PERTURBATION OF NONLINEAR DIRICHLET PROBLEMS

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ABSTRACT

The solutions of weakly-formulated non-linear Dirichlet problems are studied when the data of the problem are perturbed in various ways. The data which undergo perturbations include the Lagrangian, the boundary condition, the basic domain, and the constraints, if present.

The main conclusion states that the solution of the Dirichlet problem which minimizes the Dirichlet integral varies continuously with the data so long as it is unique. Detailed hypotheses are formulated to insure the validity of this conclusion for several large classes of problem. The hypotheses are not much stronger than the standard sufficient conditions for existence, in the generalized Lusternik-Schnirelman theory of these problems.

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INTRODUCTION

The purpose of this thesis is to study the behaviour of the solutions to nonlinear differential boundary value problems of Dirichlet type, when the data defining the problem are subjected to various perturbations. The basic result we shall obtain states that, under suitable restrictions, the solution of such a problem changes continuously with the data as long as the solution is unique. The conditions under which this conclusion is valid are essentially that the usual sufficient conditions for existence in the variational theory of Dirichlet problems should hold uniformly in some sense as the given problem is perturbed.

The differential equations which appear in the boundary value problems considered here are the Euler-Lagrange equations of multiple-integral 'Dirichlet' functionals defined on suitable Sobolev spaces. We consider such problems in their weak formulation, in which a solution is taken to be a distribution which satisfies the given boundary condition in an appropriate generalized sense, and which is a critical point of the Dirichlet integral restricted to such distributions. The nonlinearity of the Euler-Lagrange equation arises from the fact that the integrand defining the Dirichlet functional need not be quadratic in its arguments, but need only have a certain convexity in its dependence on the functions on which the functional is defined. For such problems, there are well known 'regularity' theorems asserting when a weak solution is in fact a smooth function and hence is, by a standard integration by parts argument, a classical solution of the Euler-Lagrange equation. We shall not be concerned with this question, and we work entirely with weak solutions.

The data defining such a problem appear to be of four kinds :

(i) The 'Lagrangian', or integrand, of the Dirichlet functional;

(ii) the 'Dirichlet data', or boundary conditions;

(iii) the 'domain', i.e. the set over which the independent variables in the differential equation are allowed to run;

(iv) the constraints, i.e. the set in which the dependent variables are required to lie.

In principle we could formulate a global problem on sections of a subbundle of a smooth fibre bundle where the target subfibre represents (iv) and the base manifold represents (iii); and we ought to vary (iii) by pulling back along various embeddings into the base, vary (iv) by allowing the subbundle to vary, and vary (i) and (ii) at will, all simultaneously. This amount of generality presents technical obstacles which obscure the main phenomenon, and, in addition, treating particular cases of the above allows us to dispense with some of the assumptions in each case which are needed in the general case.

Thus we shall take the following less general approach. In Chapter 1, we formulate and prove a theorem in the setting of a fixed fibre bundle over a fixed base manifold, where data (i) and (ii) are allowed to vary. In Chapter 2 we suppress (iv) by considering sections of a trivial vector bundle, and we hold (i) fixed, but we allow (ii) and (iii) to vary. In Chapter 3 we fix (i) and (iii) and allow (ii) and (iv) to vary. A simple example of the situation in Chapter 1 is the problem of minimal surfaces in ordinary Euclidean space, under variation of the boundary curve. The situation of Chapter 2 is illustrated by the problem of domain-perturbations for nonlinear elliptic boundary problems on domains of $\mathbb{R}^n$. The situation of Chapter 3 is illustrated in the study of geodesics, or more general harmonic maps, in imbedded submanifolds of $\mathbb{R}^\ell$. We conclude our discussion by spelling out these examples in a little more detail in Chapter 4. It should be noted that the significance of the uniqueness assumption in our main result is illustrated in all these cases, by well-known phenomena of jumping of the minimizing solution when uniqueness breaks down.

CHAPTER 1

PARAMETRIZED NONLINEAR DIRICHLET PROBLEMS IN STANDARD FORM

1.1 Notation

Generally we shall follow the notation of Palais [4]. For the reader's convenience we supply the following brief list :

M	compact C^∞ manifold of dimension n, possibly with nonempty boundary ∂M ;
μ	strictly positive smooth measure on M ;
E	(total space of) C^∞ fibre bundle over M ;
ξ	(total space of) C^∞ vector bundle over M ;
$\mathbb{R}^\ell_M$	product bundle $M \times \mathbb{R}^\ell$;
k	integer ≥ 0 ;
$J^k(E)$	bundle of k-jets of sections of E ;
$C^\infty(E)$	set of all C^∞ sections of E ;
$S(E)$	set of all sections of E ;
p	real number ≥ 1 ;
$L^p_k(\xi)$	Banach space completion of $C^\infty(\xi)$ in the Sobolev p'th power norm on derivatives of order $\leq k$;

$L^p_k(E)$	(for $pk > n$) Banach manifold of all sections of E, each of which belongs to $L^p_k(\xi)$ for some open vector bundle neighbourhood ξ in E ;
b, s	elements of $L^p_k(E)$;
$L^p_k(E)_b$	closed C^∞ submanifold of $L^p_k(E)$ consisting of the closure in $L^p_k(E)$ of the set of all sections $s \in L^p_k(E)$ which agree with b in some neighbourhood (depending on the particular s) of ∂M;
$\|\| \ \|\|$	Finsler structure on the tangent bundle $T(L^p_k(E))$; also, by abuse, induced structure on $T^*(L^p_k(E))$;
δ	Finsler metric on $L^p_k(E)$ induced by $\|\| \ \|\|$;
j_k	k-jet extension map $L^p_k(E) \longrightarrow L^p_0(J^k(E))$;
$FB(E,E')$	set of all fibre-preserving maps of E to E' ;
F_*	map of $C^\infty(E)$ to $S(E')$ induced by composition with $F \in FB(E,E')$;
$Lgn_k(E)$	set of all k^{th} order Lagrangians on E, i.e. maps $L : C^\infty(E) \longrightarrow S(\mathbb{R}_M)$ of the form $L = F_* \circ j_k$ for some $F \in FB(J^k(E), \mathbb{R}_M)$; F <u>represents</u> L .

It should be noted that the norm in $L^p_k(\xi)$ depends on the volume element $d\mu$ on M, but, by the meaning of 'strictly positive smooth measure', all choices of μ are equivalent to Lebesgue measure in all charts of M, so all induce equivalent norms in $L^p_k(\xi)$, and equivalent Finsler structures on $L^p_k(E)$.

The Finsler metric is defined, technically, only on the path components of $L_k^p(E)$. We set $\delta(s,s') = 2$ whenever s, s' belong to distinct path components.

The reader should also note that no a priori smoothness assumptions are made on the maps F, hence on the Lagrangians L. Such assumptions will be made below as they are needed.

1.2 Formulation of the Problem

We begin by assuming several pieces of data to be given and held fixed throughout the chapter :

[1α] A choice of M, μ, and E ;

[1β] Choices of $p > 1$ and $k \geq 1$ ($pk > n$ if E is not a vector bundle); and a choice of $|| \ ||$ on $T(L_k^p(E))$, with induced δ on $L_k^p(E)$;

[1γ] A locally compact Hausdorff space B ;

[1δ] An element $b \in L_k^p(E)$, a map $r \longmapsto b_r$ of $B \longrightarrow L_k^p(E)$, and a homeomorphism

$$\Phi : B \times L_k^p(E)_b \longrightarrow \mathbb{E} \equiv \left\{(r,s) : r \in B,\ s \in L_k^p(E)_{b_r}\right\}$$

denoted by $(r,t) \longrightarrow (r,\phi_r(t))$, such that each ϕ_r is a C^1 diffeomorphism of $\mathbb{F} \equiv L_k^p(E)_b$ onto $\mathbb{E}_r = L_k^p(E)_{b_r}$ which takes

δ-bounded sets to δ-bounded sets;

[1ε] A map $r \longmapsto L_r$ of $B \longrightarrow Lgn_k(E)$, such that each L_r extends to a C^1 map $L_k^p(E) \longrightarrow L_0^1(\mathbb{R}_M)$.

We think of B in [1γ] as a space of parameters r . Then [1δ] specifies an r-dependent family of Dirichlet-type boundary values on sections of E, and a C^o global trivialization of the map $\pi : \mathbb{E} \longrightarrow B$, $(r,s) \longmapsto r$, with the extra property that Φ^{-1} is smooth on each fibre $\{r\} \times \mathbb{E}_r$. We shall sometimes deviate from strict consistency, and identify the subset $\mathbb{E}_r$ of $L_k^p(E)$ with the fibre $\{r\} \times \mathbb{E}_r$, which is a subset of $\mathbb{E} \subset B \times L_k^p(E)$.

Item [1ε] specifies an r-dependent family of Lagrangians L_r, such that the integral

$$f(r,s) = \int_M L_r s \, d\mu \qquad (r \in B,\ s \in L_k^p(E))$$

defines a function $f : B \times L_k^p(E) \longrightarrow \mathbb{R}$ whose partial-function $f_r \equiv f(r,\cdot)$ is C^1 on $L_k^p(E)$ for each r . The restriction $g = f|_{\mathbb{E}}$ then has a C^1 partial-function $g_r = f_r|_{\mathbb{E}_r}$ for each r ; and the critical locus of g_r, namely

$$K_r = \left\{ s \in \mathbb{E}_r : dg_r(s) = 0 \right\},$$

is a well defined closed subset of $\mathbb{E}_r$. By the <u>standard parametrized Dirichlet problem with data</u> [1α] - [1ε], we mean the problem, to describe the (disjoint) union

$$K = \left\{ (r,s) : r \varepsilon B, \quad s \varepsilon K_r \right\}$$

of the critical loci of all the g_r, as a subset of $\mathbb{E} \subset B \times L_k^p(E)$.

Of course the problem as posed is too general, and in the next section we shall put conditions on the data, in particular on their behaviour with respect to the parameter r in B, which will enable us to prove conclusions of a similar kind about the sets K_r . However, an important restriction has already been built in, by the provision in [1δ] that the disjoint union $\mathbb{E}$ of the spaces $\mathbb{E}_r$ of the r-dependent variational problems should come to us equipped with the structure of a fibre space over B, trivialized by Φ^{-1} . We are thus able to form a 'partial-function' g_t on B, for each t in the model fibre $\mathbb{F}$, namely by

$$g_t(r) = (g\circ\Phi)(r,t) = g_r(\phi_r(t)) .$$

Certain key technical considerations in what follows will be organized around these functions g_t .

1.3 Formulation of Theorem I

Given a standard parametrized Dirichlet problem as defined in §1.2, we shall study the subset of each critical locus K_r on which the function g_r takes its minimum value.

To be precise, we begin with a family $\mathcal{F}$ of subsets of $\mathbb{F} = L^p_k(E)_b$ which is <u>deformation invariant</u>. This means invariant under ambient homotopy of $\mathbb{F}$, i.e., for any continuous map

$$H : [0,1] \times \mathbb{F} \longrightarrow \mathbb{F}$$

such that $H(0,\cdot)$ is the identity on $\mathbb{F}$, and for any $T \in \mathcal{F}$, the set

$$H_1(T) = \left\{ H(1,t) : t \in T \right\}$$

is also a member of the family $\mathcal{F}$ (cf. Browder [2 , Def(1.1)]). Given $\mathcal{F}$, we define a corresponding family $\mathcal{F}(r)$ of subsets of each $\mathbb{E}_r$, by

$$\mathcal{F}(r) = \left\{ S \subset E_r : S = \phi_r(T) \text{ for some } T \in \mathcal{F} \right\} .$$

Then the <u>$\mathcal{F}$-minimax of</u> g is defined to be the function $m_{\mathcal{F}}$ on B, where for any $r \in B$,

$$m_{\mathcal{F}}(r) = \inf_{S \in \mathcal{F}(r)} \sup_{s \in S} g_r(s) ;$$

and the <u>$m_{\mathcal{F}}$-realizing subset of</u> K <u>over</u> r is defined to be

$$K_{\mathcal{F}}(r) = \left\{ (r,s) : s \in K_r \text{ and } g_r(s) = m_{\mathcal{F}}(r) \right\} .$$

More generally, for any subset $C \subset B$, the <u>$m_{\mathcal{F}}$-realizing subset of</u> K <u>over</u> C is the disjoint union

$$K_{\mathcal{F}}(C) = \bigcup_{r \in C} K_{\mathcal{F}}(r) .$$

Our aim is to give conditions on the data under which m_F will be continuous on B, $K_F(r)$ will be nonempty for each r, and $K_F(C)$ will be compact in $\mathbb{E}$ for any compact subset C of B.

