SUB-RINGS OF $C(R^n)$

by

CHEONG KUOON GAN

B.Sc. (Hons), University of Malaya, Malaysia, 1973

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Department of MATHEMATICS

The University of British Columbia
Vancouver 8, Canada

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Supervisor : J.V. Whittaker

Abstract

The content of this thesis contains a study of the rings $C(R^n)$, $L_c(R^n)$, $C^m(R^n)$, $C^\infty(R^n)$, $A(R^n)$ and $P(R^n)$. We obtain the result that no two of the rings above can be isomorphic : in fact we prove the following : if $\phi : A \longrightarrow B$ is a ring homomorphism where A, B are any two of the rings and $A \subset B$, then $\phi(f) = f(p)$ for some $p \in R^n$.

We also characterise $C(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$ as rings.

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Introduction

In one of the first systematic studies of results on the rings of continuous functions, L. Gillman and M. Jerison [5] considered $C(X)$, the ring of continuous functions under pointwise addition and multiplication on an arbitrary topological space X and its sub-ring $C^*(X)$, the ring of bounded functions in $C(X)$. They showed among other things that when X is compact, then it is uniquely determined as a topological space by the ring $C(X)$ or $C^*(X)$; in fact, X is homeomorphic to the space of (fixed) maximal ideals in $C(X)$, with the Stone topology [5, Theorem 4.9] and when X is realcompact, X will be uniquely determined by the space of real maximal ideals on $C(X)$ [5, Theorem 8.3].

Later, in 1966, in her thesis [11], L.P. Su obtained parallel results on the rings $C^m(X)$, the ring of all m-times continuously differentiable functions on a C^m-differentiable n-manifold X, $L_c(X)$ the ring of all functions satisfying a Lipschitz condition on a metric space and $A(x)$, the ring of all analytic functions on a subset X of the complex plane, using the notions of m-realcompactness and L_c-realcompactness.

As part of the thesis, we make a cross-section study on the ring $C(R^n)$ and consider its sub-rings $C^m(R^n)$, $C^\infty(R^n)$, $L_c(R^n)$, $P(R^n)$ and $A(R^n)$, defined in 2.1. We obtain quite incidentally that R^n is uniquely determined as a topological space by each of the sub-rings $L_c(R^n)$, $C^m(R^n)$, $C^\infty(R^n)$ and of course (in accordance with above), $C(R^n)$ since R^n is realcompact.

In our study of these sub-rings of $C(R^n)$, we shall show in Section 2 that these sub-rings enjoy a strong common algebraic property that the set of real maximal ideals (Definition 1.4) is the same as the set of points in R^n.

Another interesting observation is that there are not "many" homomorphisms from one sub-ring into another. For example, how does one define a non-zero ring-homomorphism from $C(R^n)$ into $C^\infty(R^n)$? With the intention of distinguishing these sub-rings among themselves, we prove in Section 3 that there can only be one type of non-zero homomorphism from one sub-ring A into another sub-ring B, where $B \subset A$ (strictly), namely, those of the form $\phi(f) = f(p)$, $f \in A$, $p \in R^n$. This, then, settles the question of whether any two of these sub-rings can be isomorphic. In fact, we distinguish these sub-rings (as rings) not by any algebraic property but by utilising the existence of certain functions in one ring but not in the other.

In Section 4, we consider $C(R)$, $C^m(R)$, $L_c(R)$, $P(R)$ and $C^\infty(R)$ as semi-group under composition of functions. Though the ring structures and semi-group structures on them are in general very different, we show that at least in $C(R)$ and $P(R)$, these two structures agree in one sense, that the group of ring-automorphisms and the group of semi-group-automorphisms are essentially the same.

Finally in Section 5, which is one of our main objects of the thesis, we develop in detail a machinery that enables us to characterise $C(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$ as rings.

1. PRELIMINARIES

1.1 We begin by showing some standard results which will be needed later. We are only interested in real-valued function rings under pointwise addition and multiplication on R^n which contain the constant functions and the projections. Though our specific interest lies in sub-rings of $C(R^n)$, namely, $C^m(R^n)$, $m = 1, 2, \ldots$, $C^\infty(R^n)$, $P(R^n)$, $L_c(R^n)$ and $A(R^n)$ (see 2.1), most of the following results remain true for a ring of real-valued functions on any topological space and in some instances, even for an arbitrary ring with unity.

Throughout this section, we shall let A be a sub-ring of real-valued functions on R^n containing the constant functions and the projections on the axes. The set of constant functions will always be identified with the set of real numbers. The projections are denoted by $u_1, u_2, \ldots, u_n$.

1.2 <u>Proposition</u> : The only non-zero ring homomorphism from R, the reals, into itself is the identity homomorphism.

<u>Proof</u> : See [5, §0.22].

1.3 <u>Proposition</u> : For each $p \in R^n$, $M_p = \{f \in A : f(p) = 0\}$ is a maximal ideal of A.

<u>Proof</u> : If $f \notin M_p$, then $f(p) \neq 0$. For any $g \in A$, $g - g(p)f/f(p) \in M_p$, hence $g = g(p)f/f(p) + g - g(p)f/f(p) \in (f, M_p)$, implying $(f, M_p) = A$. Therefore M_p is a maximal ideal.

1.4.1 <u>Definition</u> : A maximal ideal M in A is called a real ideal if A/M, the residue class ring of A modulo M, is isomorphic to R .

1.4.2 <u>Proposition</u> : Every ideal of the form M_p is a real ideal.

<u>Proof</u> : We note that $\phi(f) = f(p)$ is a ring homomorphism from A onto R since A contains R, the constant functions. So, $A/\ker \phi$ is isomorphic with R . Therefore $M_p = \ker \phi$ is real.

1.5 <u>Proposition</u> : If $\phi : A \longrightarrow R$ is a non-zero real-valued homomorphism on A, then $\phi(r) = r$.

<u>Proof</u> : Observe first that $\phi(g) = \phi(g)\phi(1)$ for all $g \in A$. Since $\phi(g) \neq 0$ for some $g \in A$, we have $\phi(1) = 1$. So the restriction of ϕ to R is a non-zero homomorphism on R ; by Proposition 1.2, $\phi(r) = r$. This completes the proof.

1.6 We see that every non-zero real-valued homomorphism on A is in fact onto, therefore, $A/\ker\phi\phi$, the residue class ring of A modulo $\ker \phi$ is isomorphic to R . Since $\ker \phi$ is necessarily maximal, the above shows that it is a real ideal. Conversely, if M is a real maximal ideal, then for each $f \in A$, we can identify $M(f)$, the residue class of f modulo M, with a real number. One can easily check that $\phi : A \longrightarrow R$ given by $\phi(f) = M(f)$ is a non-zero homomorphism and $M(r) = \phi(r) = r$. Explicitly, we have

1.6.1 <u>Proposition</u> : There exists a one-to-one correspondence between real ideals of A and non-zero real-valued homomorphisms on A .

1.7.1 <u>Definition</u> : We say that a maximal ideal M in A is fixed if there exists $p \in R^n$ such that $M = M_p = \{f \in A, f(p) = 0\}$.

1.7.2 <u>Proposition</u> : Suppose A has the additional property that for $f \in A$, $1/f = f^{-1} \in A$ whenever $Z(f) = \{x \in R^n : f(x) = 0\}$ is empty. Then a maximal ideal M in A is fixed if there exists $f \in M$ whose zero set $Z(f)$ is compact.

<u>Proof</u> : $Z(M) = \{Z(g) : g \in M\}$ has the finite intersection property : for if there exist $g_i \in M$, $i = 1, 2, \ldots, n$ and $\bigcap_{i=1}^{n} Z(g_i) = \emptyset$ then

$$Z\left(\sum_{i=1}^{n} g_i^2\right) = \bigcap_{i=1}^{n} Z(g_i) = \emptyset \text{ implying } \left(\sum_{i=1}^{n} g_i^2\right)^{-1} \in A ,$$

and so

$$1 = \left(\sum_{i=1}^{n} g_i^2\right)^{-1}\left(\sum_{i=1}^{n} g_i^2\right) \in M$$

which is not possible. Obviously if there exist $f \in M$ whose zero-set $Z(f)$ is compact then $\bigcap_{g \in M} Z(g) \supset \bigcap_{g \in M} \{Z(g) \cap Z(f)\} \neq \emptyset$ and there is an $p \in R^n$ such that $g(p) = 0$ for all $g \in M$, i.e. $M \subset M_p$. By maximality, we have $M = M_p$.

1.8 <u>Proposition</u> : If every real ideal in A is fixed, then every non-zero real-valued homomorphism ϕ on A is an evaluation, i.e. there exists $p \in R^n$ such that $\phi(f) = f(p)$ for all $f \in A$.

<u>Proof</u> : From 1.6, $\ker \phi$ is a real ideal. So

$$\ker \phi = M_p = \{f \in A, \quad f(p) = 0\}$$

for some $p \in R^n$. For $f \in A$, $f - \phi(f) \in \ker \phi$ since $\phi(f) \in R$ and $\phi(\phi(f)) = \phi(f)$ (Proposition 1.5), so $f - \phi(f) \in M_p$ and $\phi(f) = f(p)$. This completes the proof.

We remark that p is unique, for if $\phi(f) = f(q)$ for some $q \in R^n$ then $u_i(p) = \phi(u_i) = u_i(q)$, $i = 1, 2, \ldots, n$, hence $p = q$. We also note that the set of non-zero real-valued homomorphisms is the same as the set of evaluations.

1.9 <u>Proposition</u> : If every real ideal in A is fixed, then there is a natural one-to-one correspondence between any two of the following :

(a) R^n ;

(b) $\mathcal{R}_A$, the set of real ideals on A ;

(c) $\Omega(A)$, the set of non-zero real-valued homomorphisms on A.

<u>Proof</u> : The proof follows from Propositions 1.6 and 1.8 and the fact that R^n is equipotent with the set of evaluations on A. Note that the correspondence is given by

$$p \longleftrightarrow M_p \longleftrightarrow \phi_p$$

where $p \in R^n$, and $\phi_p(f) = f(p) = M_p(f)$.

2. SUB-RINGS OF $C(R^n)$ (I)

2.1 We now focus our attention on the rings $C(R^n)$, $C^m(R^n)$, $C^\infty(R^n)$, $P(R^n)$, $L_c(R^n)$ and $A(R^n)$ under pointwise addition and multiplication. We adopt the following definitions. Let $f : R^n \longrightarrow R$ be a continuous function.

(i) f is said to belong to the class C^m if all partial derivatives of f of order (less than or equal to) m exist $(m = 1, 2, \ldots)$ and are continuous.

(ii) f is said to belong to the class C^∞ if all partial derivatives of f of all orders exist.

(iii) f is said to be an L_c-function if it satisfies a Lipschitz condition on each compact subset K of R^n, i.e. there exists a positive number M_K for which $|f(x) - f(y)| \leq M_K||x - y||$ whenever $x, y \in K$.

(iv) f is said to be analytic if f has a power series expansion about each point $y \in R^n$, i.e.

$$f(x) = f(y) + \sum_{i=1}^{n} a_i(x_i - y_i) + \sum_{i,j=1}^{n} a_{ij}(x_i - y_i)(x_j - y_j) +$$

$$+ \sum_{i,j,k=1}^{n} a_{ijk}(x_i - y_i)(x_j - y_j)(x_k - y_k) + \ldots$$

for x in a neighbourhood of y.

In this section we show some properties of

$C(R^n)$, the ring of continuous functions

$C^m(R^n)$, the ring of functions of class C^m

$C^\infty(R^n)$, the ring of functions of class C^∞

$P(R^n)$, the ring of polynomials in n indeterminates $x_1, x_2, \ldots, x_n$

$L_c(R^n)$, the ring of L_c-functions

$A(R^n)$, the ring of analytic functions

which form a chain $C \supset L_c \supset C^1 \supset C^2 \supset \ldots \supset C^m \supset \ldots \supset C^\infty \supset A \supset P$.

2.2 From Proposition 1.3, we know that the rings mentioned above contained real maximal ideals of the form $M_p = \{f : f(p) = 0\}$, one for each $p \in R^n$. We will soon see that in fact all real ideals in C, C^m, C^∞, P, L_c and A are of this form.

2.3 It is trivially true that f^{-1} exists in C, C^m or C^∞ whenever $Z(f)$ is empty, $f \in C$, C^m or C^∞.

In L_c, if $Z(f)$ is empty, $f \in\in L_c$, then $|f|$ is bounded below on each compact subset K of R^n. Therefore

$$|f^{-1}(x) - f^{-1}(y)| = |f(y) - f(x)|/|f(y)f(x)|$$

$$\leq \varepsilon^{-2}|f(y) - f(x)| \leq \varepsilon^{-2}M_K||x - y||$$

for any $x, y \in K$. Hence $f^{-1} \in L_c$.

