CONTRIBUTION TO THE THEORY OF STABLY TRIVIAL VECTOR BUNDLES

by

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Abstract

A vector bundle ξ over a CW-complex X is said to be stably trivial of type (n,k) if $\xi \oplus k\varepsilon \cong n\varepsilon$, where ε denotes the trivial line bundle. Let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frame in euclidian n-space R^n and let $\eta_{n,k}$ be the real $(n-k)$-dimensional vector bundle over $V_{n,k}$ whose fiber over a k-frame x is the subspace of R^n orthogonal to the span of the vectors in x. The vector bundle $\eta_{n,k}$ is "weakly universal" for stably trivial vector bundles of type (n,k), i.e. for any stably trivial vector bundle ξ of type (n,k), there is a map $f: X \longrightarrow V_{n,k}$, not necessarily unique up to homotopy, such that $f^* \eta_{n,k} \cong \xi$.

We study the following questions: (a) for which values of r is the r-fold Whitney sum $r\eta_{n,k}$ trivial, and (b) what is the maximum number of linearly independent cross-sections of $\eta_{n,k} \oplus s\varepsilon$ $(0 \leq s \leq k - 1)$. Among the results obtained are: (1) $2\eta_{n,2}$ is trivial iff n is even or $n = 3$; (2) $3\eta_{n,2}$ is trivial if n is even; (3) $r\eta_{n,k}$ is not trivial if r is odd and $< (n-2)/(n-k)$; (4) $\eta_{n,k} \oplus (k-1)\varepsilon$ is not trivial if $n \neq 2,4,8$ and $1 \leq k \leq n - 3$; (5) $\eta_{n,k} \oplus s\varepsilon$ admits exactly s linearly independent cross-sections if n and k are odd; (6) $\eta_{n,k} \oplus (k-2)\varepsilon$ admits at most $(k-1)$ linearly independent sections if $2 \leq k \leq n - 3$.

These results are used to construct examples of stably free modules and unimodular matrices over commutative noetherian rings.

The techniques used are those of homotopy theory, including Postnikov systems, K-theory and, specially, Spin operations on vector bundles. A chapter of the thesis is devoted to defining the Spin operations formally as a type of K-theoretic characteristic classes for a certain type of real vector bundles. Formulae to compute the Spin operations on a Whitney sum of vector bundles are given.

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To Miyako

Introduction

A real vector bundle ξ over a CW-complex X is said to be stably trivial of type (n,k), or, simply, of type (n,k) if $\xi \oplus k\varepsilon \cong n\varepsilon$ where ε denotes a trivial line bundle. The object of this thesis is to study such vector bundles.

For $1 \leq k \leq n-1$, let $V_{n,k}$ denote the Stiefel manifold of orthonormal k-frames in euclidian n-space R^n, and let $\eta_{n,k}$ be the $(n-k)$-dimensional real vector bundle over $V_{n,k}$ whose fiber over a k-frame $x = (x_1,\ldots,x_k)$ consists of the vector space $x = \{u: u \in R^n$ and $u \perp x_i, i = 1,\ldots,k\}$. The vector bundle $\eta_{n,k}$ is stably trivial of type (n,k). In fact, for any vector bundle ξ of type (n,k) over a CW-complex X, there is a map $f: X \longrightarrow V_{n,k}$ such that $\xi \cong f^* \eta_{n,k}$. The map f is not necessarily unique up to homotopy. Therefore, we say that $\eta_{n,k}$ is "weakly universal" for vector bundles of type (n,k). It is clear that a general study of stably trivial vector bundles should begin with a study of $\eta_{n,k}$.

In this thesis, we have concentrated our attention on the following questions.

(1) For which values of r is the r-fold Whitney sum $r\eta_{n,k}$ trivial ?

(2) What is the maximum number of linearly independent cross-sections of $\eta_{n,k} \oplus s\varepsilon$, for $0 \leq s \leq k-1$?

Concerning the first question, a purely algebraic result of T.Y. Lam implies that $r\eta_{n,k}$ is trivial for $r \geq k + k/(n-k)$. In chapter III, we prove the following: (i) $2\eta_{n,2}$ is trivial if and only if n is even or $n = 3$; (ii) $2\eta_{n,k}$ is not trivial if $n-k$ is odd and ≥ 3;

(iii) $3\eta_{n,3}$ is trivial if n is even. We also obtain some results for large values of k (i.e. $k > \frac{1}{2}n$). In particular, we prove: (iv) $r\eta_{n,k}$ is not trivial if r is odd and $< (n-2)/(n-k)$. There is a gap of approximately k units between the "positive" result deduced from the theorem of T.Y. Lam and our "negative" result (iv). However, our result suggests that even and odd multiples of $\eta_{n,k}$ may behave in very different ways.

We study the second question in Chapter IV. Assume that $1 \leq s < k \leq n - 3$ if n is even and $1 \leq s < k \leq n - 2$ if k is odd. If $n \neq 2,4,8$, we show that $\eta_{n,k} \oplus (k-1)\varepsilon$ is not trivial. Moreover, if n is odd, we show that $\eta_{n,k} \oplus (k-1)\varepsilon$ admits exactly $k - 1$ linearly independent cross-sections except possibly if k is even and smaller or equal to the Radon-Hurwitz number $\rho(n-k-1)$. Using elementary arguments, these results already imply very strong results about the vector bundles $\eta_{n,k} \oplus s\varepsilon$. For instance, it follows that $\eta_{n,k} \oplus (k-2)\varepsilon$ never admits k linearly independent cross-sections. The complete solution of question (2) will require information lying outside the truncated projective space $RP^{n-1}/RP^{n-k-1} \subset V_{n,k}$ and will therefore require further study.

In chapter V, we use the results of chapter III and IV to obtain examples of modules with special properties. We construct a family of commutative noetherian rings $\Lambda = \Lambda(n,k)$, $n = 2,3,\ldots$, $1 \leq k \leq n - 1$, such that there is a finitely generated projective Λ-module $P = P(n,k)$ with the following properties: (i) $P \oplus \Lambda^k$ is free; (ii) $P \oplus \Lambda^{k-1}$ is not free (with some possible exceptions); (iii) $P(n,2) \oplus P(n,2)$ is not free if n is odd and ≥ 5. The family of modules $P(n,k)$

generalizes an example of [Swan 1962]. Moreover, (iii) shows that a theorem of T.Y. Lam (mentioned above) is best possible in some cases. Finally, we use the modules $P(n,k)$ to give examples of unimodular matrices which are ℓ-stable but not $(\ell+1)$-stable in the sense of [Gabel-Geramita 1974], for various values of ℓ.

Our main technical result, to be found in chapter II of the thesis, is concerned with Spin operations on vector bundles. A real ℓ-dimensional vector bundle ξ over a finite CW complex X is said to have a Spin reduction if the structure group of ξ can be taken to be the spinor group $\mathrm{Spin}(\ell)$. When this can be done in an essentially unique way, we say that ξ has a unique Spin reduction. If ξ has a unique Spin reduction, the so-called Spin representations of the spinor group can be used to construct elements $\Delta(\xi)$ and, for ℓ even, $\Delta^{\pm}(\xi)$, in $KU(X)$ (by the α-construction). We view $\Delta(\xi)$ and $\Delta^{\pm}(\xi)$ as a kind of characteristic class for the vector bundle ξ. These classes are not necessarily trivial even if (a) ξ admits a non-zero section, or (b) ξ is stably trivial. These properties are clearly an advantage for the study of sectioning problems for stably trivial vector bundles. In order to make efficient use of the Spin operations $\Delta(-)$ and $\Delta^{\pm}(-)$, we develop formulae relating $\Delta(\xi_1 \oplus \xi_2)$ to $\Delta(\xi_1)$ and $\Delta(\xi_2)$ and similarly for $\Delta^{\pm}(\xi_1 \oplus \xi_2)$. We also complete this program for the real and quaternionic Spin operations $\Delta_R(-)$ and $\Delta_H(-)$ which take values in KO-theory and KSp-theory respectively.

Another technical aspect of the thesis which we would like to point out concerns the use of Postnikov systems to analyze maps having certain symmetry properties. Let $Y = X \times \ldots \times X$, and let $T_\sigma : Y \to Y$

be the map permuting the t factors of Y according to a permutation $\sigma \in S_t$. Suppose that we have a map $f: Y \longrightarrow Z$ such that $f \circ T_\sigma \simeq f$ for any $\sigma \in S_t$. Then the obstructions to lifting f into the Postnikov tower over Z must satisfy certain invariance properties relatively to some S_t-actions. These observations (and, in one case, the Spin operations) are essential in our evaluation of some k-invariants. These symmetry properties play an important role in the study of the vector bundles $r\eta_{n,k}$.