The existence of a point in $K_F(r)$ is the main assertion of the generalized Lusternik-Schnirelman theory developed by Palais [5, 6], and Browder [1, 2, 3], where incidentally several examples of families F are given. In the following list of conditions on the data [1α] - [1ε], the first states the standard hypotheses for existence in the Lusternik-Schnirelman theory, and the others mainly require that the existence hypotheses hold uniformly, in various senses, with respect to the parameter r. Note that some conditions refer to the functions g_r, and some to the unrestricted f or f_r.

[1.1] For each $r \in B$, the function $g_r : \mathbb{E}_r \longrightarrow \mathbb{R}$ is bounded below and satisfies condition (C) of Palais-Smale (i.e., any sequence $\{s_i\} \in E_r$ for which $g_r(s_i)$ is bounded and $||dg_r(s_i)||$ converges to zero contains a convergent subsequence).

[1.2] For each $R \in \mathbb{R}$ and each compact subset C of B, the subset

$$K \cap \left\{(r,s) : r \in C \text{ and } g_r(s) \leq R\right\}$$

is compact in $\mathbb{E}$.

[1.3] For each $r \in B$ and each $R \in \mathbb{R}$, there exists a neighbourhood V of r in B such that the subset

$$\mathbb{F}_{R,V} = \left\{ t \in \mathbb{F} : g_t(r') \leq R \quad \text{for some} \quad r' \in V \right\}$$

is bounded in $\mathbb{F}$.

[1.4] For each $r \in B$ and each bounded subset T of $\mathbb{F}$, there exists a neighbourhood V of r in B such that

$$\delta(\phi_r(t), \phi_{r'}(t)) < \varepsilon$$

for all $r' \in V$ and all $t \in T$.

[1.5] For each $r \in B$, each bounded subset S of $L_k^p(E)$, and each $\varepsilon > 0$, there exists a neighbourhood N of r in B such that

$$|f(r,s) - f(r',s)| < \varepsilon$$

for all $r' \in N$ and all $s \in S$.

[1.6] For each $r \in B$ and each bounded subset S of $L_k^p(E)$, there exists $A \in \mathbb{R}$ such that

$$||df_r(s)|| \leq A$$

for all $s \in S$.

With these preparations done, we can state our result :

Theorem I

Suppose a standard parametrized Dirichlet problem is given by data [1α] - [1ε] satisfying conditions [1.1] - [1.6]. Let $\mathcal{F}$ be a deformation-invariant family of subsets of a single path component of $\mathbb{F}$, such that at least one element of $\mathcal{F}$ is compact and nonvoid. Then :

(a) The $\mathcal{F}$-minimax function $m_{\mathcal{F}}$ of g is finite and continuous on B ;

(b) For each $r \in B$, the $m_{\mathcal{F}}$-realizing subset $K_{\mathcal{F}}(r)$ of K over r is not empty;

(c) For each compact subset C of B, the $m_{\mathcal{F}}$-realizing subset $K_{\mathcal{F}}(C)$ of K over C is compact.

REMARK. In case $K_{\mathcal{F}}(r)$ is a singleton $\{s_{\mathcal{F}}(r)\}$ for each r in some open set N in B, so that $s_{\mathcal{F}}$ is a section of $\pi : \mathbb{E} \longrightarrow B$ over N, it follows easily from (c) that $s_{\mathcal{F}}$ is continuous. Thus the critical point of g_r which realizes a particular minimax value $m_{\mathcal{F}}(r)$ varies continuously with r so long as it is unique. In the examples which we have in mind, this is the conclusion of principal interest.

1.4 Proof of Theorem I

We first prove conclusion (b), by assembling several results in Browder [2]. For fixed $r \in B$, let X denote the connected component of $\mathbb{E}_r$ which contains all the sets in the family $\mathcal{F}(r)$, and let h denote the restriction of g_r to X. Then X is a connected submanifold of the C^∞ Finsler manifold $\mathbb{E}_r$, and is complete in the induced metric. Hence [2, Proposition 5.2, p.32] there exists a quasigradient field for h on X. The C^1 function h is bounded below and satisfies condition (C) on

X , on account of condition [1.1], and so by [2, Proposition(5.1), p.27], and the remarks following [2, Theorem 1, p.8 and Definition(2.1), p.18], all the hypotheses of [2, Theorem 1] are satisfied, except for the finiteness of $m_{\mathcal{F}}(r)$. But this follows from the existence of a compact nonvoid element in $\mathcal{F}$, and Browder's Theorem 1 applies to establish (b).

To prove (a) and (c), we apply [7, Theorem 1.2] with our quantities $\mathbb{E}$, $\mathbb{F}$, π, g, and K in place of the quantities E, F, p, f, D of that theorem. It requires that $\mathcal{F}$ be invariant under homeomorphisms of $\mathbb{F}$, but this is used in the proof only to insure that the families $\mathcal{F}(r)$ are independent of the choice of trivialization τ of p. Since we have defined our $\mathcal{F}(r)$ by a fixed trivialization Φ^{-1} of π , we do not need the homeomorphism-invariance of $\mathcal{F}$ to apply the cited result. Its conclusions are precisely the conclusions (a) and (c) of Theorem I; and one of its two hypotheses is just condition [1.2]. The other hypothesis, in our notation, becomes the following : for each $r \in B$ and each $R \in \mathbb{R}$, there exists a neighbourhood V of r in B such that the family of functions

$$\left\{ g_t \ : \ t \in \mathbb{F}_{R,V} \right\}$$

is equicontinuous at r, where, as in [1.3] above,

$$\mathbb{F}_{R,V} = \left\{ t \in F : g_t(r') \leq R \ \text{ for some } \ r' \in V \right\} .$$

Thus the proof of Theorem I will be complete as soon as we have shown how to choose V so that this equicontinuity assertion holds. Here we shall use conditions [1.3] - [1.6].

Fix $r \in B$ and $R \in \mathbb{R}$. By [1.3], there is a neighbourhood V of r in B such that $T = \mathbb{F}_{R,V}$ is bounded in $\mathbb{F}$. By assumption in [1δ], $\phi_r(T)$ is bounded in $\mathbb{E}_r$. By [1.4] with $\varepsilon = 1$, there is a neighbourhood N_1 of r such that $\delta(\phi_r(t), \phi_{r'}(t)) < 1$ for all $r' \in N_1$ and all $t \in T$. Hence the set

$$S_1 = \left\{ \phi_{r'}(t) \; : \; r' \in N_1, \, t \in T \right\}$$

is bounded in $L^p_k(E)$ along with $\phi_r(T)$; and so also is

$$S_2 = \left\{ s \in L^p_k(E) \; : \; \delta(s,s') < 1 \quad \text{for some} \quad s' \in S_1 \right\}.$$

Given $\varepsilon > 0$, we apply [1.5] and [1.6] with $S = S_2$ to find a neighbourhood $N_2 \subset N_1$ of r such that

$$|f(r,s) - f(r',s)| < \frac{\varepsilon}{2} \qquad (r' \in N_2, \, s \in S_2),$$

and a number $A \geq 1$ such that

$$||df_r(s)|| \leq A \qquad (s \in S_2).$$

The we apply [1.4] to find a neighbourhood $N_3 \subset N_2$ of r such that

$$\delta(\phi_r(t), \phi_{r'}(t)) < \min \left\{ 1, \frac{\varepsilon}{2A} \right\}$$

for $r' \in N_3$, $t \in T$.

Now choose any $t \varepsilon T = \mathbb{F}_{R,V}$ and any $r' \varepsilon N_3$. In view of the convention adopted in §1.1, that points s, s' in different path components of $L^p_k(E)$ have $\delta(s,s') = 2$, the last inequality implies, in particular, that the points $s = \phi_r(t)$ and $s' = \phi_{r'}(t)$ can be joined by a C^1 path γ in $L^p_k(E)$ with

$$\text{length } \gamma < \min\left\{1, \frac{\varepsilon}{2A}\right\}.$$

Any point on such a path can be no further than 1 from either endpoint, so lies in S_2. Accordingly we can apply the first, second and last of the preceeding inequalities, to get

$$\begin{aligned}
&|g_t(r') - g_t(r)| \\
&= |g_{r'}(\phi_{r'}(t)) - g_r(\phi_r(t))| \\
&= |f_{r'}(s') - f_r(s')| + |f_r(s') - f_r(s)| \\
&< \frac{\varepsilon}{2} + \int_0^1 ||df_r(\gamma(u))||\ ||\gamma'(u)||\ du \\
&\leq \frac{\varepsilon}{2} + A\cdot(\text{length } \gamma) < \varepsilon .
\end{aligned}$$

Thus $\{g_t : t \varepsilon \mathbb{F}_{R,V}\}$ are equicontinuous at r as claimed, and the proof of Theorem I is complete .

CHAPTER 2

DIRICHLET PROBLEMS WITH VARIABLE DOMAINS

2.1 Formulation of Theorem II

In this chapter we shall consider nonlinear Dirichlet problems in which the candidates for solutions are real-valued functions on a bounded domain $\Omega \subset \mathbb{R}^n$, agreeing on the boundary $\partial\Omega$ with a prescribed function a. Our interest is in the behaviour of the solution when both Ω and a are perturbed. The restriction to real valued functions is for convenience of notation only, and the reader will easily see how to modify our statements to cover the case of vector-valued functions.

We start with a bounded open set $\Omega^o \subset \mathbb{R}^n$, with typical point $x = (x_1,\ldots,x_n)$ and with Lebesgue measure $d\omega(x) = dx_1 \ldots dx_N$. An n-multi-index is an n-tuple $\alpha = (\alpha_1,\ldots,\alpha_n)$ of integers $\alpha_i \geq 0$, and we will write $|\alpha|$ for the sum $\sum_i \alpha_i$, and D^α for the partial derivative operator $\prod(\partial/\partial x_i)^{\alpha_i}$. For any integer $k \geq 0$, s_k will denote the total number of α's having $|\alpha| \leq k$, and $u = (u_\alpha)_{|\alpha|\leq k}$ will denote a typical point in $\mathbb{R}^{s_k}$. Note that $\mathbb{R}^{s_k}$ is just the fibre of the k-jet bundle $J^k(\mathbb{R}_{\Omega_o})$ of the product bundle $\mathbb{R}_{\Omega_o} = \Omega^o \times \mathbb{R}$ over Ω^o, whose sections we identify with the real valued functions on Ω^o. More generally, for any product bundle $\mathbb{R}^\ell_{\Omega^o}$ over Ω^o with fibre $\mathbb{R}^\ell$ we have a natural identification of $L^p_k(\mathbb{R}^\ell_{\Omega^o})$ with $L^p_k(\Omega^o, \mathbb{R}^\ell)$ and we shall use this identification freely in the following.

