SULLIVAN'S THEORY OF MINIMAL MODELS

by

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B.Sc., University of British Columbia, 1973

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April, 1976

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Date May 4,1976

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Abstract:

For a simplicial complex K, the de Rham algebra $\mathcal{E}^*(K)$ is the differential graded algebra (DGA) of $\mathbb{Q}$-coefficient polynomial forms in the barycentric coordinates of the simplices of K which agree as differential forms on common faces. The associated de Rham cohomology algebra is isomorphic to the simplicial cohomology of K with $\mathbb{Q}$-coefficients by integration of forms over simplices.

Given a 1-connected DGA, A, the minimal model of A is a DGA, M, which is free as an algebra, has a differential which decomposes the generators, and which computes the cohomology of A. Such minimal models exist and are unique up to isomorphism.

The minimal model M(X) of a 1-connected simplicial complex X is the minimal model of $\mathcal{E}^*(X)$. It depends only on the rational homotopy type of X. For a fibration $K(\pi,n) \longrightarrow E \longrightarrow Y$, with E and Y 1-connected, we have (under mild hypothesis)

$$M(E) = M(Y) \otimes H^*(K(\pi,n);\mathbb{Q})$$

with a suitably defined differential. This is applied inductively to the Postnikov decomposition of X to show that the free generators of M(X) correspond to the generators of $\pi_*(X) \otimes \mathbb{Q}$. The number of these generators which are cocycles is the rank of the rational Hurewicz homomorphism.

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Acknowledgments

The author thanks his supervisor, Dr. Roy R. Douglas, whose help and encouragement has made this thesis possible.

The author also thanks the University of British Columbia and the National Research Council of Canada for their financial support (1973-1975).

Introduction

This thesis is a presentation of Dennis Sullivan's theory of minimal models of rational homotopy type. Our purpose is to give a comprehensive development of the basic theory in the simply connected case as first outlined by Sullivan in [13], and to prove Theorems A and B of that paper. Further results in the theory and applications to manifolds can be found in [1].

Chapter 1 develops rational de Rham theory for simplicial complexes. The classical theory considers the differential graded algebra (DGA) of smooth differential forms on a manifold, and shows that the resulting cohomology agrees with simplicial theory. We achieve the generalization to simplicial complexes and rational coefficients by using rational coefficient polynomial forms defined on the various simplices in the space, and requiring that they "patch together" along common faces. The collection of such forms is a DGA whose cohomology algebra is naturally isomorphic by integration to simplicial cohomology with rational coefficients. The proof of this equivalence, which occupies the major part of the chapter, is essentially the same as that of the classical de Rham theorem as presented by H. Whitney [15]. One advantage of using differential forms rather than simplicial cochains is that the wedge product of forms is graded commutative, whereas the cup product of cochains is not. This property is needed in Chapter 3.

In Chapter 2 we discuss Sullivan's minimal models from a purely algebraic point of view. For a given simply connected DGA we construct its minimal model, a DGA which is free as a graded algebra, has a decomposable differential, and which computes the cohomology of the original DGA . The purpose of this chapter is to prove the existence and uniqueness of the minimal model. To this end we must study induced maps of minimal models, and this leads to a discussion of homotopy in the category of DGA's . The uniqueness of the minimal model is essential in Chapter 3.

Chapter 3, the main chapter in the thesis, relates the rational homotopy theory of a space to the algebraic construction of Chapter 2. The minimal model of a simply connected, simplicial complex is defined to be the minimal model of the rational de Rham algebra of the complex; it depends only on the rational homotopy type of the space (Theorem 3.4). The construction of the model parallels the Postnikov decomposition of the space in such a way that the algebra generators of degree n correspond to the generators of $\pi_n(X) \otimes \mathbb{Q}$, and the differentials of these generators correspond to the rational k-invariants in the decomposition. The proof is based on the Guy Hirsch method for computing the cohomology of a principal $K(\pi,n)$-fibration, which is essentially our Theorem 3.9 . This method of attack was suggested by René Thom in the Cartan Seminar, 1954, and our proof is a corrected version of that given by Dennis Sullivan and Roy Douglas in the summer of 1975. Our treatment stops short of the complete result [13: Theorem C], which states that the rational homotopy type of a simply connected space is uniquely determined by its minimal model. The main technique used for this result is the Hirsch method, which we present in detail.

Chapter 0 Algebraic Preliminaries

In this chapter, we define the algebraic objects which will be used in the remainder of the thesis.

0.1 Differential Graded Algebras and Cohomology: Let $\mathbb{Q}$ denote the rational numbers. By a graded algebra, A, over $\mathbb{Q}$ we mean a graded $\mathbb{Q}$-vector space $A = \bigoplus_{n\geq 0} A^n$ together with an associative multiplication $\mu : A \otimes A \to A$ which is graded $(\mu(A^n \otimes A^m) \subset A^{n+m})$ and graded commutative $(a\cdot b = (-1)^{nm}\, b\cdot a$ when $a \in A^n$ and $b \in A^m)$. We also assume, unless otherwise stated, that A has an identity element $1 \in A^0$. The elements of A^n are said to be homogeneous of degree n (or dimension n).

A differential graded algebra (or DGA) is a graded algebra, A, together with a differential, d, of degree $+1$, which is a derivation. This means that for each n there is a vector space homomorphism $d = d_n : A^n \to A^{n+1}$ satisfying $d\circ d = 0$ (differential) and $d(a\cdot b) = d(a)\cdot b + (-1)^n\, a\cdot d(b)$ for $a \in A^n$ (derivation).

If A is a DGA, let

$$Z^n(A) = \mathrm{Ker}\{d : A^n \to A^{n+1}\} = \text{subspace of cocycles of } A^n ,$$

$$B^n(A) = \mathrm{Im}\{d : A^{n-1} \to A^n\} = \text{subspace of coboundaries of } A^n .$$

$$Z^*(A) = \bigoplus_{n\geq 0} Z^n(A) \quad , \qquad B^*(A) = \bigoplus_{n\geq 0} B^n(A) .$$

As $d^2 = 0$, we have $B^n(A) \subset Z^n(A)$. Define the n^{th} cohomology space of A to be the quotient vector space $Z^n(A)/B^n(A)$, which we denote by $H^n(A)$. As d is a derivation, we see that $Z^*(A)$ is a subalgebra of A, and $B^*(A)$ is an ideal in $Z^*(A)$. Hence $H^*(A) = \bigoplus_{n\geq 0} H^n(A) = Z^*(A)/B^*(A)$ is a graded algebra (with identity if A is), called the cohomology algebra of A.

We say that the DGA, A, is connected if $H^0(A) = \mathbb{Q}$, and that A is simply connected if it is connected and $H^1(A) = 0$. We will be mainly concerned with simply connected DGA's.

If A and B are graded algebras, a function $f : A \to B$ is a homomorphism if it preserves all the algebraic structure; that is, $f(A^n) \subset B^n$, $f(a+b) = f(a) + f(b)$, and $f(a\cdot b) = f(a)\cdot f(b)$. We also assume that $f(1) = 1$. If A and B are DGA's, we require also that f commute with the differentials; $f\circ d_A = d_B\circ f$.

If $f : A \to B$ is a DGA homomorphism, then f induces a map $f^* : H^*(A) \to H^*(B)$ by the rule $f^*([z]) = [f(z)]$, where $[z]$ denotes the cohomology class of the element $z \in Z^*(A)$. Clearly f^* is a homomorphism of graded algebras. So we have categories $\mathcal{GA}$ and $\mathcal{DGA}$ of graded and differential graded algebras, respectively, and the cohomology functor

$$H^* : \mathcal{DGA} \to \mathcal{GA}.$$

0.2 <u>Tensor Products and Free Algebras</u>: If A and B are objects in $\mathcal{GA}$, we may form their tensor product $A \otimes B$ which, as a graded vector space, is $(A\otimes B)^n = \bigoplus_{i=0}^{n} (A^i \otimes B^{n-i})$, the tensor product on the right being the usual one for vector spaces. We define the multiplication in $A \otimes B$ by $(a \otimes b) \cdot (a' \otimes b') = (-1)^{nm} (a \cdot a') \otimes (b \cdot b')$ when $b \in B^n$ and $a' \in A^m$. One verifies that $A \otimes B$ is a graded algebra.

Note that $A \otimes B$ is the coproduct of A and B in the category $\mathcal{GA}$. To see this, note that there are canonical "inclusions"

$A \longrightarrow A \otimes B$ by $a \longmapsto a \otimes 1$, and

$B \longrightarrow A \otimes B$ by $b \longmapsto 1 \otimes b$. Now given any diagram with C, f, g arbitrary in $\mathcal{GA}$,

$$\begin{tikzcd}
A \arrow[d] \arrow[dr, "f"] & \\
A \otimes B \arrow[r, dashed, "h"] & C \\
B \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

there is a unique $h : A \otimes B \to C$ in $\mathcal{GA}$ given by $h(\sum_i a_i \otimes b_i) = \sum_i f(a_i) \cdot g(b_i)$ which makes the triangles commute.

If A and B are DGA's, we can define a differential in the graded algebra $A \otimes B$ as follows: for $a \in A^n$ and $b \in B$, set $d(a \otimes b) = d(a) \otimes b + (-1)^n a \otimes d(b)$, and extend by linearity. One checks that this makes $A \otimes B$ a DGA, which is the coproduct in the category $\mathcal{DGA}$. However, we will usually consider $A \otimes B$ as a graded algebra, and define differentials different from the one above.

The free graded algebra $\Lambda_n(x)$ on a generator, x, of degree n, is the polynomial algebra on x if n is even, and the exterior algebra on x if n is odd. That is, if n is even, $(\Lambda_n(x))^k = 0$ if $k \not\equiv 0 \pmod n$ and is the one dimensional vector space spanned by x^α if $k = \alpha n$. If n is odd, we add the relation $x^2 = 0$ (graded-commutativity) so $(\Lambda_n(x))^k = 0$ if $k \neq 0$, n. A graded algebra A is <u>free</u> on a set of generators $\{x_1, x_2, \ldots\}$ if A is a tensor product of free algebras on each of the generators. We write $A = \Lambda_n(x_1, x_2, \ldots)$ if all the generators are of degree n.

If A is a graded algebra or a DGA, we will write $A = \mathbb{Q}$ if $A^0 = \mathbb{Q}$ and $A^n = 0$ for $n \geq 1$; A is called the trivial DGA (with identity).

0.3 <u>Examples</u>:

(a) If K is a simplicial complex, the simplicial cohomology algebra $H^*(K;\mathbb{Q})$ of K with coefficients in $\mathbb{Q}$ is a graded algebra.

(b) If M is a smooth manifold, the collection $E^*(M)$ of smooth differential forms on M is a differential graded algebra (over the reals) with the wedge product as multiplication and the exterior derivative as diffferential. De Rham's theorem states that the cohomology of $E^*(M)$ is isomorphic to the ordinary simplicial cohomology $H^*(M;R)$ of a smooth triangulation of M [see 15]. We generalize this example in Chapter 1.

(c) The simplicial cochain complex $C^*(K;\mathbb{Q})$ of a simplicial complex

K is <u>not</u> a DGA ; the multiplication is <u>not</u> graded commutative, even though it is on the cohomology level. This is why we use differential forms instead of cochains.

Chapter 1. De Rham's Theorem for Simplicial Complexes

In this chapter, we extend the concepts of de Rham cohomology from a real theory on smooth manifolds to a rational theory on simplicial complexes. We do this by considering rational-coefficient polynomial forms in each simplex that "patch together" on common faces. We will see that de Rham's theorem holds in this generalized setting.

We assume only a basic familiarity with differential forms, eg. [12]. Our treatment follows that of H. Whitney [15].

1.1 The de Rham Algebra: We will work on an oriented simplicial complex K. Recall that K is the union of oriented n-simplices $(n=0,1,\ldots)$, each of which is homeomorphic to the standard n-simplex $\Delta_n \subset R^{n+1}$:

$$\Delta_n = \{(x_0,\ldots,x_n) \in R^{n+1} \mid x_i \geq 0 \text{ and } \sum_{i=0}^{n} x_i = 1\} .$$

The x_i's are the barycentric coordinates of Δ_n. Note that the boundary $\partial\Delta_n$ of the standard n-simplex is a union of $(n-1)$-simplices, and is a simplicial complex homeomorphic to the $(n-1)$-sphere $S^{n-1} \subset R^n$.

In Δ_n we consider differential k-forms

$$\omega = \sum_{0 \leq i_1 < \ldots < i_k \leq n} \omega_{i_1,\ldots i_k}(x_0,\ldots,x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $x_0,\ldots,x_n$ are the barycentric coordinates in Δ_n and the $\omega_{i_1,\ldots,i_k}$ are polynomials in $x_0,\ldots,x_n$ with rational coefficients.

In fact, these forms are defined on the n-dimensional hyperplane in R^{n+1} determined by Δ_n . On this hyperplane, we have the relations

$$x_0 + \dots + x_n = 1 ,$$

$$dx_0 + \dots + dx_n = 0 .$$

If K is an (oriented) simplicial complex, a differential k-form ω on K is a collection $\{\omega_\sigma\}$, one for each simplex σ of K , of Q-polynomial k-forms in the barycentric coordinates of σ (as above) which satisfy the following coherence condition:

If σ is a simplex of K and τ is a face of σ , then

$$\omega_\sigma|\tau = \omega_\tau ,$$

where the left hand side is the restriction of ω_σ to τ in the sense of differential forms. Let $\mathcal{E}^k(K)$ denote the collection of all such k-forms on K , and set

$$\mathcal{E}^*(K) = \bigoplus_{k\geq 0} \mathcal{E}^k(K) .$$

We define a product $\wedge$ and a differential d in $\mathcal{E}^*(K)$ as follows:

If $\omega \in \mathcal{E}^k(K)$ and $\eta \in \mathcal{E}^\ell(K)$, then $\omega \wedge \eta \in \mathcal{E}^{k+\ell}(K)$ and $d\omega \in \mathcal{E}^{k+1}(K)$ are given on a simplex σ of K by

$$(\omega \wedge \eta)_\sigma = \omega_\sigma \wedge \eta_\sigma$$

$$(d\omega)_\sigma = d(\omega_\sigma) ,$$

where $\wedge$ and d on the right are the usual wedge product and exterior derivative on $\sigma \approx \Delta_n$ (as a subset of the n-dimensional hyperplane in R^{n+1}) . Clearly $\omega \wedge \eta$ and $d\omega$ satisfy the coherence condition of the last paragraph. All the usual properties of forms hold in $E^*(K)$:

$$(\omega_1+\omega_2) \wedge \eta = (\omega_1 \wedge \eta) + (\omega_2 \wedge \eta) ,$$

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) ,$$

$$(q\omega) \wedge \eta = q(\omega \wedge \eta) \quad \text{for } q \in \mathbb{Q} ,$$

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega \quad \text{when } \omega \in E^k(K) , \eta \in E^\ell(K) ,$$

$$d(d\theta) = 0 ,$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad \text{when } \omega \in E^k(K) .$$

Note that $E^0(K)$ consists of equivalence classes of $\mathbb{Q}$-polynomials in the barycentric coordinates of the various simplices of K ; the equivalence relation is the result of the fact that the sum of the barycentric coordinates is identically 1 at every point of K . The constant polynomial $1 \in E^0(K)$ acts as the identity for the wedge product in $E^*(K)$.

Hence $E^*(K)$ is a differential graded algebra over $\mathbb{Q}$, called the rational de Rham algebra of K . The cohomology algebra

$H^*(\mathcal{E}^*(K))$ is called the de Rham cohomology algebra of K, and will be denoted $H^*_{DR}(K)$.

If K and K' are simplicial complexes and $f : K \to K'$ is a simplicial map (i.e., takes simplices to simplices linearly), then f induces a map of DGA's

$$\mathcal{E}^*(f) = f^* : \mathcal{E}^*(K') \to \mathcal{E}^*(K)$$

by substitution of barycentric coordinates. That is, if $\omega \in \mathcal{E}^k(K')$, $f^*(\omega) \in \mathcal{E}^k(K)$ is given on a simplex σ of K by

$$(f^*\omega)_\sigma(x) = (\omega)_{f(\sigma)}(f(x)), \quad x \in \sigma .$$

One verifies that $(g\circ f)^* = f^*\circ g^*$ for $f : K \to K'$ and $g : K' \to K''$, and that (identity on $K)^*$ is the identity on $\mathcal{E}^*(K)$. So $\mathcal{E}^*$ is a contravariant functor from the category of simplicial complexes to the category $\mathcal{DGA}$.

Note that if $\omega \in \mathcal{E}^k(K)$ and σ is an ℓ-simplex of K, $\ell < k$, then $(\omega)_\sigma = 0$. Hence if $f : K \to K'$ $\omega \in \mathcal{E}^k(K')$ and σ is a simplex of K for which $f(\sigma)$ has dimension $< k$, then $(f^*\omega)_\sigma = 0$, even if σ has dimension $\geq k$.

If $i : L \longrightarrow K$ is the inclusion of a subcomplex L into the simplicial complex K, then the induced map

$$i^* : \mathcal{E}^*(K) \to \mathcal{E}^*(L)$$

corresponds to restriction of the forms on K to L. We will show (Proposition 1.9) that i^* is always an epimorphism. In other words, any form on the subcomplex L can be "extended" to a form on all of K. We use this fact to define the relative de Rham algebra of K modulo L as the kernel of i^*.

1.2 Simplicial Cohomology: We wish to establish a relationship between de Rham cohomology and simplicial cohomology. In this section we set the notation of the latter.

If K is an oriented simplicial complex, let $C_*(K;\mathbb{Q})$ denote the chain complex of simplicial chains in K, with $\mathbb{Q}$-coefficients. So $C_n(K;\mathbb{Q})$ is the vector space over $\mathbb{Q}$ with basis consisting of the oriented n-simplices of K, $-\sigma$ being identified with the opposite orientation of σ. There are boundary operators $\partial : C_n(K;\mathbb{Q}) \to C_{n-1}(K;\mathbb{Q})$, and the homology of $C_*(K;\mathbb{Q})$ with respect to ∂ is the simplicial homology of K, denoted $H_*(K;\mathbb{Q})$; it is a graded vector space over $\mathbb{Q}$.