Chapter I contains preliminary results.

Chapter I

Preliminaries

§1. Stiefel manifolds and stably trivial vector bundles.

Let $SO(n)$ be the Lie group of all $n \times n$ real orthogonal matrices with determinant $+1$ and $BSO(n)$ the classifying space for principal $SO(n)$-bundles. Γ_n denotes the universal n-dimensional real vector bundle over $BSO(n)$.

An orthogonal k-frame in R^n can be thought of as a $k \times n$ matrix x with real entries satisfying the equation $xx^t = I_k$. Therefore, let $p\colon SO(n) \longrightarrow V_{n,k}$ be the map forgetting the last $n - k$ vectors of an orthogonal matrix. The map p gives rise to the well-known identification of $V_{n,k}$ as a homogeneous space:

(1.1) $$SO(n-k) \xrightarrow{i} SO(n) \xrightarrow{p} SO(n)/SO(n-k) \simeq V_{n,k} .$$

Consequently, we can consider the sequence

(1.2) $$SO(n-k) \xrightarrow{i} SO(n) \xrightarrow{p} V_{n,k} \xrightarrow{j} BSO(n-k) \xrightarrow{Bi} BSO(n)$$

where any two consecutive maps form a fiber bundle, and the map j is a classifying map for the principal $SO(n-k)$-bundle (1.1). Since the vector bundle $\eta_{n,k}$ is isomorphic to the fiber bundle with fiber R^{n-k} associated to (1.1), we have that $\eta_{n,k} \cong j^* \Gamma_{n-k}$. Therefore, we can prove the following theorem.

<u>Theorem</u> (1.3). <u>Let</u> ξ <u>be a real (n-k)-dimensional vector bundle over a CW-complex</u> X . <u>Assume that</u> $\xi \oplus k\varepsilon \cong n\varepsilon$. <u>Then there is a map</u> $f\colon X \longrightarrow V_{n,k}$ <u>such that</u> $\xi \cong f^* \eta_{n,k}$. <u>The map</u> f <u>is not necessarily unique up to homotopy</u>.

Proof. Since ξ is stably trivial, it is orientable. Let $f_0: X \longrightarrow BSO(n-k)$ be a classifying map for ξ. Since $\xi \oplus k\varepsilon$ is trivial, the composition $Bi \circ f_0$ is null-homotopic, and since $V_{n,k} \xrightarrow{j} BSO(n-k) \xrightarrow{Bi} BSO(n)$ is a fibration, it follows that there is a map $f: X \longrightarrow V_{n,k}$ such that $j \circ f \simeq f_0$. Moreover, $\xi \cong f_0^* \Gamma_{n-k} \cong f^* j^* \Gamma_{n-k} \cong f^* \eta_{n,k}$ as desired.

The following example shows that the map f is not necessarily unique. Let $X = V_{n,1} \approx S^{n-1}$ and $\xi = \eta_{n,1} \cong \tau S^{n-1}$. Assume that n is even. Then $\xi \cong f^* \eta_{n,1}$ for any map $f: V_{n,1} \longrightarrow V_{n,1}$ of odd degree. (In general, the non-uniqueness of f is measured by the image of $p_{\#}: [X,SO(n)] \longrightarrow [X,V_{n,k}]$). ■

Definition (1.4). A stably trivial vector bundle ξ over a CW-complex X is said to be of type (n,k) if it satisfies the equation $\xi \oplus k\varepsilon \cong n\varepsilon$. We will refer to the situation described in theorem (1.3) by saying that "$V_{n,k}$ is a weak classifying space for vector bundles of type (n,k)" and that "$\eta_{n,k}$ is a weakly universal vector bundle of type (n,k)."

We now give some notation and state elementary properties of the vector bundle $\eta_{n,k}$.

Assume that $1 \leq \ell \leq k \leq n-1$. Let $p: V_{n,k} \longrightarrow V_{n,\ell}$ be the map forgetting the last $k-\ell$ vectors of a k-frame, and let $i: V_{n-\ell,k-\ell} \longrightarrow V_{n,k}$ be the map transforming a $(k-\ell)$-frame in $R^{n-\ell}$ into a k-frame in R^n by adding to it the last ℓ vectors of the standard basis of R^n, i.e., in matrix notation, $i(x) = \begin{pmatrix} x & 0 \\ 0 & I_k \end{pmatrix}$. Throughout the thesis, we will use the letters p and i to refer to the above maps. The sequence $V_{n-\ell,k-\ell} \xrightarrow{i} V_{n,k} \xrightarrow{p} V_{n,\ell}$ is a fibration.

Proposition (1.5). $i^* \eta_{n,k} \cong \eta_{n-\ell,k-\ell}$ and $p^* \eta_{n,\ell} \cong \eta_{n,k} \oplus (k-\ell)\varepsilon$.

Proof. Omitted. ■

Recall also that there are natural identifications $V_{n,1} \approx S^{n-1}$ and $V_{n,n-1} \approx SO(n)$. Moreover, $\eta_{n,1}$ is isomorphic to the tangent bundle τS^{n-1} of the unit sphere S^{n-1}. $\eta_{n,n-1}$ is a trivial line bundle.

§2. Cross-sections of $\eta_{n,k}$.

The following theorem is due to G.W. Whitehead.

Theorem (1.6). Let $2 \leq k \leq n-2$. The vector bundle $\eta_{n,k}$ does not admit a non-zero cross-section unless $(n,k) = (7,2)$ or $(8,3)$. The vector bundles $\eta_{7,2}$ and $\eta_{8,3}$ each admit exactly one linearly independent cross-section.

Proof. See [Whitehead 1963]. ■

Of course, if $k = 1$, the vector bundle $\eta_{n,1} \cong \tau S^{n-1}$ admits exactly $\rho(n) - 1$ linearly independent sections, where $\rho(n)$ is the Hurwitz-Radon number [Adams 1962].

§3. Triviality of large multiples of $\eta_{n,k}$.

A finitely generated module P over a commutative ring R is said to be stably free if there are integers n and k such that

$P \oplus R^k \cong R^n$. The following theorem is due to T.Y. Lam.

<u>Theorem</u> (1.7). <u>Let</u> R <u>be a commutative ring and</u> P <u>be a non-zero stably free</u> R<u>-module such that</u> $P \oplus R^k \cong R^n$. <u>Then the r-fold direct sum</u> rP <u>is free for</u> $r \geq k + k/(n-k)$.

<u>Proof</u>. We give an outline of the proof which can be found in [Lam, T.Y., 1976].

Let $1 \leq k \leq n$ and assume that $r \geq k + k/(n-k)$. Since $P \oplus R^k \cong R^n$, there is a $k \times n$ matrix α with entries in R such that $\alpha\colon R^n \longrightarrow R^k$ is an epimorphism and $P \cong \operatorname{Ker}\{\alpha\colon R^n \longrightarrow R^k\}$. Let

$$\alpha^{\oplus r} = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & & \dots & 0 \\ \dots & & & \\ 0\dots & & ..0 & \alpha \end{pmatrix}$$

Then $rP \cong \operatorname{Ker}\{\alpha^{\oplus r}\colon R^{rn} \longrightarrow R^{rk}\}$. We will show that there is a sequence of elementary row and column operations transforming $\alpha^{\oplus r}$ into a matrix $(\alpha^{\oplus r})'$ such that $rP \cong \operatorname{Ker}(\alpha^{\oplus r})'$ is obviously free.