In A, if $Z(f)$ is empty, then f has a power series expansion at each point of R^n, having non-zero constant term. We can find a unique power series g about the same point such that $f(x)g(x) = 1$ for all x in some neighbourhood of the point. It can be shown that g has positive radius of convergence about any point whenever f has [see 3, page 24] . So $f^{-1} = g \in A$.

By virtue of Proposition 1.7.2, to show that a real ideal M in C, C^m, C^∞, L_c or A is fixed we only have to exhibit an $f \in M$ whose zero-set $Z(f)$ is compact in R^n . Let M be a real ideal in C, C^m, C^∞, L_c or A , then $M = \ker \phi$ for some real-valued homomorphism on the respective rings (Proposition 1.6). Consider $g(x) = x_1^2 + x_2^2 + \ldots + x_n^2$, $x = (x_1, x_2, \ldots, x_n) \in R^n$ which belongs to each of the rings mentioned above. Let $r = \phi(g)$ and $f = g^2 - r^2$. Evidently $f \in M = \ker \phi$ and

$$Z(f) = \left\{ (x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \ldots + x_n^2 = |r| \right\}$$

is an $n-1$ sphere of radius $|r|$ in R^n which is compact.

Unfortunately P does not satisfy the hypothesis in Proposition 1.7.2, nevertheless we can show that every real ideal in P is fixed. Let M be a real ideal in $P(R^n)$. As before $M = \ker \phi$ for some homomorphism $\phi : R^n \longrightarrow R$. Let $r_i = \phi(u_i)$, u_i being the i-th projection $u_i(x_1, x_2, \ldots, x_n) = x_i$, $i = 1, 2, \ldots, n$, and $r = (r_1, r_2, \ldots, r_n)$. For each $f \in \ker \phi$, $f(r) = f(\phi(u_1), \ldots, \phi(u_n))$. Since each f is a finite sum of finite products of $x_1, x_2, \ldots, x_n$ and ϕ maps real numbers

identically onto themselves and preserves addition and multiplication, $f(\phi(u_1), \phi(u_2), \ldots, \phi(u_n)) = \phi(f)$. So $f(r) = 0$ for all $f \varepsilon \ker \phi$ and $\ker \phi = M = M_r$.

All these show that the following proposition is true.

2.3.1 <u>Proposition</u> : Every real ideal in C, C^m, C^∞, P, L_c and A is fixed.

2.4 Hence the only real-valued non-zero homomorphisms on C, C^m, C^∞, L_c, P and A are the evaluations, by Proposition 1.8. In the light of Proposition 1.9, the set of real ideal in each of these rings is in one-to-one correspondence with R^n, the correspondence being

$$M_p \longleftrightarrow p ,$$

$p \varepsilon R^n$.

3. SUB-RINGS OF $C(R^n)$ (II)

3.1 We now know that C, C^m, C^∞, P L_c and A all enjoy a common character described by one of the following :

(a) Every real ideal is fixed.

(b) Every real-valued non-zero homomorphism is an evaluation.

(c) There is a one-to-one correspondence between any two of R^n, $\mathcal{R}$ (the real ideals) and Ω (the non-zero real-valued homomorphisms).

Later we show further that each of the rings C, C^m, C^∞ and L_c completely determines R^n as a topological space. This non-trivial resemblance among the rings considered leads us to ask if any two of them can be isomorphic.

We show that no two of the rings C, C^m, C^∞, P, L_c and A can be isomorphic; in fact the only non-zero homomorphisms from A into B (where $B \subset A$ strictly, A, B are any of the rings mentioned above) are the evaluations.

3.2.1 <u>Theorem</u> : Let A and B be sub-rings of $C(R^n)$ containing R and u_i, $i = 1, 2, \ldots, n$. Suppose every real ideal in A is fixed. Then every non-zero homomorphism $\phi : A \longrightarrow B$ is given by $\phi(f) = f \circ \tau$ for some unique $\tau : R^n \longrightarrow R^n$.

<u>Proof</u> : We define $\tau : R^n \longrightarrow R^n$ in the following manner. For each

$x \in R^n$, let $\theta_x : g \longrightarrow g(x)$ be a non-zero real-valued homomorphism on B. Since $\theta_x \circ \phi$ is a non-zero real-valued homomorphism on A, Proposition 1.8 applies and so $\theta_x \circ \phi(f) = f(y)$, $f \in A$ for some fixed $y \in R^n$ (depending on x). Define $\tau(x) = y$. Then

$$(\phi f)(x) = \theta_x(\phi f) = \theta_x \circ \phi(f) = f(y) = f \circ \tau(x) , \qquad x \in R^n ,$$

hence $\phi(f) = f \circ \tau$, $f \in A$.

If $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, we see that τ is uniquely determined by $\phi(u_i) = u_i \circ \tau = \tau_i$, $i = 1, 2, \ldots, n$. This completes the proof.

3.2.2 <u>Remark</u> : Whenever such a τ exists, it must satisfy the condition that $f \circ \tau \in B$ for all $f \in A$. In particular $\tau_i = u_i \circ \tau \in B$, $i = 1, 2, \ldots, n$.

3.3 We consider now a homomorphism $\phi : C(R^n) \longrightarrow B$ where B is any of C^m, C^∞, A and P. From Theorem 3.2.1, we have $\phi(f) = f \circ \tau$ for all continuous f.

3.3.1 <u>Lemma</u> : If $\phi : C(R^n) \longrightarrow B$ is a non-zero homomorphism, then $\phi(f) = f \circ \tau$ for some constant function $\tau : R^n \longrightarrow R^n$.

<u>Proof</u> : We need only to show $\tau_i : R^n \longrightarrow R$, $i = 1, 2, \ldots, n$ are constant functions where $\tau(x) = (\tau_1(x), \tau_2(x), \ldots, \tau_n(x))$. Suppose τ_i is not a constant function for some i. Since $\tau_i = u_i \circ \tau \in B \subset C^1(R^n)$, there is a point $t = (t_1, t_2, \ldots, t_n) \in R^n$ at which $\partial\tau_i/\partial x_j$ is not zero for some $1 \leq j \leq n$. Let $E_j^n = \left\{(t_1, t_2, \ldots, t_{j-1}, s, t_{j+1}, \ldots, t_n) : s \in R\right\}$,

then E_j^n is connected, hence $\tau_i(E_j^n)$ is an interval in R containing $\tau_i(t)$. Define a continuous function $g : R^n \longrightarrow R$ by

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} (x_i - \tau_i(t))\sin \dfrac{1}{x_i - \tau_i(x)}, & x_i \neq \tau_i(t) \\ 0, & x_i = \tau_i(t). \end{cases}$$

We are going to show that the j-th partial derivative of $g\circ\tau$ does not exist.

Let $y = (t_1, t_2, \ldots, t_{j-1}, t_j+h, t_{j+1}, \ldots, t_n) \in E_j^n$, then

$$\frac{1}{h}\left\{g\circ\tau(y) - g\circ\tau(t)\right\} =$$

$$= \frac{1}{h}\left\{g(\tau_1(y), \tau_2(y), \ldots, \tau_n(y)) - g(\tau_1(t), \tau_2(t), \ldots, \tau_n(t))\right\}$$

$$= \frac{1}{h}(\tau_i(y) - \tau_i(t)) \sin \frac{1}{\tau_i(y) - \tau_i(t)}, \qquad \tau_i(y) \neq \tau_i(t).$$

Since τ_i is continuous we have $\tau_i(y) \longrightarrow \tau_i(t)$ as $h \longrightarrow 0$, therefore

$$\lim_{\varepsilon\to 0, |h|\leq\varepsilon} \sup \frac{1}{h}(\tau_i(y) - \tau_i(t)) \sin \frac{1}{\tau_i(y) - \tau_i(t)}$$

$$= \lim_{h\to 0} \frac{1}{h}(\tau_i(y) - \tau_i(t)) = \frac{\partial\tau_i}{\partial x_j}(t)$$

$$\lim_{\varepsilon\to 0, |h|\leq\varepsilon} \inf \frac{1}{h}(\tau_i(y) - \tau_i(t)) \sin \frac{1}{\tau_i(y) - \tau_i(t)}$$

$$= \lim_{h\to 0} \frac{1}{h}(\tau_i(y) - \tau_i(t)) = -\frac{\partial\tau_i}{\partial x_j}(t)$$

implying that $\lim_{h\to 0} \frac{1}{h}(g\circ\tau(y) - g\circ\tau(t))$ and hence $(\partial/\partial x_j)(g\circ\tau)$ does not exist at t. This shows that $g\circ\tau \notin B$, contradicting Remark 3.2.2. So τ_i $(1 \leq i \leq n)$ must be a constant function, and the proof is complete.

3.3.2 <u>Lemma</u> : If $\phi : C(R^n) \longrightarrow L_c(R^n)$ is a non-zero homomorphism then $\phi(f) = f\circ\tau$, $f \in C(R^n)$ for some constant $\tau : R^n \longrightarrow R^n$.

<u>Proof</u> : Appealing again to Theorem 3.2.1, we just have to show that such a τ must be a constant function. Assume the contrary that $\tau(x) = (\tau_1(x), \tau_2(x), \ldots, \tau_n(x))$ where $\tau_i : R^n \longrightarrow R$, $i = 1, 2, \ldots, n$ and there is $1 \leq j \leq n$ for which τ_j is not a constant function.

For each $y \in R^n$, let $x_k \in\in R^n$ $(k = 1, 2, \ldots)$ be any sequence converging to y. Consider the sequence

$$\{\sigma_k\} = \left\{ \frac{|\tau_j(x_k) - \tau_j(y)|}{||x_k - y||} \right\}$$

Let $m_x = \limsup_{k\to\infty} \sigma_k$. Since $\{x_k\}_{k=1}^{\infty} \cup \{y\}$ is a compact set in R^n and $\tau_j \in L_c(R^n)$ (Remark 3.2.2) there is an M for which

$$|\tau_j(x_k) - \tau_j(y)| \leq M||x_k - y|| \quad \text{for all } k = 1, 2, \ldots .$$

Hence the sequence $\{\sigma_k\}$ is bounded above and $0 \leq m_x < \infty$. Suppose now that $m_x = 0$, then,

$$0 \leq \liminf_{k\to\infty} \sigma_k \leq \limsup_{k\to\infty} \sigma_k = m_x = 0$$

implying $\lim_{k\to\infty} \sigma_k$ exists and $\lim_{k\to\infty} \sigma_k = 0$. Hence if for every sequence $\{x_k\}$

converging to y , $m_x = 0$, then $\partial\tau_j/\partial x_i$ exists at y and $(\partial\tau_j/\partial x_i)(y) = \lim_{k\to\infty} \sigma_k = 0$, $i = 1, 2, \ldots, n$. Since τ_j is non-constant there is a point $t \in R^n$ at which either $\partial\tau_j/\partial x_i$ does not exist for some i or $(\partial\tau_j/\partial x_i)(t) \neq 0$ for some i. So there is a sequence $x_k \longrightarrow t$ such that

$$\limsup_{k \to \infty} \frac{|\tau_j(x_k) - \tau_j(t)|}{||x_k - t||} = m > 0 .$$

We choose a subsequence $\{x_\ell\}$ of $\{x_k\}$ such that

$$\lim_{\ell\to\infty} \frac{|\tau_j(x_\ell) - \tau_j(t)|}{||x_\ell - t||} = \limsup_{k\to \infty} \frac{|\tau_j(x_k) - \tau_j(t)|}{||x_k - t||} = m > 0$$

and $|\tau_j(x_\ell) - \tau_j(t)|/||x - t|| \geq \frac{m}{2}$ for all ℓ . We now show that on the compact set $\{x_\ell\} \cup \{t\} = K$, $h\circ\tau$ does not satisfy a Lipschitz condition for the continuous function

$$h(x) = |x_j - \tau_j(t)|^{1/2}, \qquad x = (x_1, x_2, \ldots, x_n) \in R^n .$$

Indeed

$$\frac{|h\circ\tau(x_\ell) - h\circ\tau(t)|}{||x_\ell - t||} = \frac{|\tau_j(x_\ell) - \tau_j(t)|^{1/2}}{||\tau_j(x_\ell) - \tau_j(t)||} \cdot \frac{|\tau_j(x_\ell) - \tau_j(t)|}{||x_\ell - t||}$$

$$\geq \frac{m}{2} \cdot \frac{1}{|\tau_j(x_\ell) - \tau_j(t)|^{1/2}}$$

and the right hand side can be arbitrarily large since $x_\ell \longrightarrow t$, τ_j is continuous and $\tau_j(x_\ell) \longrightarrow \tau_j(t)$ as $\ell \to \infty$. Hence $h\circ\tau \notin L_c(R^n)$ contradicting Remark 3.2.2. So τ_i must be a constant function for $i = 1, 2, \ldots, n$.