Similarly, the simplicial cochain complex $C^*(K;\mathbb{Q})$ of K is given by $C^n(K;\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(C_n(K;\mathbb{Q}),\mathbb{Q})$, with coboundary operator $\delta : C^n(K;\mathbb{Q}) \to C^{n+1}(K;\mathbb{Q})$ dual to ∂. If $c \in C^n(K;\mathbb{Q})$, we denote the value of this homomorphism on a chain $z \in C_n(K;\mathbb{Q})$ by $\langle c,z\rangle \in \mathbb{Q}$; $\langle\ ,\ \rangle$ is a bilinear pairing of cochains and chains. To each oriented n-simplex σ of K, there corresponds a cochain $c_\sigma \in C^n(K;\mathbb{Q})$ whose value on a basis element (that is, an oriented n-simplex) $\tau \in C_n(K;\mathbb{Q})$ is given by

$$\langle c_\sigma, \tau\rangle = \begin{cases} 1 & \text{if } \tau = \sigma \\ 0 & \text{if } \tau \neq \sigma . \end{cases}$$

Every element of $C^n(K;\mathbb{Q})$ can be written uniquely as a (possibly infinite) linear combination of the cochains c_σ . When there is no confusion, we use $\sigma \in C^n(K;\mathbb{Q})$ to denote the cochain c_σ , as well as the chain $\sigma \in C_n(K;\mathbb{Q})$ and the n-simplex $\sigma \subset K$.

There is also a multiplication in $C^*(K;\mathbb{Q})$ given by the cup product

$$\cup \quad : C^n(K;\mathbb{Q}) \otimes C^m(K;\mathbb{Q}) \to C^{n+m}(K;\mathbb{Q}) \ ;$$

it is associative, has an identity $1 \in C^0(K;\mathbb{Q})$ given by $\langle 1, v \rangle = 1 \in Q$ for every vertex $v \in K$, and with respect to this product, δ is a derivation. However $C^*(K;\mathbb{Q})$ is <u>not</u> a DGA in our sense, as the cup product fails to be graded commutative. On passage to cohomology, we obtain the simplicial cohomology of K , denoted $H^*(K;\mathbb{Q})$; it is a graded algebra, as the cup product <u>is</u> graded commutative on the cohomology level.

1.3 <u>Definition</u>: We define a map of graded vector spaces

$$\psi : E^*(K) \longrightarrow C^*(K;\mathbb{Q}) \ ,$$

called integration, as follows: if $\omega \in E^k(K)$, $\psi(\omega)$ is the k-cochain whose value on a k-simplex, σ , of K is

$$\langle \psi(\omega), \sigma \rangle = \int_\sigma \omega_\sigma \ .$$

This is a rational number as we are using polynomial forms with rational coefficients. In fact, an easy computation shows that

$$\int_\sigma dx_1 \wedge \cdots \wedge dx_k = \frac{1}{k!} \quad ,$$

where $x_0, \ldots, x_k$ are the barycentric coordinates in the k-simplex σ (with the correct orientation).

As the integral is additive, ψ is a morphism of graded vector spaces. In fact, more is true:

1.4 Proposition: ψ is a natural homomorphism of cochain complexes.

Proof: To show that ψ is a cochain map, we verify the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{E}^k(K) & \xrightarrow{\psi} & C^k(K;\mathbb{Q}) \\ \downarrow d & & \downarrow \delta \\ \mathcal{E}^{k+1}(K) & \xrightarrow{\psi} & C^{k+1}(K;\mathbb{Q}) \end{array} \quad .$$

For $\omega \in \mathcal{E}^k(K)$ and σ a $(k+1)$-simplex of K, we have

$$\langle \delta\psi(\omega), \sigma \rangle = \langle \psi(\omega), \partial\sigma \rangle = \int_{\partial\sigma} \omega$$

$$= \int_\sigma d\omega = \langle \psi(d\omega), \sigma \rangle \quad ,$$

where the third equality is just Stokes' Theorem. As this holds for all such σ, $\delta\psi(\omega) = \psi(d\omega)$, and the diagram commutes.

For naturality, if $f : K \longrightarrow K'$ is a simplicial map, we must establish the commutativity of the diagram

$$\begin{tikzcd} \mathcal{E}^k(K) \arrow[r, "\psi"] & C^k(K;\mathbb{Q}) \\ \mathcal{E}^k(K') \arrow[u, "f^*"] \arrow[r, "\psi"] & C^k(K';\mathbb{Q}) \arrow[u, "f^*"] \end{tikzcd} .$$

Suppose $\omega \in \mathcal{E}^k(K')$ and σ is a k-simplex of K . If $f(\sigma)$ is a simplex of K' of dimension k , then

$$\left\langle f^*\psi(\omega),\sigma\right\rangle = \left\langle \psi(\omega),\ f_*(\sigma)\right\rangle = \left\langle \psi(\omega)\ ,\ \pm f(\sigma)\right\rangle =$$

$$= \pm \int_{f(\sigma)} \omega = \int_\sigma f^*\omega = \left\langle \psi(f^*\omega),\sigma\right\rangle ,$$

the $\pm$ depending on whether $f(\sigma)$ has the correct or opposite orientation in K' . If $f(\sigma)$ has dimension $< k$, then $f_*(\sigma) = 0 \in C_k(K';\mathbb{Q})$ and $(f^*\omega)_\sigma = 0$, so

$$\left\langle f^*\psi(\omega),\sigma\right\rangle = 0 = \left\langle \psi(f^*\omega),\sigma\right\rangle .$$

Hence $f^*\psi(\omega) = \psi(f^*\omega)$, and the diagram commutes. Q.E.D.

As ψ is a cochain map, it induces a map of graded vector spaces,

$$\psi^* : H^*_{DR}(K) \longrightarrow H^*(K;\mathbb{Q}) .$$

We will see that ψ^* is, in fact, a map of graded algebras, and, as is the case with C^∞-forms on a manifold, we have

1.5 de Rham's Theorem:

$\psi^* : H^*_{DR}(-) \to H^*(-;\mathbb{Q})$ is a natural equivalence of functors from the category of simplicial complexes to the category of graded algebras.

Before proving de Rham's Theorem, we need some preliminary results.

Recall that $\omega \in E^k(K)$ is closed if $d\omega = 0$, and exact if $\omega = d\eta$ for some $\eta \in E^{k-1}(K)$; $H^k_{DR}(K)$ is just the closed forms modulo the exact forms on K.

1.6 Proposition (Poincaré Lemma):

For $k \geq 1$, every closed k-form on Δ_n is exact.

Proof: For $k \geq n+1$, $E^k(\Delta_n) = 0$ and there is nothing to prove. So suppose $1 \leq k \leq n$, $\omega \in E^k(\Delta_n)$, and $d\omega = 0$. We can assume that x_0 and dx_0 do not appear in the expression for ω (if they do, rewrite $x_0 = 1 - x_1 - \ldots - x_n$, $dx_0 = -dx_1 - \ldots - dx_n$). Let ℓ be the largest integer $(k \leq \ell \leq n)$ for which dx_ℓ appears in ω. Then we can write $\omega = \xi + \theta \wedge dx_\ell$, where ξ and θ do not involve $dx_\ell, \ldots, dx_n$ (or x_0, dx_0). Write $\theta = \sum_J \theta_J \, dx_J$, where the sum is over all $J \in \{(j_1, \ldots, j_{k-1}) \mid 1 \leq j_1 < \ldots < j_{k-1} \leq \ell - 1\}$ and $dx_J = dx_{j_1} \wedge \ldots \wedge dx_{j_{k-1}}$. Now $0 = d\omega = d\xi + d\theta \wedge dx_\ell$, and

$$d\theta = \sum_J \sum_{i=1}^{n} \frac{\partial \theta_J}{\partial x_i} \, dx_i \wedge dx_J .$$

As any term in $d\xi$ can have only one factor dx_α for $\alpha \geq \ell$, and the terms $\frac{\partial\theta_J}{\partial x_i} dx_i \wedge dx_J \wedge dx_\ell$ have two such factors when $i \geq \ell + 1$, we have $\frac{\partial\theta_J}{\partial x_{\ell+1}} = \ldots = \frac{\partial\theta_J}{\partial x_n} = 0$, $\forall J$. By integrating θ_J with respect to x_ℓ, we can find rational polynomials λ_J satisfying

$\frac{\partial\lambda_J}{\partial x_\ell} = (-1)^{k-1}\theta_J$, $\frac{\partial\lambda_J}{\partial x_\alpha} = 0$ for $\alpha \geq \ell + 1$. Set

$\lambda = \sum_J \lambda_J \, dx_J \in E^{k-1}(\Delta_n)$. Then $d\lambda = \sum_J \sum_{i=1}^{\ell-1} \frac{\partial\lambda_J}{\partial x_i} dx_i \wedge dx_J +$

$\sum_J \frac{\partial\lambda_J}{\partial x_\ell} dx_\ell \wedge dx_J$. The second terms is just $\sum_J (-1)^{k-1}\theta_J \, dx_\ell \wedge dx_J =$

$\sum_J \theta_J \, dx_J \wedge dx_\ell = \theta \wedge dx_\ell$. So

$\omega - d\lambda = \xi - \sum_J \sum_{i=1}^{\ell-1} \frac{\partial\lambda_J}{\partial x_i} dx_i \wedge dx_J$, which does not involve

$dx_\ell, dx_{\ell+1}, \ldots, dx_n$. Repeating this process $\ell - k + 1$ times yields $\eta \in E^{k-1}(\Delta_n)$ for which $\omega - d\eta$ does not involve $dx_k, \ldots, dx_n$. But this leaves only $dx_1, \ldots, dx_{k-1}$, and $\omega - d\eta$ is a k-form, so $\omega = d\eta$.

Q.E.D.

1.7 <u>Corollary</u>:

$$H^k_{DR}(\Delta_n) = \begin{cases} \mathbb{Q}, & k = 0 \\ 0, & k \geq 1 \end{cases}$$

Proof: The Poincaré Lemma gives the result when $k \geq 1$. $E^0(\Delta_n)$ consists of all rational polynomials in $x_1,\ldots,x_n$. If $\omega \in E^0(\Delta_n)$ and $d\omega = 0$, we have

$$0 = d\omega = \sum_{i=1}^{n} \frac{\partial\omega}{\partial x_i} dx_i$$

So $\frac{\partial\omega}{\partial x_i} = 0$, $1 \leq i \leq n$, and hence ω is constant. As $B^0(E^*(\Delta_n)) = 0$, we have

$$H^0_{DR}(\Delta_n) = Z^0(E^*(\Delta_n)) \cong \mathbb{Q} . \qquad \text{Q.E.D.}$$

We now look at the problem of extending forms. Denote by $F_j\Delta_n$ the (n-1)-face of Δ_n given by $x_j = 0$, and by $\partial\Delta_n$ the topological boundary of Δ_n as an (n-1)-dimensional simplicial complex (with the induced orientations).

1.8 Lemma: If $\omega \in E^k(F_j\Delta_n)$, there is a form $\bar{\omega} \in E^k(\Delta_n)$ such that:

i) $\bar{\omega}|F_j\Delta_n = \omega$

ii) $\bar{\omega}|F_i\Delta_n = 0$ whenever $\omega|F_i\Delta_n \cap F_j\Delta_n = 0$.

Proof: Without loss of generality, assume $\omega \in E^k(F_n\Delta_n)$ (i.e., that $j=n$). Let $C = \{(x_0,\ldots,x_n) \in \Delta_n | x_n \neq 1\}$ be the complement of the n^{th} vertex. Let $p : C \longrightarrow F_n\Delta_n$ be the projection defined by

$$p(x_0,\ldots,x_n) = \left(\frac{x_0}{1-x_n},\ldots,\frac{x_{n-1}}{1-x_n}, 0\right) \in F_n\Delta_n$$

Then $p^*(\omega)$ is a k-form on C as a subset of R^{n+1}, but is <u>not</u> a polynomial form. In fact, for $0 \leq i \leq n-1$, we have

$$p^*(x_i) = x_i/(1-x_n)\text{ , and } p^*(dx_i) = d(p^*(x_i)) = d(x_i/(1-x_n)) =$$

$$\sum_{j=0}^{n} \frac{\partial}{\partial x_j}\left(\frac{x_i}{1-x_n}\right) dx_j = \frac{1}{1-x_n}\left(dx_i + \frac{x_i}{1-x_n}dx_n\right) .$$

So for sufficiently large N, $(1-x_n)^N p^*(\omega)$ is a Q-polynomial form on all of Δ_n, and we set

$$\bar{\omega} = (1-x_n)^N p^*(\omega) \in \mathcal{E}^k(\Delta_n) .$$

On $F_n\Delta_n$, $dx_n = 0$ and $(1-x_n) = 1$, and

$$p|F_n\Delta_n : F_n\Delta_n \to F_n\Delta_n \text{ is the identity map. Hence}$$

$\bar{\omega}|F_n\Delta_n = \omega$. If $\omega|F_i\Delta_n \cap F_n\Delta_n = 0$, we have $p^*(\omega)|C \cap F_i\Delta_n = 0$, and hence $\bar{\omega}|F_i\Delta_n = 0$. Q.E.D.

1.9 <u>Lemma</u>: If $\omega \in \mathcal{E}^k(\partial\Delta_n)$, there is an $\bar{\omega} \in \mathcal{E}^k(\Delta_n)$ such that $\bar{\omega}|\partial\Delta_n = \omega$.

<u>Proof</u>: Let $\omega_j = \omega|F_j\Delta_n \in \mathcal{E}^k(F_j\Delta_n)$. The coherence condition on ω says that, for all $0 \leq i, j \leq n$,

$$\omega_i|F_i\Delta_n \cap F_j\Delta_n = \omega_j|F_i\Delta_n \cap F_j\Delta_n .$$

We must find $\bar{\omega} \in \mathcal{E}^k(\Delta_n)$ so that $\bar{\omega}|F_j\Delta_n = \omega_j$, $j = 0,\ldots,n$. By Lemma 1.8, we can find $\bar{\omega}_0 \in \mathcal{E}^k(\Delta_n)$ so that

$\overline{\omega}_0 | F_0\Delta_n = \omega_0$. Suppose we have constructed $\overline{\omega}_j \in \mathcal{E}^k(\Delta_n)$, for some $j \leq n-1$, so that $\overline{\omega}_j | F_i\Delta_n = \omega_i$ for all $i \leq j$. Set $\omega'_{j+1} = \omega_{j+1} - \overline{\omega}_j | F_{j+1}\Delta_n \in \mathcal{E}^k(F_{j+1}\Delta_n)$. Then $\omega'_{j+1} | F_{j+1}\Delta_n \cap F_i\Delta_n = 0$ for $i \leq j$ by the coherence condition. By Lemma 1.8, we can find $\overline{\omega}'_{j+1} \in \mathcal{E}^k(\Delta_n)$ so that $\overline{\omega}'_{j+1} | F_{j+1}\Delta_n = \omega'_{j+1}$

$$\text{and} \quad \overline{\omega}'_{j+1} | F_i\Delta_n = 0 \quad \text{for} \quad i \leq j .$$

Set $\overline{\omega}_{j+1} = \overline{\omega}_j + \overline{\omega}'_{j+1} \in \mathcal{E}^k(\Delta_n)$. Then $\overline{\omega}_{j+1} | F_i\Delta_n = \omega_i$ for all $i \leq j+1$, and the induction continues. We eventually get $\overline{\omega} = \overline{\omega}_n \in \mathcal{E}^k(\Delta_n)$ with the desired property. Q.E.D.

1.10 <u>Proposition</u> : If $i : L \hookrightarrow K$ is the inclusion of a subcomplex L in a simplicial complex K , then $i^* : \mathcal{E}^*(K) \to \mathcal{E}^*(L)$ is an epimorphism.

<u>Proof</u>: Recall that, for $\overline{\omega} \in \mathcal{E}^k(K)$, $i^*(\overline{\omega}) = \overline{\omega}|L$. So, given $\omega \in \mathcal{E}^k(L)$, we must find $\overline{\omega} \in \mathcal{E}^k(K)$ for which $\overline{\omega}|L = \omega$. Let K^n denote the n-skeleton of K . Note that $\mathcal{E}^k(K^{k-1}) = 0$. We define $\overline{\omega}$ inductively over the relative skeleta. Set $\overline{\omega}|L = \omega$, $\overline{\omega}|K^{k-1} = 0$, and $\overline{\omega}|\sigma = 0$ for any k-simplex σ of $K - L$. This defines $\overline{\omega}$ coherently on $L \cup K^k$. Suppose we have defined $\overline{\omega}$ on $L \cup K^n$ for some $n \geq k$. If σ is an $(n+1)$-simplex of $K - L$, then $\partial\sigma \subset K^n$, and Lemma 1.9 allows us to extend $\overline{\omega}$ from $\partial\sigma$ to

σ . Doing this for every simplex of $K^{n+1} - L$ defines $\bar{\omega}$ on $L \cup K^{n+1}$. So, by induction, we can define $\bar{\omega}$ on all of K so that $\bar{\omega}|L = \omega$. Q.E.D.

1.11 <u>The Relative de Rham Algebra</u>:

If $i : L \hookrightarrow K$ is the inclusion of a subcomplex, we define

$$E^k(K,L) = \mathrm{Ker}\{i^* : E^k(K) \twoheadrightarrow E^k(L)\}$$

$$= \{\omega \in E^k(K) \mid \ \omega|L = 0\}$$

Then $E^*(K,L) = \bigoplus_{k \geq 0} E^k(K,L)$ is a DGA (without identity, unless $L = \emptyset$) called the relative de Rham algebra of K modulo L . If $f : (K,L) \longrightarrow (K',L')$ is a simplicial map of pairs (i.e., $L \subset K$, $L' \subset K'$, and $f(L) \subset L'$) , then f induces a map of DGA's

$$f^* : E^*(K',L') \longrightarrow E^*(K,L)$$

in the obvious way. Thus E^* is a contravariant functor from the category of simplicial pairs to the category $\mathcal{DGA}$, and $E^*(K,\emptyset) = E^*(K)$.

By Proposition 1.10, there is a short exact sequence of DGA's

$$0 \longrightarrow E^*(K,L) \xrightarrow{j^*} E^*(K) \xrightarrow{i^*} E^*(L) \longrightarrow 0$$

where $L \xrightarrow{i} K \xrightarrow{j} (K,L)$ are the simplicial inclusions. So passing to cohomology, we have a long exact sequence

$$\cdots \longrightarrow H^n_{DR}(K,L) \xrightarrow{j^*} H^n_{DR}(K) \xrightarrow{i^*} H^n_{DR}(L) \longrightarrow H^{n+1}_{DR}(K,L) \longrightarrow \cdots$$

where $H^*_{DR}(K,L)$ denotes the cohomology algebra of $E^*(K,L)$.