Write $\alpha = (M,V)$ where M is a $k \times k$ matrix and V is a $k \times (n-k)$ matrix. Using elementary row and column operations, one can show that $\alpha^{\oplus r}$ is equivalent to the matrix

$$\begin{pmatrix} M^r & M^{r-1}V & \dots & MV & V & 0 \\ 0 & 0 & \dots & & 0 & I_{(r-1)k} \end{pmatrix} .$$

(Try with $r = 2$! Use the fact that α has a right inverse α') . Now, by the Cayley-Hamilton theorem, the $k \times k$ matrix M satisfies its characteristic polynomial, which is monic of degree k . It follows easily that there is a new sequence of elementary row and column operations transforming the last matrix into the matrix

$$\begin{pmatrix} M^r & 0 & \dots & 0 & M^{k-1}V & \dots & MV & V & 0 \\ 0 & 0 & \dots & & & \dots & 0 & 0 & I_{(r-1)k} \end{pmatrix}$$

and finally into a matrix

$$(\alpha^{\oplus r})' = \begin{pmatrix} 0 & W & 0 \\ 0 & 0 & I_{(r-1)k} \end{pmatrix}$$

where W is a $k \times (k(n-k)+k)$ matrix. Let $Q = \operatorname{Ker} W$. Then $R^k \oplus Q \cong R^{k(n-k)+k}$. Also, quite clearly, $\operatorname{Ker}(\alpha^{\oplus r})' \cong R^{(r-k)(n-k)} \oplus Q$. Since $r \geq k + k/(n-k)$, $(r-k)(n-k) \geq k$. It follows that $rP \cong \operatorname{Ker}(\alpha^{\oplus r})' \cong R^{(r-k)(n-k)} \oplus Q \cong R^{r(n-k)}$. ∎

Corollary (1.8). *Let ξ be a real vector bundle of type (n,k) over a finite CW-complex X . Then the r-fold Whitney sum $r\xi$ is trivial for $r \geq k + k/(n-k)$. In particular, the vector bundle $r\eta_{n,k}$ is trivial for $r \geq k + k/(n-k)$.*

Proof. Let $\Gamma(\xi)$ denote the set of continuous sections of ξ . $\Gamma(\xi)$ is a finitely generated module over the ring $C(X)$ of continuous real-valued functions on X . Since ξ is of type (n,k), we have that $\Gamma(\xi) \oplus C(X)^k \cong C(X)^n$. By theorem (1.7), it follows that $r\Gamma(\xi) \cong \Gamma(r\xi)$ is free. This, in turn, implies that $r\xi$ is trivial [Swan 1962, cor. 4]. ∎

Chapter II

Spin Operations on Vector Bundles

§1. The α-construction and Spin-reductions.

Let Λ denote the field of real numbers R, the field of complex numbers C or the skew-field of quaternionic numbers H. Denote by $O(k,\Lambda)$ the Lie group of $k \times k$ matrices A with coefficients in Λ which satisfy the equation $A\bar{A}^t = I$.

Let G be a compact Lie group and $\theta\colon G \to O(k,\Lambda)$, a matrix representation of G. If $q\colon E \to X$ is a principal G-bundle over a finite CW-complex X, we can define a k-dimensional Λ-vector bundle $\theta(E)$ over X in the following way. Using the right action of G on E, define a right action of G on $E \times \Lambda^k$ by $(e,\lambda)g = (eg, \theta(g)^{-1}\lambda)$ $(e \in E,\ \lambda \in \Lambda^k,\ g \in G)$. The total space of $\theta(E)$ is taken to be the quotient space $(E \times \Lambda^k)/G$ and the projection map $(E \times \Lambda^k)/G \to X$ is defined by $[e,\lambda] \to q(e)$. The isomorphism class of $\theta(E)$ depends only on the equivalence class of the representation θ and on the isomorphism class of the principal G-bundle E.

Let $K_\Lambda(X)$ denote $KO(X)$, $K(X) = KU(X)$ or $KSp(X)$ respectively. The vector bundle $\theta(E)$ defined above determines an element (also denoted $\theta(E)$) in $K_\Lambda(X)$. This construction is referred to as the <u>α-construction</u> and the element $\theta(E) \in K_\Lambda(X)$ is denoted sometimes by $\alpha_E(\theta)$.

For $\Lambda = R$, C or H, let $R_\Lambda(G)$ denote the real representation ring $RO(G)$, the complex representation ring $R(G) = RU(G)$ or the quaternionic representation group $RSp(G)$, respectively. The following two properties of the α-construction will be used later.

(2.1). For fixed E, the function $\theta \mapsto \theta(E)$ defines by linear extension a group homomorphism $R_\Lambda(G) \to K_\Lambda(X)$. If $\Lambda = R$ or C, this homomorphism is also a ring homomorphism.

(2.2). The α-construction is natural in X and G (in a suitable sense).

The reader is referred to [Bott 1969, p.52] for more detailed statements.

Let $\rho\colon \mathrm{Spin}(\ell) \to SO(\ell)$ be the standard real representation of the ℓ-dimensional spinor group. The map $\rho\colon \mathrm{Spin}(\ell) \to SO(\ell)$ also exhibits $\mathrm{Spin}(\ell)$ as double covering of $SO(\ell)$.

Definition (2.3). A real ℓ-dimensional vector bundle ξ over a finite CW-complex X is said to have a Spin reduction if there exist a principal Spin (ℓ)-bundle E over X such that $\xi \cong \rho(E)$. Moreover, ξ is said to have a unique Spin reduction if the principal Spin (ℓ)-bundle E is uniquely determined by ξ (up to isomorphism).

It is well known that a vector bundle ξ has a Spin reduction if and only if the first two Stiefel-Whitney classes $w_1(\xi)$ and $w_2(\xi)$ are zero.

Let $f\colon X \to BSO(\ell)$ be a classifying map for an orientable vector bundle ξ. Of course, ξ has a Spin reduction if and only if the map f admits a lifting $\bar{f}$ to the classifying space $B\mathrm{Spin}(\ell)$:

$$\begin{tikzcd}
K(Z_2,1) = BZ_2 \arrow[r] & B\mathrm{Spin}(\ell) \arrow[r] & BSO(\ell) \\
 & & X \arrow[ul, dashed, "\bar{f}"] \arrow[u, "f"']
\end{tikzcd}$$

Moreover, ξ has a unique Spin reduction if and only if the lifting $\bar{f}$ is unique (up to homotopy).

Example (2.4). Let ξ^ℓ be a stably trivial vector bundle over a simply-connected CW-complex X. Then ξ has a unique Spin reduction.

<u>Proof</u>: Since ξ is stably trivial, $w_1(\xi)$ and $w_2(\xi)$ are 0. Hence ξ has a Spin reduction, and there is a lifting $\bar{f}: X \to BSpin(\ell)$ of the classifying map $f: X \to BSO(\ell)$ of ξ. Since $[X, K(Z_2,1)] \cong H^1(X; Z_2) = 0$, this lifting is unique in view of the fibration

$$K(Z_2,1) \to BSpin(\ell) \to BSO(\ell) . \blacksquare$$

<u>Proposition</u> (2.5). <u>Let ξ^ℓ and η^m be real vector bundles over finite CW-complexes X and Y respectively. Assume that ξ and η have unique Spin reductions. Then the vector bundle $\xi \times \eta$ over $X \times Y$ has a unique Spin reduction. If $X = Y$, the vector bundle $\xi \oplus \eta$ also has a unique Spin reduction.</u>

<u>Proof</u>. Let $b: SO(\ell) \times SO(m) \to SO(\ell+m)$ and $\bar{b}: Spin(\ell) \times Spin(m) \to Spin(\ell+m)$ be the maps induced by the natural identification $R^\ell \times R^m \cong R^{\ell+m}$ (see §3 for details). The following diagram is commutative (in the category of groups):

$$\begin{array}{ccccc} Z_2 \times Z_2 & \longrightarrow & Spin(\ell) \times Spin(m) & \longrightarrow & SO(\ell) \times SO(m) \\ \downarrow & & \downarrow{\scriptstyle \bar{b}} & & \downarrow{\scriptstyle b} \\ Z_2 & \longrightarrow & Spin(\ell+m) & \longrightarrow & SO(\ell+m) \end{array}$$

Applying the classifying space functor, we obtain again a commutative diagram:

$$\begin{array}{ccccc} BZ_2 \times BZ_2 & \longrightarrow & BSpin(\ell) \times BSpin(m) & \longrightarrow & BSO(\ell) \times BSO(m) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle B\bar{b}} & & \downarrow{\scriptstyle Bb} \\ BZ_2 & \longrightarrow & BSpin(\ell+m) & \longrightarrow & BSO(\ell+m) \end{array}$$

The map $\mu\colon BZ_2 \times BZ_2 \longrightarrow BZ_2$ is the multiplication and the two horizontal sequences of maps are principal bundles. The map $B\bar{b}$ is a principal bundle map.

Let f and g be classifying maps for ξ and η respectively. Assuming that ξ and η have Spin reductions, there are liftings $\bar{f}\colon X \to BSpin(\ell)$ and $\bar{g}\colon Y \to BSpin(m)$ of f and g respectively. The map $B\bar{b} \circ (f\times g)$ is obviously a lifting of $Bb \circ (f\times g)$. Since the latter map classifies $\xi \times \eta$, we have shown that this vector bundle has a Spin reduction.