3.3.3 <u>Theorem</u> : The only non-zero homomorphisms from $C(R^n)$ into $L_c(R^n)$, $C^m(R^n)$, $C^\infty(R^n)$, $A(R^n)$ or $P(R^n)$ are the evaluations.

<u>Proof</u> : This follows immediately from Lemmas 3.3.1 and 3.3.2.

3.3.4 <u>Corollary</u> : C is not isomorphic to any of the rings L_c, C^m, C^∞, A or P .

<u>Proof</u> : It is a consequence of Theorem 3.3.3.

3.4 We note that evidently there cannot be any isomorphism between $C(R^n)$ and $P(R^n)$ since $P(R^n)$ is an integral domain and $C(R^n)$ is not. We shall use this algebraic property of $P(R^n)$ to give another proof that the only non-trivial homomorphisms from $C(R^n)$ into $P(R^n)$ are the evaluations.

3.4.1 <u>Lemma</u> : If $\phi : C(R^n) \longrightarrow A$ is a non-zero homomorphism where A is a sub-ring of $C(R^n)$ containing R, then $\phi[C(R^n)] \cong C(F)$ for some closed set $F \subset R^n$.

<u>Proof</u> : There exists $\tau : R^n \longrightarrow R^n$ such that $\phi(f) = f\circ\tau$, $f \in C(R^n)$. Let $F = Cl_{R^n}\{\tau(R^n)\}$ and define $\alpha : \phi[C(R^n)] \longrightarrow C(F)$ by $\alpha(\phi f) = f|F$. If $\phi f = \phi g$, $f, g \in C(R^n)$, then $f\circ\tau(x) = g\circ\tau(x)$, $x \in R^n$ implying f and g agree on $\tau(R^n)$, aand hence, on $Cl_{R^n}\{\tau(R^n)\}$ since R^n is a Hausdorff space. This shows that $\phi f = \phi g$ implies $f|F = g|F$ so α is well-defined. It is easy to check that α is a ring homomorphism.

Now if $\alpha(\phi f) = 0$ then $f|F = 0$ implying $\phi f = f\circ\tau = 0$. And if

$g \in C(F)$, then there exists $f \in C(R^n)$ such that $f|F = g$ and $\alpha(\phi f) = f|F = g$. We have proved that α is an isomorphism.

3.4.2 <u>Theorem</u> : The only non-zero homomorphisms ϕ from $C(R^n)$ to $P(R^n)$ are the evaluations.

<u>Proof</u> : For each $f \in C(R^n)$,

$$0 = \phi\{(f - |f|)(f + |f|)\} = \phi(f - |f|)\phi(f + |f|) .$$

Since P is an integral domain $\phi(f) = \phi(|f|)$ or $\phi(f) = -\phi(|f|)$. From the fact that ϕ sends positive elements to positive elements [5, §1.6], $\phi[C(R^n)]$ is a totally ordered ring. By Lemma 3.4.1, $\phi[C(R^n)] \cong C(F)$ for some closed set $F = Cl_{R^n}\{\tau(R^n)\}$ in R. F must consist of a single point $p \in R^n$, for if F has more than one point we can easily construct a continuous function on F which is not comparable with 0. Hence $\phi(f) = f \circ \tau = f(p)$, $f \in C(R^n)$.

3.5 According to our tool, the absence of non-zero homomorphisms from one ring into another, other than the evaluations depends not so much on the ring properties but on the existence of certain functions. We now observe that there exist functions which are L_c-functions but which are not differentiable, e.g. $f(x) = |x_1|$; and there exist functions which are of class C^m $(m \geq 1)$ but not of class C^{m+r} $(r \geq 1)$, e.g. $f(x) = |x_i| \cdot x_i^m$. It will be shown that these functions serve to prove that the only non-zero homomorphisms from L_c into C^m, C^∞, A or P and from C^m into C^{m+r}, C^∞, A or P are evaluations. The same method of discrimination applies

among C^∞, A and P .

3.5.1 <u>Lemma</u> : Let B be any of the rings $C^m(R^n)$, $C^\infty(R^n)$, $A(R^n)$ and $P(R^n)$, $\phi : L_c(R^n) \longrightarrow B$, a non-zero homomorphism. Then there exists a constant function $\tau : R^n \longrightarrow R^n$ such that $\phi(f) = f\circ\tau$, $f \in L_c(R^n)$.

<u>Proof</u> : The existence of τ is shown in Theorem 3.2.1. We wish to prove that if $\tau(x) = (\tau_1(x), \tau_2(x), \ldots, \tau_n(x))$ where $\tau_i : R^n \longrightarrow R$ $(i = 1, 2, \ldots, n)$, then τ_i $(i = 1, 2, \ldots, n)$ are constant functions.

Suppose τ_i is not constant for some $1 \leq i \leq n$. Since $\tau_i = u_i\circ\tau = \phi(u_i) \in B$, τ_i has partial derivatives. Moreover, for some j, $1 \leq j \leq n$, $\partial\tau_i/\partial x_j$ is not zero at some point of R^n . Choose $t \in R^n$ for which $(\partial\tau_i/\partial x_j)(t) \neq 0$ and define

$$E_j^n = \left\{(t_1, t_2, \ldots, t_{j-1}, s, t_{j+1}, \ldots, t_n) : s \in R\right\} .$$

Since E_j^n is connected, $\tau_i(E_j^n)$ is an interval in R containing $\tau_i(t)$ as an interior point ($\tau_i(t)$ cannot be an end-point of the interval, for otherwise it is a local extremum point implying $(\partial\tau_i/\partial x_j)(t) = 0$). We define $g \in L_c(R^n)$ by $g(x_1, x_2, \ldots, x_n) = |x_i - |\tau_i(t)|$. We shall show that $g\circ\tau \notin C^1$. Let $y = (t_1, t_2, \ldots, t_{j-1}, t_j+h, t_{j+1}, \ldots, t_n) \in E_j^n$, then

$$\frac{1}{h}\,(g\circ\tau(y) - g\circ\tau(t))$$

$$= \frac{1}{h}\left\{g(\tau_1(y), \ldots, \tau_n(y)) - g(\tau_1(t), \ldots, \tau_n(t))\right\}$$

$$= \frac{1}{h}\,|\tau_i(y) - \tau_i(t)| = \frac{|\tau_i(y) - \tau_i(t)|}{\tau_i(y) - \tau_i(t)} \cdot \frac{\tau_i(y) - \tau_i(t)}{h} .$$

We see that

$$\lim_{\substack{h\to 0\\ \tau_i(y)\to\tau_i(t)^+}} \frac{g\circ\tau(y) - g\circ\tau(t)}{h} = \frac{\partial\tau_i}{\partial x_j}(t)$$

$$\lim_{\substack{h\to 0\\ \tau_i(y)\to\tau_i(t)^-}} \frac{g\circ\tau(y) - g\circ\tau(t)}{h} = -\frac{\partial\tau_i}{\partial x_j}(t) .$$

This shows that the j-th partial derivative of $g\circ\tau$ does not exist. Consequently $g\circ\tau \notin C^1$, i.e. $g\circ\tau \notin B$ which is a contradiction by Remark 3.2.2. Hence τ is a constant function.

3.5.2 <u>Lemma</u> : Let B be one of $C^{m+r}(R^n)$ $(r \geq 1)$, $C^\infty(R^n)$, $A(R^n)$ and $P(R^n)$, $\phi : C^m(R^n) \longrightarrow B$ a non-zero homomorphism. Then $\phi(f) = f\circ\tau$, $f \in C^m(R^n)$ for some constant $\tau : R^n \longrightarrow R^n$.

<u>Proof</u> : Following the method of proof in Theorem 3.5.1, we define $g \in C^m$ by $g(x_1, x_2, \ldots, x_n) = |x_i - \tau_i(t)|\cdot(x_i - \tau_i(x))^m$. We wish to show that $g\circ\tau \notin C^{m+1}$. First, we know that $g\circ\tau$ has all partial derivatives of first order and

$$\frac{\partial}{\partial x_j}(g\circ\tau) = \frac{\partial(g\circ\tau)}{\partial\tau_1}\cdot\frac{\partial\tau_1}{\partial x_j} + \frac{\partial(g\circ\tau)}{\partial\tau_{22}}\cdot\frac{\partial\tau_2}{\partial x_j} + \ldots + \frac{\partial(g\circ\tau)}{\partial\tau_n}\cdot\frac{\partial\tau_n}{\partial x_j}$$

$$= \frac{\partial(g\circ\tau)}{\partial\tau_i}\cdot\frac{\partial\tau_i}{\partial x_j} \qquad \text{since } \frac{\partial(g\circ\tau)}{\partial\tau_k} = 0 \text{ if } k \neq i .$$

$$= (m+1)|\tau_i(x) - \tau_i(t)|\cdot(\tau_i(x) - \tau_i(t))^{m-1}\frac{\partial\tau_i}{\partial x_j} .$$

We can continue differentiating with respect to x_j until we obtain,

$$\frac{\partial^m}{\partial x_j^m}(g\circ\tau) = (m+1)!\,|\tau_i(x) - \tau_i(t)|\left(\frac{\partial\tau_i}{\partial x_j}\right)^m + h(x), \quad x \in R^n$$

where $h(x)$ is differentiable at all points in R^n including $\tau_i(t)$ and is a polynomial in $\partial^k\tau_i/\partial x_i^k$ $(k = 1, 2, \ldots, m)$ with coefficients of the form $C\cdot|\tau_i(x) - \tau_i(t)|\cdot(\tau_i(x) - \tau_i(t))^\ell$ $(\ell = 1, 2, \ldots, m-1)$. Now setting up the j-th partial differential quotient for

$$\frac{\partial^m}{\partial x_j^m}(g\circ\tau) = \partial_j^m(g\circ\tau) ,$$

we see that for $y = (t_1, t_2, \ldots, t_{j-1}, t_j+s, t_{j+1}, \ldots, t_n) \in E_j^n$

$$\frac{1}{s}\left\{\partial_j^m\, g\circ\tau(y) - \partial_j^m\, g\circ\tau(t)\right\}$$

$$= \frac{1}{s}\left\{(m+1)!\,|\tau_i(y) - \tau_i(t)|\left(\frac{\partial\tau_i}{\partial x_j}\right)^m + h(y) - h(t)\right\}$$

$$= (m+1)!\left(\frac{\partial\tau_i}{\partial x_j}\right)^m \frac{|\tau_i(y) - \tau_i(t)|}{\tau_i(y) - \tau_i(t)}\cdot\frac{\tau_i(y) - \tau_i(t)}{s} + \frac{h(y) - h(t)}{s}$$

and

$$\lim_{\substack{s\to 0\\ \tau_i(y)\to\tau_i(t)^+}} \frac{1}{s}\left\{\partial_j^m\, g\circ\tau(y) - \partial_j^m\, g\circ\tau(t)\right\}$$

$$= (m+1)!\left(\frac{\partial\tau_i}{\partial x_j}\right)^{m+1}(t) + \frac{\partial h}{\partial x_j}(t) = (m+1)!\left(\frac{\partial\tau_i}{\partial x_j}\right)^{m+1}(t) + \frac{\partial h}{\partial x_j}(t)$$

$$\lim_{\substack{s\to 0\\ \tau_i(y)\to\tau_i(t)^-}} \frac{1}{s}\left\{\partial_j^m\, g\circ\tau(y) - \partial_j^m\, g\circ\tau(t)\right\}$$

$$= -(m+1)!\left(\frac{\partial\tau_i}{\partial x_j}\right)^{m+1}(t) + \frac{\partial h}{\partial x_j}(t) .$$

This shows that the j-th partial derivative of $g\circ\tau$ of order $m+1$ does not exist hence $g\circ\tau \notin C^{m+1}$. By the same argument as before, τ is a constant function.

3.5.3 <u>Lemma</u> : If $\tau : R^n \longrightarrow R^n$ and $f\circ\tau \in A(R^n)$ for each $f \in C^\infty(R^n)$ then τ_i $(i = 1, 2, \ldots, n)$ are constant functions, where $\tau(x) = (\tau_1(x), \tau_2(x), \ldots, \tau_n(x))$.