Another consequence of Proposition 1.10 is:

1.12 <u>Proposition</u>: Let K be a simplicial complex and I a directed set. If $\{L_\alpha : \alpha \in I\}$ is a direct system of subcomplexes of K (under inclusion) whose union is K , then there are natural isomorphisms

$$E^*(K) \cong \varprojlim_I E^*(L_\alpha) \quad \text{of DGA's,}$$

and

$$H^*_{DR}(K) \cong \varprojlim_I H^*_{DR}(L_\alpha) \quad \text{of graded algebras.}$$

<u>Proof</u>: It is clear that $\{E^*(L_\alpha) : \alpha \in I\}$ is an inverse system of DGA's in which all the maps are epimorphisms, and that the first isomorphism holds. D. W. Kahn [4; Theorem 1.1] has shown that for such systems of DGA's, cohomology commutes with inverse limits, so the second isomorphism holds. Q.E.D.

1.13 <u>A Right Inverse for Integration</u>: We now begin the proof of de Rham's Theorem by constructing a right inverse $\phi : C^*(K;\mathbb{Q}) \longrightarrow E^*(K)$ to the integration map ψ .

If τ and τ' are simplices of K , write $\tau < \tau'$ if τ is a face of τ' . Define the star of the simplex τ to be

$$St(\tau) = \bigcup \{ \langle\tau'\rangle \mid \tau < \tau' \} ,$$

where $\langle\tau'\rangle$ denotes the open simplex

$$\langle\tau'\rangle = \{(t_0,\ldots,t_n) \in \tau' \mid t_i > 0 , i = 0,\ldots,n\} .$$

Then $St(\tau)$ is an open subset of K . Also,

$$St(\tau) = \bigcap \{St(v) \mid v \text{ is a vertex of } \tau\} .$$

Let $\sigma = [p_0,\ldots,p_n]$ be an oriented n-simplex of K , where $p_0,\ldots,p_n$ are the vertices, and let $x_0,\ldots,x_n$ be the barycentric coordinates with respect to $p_0,\ldots,p_n$ considered as functions (0-forms) on all of K . So $x_i = 1$ at p_i , and $x_i \equiv 0$ on $K - St(p_i)$. Consider σ as a cochain in $C^n(K;Q)$ as in 1.2, and define an n-form $\phi(\sigma) \in E^n(K)$ by

$$(1) \quad \phi(\sigma) = n! \sum_{i=0}^{n} (-1)^i \, x_i \, dx_0 \wedge \ldots \widehat{dx_i} \ldots \wedge dx_n ,$$

where the symbol $\widehat{\ }$ over dx_i indicates that it is omitted.

<u>Sublemma</u>: $\phi(\sigma) \equiv 0$ on $K - St(\sigma)$.

<u>Proof</u>: If τ is a simplex of $K - St(\sigma)$, there is an i for which p_i is not a vertex of τ . So $x_i = 0$ and $dx_i = 0$ on τ . But every term in (1) has either x_i or dx_i as a factor, and hence $\phi(\sigma)|\tau = 0$. Q.E.D.

We wish to extend the definition of ϕ linearly over $C^*(K;Q)$. Every element of $C^n(K;Q)$ has the form $\sum_\alpha c_\alpha \sigma_\alpha$, where $c_\alpha \in Q$, σ_α is an n-simplex, and the sum is over all n-simplices of K. The problem is that an infinite number of the c_α may be non-zero, and we cannot add an infinite number of differential forms. However if τ is any simplex of K,

$$\langle \tau \rangle \subset St(\sigma_\alpha) \quad \text{iff} \quad \sigma_\alpha < \tau ,$$

and hence

$$\phi(\sigma_\alpha)|\tau \neq 0 \quad \text{iff} \quad \sigma_\alpha < \tau .$$

As τ has only a finite number of faces, all but a finite number of the forms $\phi(\sigma_\alpha)$ are zero on τ. So we can define

(2) $$\phi\left(\sum_\alpha c_\alpha \sigma_\alpha \right) = \sum_\alpha c_\alpha \phi(\sigma_\alpha) ,$$

keeping in mind that the sum on the right is finite on any simplex of K. So we have a linear map of graded vector spaces

$$\phi : C^*(K;\mathbb{Q}) \longrightarrow E^*(K) .$$

1.14 Proposition:

(a) ϕ is a homomorphism of cochain complexes.

(b) $\psi \circ \phi$ is the identity on $C^*(K;\mathbb{Q})$.

(c) ϕ preserves identity elements.

Proof: For part (a), we must show

$$d\phi = \phi\delta : C^n(K;\mathbb{Q}) \longrightarrow E^{n+1}(K) .$$

It suffices to show this on an arbitrary oriented n-simplex σ of K. Let $\{p_\alpha\}$ be the vertices of K, and $\sigma = [p_0,\dots,p_n]$.

$$d\phi(\sigma) = n! \sum_{i=0}^{n} (-1)^i \, dx_i \wedge dx_0 \wedge \dots \widehat{dx_i} \dots \wedge dx_n$$

$$= n! \sum_{i=0}^{n} dx_0 \wedge \dots \wedge dx_n = (n+1)! \, dx_0 \wedge \dots \wedge dx_n .$$

Now $\delta\sigma = \sum_\alpha^* [p_\alpha, p_0,\dots,p_n]$ where $\sum_\alpha^*$ indicates that the sum is over all α for which $[p_\alpha, p_0,\dots,p_n]$ is an $(n+1)$-simplex of K.

$$(3) \quad \frac{\phi(\delta\sigma)}{(n+1)!} = \sum_\alpha^* \{x_\alpha dx_0 \wedge \dots \wedge dx_n - \sum_{i=0}^{n} (-1)^i \, x_i \, dx_\alpha \wedge dx_0 \wedge \dots \widehat{dx_i} \dots \wedge dx_n\}$$

If $p_\alpha, p_0,\dots,p_n$ are <u>not</u> the vertices of a simplex and τ is any simplex, there is a $j \in \{\alpha, 0,\dots,n\}$ so that $x_j = 0$ and $dx_j = 0$ on τ. Hence

$$x_\alpha \, dx_0 \wedge \dots \wedge dx_n = 0 \quad \text{and}$$

$$x_i \, dx_\alpha \wedge dx_0 \wedge \dots \widehat{dx_i} \dots \wedge dx_n = 0 \quad (i=0,\dots,n) \quad \text{on } \tau .$$

As this is true for every τ in K, $\sum_\alpha^*$ in (3) can be replaced by $\sum_{\alpha \neq 0,\dots,n}$. Also $\sum_\alpha dx_\alpha = 0$ on K, so $\sum_{j=0}^{n} dx_j = - \sum_{\alpha \neq 0,\dots,n} dx_\alpha$.

So (3) becomes

$$\frac{\phi(\delta\sigma)}{(n+1)!} = \sum_{\alpha\neq 0,\ldots,n} x_\alpha \, dx_0 \wedge \cdots \wedge dx_n + \sum_{i=0}^{n} (-1)^i x_i \sum_{j=0}^{n} dx_j \wedge dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$$

$$= \sum_{\alpha\neq 0,\ldots,n} x_\alpha \, dx_0 \wedge \cdots \wedge dx_n + \sum_{i=0}^{n} (-1)^i x_i \, dx_i \wedge dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$$

$$= \sum_{\alpha\neq 0,\ldots,n} x_\alpha \, dx_0 \wedge \cdots \wedge dx_n + \sum_{i=0}^{n} x_i \, dx_0 \wedge \cdots \wedge dx_n$$

$$= \left(\sum_\alpha x_\alpha\right) dx_0 \wedge \cdots \wedge dx_n = dx_0 \wedge \cdots \wedge dx_n ,$$

as $\sum_\alpha x_\alpha = 1$. Hence $d\phi(\sigma) = \phi(\delta\sigma)$.

For part (b), we must show that $\psi\phi(\sigma) = \sigma$ for every n-simplex σ of K . This means that, for any oriented n-simplex τ ,

$$\int_\tau \phi(\sigma) = \begin{cases} 1 & \text{if } \tau = \sigma \\ 0 & \text{if } \tau \neq \sigma \end{cases}$$

If $\tau \neq \sigma$, then $\tau \cap St(\sigma) = \emptyset$, so that $\phi(\sigma)|\tau = 0$, and the result follows. If $\sigma = [p_0,\ldots,p_n]$, then the relations

$$\sum_{i=0}^{n} x_i = 1 \quad \text{and} \quad \sum_{i=0}^{n} dx_i = 0$$

are valid on σ . So we have

$$\frac{\phi(\sigma)}{n!} = x_0 \, dx_1 \wedge \cdots \wedge dx_n + \sum_{i=1}^{n} (-1)^i x_i \left(-\sum_{j=1}^{n} dx_j\right) \wedge dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$$

$$= x_0 dx_1 \wedge \cdots \wedge dx_n + \sum_{i=1}^{n} (-1)^{i+1} x_i \, dx_i \wedge dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$$

$$= x_0 dx_1 \wedge \cdots \wedge dx_n + \sum_{i=1}^{n} x_i \, dx_1 \wedge \cdots \wedge dx_n$$

$$= \left(\sum_{i=0}^{n} x_i\right) dx_1 \wedge \cdots \wedge dx_n = dx_1 \wedge \cdots \wedge dx_n$$

So $\int_\sigma \phi(\sigma) = n! \int_\sigma dx_1 \wedge \cdots \wedge dx_n = n! \cdot \frac{1}{n!} = 1$. This proves (b).

For part (c), recall that the identity element of $C^*(K;\mathbb{Q})$ is $1 = \sum_\alpha p_\alpha \in C^0(K;\mathbb{Q})$. But $\phi(p_\alpha) = x_\alpha$, so

$$\phi(1) = \phi\left(\sum_\alpha p_\alpha\right) = \sum_\alpha \phi(p_\alpha) = \sum_\alpha x_\alpha = 1 ,$$

and ϕ preserves identities. Q.E.D.

1.15 Corollary: $\psi : E^*(K) \longrightarrow C^*(K;\mathbb{Q})$

and $\psi^* : H^*_{DR}(K) \longrightarrow H^*(K;\mathbb{Q})$

are both epimorphisms.

Proof: $\psi \circ \phi = id$, so ψ is epimorphic, and $\psi^* \circ \phi^* = (\psi \circ \phi)^* = (id)^* = id$, so ψ^* is epimorphic. Q.E.D.

1.16 Proof of de Rham's Theorem:

We must show that $\psi^* : H^*_{DR}(K) \longrightarrow H^*(K;\mathbb{Q})$ is a natural isomorphism of graded algebras. By Corollary 1.15, ψ^* is an epimorphism; and naturality follows from Proposition 1.4. We leave the proof that ψ^* is an algebra homomorphism till last, and prove the isomorphism part by induction on the dimension of the complex K.

If K is 0-dimensional, we have

$$H^*_{DR}(K) = E^*(K) \cong C^*(K;\mathbb{Q}) = H^*(K;\mathbb{Q}) ,$$

and clearly ψ^* is an isomorphism.

Suppose inductively that ψ^* is an isomorphism (of graded vector spaces) for all complexes of dimension less than n, and suppose K has dimension n. As ψ is an epimorphism, we have a short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Ker}(\psi) \hookrightarrow E^*(K) \xrightarrow{\psi} C^*(K;\mathbb{Q}) \longrightarrow 0 .$$

This induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^k(\mathrm{Ker}\ \psi) \longrightarrow H^k_{DR}(K) \xrightarrow{\psi^*} H^k(K;\mathbb{Q}) \longrightarrow H^{k+1}(\mathrm{Ker}\ \psi) \longrightarrow \cdots$$

So ψ^* is an isomorphism if and only if $H^*(\mathrm{Ker}\ \psi) = 0$. The proof relies on the following:

Lemma: Suppose $\omega \in E^k(\Delta_n)$, $1 \le k \le n$, $d\omega = 0$, and $\psi(\omega) = 0$, and $\eta \in E^{k-1}(\partial\Delta_n)$, $d\eta = \omega|\partial\Delta_n$, and $\psi(\eta) = 0$. Then there is an $\overline{\eta} \in E^{k-1}(\Delta_n)$ for which $\overline{\eta}|\partial\Delta_n = \eta$, $d\overline{\eta} = \omega$, and $\psi(\overline{\eta}) = 0$.

Proof: By Lemma 1.9, we can find $\eta' \in E^{k-1}(\Delta_n)$ so that $\eta'|\partial\Delta_n = \eta$. Then $d(\omega - d\eta') = 0$, so the Poincaré Lemma gives $\theta \in E^{k-1}(\Delta_n)$ so that $d\theta = \omega - d\eta'$. We have

$$d(\theta|\partial\Delta_n) = \omega|\partial\Delta_n - d(\eta'|\partial\Delta_n) = d\eta - d\eta = 0 .$$

If $k = 1$, $\theta|\partial\Delta_n$ is a closed 0-form, and hence is constant: $\theta|\partial\Delta_n = c$, $c \in Q$. Set $\overline{\eta} = \eta' + \theta - c \in E^0(\Delta_n)$. Then

$$d\overline{\eta} = d\eta' + d\theta = \omega ,$$

$$\overline{\eta}|\partial\Delta_n = \eta'|\partial\Delta_n + \theta|\partial\Delta_n - c = \eta'|\partial\Delta_n = \eta ,$$

and $$\psi(\overline{\eta}) = \psi(\overline{\eta}|\partial\Delta_n) = \psi(\eta) = 0 ,$$ as desired.

So suppose $k \geq 2$. By the induction hypothesis:

$$\psi^* : H^{k-1}_{DR}(\partial\Delta_n) \xrightarrow{\cong} H^{k-1}(\partial\Delta_n;\mathbb{Q}) ,$$

and $H^{k-1}(\partial\Delta_n;\mathbb{Q}) = 0$ if $k \neq n$. So if $k \neq n$, $\theta|\partial\Delta_n \in E^{k-1}(\partial\Delta_n)$ is exact.

If $k = n$, we have

$$H^{n-1}_{DR}(\partial\Delta_n) \cong H^{n-1}(\partial\Delta_n;\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(H_{n-1}(\partial\Delta_n;\mathbb{Q}), \mathbb{Q})$$

by the universal coefficient theorem, and this composite isomorphism takes $[\theta|\partial\Delta_n]$ to the homomorphism given by $[z] \longrightarrow \left\langle \psi(\theta|\partial\Delta_n), z \right\rangle$ for $[z] \in H_{n-1}(\partial\Delta_n;\mathbb{Q})$. Now $H_{n-1}(\partial\Delta_n;\mathbb{Q})$ is a 1-dimensional vector space generated by $[\partial\Delta_n]$ (here, ∂ is the boundary operator in

$C_*(\Delta_n;\mathbb{Q}))$. But $\left\langle \psi(\theta|\partial\Delta_n), \partial\Delta_n \right\rangle = \left\langle \delta\psi(\theta), \Delta_n \right\rangle$

$$= \left\langle \psi(d\theta), \Delta_n \right\rangle$$

$$= \left\langle \psi(\omega), \Delta_n \right\rangle - \left\langle \psi(d\eta'), \Delta_n \right\rangle$$

$$= - \left\langle \psi(\eta'), \partial\Delta_n \right\rangle$$

$$= - \left\langle \psi(\eta), \partial\Delta_n \right\rangle = 0$$

as $\psi(\omega) = 0$ and $\psi(\eta) = 0$. So $[\theta|\partial\Delta_n]$ maps to the zero homomorphism, and hence $[\theta|\partial\Delta_n] = 0 \in H^{n-1}_{DR}(\partial\Delta_n)$.

So, for $k \geq 2$, $\theta|\partial\Delta_n$ is exact, and we have $\lambda \in E^{k-2}(\partial\Delta_n)$ such that $d\lambda = \theta|\partial\Delta_n$. Again by Lemma 1.9, we can find $\lambda' \in E^{k-2}(\Delta_n)$ so that $\lambda'|\partial\Delta_n = \lambda$. Set $\overline{\eta} = \eta' + \theta - d\lambda' \in E^{k-1}(\Delta_n)$. Then $d\overline{\eta} = d\eta' + d\theta = \omega$,

$$\overline{\eta}|\partial\Delta_n = \eta'|\partial\Delta_n + \theta|\partial\Delta_n - d(\lambda'|\partial\Delta_n)$$

$$= \eta + \theta|\partial\Delta_n - d\lambda = \eta ,$$

and $\psi(\overline{\eta}) = \psi(\overline{\eta}|\partial\Delta_n) = \psi(\eta) = 0$.

This completes the proof of the Lemma.

We now show that $H^*(\mathrm{Ker}(\psi : E^*(K) \twoheadrightarrow C^*(K;\mathbb{Q}))) = 0$ for K an n-dimensional complex. Suppose $\omega \in E^k(K)$ is such that $d\omega = 0$ and $\psi(\omega) = 0$. We must find $\eta \in E^{k-1}(K)$ for which $d\eta = \omega$ and $\psi(\eta) = 0$.

If $k = 0$, $d\omega = 0$ implies that ω is constant on each path component of K ; and $\psi(\omega) = 0$ gives $\omega \equiv 0$ on K . So $H^0(\mathrm{Ker}\ \psi) = 0$.

So suppose $k \geq 1$. Then $\omega|K^{n-1} \in E^k(K^{n-1})$ is such that $d(\omega|K^{n-1}) = 0$ and $\psi(\omega|K^{n-1}) = 0$, where K^{n-1} is the $(n-1)$-skeleton of K . By the induction hypothesis, $\psi^* : H^*_{DR}(K^{n-1}) \cong H^*(K^{n-1};\mathbb{Q})$, and hence $H^*(\mathrm{Ker}(\psi : E^*(K^{n-1}) \twoheadrightarrow C^*(K^{n-1};\mathbb{Q}))) = 0$. So there is an $\eta' \in E^{k-1}(K^{n-1})$ for which

$$\psi(\eta') = 0 \quad \text{and} \quad d\eta' = \omega|K^{n-1} .$$

Then, working one n-simplex at a time, the Lemma allows us to extend η' to $\eta \in E^{k-1}(K)$ in such a way that $\psi(\eta) = 0$ and $d\eta = \omega$. Hence, $H^*(\mathrm{Ker}\ \psi) = 0$, and

$$\psi^* : H^*_{DR}(K) \cong H^*(K;Q)$$

for n-dimensional K . This concludes the induction step, and shows that ψ^* is an isomorphism of graded vector spaces for all finite dimensional complexes.