Now assume that ξ and η have unique Spin reductions, and let $F\colon X \times Y \longrightarrow BSpin(\ell+m)$ be any lifting of $Bb \circ (f\times g)$. Let $\nu_i\colon BSpin(i) \times K(Z_2,1) \to BSpin(i)$ denote the action of $K(Z_2,1)$ on the total space of the principal fibration $K(Z_2,1) \to BSpin(i) \to BSO(i)$. Then $F \simeq \nu_{\ell+m} \circ (B\bar{b} \circ (\bar{f}\times\bar{g}),a)$ for some map $a\colon X \times Y \to K(Z_2,1)$. Now $[X\times Y,\ K(Z_2,1)] \cong H^1(X\times Y;\ Z_2) \cong H^1(X;\ Z_2) + H^1(Y;\ Z_2) \cong [X,\ K(Z_2,1)] \times [Y,\ K(Z_2,1)]$. Thus, if a_X denotes the restriction of a to $X \subset X \times Y$ and a_Y denotes the restriction of a to $Y \subset X \times Y$, we have $a \simeq \mu \circ (a_X \times a_Y)$. Consequently, $F \simeq \nu_{\ell+m} \circ (B\bar{b} \circ (\bar{f}\times\bar{g}),a) \simeq \nu_{\ell+m} \circ (B\bar{b} \circ (\bar{f}\times\bar{g}),\mu\circ(a_X\times a_Y)) \simeq B\bar{b}(\nu_\ell \circ (\bar{f},a_X) \times \nu_m \circ (\bar{g},a_Y))$. The last $\simeq$ is due to the fact that $B\bar{b}$ is a principal bundle map. Since we assume that ξ and η have unique Spin reductions, we must have $\nu_\ell \circ (\bar{f},a_X) \simeq \bar{f}$ and $\nu_m \circ (\bar{g},a_Y) \simeq \bar{g}$. We deduce that $F \simeq B\bar{b} \circ (\bar{f}\times\bar{g})$. This shows that $\xi \times \eta$ has a unique Spin reduction.

The proof of the statement for $\xi \oplus \eta$ if $X = Y$ is similar. ■

§2. Complex Spin operations.

In this section, the complex Spin representations are used to define an operation on real vector bundles which admit a unique Spin reduction. We first recall some facts about the representation ring of the spinor groups.

Let C_ℓ be the Clifford algebra over R^ℓ (given its usual negative definite quadratic form). There are natural inclusions $S^{\ell-1} \subset R^\ell \subset C_\ell$. The group $\mathrm{Spin}(\ell)$ can be taken as the subgroup of the group of invertible elements of C_ℓ consisting of those of the form $u_1 \ldots u_{2p}$ $(u_i \in S^{\ell-1})$. Let $r = [\ell/2]$, $T = R/4\pi Z$ and $T^r = T\times\ldots\times T$. Consider the map $j_\ell \colon T^r \longrightarrow \mathrm{Spin}(\ell)$ defined by:
$j_\ell(\theta_1,\ldots,\theta_r) = (\cos\tfrac{1}{2}\theta_1 - e_1e_2\sin\tfrac{1}{2}\theta_1)\ldots(\cos\tfrac{1}{2}\theta_r - e_{2r-1}e_{2r}\sin\tfrac{1}{2}\theta_r)$
$(\theta_i \in [0,4\pi)$ and $\{e_i\colon i=1,\ldots,\ell\}$ = Standard basis of $R^\ell)$. The image of j_ℓ is a maximal torus of $\mathrm{Spin}(\ell)$. Let $\alpha_k \colon T^r \longrightarrow C$ be the representation of T^r defined by $\alpha_k(\theta_1,\ldots,\theta_r) = \exp(\tfrac{1}{2}i\theta_k)$ $(1 \le k \le r)$. The (complex) representation ring $R(T^r)$ is isomorphic to the polynomial ring $Z[\alpha_1,\ldots,\alpha_r, \alpha_1^{-1},\ldots,\alpha_r^{-1}]$.

Let

$$\lambda_i = i\text{-th elementary symmetric function in } \alpha_1,\ldots,\alpha_r$$

$$\Delta_{2r}^+ = \sum_I \alpha_1^{\varepsilon_1}\ldots\,\alpha_r^{\varepsilon_r}$$

$$\Delta_{2r}^- = \sum_J \alpha_1^{\varepsilon_1}\ldots\,\alpha_r^{\varepsilon_r}$$

$$\Delta_{2r+1} = \sum_{I\cup J} \alpha_1^{\varepsilon_1}\ldots\,\alpha_r^{\varepsilon_r}$$

where $I = \{(\varepsilon_1,\ldots,\varepsilon_r);\ \varepsilon_i = \pm 1$ and $\varepsilon_1\ldots\varepsilon_r = 1\}$ and $J = \{(\varepsilon_1,\ldots,\varepsilon_r);\ \varepsilon_i = \pm 1$ and $\varepsilon_1\ldots\varepsilon_r = -1\}$.

Theorem (2.6). The inclusion of the maximal torus $j_\ell \colon T^{[\ell/2]} \longrightarrow \mathrm{Spin}(\ell)$ induces the following isomorphisms:

$$R(\mathrm{Spin}(2r)) \cong Z[\lambda_1,\ldots,\lambda_{r-2}, \Delta^+_{2r}, \Delta^-_{2r}]$$

$$R(\mathrm{Spin}(2r+1)) \cong Z[\lambda_1,\ldots,\lambda_{r-1}, \Delta_{2r+1}] \ .$$

Moreover, in $R(\mathrm{Spin}(2r))$,

$$(\Delta^+_{2r})^2 + (\Delta^-_{2r})^2 = \lambda_r + 2\lambda_{r-2} + 2\lambda_{r-4} + \ldots$$

and in $R(\mathrm{Spin}(2r+1))$,

$$(\Delta_{2r+1})^2 = \lambda_r + \lambda_{r-1} + \ldots + \lambda_1 + 1 \ .$$

Proof. See [Husemoller 1966]. ∎

The representations $\Delta^{\pm}_{2r}$, $\Delta_{2r} = \Delta^+_{2r} + \Delta^-_{2r}$ and Δ_{2r+1} are called the Spin representations. Notice that $\dim \Delta^{\pm}_{2r} = 2^{r-1}$ and $\dim \Delta_{2r} = \dim \Delta_{2r+1} = 2^r$.

Let ξ be a real ℓ-dimensional vector bundle over a finite CW-complex X . Assume that ξ has a unique Spin reduction, i.e. $\xi \cong \rho(E)$ for some principal $\mathrm{Spin}(\ell)$-bundle E over X and E is unique up to isomorphism. Define the following element(s) of $KU(X)$:

$$\Delta(\xi) = \Delta_\ell(E) \quad \text{if } \ell = 1,2,3,\ldots$$

$$\Delta^{\pm}(\xi) = \Delta^{\pm}_\ell(E) \quad \text{if } \ell = 2,4,6,\ldots \ .$$

Definition (2.7). $\Delta(-)$ and $\Delta^{\pm}(-)$ are called the (complex) Spin operations. They are defined for any real vector bundle ξ over a finite CW-complex X and such that ξ has a unique Spin reduction, and they take value in $KU(X)$.

Remark (2.8). The Spin operations have been previously used in various context without being formally defined. For instance, see [Feder 1966]. In this thesis, we view the Spin operations as a type of KU-theoritic characteristic classes defined for certain real vector bundles. To our best knowledge, this point of view has not been taken before.

Example (2.9). Let $\xi = n\varepsilon$ be the n-dimensional trivial vector bundle over a simply-connected CW-complex X. Then $\Delta(\xi) = 2^{[n/2]}$. If $n = 2s$ is even, then $\Delta^{\pm}(\xi) = 2^{s-1}$.

Proof. The condition that X is simply connected insures that ξ admits a unique Spin reduction. Clearly, $\xi \cong \rho(X \times \mathrm{Spin}(n))$ and $\Delta(X \times \mathrm{Spin}(n)) \cong X \times C^k$, $k = 2^{[n/2]}$. Similarly for $\Delta^{\pm}(\xi)$ if $n = 2s$. ■

Example (2.10). Let ξ^{ℓ} be a real vector bundle over X and let $f\colon Y \to X$ be a continuous map. Assume that ξ and $f^*\xi$ have unique Spin reductions. Then $\Delta f^*(\xi) = f^!\Delta(\xi)$ and, if ℓ is even, $\Delta^{\pm}f^*(\xi) = f^!\Delta^{\pm}(\xi)$, where $f^!\colon KU(X) \to KU(Y)$ is the homomorphism induced by f.

Proof. Naturality of the α-construction. ■

Example (2.11). Let $n \geq 1$ and let γ_{2n} denote a generator of $\widetilde{KU}(S^{2n}) \cong Z$. Then $\Delta^{+}(\tau S^{2n}) = 2^{n-1} \pm \gamma_{2n}$ and $\Delta^{-}(\tau S^{2n}) = -\Delta^{+}(\tau S^{2n})$.