With $n \geq 1$, $k \geq 1$, and Ω^o fixed, let $F : \Omega^o \times \mathbb{R}^{s_k} \longrightarrow \mathbb{R}$ be a C^2 function. (Thus F is the principal part of an element of $FB[J^k(\mathbb{R}_{\Omega^o}), \mathbb{R}_{\Omega^o}]$.) Suppose F satisfies the following conditions for a certain $p \geq 2$, certain constants $C, C_1 > 0$, and a certain continuous function $\mathbb{C} : R^n \longrightarrow R$ with $\mathbb{C}(0) = 0$:

[2.1] For all $x \in \Omega^o$ and all $u \in \mathbb{R}^{s_k}$,

$$|F(x,u)| \leq C\left(1 + \sum_{|\alpha| \leq k} |u_\alpha|^p\right).$$

[2.2] For all $x, x' \in \Omega^o$ and all $u \in \mathbb{R}^{s_k}$,

$$|F(x,u) - F(x',u)| \leq \mathbb{C}(x - x')\left(1 + \sum_{|\alpha| \leq k} |u_\alpha|^p\right).$$

[2.3] For all α, all $x \in \Omega^o$, and all $u \in \mathbb{R}^{s_k}$,

$$\left|\frac{\partial F}{\partial u_\alpha}(x,u)\right| \leq C\left(1 + \sum_{|\beta| \leq k} |u_\beta|^{p-1}\right).$$

[2.4] For all α, all $x, x' \in \Omega^o$, and all $u \in \mathbb{R}^{s_k}$,

$$\left|\frac{\partial F}{\partial u_\alpha}(x,u) - \frac{\partial F}{\partial u_\alpha}(x',u)\right| \leq \mathbb{C}(x - x')\left(1 + \sum_{|\beta| \leq k} |u_\beta|^{p-1}\right).$$

[2.5] For all α, β, all $x \in \Omega^o$, and all $u \in \mathbb{R}^{s_k}$,

$$\left|\frac{\partial^2 F}{\partial u_\alpha \partial u_\beta}(x,u)\right| \leq C\left(1 + \sum_{|\gamma|\leq k} |u_\gamma|^{p-2}\right).$$

[2.6] For all $x \in \Omega^o$ and all $u, v \in \mathbb{R}^{s_k}$,

$$\sum_{|\alpha|,|\beta|\leq k} \frac{\partial^2 F}{\partial u_\alpha \partial u_\beta}(x,u)v_\alpha v_\beta \geq C_1 \sum_{|\alpha|=k} |u_\alpha|^{p-2}|v_\alpha|^2 .$$

Conditions [2.1], [2.3], and [2.5] in particular imply that the integral

$$h^o(v) = \int_{\Omega^o} F(x, j_k(v)(x))\, d\omega(x)$$

is well defined for all v in the Sobolev space $L_k^p(\Omega^o, \mathbb{R})$, and that h^o is in fact a C^2 function (c.f. Lemma 2.1 in §2.2 below). Here the symbol $j_k(v)$ is being abused to denote the principal part of the k-jet of the section of $L_k^p(\mathbb{R}_{\Omega^o})$ whose principal part is v. We shall consider 'restrictions' $h = h^\Omega$ obtained by replacing Ω^o in the above integral by various subdomains $\Omega \subset \Omega^o$.

To be precise, for all r in a suitable parameter space B, we consider an open subdomain $\Omega_r \subset \Omega^o$ whose closure is obtained as the diffeomorphic image $\lambda(r)(M)$ of a fixed smooth manifold M. As r ranges over B, $\lambda(r)$ is supposed to change continuously in the space $\mathrm{Diffeo}[M,\Omega^o]$ of C^∞ diffeomorphisms of M into Ω^o, equipped with the topology of uniform convergence of all derivatives. We also suppose that a suitable boundary-function a_r is given for each r. Our result is the following.

Theorem II

Fix a bounded open subset $\Omega^o \subset \mathbb{R}^n$, an integer $k \geq 1$, and a real number $p \geq 2$. Let $F : \Omega^o \times \mathbb{R}^{s_k} \longrightarrow \mathbb{R}$ be C^2 and satisfy conditions [2.1] - [2.6] given above.

Fix a locally compact Hausdorff space B and a compact n-dimensional C^∞ manifold M with boundary ∂M. Let $r \longmapsto a_r$ be continuous from B into $L_k^p(\Omega^o, \mathbb{R})$, and let $r \longmapsto \lambda(r)$ be continuous from B into $\mathrm{Diffeo}[M, \Omega^o]$.

For each $r \in B$, set $\overline{\Omega}_r = \lambda(r)(M)$ and set $D_r = L_k^p(\Omega_r, \mathbb{R})_{a_r}$, and let h_r be the C^2 real valued function on D_r defined by

$$h_r(v) = \int_{\Omega_r} F(x, j_k(v)(x)) \, d\omega(x) .$$

Then the following conclusions hold :

(a) The function $m : B \longrightarrow \mathbb{R}$ defined by

$$m(r) = \inf_{v \in D_r} h_r(v)$$

is finite and continuous on B ;

(b) For each $r \in B$, the set

$$M(r) = \left\{ v \in D_r : dh_r(v) = 0 \text{ and } h_r(v) = m(r) \right\}$$

is not void;

(c) For each compact subset C of B, the set

$$\lambda^*M(C) = \left\{ (r,\ v\circ\lambda(r)) : r \in C \text{ and } v \in M(r) \right\}$$

is a compact subset of $B \times L_k^p(\mathbb{R}_M)$.

We shall deduce Theorem II from Theorem I of §1.2, by constructing a suitable standard parametrized problem. It should be noted that our conditions [2.1] - [2.6] can be weakened considerably by exploiting the Sobolev inequalities (cf. Browder [1, pp 25-29]). The techniques for doing this are well established and we shall not dwell further on the point.

Before carrying out the reduction to Theorem I, we shall establish some properties of the functions h_r .

2.2 Preliminary consequences of the assumptions

In the following we use the assumption that F satisfies [2.1]-[2.6].

Lemma 2.0

For each $\Omega \subset \Omega^o$ the map

$$F_* : C^\infty(J^k(\mathbb{R}_\Omega)) \longrightarrow S(\mathbb{R}_\Omega)$$

extends to a C^2 map

$$F_* : L_0^p(J^k(\mathbb{R}_\Omega)) \longrightarrow L_0^1(\mathbb{R}_\Omega) .$$

Furthermore, the maps

$$d(F_*) : L_0^p(J^k(\mathbb{R}_\Omega)) \longrightarrow L(L_0^p(J^k(\mathbb{R}_\Omega), L_0^1(\mathbb{R}_\Omega))$$

and

$$d^2(F_*) : L_0^p(J^k(\mathbb{R}_\Omega)) \longrightarrow L_S^2(L_0^p(J^k(\mathbb{R}_\Omega)), L_0^1(\mathbb{R}_\Omega,))$$

take bounded sets into bounded sets. (The reader is warned that the symbol L_S^2 above means symmetric bilinear maps and should not be confused with L_k^p).

Finally for v, v_1, and $v_2 \in L_0^p(J^k(\mathbb{R}_\Omega))$,

$$d(F_*)(v)(v_1) = \delta F_v(v_1)$$

and

$$d^2(F_*)(v)(v_1,v_2) = \delta^2 F_v(v_1,v_2) .$$

<u>Proof</u> : See Browder [1] .

For $\Omega \subset \Omega^o$ let $H^\Omega : L_0^p(J^k(\mathbb{R}_\Omega)) \longrightarrow \mathbb{R}$ be defined by

$$H^\Omega(v) = \int_\Omega F_* v(x)\, d\omega(x) .$$

Let $h^\Omega : L_k^p(\mathbb{R}_\Omega) \longrightarrow \mathbb{R}$ be defined by $h^\Omega = H^\Omega \circ j_k$.

<u>Lemma 2.1(i)</u>

For each $\Omega \subset \Omega^o$, $H^\Omega : L_0^p(J^k(\mathbb{R}_\Omega)) \longrightarrow \mathbb{R}$ is a C^2 function such that :

(a) $dH : L_0^p(J^k(\mathbb{R}_\Omega)) \longrightarrow L_0^p(J^k(\mathbb{R}_\Omega))^*$ takes bounded sets into bounded sets ;

(b) dH is uniformly continuous on bounded sets of $L_0^p(J^k(\mathbb{R}_\Omega))$.

<u>Proof</u> : The fact that H^Ω is C^2 follows immediately from lemma 2.0. Part (b) follows from the boundedness of $d^2(F_*)$ in lemma 2.0 together with the mean value theorem.

<u>Lemma 2.1(ii)</u>

For each $\Omega \subset \Omega^o$, $h^\Omega : L_k^p(\mathbb{R}_\Omega) \longrightarrow \mathbb{R}$ is a C^2 function such that

(a) $dh^\Omega : L_k^p(\mathbb{R}_\Omega) \longrightarrow L_k^p(\mathbb{R}_\Omega)^*$ and $d^2h^\Omega : L_k^p(\mathbb{R}_\Omega) \longrightarrow L^2(L_k^p(\mathbb{R}_\Omega),\mathbb{R})$ takes bounded sets into bounded sets ;

(b) dh^Ω is uniformly continuous on bounded subsets of $L_k^p(\mathbb{R}_\Omega)$.

<u>Proof</u> : Since $h^\Omega = H^\Omega \circ j_k$ and $j_k : L_k^p(\mathbb{R}_\Omega) \longrightarrow L_0^p(J^k(\mathbb{R}_\Omega))$ is a bounded linear map the proof follows immediately from lemma 2.1(i).

<u>Lemma 2.2</u>

For $\Omega \subset \Omega^o$ and $v_1, v_2 \in L_k^p(\mathbb{R}_\Omega)$

$$[dh^\Omega(v_1) - dh^\Omega(v_2)](v_1 - v_2)$$

$$\geq C \sum_{|\alpha|=k} \int_\Omega |D^\alpha v_1(x) - D^\alpha v_2(x)|^p \, d\omega(x)$$

for some constant C .

Proof : From lemma 2.0 it follows that

$$dh^{\Omega}(v_3)(v_1,v_2)$$

$$= \int_{\Omega} \sum_{0\le|\alpha|,|\beta|\le k} \frac{\partial^2 F}{\partial u_\alpha \partial u_\beta}(x,j_k(v_3)(x))u_\alpha(j_k(v_1)(x))u_\beta(j_k(v_2)(x)) .$$

So

$$[dh^{\Omega}(v_1) - dh^{\Omega}(v_2)](v_1 - v_2)$$

$$= \int_{\Omega} \int_0^1 \left(\sum_{0\le|\alpha|,|\beta|\le k} \frac{\partial^2 F}{\partial u_\alpha \partial u_\beta}(x,j_k(v_1) + t(j_k(v_2) - j_k(v_1)(x))\right) \times \left[u_\alpha(j_k(v_1-v_2)(x))u_\beta(j_k(v_1-v_2)(x))\right] dt\, d\omega(x)$$

and by [2.6]

$$\geq \int_{\Omega} \sum_{|\alpha|=k} \int_0^1 |D^\alpha v_1(x) + t(D^\alpha(v_2-v_1)(x))|^{p-2}|D^\alpha(v_2-v_1)(x)|^2\, dt\, d\omega(x)$$

$$\geq c \sum_{|\alpha|=k} \int_{\Omega} |D^\alpha v_1(x) - D^\alpha v_2(x)|^p\, dx$$

for some constant c .

Lemma 2.3

For $v_1, v_2, a_1, a_2 \in L_k^p(\mathbb{R}_\Omega)$ with $v_1 \in L_k^p(\mathbb{R}_\Omega)_{a_1}$ and $v_2 \in L_k^p(\mathbb{R}_\Omega)_{a_2}$,

$$[dh^{\Omega}(v_1) - dh^{\Omega}(v_2)](v_1-v_2) \geq C\left(||v_1-v_2-(a_1-a_2)||^p_{L_k^p}\right) - ||(a_1-a_2)||^p_{L_k^p}$$

for some constant C independent of Ω .