<u>Proof</u> : Suppose τ_i is non-constant, then $\tau_i = u_i\circ\tau \in A$ has a power series at some point $t \in R^n$ and for some j, $(\partial\tau_i/\partial x_j)(t) \neq 0$, i.e.,

$$\tau_i(x) = a_0 + \sum_{i=1}^{n} a_i(x_i - t_i) + \sum_{i,j=1}^{n} a_{ij}(x_i - t_i)(x_j - t_j) + \ldots ,$$

$||x - t|| < \varepsilon$. We now consider $x \in R^n$ of the form

$$x = (t_1, t_2, \ldots, t_{j-1}, t_j+x_j, t_{j+1}, \ldots, t_n),$$

then

$$\tau_i(x) = a_0 + a_jx_j + a_{jj}x_j^2 + \ldots , \qquad |x_j| < \varepsilon_1 .$$

Let

$$\sigma(x_j) = \tau_i(x) - a_0 = a_jx_j + a_{jj}x_j^2 + \ldots$$

where $a_j = (\partial\tau_i/\partial x_j)(t) \neq 0$. Then σ has a composition inverse σ^{-1} such that $\sigma\sigma^{-1}(y_j) = y_j$, $\sigma^{-1}\sigma(x_j) = x_j$ and $\sigma^{-1}(y_j)$ has a power series expansion about 0 [3, Proposition 7.1, I.1.7. and Proposition 9.1, I.2.9] . Define $g(x) = \exp\{-(x_i-a_0)^2\}$ if $x_i \neq a_0$ and $g(x) = 0$ if $x_i = a_0$, then $g \in C^\infty(R^n)$, therefore $g\circ\tau \in A(R^n)$. Now

$$\exp\{-\sigma(x_j)^2\} = \exp\{-(\tau_i(x)-a_0)^2\} = g\circ\tau(x)$$

$$= b_0 + b_j x_j + b_{jj} x_j^2 + \dots , \qquad |x_j| < \varepsilon_2 .$$

By [3, Proposition 5.1, I.2.5], composition of two convergent series has a positive radius of convergent, so,

$$\exp\{- y_j^2\} = \exp\{-\sigma\sigma^{-1}(y_j)^2\}$$

$$= b_0 + b_j(\sigma^{-1}(y_j)) + b_{jj}(\sigma^{-1}(y_j))^2 + \dots , \qquad |y_j| < \varepsilon_3$$

which contradict the fact that $\exp\{- y_j^2\}$ has no power series expansion about 0 . Therefore τ_i is a constant.

3.5.4 <u>Lemma</u> : If for all $f \in A(R^n)$ (or $C^\infty(R^n)$), $f\circ\tau \in P(R^n)$ for some $\tau : R^n \longrightarrow R^n$, then τ is a constant.

<u>Proof</u> : Since $g(x) = \sin x_i$, $x = (x_1, x_2, \dots, x_n) \in R^n$, belongs to $A(R^n)$ (or $C^\infty(R^n)$), we have

$$g\circ\tau(x) = g(\tau_1(x), \tau_2(x), \dots, \tau_n(x)) = \sin \tau_i(x) \in P(R^n).$$

Since a non-constant polynomial in n variables is unbounded $\sin \tau_i(x)$, being a bounded function must be a constant. Hence τ_i is a constant for each $1 \leq i \leq n$.

3.5.5 <u>Theorem</u> : Let A, B denote any of the rings C, C^m, L_c, C^∞, A and P, and $B \subset A$ (strictly). Then the only non-zero homomorphisms $\phi : A \longrightarrow B$ are the evaluations.

<u>Proof</u> : Theorem 3.2.1 proves that ϕ is of the form $\phi(f) = f \circ \tau$ for some $\tau : R^n \longrightarrow R^n$. Lemmas 3.3.1, 3.3.2, 3.5.1-3.5.4 show that such a τ is a constant function. If we denote $p = \tau(0) \in R^n$, then we have as required $\phi(f) = f(p)$, $f \in A$.

3.5.6 <u>Corollary</u> : No two of the above-mentioned rings can be isomorphic.

3.6 Quite unlike the results in 3.3 and 3.5, if $A \subset B$, A, B any of the rings in consideration, then there is a wealth of homomorphisms $\phi : A \longrightarrow B$. In fact if f_i, $i = 1, 2, \ldots, n$, are any n functions in B, then $\tau = (f_1, f_2, \ldots, f_n)$ induces a homomorphism $\phi : A \longrightarrow B$ given by $\phi(f) = f \circ \tau$, $f \in A$.

4. SEMI-GROUP STRUCTURES

4.1 In this section, we would like to consider the case $n = 1$, that is, to restrict our attention to $C(R)$, $L_c(R)$, $C^m(R)$, $C^\infty(R)$, $A(R)$ and $P(R)$. Of course all the results obtained previously remain true, with u_1 taken to be the identity function on R.

Let End A denote the semi-group of ring-homomorphisms on A where A is one of the groups mentioned above. Since each ring is closed under composition of functions, we can regard A as semi-group $(A, \circ)$ under composition.

4.1.1 <u>Theorem</u> : End A is anti-isomorphic to the semi-group $(A, \circ)$.

<u>Proof</u> : Each $\phi \in$ End A is of the form $\phi(f) = f\circ\tau$ for some unique $\tau = \phi(u_1) \in A$ (Theorem 3.2.1). Conversely each $\tau \in A$ induces a ring-homomorphism $\phi : A \longrightarrow A$ given by $\phi(f) = f\circ\tau$. Hence $\phi \longrightarrow \phi(u_1)$ is one-to-one and onto. Since

$$\phi_1\circ\phi_2(f) = \phi_1(f\circ\tau_2) = f\circ\tau_2\circ\tau_1$$

we have $(\phi_1\circ\phi_2)(u_1) = \phi_2(u_1)\circ\phi_1(u_1)$ and the theorem is proved.

4.2 L.E. Pursell showed in [9] that ϕ is an automorphism on $P(R)$ if and only if there exist $a\ (\neq 0)$, $b \in R$ such that $\phi(f) = f(ax + b)$ [9, Theorem 5'] and $\phi \longrightarrow \phi(x)$ is an anti-isomorphism from the group of ring-automorphisms on $P(R)$ onto the group of all non-singular affine

transformations on R [9, Theorem 6]. We can show that these results follow from the following theorem.

4.2.1 <u>Theorem</u> : Let A be any of the rings $C(R)$, $L_c(R)$, $C^m(R)$, $C^\infty(R)$, $A(R)$ and $P(R)$. Then Aut A, the group of ring-automorphisms of A is anti-isomorphic with the group

$$A' = \{\tau \in A : \text{there exists } \sigma \in A \text{ such that } \sigma\circ\tau = \tau\circ\sigma = u_1 \}$$

under composition.

<u>Proof</u> : For each $\phi \in \text{End } A$, there exist $\tau, \sigma \in A$ such that $\phi(f) = f\circ\tau$ and $\phi^{-1}(f) = f\circ\sigma$. Since

$$\phi^{-1}(\sigma\circ\tau) = \sigma\circ\tau\circ\sigma = \sigma\circ u_1 = \sigma = u_1\circ\sigma = \phi^{-1}(u_1)$$

we have $\sigma\circ\tau = u_1$. On the other hand

$$u_1 = \phi^{-1}\phi(u_1) = u_1\circ\tau\circ\sigma = \tau\circ\sigma .$$

Hence $\tau = \phi(u_1) \in A'$. Conversely if $\tau \in A'$, we can define a ring-endomorphism by $\phi(f) = f\circ\tau$. Since $f\circ\tau = g\circ\tau$ implies

$$f = f\circ u_1 = f\circ\tau\circ\sigma = g\circ\tau\circ\sigma = g\circ u_1 = g$$

and if $g \in A'$, then $\phi(g\circ\sigma) = g\circ\sigma\circ\tau = g$, this endomorphism ϕ is in fact an automorphism. Since $\phi_\mu\circ\phi_\nu \longrightarrow \phi_\nu(u_1)\circ\phi_\mu(u_1)$, $\phi \longrightarrow \phi(u_1)$ is an anti-isomorphism.

<u>Remark</u> : (i) In $P(R)$, $(P(R))'$ is the set of polynomials which have compositional inverses, so $(P(R))' = \{ax + b : a, b \in R, a \neq 0 \}$.

(ii) In $C(R)$, $(C(R))'$ is the set of all homeomorphisms on R.

(iii) In $A(R)$, $(A(R))'$ is the set of all functions

$$f(x) = a_1x + a_2x^2 + \ldots, \quad x \in R \quad \text{where} \quad a_1 \neq 0 .$$

4.3 Very often, one encounters a set on which two distinct algebraic structures can be defined. As an expected observation, the two algebraic entities always have different and unrelated characters. And this is the case with $C(R)$, $L_c(R)$, $C^m(R)$, $C^\infty(R)$ or $P(R)$, considered as a ring under pointwise multiplication and addition and as a semi-group under composition. We show eventually in this section that at least for $C(R)$ and $P(R)$, these two structures concur in one instance, namely, the group of semi-group automorphisms and the group of ring automorphisms are essentially the same. We first show a result on automorphisms of semi-group of a class of functions. We denote a semi-group of real-valued functions on R which contains the set of constant functions by A, while keeping in mind that we are actually interested in C, L_c, C^m, C^∞ and P, and in particular C and P.

4.3.1 <u>Definition</u> : An element $z \in A$ is said to be a left zero if $z \circ f = z$ for each $f \in A$.

4.3.2 <u>Definition</u> : An automorphism ϕ on A is called inner if there exists $h \in A$ whose compositional inverse $h^{-1} \in A$ exists and $\phi(f) = h \circ f \circ h^{-1}$ for each $f \in A$.

4.4.1 Lemma : The set of left zeros in A is precisely the set of constant functions, R .

Proof : Suppose h is a left zero, then for any arbitrary $x, y \in R$, $h(x) = (h \circ y)(x) = h(y)$. Therefore h is a constant function. The fact that the constant functions are left zeros is trivial.

4.4.2 Lemma : If ϕ is an automorphism on A, then there is a bijection $h : R \longrightarrow R$ such that for each $f \in A$, $\phi(f) = h \circ f \circ h^{-1}$.

Proof : If $x \in R$, then $\phi(x) \circ f = \phi(x) \circ \phi\phi^{-1}(f) = \phi(x \circ \phi^{-1}(f)) = \phi(x)$ implies that $\phi(x)$ is a left zero. By Lemma 4.4.1, $\phi(x) = y$ for some $y \in R$. Define $h : R \longrightarrow R$ by $h(x) = y$, then h is a bijection since ϕ is an automorphism. Moreover $\phi(x) = h(x)$ for $x \in R$ and

$$\phi f \circ h(x) = \phi f \circ \phi x = \phi(f \circ x) = \phi(f(x)) = h(f(x)) = h \circ f(x) .$$

Hence $\phi(f) = h \circ f \circ h^{-1}$.

4.4.3 Lemma : If $h : R \longrightarrow R$ is a continuous bijection then, h is strictly monotonic.

Proof : Immediate.

4.5.1 Theorem : Every automorphism of $C(R)$ is inner.

Proof : By Lemma 4.4.2, $\phi(f) = h \circ f \circ h^{-1}$ for some bijection h . We need to show that h is continuous. Let $x_o \in R$ and $\varepsilon > 0$ be given. Choose a number $y_o \neq h(x_o)$, and define $g \in C(R)$ as follows :

$$g(x) = \begin{cases} y_o - \frac{1}{\varepsilon}(y_o - h(x_o))\cdot|x - h(x_o)|, & |x - h(x_o)| < \varepsilon \\ h(x_o) , & |x - h(x_o)| \geq \varepsilon . \end{cases}$$

Since ϕ is an automorphism, there exists $f \in C(R)$ whose image $\phi(f) = g$. Moreover $y_o = gh(x_o) = hf(x_o)$ implies $f(x_o) \neq x_o$; by continuity there exists $\delta > 0$ such that $f(x) \neq x_o$ whenever $|x - x_o| < \delta$. So $gh(x) = hf(x) \neq h(x_o)$ whenever $|x - x_o| < \delta$, knowing that h is one-to-one. From the definition of g, we conclude that $|h(x) - h(x_o)| < \varepsilon$ whenever $|x - x_o| < \delta$. Since x_o is arbitrary, h is continuous on R .

4.5.2 <u>Theorem</u> : Every automorphism ϕ of $C^1(R)$ is of the form $\phi(f) = h\circ f\circ h^{-1}$ for some differentiable function h .

<u>Proof</u> : By Lemma 4.4.2 and Theorem 4.5.1, $\phi(f) = h\circ f\circ h^{-1}$ for some continuous h. We would like to show that h indeed has a derivative on R.