Proof. Since τS^{2n} is stably trivial and S^{2n} is simply connected, τS^{2n} has a unique Spin reduction and $\Delta^{\pm}(\tau S^{2n})$ are defined. The principal Spin(2n)-bundle $E\colon \mathrm{Spin}(2n) \to \mathrm{Spin}(2n+1) \to \mathrm{Spin}(2n+1)/\mathrm{Spin}(2n) \simeq S^{2n}$ satisfies $\rho(E) \cong \tau S^{2n}$. It is well known that $\pm\gamma_{2n} = \Delta^{\pm}_{2n}(E) - 2^{n-1}$ are generators of $\widetilde{KU}(S^{2n})$ [Bott 1969, thm. III, p.75]. ■

Example (2.12). Let $1 \leq k \leq n-2$ and let $\eta = \eta_{n,k}$ be the vector bundles over $V_{n,k}$ described in I.1. Then $\Delta(\eta_{n,k}) = \tau_{n,k} + 2^{[(n-k)/2]}$ where $\tau_{n,k}$ is a torsion element of $\widetilde{KU}(V_{n,k})$ with order 2^t and where $t = \frac{1}{2}k - 1$ if n and k are even and $t = [k/2]$ otherwise. If $n-k$ is even, $\Delta^{\pm}(\eta) = \pm\gamma_{n-k} + 2^{\frac{1}{2}(n-k)-1}$ where γ_{n-k} generates an infinite cyclic summand of $\widetilde{KU}(V_{n,k})$.

Proof. Recall that $V_{n,k}$ is $(n-k-1)$-connected. Hence, for $k \leq n-2$, $V_{n,k}$ is simply connected. Since $\eta = \eta_{n,k}$ is stably trivial, we deduce that η has a unique Spin reduction. Let E be the principal Spin(n-k)-bundle $\mathrm{Spin}(n-k) \longrightarrow \mathrm{Spin}(n) \longrightarrow V_{n,k}$. We have $\rho(E) \cong \eta_{n,k}$. Gitler and Lam [Gitler and Lam 1970, pp.45-46] have shown that $\tau_{n,k} = \Delta_{n-k}(E) - 2^{[(n-k)/2]}$ is a torsion element of $\widetilde{KU}(V_{n,k})$ with order as described above. If $n-k$ is even, they have shown that $\gamma_{n-k} = \Delta^{\pm}_{n-k}(E) - 2^{\frac{1}{2}(n-k)-1}$ are the generators of an infinite cyclic summand of $\widetilde{KU}(V_{n,k})$. ∎

§3. An equivalence of representations.

The result of this section is used to prove theorem (2.14) and (2.25) below.

Let $b\colon SO(\ell) \times SO(m) \longrightarrow SO(\ell+m)$ and $\bar{b}\colon \mathrm{Spin}(\ell) \times \mathrm{Spin}(m) \longrightarrow \mathrm{Spin}(\ell+m)$ be the group homomorphisms induced by the natural identification $R^{\ell} \times R^m \cong R^{\ell+m}$. For matrices A in $SO(\ell)$ and B in $SO(m)$, we have $b(A,B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. For $u_1 \cdots u_{2p} \in \mathrm{Spin}(\ell)$ and $v_1 \cdots v_{2q} \in \mathrm{Spin}(m)$ $(u_i \in S^{\ell-1} \subset R^{\ell} \cong R^{\ell} \times 0$ and $v_i \in S^{m-1} \subset R^m \cong 0 \times R^m)$, we have $\bar{b}(u_1 \cdots u_{2p}, v_1 \cdots v_{2q}) = u_1 \cdots u_{2p} v_1 \cdots v_{2q}$. Moreover, the homomorphism

$\bar{b}$ is the lifting of b to the covering spaces, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}(\ell) \times \mathrm{Spin}(m) & \xrightarrow{\bar{b}} & \mathrm{Spin}(\ell+m) \\ \downarrow & & \downarrow \\ SO(\ell) \times SO(m) & \xrightarrow{b} & SO(\ell+m) \end{array} .$$

Recall that for compact Lie groups G and H, there is a natural isomorphism $R(G\times H) \cong R(G) \otimes R(H)$ induced by the tensor product of representations.

<u>Theorem</u> (2.13). <u>We have the following equalities in</u> $R(\mathrm{Spin}(\ell) \times \mathrm{Spin}(m))$:

(a) <u>for ℓ,m even</u>: $\bar{b}^*(\Delta^{\pm}_{\ell+m}) = \Delta^{\pm}_{\ell} \otimes \Delta^{+}_{m} + \Delta^{\mp}_{\ell} \otimes \Delta^{-}_{m}$;

(b) <u>for ℓ even and m odd</u>: $\bar{b}^*(\Delta_{\ell+m}) = \Delta_{\ell} \otimes \Delta_{m}$;

(c) <u>for ℓ,m odd</u>: $\bar{b}^*(\Delta^{\pm}_{\ell+m}) = \Delta_{\ell} \otimes \Delta_{m}$.

<u>Proof</u>. We prove only (a). The proofs of (b) and (c) are similar. We use the notation of §2. Assume that $\ell = 2r$ and $m = 2s$ and consider the map $c\colon T^r \times T^s \xrightarrow{\cong} T^{r+s}$ induced by the natural isomorphism $R^r \times R^s \longrightarrow R^{r+s}$. The following diagram commutes:

$$\begin{array}{ccc} T^r \times T^s & \xrightarrow{\quad c \quad} & T^{r+s} \\ \downarrow{\scriptstyle j_\ell \times j_m} & & \downarrow{\scriptstyle j_{\ell+m}} \\ \mathrm{Spin}(\ell) \times \mathrm{Spin}(m) & \xrightarrow{\quad \bar{b} \quad} & \mathrm{Spin}(\ell+m) \end{array} .$$

Identify $R(T^{r+s}) \cong Z[\alpha_1,\ldots,\alpha_{r+s}, \alpha_1^{-1},\ldots,\alpha_{r+s}^{-1}] \cong Z[\alpha_1,\ldots,\alpha_r, \alpha_1^{-1},\ldots,\alpha_r^{-1}] \otimes Z[\alpha_{r+1},\ldots,\alpha_{r+s}, \alpha_{r+1}^{-1},\ldots,\alpha_{r+s}^{-1}] \cong R(T^r \times T^s)$. We have:

$$(j_\ell \times j_m)^* \bar{b}^* (\Delta^+_{\ell+m}) = c^* j^*_{\ell+m} (\Delta^+_{\ell+m})$$

$$= c^* \sum_{\varepsilon_1 \ldots \varepsilon_{r+s}=1} \alpha_1^{\varepsilon_1} \ldots \alpha_{r+s}^{\varepsilon_{r+s}}$$

$$= \left(\sum_{\varepsilon_1 \ldots \varepsilon_r = 1} \alpha_1^{\varepsilon_1} \ldots \alpha_r^{\varepsilon_r} \right) \otimes \left(\sum_{\varepsilon_{r+1} \ldots \varepsilon_{r+s}=1} \alpha_{r+1}^{\varepsilon_{r+1}} \ldots \alpha_{r+s}^{\varepsilon_{r+s}} \right)$$

$$+ \left(\sum_{\varepsilon_1 \ldots \varepsilon_r = -1} \alpha_1^{\varepsilon_1} \ldots \alpha_r^{\varepsilon_r} \right) \otimes \left(\sum_{\varepsilon_{r+1} \ldots \varepsilon_{r+s}=-1} \alpha_{r+1}^{\varepsilon_{r+1}} \ldots \alpha_{r+s}^{\varepsilon_{r+s}} \right)$$

$$= (j^*_\ell \Delta^+_\ell) \otimes (j^*_m \Delta^+_m) + (j^*_\ell \Delta^-_\ell) \otimes (j^*_m \Delta^-_m)$$

$$= (j_\ell \times j_m)^* (\Delta^+_\ell \otimes \Delta^+_m + \Delta^-_\ell \otimes \Delta^-_m) .$$

Since $(j_\ell \times j_m)^*$ is a monomorphism, we deduce that $\bar{b}^*(\Delta^+_{\ell+m}) = \Delta^+_\ell \otimes \Delta^+_m + \Delta^-_\ell \otimes \Delta^-_m$ as desired. Similarly for $\bar{b}^*(\Delta^-_{\ell+m})$. ■

§4. Complex Spin operations and Whitney sums.

Recall that the external tensor product of compact vector bundles induces a bilinear pairing $\otimes$: $KU(X) \otimes KU(Y) \longrightarrow KU(X\times Y)$. We prove the following theorem.

