from $\mathfrak{A}$ to $\mathfrak{A}$, however if the target becomes $\mathfrak{A}/\mathbb{P}$ then Z is a functor.

$$\begin{array}{ccccc} K & \to & P & \to & A \\ \downarrow & & \downarrow & & \downarrow f \\ L & \to & Q & \to & B \end{array}$$

, let $Z(f)$ be the representative of the induced map $K \to L$ in $\pi(K,L)$.

If $Z(f)$ is well-defined, then Z is clearly an additive functor $\mathfrak{A} \to \mathfrak{A}/P$. To do this, it will suffice to consider the case $f = 0$ and prove $Z(f) = 0$, that is $Z(f)$ factors over a projective. However even more is true, if f factors over a projective, so does the induced map. In fact, if f factors over Q in the above diagram, then by Prop. 6.4, $K \to L$ factors over P. Thus Z factors :

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "Z"] \arrow[dr] & \mathfrak{A}/\mathbb{P} \\ & \mathfrak{A}/\mathbb{P} \arrow[u] \end{tikzcd}$$

The functors F and Z can be related by

$F(A) = \big(0 \to Z(A) \to P \to A \to 0\big)$

Suppose Z and Z' are different syzygy functors, arising from different presentations chosen.

Consider

$$\begin{array}{ccccc} & & K' & = & K' \\ & & \downarrow & & \downarrow \\ K & \to & N & \to & P' \\ \| & & \downarrow & & \downarrow \\ K & \to & P & \to & A \end{array}$$

Let $n_A : K \rightarrowtail K \oplus P' \cong K' \oplus P \twoheadrightarrow K'$, n_A is an isomorphism in $\mathfrak{A}/\mathbb{P}$, and determines a natural equivalence between Z and Z'.

The n^{th} syzygy functor can be defined by $Z_n(A) = Z(Z_{n-1}(A))$, (where Z is now regarded as a functor $\mathfrak{A}/\mathbb{P} \to \mathfrak{A}/\mathbb{P}$)

<u>9.2</u> ((8)Prop. 1.2) For any Abelian category $\mathbb{C}$, if $\mathbb{B}$ is a full category of resolving projectives, then any functor $G : \mathbb{B} \to \mathbb{D}$, $\mathbb{D}$ abelian, has a unique extension to a right exact functor $\mathbb{C} \to \mathbb{D}$. Explicitly for each $C \epsilon \mathbb{C}$, chose $B' \to B \to C \to 0$, B', B in $\mathbb{B}$, and define $G(C) = \text{coker} : G(B') \to G(B)$.

<u>Theorem 9.3</u> Let $\underline{A} = 0 \to A'' \to A \to A' \to 0$, the following is an exact sequence in $\mathcal{E}/\mathcal{S}$, with P, P', P'' projective,

$$\begin{tikzcd}
0 \arrow[r] & K'' \arrow[r, equal] \arrow[d] & K'' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K' \arrow[r, "W"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r, "-W"] & A'' \arrow[r] & A \arrow[r] & A' \arrow[r, equal] & A' \arrow[r] & 0
\end{tikzcd} \qquad (\#)$$

<u>Proof</u> Let $P' \twoheadrightarrow A'$, and $P'' \twoheadrightarrow A''$, set $P = P' \oplus P''$, then

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K'' \arrow[r, "k"] \arrow[d, "b''"'] & K \arrow[r, "k'"] \arrow[d, "b"] \arrow[dl, "u"'] & K' \arrow[r] \arrow[d, "b'"] \arrow[dl, "v"'] & 0 \\
0 \arrow[r] & P'' \arrow[r, "i"] \arrow[d, "a''"'] & P \arrow[r, "p"] \arrow[d, "a"] \arrow[dl, "r"'] & P' \arrow[r] \arrow[d, "a'"] \arrow[dl, "s"'] & 0 \\
0 \arrow[r] & A'' \arrow[r, "f"] \arrow[d] & A \arrow[r, "f'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \qquad (+)$$

The maps r, s, u, v result from the splitting of $P'' \to P \to P'$, via canonical projections and injections, and give the properties of Prop. 6.4.