Lemma 4.4.3 states that h is monotonic and so is h^{-1}. Since a monotonic function has a finite derivative almost everywhere [7, Theorem 4, p. 211], there is a point x_o at which h has a finite derivative. This last fact will allow us to conclude that h has a finite derivative at every other point of R . Let $x_1 \in R$, then

$$(1) \qquad \frac{h(x) - h(x_1)}{x - x_1} = \frac{hf(y) - hf(x_o)}{x - x_1}$$

$$= \frac{(\phi f)\circ h(y) - (\phi f)\circ h(x_o)}{h(y) - h(x_o)} \cdot \frac{h(y) - h(x_o)}{y - x_o} \cdot \frac{y - x_o}{x - x_1} , \qquad (y \neq x_o)$$

if $x = f(y)$ for some $f \in C^1(R)$ such that $f(x_o) = x_1$. From (1) it is

clear that the derivative of h at x_1 exists if we have $x \longrightarrow x_1$ implies $y \longrightarrow x_o$ and $(y - x_o)/(x - x_1)$ is finite as $x \longrightarrow x_1$. We see that $x = f(y) = y + x_1 - x_o$ does satisfy the conditions, hence h is differentiable on R. This completes the proof.

We remark that whether or not h has to have a continuous derivative still remained unanswered by us. In fact whether every automorphism on $C^m(R)$ or $C^\infty(R)$ is inner remains hitherto an open question.

4.5.3 <u>Theorem</u> : Every automorphism ϕ on $P(R)$ is inner.

<u>Proof</u> : Again by Lemma 4.4.2, $\phi(Q) = h \circ Q \circ h^{-1}$ for some bijection h on R. To show h is continuous we note that $Q(R)$ is either all of R or has the form $[a, \infty)$ or $(-\infty, b]$. Since $\phi(Q)(R) = \phi(Q) \circ h(R) = h(Q(R))$ we see that h maps closed subbasic sets of R into closed subbasic sets. This shows that h is continuous. By Lemma 4.4.3, h is monotonic and as in the proof of Theorem 4.5.2, h is differentiable at some $x_o \in R$. We show that it is in fact differentiable at any other point of R. Let $x_1 \in R$, $x = Q(y) = y + x_1 - x_o$ then

$$\frac{h(x) - h(x_1)}{x - x_1} = \frac{hQ(y) - hQ(x_o)}{x - x_1}$$

$$= \frac{(\phi Q) \circ h(y) - (\phi Q) \circ h(x_o)}{h(y) - h(x_o)} \cdot \frac{h(y) - h(x_o)}{y - x_o} \cdot \frac{y - x_o}{x - x_1}$$

and $h'(x_1) = (\phi Q)'(h(x_o)) \cdot h'(x_o)$. Now Q is invertible implies ϕQ is invertible; but the only invertible elements in the semi-group of polynomials

are the linear ones. So $(\phi Q)' = c$ where $0 \neq c \in R$. The equation $h'(x_1) = (\phi Q)' \circ h(x_o) \cdot h'(x_o)$ then suggests that $(\phi Q)' = 1$ since h' cannot have a jump discontinuity. This shows that h' is a constant function and therefore h is a linear polynomial.

4.6.1 <u>Theorem</u> : Let A denote the semi-group $L_c(R)$, $C^m(R)$, $C^\infty(R)$ or $P(R)$. Then every automorphism ϕ on A can be extended uniquely to an automorphism on $C(R)$.

<u>Proof</u> : From the proofs of Theorems 4.5.1 and 4.5.3, it is clear that $\phi(f) = h \circ f \circ h^{-1}$ for some homeomorphism h on R. Defining $\phi^*(f) = h \circ f \circ h^{-1}$ for $f \in C(R)$ we see that ϕ^* is an automorphism which extends ϕ. If ψ is another automorphism on $C(R)$ which extends ϕ then $\psi(f) = k \circ f \circ k^{-1}$ for some homeomorphism k; and by definition of h and k, $h(x) = \phi(x) = \psi(x) = k(x)$ for each $x \in R$. Hence $h = k$ and $\phi^* = \psi$.

4.6.2 <u>Theorem</u> : Let A be the semi-group C, L_c, C^m, C^∞ or P. Then every automorphism on A is determined uniquely by its action on A', the sub-semigroup of invertible elements in A.

<u>Proof</u> : We first show that if an automorphism ϕ maps A' onto A' identically, then ϕ is the identity on A. We know that $\phi(f) = h \circ f \circ h^{-1}$ for some homeomorphism h. Suppose there exists $x_o \in R$ such that $h(x_o) \neq x_o$ and define

$$f(x) = \frac{h(x_o) - y_o}{h(x_o) - x_o}(x - x_o) + y_o \qquad \text{where } y_o = h^{-1}(x_o) .$$

We see that f is an invertible element in C, L_c, C^m, C^∞ and P. From the functional identity $(\phi f)\circ h = h\circ f$ we arrive at

$$x_o = h(y_o) = hf(x_o) = (\phi f)h(x_o) = fh(x_o) = h(x_o)$$

which is a contradiction. Hence $h(x) = x$ for all $x \in R$. This shows that $\phi(f) = f$ for each $f \in A$.

Suppose now that ϕ and ψ are two automorphisms on A which agree on A'. The composition $\psi^{-1}\circ\phi$ is an automorphism on A that maps A' identically onto itself, so $\psi^{-1}\circ\phi$ is the identity on the whole of A, implying $\psi = \phi$.

4.6.3 <u>Theorem</u> : Let A denote $C(R)$ or $P(R)$. Then the group of ring-automorphisms on A is anti-isomorphic with the group of semi-group automorphisms on A.

<u>Proof</u> : By Theorem 4.2.1, the group of ring automorphisms of A is anti-isomorphic with A', the group of invertible elements of A. Hence we need only to show that the group of semi-group automorphisms on A is isomorphic to A'.

Indeed there exists a one-to-one correspondence given by $\phi_\tau \longrightarrow \tau$ where $\phi_\tau(f) = \tau\circ f\circ\tau^{-1}$. Since

$$\phi_\tau\circ\phi_\mu(f) = \phi_\tau(\mu\circ f\circ\mu^{-1}) = \tau\circ\mu\circ f\circ\mu^{-1}\circ\tau^{-1} = (\tau\circ\mu)\circ f\circ(\tau\circ\mu)^{-1} = \phi_{\tau\mu}(f) ,$$

it follows that the correspondence does define a group isomorphism hence the proof is complete.

5. CHARACTERISTIONS OF $C(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$

5.1 In this section, we set out to characterise $C(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$ as rings. Let X be a topological space and $C(X)$, the ring of continuous functions on X. We shall consider a sub-ring A of $C(X)$ which contains the set of constant functions.

5.1.1 <u>Definition</u> : A is called a regular sub-ring of $C(X)$ if $Z(A) = \{Z(f) : f \in A\}$ forms a base for closed sets in X.

5.1.2 <u>Definition</u> : An ideal $M \subset A$ is real if $A/M \cong R$. We denote the set of all real ideals in A by $\mathcal{R}_A$.

5.1.3 <u>Definition</u> : A is said to be a point-determining sub-ring of $C(X)$ if for each $M \in \mathcal{R}_A$, $M = M_x = \{f \in A : f(x) = 0\}$ for some $x \in X$, i.e. if every real ideal in A is fixed.

5.1.4 <u>Remark</u> (i) If $C(X)$ contains a regular sub-ring, then X is necessarily completely regular [5, Theorem 3.7].

(ii) We wish to point out that since there exists a one-to-one correspondence between real ideals and non-zero homomorphisms (1.6), A is point-determining if and only if every non-zero homomorphism is an evaluation. For each $M \in \mathcal{R}_A$, the corresponding homomorphism is denoted by $f \longrightarrow M(f)$ where $M(f)$ is identified with a real number. From Proposition 1.5, if $r \in R$, then $M(r) = r$ for each $M \in \mathcal{R}_A$.

(iii) If A is a regular sub-ring of $C(X)$, then any sub-ring of $C(X)$ containing A is also regular.

(iv) As a note, $L_c(R^n)$, $C^m(R^n)$, $C^\infty(R^n)$ and $P(R^n)$ are all point-determining sub-rings of $C(R^n)$, by Proposition 2.3.1.

5.2 Here we would like to show that $L_c(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$ are regular sub-rings of $C(R^n)$. By Remark 5.1.4(iii) it is sufficient to show that $C^\infty(R^n)$ is a regular sub-ring of $C(R^n)$, i.e., for any $p \in R^n$ and closed set $F \subset R^n$ such that $p \notin F$, there exists $f \in C(R^n)$ whose zero set $Z(f) \supset F$ but $p \notin Z(f)$. In fact something more can be shown.

5.2.1 <u>Theorem</u> : Let F be an arbitrary closed subset of R^n. Then $F = Z(f)$ for some $f \in C^\infty(R^n)$.

<u>Proof</u> : See [11, Theorem 2.2].

5.2.2 <u>Corollary</u> : $L_c(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$ are regular sub-rings of $C(R^n)$.

<u>Proof</u> : Follows from Theorem 5.2.1 and Remark 5.1.4(iii).

5.3.1 <u>Theorem</u> : If A is a point-determining, regular sub-ring of $C(X)$, then A determines X uniquely.

<u>Proof</u> : We would like to show that X in fact is homeomorphic with $\mathcal{R}_A$ with a suitable topology. Let $\{M \in \mathcal{R}_A : M(f) = 0\}$, $f \in A$ be a base for the closed sets in $\mathcal{R}_A$. Since $f(x) = 0$ if and only if $M_x(f) = 0$, the

correspondence $Z(f) \longleftrightarrow \{M : M(f) = 0\}$ is one-to-one from $Z(A)$ onto the base for the closed sets in $\mathcal{R}_A$, so $x \longrightarrow M_x$ is clearly a homeomorphism between X and $\mathcal{R}_A$. This completes the proof.

The topology on $\mathcal{R}_A$ described above is actually the Stone topology on $\mathcal{R}_A$ and is the same as the hull-kernel topology [5, p. 111]. We denote this space by $s\mathcal{R}_A$.

For $f \in A$, we define $f^* : \mathcal{R}_A \longrightarrow R$ by $f^*(M) = M(f)$. The family $\{f^* : f \in A\}$ of functions on $\mathcal{R}_A$ induces a weak topology on $\mathcal{R}_A$ which we denote by $w\mathcal{R}_A$.

It is easy to see that the weak topology is finer than the Stone topology, since from the equality

$$\{M \in \mathcal{R}_A : M(f) = 0\} = (f^*)^{-1}\{0\}, \qquad f \in A,$$

every member of the base in $s\mathcal{R}_A$ is weakly closed. In case A is a point-determining regular sub-ring of $C(X)$, we can show that the two topologies coincide : Let $\{M_y : |M_x(f) - M_y(f)| \geq \varepsilon\}$, $f \in A$, be a subbasic weakly closed set in $\mathcal{R}_A$. Since $\{y \in X : |f(x) - f(y)| \geq \varepsilon\}$ is closed in X (note that $f \in A$ implies f is continuous on X), by Theorem 5.3.1, in which it is proved that $x \longrightarrow M_x$ is a homeomorphism, $\{M_y : |M_x(f) - M_y(f)| \geq \varepsilon\}$ is closed in the Stone topology. Thus we have :

5.3.2 <u>Corollary</u> : If A is a point-determining, regular sub-ring of $C(X)$, then $s\mathcal{R}_A = w\mathcal{R}_A$.

5.3.3 <u>Corollary</u> : $L_c(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$ determine R^n as a topological space.

<u>Proof</u> : Since all the rings mentioned are point-determining regular sub-rings of $C(X)$ (Remark 5.1.4(iv) and Corollary 5.2.2), the corollary follows from Theorem 5.3.1.

5.4 We now consider an arbitrary ring A containing a sub-ring isomorphic to the ring of real numbers, R . (Note that A need not be a sub-ring of any ring of continuous functions). We shall always identify this sub-ring with R. Such a ring A is the same as an algebra A over R where A is an algebra containing unity. However since such an algebra has no different algebraic properties, we shall only concern ourselves with ring structures in a ring A (containing R).

The set of real ideals on A is defined as in Definition 1.4.1 and will be denoted by $\mathcal{R}_A$. This set with the Stone topology and weak topology described in 5.3 will play a central role in the characterisation of $C(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$.

5.4.1 <u>Definition</u> : An (arbitrary) ring A with unity is said to be regular if $\bigcap \mathcal{R}_A = \bigcap_{M \in \mathcal{R}_A} \{f : f \in M\} = 0$ and for every $M \in \mathcal{R}_A$ and $a \in M$, there is a $b \in M$ such that $N\{(b-1)^2(1-a)\} \geq 0$ for each $N \in \mathcal{R}_A$.