Theorem (2.14). Let ξ^ℓ and η^m be vector bundles over finite CW-complexes X and Y respectively. Assume that ξ and η have unique Spin reductions. The Spin operations satisfy the following equalities:

(a) for ℓ,m even: $\Delta^\pm(\xi\times\eta) = \Delta^\pm(\xi) \otimes \Delta^+(\eta) + \Delta^\mp(\xi) \otimes \Delta^-(\eta)$;

(b) for ℓ even and m odd: $\Delta(\xi\times\eta) = \Delta(\xi) \otimes \Delta(\eta)$;

(c) for ℓ,m odd: $\Delta^\pm(\xi\times\eta) = \Delta(\xi) \otimes \Delta(\eta)$.

Proof. The vector bundle $\xi \times \eta$ has a unique Spin reduction by proposition (2.5). Consequently, the Spin operations are defined on $\xi \times \eta$. Let E be a principal Spin(ℓ)-bundle over X such that $\xi \cong \rho(E)$ and similarly for F over X with $\eta \cong \rho(F)$. Then $E \times F$ is a principal $\mathrm{Spin}(\ell) \times \mathrm{Spin}(m)$-bundle over $X \times Y$. Let $\bar{b}\colon \mathrm{Spin}(\ell) \times \mathrm{Spin}(m) \longrightarrow \mathrm{Spin}(\ell+m)$ be as in the previous section and let $\bar{b}(E{\times}F)$ denote the fiber bundle with fiber $\mathrm{Spin}(\ell+m)$ (considered as $\mathrm{Spin}(\ell) \times \mathrm{Spin}(m)$ - space through $\bar{b}$) associated to $E \times F$. The fiber bundle $\bar{b}(E{\times}F)$ is a principal $\mathrm{Spin}(\ell+m)$-bundle and $\rho(\bar{b}(E{\times}F)) \cong \xi \times \eta$.

Now, assume that ℓ and m are even. By naturality of the constructions involved, we have:

$$\Delta^{\pm}(\xi\times\eta) = \Delta^{\pm}_{\ell+m}(\bar{b}(E\times F)) = (\Delta^{\pm}_{\ell+m} \circ \bar{b})(E\times F)$$
$$= (\bar{b}^{*}\, \Delta_{\ell+m})(E\times F) .$$

Since the α-construction induces a ring homomorphism $R(-) \to KU(-)$, we also have the following equalities:

$$\Delta^{\pm}(\xi) \otimes \Delta^{+}(\eta) + \Delta^{\mp}(\xi) \otimes \Delta^{-}(\eta) =$$
$$\Delta^{\pm}_{\ell}(E) \otimes \Delta^{+}_{m}(F) + \Delta^{\mp}_{\ell}(E) \otimes \Delta^{-}(F) =$$
$$(\Delta^{\pm}_{\ell} \otimes \Delta^{+}_{m} + \Delta^{\mp}_{\ell} \otimes \Delta^{-}_{m})(E\times F) .$$

By theorem (2.13), $\bar{b}^{*}\, \Delta^{\pm}_{\ell+m} = \Delta^{\pm}_{\ell} \otimes \Delta^{+}_{m} + \Delta^{\mp}_{\ell} \otimes \Delta^{-}_{m}$. Consequently $\bar{b}^{*}\, \Delta^{\pm}_{\ell+m}(E\times F) \cong (\Delta^{\pm}_{\ell} \otimes \Delta^{+}_{m} + \Delta^{\mp}_{\ell} \otimes \Delta^{-}_{m})(E\times F)$. The equalities above then imply that $\Delta^{\pm}(\xi\times\eta) = \Delta^{\pm}(\xi) \otimes \Delta^{+}(\eta) + \Delta^{\mp}(\xi) \otimes \Delta^{-}(\eta)$, as desired.

The proof of (b) and (c) is similar. ■

The following corollary is immediate.

Corollary (2.15). Let ξ^{ℓ} and η^{m} be real vector bundles over a finite CW-complex X. Assume that ξ and η have unique Spin reductions. The Spin operations satisfy the following identities:

(a) for ℓ, m even: $\Delta^{\pm}(\xi \oplus \eta) = \Delta^{\pm}(\xi)\Delta^{+}(\eta) + \Delta^{\mp}(\xi)\Delta^{-}(\eta)$;

(b) for ℓ even, m odd: $\Delta(\xi \oplus \eta) = \Delta(\xi)\Delta(\eta)$;

(c) for ℓ, m odd: $\Delta^{\pm}(\xi \oplus \eta) = \Delta(\xi)\Delta(\eta)$. ∎

§5. Real and quaternionic Spin operations.

In this section, we define real and quaternionic Spin operations, i.e. Spin operations defined on real vector bundles having a unique Spin reduction and taking values in KO-theory and KSp-theory. Formulae to compute these operations on Whitney sums of vector bundles are also given. This section is included for the sake of completeness, since the theoretical results obtained are not used in the rest of this thesis. However, the author hopes that example (2.27) will convince the reader that they are of some value. We begin by recalling some facts about real and quaternionic representation rings and real and quaternionic K-theory. Most details are omitted. The reader who is not familiar with the approach taken should consult [Adams 1967] and [Atiyah 1969, §1.5].

Recall that a real (resp.: quaternionic) representation of a compact Lie group G can be regarded as a pair (V,j) where V is a complex representation and $j: V \longrightarrow V$ is a conjugate-linear G-map such that $j^2 = 1$ (resp.: $j^2 = -1$). The map j is called a structure map. Forgetting the structure maps induces the "complexification" homomorphisms:

$$c\colon RO(G) \longrightarrow R(G)$$

$$c'\colon RSp(G) \longrightarrow R(G) .$$

The homomorphisms c and c' are monomorphisms [Adams 1969, 3.27]. There are also "realification" and "quaternionification" homomorphisms:

$$r\colon R(G) \longrightarrow RO(G)$$

$$q\colon R(G) \longrightarrow RSp(G)$$

and a "conjugation" homomorphism:

$$t\colon R(G) \longrightarrow R(G)$$

[Adams 1969, §3.5 (i),(iv),(v)]. The following identities are satisfied:

$$\begin{aligned} rc &= 2 \\ cr &= 1 + t \\ qc' &= 2 \\ c'q &= 1 + t \end{aligned} \qquad (2.16)$$

Let $\alpha = (V,j)$ and $\alpha' = (V',j')$ be real representations of G (i.e. $j^2 = 1_V$ and $j'^2 = 1_{V'}$) . Let also $\beta = (W,k)$ and $\beta' = (W',k')$ be quaternionic representations of G (i.e. $k^2 = -1_W$ and $k'^2 = -1_{W'}$) . Then $(j\otimes j')^2 = 1_{V\otimes V'}$ and $(k\otimes k')^2 = -1_{W\otimes W'}$. Thus $\alpha \otimes \alpha' = (V\otimes V', j\otimes j')$ and $\beta \otimes \beta' = (W\otimes W', k\otimes k')$ are real representations of G . Similarly, $\alpha \otimes \beta' = (V\otimes W', j\otimes k')$ and $\beta \otimes \alpha' = (W\otimes V', k\otimes j')$ are quaternionic representations of G . Therefore, we can define a multiplication in $RO(G) \oplus RSp(G)$ by setting $([\alpha],[\beta])\cdot([\alpha'],[\beta']) = ([\alpha\otimes\alpha'] + [\beta\otimes\beta'], [\alpha\otimes\beta'] + [\beta\otimes\alpha'])$ and extending linearly to

$RO(G) \oplus RSp(G)$. The group $RO(G) \oplus RSp(G)$ is endowed in this way of a natural structure of Z_2-graded ring. Moreover, we can give a structure of Z_2-graded ring to $R(G) \oplus R(G)$ by defining $(u,v)(u',v') = (uu' + vv', uv' + vu')$ for $u,u',v,v' \in R(G)$. Then the map

$$(2.17) \qquad c'' = c \oplus c': RO(G) \oplus RSp(G) \longrightarrow RU(G) \oplus RU(G)$$

is a <u>natural monomorphisn of Z_2-graded rings</u>.

If G and H are compact Lie groups, let $\pi_G: G \times H \longrightarrow G$ and $\pi_H: G \times H \longrightarrow H$ be the projection maps. The reader will easily see that π_G and π_H can be used in the usual way to define an external product:

$$(2.18) \qquad [RO(G) \oplus RSp(G)] \hat{\otimes} [RO(H) \oplus RSp(H)] \longrightarrow [RO(G{\times}H) \oplus RSp(G{\times}H)]$$

where $\hat{\otimes}$ denotes the tensor product of Z_2-graded rings. It should be noticed that (2.18) is <u>not</u> an isomorphism.