Proof : By lemma 2.2 we need only show that there exists a constant c independent of Ω such that

$$\sum_{|\alpha|=k} \int_\Omega |D^\alpha(v_1-v_2)(x)|^p \, d\omega(x)$$

$$\geq c \left(||(v_1 - v_2) - (a_1 - a_2)||^p_{L^p_k} \right) - ||a_1 - a_2||^p_{L^p_k} .$$

Now $v_1 - v_2 - (a_1 - a_2) \in L^p_k(\mathbb{R})_0$. Therefore there exists a constant c independent of Ω such that

$$\sum_{|\alpha|=k} \int_\Omega |D^\alpha(v_1-v_2)(x) - D^\alpha(a_1-a_2)(x)|^p \, d\omega(x)$$

$$\geq c \, ||v_1 - v_2 - (a_1 - a_2)||^p_{L^p_k} .$$

Now there exists a constant C_1 independent of Ω such that

$$C_1 \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha(v_1-v_2)(x) - D^\alpha(a_1-a_2)(x)|^p \, d\omega(x) \right)$$

$$\leq \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha(v_1-v_2)(x)|^p \, d\omega(x) + \int_\Omega |D^\alpha(a_1-a_2)(x)|^p \, d\omega(x) \right.$$

$$\leq \sum_{|\alpha|=k} \int_\Omega |D^\alpha(v_1-v_2)(x)|^p \, d\omega(x) + ||a_1 - a_2||^p_{L^p_k} .$$

Therefore there exists a constant C independent of Ω such that

$$\sum_{|\alpha|=k} \int_\Omega |D^\alpha(v_1-v_2)(x)|^p \, d\omega(x) + ||a_1-a_2||^p_{L^p_k} \geq C||v_1-v_2-(a_1-a_2)||^p_{L^p_k} ,$$

and the result follows.

Lemma 2.4

Let $\{v_i\}$ be a bounded sequence in $L_k^p(\mathbb{R}_\Omega)$ such that $v_i \in L_k^p(\mathbb{R}_\Omega)_{a_i}$ with $\{a_i\}$ converging to a. Then if $dh^\Omega(v_i) \longrightarrow 0$, it follows that $\{v_i\}$ is a convergent sequence in $L_k^p(\mathbb{R}_\Omega)$.

Proof : Fix $\varepsilon > 0$. Pick A so large that $i, j > A$ implies that

$$||a_i - a_j||^p_{L_k^p} \leq \delta_1 ,$$

and

$$[dh^\Omega(v_1) - dh^\Omega(v_2)](v_1 - v_2) < \delta_2$$

for some δ_1, δ_2 to be determined. By lemma 2.3

$$\delta_2 \geq C\left(||v_1-v_2-(a_1-a_2)||^p_{L_k^p} \right) - ||a_1-a_2||^p_{L_k^p} .$$

Since

$$||v_1-v_2-(a_1-a_2)||_{L_k^p} \geq ||v_1-v_2||_{L_k^p} - ||a_1-a_2||_{L_k^p} ,$$

we get

$$\delta_2 \geq C\left(||v_1-v_2||_{L_k^p} - ||a_1-a_2||_{L_k^p} \right)^p - ||a_1-a_2||^p_{L_k^p} .$$

Since $\{v_i\}$ is a bounded sequence there exists δ_1 such that

$$||a_1 - a_2||_{L_k^p} < \delta_1$$

implies that

$$\left| C\left(||v_1-v_2||_{L^p_k} - ||a_1-a_2||_{L^p_k} \right)^p - ||a_1-a_2||^p_{L^p_k} - C||v_1-v_2||^p_{L^p_k} \right| < \frac{C\varepsilon}{2} .$$

It follows that

$$C||v_1 - v_2||^p_{L^p_k} < \frac{C\varepsilon}{2} + \delta_2 .$$

Setting $\delta_2 = \frac{C\varepsilon}{2}$ we get

$$||v_i - v_j||_{L^p_k} < \varepsilon .$$

<u>Lemma 2.5</u>

Fix $S \subset L^p_k(\mathbb{R}_\Omega)$ bounded and $R \in \mathbb{R}$. Let h denote h^{Ω^o}. Then the set

$$\left\{ v \in \bigcup_{w \in S} L^p_k(\mathbb{R}_{\Omega^o})_w : \quad h(v) \leq R \right\}$$

is bounded in $L^p_k(\mathbb{R}_{\Omega^o})$.

<u>Proof</u> : Fix $v \in L^p_k(\mathbb{R}_{\Omega^o})_w$. Suppose $v = t + w$ with $t \in L^p_k(\mathbb{R}_{\Omega^o})_0$. Then

$$h(v) = h(t + w) = h(w) + \int_0^1 dh(w + ut)(t)\, du$$

$$= h(w) + dh(w)(t) + \int_0^1 [dh(w+ut) - dh(w)](t)\, du$$

$$= h(w) + dh(w)(t) + \int_0^1 \frac{1}{u} [dh(w+ut) - dh(w)](ut)\, du$$

By lemma 2.2

$$\geq \; h(w) + dh(w)(t) + C\int_0^1 \frac{1}{u}||ut||^p_{L^p_k}\,du$$

$$= \; h(w) + dh(w)(t) + \frac{C}{p}\,||t||^p_{L^p_k} \; .$$

Since S is bounded, by lemma 2.1(ii)

$$\sup_{w\varepsilon S} ||dh(w)|| = A < \infty \; ,$$

$$\sup_{w\varepsilon S} |h(w)| = B < \infty \; .$$

Therefore

$$h(w + t) \geq -B - A||t||_{L^p_k} + \frac{C}{p}\,||t||^p_{L^p_k}$$

for all $w \varepsilon S$, where $C > 0$, $p > 0$. It follows immediately that $h(w + t) < R$ with $w \varepsilon S$ implies that $||t||_{L^p_k}$ is bounded.

<u>Lemma</u> <u>2.6</u>

Fix $S \subset L^p_k(\mathbb{R}_{\Omega^o})$ bounded and $R \varepsilon \mathbb{R}$. For any $\Omega \subset \Omega^o$ let

$$T = \left\{ w|_{\Omega} : w \varepsilon S \right\}$$

i.e. T consists of the "restriction" of S to Ω . Then there exists a constant $B \varepsilon \mathbb{R}$, independent of the choice of Ω such that

$$v \varepsilon \left\{ v \varepsilon \bigcup_{w\varepsilon T} L^p_k(\mathbb{R}_\Omega)_w : h^\Omega(v) < R \right\}$$

implies that $||v||_{L^p_k} < B$.

Proof : Pick $v \varepsilon L_k^p(\mathbb{R}_\Omega)_w$. Then $v = w + t$ for $t \varepsilon L_k^p(\mathbb{R}_\Omega)_0$, and w is the restriction of some $w_1 \varepsilon L_k^p(\mathbb{R}_{\Omega^o})$ to Ω . Extend v to Ω^o by setting it equal to w_1 on $\Omega^o - \Omega$. Call the extension v_1 . Then

$$h^{\Omega^o}(v_1) = h^\Omega(v) + \int_{\Omega^o - \Omega} F \circ j_k(w_1)\, d\omega$$

$$\leq R + \left| \int_{\Omega^o} F \circ j_k(w_1)\, d\omega \right| .$$

By lemma 2.1(ii) there exists a constant $c \varepsilon \mathbb{R}$ such that

$$\left| \int_{\Omega^o} F \circ j_k(w_1) d\omega \right| < c$$

for all $w_1 \varepsilon S$. Therefore

$$h^{\Omega^o}(v_1) \leq R + c .$$

By lemma 2.5 there exists a constant $A \varepsilon \mathbb{R}$ with $||v_1||_{L_k^p} < A$. Then it follows that

$$||v||_{L_k^p} \leq A + \sup_{w_1 \varepsilon S} ||w_1||_{L_k^p}$$

and the result follows for

$$B = A + \sup_{w_1 \varepsilon S} ||w_1||_{L_k^p} .$$

Lemma 2.7

For each Ω, and $w \in L_k^p(\mathbb{R}_\Omega)$, the restriction of h^Ω to $L_k^p(\mathbb{R}_\Omega)_w$ is bounded below.

Proof : In the course of verifying lemma 2.5 we derived the inequality

$$h(t + w) \geq - B - A||t||_{L_k^p} + \frac{c}{p}||t||_{L_k^p}^p$$

for $t \in L_k^p(\mathbb{R}_\Omega)_0$, and lemma 2.7 follows immediately from this inequality.

We are now ready to proceed with the reduction from Theorem II to Theorem I .

2.3 Construction of the Associated Standard Problem

We construct the standard problem by defining the data [1α] - [1ε].

[1α] M has already been determined. Let μ be any strictly positive smooth measure on M, and let E be the product bundle $\mathbb{R}_M$.

[1β] p, k are already determined. Let $||\ ||$ be the standard norm on $L_k^p(\mathbb{R}_M)$, so that $\delta(s_1,s_2) = ||s_1 - s_2||_{L_k^p}$.

[1γ] B has been determined.

For each $r \in B$ the map $\lambda(r) : M \longrightarrow \Omega_r \subset \Omega^o$ induces a map $\lambda(r)^* : L_k^p(\Omega_r,\mathbb{R}) \longrightarrow L_k^p(M,\mathbb{R})$ given by $\lambda(r)^*(v) = v \circ \lambda(r)$. Recall that we have a map $B \longrightarrow L^p_k(\Omega^o,\mathbb{R})$ given by $r \longmapsto a_r$. We define a map $B \longrightarrow L^p_k(M,\mathbb{R})$ by

$$r \longmapsto b_r \overset{\text{def}}{\equiv} \lambda(r)^*(a_r) .$$

[1δ] Let b = the zero section of $L^p_k(\mathbb{R}_M)$, and the map $r \longmapsto b_r$ as given above. Let $\Phi : B \times L^p_k(\mathbb{R}_M)_0 \longrightarrow \mathbb{E}$ be defined by $\Phi(r,s) = (r,s+b_r)$ so that $\phi_r(s) = s + b_r$.

For each $r \in M$, let $\alpha_r \in C^\infty(M,\mathbb{R})$ be defined by

$$\alpha_r(y)\lambda(r)^*\omega(y) = \mu(y) .$$

[1ε] For $s \in L^p_k(\mathbb{R}_M)$ and $r \in B$, let

$$L_r s(y) = \alpha_r(y)F(\lambda(r)(y),j_k(s\circ\lambda(r)^{-1})(\lambda(r)(y)))$$

where we repeat that the symbol $j_k(\cdot)$ is being abused to denote the principal part of the k-jet of the section $s\circ\lambda(r)^{-1} \in L^p_k(\mathbb{R}_{\Omega_r})$.

Assuming for the moment that the standard parametrized problem with the above data satisfies the hypotheses of Theorem I, we indicate how Theorem II follows.

From the definition of L_r it follows that for $s \in L^p_k(\mathbb{R}_M)$ that

$$\int_M L_r s(y)\, d\mu(y) = \int_{\Omega_r} L\circ(s\circ\lambda(r)^{-1})(v)\, d\mu(x)$$

so that $g_r(t) = h_r(t\circ\lambda(r)^{-1})$. Now in Theorem I let

$$\mathcal{F} = \left\{ \{s\} : s \in L_k^p(\mathbb{R}_M)_0 \right\} .$$

Then

$$m_{\mathcal{F}}(r) = \inf_{t\in E_r} g_r(t) = \inf_{v\in D_r} h_r(v) = m(r)$$

So $m_{\mathcal{F}}(r) = m(r)$ and hence $m(r)$ is finite and continuous on B. Also,

$$\begin{aligned} M(r) &= \left\{ v \in D_r : dh_r(v) = 0 \text{ and } h_r(v) = m(r) \right\} \\ &= \left\{ v \in D_r : dg_r(v\circ\lambda(r) = 0 \text{ and } g_r(t\circ\lambda(r)) = m_{\mathcal{F}}(r) \right\} \end{aligned}$$

Therefore $v \in M(r)$ iff $v\circ\lambda(r) \in K_{\mathcal{F}}(r)$ from which it follows that $M(r)$ is not void.

Finally $\lambda^*M(C) = \left\{(r, v\circ\lambda(r)) : r \in C \text{ and } v \in M(r)\right\}$ and from the above discussion it follows that $\lambda^*M(C) = K_{\mathcal{F}}(C)$ and from Theorem I it follows that $\lambda^*M(C)$ is compact.