We point out that the definition of regularity for a sub-ring A of $C(X)$ (for some X) (Definition 5.1.1) and the definition of regularity for an arbitrary ring A (Definition 5.4.1) are different. Nevertheless,

these two definitions are equivalent in case A is a point-determining sub-ring of $C(X)$ for some completely regular topological space X. This is to be justified in Theorem 5.5. Furthermore, and of more importance, we shall see shortly that if A is a regular ring, then $w\mathcal{R}_A = s\mathcal{R}_A$. (Note : this fact is proved in Corollary 5.3.2 when A is a point-determining sub-ring of $C(X)$). To this end we first prove the following.

5.4.2 <u>Lemma</u> : If A is any ring, then for each $M_o \in \mathcal{R}_A$ contained in a weakly open set $\mathcal{U}$ in $w\mathcal{R}_A$, there exists an $a \in A$ such that $a^*(M_o) = 0$, $a^*(N) \geq 1$ for $N \notin \mathcal{U}$.

<u>Proof</u> : By definition of the weak topology on $\mathcal{R}_A$, there exist $a_i \in A$, $\varepsilon_i > 0$, $i = 1, 2, \ldots, n$, such that

$$M_o \in \bigcap_{i=1}^{n} \left\{ M : |a_i^*(M) - a_i^*(M_o)| < \varepsilon_i \right\} \subset \mathcal{U}$$

Let $\alpha = \min \{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \}$ and $a = \frac{1}{\alpha^2} \sum_{i=1}^{n} \{a_i - M_o(a_i)\}^2$. Then $a^*(M_o) = 0$ and for $N \notin \mathcal{U}$, $|a_j^*(N) - a_j^*(M_o)| \geq \varepsilon_j$ for some j, so

$$a^*(N) \geq \frac{1}{\alpha^2} |a_j^*(N) - a_j^*(M_o)| \geq 1 .$$

5.4.3 <u>Theorem</u> : Suppose $\bigcap \mathcal{R}_A = 0$, then A is a regular ring if and only if $w\mathcal{R}_A = s\mathcal{R}_A$.

<u>Proof</u> : (Necessity) Assume that A is a regular ring (i.e., Definition 5.4.1 is true). From the equality

$$\{ M \in \mathcal{R}_A : M(a) = 0 \} = (a^*)^{-1}\{0\} , \qquad a \in A ,$$

we see that the Stone topology is contained in the weak topology for any ring. To show $s\mathcal{R}_A = w\mathcal{R}_A$, we only need to show that every closed set in $w\mathcal{R}_A$ is closed in the Stone topology.

Let $\mathcal{F}$ be a closed set in $w\mathcal{R}_A$ and $M_o \in \mathcal{R}_A$ such that $M_o \notin \mathcal{F}$. By Lemma 5.4.2, there exists an $a \in A$ such that $a^*(M_o) = 0$ (i.e. $a \in M_o$), $a^*(N) \geq 1$ for $N \in \mathcal{F}$. Regularity of A now ensures the existence of a $b \in A$ such that $b \in M_o$ and

$$M\{(b-1)^2(1-a)\} \geq 0 \qquad \text{for every } M \in \mathcal{R}_A ;$$

$$\text{i.e.} \quad M\{(b-1)^2\} \cdot M\{(1-a)\} \geq 0 \qquad \text{for every } M \in \mathcal{R}_A .$$

As $N(1 - a) = N(1) - N(a) = 1 - a^*(N) \leq 0$ for all $N \in \mathcal{F}$, we obtain from the inequality above that $N\{(b-1)^2\} = 0$ for any $N \in \mathcal{F}$. Moreover, $M_o\{(b-1)^2\} = 1 \neq 0$ so,

$$M_o \notin \mathcal{K} , \qquad \text{where } \mathcal{K} = \{ M : (b-1)^*(M) = 0 \}$$

and $\mathcal{K} \in Z(A)$ is closed in $s\mathcal{R}_A$. We have $\mathcal{K} \supset \mathcal{F}$ but $M_o \notin \mathcal{K}$. This shows that $\mathcal{F}$ is closed in the Stone topology. So $s\mathcal{R}_A = w\mathcal{R}_A$.

(Sufficiency) Let $M_o \in \mathcal{R}_A$ and $a \in M_o$, we wish to find a $b \in A$ for which Definition 5.4.1 is satisfied. Choose $\mathcal{U} = \{ M : (1-a)^*(M) > 0 \}$ to be an open set in $w\mathcal{R}_A$ containing M_o . By assumption $\mathcal{U}$ is open in $s\mathcal{R}_A$, and there exists $c \in A$ such that

$$M_o \notin \{ M : c^*(M) = 0 \} \quad \text{and} \quad M_1 \in \{ M : c^*(M) = 0 \}$$

for all $M_1 \notin \mathcal{U}$. Hence for any $N \in \mathcal{R}_A$,

$$\text{if } N \in \mathcal{U} \quad \text{then } N\{c^2(1-a)\} = \{N(c)\}^2 \cdot N(1-a) \geq 0$$

$$\text{if } N \notin \mathcal{U} \quad \text{then } N\{c^2(1-a)\} = \{N(c)\}^2 \cdot N(1-a) = 0 \cdot N(1-a) = 0 .$$

Let $b = 1 - (c/c^*(M_o))$, then $b \in M_o$ and for all $N \in \mathcal{R}_A$

$$N\{(b-1)^2(1-a)\} = c^*(M_o)^{-2}\, N\{c^2(1-a)\} \geq 0 .$$

Therefore A is a regular ring!

5.5 <u>Theorem</u> : Let A be an arbitrary ring (containing R). A is regular if and only if it is isomorphic to a point-determining regular sub-ring of $C(X)$ for some topologically unique completely regular space X .

<u>Proof</u> : (Sufficiency) Let $f \in \bigcap \mathcal{R}_A$, $x \in X$. By Proposition 1.4.2, $M_x \in \mathcal{R}_A$ and so $f \in M_x$, implying $f(x) = 0$. Therefore $f = 0$ on X , $\bigcap \mathcal{R}_A = 0$. From Theorem 5.3.1 and Corollary 5.3.2, X is homeomorphic to $s\mathcal{R}_A = w\mathcal{R}_A$. Theorem 5.4.3 then states that A is a regular ring.

(Necessity) For each $a \in A$, define $a^* : \mathcal{R}_A \longrightarrow A$ by $a^*(M) = M(a)$ and endow $\mathcal{R}_A$ with the weak topology induced by $A^* = \{a^* : a \in A\}$. Because $\bigcap \mathcal{R}_A = 0$, we can show that A and A^* are isomorphic as rings. Defining

$$a^*b^* = (ab)^* , \qquad a^* + b^* = (a + b)^* ,$$

we see that $a \longrightarrow a^*$ is a ring homomorphism, and we only have to show that this homomorphism is one-to-one. Let $a^* = b^*$, then $a^*(M) = b^*(M)$ for all $M \in \mathcal{R}_A$, i.e.

$$M(b - a) = M(b) - M(a) = b^*(M) - a^*(M) = 0$$

implying $b - a \in M$ for all $M \in \mathcal{R}_A$. Hence $b - a = 0$ and $b = a$ showing that A and A^* are isomorphic.

Now A is regular implies $w\mathcal{R}_A = s\mathcal{R}_A$ (Theorem 5.4.3). Since by definition, $Z(A^*) = \{Z(a^*) : a \in A\}$ forms a base for $s\mathcal{R}_A$ where

$$Z(a^*) = \{M \in \mathcal{R}_A : a^*(M) = 0\} = \{M \in \mathcal{R}_A : M(a) = 0\} ,$$

it also forms a base of $w\mathcal{R}_A$. Hence A^* is a regular sub-ring of $C(w\mathcal{R}_A)$ by Definition 5.1.1.

Next, we show that A^* is point-determining. Let $M^* \in \mathcal{R}_{A^*}$, denote $\{a \in A : a^* \in M^*\}$ by M. From the isomorphism $a \longrightarrow a^*$ it is clear that $M \in \mathcal{R}_A$. We have $M^* = \{a^* : a^*(M) = 0\} = M^*_M \subset C(\mathcal{R}_A)$ by definition of M. Hence A^* is a point-determining, regular sub-ring of $C(\mathcal{R}_A)$. Uniqueness of $\mathcal{R}_A$ follows from Theorem 5.3.1 and from Remark 5.1.4(i), $\mathcal{R}_A$ is completely regular.

5.6 We are now in a position to proceed with the characterisation of $C(R^n)$, $C^m(R^n)$ and $C^\infty(R^n)$ as regular rings with certain properties on the real ideal space that can be used to identify $\mathcal{R}_A$ with R^n.

5.6.1 <u>Lemma</u> : Suppose A is a regular ring and there exist $u_1, u_2, \ldots, u_n \in A$ such that

(i) for $r_i \in R$, $i = 1, 2, \ldots, n$, $u_1-r_1, u_2-r_2, \ldots, u_n-r_n$ are contained in one unique $M \in \mathcal{R}_A$;

(ii) if $a \in A$ and $M \in \mathcal{R}_A$ such that $M(a) \neq 0$, then there exist $\alpha \in R$, $b \in A$ for which

$$\sum_{i=1}^{n} (u_i - M(u_i))^2 + a^2 = \alpha^2 + b^2 .$$

Then $\mathcal{R}_A$ is homeomorphic with R^n. Hence A is isomorphic to a point-determining, regular sub-ring of $C(R^n)$.

<u>Proof</u> : Define $\mu : \mathcal{R}_A \longrightarrow R^n$ by $\mu(M) = (M(u_1), M(u_2), \ldots, M(u_n))$. Suppose for $M, N \in \mathcal{R}_A$, $\mu(M) = \mu(N)$, then we have for $i = 1, 2, \ldots, n$, $M(u_i) = N(u_i)$. Let $r_i = M(u_i) = N(u_i)$, $i = 1, 2, \ldots, n$. Then $u_i - r_i \in M$ and $u_i - r_i \in N$ for $1 \leq i \leq n$, by condition (i), we have $M = N$. And for $s = (s_1, s_2, \ldots, s_n) \in R^n$, let M_s be the unique real ideal containing $u_1 - s_1, u_2 - s_2, \ldots, u_n - s_n$, then

$$\mu(M_s) = (M_s(u_1), M_s(u_2), \ldots, M_s(u_n))$$

$$= (s_1, s_2, \ldots, s_n) = s .$$

This shows that μ is one-to-one and onto.

We topologise $\mathcal{R}_A$ by the weak topology induced by $A^* = \{a^* : a \in A\}$ as in Theorem 5.5. Since $\mu = (u_1^*, u_2^*, \ldots, u_n^*)$ and each u_i^* is continuous by definition of weak topology, we see that μ is also continuous.

We would like to show that $\mu(F)$ is closed in R^n for any closed set F in $w\mathcal{R}_A$. Knowing that A is regular and by Theorem 5.4.3, we need to consider only closed sets in $\mathcal{R}_A$ of the form $\{F = M \in \mathcal{R}_A : M(a) = 0\}$, $a \in A$. Suppose

$$s \notin \mu(F) = \left\{ (M(u_1), M(u_2), \ldots, M(u_n)) : M(a) = 0 \right\}$$

then $M_s(a) \neq 0$ where $s = \mu(M_s) = (M_s(u_1), M_s(u_2), \ldots, M_s(u_n))$. By (ii) there exist $\alpha \in R$, $b \in A$ such that

$$\sum_{i=1}^{n} (u_i - M_s(u_i))^2 + a^2 = \alpha^2 + b^2$$

then for $N \in \mathcal{R}_A$,

$$\alpha^2 \leq \alpha^2 + N(b)^2 = \sqrt{\sum_{i=1}^{n} (N(u_i) - M_s(u_i))^2} + N(a)^2 .$$

Let

$$B_{\alpha/2}(s) = \left\{ t \in R^n : ||t - s|| = \sqrt{\sum_{i=1}^{n} (t_i - s_i)^2} < \frac{\alpha}{2} \right\} .$$

Then for $N \in \mu^{-1}\{B_{\alpha/2}(s)\}$, i.e. for $N \in \mathcal{R}_A$ satisfying

$$\sqrt{\sum_{i=1}^{n} (N(u_i) - M_s(u_i))^2} < \frac{\alpha}{2}$$

we have, by the inequality above $\alpha^2 < \alpha^2/4 + N(a)^2$ implying $N(a) \neq 0$ or $N \notin F$. Since $\mu^{-1}\{B_{\alpha/2}(s)\} \cap F \neq \emptyset$ and μ is one-to-one and onto, we must have $B_{\alpha/2}(s) \cap \mu(F) = \emptyset$. This argument shows that $\mu(F)$ is a closed subset in R^n and hence $\mu : \mathcal{R}_A \longrightarrow R^n$ is a homeomorphism.