(I) Making use of Thm. 6.7, to start a projective resolution of $\underline{A}$:

$$\begin{tikzcd}
K' \arrow[r, "{(b', w)}"] \arrow[d, equal] & A''\oplus P' \arrow[r, "{\langle -f, s\rangle}"] \arrow[d] & A \arrow[d, "f'"] \\
K' \arrow[r, "b'"] \arrow[d, "w"] & P' \arrow[r, "a'"] \arrow[d, "s"] & A' \arrow[d, equal] \\
A'' \arrow[r, "f"] & A \arrow[r, "f'"] & A'
\end{tikzcd}$$

w the induced map. Lower right square commutes by (+).

(II) Building on the kernel of (I) and using Thm. 6.7 again

$$\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & K'\oplus P \arrow[r] \arrow[d] & A''\oplus P' \arrow[d] \\
K \arrow[r, "b"] \arrow[d, "k'"'] & P \arrow[r, "a"] \arrow[d, "{(-r, p)}"] & A \arrow[d, equal] \\
K' \arrow[r, "{(b', w)}"'] & A''\oplus P \arrow[r, "{\langle -f, s\rangle}"'] & A
\end{tikzcd}$$

however this time we must check bottom squares commute. For the lower left, it is required that $b'k' = pb$, which is clear from (+), and $wk' = -rb$. For the second equality apply the monic f :

$f(wk' + rb) = sb'k' + (a - sp)b = sb'k' - spb = 0$

a similiar calculation for lower right square.

(III) Continuing to project on the kernel, the obvious choice is

$$\begin{tikzcd}
K'' \arrow[r, "{(0, b'')}"] \arrow[d] & P \arrow[r, "{(p, r)}"] \arrow[d] & P'\oplus A'' \arrow[d, equal] \\
K \arrow[r] & K'\oplus P \arrow[r] & P'\oplus A
\end{tikzcd}$$

However, $(p,r) : P = P'\oplus P'' \longrightarrow P'\oplus A''$, $(p,r) = 1\oplus a''$ so the top row is isomorphic, in $\mathcal{E}/\mathcal{S}$, to $0 \to K'' \to P'' \to A'' \to 0$, and we use this for the projective:

$$\begin{tikzcd}
K'' \arrow[r, "{(k,b'')}"] \arrow[d, equal] & K\oplus P'' \arrow[r, "{\langle (k',b),-(0,i)\rangle}"] \arrow[d] & K'\oplus P \arrow[d, "{\langle -w,-r\rangle}"] \\
K'' \arrow[r, "b''"] \arrow[d, "k"] & P'' \arrow[r, "a''"] \arrow[d, "{(0,i)}"] & A'' \arrow[d, "{(0,-1)}"] \\
K \arrow[r, "{(k',b)}"'] & K'\oplus P \arrow[r, "{\langle -(w,b'),(-r,p)\rangle}"'] & P'\oplus A''
\end{tikzcd}$$

Lower left commutes by (+). Lower right

$\langle -(w\ b'), (p,-r)\rangle\ (0,i) = (p,-r)i = (0,-ri) = (0,-a'')$

using (+).

(IV) The kernel of (III) is isomorphic to $K'' \to K \to K'$ via

$$\begin{tikzcd}
\underline{K} \arrow[d, "\theta"] & K'' \arrow[r, "k"] \arrow[d, equal] & K \arrow[r, "k'"] \arrow[d, "{(1,u)}"] & K' \arrow[d, "{(1,v)}"] \\
\underline{K}^{\circ} \arrow[d, "\Psi"] & K'' \arrow[r, "{(k,b'')}"'] \arrow[d, equal] & K\oplus P'' \arrow[r, "t"'] \arrow[d] & K'\oplus P \arrow[d] \\
\underline{K} & K'' \arrow[r] & K \arrow[r] & K'
\end{tikzcd}$$

$t = \langle (k',b),\ -\ (0,p)\rangle$

For θ, Ψ to be inverse isomorphisms, one need only check θ and Ψ are well-defined, because $(1 - \theta\Psi)\sim 0$ and $(1 - \Psi\theta)\sim 0$ since left legs are the zero maps. For Ψ this is clear, and for θ, upper left square commutes directly from (+).

Finally $t(1,u) = \langle (k',b),\ -\ (0,p)\rangle(1,u) = (k',b) - (0,p)u$

$= (k',b-pu) = (k',vk') = (1,v)k'$.