Therefore in order to prove Theorem II we need only verify conditions [1.1] - [1.6] for the standard problem just constructed.

2.4 Verification of Conditions [1.1] - [1.6]

We shall carry out the verifications in the order 1, 4, 3, 5, 6, 2.

Verification of [1.1] :

Fix $r \in B$ and $t \in L^p_k(\mathbb{R}_M)_{b_r}$. Then $g_r(t) = h^{\Omega_r}(t \circ \lambda(r)^{-1})$ and by lemma 2.6 g_r is bounded below.

Now assume that $\{t_i\}$ is a bounded sequence with $||dg_r(t_i)|| \to 0$. It follows that $||dh^{\Omega_r}(t_i \circ \lambda(r)^{-1})|| \to 0$. By lemma 2.4 $\{t_i \circ \lambda(r)^{-1}\}$ is a Cauchy sequence. Hence $\{t_i\}$ is a Cauchy sequence.

Verification of [1.4] :

Since $\phi_r(s) = s + b_r$,

$$\delta(\phi_r(s), \phi_{r'}(s)) = ||s + b_r - s + b_{r'}||_{L^p_k}$$

$$= ||b_r - b_{r'}||_{L^p_k}$$

$$= \sum_\alpha \int_M |D^\alpha(a_r \circ \lambda(r))(y) - D^\alpha(a_{r'} \circ \lambda(r'))(y)|^p \, d\mu$$

$$\leq \sum_\alpha \int_M |D^\alpha(a_r \circ \lambda(r))(y) - D^\alpha(a_r \circ \lambda(r'))(y)|^p \, d\mu$$

$$+ \sum_\alpha \int_M |D^\alpha(a_r \circ \lambda(r'))(y) - D^\alpha(a_{r'} \circ \lambda(r'))(y)|^p \, d\mu .$$

It follows from the continuity of the functions $r \longmapsto a_r$ and $r \longmapsto \lambda(r)$

that the above terms can be made arbitrarily small for r' in some neighbourhood V_r of r.

Verification of [1.3] :

Fix $r \varepsilon B$ and $R \varepsilon \mathbb{R}$. Let W be a neighbourhood of r such that the set $\{a_{r'} : r' \varepsilon W\}$ is bounded in $L^p_k(\mathbb{R}_{\Omega^0})$. Now $g_t(r') \leq R$ implies that $h^{\Omega_{r'}}((t+b_{r'})\circ\lambda(r')^{-1}) \leq R$. From lemma 2.6 it follows that there exists a $B \varepsilon \mathbb{R}$ such that

$$||(t + b_{r'})\circ\lambda(r')^{-1}||_{L^p_k} < B .$$

Then

$$||t\circ\lambda(r')^{-1}||_{L^p_k} \leq B + ||b_{r'}\circ\lambda(r')^{-1}||_{L^p_k} .$$

But since the sections $b_{r'}\circ\lambda(r')^{-1}$ are just the restrictions of the $a_{r'}$'s to the various $\Omega_{r'}$'s it follows that

$$\sup_{r'\varepsilon V} ||b_{r'}\circ\lambda(r')^{-1}||_{L^p_k} < \infty$$

and that there exists an $A \varepsilon \mathbb{R}$ with

$$||t\circ\lambda(r')^{-1}||_{L^p_k} < A$$

for all $t \varepsilon F_{R,W}$ and all $r' \varepsilon W$.

From the continuity of the map

$$r' \longmapsto \lambda(r')$$

it is easily seen that there exists a neighbourhood $V \subset W$ of r and a constant $A_1 \in \mathbb{R}$ such that

$$||t \circ \lambda(r')^{-1}||_{L^p_k} < A \quad \text{and} \quad r' \in V$$

implies that $||t||_{L^p_k} < A_1$. From this it follows that $\mathbb{F}_{R,V}$ is bounded.

<u>Verification of [1.5]</u> :

For $s \in L^p_k(\mathbb{R}_M)$ let $s_r \in L^p_k(\mathbb{R}_{\Omega_r})$ denote $s \circ \lambda(r)^{-1}$. For $y \in M$, let $x_r = \lambda(r)(y)$. Then for $r, r' \in B$

$$|f(r,s) - f(r',s)| = \left| \int_M [L_r s - L_{r'} s] \, d\mu \right|$$

$$= \left| \int_{\Omega_r} Ls_r \, d\omega - \int_{\Omega_{r'}} Ls_{r'} \, d\omega \right|$$

$$= \left| \int_{\Omega_r} F(x_r, j_k(s_r)(x_r)) \, d\omega(x_r) - \int_{\Omega_{r'}} F(x_{r'}, j_k(s_{r'})(x_{r'})) \, d\omega(x_{r'}) \right|$$

Let $\beta : \Omega_r \longrightarrow \Omega_{r'}$ be given by

$$\beta = \lambda(r') \circ \lambda(r)^{-1} .$$

Then $x_{r'} = \beta(x_r)$ and we rewrite the above as

$$\left| \int_{\Omega_r} F(x_r, j_k(s_r)(x_r))\, d\omega(x_r) \right.$$

$$\left. - \int_{\Omega_r} F(\beta(x_r), j_k(s_{r'})(\beta(x_r)))\, d(\beta^*\omega)(x_r) \right|$$

$$\text{(A)} \qquad \leq \left| \int_{\Omega_r} \left[F(x_r, j_k(s_r)(x_r)) - F(x_r, j_k(s_{r'})(\beta(x_r))) \right] d\omega(x_r) \right|$$

$$\text{(B)} \qquad + \left| \int_{\Omega_r} \left[F(x_r, j_k(s_{r'})(\beta(x_r))) - F(\beta(x_r), j_k(s_{r'})(\beta(x_r))) \right] d\omega(x_r) \right|$$

$$\text{(C)} \qquad + \left| \int_{\Omega_r} F(\beta(x_r), j_k(s_{r'})(\beta(x_r)))\, d[\omega - \beta^*\omega](x_r) \right| .$$

We shall deal with each of the above terms separately.

(A) $x_r \longmapsto (x_r, j_k(s_{r'})(\beta(x_r)))$ is a section in $L_0^p(J^k(\mathbb{R}_{\Omega_r}))$. In fact it is just

$$\beta^*(j_k(s_{r'})) .$$

Furthermore, for $s \in S \subset L_k^p(\mathbb{R}_M)$ bounded and s_r, $s_{r'}$ as defined above there exists a neighbourhood V_r of r such that

$$||\beta^*(j_k(s_{r'})) - s_r||_{L_0^p} < \varepsilon$$

for all $s \in S$ and $r' \in V_r$. Now (A) is the same as

$$H^{\Omega_r}(j_k(s_r)) - H^{\Omega_r}(\beta^*(j_k(s_{r'})))$$

And from the boundedness of dH^{Ω_r} which follows from lemma 2.1(i) we get a bound on (A) which can be made arbitrarily small.

(B) By condition [2.2] (B) is bounded by

$$\int_{\Omega_r} |\mathbb{C}(x-\beta(x_r))| \left(1 + \sum_{\alpha} |u_\alpha(j_k(s_{r'}))(\beta(x_r))|^p\right) d\omega .$$

Now for any $\varepsilon > 0$ there exists a neighbourhood W of r such that for $r' \varepsilon W$,

$$\sup_{x_r \varepsilon \Omega_r} |\mathbb{C}(x_r - \beta(x_r))| < \varepsilon .$$

(Recall that β depends on r'). Therefore (B) is bounded by

$$\varepsilon \int_{\Omega_r} \left(1 + \sum_{\alpha} |u_\alpha(j_k(s_{r'})(\beta(x_r))|^p \right) d\omega .$$

Since $s_{r'}$ is derived from $s \varepsilon S$ a bounded set, there exists a constant $A \varepsilon \mathbb{R}$ such that

$$\int_{\Omega_r} \left(1 + \sum_{\alpha} |u_\alpha(j_k(s_{r'})(\beta(x_r))|^p \right) d\omega \leq A$$

for all $s \varepsilon S$. It follows that (B) can be made arbitrarily small.

(C) From the continuity of the map $r \longmapsto \lambda(r)$ it is easily seen that the difference of the measure $\omega - \beta^*\omega$ can be made uniformly small over Ω_r by restricting r' to lie in a suitable neighbourhood of r . It follows that (C) can be made arbitrarily small.

<u>Verification of [1.6]</u> :

Let $S \subset L^p_k(\mathbb{R}_M)$ be bounded. Then the set $\{ s \circ \lambda(r)^{-1} : s \varepsilon S \}$

is bounded in $L_k^p(\mathbb{R}_{\Omega_r})$. Now

$$df_r(s)(t) = dH^{\Omega'}(j_k(s\circ\lambda(r)^{-1})\circ j_k(t\circ\lambda(r)^{-1})).$$

From lemma 2.1(ii) it follows that

$$\sup_{s\varepsilon S} ||dH^{\Omega'}(j_k(s\circ\lambda(r)^{-1}))|| < \infty ,$$

and since $L_k^p(\mathbb{R}_M) \longrightarrow L_0^p(J^k(\mathbb{R}_{\Omega_r}))$ given by $t \longmapsto j_k(t\circ\lambda(r)^{-1})$ is bounded it follows that

$$\sup_{s\varepsilon S} ||df_r(s)|| < \infty .$$

Verification of [1.2] :

Fix $R \varepsilon \mathbb{R}$ and $C \subset B$ compact. Let $\{(r_i,s_i)\}$ be a sequence with

$$\{(r_i,s_i)\} \subset K \cap \left\{(r',s) : r' \varepsilon C \text{ and } g_{r'}(s) \leq R\right\} .$$

We need to show that $\{(r_i,s_i)\}$ has a convergent subsequence.

Since C is compact we can assume that $\{r_i\}$ is a convergent sequence and we assume that $\{r_i\}$ converges to $r \varepsilon B$. From conditions [1.3] and [1.4] it follows that $\{s_i\}$ is bounded. In order to proceed we need the following proposition whose proof will be found below.

Lemma 2.8

Fix $S \subset L_k^p(R_M)$ bounded and $\varepsilon > 0$. Then for each $r \varepsilon B$ there exists a neighbourhood V of r such that $||dg_r(s) - dg_{r'}(s)|| < \varepsilon$ for all $r' \varepsilon V$ and $s \varepsilon S$.

From this it follows that since $\{r_i\}$ converges to r, $\{s_i\}$ is bounded, and $dg_{r_i}(s_i) = 0$, that $dg_r(s_i)$ converges to zero. From this it follows that $dh^{\Omega_r}(s_i \circ \lambda(r)^{-1})$ converges to zero. Finally from lemma 2.4 we get that $\{s_i \circ \lambda(r)^{-1}\}$ is a convergent sequence and therefore $\{s_i\}$ is a convergent sequence.