Under conditions of the Lemma, an application of Theorem 5.5 will enable us to conclude that A is isomorphic to a point-determining, regular sub-ring of $C(R^n)$. This completes the proof.

Now if we impose a maximal condition, we obtain the following characterisation.

5.6.2 <u>Theorem</u> : Let A be a ring containing R . Then $A = C(R^n)$ if and only if A is regular and there exist $u_1, u_2, \ldots, u_n$ such that

(i) for $r_i \in R$, $i = 1, 2, \ldots, n$, $u_1-r_1, u_2-r_2, \ldots, u_n-r_n$ are contained in one unique $M \in \mathcal{R}_A$;

(ii) if $a \in A$ and $M \in \mathcal{R}_A$ such that $M(a) \neq 0$, then there exist $\alpha \in R$, $b \in A$ for which

$$\sum_{i=1}^{n} (u_i - M(u_i))^2 + a^2 = \alpha^2 + b^2 ;$$

(iii) A has no ring extension that is regular and satisfies conditions (i) and (ii) above for $u_1, u_2, \ldots, u_n$.

<u>Proof</u> : (Necessity) From 2.4, there is a one-to-one correspondence between R^n and $\mathcal{R}_{C(R^n)}$ given by $x \longrightarrow M_x = \{f \in C(R^n) : f(x) = 0 \}$. Suppose $f \in \bigcap \mathcal{R}_{C(R^n)} = \bigcap \{ M : M \in \mathcal{R}_{C(R^n)} \}$, then for each $x \in R^n$, $M_x \in \mathcal{R}_{C(R^n)}$ and $f \in M_x$ implies $f(x) = M(x) = 0$. Hence $f = 0$. Now the complete regularity of R^n will testify that $C(R^n)$ is a regular ring.

If u_i $(1 \leq i \leq n)$ is taken to be the i-th projection on R^n , then for $r_i \in R$, $i = 1, 2, \ldots, n$, let $r = (r_1, r_2, \ldots, r_n) \in R^n$ then

$$u_i - r_i \in M_r = \{f \in C(R^n) : f(r) = 0 \} , \quad i = 1, 2, \ldots, n$$

and M_r is unique, since the correspondence $r \longrightarrow M_r$ is one-to-one.

And if $f \in C(R^n)$, $s \in R^n$ are such that $f(s) \neq 0$, then

$$\sum_{i=1}^{n} (u_i - s_i)^2 + f^2 > 0 \quad \text{on } R^n ,$$

in particular on $S = \{ t \in R^n : ||t - s|| \leq 1 \}$. Let

$$\beta = \min_{x \in S} \left\{ \sum_{i=1}^{n} (x_i - s_i)^2 + f(x)^2 \right\} > 0$$

and choose $\alpha^2 = \frac{1}{2} \min\{\beta, 1\}$. Then

$$\sum_{i=1}^{n} (u_i - s_i)^2 + f^2 - \alpha^2 > 0 \quad \text{on } R^n .$$

Defining

$$g(x) = \left\{ \sum_{i=1}^{n} (x_i - s_i)^2 + f(x)^2 - \alpha^2 \right\}^{1/2}$$

we see that $g \in C(R^n)$ and

$$\sum_{i=1}^{n} (u_i - s_i)^2 + f^2 = \alpha^2 + g^2 \quad \text{where } s_i = M_s(u_i) .$$

We have thus proved that conditions (i) and (ii) are satisfied.

Now let B be a regular ring containing $C(R^n)$ satisfying (i) and (ii). By the proof of Lemma 5.6.1, there is a homeomorphism $\mu : w\mathcal{R}_B \longrightarrow R^n$. We define a function $\phi : B \longrightarrow C(R^n)$ by $\phi(b) = b^* \circ \mu^{-1}$. Since μ^{-1} and b^* are continuous, $\phi(b)$ is also continuous on R^n , moreover if $\phi(b) = \phi(c)$ then

$$b^* = b^* \circ \mu^{-1} \circ \mu = \phi(b) \circ \mu = \phi(c) \circ \mu = c^* \circ \mu^{-1} \circ \mu = c^*$$

and since $a \longrightarrow a^*$ is one-to-one (see proof of Theorem 5.5), $b = c$. To

show that ϕ is a monomorphism, for $x \in R^n$, let $\mu^{-1}(x) = M \in \mathcal{R}_B$, then

$$\phi(b \,\square\, c)(x) = (b \,\square\, c)^* \circ \mu^{-1}(x) = (b \,\square\, c)^*(M) = M(b \,\square\, c)$$

$$= M(b) \,\square\, M(c) = b^*(M) \,\square\, c^*(M)$$

$$= b^* \circ \mu^{-1}(x) \,\square\, c^* \circ \mu^{-1}(x)$$

$$= (\phi(b) \,\square\, \phi(c))(x) \ .$$

Hence $\phi(b \,\square\, c) = \phi(b) \,\square\, \phi(c)$ where $\square$ represents pointwise addition or multiplication, so ϕ is indeed a monomorphism.

Let $\psi = \phi | C(R^n)$ be the restriction of ϕ to $C(R^n)$, then

$$(\psi f)(x) = f^* \circ \mu^{-1}(x) = f^*(M_x) = M_x(f) = f(x),$$

$x \in R^n$, $f \in C(R^n)$, implying $\psi f = f$ for $f \in C(R^n)$. Now let $b \in B$, then $\phi(b) = f$ for some $f \in C(R^n)$ and $\phi(b) = f = \psi(f) = \phi(f)$. Since ϕ is a monomorphism $b = f \in C(R^n)$. This shows that $B = C(R^n)$ as a ring.

(Sufficiency) By Lemma 5.6.1, we know that A is isomorphic to a point-determining, regular sub-ring of $C(R^n)$. By (iii), $A \cong C(R^n)$, since it has been proved above that $C(R^n)$ does satisfy the stated conditions.

5.7 As proved in Theorem 5.5, if A is a regular ring then $a \longrightarrow a^*$ is a ring isomorphism between A and $A^* = \{a^* : a \in A\}$ where $a^* : \mathcal{R}_A \longrightarrow R$ is given by $a^*(M) = M(a)$. In what follows, $\mathcal{R}_A$, the real ideal space of a regular ring A is granted the weak topology induced by A^* (which coincides with the Stone topology by Theorem 5.4.3). So A^* is a sub-ring of $C(\mathcal{R}_A)$, the ring of continuous functions on $\mathcal{R}_A$. Letting

$M^* = \{a^* : a \in M\}$ for $M \in \mathcal{R}_A$

$$C(\mathcal{R}_A)\cdot(u^*-r) = \{g\cdot(u^*-r) : g \in C(\mathcal{R}_A)\} \quad \text{for } u \in A$$

$$C(\mathcal{R}_A)\cdot M^* = \{g\cdot a^* : g \in C(\mathcal{R}_A),\ a^* \in M^*\}$$

and noting the $M^* \subset C(\mathcal{R}_A)$, we first prove this lemma.

5.7.1 <u>Lemma</u> : If A is a regular ring containing elements $u_1, u_2, \ldots, u_n$ such that

(i) A has ring extensions $A = A_m \subset A_{m-1} \subset \ldots \subset A_1 \subset A_o = C(R^n)$ which are regular.;

(ii) If $M_k \in \mathcal{R}_{A_k}$, $k = 1, 2, \ldots, m$, then

$$M_k^* \subset \sum_{i=1}^{n} C(\mathcal{R}_{A_k})(u_i^*-r_i) \subset C(\mathcal{R}_{A_k})M_k^*$$

for some $r_1, r_2, \ldots, r_n \in R$.

(iii) For each k, if $a \in A_k$ and $M_k \in \mathcal{R}_{A_k}$ are such that $M_k(a) \neq 0$, then there exist $\alpha \in R$, $b \in A_k$ for which

$$\sum_{i=1}^{n} (u_i - M_k(u_i))^2 + a^2 = \alpha^2 + b^2 .$$

(iv) For each k $(1 \leq k \leq m)$, there exist linear mappings $\partial_i^k : A_k^* \longrightarrow A_{k-1}^*$, $i = 1, 2, \ldots, n$ satisfying

$$\partial_i^k(rf) = r\partial_i^k(f) , \qquad r \in R,\ f \in A_k^*$$

$$\partial_i^k(fg) = f\cdot\partial_i^k(g) + g\cdot\partial_i^k(f) , \qquad f, g \in A_k^*$$

$$\partial_i^k(u_j) = \begin{cases} 0 & \text{if } i \neq j, \quad 1 \leq j \leq n \\ 1 & \text{if } i = j . \end{cases}$$

Then A is isomorphic to a regular sub-ring of $C^m(R^n)$.

<u>Proof</u> : Let $M_k \in \mathcal{R}_{A_k}$, then for $r_1, r_2, \ldots, r_n \in R$ we have by (ii),

$$(u_i^* - r_i) = g \cdot a^* \quad (i = 1, 2, \ldots, n)$$

for some $g \in C(\mathcal{R}_{A_k})$ and $a^* \in M_{A_k}^*$. But $a^* \in M_{A_k}^*$ implies $a \in M_k$ and $a^*(M_k) = M_k(a) = 0$. Therefore $(u_i^* - r_i)(M_k) = 0$, hence $(u_i^* - r_i) \in M_k^*$ or $u_i - r_i \in M_k$. Now M_k with these properties is unique, for if there exists $N \in \mathcal{R}_{A_k}$ such that $u_i - r_i \in N$, $i = 1, 2, \ldots, n$, and if $M_k \neq N$, then we can assume without loss of generality that there exists an $a \in M_k \setminus N$, that is $M_k(a) = 0$, but $N(a) \neq 0$. However since by (ii)

$$a^* = \sum_{i=1}^{n} g_i \cdot (u_i^* - r_i) , \qquad g_1, g_2, \ldots, g_n \in C(\mathcal{R}_{A_k})$$

we have

$$N(a) = a^*(N) = \sum_{i=1}^{n} g_i(N) \cdot (u_i^* - r_i)(N) = \sum_{i=1}^{n} g_i(N) \cdot N(u_i - r_i) = 0$$

which is a contradiction. Therefore $u_i - r_i$, $i = 1, 2, \ldots, n$, belong to a unique $M_k \in \mathcal{R}_{A_k}$. Now with (iii), Lemma 5.6.1 applies and we have for $k = 1, 2, \ldots, m$, $\mathcal{R}_{A_k} \overset{\mu}{\cong} R^n$ where

$$\mu(M) = (M(u_1), M(u_2), \ldots, M(u_n)) , \qquad M \in \mathcal{R}_{A_k} .$$

For each $M \varepsilon \mathcal{R}_{A_k}$, M is of the form $M = M_r$ where $\mu(M_r) = r \varepsilon R^n$. At this juncture to overcome impending symbolic difficulties, we shall agree to identify $\mathcal{R}_{A_k}$ with R^n, that is, $M_r \varepsilon \mathcal{R}_{A_k}$ with $r \varepsilon R^n$ and each $f^* \varepsilon M^*$ $(f \varepsilon M)$ can be identified with $f \varepsilon C(R^n)$ where $f^*(M_r) = M_r(f) = f(r)$. Upon doing so we shall see that u_i^* is in fact the continuous i-th projection on R^n. We have by (ii) $u_i^*(y) - x_i = g(y)f^*(y)$ for some $g \varepsilon C(R^n)$, $f \varepsilon M_x$. Since $u_i^*(x) - x_i = g(x)f^*(x) = g(x)\cdot 0 = 0$, we have $u_i^*(x) = x_i$ as required.

Consider now $f \varepsilon A_k^*$ and $r \varepsilon R^n$. Since $M_r(f - f(r)) = 0$, $f - f(r) \varepsilon M_r = \{g \varepsilon A_k : g(r) = 0\}$ and by (ii),

$$f - f(r) = \sum_{j=1}^{n} g_j\cdot(u_j - r_j) \tag{2}$$

from which we see that

$$\lim_{x_i \to r_i} \frac{f(r_1,\ldots,r_{i-1},x_i,r_{i+1},\ldots,r_n) - f(r_1,\ldots,r_n)}{x_i - r_i}$$

$$= \lim_{x_i \to r_i} g_i(r_1,\ldots,r_{i-1},x_i,r_{i+1},\ldots,r_n) = g_i(r)$$

so $(\partial f/\partial x_i)(r)$ exists for all $r \varepsilon R^n$. Moreover from (2)

$$\partial_i^k(f) = \sum_{j=1}^{n} \partial_i^k(g_j)\cdot(u_j - r_j) + g_i$$

and evaluating this at $r = (r_1, \ldots, r_n)$, we obtain

$$(\partial_1^k f)(r) = g_i(r) = (\partial f/\partial x_i)(r)$$

so $\partial_i^k = \partial f/\partial x_i$. This shows that all the ∂_i^k are partial differentiation operators. Since for $f \in A_1^*$, $\partial_i^1(f) \in C(R^n)$, we see that $A_1^* \subset C^1(R^n)$ and from the fact that $\partial_i^2(f) \in A_1^*$ for $f \in A_2^*$ we conclude that $A_2^* \subset C^2(R^n)$. Inductively we obtain $A_m^* \subset C^m(R^n)$. Finally we have that A is isomorphic to a sub-ring of $C^m(R^n)$ via the isomorphism $a \longrightarrow a^*$.