(V) Putting the pieces together gives required sequence. //

<u>Corollary 9.4</u> (i) (remark of Freyd, (8)page 109)

The extension of the syzygy functor Z to $\mathcal{E}/\mathcal{S}$ is given by $Z(A'' \to A \to A') = Z(A'') \to Z(A) \to Z(A')$.

Hence $0 \to Z(\underline{A}) \to F(A'') \to F(A) \to F(A') \to \underline{A} \to 0$ is exact in $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> By the theorem $F(A'') \to F(A) \to F(A') \to \underline{A} \to 0$ is exact. $Z(\underline{A}) = \text{coker}\,(FZ(A) \to FZ(A')) = Z(A'') \to Z(A) \to Z(A')$ by the theorem applied to $\underline{K} = K'' \to K \to K'$. $(K = Z(A))$. //

<u>Remarks</u> (a) With this extension of Z (unique up to natural equivalence) $ZF = FZ : \mathcal{O} \to \mathcal{E}/\mathcal{S}$.

(b) This extension is not the syzygy functor associated with $\mathcal{E}/\mathcal{S}$, but the third syzygy functor.

<u>Corollary 9.5</u>

(i) F is a half-exact functor $\mathcal{O} \to \mathcal{E}/\mathcal{S}$.

(ii) F is left exact $\Leftrightarrow \mathcal{E}/\mathcal{S} = 0 \Leftrightarrow$ all short exact sequences in $\mathcal{O}$ split.

(iii) F right exact implies p.d. $\mathcal{E}/\mathcal{S} \leq 2$. //

<u>Corollary 9.6</u> (remark of Freyd (8)page 109)

Z is an exact functor $\mathcal{E}/\mathcal{S} \to \mathcal{E}/\mathcal{S}$.

<u>Proof</u> $Z : \mathcal{E}/\mathcal{S} \to \mathcal{E}/\mathcal{S}$, is the unique right exact extension of $Z : \mathcal{O}/\mathbb{P} \to \mathcal{E}/\mathcal{S}$, hence it suffices to prove Z preserves monics. Let $\underline{f} : \underline{A} \to \underline{B}$ be monic in $\mathcal{E}/\mathcal{S}$.

$$\begin{array}{ccccc} A'' & \to & A & \to & A' \\ {\scriptstyle f''}\downarrow & & \downarrow & & \downarrow \\ B'' & \to & B & \to & B' \end{array}$$

Using the canonical factorization of $\underline{f}$ given in Thm. 6.7, one can assume $A'' = B''$ and f'' is the identity, but then

$$\begin{array}{ccccccc} 0 & \to & Z(\underline{A}) & \to & F(A'') & \to & F(A) \\ & & \downarrow & & \| & & \downarrow \\ 0 & \to & Z(\underline{B}) & \to & F(A'') & \to & F(B) \end{array}$$

, hence $Z(\underline{A}) \to Z(\underline{B})$ is monic. //

<u>Corollary 9.7</u> Following is a projective resolution of $\underline{A}$

$$\ldots F(Z_nA) \to F(Z_nA') \to F(Z_{n-1}A'') \to \ldots$$

$$\ldots \to F(ZA') \to F(A'') \to F(A) \to F(A') \to \underline{A} \to 0 \;. \;//$$

$$\begin{array}{ccccc} Z(A') & \longrightarrow & P' & \longrightarrow & A' \\ {\scriptstyle w}\downarrow & & \downarrow & & \| \\ A'' & \xrightarrow{f} & A & \xrightarrow{f'} & A' \end{array}$$

, let $f_1 = f'$, $f_2 = f$, and $f_3 = -w$, this gives rise to an infinite sequence .

$$\ldots Z_n(A) \to Z_n(A') \xrightarrow[f_{3n}]{} Z_{n-1}(A'') \to \ldots$$

$$\ldots Z(A) \xrightarrow[f_4]{} Z(A') \xrightarrow[f_3]{} A'' \xrightarrow[f_2]{} A \xrightarrow[f_1]{} A'$$

Corollary 9.8 (i) If f_m factors over a projective, then p.d. $\underline{A} \leq m - 1$.