<u>Proof of Lemma 2.8</u> :

We will employ the same notation as in the verification of [1.5]. That is for $s, t \;\varepsilon\; L^p_k(\mathbb{R}_M)$, we let $s_r, t_r \;\varepsilon\; L^p_k(\mathbb{R}_{\Omega_r})$ denote $s \circ \lambda(r)^{-1}$, $t \circ \lambda(r)^{-1}$, and for $r, r' \;\varepsilon\; B$ we let $\beta : \Omega_r \longrightarrow \Omega_{r'}$ be given by $\beta = \lambda(r') \circ \lambda(r)^{-1}$, and for $x_r \;\varepsilon\; \Omega_r$ we let $x_{r'} = \beta(x_r)$. Now

$$dg_{r'}(s)(t) = \int_{\Omega_{r'}} \delta F_{j_k(s_{r'})(x_{r'})} \, j_k(t_{r'})(x_{r'}) \, d\omega(x_{r'})$$

while

$$dg_r(s)(t) = \int_{\Omega_r} \delta F_{j_k(s_r)(x_r)} \, j_k(t_r)(x_r) \, d\omega(x_r) .$$

In order to compare $dg_{r'}(s)(t)$ and $dg_r(s)(t)$ we need to employ the coordinate system on $J^k(\mathbb{R}_{\Omega^o})$. In coordinates

$$dg_{r'}(s)(t)$$

$$= \sum_\alpha \int_{\Omega_{r'}} \frac{\partial F}{\partial u_\alpha}(x_{r'}, j_k(s_{r'})(x_{r'})) \circ u_\alpha(j_k(t_{r'})(x_{r'}) \; d\omega(x_{r'})$$

$$= \sum_\alpha \int_{\Omega_r} \frac{\partial F}{\partial u_\alpha}(\beta(x_r), j_k(s_{r'})(\beta(x_r))) \circ u_\alpha(j_k(t_{r'})(\beta(x_r)) \; d(\beta^*\omega)(x_r) \; .$$

So

$$|dg_r(s)(t) - dg_{r'}(s)(t)|$$

$$\text{(A)} \quad \le \left| \sum_\alpha \int_{\Omega_r} \frac{\partial F}{\partial u_\alpha}(x_r, j_k(s_r)(x_r)) \left[u_\alpha(j_k(t_r)(x_r)) - u_\alpha(j_k(t_{r'})(\beta(x_r)))\right] \; d\omega(x_r) \right|$$

$$\text{(B)} \quad + \left| \sum_\alpha \int_{\Omega_r} \left[\frac{\partial F}{\partial u_\alpha}(x_r, j_k(s_r)(x_r)) - \frac{\partial F}{\partial u_\alpha}(x_r, j_k(s_{r'})(\beta(x_r)))\right] \times u_\alpha(j_k(t_{r'})(\beta(x_r)) \; d\omega(x_r) \right|$$

$$\text{(C)} \quad + \left| \sum_\alpha \int_{\Omega_r} \left[\frac{\partial F}{\partial u_\alpha}(x_r, j_k(s_{r'})(\beta(x_r))) - \frac{\partial F}{\partial u_\alpha}(\beta(x_r), j_k(s_{r'})(\beta(x_r)))\right] \times u_\alpha(j_k(t_{r'})(\beta(x_r)) \; d\omega(x_r) \right|$$

$$\text{(D)} \quad + \left| \sum_\alpha \int_{\Omega_r} \frac{\partial F}{\partial u_\alpha}(\beta(x_r), j_k(s_{r'})(\beta(x_r))) \cdot u_\alpha(j_k(t_{r'})(\beta(x_r)) \; d[\omega - \beta^*\omega](x_r) \right|$$

As before we deal with each of these terms separately.

Now (A) is the same as

$$dH^{\Omega_r}(j_k(s_r))(j_k(t_r) - \beta^* j_k(t_{r'}))$$

where

$$\beta^* j_k(t_{r'})(x_r) = j_k(t_{r'})(\beta(x_r)) .$$

It is easily seen that for any $\delta > 0$ there exists a neighbourhood W of r such that

$$||j_k(t_r) - \beta^* j_k(t_{r'})||_{L_0^p} < \delta$$

for t_r, $t_{r'}$ derived from t with $||t||_{L_k^p} = 1$. From the boundedness of dH^{Ω_r} which follows from lemma 2.1 it follows that (A) can be made arbitrarily small.

(B) This term is the same as

$$\left(dH^{\Omega_r}(j_k(s_r)) - dH^{\Omega_r}(\beta^* j_k(s_{r'}))\right)\beta^* j_k(t_{r'}) .$$

From the uniform continuity of dH^{Ω_r}, the boundedness of $\beta^* j_k(t_{r'})$, $j_k(s_r)$, and $\beta^* j_k(s_{r'})$ and the fact that $||j_k(s_r) - \beta^* j_k(s_{r'})||_{L_0^p}$ can be made arbitrarily small it follows that (B) can be made arbitrarily small.

(C) It is easily seen that for $\delta > 0$ there exists a neighborhood W of r such that

$$|\beta(x_r) - x_r| = |\lambda(r')\circ\lambda(r)^{-1}(x_r) - x_r| < \delta$$

for all $x_r \in \Omega_r$ and all $r' \in W$. From condition [2.4] it follows that (C) is dominated by

$$\sum_{\alpha} \int_{\Omega_r} \left| \mathbb{C}(\beta(x_r)-x_r)\left[1 + \sum_{\beta} |u_\beta(j_k(s_{r'})(\beta(x_r)))|^{p-1}\right] u_\alpha(j_k(t_{r'})(\beta(x_r))) \right| \, d\omega \; .$$

It follows from Holder's inequality and the above remark that (C) can be made arbitrarily small.

(D) It follows from the fact that $\omega - \beta*\omega$ can be made uniformly small that (D) can be made arbitrarily small.

This completes the verification of conditions [1.1] - [1.6], and hence Theorem II is proved.

CHAPTER 3

DIRICHLET PROBLEMS WITH VARIABLE HOLONOMIC CONSTRAINTS

3.1 Formulation of Theorem III

In this chapter we shall consider a parametrized version of the Dirichlet problem described by Palais [4, pp 104-105, p 109]. The solution candidates are vector valued functions on a manifold M, whose values are constrained to lie in a given submanifold $W \subset \mathbb{R}^{\ell}$, as well as to agree with those of a given function a on ∂M, in case ∂M is not empty. We shall study the solution when both W and a are permitted to vary.

To be precise, we begin with a compact C^{∞} manifold M, with positive smooth measure μ and possibly with boundary, and a k^{th} order Lagrangian $L \in Lgn_k(\xi)$, where ξ is the product vector bundle $\mathbb{R}^{\ell}_M$ for some $\ell \geq 2$. We suppose that L satisfies the following conditions :

[3.1] For some p with $pk > n$, L extends to a C^1 map : $L^p_k(\xi) \longrightarrow L^1_0(\mathbb{R}_M)$ so that the integral

$$h(v) = \int_M Lv \, d\mu$$

defines a C^1 real-valued function on $L^p_k(\xi)$.

[3.2] For any $a_1, a_2 \in L^p_k(\xi)$ and any $v_1 \in L^p_k(\xi)_{a_1}$, $v_2 \in L^p_k(\xi)_{a_2}$,

$$[dh(v_1)-dh(v_2)](v_1-v_2) \geq c||(v_1-v_2)-(a_1-a_2)||^p_{L^p_k} - ||(a_1-a_2)||^p_{L^p_k}$$

for some constant c .

[3.3] The map $dh : L^p_k(\xi) \longrightarrow L^p_k(\xi)^*$ takes bounded sets to bounded sets.

Next, let W be a closed C^∞ submanifold of $\mathbb{R}^\ell$ with $\partial W = \emptyset$. For each r in a parameter space B, the varied constraint manifold W_r will be obtained by acting on W with a diffeomorphism $\Lambda(r)$ of the ambient Euclidean space $\mathbb{R}^\ell$. Also, fixing a boundary-value function b on M with values in W, we obtain independently - varied boundary functions a_r by first composing b with another diffeomorphism $\Psi(r)$ of $\mathbb{R}^\ell$ which carries W onto W, then composing the result with $\Lambda(r)$. Notations like $\Psi(r)_*b$ will be used for such a composite function, and by abuse, b may denote a section of the trivial bundle W_M and $\Psi(r)_*b$ the induced section.

We shall require the maps Λ and Ψ to be continuous from B into the space $\mathrm{Diffeo}(\mathbb{R}^\ell)$ of all C^∞ diffeomorphisms of $\mathbb{R}^\ell$ onto itself, with the topology of uniform convergence of each derivative on each compact set. Our result is the following.

<u>Theorem III</u>

Let M be a compact C^∞ manifold of dimension n, possibly with boundary, and with a strictly positive smooth measure μ . With $\xi = \mathbb{R}^\ell_M$, $\ell \geq 2$, let $L \in \mathrm{Lgn}_k(\xi)$ satisfy conditions [3.1] - [3.3] for some p with $pk > n$, and set

$$h(v) = \int_M Lv \, d\mu \qquad \text{for} \qquad v \in L_k^p(\xi) .$$

Let W be a closed C^∞ submanifold of $\mathbb{R}^\ell$ without boundary, (W compact if $\partial M = \emptyset$). Let $E = W_M$ and $b \in L_k^p(E)$. Let B be a locally compact Hausdorff space and let

$$\Psi : B \longrightarrow \text{Diffeo}(\mathbb{R}^\ell) \qquad \text{and} \qquad \Lambda : B \longrightarrow \text{Diffeo}(\mathbb{R}^\ell)$$

be continuous maps such that, for each $r \in B$, $\Psi(r)(W) = W$ and $\Lambda(r)(W)$ is a closed C^∞ submanifold of $\mathbb{R}^\ell$.

Set $W(r) = \Lambda(r)(W)$. Then $E(r) \overset{\text{def}}{\equiv} W(r)_M$ is a C^∞ subbundle of $\mathbb{R}^\ell_M$. Let $a_r = \Lambda(r)_*\Psi(r)_*b$, and $D_r = L_k^p(E(r))_{a_r}$. Let h_r denote the restriction of the C^1 function h to D_r.

Let F be a deformation-invariant family of subsets of a single path component of $L_k^p(E)_b$, such that F contains at least one compact non-void element. For each $r \in B$, set

$$\mathcal{D}(r) = \left\{ V \subset D_r : V = \Lambda(r)_*\Psi(r)_*(T), \quad \text{some} \quad T \in F \right\} .$$

Then the following conclusions hold :

(a) The function $m : B \longrightarrow R$ defined by

$$m(r) = \inf_{V \in \mathcal{D}(r)} \ \sup_{v \in V} h_r(v)$$

is finite and continuous on B.

(b) For each $r \in B$, the set

$$M(r) = \left\{(r, v) : v \in D_r, \ dh_r(v) = 0, \text{ and } h_r(v) = m(r)\right\}$$

is not empty.

(c) For each compact subset $C \subset B$, the set

$$M(C) = \left\{(r, v) : r \in C \text{ and } (r, v) \in M(r)\right\}$$

is a compact subset of $B \times L_k^p(E)$.

Theorem III will be deduced from Theorem I of §1.3 by constructing a suitable standard problem.

3.2 Construction of the Standard Problem

[1α] Let M, μ as given in §3.1, and $E = W_M$.

[1β] p, k as determined in §3.1. Since W is a submanifold of $\mathbb{R}^\ell$, the inclusion $W \longrightarrow \mathbb{R}^\ell$ induces an inclusion $L_k^p(W_M) \longrightarrow L_k^p(\mathbb{R}_M^\ell)$. We give $L_k^p(W_M)$ the induced Finsler structure $\|\ \|$, and the corresponding Finsler metric.

[1γ] B as given in §3.1 .

[1δ] b as given in §3.1 with $b_r = \Psi(r)_* b$ and ϕ_r equal to the restriction of $\Psi(r)_*$ to $L_k^p(E)_b$.

[1ε] $L_r = L \circ \Lambda(r)_*$.

We deduce Theorem III from the application of Theorem I to the standard parametrized problem determined by the data [1α] - [1ε] . The passage from Theorem III to Theorem I is more direct than the passage from Theorem II to Theorem I because we deal here with a fixed domain which enables us to construct a parametrized standard problem more closely related to the original problem. In fact since

$$\mathcal{D}(r) = \left\{V \subset D_r : V = \Lambda(r)_*\Psi(r)_*(T), \quad \text{some} \quad T \in \mathcal{F}\right\},$$

and

$$\mathcal{F}(r) = \left\{S \subset L_k^p(E)_{b_r} : S = \Psi(r)_*(T), \quad \text{some} \quad T \in \mathcal{F}\right\},$$

and

$$g_r(s) = h_r(\Lambda(r)_*(s))$$

it follows that

$$\begin{aligned} m(r) &= \inf_{V\in\mathcal{D}(r)} \sup_{v\in V} h_r(v) \\ &= \inf_{S\in\mathcal{F}(r)} \sup_{s\in S} g_r(s) = m_{\mathcal{F}}(r) \end{aligned}$$

and (a) of Theorem III follows from (a) of Theorem I .