5.7.2 <u>Theorem</u> : $A \cong C^m(R^n)$ if and only if A is a regular ring containing elements $u_1, u_2, \ldots, u_n \in A$ such that conditions (i), (ii), (iii), (iv) of Lemma 5.7.1 are satisfied and (v) A has no ring extension which is regular and satisfies the same four conditions.

<u>Proof</u> : (Necessity) It is clear that $C^m(R^n)$ is a regular ring from Corollary 5.2.2 and Theorem 5.5. Taking u_i to be the i-th projection, we shall prove that all the conditions are satisfied.

(i) $C^m(R^n)$ has ring extensions $C^k(R^n)$, $k = 0, 1, 2, \ldots, m-1$ which are also regular by the same reason as that for $C^m(R^n)$.

(ii) By Proposition 2.3.1, for each k $(1 \leq k \leq m)$, $M_k \in \mathcal{R}_{A_k}$ is fixed at some point $r \in R^n$, i.e. $M_k = \{f \in C^k(R^n) : f(r) = 0\}$. Now if $f \in M_k$, then

$$
\begin{aligned}
f(x_1,x_2,\ldots,x_n) &= f(x_1,x_2,x_3,\ldots,x_{n-1},x_n) - f(r_1,r_2,r_3,\ldots,r_{n-1},r_n) \\
&= f(x_1,x_2,x_3,\ldots,x_{n-1},x_n) - f(r_1,x_2,x_3,\ldots,x_{n-1},x_n) \\
&\quad + f(r_1,x_2,x_3,\ldots,x_{n-1},x_n) - f(r_1,r_2,x_3,\ldots,x_{n-1},x_n) \\
&\quad + f(r_1,r_2,x_3,\ldots,x_{n-1},x_n) - f(r_1,r_2,r_3,\ldots,x_{n-1},x_n) \\
&\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\quad + f(r_1,r_2,r_3,\ldots,r_{n-1},x_n) - f(r_1,r_2,r_3,\ldots,r_{n-1},r_n)
\end{aligned}
$$

and

$$
\begin{aligned}
&f(x_1,x_2,\ldots,x_n) \\
&= \sum_{i=1}^{n} \frac{f(r_1,r_2,\ldots,r_{i-1},x_i,\ldots,x_n) - f(r_1,r_2,\ldots,r_i,x_{i+1},\ldots,x_n)}{x_i - r_i}(x_i - r_i)
\end{aligned}
$$

where $x_i \neq r_i$ $(i = 1, 2, \ldots, n)$. Now since f is differentiable on R^n, we infer that

$$
\begin{aligned}
&g_i(x_1,x_2,\ldots,x_n) \\
&= \begin{cases} \dfrac{f(r_1,r_2,\ldots,r_{i-1},x_i,\ldots,x_n) - f(r_1,r_2,\ldots,r_i,x_{i+1},\ldots,x_n)}{x_i - r_i}, & x_i \neq r_i \\[2ex] \dfrac{\partial f}{\partial x_i}(r), & x_i = r_i \end{cases}
\end{aligned}
$$

is continuous and we have $f = \sum_{i=1}^{n} g_i \cdot (u_i - r_i)$. Hence

$$
M_k \subset \sum_{i=1}^{n} C(R^n)(u_i - r_i) .
$$

The other inclusion is trivial.

(iii) This can be shown in exactly the same way as in the proof of Theorem 5.6.2.

(iv) ∂_i^k , $(1 \leq k \leq m)$ is just the i-th partial differentiation operator so (iv) is satisfied.

Now if B is a regular ring which contains $C^m(R^n)$ and satisfies conditions (i) to (iv), then by Lemma 5.7.1, B^* is isomorphic to a sub-ring of $C^m(R^n)$, so each $b^* \varepsilon B^*$ has continuous partial derivatives of all orders up to m. As done in Theorem 5.6.2, we define a monomorphism $\phi : B \longrightarrow C^m(R^n)$ by $\phi(b) = b^* \circ \mu^{-1}$ where $\mu : \mathcal{R}_B \longrightarrow R^n$ is a homeomorphism. Moreover if $\psi = \phi | C^m(R^n)$ is the restriction of ϕ to $C^m(R^n)$ then

$$(\psi f)(x) = f^* \circ \mu^{-1}(x) = f^*(M_x) = M_x(f) = f(x)$$

implying $\psi f = f$ for $f \varepsilon C^m(R^n)$. Now let $b \varepsilon B$ then $\phi(b) = f$ for some $f \varepsilon C^m(R^n)$, but $\phi(b) = f = \psi(f) = \phi(f)$, therefore $b = f$ since ϕ is a monomorphism. This shows that ϕ maps B identically onto $C^m(R^n)$, i.e. $B = C^m(R^n)$ as a ring.

(Sufficiency) Suppose A has all the conditions stated, by Lemma 5.7.1, A is isomorphic to a regular sub-ring of $C^m(R^n)$. Since $C^m(R^n)$ satisfies (i) to (iv), condition (v) asserts that $A \cong C^m(R^n)$.

5.8 The conditions imposed on A in order to characterise $C^m(R^n)$ will turn out to be less formidable in the case of $C^\infty(R^n)$. It works out that differentiation in this case is characterised algebraically by means of derivation.

5.8.1 <u>Definition</u> : For an arbitrary ring A (containing R), a derivation on A is a mapping $\partial : A \longrightarrow A$ such that

(i) $\partial(ra + sb) = r\partial(a) + s\partial(b)$

(ii) $\partial(ab) = a\partial(b) + b\partial(a)$

where $a, b \in A$, $r, s \in R$.

As a consequence of this definition, $\partial(r) = 0$ for each $r \in R$. Typical examples are given by partial differentiation operators $\partial/\partial x_i$ on $C^\infty(R^n)$, $i = 1, 2, \ldots, n$. The following lemma shows that under certain conditions the differential operators are completely determined by derivations.

5.8.2 <u>Lemma</u> : Let A be a sub-ring of $C(R^n)$ containing the projections $u_1, u_2, \ldots, u_n$, and the constant functions. Suppose

$$M_r = \{f \in A : f(r) = 0\} = \sum_{i=1}^{n} A(u_i - r_i) ,$$

$r = (r_1, r_2, \ldots, r_n) \in R^n$ and there exist derivation ∂_i, $i = 1, 2, \ldots, n$ on A such that $\partial_i(u_j) = 0$ if $i \neq j$ and $\partial_i(u_i) = 1$. Then $A \subset C^\infty(R^n)$ and $\partial_i f = \partial f/\partial x_i)$, $f \in A$.

<u>Proof</u> : Let $f \in A$ and $r \in R^n$ then $f - f(r) \in M_r$, so there exist $g_i \in A$, $i = 1, 2, \ldots, n$ for which

$$f - f(r) = \sum_{j=1}^{n} g_j \cdot (u_j - r_j) . \tag{3}$$

We consider the i-th partial differential quotient

$$\frac{f(r_1,r_2,\ldots,r_{i-1},x_i,r_{i+1},\ldots,r_n) - f(r)}{x_i - r_i}$$

$$= g_i(r_1,r_2,\ldots,r_{i-1},x_i,r_{i+1},\ldots,r_n) , \qquad x_i \neq r_i .$$

Since $g \in C(R^n)$, we see that the i-th partial derivative of f exists and $(\partial f/\partial x_i)(r) = g_i(r)$. However from (3)

$$\partial_i f = \sum_{j=1}^{n} \{\partial_i g_j.(u_j - r_j)\} + g_i$$

and evaluating at $r = (r_1, r_2, \ldots, r_n)$, we obtain $(\partial_i f)(r) = g_i(r)$. So $\partial_i f = \partial f/\partial x_i$. By definition of derivations all partial derivatives of f of all orders exist, implying $f \in C^\infty(R^n)$. Hence $A \subset C^\infty(R^n)$. This completes the proof.

It is not difficult now to give an algebraic characterisation of $C^\infty(R^n)$, relying heavily on the methods of proof employed in Theorems 5.6.2 and 5.7.2.

5.8.3 <u>Theorem</u> : A ring A is isomorphic to $C(R^n)$ if and only if A is regular and there exist $u_1, u_2, \ldots, u_n \in A$ such that

(i) $M \in \mathcal{R}_A$ implies $M = \sum_{i=1}^{n} A(u_i - r_i)$ for some $r_1, r_2, \ldots, r_n \in R$.

(ii) If $M \in \mathcal{R}_A$, $a \in A$ and $M(a) \neq 0$, then there exist $\alpha \in R$, $b \in A$ such that

$$\sum_{i=1}^{n} (u_i - M(u_i))^2 + a^2 = \alpha^2 + b^2 .$$

(iii) There exist derivations ∂_i, $i = 1, 2, \ldots, n$, on A such that $\partial_i(u_j) = 0$ if $i \neq j$ and $\partial_i(u_i) = 1$.

(iv) A has no ring extension which is regular and satisfies conditions (i) to (iii) above.

<u>Proof</u> : (Necessity) It is evident that $C^\infty(R^n)$ is a regular ring (by Corollary 5.2.2 and Theorem 5.5). Taking u_i to be the i-th projection and ∂_i to be $\partial/\partial x_i$, we see that (iii) is clearly satisfied. A repetition of the proof in Theorem 5.6.2 will prove (ii).

For $M \in \mathcal{R}_{C^\infty(R^n)}$, $M = M_r = \{f \in C^\infty(R^n) : f(r) = 0\}$, so $\sum_{i=1}^{n} C^\infty(R^n)(u_i - r_i) \subset M_r$. If $f \in M_r$, then $f = \sum_{i=1}^{n} g_i \cdot (u_i - r_i)$ for some $g_1, g_2, \ldots, g_n \in C(R^n)$ as in the proof of Theorem 5.7.2. Now since $f \in C^\infty(R^n)$ it can be shown that each $g_i \in C^\infty(R^n)$, so $f \in \sum_{i=1}^{n} C^\infty(R^n)(u_i - r_i)$ and (i) is verified.

Next, if B is a regular ring containing $C^\infty(R^n)$ and satisfies (i) to (iii), then by a repetition of an argument in Theorem 5.6.2, there exists a monomorphism $\phi : B \longrightarrow C^\infty(R^n)$ and one can similarly show that $\phi f = f$ for every $f \in C^\infty(R^n)$. Therefore $B = C(R^n)$ as a ring.

(Sufficiency) (i) and (ii) and Lemma 5.6.1 show that A is isomorphic to a point-determining sub-ring of $C(R^n)$. Lemma 5.8.2 ensures that A is isomorphic to a sub-ring of $C^\infty(R^n)$. Condition (iv) says that $A \cong C^\infty(R^n)$.

Bibliography

[1] Frank W. Anderson and Robert L. Blair, Characterisations of the algebra of all real-valued continuous functions on a completely regular space, Illinois J. Math. 3 (1959), pp 121-133.

[2] B. Banaschewski, An algebraic characterisation of $C^{\infty}(R^n)$, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astro., Phys., 16 (1968), pp 169-174.

[3] H. Cartan, Elementary Theory of Analytic Functions of one or several complex variables, Addison-Wesley Publishing Co., Inc. 1963.

[4] A.G. Fadell and K.D. Magill, Jr., Automorphisms of semi-groups of polynomials, Compositio Mathematica, Vol. 21, Fasc. 3, 1969, pp 233-239.

[5] L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand, Princeton, N.Y. 1960.

[6] Kenneth D. Magill, Jr., Automorphisms of the semi-group of all differentiable functions, Glasgow Math. J. 8 (1967), pp 63-66.

[7] I.P. Natanson, Theory of functions of a real variable, Vol. I. Frederick Ungar Publishing Co. N.Y. 1955.

[8] L.E. Pursell, An algebraic characterisation of fixed ideals in certain function rings, Pacific J. Math. 5 (1955), pp 963-969.

[9] L.E. Pursell, The ring of real polynomials, Amer. Math. Monthly (1969) 76, pp 509-514.

[10] Boris M. Schein, Automorphisms of polynomial semi-groups, Semigroup Forum, Vol. 1 (1970), pp279-281.

[11] L.P. Su, Ph.D. thesis, The University of British Columbia, 1966.