In a very similar way, real and quaternionic vector bundles over finite CW-complexes can be regarded as complex vector bundles with structure maps. A real (Resp.: quaternionic) vector bundle over a finite CW-complex X can be thought of as a pair (ξ,T) where ξ is a complex vector bundle over X and $T: \xi \longrightarrow \xi$ is a conjugate-linear bundle map such that $T^2 = 1$ (resp.: $T^2 = -1$). One can proceed as we did for group representations and define maps $c: KO(X) \longrightarrow K(X)$, $c': KSp(X) \longrightarrow K(X)$, etc... . The identities (2.16) also hold. The tensor product of vector bundles and structure maps induces a natural structure of Z_2-graded ring on $KO(X) \oplus KSp(X)$. If Y is an other

CW-complex, the exterior tensor product induces a bilinear pairing

$$(2.19)\qquad \otimes\colon [KO(X) \oplus KSp(X)] \hat{\otimes} [KO(Y) \oplus KSp(Y)] \longrightarrow KO(X\times Y) + KSp(X\times Y) .$$

<u>Remark</u> (2.20). The existence of this natural Z_2-graded ring structure on $KO(X) \oplus KSp(X)$ was made obvious by the work of Atiyah, Bott,... . However, to our best knowledge, it was first used explicitly in [Sigrist and Suter 1972]. See also [Allard 1974].

Let $q\colon E \longrightarrow X$ and $\theta\colon G \longrightarrow O(k,\Lambda)$ $(\Lambda = R$ or $H)$ be as in §1. The element $\theta(E)$ obtained by the α-construction can be regarded as an element of $KO(X) \oplus KSp(X)$ via the natural inclusion $KO(X) \longrightarrow KO(X) \oplus KSp(X)$ or $KSp(X) \longrightarrow KO(X) \oplus KSp(X)$. The reader will easily prove the following property of the α-construction (compare (2.1))

(2.21) For fixed E, the α-construction $\theta \longmapsto \theta(E)$ induces a homomorphism of Z_2-graded ring $RO(G) \oplus RSp(G) \longrightarrow KO(X) \oplus KSp(X)$.

We now describe the real and quaternionic Spin representations of the group $Spin(\ell)$. In view of the monomorphism (2.17), it suffices to give their images under c and c' respectively. Table (I) contains this list.

$\ell(8)$	Real Spin representations	Quaternionic Spin representations
0	$\Delta_\ell^{\pm}$	$2\Delta_\ell^{\pm}$
1	Δ_ℓ	$2\Delta_\ell$
2	$\dot{\Delta}_\ell$	$\dot{\Delta}_\ell$
3	$2\Delta_\ell$	Δ_ℓ
4	$2\Delta_\ell^{\pm}$	$\Delta_\ell^{\pm}$
5	$2\Delta_\ell$	$\dot{\Delta}_\ell$
6	Δ_ℓ	Δ_ℓ
7	Δ_ℓ	$2\Delta_\ell$

TABLE I: Real and quaternionic Spin operations

The list of real Spin representations is taken from [Bott 1969, p.66]. Knowing the real Spin representations, one can deduce the list of quaternionic Spin representations using elementary representation theory. For example, it is a theorem that a self-conjugate irreducible complex representation is either real or quaternionic (i.e. in Im c or in Im c'), but not both (see [Adams 1969, 3.56]). If $\ell \equiv 4(8)$, the irreducible complex representations $\Delta_\ell^{\pm}$ are self-conjugate, but not real (i.e. not in Bott's list). Hence they must be quaternionic ...

We are now ready to define real and quaternionic Spin operations. Let ξ^ℓ be a ℓ-dimensional real vector bundle over a compact connected CW complex X, and assume that ξ^ℓ has a unique Spin reduction, i.e. $\xi = \rho(E)$ for some Spin(ℓ)-bundle E over X and E is unique up to isomorphism. Denote the real Spin representation(s) of Spin(ℓ) by ϕ_ℓ if $\ell \not\equiv 0(4)$ and by $\phi_\ell^{\pm}$ if $\ell \equiv 0(4)$. In the latter case define also $\phi_\ell = \phi_\ell^{+} + \phi_\ell^{-}$. Define the following element(s) of KO(X):

$$\Delta_R(\xi) = \phi_\ell(E) \quad \text{for} \quad \ell = 1,2,\ldots$$

$$\Delta_R^{\pm}(\xi) = \phi_\ell^{\pm}(E) \quad \text{for} \quad \ell = 4,8,\ldots \ .$$

Similarly, denote the quaternionic Spin representations of $\mathrm{Spin}(\ell)$ by ψ_ℓ if $\ell \not\equiv 0(4)$ and by $\psi_\ell^{\pm}$ if $\ell \equiv 0(4)$. In the latter case also define $\psi_\ell = \psi_\ell^{+} + \psi_\ell^{-}$. Define the following element(s) of $KSp(X)$:

$$\Delta_H(\xi) = \psi_\ell(E) \quad \text{if} \quad \ell = 1,2,\ldots$$

$$\Delta_H^{\pm}(\xi) = \psi_\ell^{\pm}(E) \quad \text{if} \quad \ell = 4,8,\ldots \ .$$

<u>Definition</u> (2.22). <u>$\Delta_R(-)$ and $\Delta_R^{\pm}(-)$ are called the real Spin operations and $\Delta_H(-)$ and $\Delta_H^{\pm}(-)$ are called the quaternionic Spin operations. These operations are defined for any real vector bundle ξ over a finite CW-complex X and such that ξ has a unique Spin reduction, and they take values in $KO(X)$ and $KSp(X)$ respectively. They can also be regarded as taking values in the Z_2-graded ring $KO(X) \oplus KSp(X)$ via the natural inclusions $KO(X) \longrightarrow KO(X) \oplus KSp(X)$ and $KSp(X) \longrightarrow KO(X) \oplus KSp(X)$.</u>

<u>Example</u> (2.23). Let $\xi \cong \ell\varepsilon$ be a trivial ℓ-dimensional vector bundle over a simply connected CW-complex X. Then $\Delta_R(\xi) = 2^{(\ell-r)/2} \in KO(X)$ with $r = 0,-1,0,1,2,1,0,-1$ for $\ell \equiv 0,1,\ldots,7(8)$ respectively. If $\ell \equiv 0(8)$, $\Delta_R^{\pm}(\xi) = 2^{(\ell-2)/2}$ and if $\ell \equiv 4(8)$, $\Delta_R^{\pm}(\xi) = 2^{\ell/2}$.

<u>Example</u> (2.24). Recall that $\widetilde{KO}(S^n) \cong Z$ if $n \cong 0(4)$, $\widetilde{KO}(S^n) \cong Z_2$ if $n \equiv 1,2(8)$ and $\widetilde{KO}(S^n) \cong 0$ otherwise. Let β_n denote a generator of $\widetilde{KO}(S^n)$ if this group is not trivial. Then $\Delta_R^{+}(\tau S^n) = \pm\beta_n + 2^{(n-2)/2}$ if $n \equiv 0(8)$ and $\Delta_R^{+}(\tau S^n) = \beta_n + 2^{n/2}$ if $n \equiv 4(8)$.

In both cases, $\Delta_R^-(\tau S^n) = -\Delta_R^+(\tau S^n)$. If $n \equiv 1(8)$, $\Delta_R(\tau S^n) = \beta_n + 2^{(n-1)/2}$ and if $n \equiv 2(8)$, $\Delta_R(\tau S^n) = \beta_n + 2^{n/2}$. The proof of these facts is contained in [Atiyah, Bott and Shapiro 1963].