Also we have

$$M(r) = \left\{(r, v) : v \in D_r, \ dh_r(v) = 0 \ \text{ and } \ h_r(v) = m(r)\right\}$$

$$= \left\{(r, \Lambda(r)_*s) \ : \ s \in L_k^p(E)_{b_r}, \ dg_r(s) = 0 \ \text{ and } \ g_r(s) = m_{\mathcal{F}}(r)\right\}$$

which is non-empty by (b) of Theorem I. Finally,

$$M(C) = \left\{ (r, v) : r \varepsilon C \text{ and } (r, v) \varepsilon M(r) \right\}$$

$$= \left\{ (r, \Lambda(r)_* s) : r \varepsilon C, s \varepsilon L^p_k(E)_{b_r}, dg_r(s) = 0 \right.$$
$$\left. \text{and } g_r(s) = m_F(r) \right\}$$

which is comapct when C is compact by (c) of Theorem I .

Before proceeding with the verifications we prove a few lemmas about the map $r \longmapsto \Lambda(r)_*$. A standard assumption will be the continuity of the map $r \longmapsto \Lambda(r)$.

<u>Lemma 3.1</u>

Fix $S \subset L^p_k(\xi)$ bounded and $\varepsilon > 0$. Then there exists a neighbourhood V of r such that

$$||\Lambda(r')_* s - \Lambda(r)_* s||_{L^p_k} < \varepsilon$$

for each $r' \varepsilon V$ and $s \varepsilon S$.

<u>Proof</u> : The proof follows easily from Palais [4, Lemma 9.9, p.31].

<u>Lemma 3.2</u>

If $S \subset L^p_k(\xi)$ is bounded then $\Lambda(r)_*(S)$ is bounded.

<u>Proof</u> : This also follows from the above cited lemma 9.9.

Lemma 3.3

Fix $r \in B$ and $S \subset L_k^p(\xi)$ bounded. Then there exists a neighbourhood V of r such that $\bigcup_{r' \in V} \Lambda(r')_*(S)$ is bounded in $L_k^p(\xi)$.

Proof : This follows from lemmas 3.1 and 3.2.

Lemma 3.4

Fix $r \in B$ and $S \subset L_k^p(\xi)$ bounded. Then

$$\inf_{s \in S} ||d(\Lambda(r)_*)(s)|| > 0 .$$

Proof : By lemma 3.2, $\Lambda(r)(S)$ is bounded. Now

$$||d(\Lambda(r)_*)(s)|| = ||d(\Lambda(r)_*^{-1})(\Lambda(r)_*)(s)||^{-1} .$$

Therefore we need only prove that for $S \subset L_k^p(\xi)$ bounded, and $r \in B$ that $\sup_{s \in S} ||d(\Lambda(r)_*)(s)|| < \infty$. Now for $s \in S$ and $t \in L_k^p(\xi)$,

$$d(\Lambda(r)_*)(s)(t)(x) = [\delta\Lambda(r) \circ s(x)](t(x)) ,$$

by Palais [4, Theorem 11.3, p.41]. Again by [4, Lemma 9.9, p.31], it follows that for $||t||_{L_k^p} = 1$,

$$||(\delta\Lambda(r) \circ s)_* t||_{L_k^p} \leq A < \infty$$

for some $A \in \mathbb{R}$, independent of $s \in S$. This implies that

$$\sup_{s \in S} ||d(\Lambda(r)_*)(s)|| < \infty .$$

3.3 Verification of conditions [1.1] - [1.6]

It follows by the same techniques as employed in Chapter 2 that condition [3.2] implies that $h : L_k^p(\xi) \longrightarrow \mathbb{R}$ is bounded below and satisfies condition (C), and that for $S \subset L_k^p(\xi)$ bounded and $R \in \mathbb{R}$, the set

$$\left\{ s \in L_k^p(\xi)_a \ : \ a \in S \ \text{ and } \ h(s) \leq R \right\}$$

is bounded in $L_k^p(\xi)$.

Verification of [1.1] :

Since $g_r = h \circ \Lambda(r)_*$, g_r is bounded below for each r, and by the above remarks combined with lemma 3.2 it follows that for $b \in L_k^p(E)$, and $R \in \mathbb{R}$, the set

$$\left\{ s \in L_k^p(E)_b \ : \ g_r(s) \leq R \right\}$$

is bounded in $L_k^p(\xi)$ and hence bounded in the Finsler metric on $L_k^p(E)$ (Uhlenbeck [8]).

Therefore in order to show that g_r satisfies condition (C) we need only show that if $\{s_i\}$ is a bounded sequence in $L_k^p(E)_b$ (and hence bounded in $L_k^p(\xi)_b$), such that $dg_r(s_i) \longrightarrow 0$, then $\{s_i\}$ is convergent. Now

$$dg_r(s_i) \ = \ dh(\Lambda(r)_*(s_i)) \circ d(\Lambda(r)_*)(s_i) \ .$$

If $dg_r(s_i) \longrightarrow 0$ it follows from lemma 3.4 that $dh(\Lambda(r)_*(s_i)) \longrightarrow 0$, that $\{\Lambda(r)_* s_i\}$ is convergent and therefore so is $\{s_i\}$.

Verification of [1.2] :

Fix $C \subset B$ compact and $R \in \mathbb{R}$. Let $\{(r_i, s_i)\}$ be a sequence in $K \cap \{(r, s) : r \in C$ and $g_r(s) \leq R\}$. Since B is compact we can assume that $\{r_i\}$ converges to $r \in B$. From lemma 3.3 it follows that the set $\{b_{r_i}\}$ is bounded in $L_k^p(\xi)$.

Now $g_{r_i}(s) \leq R$ implies that $h(\Lambda(r_i)_*(s)) \leq R$, where $\Lambda(r_i)_*(s) \in L_k^p(\xi)_{b_{r_i}}$. From condition [3.2] it follows that the set $\{\Lambda(r_i)_*(s)\}$ is bounded in $L_k^p(\xi)$.

In order to proceed with the verification of [1.2] we need the following extension of the construction in Palais [4, pp 112-114].

Fix $r \in B$. Then $\Lambda(r)(W)$ is a closed C^∞ submanifold of $\mathbb{R}^\ell$. For each $w \in W$ let $q^r(w)$ denote the orthogonal projection of $\mathbb{R}^\ell = T\mathbb{R}^\ell_w$ onto TW_w. Then q is a C^∞ map of W into the vector space $L(\mathbb{R}^\ell, \mathbb{R}^\ell)$, and since W is a closed C^∞ submanifold of $\mathbb{R}^\ell$, it extends to a C^∞ map of $\mathbb{R}^\ell$ into the vector space $L(\mathbb{R}^\ell, \mathbb{R}^\ell)$. If we define $Q^r(x, v) = (x, q^r(v))$, Q^r is a C^∞ fibre bundle morphism of $\xi = \mathbb{R}^\ell_M$ into $L(\xi, \xi)$.

As in [4, theorem 19.14], we define a map $L_k^p(\xi) \longrightarrow L(L_k^p(\xi), L_k^p(\xi))$ denoted by $s \longmapsto P_s^r$ and given by

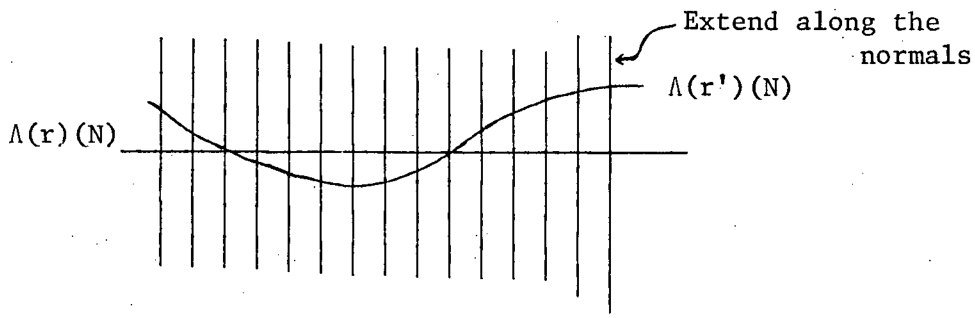
$$P_s^r(t)(x) = Q^r(s(x))(t(x)) .$$

In the above construction the map P_s^r was constructed by appealing to a general extension theorem. We wish to show that these maps can be constructed for r' in some neighbourhood of r such that $P_s^{r'}$ is "close" to P_s^r if r' is close to r. More precisely we have the following.

Lemma 3.5

Fix $r \varepsilon B$ and $S \subset L_k^p(\xi)$ bounded. There exists $\delta > 0$ and a method of defining the extensions of the $q^{r'}$ such that for each $\varepsilon > 0$ there exists a neighbourhood V of r with $||P_s^r - P_s^{r'}|| < \varepsilon$ for all $r' \varepsilon V$, and all s with distance$(s, L_k^p(E(r))) < \delta$.

Proof : Let N be an arbitrarily large compact subset of W to be determined. For $r \varepsilon B$ define the projection $q^r : \Lambda(r)(N) \longrightarrow L(\mathbb{R}^\ell, \mathbb{R}^\ell)$, as described above. For each point $z \varepsilon \Lambda(r)(N)$ extend q^r a finite distance along the normal directions to $\Lambda(r)(N)$ at z in $\mathbb{R}$ by making it constant. Now for r' "close" to r define $q^{r'} : \Lambda(r')(N) \to L(\mathbb{R}^\ell, \mathbb{R}^\ell)$ as above and extend it (shrinking N slightly if necessary) by making it constant along the normal directions determined by $\Lambda(r)(N)$. The following diagram should clarify this argument.



It is then easily verified that the maps $P^{r'}$ have the required property. This completes the proof of lemma 3.5.

We resume the verification of [1.2]. First, $dg_{r_i}(s_i) = 0$ implies that

$$dh(\Lambda(r_i)_*(s_i)) \circ P^{r_i}_{\Lambda(r_i)_*(s_i)} = 0 \quad .$$

Let $t_i = \Lambda(r)_*(s_i)$. Then

$$dh(t_i)(t_i - t_j)$$

$$= dh(t_i)\left[(P^{r_i}_{t_i})(t_i - t_j)\right] + dh(t_i)\left[(I - P^{r_i}_{t_i})(t_i - t_j)\right] \quad .$$

Now $\{t_i\}$ is bounded in $L^p_k(\xi)$, and distance$(t_i, L^p_k(\Lambda(r)(E_r))_{b_r} \longrightarrow 0$. Therefore, there exists a sequence $\{u_i\}$ in $L^p_k(\Lambda(r)(E_r))_{b_r}$ such that $||u_i - t_i||_{L^p_k} \longrightarrow 0$. Consider the difference

$$||(I - P^{r_i}_{t_i})(t_i - t_j) - (I - P^{r}_{u_i})(u_i - u_j)||_{L^p_k}$$

$$\leq \; ||(I - P^{r_i}_{t_i})(t_i - u_i) - (t_j - u_j))||_{L^p_k}$$

$$+ ||(P^{r_i}_{t_i} - P^{r}_{t_i})(u_i - u_j)||_{L^p_k} + ||(P^{r}_{t_i} - P^{r}_{u_i})(u_i - u_j)||_{L^p_k} \; .$$

Now [4, Theorem 19.14, p.112] combined with lemma 3.5 above implies that the above terms converge to zero. By [4, Theorem 19.15, p.113].

$$||(I - P^{r}_{u_i})(u_i - u_j)||_{L^p_k} \longrightarrow 0$$

for a subsequence of $\{u_i\}$ which we assume is $\{u_i\}$, and it follows that

$$||(I - P^{r_i}_{t_i})(t_i - t_j)||_{L^p_k} \longrightarrow 0 .$$

It follows that the difference

$$\left| dh(t_i)(t_i - t_j) - dh(t_i)\left[(P^{r_i}_{t_i})(t_i - t_j)\right]\right|$$

tends to zero. Now $t_i - t_j = t_{i,j} + (b_i - b_j)$ for $t_{i,j} \in L^p_k(\xi)_0$. Therefore

$$dh(t_i)\left[(P^{r_i}_{t_i})(t_i - t_j)\right]$$

$$= dh(t_i)\left[(P^{r_i}_{t_i})(t_{i,j})\right] + dh(t_i)\left[(P^{r_i}_{t_i})(b_i - b_j)\right]$$

$$= dh(t_i)\left[(P^{r_i}_{t_i})(b_i - b_j)\right] .$$

Since $||b_i - b_j||_{L^p_k} \longrightarrow 0$ we get finally that $|dh(t_i)(t_i - t_j)| \longrightarrow 0$.