If K is infinite dimensional, we have a map of inverse systems of graded vector spaces:

$$\begin{array}{ccccccccc}
H^*_{DR}(K^0) & \longleftarrow & H^*_{DR}(K^1) & \longleftarrow & \cdots & \longleftarrow & H^*_{DR}(K^n) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \psi^*} & & & & \downarrow{\scriptstyle \psi^*} & & \\
H^*(K^0;\mathbb{Q}) & \longleftarrow & H^*(K^1;\mathbb{Q}) & \longleftarrow & \cdots & \longleftarrow & H^*(K^n;\mathbb{Q}) & \longleftarrow & \cdots
\end{array}$$

By Proposition 1.12, $\varprojlim H^*_{DR}(K^n) \cong H^*_{DR}(K)$. Also $H^n(K;\mathbb{Q}) \cong H^n(K^m;\mathbb{Q})$ when $m \geq n+1$, so $\varprojlim H^*(K^n;\mathbb{Q}) \cong H^*(K;\mathbb{Q})$. As each K^n is a finite dimensional complex, the vertical maps in the diagram are isomorphisms, and hence the induced map on the inverse limits is an isomorphism. So

$$\psi^* : H^*_{DR}(K) \cong H^*(K;\mathbb{Q}) .$$

Note that this also shows $\phi^* : H^*(K;\mathbb{Q}) \longrightarrow H^*_{DR}(K)$ is an isomorphism, and that $\phi^* = (\psi^*)^{-1}$.

We conclude the proof of de Rham's theorem by showing that ψ^* preserves products. We define a product, $\wedge$, in $C^*(K;\mathbb{Q})$ as follows: for $c_1, c_2 \in C^*(K;\mathbb{Q})$,

$$c_1 \wedge c_2 = \psi(\phi(c_1) \wedge \phi(c_2)) ,$$

where $\wedge$ on the right is the product in $E^*(K)$. This product is graded and graded-commutative, but is not associative (as $\phi \circ \psi \neq \mathrm{id}$) . Whitney [14] has shown that a graded product, $\wedge$, on $C^*(K;\mathbb{Q})$ induces the cup product on $H^*(K;\mathbb{Q})$ provided it satisfies the following properties:

(i) if σ_1 and σ_2 are k- and ℓ-simplices, resp., considered as cochains, and if τ is a $(k+\ell)$-simplex, $\left\langle \sigma_1 \wedge \sigma_2 , \tau \right\rangle \neq 0$

implies $\sigma_1 < \tau$ and $\sigma_2 < \tau$,

(ii) if $c_1 \in C^k(K;\mathbb{Q})$ and $c_2 \in C^*(K;\mathbb{Q})$,

$$\delta(c_1 \wedge c_2) = \delta c_1 \wedge c_2 + (-1)^k c_1 \wedge \delta c_2$$

(iii) $1 \wedge c = c = c \wedge 1$ for all $c \in C^*(K;\mathbb{Q})$.

We verify these properties for the product, $\wedge$, defined above. Clearly, $\phi(\sigma_1) \wedge \phi(\sigma_2) \equiv 0$ on $K - (St(\sigma_1) \cap St(\sigma_2))$. So if $0 \neq \langle \sigma_1 \wedge \sigma_2, \tau \rangle = \int_\tau \phi(\sigma_1) \wedge \phi(\sigma_2)$, we must have $\langle \tau \rangle \subset St(\sigma_1) \cap St(\sigma_2)$, and hence $\sigma_1 < \tau$ and $\sigma_2 < \tau$. This proves (i). Property (ii) follows immediately from the facts that d is a derivation on $E^*(K)$, and ψ and ϕ are chain maps. Property (iii) follows from Proposition 1.14(c):

$$1 \wedge c = \psi(\phi(1) \wedge \phi(c)) = \psi(1 \wedge \phi(c)) = \psi(\phi(c)) = c .$$

So $\wedge$ induces the cup product in $H^*(K;\mathbb{Q})$. That is, if c_1 and c_2 are cocycles,

$$[c_1] \cup [c_2] = [c_1 \wedge c_2] .$$

To see that ψ^* is multiplicative, take any closed forms $\omega_1 , \omega_2 \in E^*(K)$, and set $c_i = \psi(\omega_i)$, $i = 1, 2$. Then $\phi^*([c_i]) = [\omega_i]$, $i = 1, 2$, and we have

$$\psi^*([\omega_1] \wedge [\omega_2]) = \psi^*(\phi^*([c_1]) \wedge \phi^*([c_2]))$$

$$= \psi^*([\phi(c_1) \wedge \phi(c_2)])$$

$$= [\psi(\phi(c_1) \wedge \phi(c_2))]$$

$$= [c_1 \wedge c_2]$$

$$= [c_1] \cup [c_2]$$

$$= \psi^*([\omega_1]) \cup \psi^*([\omega_2]) .$$

This completes the proof of de Rham's Theorem.

If L is a subcomplex of K, then clearly

$$\psi(E^*(K,L)) \subset C^*(K,L;\mathbb{Q}) ,$$

and we have

1.17 <u>Corollary</u>: $\psi^* : H^*_{DR}(K,L) \cong H^*(K,L;\mathbb{Q})$ is a natural equivalence of functors from the category of simplicial pairs to the category of graded algebras.

<u>Proof</u>: Naturality is clear. We have a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & E^*(K,L) \arrow[r, hook] \arrow[d, "\psi"] & E^*(K) \arrow[r, two heads] \arrow[d, two heads, "\psi"] & E^*(L) \arrow[r] \arrow[d, two heads, "\psi"] & 0 \\
0 \arrow[r] & C^*(K,L;\mathbb{Q}) \arrow[r, hook] & C^*(K;\mathbb{Q}) \arrow[r, two heads] & C^*(L;\mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

which induces a map of long exact cohomology sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^n_{DR}(K) \arrow[r] \arrow[d, "\cong"] & H^n_{DR}(L) \arrow[r] \arrow[d, "\cong"] & H^{n+1}_{DR}(K,L) \arrow[r] \arrow[d] & H^{n+1}_{DR}(K) \arrow[r] \arrow[d, "\cong"] & H^{n+1}_{DR}(L) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H^n(K;Q) \arrow[r] & H^n(L;Q) \arrow[r] & H^{n+1}(K,L;Q) \arrow[r] & H^{n+1}(K;Q) \arrow[r] & H^{n+1}(L;Q) \arrow[r] & \cdots
\end{tikzcd}$$

The result follows by the 5-lemma. Q.E.D.

Chapter 2 The Minimal Model

In this chapter, we describe an algebraic construction on simply-connected DGA's which is, in some sense, minimal. This will be applied in Chapter 3 to the de Rham algebra $\mathcal{E}^*(K)$ of a simply-connected simplicial complex K, and we shall see that the construction parallels the Postnikov decomposition of K.

We first need some definitions.

2.1 Definition: If $A = \bigoplus_{n=0}^{\infty} A^n$ is a graded algebra, set $A^+ = \bigoplus_{n=1}^{\infty} A^n$. Define $D(A)$ to be the image of $A^+ \otimes A^+$ under multiplication. $D(A)$ is clearly an ideal of A, called the ideal of decomposables; it consists of all sums of non-trivial products in A.

If A is a DGA, we say that A has a decomposable differential if the image of the differential is contained in the ideal of decomposables; $B^*(A) \subset D(A)$.

2.2 Minimal Algebras: A DGA, M, is called a minimal algebra if it satisfies the following four properties:

(i) M is free as a graded algebra,

(ii) M has a decomposable differential,

(iii) $M^0 = \mathbb{Q}$, $M^1 = 0$,

(iv) M has cohomology of finite type; i.e., for each n, $H^n(M)$ is a finite dimensional vector space.

Note that properties (ii) - (iv) imply

(v) for each n, M^n is a finite dimensional vector space. Let $\mathcal{M}$ denote the full subcategory of $\mathcal{DGA}$ consisting of all minimal algebras and all DGA maps between them. By an abuse of notation, we write $M \in \mathcal{M}$ for "M is an object of $\mathcal{M}$."

Let M_n be the free algebra on the (free) generators of M of degree $\leq n$; this is a subalgebra of M (as a graded algebra), and $M_n^k = M^k$ for $k \leq n$. As $M^1 = 0$ and the differential is decomposable, we have $d(M_n) \subset M_n$, so that M_n is a sub-DGA of M, and $M_n \in \mathcal{M}$. Moreover,

$$d(M_n^k) \subset M_{n-1}^{k+1} \quad \text{for } k \leq n .$$

If $f : M \to N$ is a map in $\mathcal{M}$, the fact that f is an algebra map gives $f(M_n) \subset N_n$, so that

$$f_n = f|M_n : M_n \longrightarrow N_n$$

is also a map in $\mathcal{M}$.

So, to each $M \in \mathcal{M}$ there is associated a canonical sequence of sub-DGA's,

$$\mathbb{Q} = M_1 \subset M_2 \subset \dots \subset M_n \subset \dots \subset \bigcup_n M_n = M$$

so that $M_n \in \mathcal{M}$ and has no generators of degree $> n$. We also have $M_n \cong M_{n-1} \otimes \Lambda_n(x_1,\dots,x_k)$ as a graded algebra, and $d(x_i) \in M_{n-1}^{n+1}$.

2.3 Minimal Models: In the remainder of this chapter, all DGA's considered are assumed to have cohomology of finite type.

Suppose A is a simply connected DGA; recall that this means $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$. A DGA $M = M(A)$ is called a minimal model for A if

(i) $M \in \mathcal{M}$

(ii) there is a DGA map $\rho : M \longrightarrow A$ which induces an isomorphism on cohomology; $\rho^* : H^*(M) \cong H^*(A)$.

Note that many such maps ρ may exist.

2.4 <u>Examples</u>:

(a) If $M \in \mathcal{M}$, it is its own minimal model, as the identity map satisfies condition (ii) of the definition.

(b) Suppose A is a simply connected DGA and that $H^*(A)$ is a free algebra. Then $H^*(A)$ with zero differential is a minimal model for A. Clearly the zero differential is decomposable, so $H^*(A) \in \mathcal{M}$. To verify condition (ii), pick cocycles $z_1, z_2, \ldots$ in A in such a way that $[z_1], [z_2], \ldots$ generate $H^*(A)$ freely, and define $\rho : H^*(A) \longrightarrow A$ by $\rho([z_i]) = z_i$. Then ρ is a DGA map, and ρ^* is the identity on $H^*(A)$.

The main result of this chapter is:

2.5 <u>Theorem</u>: Every simply connected DGA (with cohomology of finite type) has a minimal model which is unique up to isomorphism.

<u>Proof (Existence)</u>: Let A be a simply connected DGA. We construct a minimal model M for A inductively as follows:

Set $M_1 = \mathbb{Q}$ (i.e., $M_1^0 = \mathbb{Q}$ and $M_1^n = 0$ for $n \geq 1$). As $\mathbb{Q} = H^0(A) = Z^0(A) \subset A^0$, we can define $\rho_1 : M_1 \longrightarrow A$ which maps M_1^0 identically onto $Z^0(A)$. Then $\rho_1^* : H^n(M_1) \longrightarrow H^n(A)$ is an isomorphism for $n = 0, 1$ and a monomorphism for $n = 2$.

Suppose we have constructed $M_{n-1} \in \mathcal{M}$ and $\rho_{n-1} : M_{n-1} \longrightarrow A$ in such a way that M_{n-1} has no algebra generators of degree $\geq n$, and that $\rho_{n-1}^* : H^k(M_{n-1}) \longrightarrow H^k(A)$ is an isomorphism for $k \leq n - 1$ and a monomorphism for $k = n$. We have a short exact sequence

$$0 \longrightarrow H^n(M_{n-1}) \overset{\rho_{n-1}^*}{\rightarrowtail} H^n(A) \overset{\varepsilon}{\twoheadrightarrow} \mathrm{Coker}(\rho_{n-1}^*) \longrightarrow 0 .$$

Choose $z_1, \ldots, z_\ell \in Z^n(A)$ so that $\{\varepsilon([z_i])\}_{i=1}^{\ell}$ forms a basis for $\mathrm{Coker}(\rho_{n-1}^*)$. We also have an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\rho_{n-1}^*) \longrightarrow H^{n+1}(M_{n-1}) \xrightarrow{\rho_{n-1}^*} H^{n+1}(A) .$$

Choose $w_1, \ldots, w_m \in Z^{n+1}(M_{n-1})$ so that $\{[w_i]\}_{i=1}^m$ is a basis for $\mathrm{Ker}(\rho_{n-1}^*)$. Clearly $w_i \in M_{n-1}^{n+1}$ i s decomposable. As $\rho_{n-1}^*([w_i]) = 0 \in H^{n+1}(A)$, we must have $\rho_{n-1}(w_i) \in B^{n+1}(A)$, and we can choose $v_1, \ldots, v_m \in A^n$ so that $d(v_i) = \rho_{n-1}(w_i)$.

Define a DGA, M_n, to be

$$M_n = M_{n-1} \otimes \Lambda_n(x_1, \ldots, x_\ell, y_1, \ldots, y_m)$$

as a graded algebra, and extend the differential in M_{n-1} to M_n by setting $d(1 \otimes x_i) = 0$ and $d(1 \otimes y_i) = w_i \otimes 1$ (and requiring d to be a derivation). The choice of w_i shows that $M_n \in \mathcal{M}$.

Extend ρ_{n-1} to $\rho_n : M_n \longrightarrow A$ by setting $\rho_n(1 \otimes x_i) = z_i$ and $\rho_n(1 \otimes y_i) = v_i$. Then

$d\rho_n(1 \otimes y_i) = d(v_i) = \rho_{n-1}(w_i) = \rho_n(dy_i)$, so ρ_n is a map of DGA's. By the choice of the z_i's and w_i's, it is now clear that $\rho_n^* : H^k(M_n) \longrightarrow H^k(A)$ is an isomorphism for $k \leq n$ and a monomorphism for $k = n + 1$. This completes the induction step, and setting $M = \varinjlim M_n$, $\rho = \varinjlim \rho_n$ completes the construction. Q.E.D.

To prove the uniqueness part of Theorem 2.5, we need some other results about the category $\mathcal{M}$.

2.6 <u>Theorem</u>: Suppose $f : M \to N$ is a map in $\mathcal{M}$ and $f^* : H^k(M) \longrightarrow H^k(N)$ is an isomorphism for $k \leq m$ and a monomorphism for $k = m + 1$. Then $f_m : M_m \longrightarrow N_m$ is an isomorphism.

<u>Proof</u>: For $m = 1$, the theorem is clear, as $M_1 = \mathbb{Q} = N_1$ and $f(1) = 1$.

Assume inductively that $f_{n-1} : M_{n-1} \cong N_{n-1}$, for some $n \leq m$. It follows that

$$f_{n-1} : Z^*(M_{n-1}) \cong Z^*(N_{n-1})$$

and

$$f_{n-1} : B^*(M_{n-1}) \cong B^*(N_{n-1}) .$$

Moreover, as $M \in \mathcal{M}$, we have

$$Z^{n+1}(M_{n-1}) = Z^{n+1}(M_n) \subset Z^{n+1}(M)$$

and
$$B^{n+1}(M_{n-1}) \subset B^{n+1}(M_n) = B^{n+1}(M) ,$$

and the same is true for N .

We first show that $f : B^{n+1}(M) \cong B^{n+1}(N)$. The commutative diagram

$$\begin{tikzcd}
B^{n+1}(M) \arrow[r, hook] \arrow[d, "f"'] & M^{n+1}_{n-1} \arrow[d, tail, two heads, "f_{n-1}", "\cong"'] \\
B^{n+1}(N) \arrow[r, hook] & N^{n+1}_{n-1}
\end{tikzcd}$$

shows that $f|B^{n+1}(M)$ is monic. To see that it is also epic, choose any $w \in B^{n+1}(N)$. As

$$w \in B^{n+1}(N) = B^{n+1}(N_n) \subset Z^{n+1}(N_n) = Z^{n+1}(N_{n-1}) ,$$

and f_{n-1} is an isomorphism, we can find (a unique) $x \in Z^{n+1}(M_{n-1})$ for which $f(x) = f_{n-1}(x) = w$. Now $Z^{n+1}(M_{n-1}) \subset Z^{n+1}(M)$, so consider $[x] \in H^{n+1}(M)$; we have $f^*([x]) = [f(x)] = [w] = 0 \in H^{n+1}(N)$. But f^* is monic in dimension $n + 1$, so $[x] = 0$, and $x \in B^{n+1}(M)$. Hence $f(B^{n+1}(M)) = B^{n+1}(N)$, and $f : B^{n+1}(M) \cong B^{n+1}(N)$, as desired.

We now show that $f : Z^n(M) \cong Z^n(N)$. From the commutative square

$$\begin{tikzcd} B^n(M_{n-1}) \arrow[r, equal] \arrow[d, tail, twoheadrightarrow, "f_{n-1}"', "\cong"] & B^n(M) \arrow[d, "f"] \\ B^n(N_{n-1}) \arrow[r, equal] & B^n(N) \end{tikzcd}$$

we get that $f : B^n(M) \cong B^n(N)$. We have a map of short exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & B^n(M) \arrow[r] \arrow[d, "f"', "\cong"] & Z^n(M) \arrow[r, twoheadrightarrow] \arrow[d, "f"] & H^n(M) \arrow[r] \arrow[d, "f^*"', "\cong"] & 0 \\ 0 \arrow[r] & B^n(N) \arrow[r] & Z^n(N) \arrow[r, twoheadrightarrow] & H^n(N) \arrow[r] & 0 \end{tikzcd} ,$$

so the 5-lemma gives the result $f : Z^n(M) \cong Z^n(N)$.

We have another map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z^n(M) \arrow[r] \arrow[d, "f"', "\cong"] & M^n \arrow[r, twoheadrightarrow, "d"] \arrow[d, "f"] & B^{n+1}(M) \arrow[r] \arrow[d, "f"', "\cong"] & 0 \\ 0 \arrow[r] & Z^n(N) \arrow[r] & N^n \arrow[r, twoheadrightarrow] & B^{n+1}(N) \arrow[r] & 0 \end{tikzcd} ,$$

and, again by the 5-lemma, $f : M^n \cong N^n$.