(ii) In particular if f_1 factors over a projective then $\underline{A} \cong F(A')$, so $\underline{A}$ is projective.

Proof Extend (#) in Thm. 9.3 to the projective resolution given in Cor. 9.7. Then the sequence of maps f_m is the bottom row. If f_m factors over a projective then the corresponding map between exact sequences is zero. //

Remark 9.9 If $f : A \to B$ in $\mathfrak{A}$, and $g : Q \twoheadrightarrow B$, Q projective, then $\langle f,-g\rangle : A \oplus Q \twoheadrightarrow B$, (see 8.6); one can define the projective dimension of f, as the projective dimension of $0 \to K \to A\oplus Q \to B \to 0$ in $\mathcal{E}/\mathcal{S}$.

Corollary 9.10 If p.d. $A'' \leq n \Rightarrow$ p.d. $\underline{A} \leq 3n+1$, $n \geq 0$

p.d. $A \leq n \Rightarrow$ p.d. $\underline{A} \leq 3n$, $n \geq 0$

p.d. $A' \leq n \Rightarrow$ p.d. $\underline{A} \leq 3n-1$, $n \geq 1$

p.d. $A' = 0 \Rightarrow \underline{A} = 0$.

If p.d. $\mathfrak{A} \leq n \Rightarrow$ p.d. $\mathcal{E}/\mathcal{S} \leq 3n-1$, $n \geq 1$; p.d. $\mathfrak{A} = 0 \Rightarrow \mathcal{E}/\mathcal{S} = 0$.

Proof F kills projectives, apply Cor. 9.7. //

10/ <u>Auslander's f.p. Duality Functor</u>

10.1 Let M_R' denote the subcategory of f.p. right modules. For each A of M_R'; choose $P' \to P \to A \to 0$, P', P in P_R' (f.g. proj.) (for convenience select $0 \to P = P \to 0$ for $A = P$ proj.)

Define $D(A) = \operatorname{coker} P^* \to P'^*$ ($\hat{A}$ of Part one).

If $f : A \to B$ then

$$\begin{tikzcd} P' \arrow[r] \arrow[d, dashed] & P \arrow[r] \arrow[d, dashed] & A \arrow[d, "f"] \\ Q' \arrow[r] & Q \arrow[r] & B \end{tikzcd} \qquad \text{(I)}$$

dualize

$$\begin{tikzcd} Q^* \arrow[r] \arrow[d] & Q'^* \arrow[r] \arrow[d] & D(B) \arrow[d, dashed, "g"] \\ P^* \arrow[r] & P'^* \arrow[r] & D(A) \end{tikzcd}$$

, let $D(f)$ be the representative of g in $\pi(D(B), D(A))$.

As for the syzygy functor, if D is well-defined, it will be an additive functor $M_R' \to {}_R M'/{}_R P'$; so required that $f = 0$ implies $D(f) = 0$. Even more is true, if f factors over a projective, then $D(f) = 0$ so D factors through M_R'/P_R'.

If f factors over Q in (I) via θ_o, then by Prop. 4.2 the induced chain map $\underline{f} : \mathbb{P}_A \to \mathbb{P}_B$ is homotopic to zero, so θ_1 exists

$$\begin{tikzcd} B^* \arrow[r] \arrow[d] & Q^* \arrow[r, two heads] \arrow[d] \arrow[dl, "\theta_o^*"'] & N \arrow[r, tail] \arrow[d] & Q'^* \arrow[r] \arrow[d] \arrow[dll, "\theta_1^*"'] & D(B) \\ A^* \arrow[r] & P^* \arrow[r, two heads] & M \arrow[r, tail] & P'^* \arrow[r] & D(A) \end{tikzcd}$$

$$\begin{tikzcd} Q^* \arrow[r] & N \arrow[r] & Q'^* \arrow[dll] \\ P^* \arrow[r] & M \arrow[r] & P'^* \end{tikzcd} = \begin{tikzcd} Q^* \arrow[r] & N \arrow[r] & Q'^* \arrow[dll] \\ P^* \arrow[r] & M \arrow[r] & P'^* \end{tikzcd} + \begin{tikzcd} & Q^* \arrow[dl] & & \\ A^* \arrow[r] & P^* \arrow[r] & M \arrow[r] & P'^* \end{tikzcd}$$ (last term is zero.)

$$= \begin{tikzcd} Q^* \arrow[d] & & \\ P^* \arrow[r] & M \arrow[r] & P'^* \end{tikzcd} = \begin{tikzcd} Q^* \arrow[r] & N \arrow[d] & \\ & M \arrow[r] & P'^* \end{tikzcd}$$

since $Q^* \rightarrowtail N$ and $M \twoheadrightarrow P'^*$, cancel to obtain $N \to M = N \to Q'^* \to P^* \to M$, a factorization over Q'^*, hence $D(B) \to D(A)$ factors over P'^* by Prop. 6.4., and is thus the zero map.