By [3.2] we can conclude that $\{t_i\}$ is a Cauchy sequence. Therefore $\{s_i\}$ is a Cauchy sequence, and Condition [1.2] is verified.

We complete the verifications in the order [1.4], [1.3], [1.5], and [1.6].

Verification of [1.4] :

Let $T \subset L_k^p(E)$ be bounded. Then T is intrinsically bounded and by Uhlenbeck [8], T is contained in a finite number of vector bundle neighbourhoods $L_k^p(\xi_i)$. Suppose that $t \in L_k^p(\xi_i)$. Then $\Psi(r)_*(t) \in L_k^p(\eta_i)$ where η_i is the vector bundle neighbourhood in E obtained by composing the map $\xi_i \longrightarrow E$ with the map $\Psi(r) : E \longrightarrow E$. This induces a map $L_k^p(\xi_i) \longrightarrow L_k^p(\eta_i)$ and by lemma 3.1 there exists a neighbourhood V of r such that

$$||\Psi(r)_*t - \Psi(r')_*t||_{L_k^p(\eta_i)} < \varepsilon$$

for all $r' \in V$ and $t \in T \cap L_k^p(\xi_i)$. Since $||\ ||$ on E is an admissable Finsler structure (Uhlenbeck [8]) the result follows.

Verification of [1.3] :

Fix $r \in B$ and $R \in \mathbb{R}$. Then $g_t(r') \leq R$ implies that $h(\Lambda(r')_* \circ \phi_{r'}(t)) \leq R$. By the remarks before the verification of [1.1] together with lemma 3.2 there exists a neighbourhood V_1 of r such that the set

$$\left\{ \phi_{r'}(t) \,:\, h(\Lambda(r')_* \circ \phi_{r'}(t)) \leq R \right\}$$

is bounded in $L_k^p(\xi)$. By the argument used in the verification of [1.4], we can find a neighbourhood $V \subset V_1$ such that [1.3] holds.

Verification of [1.5] :

We have

$$|f(r, s) - f(r', s)|$$

$$= |h(\Lambda(r)_*(s)) - h(\Lambda(r')_*(s))|$$

Now combining [3.3] with lemma 3.1 and the mean value theorem the result follows.

Verification of [1.6] :

This follows from [3.3] combined with the proof of lemma 3.4.

This completes the verifications, and hence Theorem III is proved.

CHAPTER 4

EXAMPLES

4.1 Perturbation of Minimal Surfaces

In this example we have a fixed domain and varying boundary conditions. The functions are vector $(\mathbb{R}^3)$ valued.

Let $M \subset \mathbb{R}^2$ be a compact C^∞ two dimensional submanifold of $\mathbb{R}^2$, and B be a locally compact topological space.

Let $F \in FB[J^1(\mathbb{R}^3_M), \mathbb{R}_M]$ be given by

$$F(x_i, u^j, u^j_{x_i}) = \frac{1}{2} \sum_{i,j} (u^j_{x_i})^2 .$$

Let the map $B \longrightarrow L^2_1(\mathbb{R}^3_M)$ given by $r \longmapsto b_r$ be continuous, and let $\phi_r : L^2_1(\mathbb{R}^3_M)_0 \longrightarrow L^2_1(\mathbb{R}^3_M)_{b_r}$ be given by $\phi_r(s) = s + b_r$. Then it is easily seen that the standard problem determined by L_r, ϕ_r, b_r satisfies [1.1] - [1.6]. Now if the set $\mathcal{F}$ of Chapter 1 is the set of singletons in $L^2_1(\mathbb{R}^3_M)_0$, it follows that for each r we are considering minimum values of the function g_r .

If b_r is a smooth section, so that its principal part carries ∂M to a smooth curve Γ_r in $\mathbb{R}^3$ then a section belonging to $L^2_1(\mathbb{R}^3_M)b_r$ defines a (generalized) surface in $\mathbb{R}^3$ whose boundary is Γ_r . It is well known, in this case, that a section which minimizes our Dirichlet integral g_r corresponds to a surface of minimum area spanning Γ_r. Moreover, the value of g_r agrees with the surface area just in this case. Hence Theorem I applies to give conclusions :

(a) There is at least one (generalized) minimal surface for each r,

(b) the minimum surface area varies continuously with r, and

(c) if for each r in a neighbourhood $V \subset B$ the minimizing section s_r is unique, then the map $V \longrightarrow L_1^2(\mathbb{R}_M^3)$ given by $r \longmapsto s_r$ is continuous.

4.2 Perturbation of the Operator

Let M be a smooth submanifold of $\mathbb{R}^n$ with boundary ∂M and Lebesgue measure μ. Let $B = [0, \infty)$. Let $\alpha : M \times \mathbb{R} \times B \longrightarrow \mathbb{R}$ be C^2 in its second and third arguments and be denoted by

$$(x^i, u, r) \longmapsto \alpha(x^i, u, r) .$$

Let $F_r \in FB[J^1(\mathbb{R}_M), \mathbb{R}_M]$ be given by

$$F_r(x^i, u, u_{x_i}) = \frac{1}{2}\left\{ \sum_i (u_{x_i})^2 + \alpha(x, u, r) \right\} .$$

Let $L_r \in Lgn_k(\mathbb{R}_M)$ be represented by F_r. Let $B \longrightarrow L_1^2(\mathbb{R}_M)$ given by $r \longmapsto b_r$ be continuous. Let $\phi_r : L_1^2(\mathbb{R}_M)_0 \longrightarrow L_1^2(\mathbb{R}_M)b_r$ be given by

$\phi_r(s) = s + b_r$. Assume that α satisfies the following conditions :

(i) $|\alpha(x, u, r)| \leq C(r)\left(1 + |u|^2\right)$

(ii) $|\alpha(x, u, r) - \alpha(x, u, r')| \leq C_1(r - r')\left(1 + |u|^2\right)$

(iii) $\left|\frac{\partial\alpha}{\partial u}(x, u, r)\right| \leq C(r)\left(1 + |u|\right)$

(iv) $0 \leq \frac{\partial^2\alpha}{\partial u^2}(x, u, r) < C(r)$

where C and C_1 are continuous functions of r, with $C_1(0) = 0$. It is then easily verified that the standard problem determined by L_r, b_r, ϕ_r satisfies conditions [1.1] - [1.6].

The Euler-Lagrange operator associated with L_r is

$$- \sum_i u_{x_i x_i} + \frac{1}{2}\frac{\partial\alpha}{\partial u}(r, x_i, u)$$

Of course if $\alpha(r, x_i, u) = \gamma(r)u^2$ for $\gamma : [0, \infty) \longrightarrow \mathbb{R}$ continuous we have the parametrized linear Euler Lagrange equation

$$- \sum u_{x_i x_i} + \gamma(r)u = 0 .$$

In any case Theorem I applies again as in 4.1.

4.3 Domain Perturbations

We shall employ the notation of Chapter 2 . Fix $\Omega^o \subset \mathbb{R}^n$ and let $F \in FB[J^1(\mathbb{R}_{\Omega^o}), \mathbb{R}_{\Omega^o}]$ by given by

$$F(x^i, u, u_{x_i}) = \sum_i (u_{x_i})^2 + \alpha(x_i, u)$$

where $\alpha : \Omega^o \times \mathbb{R} \longrightarrow \mathbb{R}$ is C^2 and satisfies the conditions :

(i) $\quad |\alpha(x_i, u)| < C\left(1 + |u|^2\right)$

(ii) $\quad |\alpha(x_i, u) - \alpha(x_i', u)| < C_1(x_i - x_i')\left(1 + |u|^2\right)$

(iii) $\quad \left|\frac{\partial\alpha}{\partial u}(x_i, u)\right| < C\left(1 + |u|\right)$

(iv) $\quad \left|\frac{\partial\alpha}{\partial u}(x_i, u) - \frac{\partial\alpha}{\partial u}(x_i', u)\right| \leq C\left(1 + |u|\right)$

(v) $\quad 0 \leq \frac{\partial^2\alpha}{\partial u^2}(x_i, u) \leq C$

where C is a constant and C_1 a continuous function on $\mathbb{R}^n$ with $C_1(0) = 0$.

Let $L \in Lgn_1(\mathbb{R}_{\Omega^o})$ be represented by F . Then it is clear that F satisfies [2.1] - [2.6] of Chapter 2 with $p = 2$. Let $L(\mathbb{R}^n)$ be the set of linear isomorphisms of $\mathbb{R}^n$ over $\mathbb{R}$. Then for each $r \in L(\mathbb{R}^n)$ the restriction of r to the closure of a smooth subdomain $\bar{\Omega} \subset \Omega^o$ is a diffeomorphism of $\bar{\Omega}$ into $\mathbb{R}^n$. If $\bar{\Omega}$ is strictly contained in Ω^o , there exists a neighbourhood V of the identity in $L(\mathbb{R}^n)$ such that

$r(\overline{\Omega}) \subset \Omega^o$ for $r \varepsilon V$. Let $B = V$, $M = \overline{\Omega}$ and $\lambda(r) = r|_{\overline{\Omega}}$. It is clear that λ is continuous from B into $\mathrm{Diffeo}(\overline{\Omega}, \mathbb{R}^n)$. Let $r \longmapsto b_r$ be a continuous map of $B \longrightarrow L^p_k(\mathbb{R}_{\Omega^o})$.

It is easily shown that the above problem satisfies the conditions of Chapter 2 and that Theorem II applies to it.

4.4 Perturbation of Geodesics

This is an example of the type of problem treated in Chapter 3.

Let $M = [0, 1]$. Let $W \subset \mathbb{R}^\ell$ be a closed q dimensional submanifold of $\mathbb{R}^\ell$. Let $F \varepsilon FB[J^1(\mathbb{R}^\ell_M), \mathbb{R}_M]$ be given by

$$F(x, u^j, u^j_x) = \sum_j (u^j_x)^2 .$$

It is easily verified that F satisfies the conditions of Chapter 4. The critical points of the map constructed with F correspond to geodesics on W in the Riemannian structure induced on W by the inclusion of W into $\mathbb{R}^q$.

Via the map $\Lambda : B \longrightarrow \mathrm{Diffeo}(\mathbb{R}^q)$ defined in Chapter 4 we induce a continuous change in the Riemannian structure of W. The map $\Phi : B \longrightarrow \mathrm{Diffeo}(\mathbb{R}^q)$ varies the endpoints of the geodesics.

Fix a path component of $L^2_1(W_M)_b$. By Palais [4, Thm. 13.14, p.54], this is the same as picking a homotopy class of continuous maps $M \longrightarrow W$, which we denote by $\mathcal{H}$. Fix $r \in B$ and assume that for each r in some neighbourhood $V \subset B$, the minimizing geodesic assured by Theorem III(b) is unique, say v_r . Then we have shown that v_r varies continuously with r , where variations of r in V correspond to variations of the Riemannian structure on W and of the end points of the geodesics, corresponding to $b_r(0)$ and $b_r(1)$.

It is possible to change the above example to the case where $M = S^1$. In this case we must assume that W is compact. We can also increase the dimension of M and let F represent "powers" of the Laplace-Betrami operator on W . For details see Palais [4, p.127].

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