Define an algebra map $g : N_n \longrightarrow M_n$ as follows: select a set $\{y_i\}$ that generate N_n freely and set $g(y_i) = f^{-1}(y_i)$. Then clearly $f_n \circ g$ is the identity on N_n , and f_n is an epimorphism. But M_n and N_n are free algebras with the same number of free generators of each degree, so that, for all $k \geq 0$,

$$\dim_{\mathbb{Q}} M_n^k = \dim_{\mathbb{Q}} N_n^k < \infty .$$

Hence $f_n : M_n \cong N_n$. Q.E.D.

2.7 Corollary: If $f : M \longrightarrow N$ is a map in $\mathcal{M}$ and $f^* : H^*(M) \longrightarrow H^*(N)$ is an isomorphism, then $f : M \longrightarrow N$ is an isomorphism.

Proof: Theorem 2.6 shows that $f : M^m \cong N^m$ for all m , and hence $f : M \cong N$. Q.E.D.

To study the problem of induced maps between minimal models, we must first develop the notion of homotopy for DGA's .

2.8 Definitions: Define I to be the DGA freely generated by t in degree 0 and dt (the differential of t) in degree 1 . So $I^0 = \mathbb{Q}[t]$, all rational coefficient polynomials in t , $I^1 \cong \mathbb{Q}[t]$ consists of all products $p(t)dt$, where $p(t) \in I^0$, and $I^n = 0$ for $n \geqq 2$. For $p(t) \in I^0$ we have

$$d(p(t)) = p'(t)dt ,$$

where $p'(t)$ is the ordinary derivative of $p(t)$. Note that

$$I \cong \mathcal{E}^*([0,1]) ,$$

the de Rham algebra of the unit interval. In fact, the rôle played by I in DGA homotopies is analogous to that played by $[0,1]$ in topological homotopies. Note also that, as the indefinite integral of a polynomial is again a polynomial, we have $H^*(I) = \mathbb{Q}$.

Let A be a DGA, and consider $A \otimes I$ to be a DGA as described in section 0.2. A typical element $x \in (A \otimes I)^n = (A^n \otimes I^0) \oplus (A^{n-1} \otimes I^1)$ has the form

$$x = \sum_{i=1}^{k} a_i \otimes p_i(t) + \sum_{j=1}^{\ell} b_j \otimes q_j(t)dt ,$$

where $a_i \in A^n$, $b_j \in A^{n-1}$ and $p_i(t)$, $q_j(t) \in I^0 = Q[t]$. For such an element x we have

$$d(x) = \sum_{i=1}^{k} d(a_i) \otimes p_i(t) + (-1)^n \sum_{i=1}^{k} a_i \otimes p_i'(t)dt$$
$$+ \sum_{j=1}^{\ell} d(b_j) \otimes q_j(t)dt .$$

For each $\alpha \in \mathbb{Q}$, define a DGA map $e_\alpha : A \otimes I \longrightarrow A$ by insisting that

$$\begin{cases} e_\alpha(a \otimes 1) = a \quad , \text{ for all } a \in A , \\ e_\alpha(1 \otimes t) = \alpha \in \mathbb{Q} \subset A^0 \\ e_\alpha(1 \otimes dt) = 0 . \end{cases}$$

So, for the typical element x above, we have

$$e_\alpha(x) = \sum_{i=1}^{k} p_i(\alpha) \cdot a_i ,$$

Clearly e_α is a DGA map. We are especially concerned with the cases $\alpha = 0$ and $\alpha = 1$.

By the Künneth formula, we have

$$H^*(A \otimes I) \cong H^*(A) \otimes H^*(I) \cong H^*(A) .$$

We wish to know that e_α^* is an isomorphism which is independent of the choice of α .

2.9 Lemma: Every element of $H^n(A \otimes I)$ has a representation as $[a \otimes 1]$ for a unique $[a] \in H^n(A)$.

Proof: We first show that every element $x \in (A \otimes I)^n$ can be written in the form

$$x = \sum_{i=1}^{k} a_i \otimes p_i(t) + d(y) ,$$

where $a_i \in A^n$, $p_i(t) \in Q[t]$, and $y \in (A \otimes I)^{n-1}$. For suppose x has a term of the form $b \otimes q(t)dt$, where $b \in A^{n-1}$ and $q(t) \in Q[t]$. Choose an indefinite integral $r(t) \in Q[t]$ for the polynomial $q(t)$; so $r'(t) = q(t)$. Then

$$d(b \otimes r(t)) = d(b) \otimes r(t) + (-1)^{n-1}\, b \otimes q(t)dt ,$$

so that

$$b \otimes q(t)dt = (-1)^n\, d(b) \otimes r(t) + (-1)^{n-1}\, d(b \otimes r(t)) ,$$

and the result follows.

Now, for $[x] \in H^n(A \otimes I)$, we can write

$$x = \sum_{i=1}^{k} a_i \otimes p_i(t) + d(y)$$

as above, where

$$p_i(t) = \sum_{m=0}^{\ell} c_{im}\, t^m \in \mathbb{Q}[t] .$$

As $d(x) = 0$, we have

$$0 = d(x) = \sum_{i=1}^{k} d(a_i) \otimes p_i(t) + (-1)^n \sum_{i=1}^{k} a_i \otimes p_i'(t)dt .$$

As the first sum is in $A^{n+1} \otimes I^0$ and the second in $A^n \otimes I^1$, each sum is zero, and in particular

$$0 = \sum_{i=1}^{k} a_i \otimes p_i'(t)dt = \sum_{i=1}^{k} a_i \otimes \left(\sum_{m=1}^{\ell} m\, c_{im}\, t^{m-1}\right)dt$$

$$= \sum_{m=1}^{\ell} m \cdot \left(\sum_{i=1}^{k} c_{im} \cdot a_i\right) \otimes t^{m-1}dt$$

So, as the powers of t are independent, we must have

$$\sum_{i=1}^{k} c_{im} \cdot a_i = 0 \quad \text{for} \quad 1 \leq m \leq \ell .$$

Hence $x = \sum_{i=1}^{k} a_i \otimes \left(\sum_{m=0}^{\ell} c_{im}\, t^m\right) + d(y)$

$$= \sum_{m=0}^{\ell} \left(\sum_{i=1}^{k} c_{im} \cdot a_i\right) \otimes t^m + d(y)$$

$$= \left(\sum_{i=1}^{k} c_{i0} \cdot a_i\right) \otimes 1 + d(y)$$

$$= a \otimes 1 + d(y)$$

where $a = \sum_{i=1}^{k} c_{i0} \cdot a_i$. Also,

$$0 = d(x) = d(a \otimes 1) = d(a) \otimes 1 ,$$

so that $a \in Z^n(A)$, and

$$[x] = [a \otimes 1] \in H^n(A \otimes I) ,$$

as desired.

To show uniqueness, suppose

$$[a \otimes 1] = [b \otimes 1] ,$$

so that $(a-b) \otimes 1 \in B^n(A \otimes I)$. By the first result of this proof, we can find $a_i \in A^{n-1}$ and $p_i(t) \in I^0$ $(1 \leq i \leq k)$ such that

$$(a-b) \otimes 1 = d\left(\sum_{i=1}^{k} a_i \otimes p_i(t)\right)$$

$$= \sum_{i=1}^{k} d(a_i) \otimes p_i(t) + (-1)^{n-1} \sum_{i=1}^{k} a_i \otimes p_i'(t)dt .$$

As before we have

$$\text{(i)} \quad (a-b) \otimes 1 = \sum_{i=1}^{k} d(a_i) \otimes p_i(t) \in A^n \otimes I^0$$

$$\text{(ii)} \quad 0 = \sum_{i=1}^{k} a_i \otimes p_i'(t)dt \in A^{n-1} \otimes I^1 .$$

Again writing $p_i(t) = \sum_{m=0}^{\ell} c_{im} t^m$ for $c_{im} \in \mathbb{Q}$, equation (ii) becomes

$$0 = \sum_{i=1}^{k} a_i \otimes \left(\sum_{m=1}^{\ell} m \cdot c_{im}\, t^{m-1}\right) = \sum_{m=1}^{\ell} m \cdot \left(\sum_{i=1}^{k} c_{im} \cdot a_i\right) \otimes t^{m-1} .$$

Hence $\sum_{i=1}^{k} c_{im} \cdot a_i = 0$ for $1 \leq m \leq \ell$, and (i) becomes

$$\begin{aligned}(a-b) \otimes 1 &= \sum_{i=1}^{k} d(a_i) \otimes \left(\sum_{m=0}^{\ell} c_{im}\, t^m\right) \\ &= \sum_{m=0}^{\ell} d\left(\sum_{i=1}^{k} c_{im} \cdot a_i\right) \otimes t^m \\ &= d\left(\sum_{i=1}^{k} c_{i0} \cdot a_i\right) \otimes 1 .\end{aligned}$$

So $a - b \in B^n(A)$, and $[a] = [b] \in H^n(A)$. Q.E.D.

2.10 <u>Proposition</u>: For all $\alpha, \beta \in \mathbb{Q}$,

$$e_\alpha^* = e_\beta^* : H^*(A \otimes I) \longrightarrow H^*(A)$$

is an isomorphism of graded algebras. In particular, $e_0^* = e_1^*$ is an isomorphism.

<u>Proof</u>: This follows easily from Lemma 2.9, as

$$e_\alpha^*([a \otimes 1]) = [a] \quad \text{for any} \quad \alpha \in \mathbb{Q} . \qquad \text{Q.E.D.}$$

2.11 <u>DGA Homotopy</u>: Suppose $f, g : A \longrightarrow B$ are maps in $\mathcal{DGA}$. We say that f is homotopic to g , written $f \simeq g$, if there is a DGA-map $F : A \longrightarrow B \otimes I$ such that $e_0 \circ F = f$ and $e_1 \circ F = g$;

that is, the following diagram commutes:

$$\begin{tikzcd} & & B \\ A \arrow[urr, "f"] \arrow[rr, "F"] \arrow[drr, "g"'] & & B \otimes I \arrow[u, "e_0"'] \arrow[d, "e_1"] \\ & & B \end{tikzcd}$$

The question arises whether DGA-homotopy is an equivalence relation. It is clearly reflexive, as $F(a) = f(a) \otimes 1$ defines a homotopy from f to itself. For symmetry, define a DGA map $r : B \otimes I \longrightarrow B \otimes I$ by requiring

$$\begin{cases} r(b \otimes 1) = b \otimes 1 \quad \text{for all} \quad b \in B \ , \\ r(1 \otimes t) = 1 \otimes (1-t) \ , \\ r(1 \otimes dt) = 1 \otimes (-dt) \ . \end{cases}$$

Then $e_0 \circ r = e_1$ and $e_1 \circ r = e_0$, so if $F : A \longrightarrow B \otimes I$ is a homotopy from f to g, then $r \circ F$ is a homotopy from g to f. I do not know if the homotopy relation is transitive in general, but Sullivan [1] has shown that it is, when the domain DGA is in $\mathcal{M}$. His proof uses our Theorem 2.13, but we omit the details, as we will not use the result.

The justification for the name "homotopy" is that, as in the case of topological homotopies, we have:

2.12 Proposition: If $f \simeq g : A \longrightarrow B$ are homotopic maps in $\mathcal{DGA}$, then

$$f^* = g^* : H^*(A) \longrightarrow H^*(B) .$$

Proof: Suppose $F : A \longrightarrow B \otimes I$ is a homotopy from f to g. Passing to cohomology, Proposition 2.10 implies

$$f^* = (e_0 \circ F)^* = e_0^* \circ F^* = e_1^* \circ F^* = (e_1 \circ F)^* = g^* . \qquad \text{Q.E.D.}$$

The reason we introduce DGA homotopies is to study induced maps between minimal models. We have the following lifting theorem:

2.13 Theorem: Suppose $\rho : A \longrightarrow C$ and $f : M \longrightarrow C$ are maps in $\mathcal{DGA}$, $M \in \mathcal{M}$, and $\rho^* : H^*(A) \longrightarrow H^*(C)$ is an isomorphism. Then there is a DGA map $g : M \longrightarrow A$ such that $f \simeq \rho \cdot g$, so that the diagram

$$\begin{tikzcd} & M \arrow[dl, dashed, "g"'] \arrow[dr, "f"] & \\ A \arrow[rr, "\rho"'] & & C \end{tikzcd}$$

commutes up to homotopy. Furthermore, if ρ is an epimorphism, we can choose g so that $f = \rho \circ g$.

Proof: Assume inductively that we have constructed $g_{n-1} : M_{n-1} \longrightarrow A$ and a homotopy $F_{n-1} : M_{n-1} \longrightarrow C \otimes I$ from $f|M_{n-1}$ to $\rho \circ g_{n-1}$. We will extend these maps to M_n one free n-dimensional algebra generator at a time. So assume M_n has only one such generator $m \in M_n^n$, so that $M_n \cong M_{n-1} \otimes \Lambda_n(m)$ as algebras.

As $d(m) \in M_{n-1}^{n+1}$, we have

$$e_0F_{n-1}(dm) = f(dm) = d(f(m)) \in B^{n+1}(C) \ .$$

But $e_0^* : H^*(C \otimes I) \cong H^*(C)$ is an isomorphism, so $F_{n-1}(dm) \in B^{n+1}(C \otimes I)$. Choose $h' \in (C \otimes I)^n$ for which $d(h') = F_{n-1}(dm)$, and set $c = e_0(h') - f(m) \in C^n$. Then $d(c) = e_0F_{n-1}(dm) - f(dm) = 0$ as $e_0F_{n-1} = f|M_{n-1}$, so that $c \in Z^n(C)$. Setting $h = h' - c \otimes 1 \in (C \otimes I)^n$, one verifies that

$$\begin{cases} d(h) = F_{n-1}(dm) \ , \\ e_0(h) = f(m) \ . \end{cases}$$

By Proposition 2.12, $f^* = \rho^* \circ g_{n-1}^* : H^*(M_{n-1}) \longrightarrow H^*(C)$, so $\rho^*([g_{n-1}(dm)]) = f^*([dm]) = [d(f(m))] = 0$ in $H^{n+1}(C)$. As ρ^* is an isomorphism, $g_{n-1}(dm) \in B^{n+1}(A)$, and we can find $a \in A^n$ such that $d(a) = g_{n-1}(dm)$. Now

$$\begin{aligned} d(\rho(a)) &= \rho(da) = \rho g_{n-1}(dm) = \\ &= e_1F_{n-1}(dm) = e_1(dh) = \\ &= d(e_1(h)) \ , \end{aligned}$$

so $\rho(a) - e_1(h) \in Z^n(C)$. As ρ^* is an isomorphism, we can find $b \in Z^n(A)$ such that

$$\rho^*([b]) = [\rho(a) - e_1(h)] \in H^n(C) .$$

Extend g_{n-1} to $g_n : M_n \longrightarrow A$ by setting

$$g_n(m) = a - b ;$$

this is well defined as M_n is free. Now we have

$$d(g_n(m)) = d(a) = g_{n-1}(dm) = g_n(dm) ,$$

so g_n is a map of DGA's.

Observe that $\rho g_n(m) - e_1(h) \in Z^n(C)$, as

$$d(\rho g_n(m) - e_1(h)) = \rho g_{n-1}(dm) - e_1(dh) =$$

$$= e_1 F_{n-1}(dm) - e_1(F_{n-1}(dm)) = 0 .$$

But then we have

$$[\rho g_n(m) - e_1(h)] = [\rho(a) - \rho(b) - e_1(h)] =$$

$$= 0 \in H^n(C)$$

by the choice of b . Hence, we can find $x \in C^{n-1}$ such that

$$d(x) = \rho g_n(m) - e_1(h) .$$

Extend F_{n-1} to $F_n : M_n \longrightarrow C \otimes I$ by setting

$$F_n(m) = h + d(x \otimes t) .$$

Then $d(F_n(m)) = d(h) = F_{n-1}(dm) = F_n(dm)$, so F_n is a DGA map. Also

$$e_0 F_n(m) = e_0(h) + d(e_0(x \otimes t)) = f(m) ,$$

$$\begin{aligned} e_1 F_n(m) &= e_1(h) + d(e_1(x \otimes t)) \\ &= e_1(h) + d(x) \\ &= e_1(h) + \rho g_n(m) - e_1(h) = \rho g_n(m) . \end{aligned}$$

So F_n is the desired homotopy from $f|M_n$ to $\rho \circ g_n$ and the induction continues. Note that the only information about m that we used is that it is a free generator and $dm \in M_{n-1}$. So if there are several n-dimensional generators in M_n , the above construction can be applied one generator at a time to yield g_n and F_n on M_n with the desired properties. This proves the first part of the theorem.

Now suppose ρ is an epimorphism, and assume inductively that we have constructed $g_{n-1} : M_{n-1} \longrightarrow A$ so that $\rho \circ g_{n-1} = f|M_{n-1}$. As before, assume that $m \in M_n^n$ is the only free generator of degree n . Now, as above, $dm \in M_{n-1}^{n+1}$ and $g_{n-1}(dm) \in B^{n+1}(A)$. Choose $a \in A^n$ such that $d(a) = g_{n-1}(dm)$. Then

$d(\rho(a)) = \rho g_{n-1}(dm) = f(dm) = d(f(m))$, so $\rho(a) - f(m) \in Z^n(C)$.

As ρ^* is an isomorphism, we can find $b \in Z^n(A)$ such that

$$\rho^*([b]) = [\rho(a) - f(m)] \in H^n(C) .$$

Hence $\rho(b) - \rho(a) + f(m) \in B^n(C)$, and we can find $x \in C^{n-1}$ such that $d(x) = \rho(b) - \rho(a) + f(m)$. As ρ is an epimorphism, we can choose $c \in A^{n-1}$ such that $\rho(c) = x$.

Now extend g_{n-1} to $g_n : M_n \longrightarrow A$ by defining

$$g_n(m) = a - b + d(c) .$$

Then $d(g_n(m)) = d(a) = g_{n-1}(dm) = g_n(dm)$, so g_n is a DGA map. Also,

$$\begin{aligned} \rho g_n(m) &= \rho(a) - \rho(b) + d(\rho(c)) \\ &= \rho(a) - \rho(b) + d(x) = f(m) , \end{aligned}$$

and $\rho \circ g_n = f|M_n$ as desired. Q.E.D.

We are now in a position to prove the uniqueness part of Theorem 2.5. However, we will prove a slightly stronger result which will be needed in Chapter 3.