$f : A \to B$ induces a unique map Ψ between kernels of

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Ext}^1(D(A),-) & \to & A\otimes - & \to & \mathrm{Hom}(A^*,-) \\ & & \downarrow \Psi & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Ext}^1(D(B),-) & \to & B\otimes - & \to & \mathrm{Hom}(B^*,-) \end{array} \qquad \text{(II)}$$

By Prop. 7.9 there is a unique map of $\pi(D(B),D(A))$ corresponding to Ψ, that this is the map $D(f)$ can be seen from (#1) of Thm. 2.1 (the left leg).

Suppose different presentations were used, giving a duality functor $D°$. Replace B by A and $D(B)$ by $D°(A)$ in (II), then the induced map $\mathrm{Ext}^1(D(A),-) \to \mathrm{Ext}^1(D°(A),-)$ is an isomorphism; hence by Thm. 8.4, arises from a natural isomorphism in M_R/P_R, $n_A : D°(A) \to D(A)$. There is then projectives (which can be taken to be f.g.) P, Q such that $D°(A) \oplus Q = D(A) \oplus P$, and the functors D and $D°$ are equivalent.

D has a left inverse D' (use starred sequences as presentations when constructing D'); then again determine D'' such that $D''D' = 1$, by the above $D \cong D''$, hence $DD' \cong D''D' = 1$. We now have:

<u>Thm. 10.2</u> (a) The functor D determines a contravariant equivalence between M'_R/P'_R and ${}_RM'/{}_RP'$. ((2)Prop.2.6, page 52.)

(b) $\pi(A,D(B)) \cong \pi(B,D(A))$, $\pi(D(A),B) \cong \pi(D(B),A)$, so

D is its' own adjoint on left and right (viewed as a contravariant functor. //

11/ Odds and Ends

Consider the map $\theta : A^*\otimes - \to \mathrm{Hom}(A,-)$ (see Thm. 2.1)

Proposition 11.1 Im θ_B are those f of Hom(A,B) which factor through a f.g. projective.

Proof If $f = \sum_i^n f_i \otimes b_i \in A^*\otimes B$, then

$$\begin{tikzcd} A \arrow[r, "\theta(f)"] \arrow[d, "\pi f_i"'] & B \\ R^n \arrow[ur, "\oplus b_i"'] & \end{tikzcd}$$

Conversely if f factors over a f.g. proj., it factors over a f.g. free. Let $f_i = A \to R^n \xrightarrow[p_i]{} R$ and $b_i = h(e_i)$, $\{e_i\}$ a basis of R^n, where $\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ R^n \arrow[ur, "h"'] & \end{tikzcd}$, then $\theta : \sum f_i \otimes b_i \mapsto f$. //

Remark Im $\theta_B \subseteq P(A,B)$ and if A or B is f.g. there is equality.

Proposition 11.2 (i) $\pi(A,-) \cong \mathrm{Tor}_1(D(A),-)$ for A f.p.

(ii) $\pi_{n-1}(A,-) \cong \mathrm{Tor}_n(D(A),-)$ for A f.p.

Proof (i) Both are coker θ, by Thm. 2.1 and Prop. 11.1.

(ii) $\pi_{n-1}(A,B) = \pi(A,Z_{n-1}(B)) = \mathrm{Tor}_1(D(A), Z_{n-1}(B)) = \mathrm{Tor}_n(D(A),B)$//

For a contravariant functor G, the first right satellite of G is $S^1G(A) = \mathrm{Coker}\ G(Q) \to G(K)$ where $0 \to K \to Q \to A \to 0$, Q proj. The n^{th} right satellite $S^nG = S^1(S^{n-1}(G))$. We specialize $G = P(-,B)$.