2.14 <u>Theorem</u>: Suppose $f : A \longrightarrow C$ is a map of simply connected DGA's such that $f^* : H^*(A) \longrightarrow H^*(C)$ is an isomorphism. If M and N are minimal models for A and C , respectively, then $M \cong N$.

<u>Proof</u>: Suppose $\rho : M \longrightarrow A$ and $\lambda : N \longrightarrow C$ induce isomorphisms on cohomology. Theorem 2.13 gives a map $g : M \longrightarrow N$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & A \\ g \downarrow & \simeq & \downarrow f \\ N & \xrightarrow{\lambda} & C \end{array}$$

So $f \circ \rho \simeq \lambda \circ g$, and Proposition 2.12 implies that

$$f^* \circ \rho^* = \lambda^* \circ g^* : H^*(M) \longrightarrow H^*(C) .$$

But f^*, ρ^* and λ^* are isomorphisms, so

$$g^* : H^*(M) \cong H^*(N) .$$

Now Corollary 2.7 implies that $g : M \cong N$. Q.E.D.

2.15 Remarks: Throughout this chapter we have considered only DGA's with cohomology of finite type. If we drop this requirement, and condition (iv) in the definition of a minimal algebra, the results of this chapter remain valid. We included the finiteness condition to streamline some of the proofs, and because one rarely encounters a DGA (or topological space) whose cohomology doesn't have finite type.

For a generalization of this theory to the non-simply connected case, see [1].

Chapter 3 The Minimal Model in Rational Homotopy Theory

In this chapter, we discuss the relationship between the algebraic construction of Chapter 2 and the rational homotopy theory of a topological space.

3.1 Definitions: By a space we mean a topological space, and maps between spaces are assumed to be continuous. For a space X, we let $H^*(X;G)$ denote the singular cohomology algebra of X with coefficients in the abelian group G. If X is a simplicial complex, we make free use of the natural isomorphism between singular and simplicial cohomology, denoting both by $H^*(X;G)$; it will be clear from context which theory we are using.

Let $f : X \to Y$ be a map of simply connected spaces. We say that f is a rational homotopy equivalence if one (and hence all) of the following conditions hold:

(i) $f_* : H_*(X;\mathbb{Q}) \longrightarrow H_*(Y;\mathbb{Q})$ is an isomorphism;

(ii) $f^* : H^*(Y;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$ is an isomorphism;

(iii) $(f_\# \otimes 1) : \pi_*(X) \otimes \mathbb{Q} \longrightarrow \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism. The equivalence of these conditions follows from the mod-C Whitehead theorem [11; Theorem 9.6.22]. Note that if f is a rational homotopy equivalence, we are <u>not</u> guaranteed the existence of a map $g: Y \to X$ for which $g^* = (f^*)^{-1}$.

Two simply connected spaces, X and Y, are said to have the same rational homotopy type if there is a third simply connected space, Z, and rational homotopy equivalences $f : Z \longrightarrow X$ and

$g : Z \longrightarrow Y$. Rational homotopy type induces an equivalence relation on simply connected spaces; reflexivity and symmetry are clear, and transitivity is a pullback argument. Spaces of the same (weak) homotopy type clearly have the same rational homotopy type.

3.2 Geometric Realization: We now outline a procedure, due to Milnor, which allows us to replace a space with a simplicial complex that contains the same homotopy-theoretic information. The details can be found in [3], [5], [6], and [7].

If X is a topological space, let $S(X)$ be the graded set of all singular simplices in X ; that is, $S_n(X)$ consists of all continuous maps $\sigma : \Delta_n \to X$. $S(X)$ becomes a semi-simplicial complex of defining the face and degeneracy operators in the obvious way (see [6]).

From $S(X)$ we can build a topological space $|S(X)|$, called the geometric realization of $S(X)$, as follows: to each $\sigma \varepsilon S(X)$ we associate an n-cell

$$|\sigma| = \{\sigma\} \times \Delta_n ,$$

where $\{\sigma\}$ is the set with one element. Then $|S(X)|$ is obtained from the disjoint union of all such cells by identifying $\{\sigma|F_i\Delta_n\} \times \Delta_{n-1}$ with $\{\sigma\} \times F_i\Delta_n$, $0 \leq i \leq n$. We give $|S(X)|$ the identification topology: a subset $W \subset |S(X)|$ is open iff $W \cap |\sigma|$ is open in $|\sigma|$ for every $\sigma \varepsilon S(X)$. One verifies that $|S(X)|$ is a CW-complex with one open n-cell

$$<\sigma> = \{\sigma\} \times \mathrm{int}(\Delta_n)$$

for each non-degenerate simplex $\sigma \in S_n(X)$. Also, for $n \geq 2$, the n^{th} barycentric subdivision of $|S(X)|$, defined in the obvious way, is a simplicial complex.

We now define the natural projection

$$\omega_X : |S(X)| \longrightarrow X .$$

If $y \in |S(X)|$, we have $y \in <\sigma> = \{\sigma\} \times \mathrm{int}(\Delta_n)$ for some unique $\sigma : \Delta_n \to X$, $y = (\sigma, t)$ for some $t \in \mathrm{int}(\Delta_n)$. Define

$$\omega_X(y) = \sigma(t) .$$

It is clear that ω_X is a continuous surjection. Also, if $A \subset X$ is a subspace, $|S(A)| \subset |S(X)|$ as a subcomplex, and $\omega_X(|S(A)|) = A$.

The most important fact about ω is that

$$(\omega_X)_\# : \pi_n(|S(X)|) \longrightarrow \pi_n(X)$$

is an isomorphism for all n, so that ω_X is a weak homotopy equivalence. Hence, if X has the homotopy type of a CW-complex the Whitehead theorem implies that ω_X is a homotopy equivalence.

The above construction can be made functorial. If $f : X \longrightarrow Y$ is a map, we define a map

$$|S(f)| : |S(X)| \longrightarrow |S(Y)|$$

as follows: for an arbitrary point $(\sigma, t) \in |S(X)|$, where $\sigma : \Delta_n \to X$

and $t \in \mathrm{int}(\Delta_n)$, we set

$$|S(f)|\ (\sigma,t) = (f\circ\sigma,\ t)\ .$$

One verifies that $|S(-)|$ is a functor from the category of topological spaces to itself, and that ω is a natural transformation from $|S(-)|$ to the identity functor.

3.3 <u>The Minimal Model</u>: Let K be a simply connected simplicial complex with rational cohomology of finite type. Then $E^*(K)$, the de Rham algebra of K , is a simply connected DGA, and by Theorem 2.5 we can build its minimal model

$$M(K) = M(E^*(K))\ .$$

$M(K)$ will be called the minimal model of K .

If X is a simply connected space (with rational cohomology of finite type), we may triangulate the geometric realization $|S(X)|$ of $S(X)$ to obtain a simplicial complex $|S(X)|'$. We define the minimal model, $M(X)$, of X to be the minimal model of $|S(X)|'$:

$$M(X) = M(E^*(|S(X)|'))\ .$$

Note that if K is a simplicial complex, we have two definitions of its minimal model, namely $M(E^*(K))$ and $M(E^*(|S(K)|'))$. The fact that these DGA's are isomorphic follows from the proof of the next theorem. The proof will also show that $M(X)$ doesn't depend on the way we triangulate $|S(X)|$.

3.4 <u>Theorem</u>: Simply connected spaces of the same rational homotopy type have isomorphic minimal models.

<u>Proof</u>: Suppose X and Y have the same rational homotopy type. Then there is a space Z and rational homotopy equivalences $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$. Triangulate the geometric realizations of these spaces to get a commutative diagram:

$$\begin{array}{ccccc} |S(X)|' & \xleftarrow{|S(f)|} & |S(Z)|' & \xrightarrow{|S(g)|} & |S(Y)|' \\ \downarrow{\scriptstyle \omega_X} & & \downarrow{\scriptstyle \omega_Z} & & \downarrow{\scriptstyle \omega_Y} \\ X & \xleftarrow{f} & Z & \xrightarrow{g} & Y \end{array} .$$

As the vertical maps are weak homotopy equivalences, we have that $|S(f)|$ and $|S(g)|$ are rational homotopy equivalences.

By the simplicial approximation theorem (subdividing $|S(Z)|'$ if necessary) we can find simplicial maps $\phi : |S(Z)|' \longrightarrow |S(X)|'$ and $\psi : |S(Z)|' \longrightarrow |S(Y)|'$ which are also rational homotopy equivalences. Now applying the de Rham functor E^*, we obtain DGA-maps

$$\phi^* : E^*(|S(X)|') \longrightarrow E^*(|S(Z)|')$$

and

$$\psi^* : E^*(|S(Y)|') \longrightarrow E^*(|S(Z)|')$$

which induce isomorphisms on cohomology. Hence, by Theorem 2.14 we have

$$M(X) \cong M(Z) \cong M(Y) .$$

Q.E.D.

3.5 Examples:

(a) Let S^n denote the n-sphere. If n is odd, $n \geq 3$, the rational cohomology of S^n is free, so by Example 2.4(b) the minimal model is

$$M(S^n) = \Lambda_n(x) \ , \ n \text{ odd}$$

with zero differential. If n is even, we must kill the cocycle x^2 in degree $2n$, so

$$M(S^n) = \Lambda_n(x) \otimes \Lambda_{2n-1}(y) \ , \ n \text{ even}$$

with the differential given by $d(x\otimes 1) = 0$, $d(1\otimes y) = x^2 \otimes 1$.

(b) Let CP^n denote complex projective n-space. The rational cohomology of CP^n is a truncated polynomial algebra on a single generator of degree 2, truncated a height $n + 1$. The minimal model is

$$M(CP^n) = \Lambda_2(x) \otimes \Lambda_{2n+1}(y) \ ,$$

with differential $d(x\otimes 1) = 0$, $d(1\otimes y) = x^{n+1} \otimes 1$. In both of these examples it can be shown directly that the given model is the only minimal algebra with the correct cohomology algebra.

(c) Let π be a finitely generated abelian group, and $n \geq 2$ an integer. An Eilenberg-MacLane space of type (π,n) is a space $K(\pi,n)$ for which

$$\pi_k(K(\pi,n)) = \begin{cases} \pi \ , & k = n \\ 0 \ , & k \neq n \ . \end{cases}$$

Any two Eilenberg-MacLane spaces of type (π,n) have the same weak homotopy type; furthermore we may choose $K(\pi,n)$ to be a simplicial complex. The rational cohomology of $K(\pi,n)$ is the free algebra on ℓ generators of degree n, where ℓ is the rank of π. Hence

$$\begin{aligned} M(K(\pi,n)) &= H^*(K(\pi,n);Q) \\ &= \Lambda_n(x_1,\ldots,x_\ell)\ , \quad \ell = \mathrm{rank}(\pi)\ , \end{aligned}$$

with zero differential.

Notice that, in all these examples, the number of free generators of degree n in $M(X)$ is exactly the rank of $\pi_n(X)$. The proof of this result in the general case occupies the remainder of the thesis. We first discuss the two main tools to be used: spectral sequences and the Postnikov decomposition.

3.6 <u>Spectral Sequences</u>: In this section we describe the two spectral sequences used in this chapter. The first is the Serre spectral sequence of a fibration, and the second is a special case of the spectral sequence of a filtered DGA. Details can be found in [11; Chapter 9].

All fibrations considered are orientable (see [11; Section 9.2]). Let $F \xrightarrow{i} E \xrightarrow{p} B$ be an orientable fibration over a connected CW-complex B, $A \subset B$ a subcomplex, and G an abelian group. The Serre spectral sequence of this fibration is given by

$$E_2^{p,q} = H^p(B,A;H^q(F;G))\ ,$$

and the differential on the E_r-level has bidegree $(r,1-r)$. E_∞^{**} is the bigraded module associated with a decreasing filtration of $H^*(E,E_A;G)$, where $E_A = p^{-1}(A)$. Letting B^n denote the n-skeleton of B and setting $E_n = p^{-1}(A \cup B^n)$, the filtration is given by:

$$F^pH^n(E,E_A;G) = \mathrm{Ker}\{H^n(E,E_A;G) \longrightarrow H^n(E_{p-1},E_A;G)\} .$$

We have
$$F^{n+1}H^n(E,E_A;G) = 0 ,$$
$$F^0H^n(E,E_A;G) = H^n(E,E_A;G)$$
and
$$E_\infty^{p,q} = F^pH^{p+q}(E,E_A;G)/F^{p+1}H^{p+q}(E,E_A;G) .$$

The map $p^* : H^n(B,A;G) \longrightarrow H^n(E,E_A;G)$ can be factored as $H^n(B,A;G) \cong E_2^{n,0} \twoheadrightarrow E_\infty^{n,0} \cong F^nH^n(E,E_A;G) \hookrightarrow H^n(E,E_A;G)$. Also, if $A = \phi$, $i^* : H^n(E;G) \longrightarrow H^n(F;G)$ can be factored as

$$H^n(E;G) = F^0H^n(E;G) \twoheadrightarrow E_\infty^{0,n} \rightarrowtail E_2^{0,n} \cong H^n(F;G) .$$

These factorizations can be used to derive the Serre exact sequence: if B is n-connected and F is m-connected, there is an exact sequence

$$\cdots \longrightarrow H^q(B;G) \xrightarrow{p^*} H^q(E;G) \xrightarrow{i^*} H^q(F;G) \xrightarrow{\tau} H^{q+1}(B;G) \longrightarrow \cdots \longrightarrow H^{n+m+1}(F;G)$$

where τ is the transgression.

Now let A be a DGA, and set $\Lambda_n = \Lambda_n(x_1,\ldots,x_\ell)$, $n \geq 2$. Suppose that the graded algebra $C = A \otimes \Lambda_n$ is equipped with a differential in such a way that $d(a\otimes 1) = d(a) \otimes 1$ for $a \in A$ and

$d(1 \otimes x_i) \in A^{n+1} \otimes 1$. We define a decreasing filtration on C by

$$F^pC^k = \bigoplus_{q \geq p} A^q \otimes \Lambda_n^{k-q} .$$

We have

$$0 = F^{k+1}C^k \subset F^kC^k \subset \ldots \subset F^0C^k = C^k .$$

Also, this filtration is preserved by the multiplication and differential in C :

$$F^pC \cdot F^qC \subset F^{p+q}C ,$$

$$d(F^pC) \subset F^{p+1}C \subset F^pC .$$

Hence there is a convergent E_1-spectral sequence given by

$$E_1^{p,q} = H^{p+q}(F^pC/F^{p+1}C) ,$$

which converges to some filtration of $H^*(C)$. As $d(F^pC) \subset F^{p+1}C$, we have

$$E_1^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q} \cong A^p \otimes \Lambda_n^q .$$

We wish to compute the E_2-level of this spectral sequence. In the general case, the elements of $E_1^{p,q} = H^{p+q}(F^pC/F^{p+1}C)$ have the form $[c + F^{p+1}C]$ where $c \in F^pC$ and $dc \in F^{p+1}C$. The differential $d_1 : E_1^{p,q} \longrightarrow E_1^{p+1,q}$ is then given by

$$d_1([c + F^{p+1}C]) = [dc + F^{p+2}C] .$$

Hence, for $a \otimes b \; \varepsilon \; A^p \otimes {}^q_n = E_1^{p,q}$, $d_1(a\otimes b)$ is just the sum of terms of $d(a\otimes b)$ that are in $A^{p+1} \otimes \Lambda^q_n$; that is

$$d_1(a\otimes b) = d(a) \otimes b \quad .$$

Therefore, $E_2^{p,q} \cong H^p(A) \otimes \Lambda^q_n$.

3.7 <u>The Postnikov Decomposition</u>: If X is any pointed space, we denote the space of paths to the base point by PX , and the space of loops at the base point by ΩX . Recall that PX is contractable, and that there is the standard path fibration $PX \longrightarrow X$ with fibre ΩX .

Let X be a simply connected space. The Postnikov decomposition of X is a tower of spaces and maps:

$$\begin{tikzcd}
& \vdots \arrow[d] \\
& X_n \arrow[d, "p_n"] \\
& X_{n-1} \arrow[d, "p_{n-1}"] \\
& \vdots \arrow[d] \\
X \arrow[uuur, "f_n"] \arrow[uur, "f_{n-1}"'] \arrow[r, "f_1"'] & X_1 = *
\end{tikzcd}$$

with the following properties:

(i) $p_n \circ f_n = f_{n-1}$;

(ii) $\pi_k(X_n) = 0$ for $k > n$;

(iii) $(f_n)_\# : \pi_k(X) \longrightarrow \pi_k(X_n)$ is an isomorphism for $k \leq n$;

(iv) p_n is a principal fibration with fibre $K(\pi_n(X),n)$

In fact, there is a pullback diagram:

$$\begin{tikzcd}
K(\pi_n(X),n) \arrow[r, equal] \arrow[d] & \Omega K(\pi_n(X),n+1) \arrow[d] \\
X_n \arrow[r] \arrow[d, "p_n"'] \arrow[dr, phantom, "\text{Pullback}"] & PK(\pi_n(X),n+1) \arrow[d] \\
X_{n-1} \arrow[r, "k_n"] & K(\pi_n(X),n+1)
\end{tikzcd}$$

where the right column is the path fibration. The above properties determine the spaces X_n up to weak homotopy type. As X_2 is a $K(\pi_2(X),2)$, we may assume that X_n has the homotopy type of a CW-complex (see [8]). The map $k_n : X_{n-1} \longrightarrow K(\pi_n(X),n+1)$ is called the n^{th} k-invariant of X and is determined up to homotopy.

Details of the construction of the Postnikov decomposition can be found in [9; Chapter 13].

The Postnikov decomposition of X allows us to focus attention on one homotopy group at a time. We now study the principal fibrations p_n in the decomposition and show that the minimal model of the total space X_n is just the tensor product of the minimal models of the base X_{n-1} and the fibre $K(\pi_n(X),n)$ with a suitable differential.

3.8 <u>The Main Construction</u>: Let $p : E \longrightarrow Y$ be a fibration with fibre $K(\pi,n)$, where Y is a simply connected simplicial complex, E has the homotopy type of a simply connected CW-complex, π is a finitely generated abelian group, and $n \geq 2$. For each subcomplex $B \subset Y$, let $E_B = p^{-1}(B)$. Then the map $p|E_B : E_B \longrightarrow B$ is also an orientable fibration with fibre $K(\pi,n)$. We denote the m-skeleton of Y by Y^m, and set $E_m = p^{-1}(Y^m) = E_{Y^m}$. Note that, for each vertex $y \in Y$, E_y is an Eilenberg-MacLane space of type (π,n).

As in 3.2, let $|S(E)|$ be the geometric realization of $S(E)$; this is a CW-complex which is triangulable, and the evaluation map $\omega : |S(E)| \longrightarrow\!\!> E$ is a homotopy equivalence. For any subcomplex $B \subset Y$ we have $E_B \subset E$, and hence $|S(E_B)|$ is a subcomplex of $|S(E)|$. Now barycentrically subdivide $|S(E)|$ twice to get a simplicial complex. Let $f : K \longrightarrow Y$ be a simplicial approximation to the composite

$$|S(E)| \xrightarrow{\omega}\!\!> E \xrightarrow{p} Y ,$$

where K is a further subdivision of $|S(E)|$. For a subcomplex $B \subset Y$, let K_B denote $|S(E_B)|$ with this subdivision; clearly K_B is a simplicial subcomplex of K, and $f|K_B$ is a simplicial approximation to the composite

$$|S(E_B)| \xrightarrow{\omega}\!\!> E_B \xrightarrow{p} B .$$

Also, if $B = Y^m$, we let $K_m = K_B$. So, for each subcomplex $B \subset Y$ and vertex $y \in B$, we have a diagram

$$\begin{tikzcd}
E_y \arrow[r, hook] & E_B \arrow[r] & B \\
K_y \arrow[r, hook] \arrow[u, "\omega"] \arrow[ur, phantom, "\circlearrowleft"] & K_B \arrow[r, "f"] \arrow[u, "\omega"] \arrow[ur, phantom, "\simeq"] & B \arrow[u, equal]
\end{tikzcd}$$

where the maps in the bottom row are simplicial, and the vertical maps are homotopy equivalences.

Choose a minimal model $M(Y)$ for Y and a DGA map

$$\rho : M(Y) \longrightarrow \mathcal{E}^*(Y)$$

which induces an isomorphism on cohomology. Notice that, as K_y is a $K(\pi,n)$ for any vertex $y \in Y$, we have

$$H^*_{DR}(K_y) \cong H^*(K_y;\mathbb{Q}) \cong \Lambda_n(x_1,\ldots,x_\ell) ,$$

the free algebra on generators $x_1,\ldots,x_\ell$ in degree n, where ℓ is the rank of π. Hence $\Lambda_n(x_1,\ldots,x_\ell)$ is the minimal model of K_y. We write Λ_n for $\Lambda_n(x_1,\ldots,x_\ell)$. Our aim is to equip $\mathcal{E}^*(Y) \otimes \Lambda_n$ with a differential in such a way that its minimal model is $M(Y) \otimes \Lambda_n$ with a suitable differential, and so that there is a DGA map

$$\lambda : \mathcal{E}^*(Y) \otimes \Lambda_n \longrightarrow \mathcal{E}^*(K)$$

inducing an isomorphism on cohomology. It will follow that

$$M(K) \cong M(Y) \otimes \Lambda_n .$$

<u>Lemma</u>: For any vertex $y \in Y$, the inclusion $i : K_y \longrightarrow K_1$ induces an isomorphism $i^* : H^n_{DR}(K_1) \cong H^n_{DR}(K_y)$.

<u>Proof</u>: Consider the Serre spectral sequence with rational coefficients for the fibration $E_y \xrightarrow{i} E_1 \xrightarrow{p} Y^1$. This is given by

$$E_2^{p,q} = H^p(Y^1;H^q(E_y;\mathbb{Q}))$$

and converges to $H^*(E_1;\mathbb{Q})$. As $H^p(Y^1) = 0$ for $p \geq 2$ and $H^*(E_y;\mathbb{Q}) \cong \Lambda_n$, we have that

$$E_2^{p,q} = 0 \quad \text{for} \quad p \geq 2 \quad \text{or} \quad q \not\equiv 0 \pmod{n} .$$

Hence all the differentials in the spectral sequence are zero, and $E_\infty^{p,q} \cong E_2^{p,q}$. Also, on the $p + q = n$ diagonal, the only non-zero entry is $E_\infty^{0,n}$. So the edge homomorphism $i^* : H^n(E_1;\mathbb{Q}) \longrightarrow H^n(E_y;\mathbb{Q})$ can be factored as

$$H^n(E_1;\mathbb{Q}) = F^0H^n(E_1;\mathbb{Q}) \cong E_\infty^{0,n} \cong E_2^{0,n} \cong H^n(E_y;\mathbb{Q}) ,$$

and hence i^* is an isomorphism in dimension n. The result follows from the natural equivalence of simplicial and de Rham cohomology, and the commutative diagram

$$\begin{tikzcd} E_y \arrow[r, hook] & E_1 \\ K_y \arrow[u, "\omega"] \arrow[r, hook] & K_1 \arrow[u, "\omega"'] \end{tikzcd} ,$$

where ω is a homotopy equivalence. Q.E.D.

It follows from this Lemma that $H^n_{DR}(K_1)$ is an ℓ-dimensional vector space. For $1 \leq i \leq \ell$, choose closed forms $\omega_i \in \mathcal{E}^n(K_1)$ whose cohomology classes form a basis for $H^n_{DR}(K_1)$. Then, for any vertex $y \in Y$, the collection $\{[\omega_i|K_y]\}_{i=1}^{\ell}$ is a free set of generators for the algebra $H^*_{DR}(K_y) \cong \Lambda_n$. By Proposition 1.10, we can extend ω_i to a form $\bar{\omega}_i \in \mathcal{E}^n(K)$. Then $d\bar{\omega}_i$ is a closed form on K which is zero on K_1. We consider $d\bar{\omega}_i$ as an element of $\mathcal{E}^{n+1}(K,K_0)$.

<u>Lemma</u>: $[d\bar{\omega}_i] \in \mathrm{Im}\{f^* : H^{n+1}_{DR}(Y,Y^0) \longrightarrow H^{n+1}_{DR}(K,K_0)\}$.

<u>Proof</u>: Consider the Serre spectral sequence for the fibration $E_y \longrightarrow E \xrightarrow{p} Y$ relative to Y^0. It is given by

$$E_2^{p,q} = H^p(Y,Y^0;H^q(E_y;\mathbb{Q})) ,$$

and converges to $H^*(E,E_0,\mathbb{Q})$. As $E_2^{p,q} = 0$ for $p = 0$ or $q \not\equiv 0$ (mod n), we see that all the differentials terminating at the $(n+1,0)$ position are zero, and hence $E_2^{n+1,0} \cong E_\infty^{n+1,0}$. Now the edge homomorphism $p^* : H^{n+1}(Y,Y^0;\mathbb{Q}) \longrightarrow H^{n+1}(E,E_0,\mathbb{Q})$ factors as

$$H^{n+1}(Y,Y;\mathbb{Q}) \cong E_2^{n+1,0} \cong E_\infty^{n+1,0} = F^{n+1}\, H^{n+1}(E,E_0;\mathbb{Q}) \hookrightarrow H^{n+1}(E,E_0;\mathbb{Q}) ,$$

and hence $\mathrm{Im}(p^*) = F^{n+1}\, H^{n+1}(E,E_0;\mathbb{Q})$. But on the $p + q = n + 1$ diagonal, the only non-zero entries are $E_\infty^{n+1,0}$ and $E_\infty^{n,1}$. Hence

$$\begin{aligned}\mathrm{Im}(p^*) &= F^{n+1}\, H^{n+1}(E,E_0;\mathbb{Q}) \\ &= F^2\, H^{n+1}(E,E_0;\mathbb{Q}) \\ &= \mathrm{Ker}\{H^{n+1}(E,E_0;\mathbb{Q}) \longrightarrow H^{n+1}(E_1,E_0;\mathbb{Q})\}\end{aligned}$$

Passing to the homotopy equivalent maps $K_y \longrightarrow K \xrightarrow{f} Y$, we have

$$\mathrm{Im}\{f^* : H_{DR}^{n+1}(Y,Y^0) \longrightarrow H_{DR}^{n+1}(K,K_0)\} = \mathrm{Ker}\{H_{DR}^{n+1}(K,K_0) \longrightarrow H_{DR}^{n+1}(K_1,K_0)\} .$$

Now the map on the right is induced by the restriction to K_1 of the forms on K, and as $d\bar{\omega}_i|K_1 = 0$, we have $[d\bar{\omega}_i] \in \mathrm{Im}(f^*)$. Q.E.D.

Hence we can find cohomology classes $c_i \in H_{DR}^{n+1}(Y,Y^0) = H_{DR}^{n+1}(Y)$ such that $f^*(c_i) = [d\bar{\omega}_i]$ in $H_{DR}^{n+1}(K,K_0)$. Note that c_i is the transgression of $[\omega_i|K_y]$ in the Serre spectral sequence for $K_y \longrightarrow K \xrightarrow{f} Y$, and this is the case for <u>any</u> vertex $y \in Y$.

As $\rho^* : H^*(M(Y)) \longrightarrow H_{DR}^*(Y)$ is an isomorphism, we can find $m_i \in Z^{n+1}(M(Y))$ such that $\rho^*([m_i]) = c_i$, $1 \leq i \leq \ell$. Set $\psi_i = \rho(m_i) \in E^{n+1}(Y)$, so that $[\psi_i] = c_i$. Considering ψ_i as an element of $E^{n+1}(Y,Y^0)$, we have that $[f^*(\psi_i)] = [d\bar{\omega}_i]$ in $H^{n+1}(K,K_0)$. Hence there are forms $\eta_i \in E^n(K,K_0)$, $1 \leq i \leq \ell$, such that $f^*(\psi_i) = d\bar{\omega}_i + d\eta_i$. Also, as η_i is zero on K_0 we have that $(\bar{\omega}_i + \eta_i)|K_0 = \omega_i|K_0$.

Now define a differential in the algebra $E^*(Y) \otimes \Lambda_n$ by setting

$$\begin{cases} d(\xi \otimes 1) = d(\xi) \otimes 1 & \text{for } \xi \in E^*(Y) , \\ d(1 \otimes \chi_i) = \psi_i \otimes 1 & 1 \leq i \leq \ell , \end{cases}$$

and extending to all of $E^*(Y) \otimes \Lambda_n$ by requiring that d be a linear derivation. Similarly, define a differential in $M(Y) \otimes \Lambda_n$ by setting

$$\begin{cases} d(a \otimes 1) = d(a) \otimes 1 & \text{for} \quad a \in M(Y) \,, \\ d(1 \otimes x_i) = m_i \otimes 1 \,, & 1 \leq i \leq \ell \,. \end{cases}$$

As $\rho(m_i) = \psi_i$, the algebra map

$$\rho \otimes 1 : M(Y) \otimes \Lambda_n \longrightarrow \mathcal{E}^*(Y) \otimes \Lambda_n$$

is a map of DGA's . We will later show that $\rho \otimes 1$ induces an isomorphism on cohomology.

Define a map

$$\lambda : \mathcal{E}^*(Y) \otimes \Lambda_n \longrightarrow \mathcal{E}^*(K)$$

by setting

$$\begin{cases} \lambda(\xi \otimes 1) = f^*(\xi) & \text{for} \quad \xi \in \mathcal{E}^*(Y) \\ \lambda(1 \otimes x_i) = \bar{\omega}_i + \eta_i \,, & 1 \leq i \leq \ell \,, \end{cases}$$

and extending so as to be an algebra map. We have

$$d\lambda(\xi \otimes 1) = df^*(\xi) = f^*(d\xi) = \lambda(d\xi \otimes 1) = \lambda d(\xi \otimes 1) \quad ,$$

and $\quad d\lambda(1 \otimes x_i) = d\bar{\omega}_i + d\eta_i = f^*(\psi_i) = \lambda(\psi_i \otimes 1) = \lambda d(1 \otimes x_i)$,

so λ is a map of DGA's . For each subcomplex $B \subset Y$, the map λ restricts to a DGA map

$$\lambda_B : \mathcal{E}^*(B) \otimes \Lambda_n \longrightarrow \mathcal{E}^*(K_B)$$

by setting $d(1 \otimes x_i) = (\psi_i | B) \otimes 1$ in $\mathcal{E}^*(B) \otimes \Lambda_n$, and

$\lambda_B(1 \otimes x_i) = (\bar{\omega}_i + \eta_i)|K_B$. Then, for subcomplexes $B \subset C \subset Y$, there is a commutative diagram of DGA maps

$$\begin{tikzcd} E^*(C) \otimes \Lambda_n \arrow[r, "\lambda_C"] \arrow[d, "j_1^* \otimes 1"'] \arrow[dr, phantom, "\circlearrowright"] & E^*(K_C) \arrow[d, "j_2^*"] \\ E^*(B) \otimes \Lambda_n \arrow[r, "\lambda_B"'] & E^*(K_B) \end{tikzcd}$$

induced by

$$\begin{tikzcd} K_B \arrow[r, hook, "j_2"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & K_C \arrow[d, "f"] \\ B \arrow[r, hook, "j_1"'] & C \end{tikzcd}$$

The main technical result of the chapter is:

3.9 <u>Theorem</u>: The induced map

$$\lambda^* : H^*(E^*(Y) \otimes \Lambda_n) \longrightarrow H^*_{DR}(K)$$

is an isomorphism.

<u>Proof</u>: We show that λ_B^* is an isomorphism for every finite dimensional subcomplex $B \subset Y$; this we do by induction on the dimension of B .

If $y \in Y$ is a vertex, we have

$$E^*(y) \otimes \Lambda_n = \mathbb{Q} \otimes \Lambda_n \cong \Lambda_n$$

with zero differential, and λ_y is just the map $\Lambda_n \longrightarrow E^*(K_y)$ that takes x_i to $(\bar{\omega}_i + \eta_i)|K_y = \omega_i|K_y$. But $\{[\omega_i|K_y]\}_{i=1}^{\ell}$ is a system of free generators for $H^*_{DR}(K_y) = \Lambda_n$, and hence λ_y^* is an isomorphism. If B is any o-dimensional subcomplex, then $K_B = \bigcup_{y \varepsilon B} K_y$, and $K_y \cap K_z = \emptyset$ for distinct vertices y and z of B . Hence there is a direct product decomposition

$$E^*(K_B) \cong \prod_{y \varepsilon B} E^*(K_y) \quad .$$

We also have

$$E^*(B) \otimes \Lambda_n \cong (\prod_{y \varepsilon B} E^*(y)) \otimes \Lambda_n \cong \prod_{y \varepsilon B} (E^*(y) \otimes \Lambda_n) ,$$

and λ_B is just the direct product of the maps λ_y . As cohomology commutes with direct products and each λ_y^* is an isomorphism, λ_B^* is an isomorphism.

Now assume that λ_B^* is an isomorphism for all subcomplexes $B \subset Y$ with $\dim(B) \leq m - 1$, for some $m \geq 1$. Let B be any m-dimensional subcomplex of Y, and let $\mathcal{C}$ denote the collection of subcomplexes C such that $B^{m-1} \subset C \subset B$ and λ_C^* is an isomorphism. By the induction hypothesis, $B^{m-1} \varepsilon \mathcal{C}$, so $\mathcal{C}$ is non-empty. We show that Zorn's Lemma applies.

Suppose $\{C_\alpha : \alpha \varepsilon J\}$ is a chain in $\mathcal{C}$ with $C_\alpha \subset C_\beta$ when $\alpha \leq \beta$ in J . Set $C = \bigcup_\alpha C_\alpha$; then $K_C = \bigcup_\alpha K_{C_\alpha}$. For each $\alpha \leq \beta$, we have a commutative diagram:

$$\begin{array}{ccc} E^*(C_\beta)\otimes\Lambda_n & \xrightarrow{\lambda_{C_\beta}} & E^*(K_{C_\beta}) \\ \downarrow & & \downarrow \\ E^*(C_\alpha)\otimes\Lambda_n & \xrightarrow[\lambda_{C_\alpha}]{} & E^*(K_{C_\alpha}) \end{array} \quad ,$$

where the vertical maps are induced by inclusion. So the collection $\{\lambda_{C_\alpha} : \alpha \in J\}$ is a map between inverse systems of DGA's. By Proposition 1.12 we have isomorphisms

$$\varprojlim_\alpha (E^*(C_\alpha)\otimes\Lambda_n) \cong \left(\varprojlim_\alpha E^*(C_\alpha)\right)\otimes\Lambda_n \cong E^*(C)\otimes\Lambda_n \ ,$$

and

$$\varprojlim_\alpha E^*(K_{C_\alpha}) \cong E^*(K_C) \ ;$$

and the induced map on the limits corresponds under these isomorphisms to $\lambda_C : E^*(C)\otimes\Lambda_n \longrightarrow E^*(K_C)$. Again by 1.12, cohomology commutes with inverse limits, and as each λ^*_C is an isomorphism, λ^*_C is also an isomorphism. Hence $C \in \mathcal{C}$.

We have shown that every chain in $\mathcal{C}$ has an upper bound and hence, by Zorn's Lemma, there is a maximal element $C \in \mathcal{C}$. If $C = B$, this completes the induction step. So suppose $C \neq B$. Then there is an m-simplex $\sigma \subset B$ such that $C \cap \sigma = \partial\sigma$. Let $D = C \cup \sigma$; we show that $D \in \mathcal{C}$, contradicting the maximality of C.

The differential in $E^*(D)\otimes\Lambda_n$ restricts to a differential in $E^*(D,C)\otimes\Lambda_n$, and we define a map of DGA's

$$\lambda' : E^*(D,C) \otimes \Lambda_n \longrightarrow E^*(K_D,K_C)$$

by requiring that the following diagram be commutative (with exact rows);

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^*(D,C) \otimes \Lambda_n & \longrightarrow & E^*(D) \otimes \Lambda_n & \twoheadrightarrow & E^*(C) \otimes \Lambda_n & \longrightarrow & 0 \\ & & \Big\downarrow \lambda' & & \Big\downarrow \lambda_D & & \Big\downarrow \lambda_C & & \\ 0 & \longrightarrow & E^*(K_D,K_C) & \longrightarrow & E^*(K_D) & \twoheadrightarrow & E^*(K_C) & \longrightarrow & 0 \end{array} \tag{1}$$

This diagram induces a map of long exact cohomology sequences, and λ_C^* is an isomorphism as $C \in \mathcal{C}$. So, by the 5-lemma, to show that λ_D^* is an isomorphism it suffices to show that $(\lambda')^*$ is an isomorphism.