Proposition 11.3

(i) $0 \to S^1P(A,B) \to \mathrm{Ext}^1(A,B) \to \pi(Z(A),B) \to 0$

(ii) $0 \to S^nP(A,B) \to \mathrm{Ext}^n(A,B) \to \pi(Z_n(A),B) \to 0$

Proof (i)

$$\begin{tikzcd} 0 \arrow[r] & P(Q,B) \arrow[r] \arrow[d] & \mathrm{Hom}(P,B) \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & P(K,B) \arrow[r] \arrow[d] & \mathrm{Hom}(K,B) \arrow[r] \arrow[d] & (Z(A),B) \arrow[r] & 0 \\ & S^1P(A,B) & \mathrm{Ext}^1(A,B) & & \end{tikzcd}$$

apply Snake Lemma.

(ii) $S^nP(A,B) = S^1P(Z_{n-1}(A),B)$

$\mathrm{Ext}^n(A,B) = \mathrm{Ext}^1(Z_{n-1}(A),B)$ apply (i). //

<u>Remarks</u> (a) Since $\mathcal{A}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ is a full embedding we have $0 \to S^1P(A,B) \to \mathrm{Ext}^1(A,B) \to \mathcal{E}/\mathcal{S}\,(Z(A),B) \to 0$. So Freyd's Prop. 2.10 (8), $(\mathrm{Ext}^n(A,B) \cong \mathcal{E}/\mathcal{S}(Z_n(A),B))$ is incorrect, and Prop. 11.3 is the corrected version (for counter example $\mathcal{A}=$ Ab, then $\pi(Z(A),B) \doteq 0$ since $Z(A)$ is free.)

(b) Prop. 11.3 should be taken in conjunction with Prop.7.5.

Since right satellites have been introduced, we may as well introduce left satellites of covariant functor H, (from an abelian category to Ab)

$S_1H(A) = \ker H(K) \to H(P)$, where $0 \to K \to P \to A \to 0$,

P projective. $S_{n+1}H = S_1(S_nH)$.

<u>Proposition 11.4</u> $[\mathrm{Ext}^n(A,-), H] = S_nH(A)$, $n>0$. ((10)Thm.1.2)

<u>Proof</u> If $0 \to K \to P \to A \to 0$, then $(P,-) \to (K,-) \to \mathrm{Ext}^1(A,-) \to 0$ apply $[-,H]$ and use Yoneda's lemma, $[(M,-),H] \cong H(M)$;

$0 \to [\mathrm{Ext}^1(A,-),H] \to H(K) \to H(P)$, so result holds for n =1.

Since $\mathrm{Ext}^n(K,-) \cong \mathrm{Ext}^{n+1}(A,-)$ and $S_nH(K) \cong S_{n+1}H(A)$ result follows by induction. //

<u>Corollary 11.5</u> ((10)Cor. to Thm. 1.3)

$S_1\mathrm{Ext}^1(B,-)\,(A) \cong [\mathrm{Ext}^1(A,-), \mathrm{Ext}^1(B,-)] \cong \pi(B,A)$.

<u>Proof</u> By Prop. 11.4 and Prop. 7.9. //

<u>11.6 Problems</u>

(a) In order to make the assignment $A \to \hat{A}$ functorial, one was forced to pass to the quotient category $\mathcal{A}/\mathcal{P}$, which unfortunately was not in general abelian (Example 8.8). When is $\mathfrak{A}/\mathbb{P}$ abelain? (Probably only when $\mathbb{P} = \mathfrak{A}$).

(b) By Cor. 9.10 p.d $\mathfrak{A} = n \Rightarrow$ p.d.$\mathcal{E}/\mathcal{S} \leq 3n-1$, is this actually an equality, what is the exact relationship between projective dimensions of $\mathfrak{A}$ and $\mathcal{E}/\mathcal{S}$?

(c) $\mathfrak{A}/\mathbb{P} \to \mathcal{E}/\mathcal{S}$, embeds $\mathfrak{A}/\mathbb{P}$ as a resolving class of projectives. Are there any other projectives in $\mathcal{E}/\mathcal{S}$? Are direct summands of F(A) isomorphic to F(A') some A'?

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