Similarly for the pair $(\sigma,\partial\sigma)$, there is a DGA-map λ'' in the following commutative diagram (with exact rows);

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^*(\sigma,\partial\sigma) \otimes \Lambda_n & \longrightarrow & E^*(\sigma) \otimes \Lambda_n & \twoheadrightarrow & E^*(\partial\sigma) \otimes \Lambda_n & \longrightarrow & 0 \\ & & \Big\downarrow \lambda'' & & \Big\downarrow \lambda_\sigma & & \Big\downarrow \lambda_{\partial\sigma} & & \\ 0 & \longrightarrow & E^*(K_\sigma,K_{\partial\sigma}) & \longrightarrow & E^*(K_\sigma) & \twoheadrightarrow & E^*(K_{\partial\sigma}) & \longrightarrow & 0 \end{array} \tag{2}$$

Also, the inclusions $h_1 : (\sigma,\partial\sigma) \longrightarrow (D,C)$ and $h_2 : (K_\sigma, K_{\partial\sigma}) \longrightarrow (K_D,K_C)$ induce maps on the relative de Rham algebras, and we have the following commutative square:

$$\begin{array}{ccc} \mathcal{E}^*(D,C) \otimes \Lambda_n & \xrightarrow{h_1^* \otimes 1} & \mathcal{E}^*(\sigma,\partial\sigma) \otimes \Lambda_n \\ \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \lambda''} \\ \mathcal{E}^*(K_D,K_C) & \xrightarrow{h_2^*} & \mathcal{E}^*(K_\sigma, K_{\partial\sigma}) \end{array} \tag{3}$$

Clearly $h_1^* : \mathcal{E}^*(D,C) \longrightarrow \mathcal{E}^*(\sigma,\partial\sigma)$ is an isomorphism, and hence $h_1^* \otimes 1$ is also an isomorphism. By [11; Lemma 9.2.2], the inclusion $(E_\sigma,E_{\partial\sigma}) \longrightarrow (E_D,E_C)$ induces an isomorphism $H^*(E_D,E_C) \cong H^{*'}(E_\sigma,E_{\partial\sigma})$ on integral cohomology, and hence h_2^* in diagram (3) induces an isomorphism on de Rham cohomology. So to prove that $(\lambda')^*$ is an isomorphism, it suffices to show that $(\lambda'')^*$ is an isomorphism.

Diagram (2) induces a map of long exact cohomology sequences and, again by the 5-lemma, $(\lambda'')^*$ is an isomorphism if both λ_σ^* and $\lambda_{\partial\sigma}^*$ are isomorphisms. As $\partial\sigma$ is an $(m - 1)$-dimensional subcomplex, $\lambda_{\partial\sigma}^*$ is an isomorphism by the induction hypothesis. So we need only show that λ_σ^* is an isomorphism. Let y be a vertex of σ, and denote the inclusions by $j_1 : y \longrightarrow \sigma$ and $j_2 : K_y \longrightarrow K_\sigma$. Then there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{E}^*(\sigma) \otimes \Lambda_n & \xrightarrow{j_1^* \otimes 1} & \mathcal{E}^*(y) \otimes \Lambda_n \\ \downarrow{\scriptstyle \lambda_\sigma} & & \downarrow{\scriptstyle \lambda_y} \\ \mathcal{E}^*(K_\sigma) & \xrightarrow{j_2^*} & \mathcal{E}^*(K_y) \end{array} \tag{4}$$

As σ is contractible, j_2 is a homotopy equivalence, and j_2^* is an isomorphism on cohomology. Also λ_y^* is an isomorphism by the induction hypothesis. So to show that λ_σ^* is an isomorphism, it suffices to show that $j_1^* \otimes 1$ induces an isomorphism on cohomology.

As in section 3.6, there are filtrations on the DGA's $\mathcal{E}^*(\sigma) \otimes \Lambda_n$ and $\mathcal{E}^*(y) \otimes \Lambda_n$ and the associated spectral sequences are given by

$$E_2^{p,q} \cong H_{DR}^p(\sigma) \otimes \Lambda_n^q \quad ,$$

$$\hat{E}_2^{p,q} \cong H_{DR}^p(y) \otimes \Lambda_n^q \quad .$$

These spectral sequences converge to $H^*(\mathcal{E}^*(\sigma) \otimes \Lambda_n)$ and $H^*(\mathcal{E}^*(y) \otimes \Lambda_n)$, respectively. Now $j_1^* \otimes 1 : \mathcal{E}^*(\sigma) \otimes \Lambda_n \longrightarrow \mathcal{E}^*(y) \otimes \Lambda_n$ preserves filtrations, and hence induces a morphism of spectral sequences $E \longrightarrow \hat{E}$. On the E_2-level, this morphism is just

$$j_1^* \otimes 1 : H_{DR}^p(\sigma) \otimes \Lambda_n^q \longrightarrow H_{DR}^p(y) \otimes \Lambda_n^q \; ,$$

which is clearly an isomorphism as σ is contractible. So, by the comparison spectral sequence theorem,

$$(j_1^* \otimes 1)^* : H^*(\mathcal{E}^*(\sigma) \otimes \Lambda_n) \longrightarrow H^*(\mathcal{E}^*(y) \otimes \Lambda_n)$$

is an isomorphism, as desired.

Retracing our steps, we have shown that λ_D^* is an isomorphism and hence $D \in \mathcal{C}$, contradicting the maximality of $C \in \mathcal{C}$. Hence $C = B$, and λ_B^* is an isomorphism, as desired. This concludes the induction step, and gives the result for all finite dimensional subcomplexes of Y. As $Y = \bigcup_{m \geq 0} Y^m$, the same inverse limit argument as in the induction step shows that $\lambda^* : H^*(\mathcal{E}^*(Y) \otimes \Lambda_n) \longrightarrow H^*_{DR}(K)$ is an isomorphism. Q.E.D.

3.10 Proposition: The induced map

$$(\rho \otimes 1)^* : H^*(M(Y) \otimes \Lambda_n) \longrightarrow H^*(\mathcal{E}^*(Y) \otimes \Lambda_n)$$

is an isomorphism.

Proof. Again using the results of 3.6, there are filtrations on the DGA's $M(Y) \otimes \Lambda_n$ and $\mathcal{E}^*(Y) \otimes \Lambda_n$, and the associated spectral sequences are given by:

$$E_2^{p,q} = H^p(M(Y)) \otimes \Lambda_n^q \quad ,$$

$$\hat{E}_2^{p,q} = H^p_{DR}(Y) \otimes \Lambda_n^q \quad .$$

These spectral sequences converge to $H^*(M(Y) \otimes \Lambda_n)$ and $H^*(\mathcal{E}^*(Y) \otimes \Lambda_n)$, respectively. As $\rho \otimes 1$ preserves filtrations, it induces a morphism of spectral sequences $E \longrightarrow \hat{E}$ which, on the E_2-level, is given by

$$\rho^* \otimes 1 : H^p(M(Y)) \otimes \Lambda_n^q \to H^p_{DR}(Y) \otimes \Lambda_n^q \ .$$

As ρ^* is an isomorphism, so is $\rho^* \otimes 1$. Hence, by the comparison spectral sequence theorem, $(\rho \otimes 1)^*$ is an isomorphism. Q.E.D.

Still in the notation of 3.8, we have:

3.11 Corollary: Suppose $M(Y)$ has no algebra generators of degree $n + 1$. Then $M(Y) \otimes \Lambda_n$ with the above differential is the minimal model of E.

<u>Proof</u>: By definition, $M(E) = M(\mathcal{E}^*(K))$. Now the differential in $M(Y) \otimes \Lambda_n$ is decomposable because

$$d(1 \otimes x_i) = m_i \otimes 1 \in M^{n+1}(Y) \otimes \Lambda_n^0 ,$$

and the elements of $M^{n+1}(Y)$ are decomposable by hypothesis. Hence $M(Y) \otimes \Lambda_n$ is a minimal algebra. By Theorem 3.9 and Proposition 3.10, the composite

$$\lambda \circ (\rho \otimes 1) : M(Y) \otimes \Lambda_n \longrightarrow \mathcal{E}^*(K)$$

is a DGA map which induces an isomorphism on cohomology, and the result follows. Q.E.D.

We now come to the main theorem of the thesis. The notation for the Postnikov decomposition is described in section 3.7 .

3.12 <u>Theorem</u>: Let X be a simply connected space and X_n the n^{th} space in its Postnikov decomposition. Then

$$M_n(X) \cong M(X_n) \cong M(X_{n-1}) \otimes \Lambda_n(x_1, \ldots, x_\ell)$$

with a suitably defined differential, where $\ell = \text{rank}(\pi_n(X))$. Hence the number of free generators of degree n in $M(X)$ is the rank of $\pi_n(X)$.

<u>Proof</u>: First, because the map $f_n : X \to X_n$ induces an isomorphism on homotopy groups through degree n and an epimorphism in degree $n + 1$, the same is true on homology by the Whitehead Theorem (see [11; Theorem 7.5.9.]) Hence, by the universal coefficient theorem, f_n induces an isomorphism on rational cohomology through degree n and a monomorphism in degree $n + 1$.

Now we geometrically realize X and X_n and simplicially approximate f_n, obtaining a homotopy equivalent simplicial map, which we also denote $f_n : X \longrightarrow X_n$. Let $\rho : M(X) \longrightarrow E^*(X)$ and $\rho' : M(X_n) \longrightarrow E^*(X_n)$ be minimal models for X and X_n. By Theorem 2.13, there is a DGA map $g : M(X_n) \longrightarrow M(X)$ such that the following diagram commutes up to DGA-homotopy:

$$\begin{tikzcd}
M(X_n) \arrow[r, "\rho'"] \arrow[d, "g"'] \arrow[dr, phantom, "\simeq"] & E^*(X_n) \arrow[d, "f_n^*"] \\
M(X) \arrow[r, "\rho"] & E^*(X)
\end{tikzcd}$$

As ρ^* and $(\rho')^*$ are isomorphisms, $g^* : H^k(M(X_n)) \longrightarrow H^k(M(X))$ is an isomorphism for $k \leq n$ and a monomorphism for $k = n + 1$. Hence by Theorem 2.6, $M_n(X) \cong M_n(X_n)$.

We now prove the theorem by inducation on n. For $n = 2$, $X_2 = K(\pi_2(X),2)$, and by Example 3.5(c) we have

$$M(X_2) = \Lambda_2(x_1, \ldots, x_2) \quad , \quad \ell = \text{rank } (\pi_2(X)) .$$

Hence $\qquad M_2(X) \cong M_2(X_2) = M(X_2) \quad$ as desired.

Assume inductively that $M_{n-1}(X) \cong M(X_{n-1})$ for some $n \geq 3$. Let Y denote a triangulation of the geometric realization of X_{n-1}. Pull back the fibration $p_n : X_n \longrightarrow X_{n-1}$ over the evaluation map $\omega : Y \longrightarrow\!\!> X_{n-1}$ to get a fibration $p : E \longrightarrow Y$ with fibre $K(\pi_n(X),n)$. As ω is a homotopy equivalence, E has the same homotopy type as X_n

and p is equivalent to p_n in the homotopy category. We can now apply the construction in section 3.8 . As

$$M(Y) \cong M(X_{n-1}) \cong M_{n-1}(X) ,$$

there are no generators of degree $n + 1$ in $M(Y)$, so Corollary 3.11 yields

$$\begin{aligned} M(X_n) &\cong M(E) \\ &\cong M(Y) \otimes \Lambda_n(x_1, \ldots, x_\ell) \\ &\cong M(X_{n-1}) \otimes \Lambda_n(x_1, \ldots, x_\ell) \quad , \end{aligned}$$

where $\ell = \operatorname{rank}(\pi_n(X))$. This also shows that $M(X_n)$ has no generators above degree n, so

$$M_n(X) \cong M_n(X_n) = M(X_n) \quad .$$

This proves the theorem. Q.E.D.

We prove one last result about the minimal model. Recall that the Hurewicz homomorphism is a natural transformation from homotopy to homology:

$$h_n : \pi_n(X) \longrightarrow H_n(X) .$$

If X is $(n - 1)$-connected, $n \geq 2$, then h_k is an isomorphism for $k \leq n$ and an epimorphism for $k = n + 1$.

As the minimal model contains information about both homotopy and (co)-homology, it is reasonable to expect information about the rational Hurewicz homomorphism

$$h_n \otimes 1 : \pi_n(X) \otimes \mathbb{Q} \longrightarrow H_n(X) \otimes \mathbb{Q} \cong H_n(X;\mathbb{Q}) .$$

Roughly speaking, the rank of $h_n \otimes 1$ is the number of d-closed algebra generators of degree n in $M(X)$. As there are many ways to choose a system of generators, the precise statement is:

3.13 Theorem: Let X be a simply connected space with cohomology of finite type, and $n \geq 2$ an integer. Then the rank of $h_n \otimes 1$ is the vector space dimension of $Z^n(M_n(X))/Z^n(M_{n-1}(X))$.

Proof: Consider the Postnikov decomposition of X as described in section 3.7 . Let $g : F \longrightarrow X$ be the homotopy-theoretic fibre of $f_{n-1} : X \longrightarrow X_{n-1}$. From the associated exact sequence in homotopy we have that F is $(n-1)$-connected and $g_\# : \pi_n(F) \longrightarrow \pi_n(X)$ is an isomorphism. Therefore the Hurewicz homomorphism $h : \pi_n(F) \longrightarrow H_n(F)$ is an isomorphism. By the naturality of the Hurewicz homomorphism, we have a commutative diagram

$$\begin{tikzcd}
\pi_n(F) \arrow[r, "g_\#", "\cong"'] \arrow[d, "h", "\cong"'] & \pi_n(X) \arrow[d, "h_n"] & & \\
H_n(F) \arrow[r, "g_*"'] & H_n(X) \arrow[r, two heads, "(f_{n-1})_*"'] & H_n(X_{n-1}) \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is a portion of the Serre exact sequence for the homotopy fibration $F \longrightarrow X \longrightarrow X_{n-1}$. Hence we have

$$\mathrm{Im}(h_n) = \mathrm{Im}(g_*) = \mathrm{Ker}(f_{n-1})_*$$

As tensoring with $\mathbb{Q}$ is exact,

$$\begin{aligned}\operatorname{Im}(h_n \otimes 1) &\cong \operatorname{Im}(h_n) \otimes \mathbb{Q} = \operatorname{Ker}(f_{n-1})_* \otimes \mathbb{Q} \\ &\cong \operatorname{Ker}\{(f_{n-1})_* : H_n(X;\mathbb{Q}) \twoheadrightarrow H_n(X_{n-1};\mathbb{Q})\} .\end{aligned}$$

So
$$\begin{aligned}\operatorname{rank}(h_n \otimes 1) &= \dim_{\mathbb{Q}} H_n(X;\mathbb{Q}) - \dim_{\mathbb{Q}} H_n(X_{n-1};\mathbb{Q}) \\ &= \dim_{\mathbb{Q}} H^n(X;\mathbb{Q}) - \dim_{\mathbb{Q}} H^n(X_{n-1};\mathbb{Q}) ,\end{aligned}$$

as rational homology and cohomology have the same (finite) dimension. Now $H^n(X;\mathbb{Q}) \cong H^n(M(X))$, and by Theorem 3.12,

$$H^n(X_{n-1};\mathbb{Q}) \cong H^n(M(X_{n-1})) \cong H^n(M_{n-1}(X)) .$$

As the differential in $M(X)$ is decomposable, we have

$$B^n(M(X)) = B^n(M_{n-1}(X)) ,$$

$$Z^n(M(X)) = Z^n(M_n(X)) .$$

Hence
$$\begin{aligned}\operatorname{rank}(h_n \otimes 1) &= \dim Z^n(M_n(X)) - \dim Z^n(M_{n-1}(X)) \\ &= \dim Z^n(M_n(X))/Z^n(M_{n-1}(X)) .\end{aligned}$$
Q.E.D.

3.14 <u>Remarks</u>: We conclude by noting some further results concerning the minimal model.

By Theorem 3.12, we can identify the free generators of $M(X)$ with the generators of $\pi_*(X) \otimes \mathbb{Q}$. The rational Whitehead products are then given by the quadratic terms in the differential. For example, if $d(x) = a \cdot b$ for free generators x, a, b in $M(X)$, then x is the

Whitehead product of a and b under the above identification. The precise statements can be found in [1]. The proof of this relationship uses the universality of the Whitehead product on the wedge of two spheres.

We have shown how the minimal model can be built from the Postnikov decomposition. There is also a construction associating a rational Postnikov tower to each minimal algebra. Applying this construction to $M(X)$ yields the Postnikov decomposition of X "tensored with $\mathbb{Q}$". This can be used to show that isomorphism types of minimal algebras correspond bijectively with rational homotopy types of spaces. Again, further details may be found in [1].

BIBLIOGRAPHY

[1] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, "Real Homotopy Theory of Kähler Manifolds", Inventiones Math. 29(1975).

[2] E. Friedlander, P. Griffiths and J. Morgan, "Homotopy Theory and Differential Forms", Lecture Notes, Florence, 1972.

[3] S.T. Hu, "Homotopy Theory", Academic Press, 1959.

[4] D.W. Kahn, "The Existence and Applications of Anticommutative Cochain Algebras", Illinois Jour. of Math. 7(1963).

[5] A.T. Lundell and S. Weingram, "The Topology of CW-Complexes", Van Nostrand, 1969.

[6] J.P. May, "Simplicial Objects in Algebraic Topology", Van Nostrand, 1967.

[7] J. Milnor, "The Geometric Realization of a Semi-simplicial Complex", Annals of Math. 65(1957).

[8] J. Milnor, "On Spaces Having the Homotopy Type of a CW-complex", Trans. Amer. Math. Soc. 90(1959).

[9] R.E. Mosher and M.C. Tangora, "Cohomology Operations and Applications in Homotopy Theory", Harper & Row, 1968.

[10] D. Quillen, "Rational Homotopy Theory", Annals of Math 90(1969).

[11] E.H. Spanier, "Algebraic Topology", McGraw-Hill, 1966.

[12] M. Spivak, "Calculus on Manifolds", Benjamin, 1965.

[13] D. Sullivan, "Differential Forms and the Topology of Manifolds", Proceedings of Conference on Manifolds, Tokyo, 1973.

[14] H. Whitney, "On Products in a Complex", Annals of Math. 39(1938).

[15] H. Whitney, "Geometric Integration Theory", Princeton University Press, 1957.