__Theorem__ (2.25). __Let__ X, Y, ξ^ℓ __and__ η^m __be as in theorem (2.14). Let__ $\Delta_R(-)$ __and__ $\Delta_H(-)$ __take values in the__ Z_2-__graded ring__ $KO(-) \oplus KSp(-)$. __Then__ $\Delta_R(\xi\times\eta)$ $(\Delta_R^{\pm}(\xi\times\eta)$ __if__ $\ell + m \equiv 0(4))$ __is equal to the expression given in Table (II), and__ $\Delta_H(\xi\times\eta)$ $(\Delta_H^{\pm}(\xi\times\eta)$ __if__ $\ell + m \equiv 0(4))$ __is equal to the expression given in Table (III). The product__ $\circ$ __should be taken to be the external product__ $\otimes$ __(2.19).__

__Proof__. We give the proof of the theorem for $\Delta_H^{\pm}(-)$ and $\ell, m \equiv 0(8)$. The other cases are similar. Moreover, the proof is similar to that of theorem (2.14). Consequently, let b, E, F and $\bar{b}(E\times F)$ be as in the proof of (2.14). By the naturality of the constructions involved, we have that $\Delta_H^{\pm}(\xi\times\eta) = \psi_{\ell+m}^{\pm}(\bar{b}(E\times F)) = (\psi_{\ell+m}^{\pm} \circ \bar{b})(E\times F) = \bar{b}^* \psi_{\ell+m}^{\pm}(E\times F)$, where b^* now stands for the group homomorphism induced by $\bar{b}$ on the quaternionic representation rings. On the other hand, using (2.21), we deduce that $\Delta_H^{\pm}(\xi) \otimes \Delta_R^+(\eta) + \Delta_H^{\mp}(\xi) \otimes \Delta_R^-(\eta) = \psi_\ell^{\pm}(E) \otimes \phi_m^+(F) + \psi_\ell^{\mp}(E) \otimes \phi_m^-(F) = (\psi_\ell^{\pm} \otimes \phi_m^+ + \psi_\ell^{\mp} \otimes \phi_m^-)(E\times F)$. (Given groups A and B we will write a for $(a,0) \in A \times B$ and b for $(0,b) \in A \times B$ if there is no danger of confusion).

We have the following equalities in $RO(G) \oplus RSp(G)$, $G = Spin(\ell) \times Spin(m)$. Firstly, from the naturality of the homomorphism c'', we have that $c'' \bar{b}^* \psi_{\ell+m}^{\pm} = \bar{b}^* c'' \psi_{\ell+m}^{\pm} = \bar{b}^* (0, 2\Delta_{\ell+m}^{\pm})$. Using theorem (2.13) and the Z_2-graded ring structure given to $R(-) \oplus R(-)$, we have that $\bar{b}^*(0,2\Delta_{\ell+m}^{\pm}) = (0,2\Delta_\ell^{\pm}) \otimes (\Delta_m^+,0) + (0,2\Delta_\ell^{\mp}) \otimes (\Delta_m^-,0)$. Finally, since c'' is a homomorphism of Z_2-

ℓ (8) \ m(8)	0	1	2	3	4	5	6	7
0	$\Delta_R^{\pm}(\xi)\circ\Delta_R^{+}(\eta) + \Delta_R^{\mp}(\xi)\circ\Delta_R^{-}(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R^{\pm}(\xi)\circ\Delta_R^{+}(\eta) + \Delta_R^{\mp}(\xi)\circ\Delta_R^{-}(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$
1	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$2\Delta_R(\xi)\circ\Delta_R(\eta)$	$2\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$
2	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$2\Delta_R(\xi)\circ\Delta_R(\eta)$	$2r(\Delta^{\pm}(\xi)\circ\Delta^{+}(\eta))$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$r(\Delta^{\pm}(\xi)\circ\Delta^{+}(\eta))$	$\Delta_R(\xi)\circ\Delta_R(\eta)$
3	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$2\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$
4	$\Delta_R^{\pm}(\xi)\circ\Delta_R^{+}(\eta) + \Delta_R^{\mp}(\xi)\ \Delta_R^{-}(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H^{\pm}(\xi)\circ\Delta_H(\eta) + \Delta_H^{\mp}(\xi)\circ\Delta_H^{-}(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$
5	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$2\Delta_H(\xi)\circ\Delta_H(\eta)$	$2\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$
6	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$r(\Delta^{\pm}(\xi)\circ\Delta^{+}(\eta))$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$2\Delta_H(\xi)\circ\Delta_H(\eta)$	$2r(\Delta^{\pm}(\xi)\circ\Delta^{+}(\eta))$	$2\Delta_R(\xi)\circ\Delta_R(\eta)$
7	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_H(\eta)$	$2\Delta_R(\xi)\circ\Delta_R(\eta)$	$2\Delta_R(\xi)\circ\Delta_R(\eta)$

TABLE II: Product formulae for Real Spin Operations

ℓ (8) \ m(8)	0	1	2	3	4	5	6	7
0	$\Delta_H^{\pm}(\xi)\circ\Delta_R^{+}(\eta)$ $+\Delta_H^{\mp}(\xi)\circ\Delta_R^{-}(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R^{\pm}(\xi)\circ\Delta_H(\eta)$ $+\Delta_R^{\mp}(\xi)\circ\Delta_H^{+}(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$
1	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$2\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$
2	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$q(\Delta^{\pm}(\xi)\circ\Delta^{+}(\eta))$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$2\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$q(\Delta^{\pm}(\xi)\circ\Delta(\eta))$	$\Delta_R(\xi)\circ\Delta_H(\eta)$
3	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$2\Delta_H(\xi)\circ\Delta_R(\eta)$
4	$\Delta_H^{\pm}(\xi)\circ\Delta^{+}(\eta)$ $+\Delta_H^{\mp}(\xi)\circ\Delta_R^{-}(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$2\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H^{\pm}(\xi)\circ\Delta_R^{+}(\eta)$ $+\Delta_H^{\mp}(\xi)\circ\Delta_R^{-}(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$
5	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$
6	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$2\Delta_H(\xi)\circ\Delta_R(\eta)$	$q(\Delta^{\pm}(\xi)\circ\Delta(\eta))$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$q(\Delta^{\pm}(\xi)\circ\Delta^{+}(\eta))$	$\Delta_H(\xi)\circ\Delta_R(\eta)$
7	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$\Delta_H(\xi)\circ\Delta_R(\eta)$	$2\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$	$\Delta_R(\xi)\circ\Delta_H(\eta)$

TABLE III: Product formulae for Quaternionic Spin Operations graded ring, the last expression is equal to $c''(\psi_\ell^\pm \otimes \phi_m^+ + \psi_\ell^\mp \otimes \phi_m^-)$. Thus we have shown that $c''\bar{b}^* \psi_{\ell+m}^\pm = c''(\psi_\ell^\pm \otimes \phi_m^+ + \psi_\ell^\mp \otimes \phi_m^-)$. Since c'' is a monomorphism, we deduce that $\bar{b}^*\psi_{\ell+m}^\pm = \psi_\ell^\pm \otimes \phi_m^+ + \psi_\ell^\mp \otimes \phi_m^-$. It follows that $\bar{b}^*\psi_{\ell+m}^\pm(E\times F) = (\psi_\ell^\pm \otimes \phi_m^+ + \psi_\ell^\mp \otimes \phi_m^-)(E\times F)$. In view of the last paragraph, this implies the result wanted, i.e. $\Delta_H^\pm(\xi\times\eta) = \Delta_H^\pm(\xi) \otimes \Delta_R^+(\eta) + \Delta_H^\mp(\xi) \otimes \Delta_R^-(\eta)$. ■

Corollary (2.26). *Let* X, ξ^ℓ *and* η^m *be as in corollary* (2.15). *Let* $\Delta_R(-)$ *and* $\Delta_H(-)$ *take values in the* Z_2-*graded ring* $KO(-) \oplus KSp(-)$. *Then* $\Delta_R(\xi\oplus\eta)$ $(\Delta_R^\pm(\xi\oplus\eta)$ *if* $\ell + m \equiv 0(4))$ *is equal to the expression given in Table (II) and* $\Delta_H(\xi\oplus\eta)$ $(\Delta_H^\pm(\xi\oplus\eta)$ *if* $\ell + m \equiv 0(4))$ *is equal to the expression given in Table (III). The product* $\circ$ *should be taken to be the* Z_2-*graded product in* $KO(X) \oplus KSp(X)$.

Proof. Obvious. ■

Example (2.27). Let $\tau^n = \tau S^n$ and let ζ^{8k} be a real vector bundle over S^{8k+1} such that $\zeta^{8k} \oplus \varepsilon \cong \tau^{8k+1}$. Then $\zeta^{8k} \times \tau^{8\ell}$ is not trivial over $S^{8k+1} \times S^{8\ell}$.

Proof. The result is trivial if $k = 0$. Therefore, we assume $k > 0$. Recall examples (2.23) and (2.24). By corollary (2.26), we have that: $\beta_{8k+1} = \Delta_R(\tau^{8k+1}) - 2^{4k} = \Delta_R(\zeta + \varepsilon) - 2^{4k} = (\Delta_R^+(\zeta) + \Delta_R^-(\zeta))\Delta_R(\varepsilon) - 2^{4k} = (\Delta_R^+(\zeta) - 2^{4k-1}) + (\Delta_R^-(\zeta) - 2^{4k-1})$. Since $\widetilde{KO}(S^{8k+1}) \cong Z_2$, it follows that $\Delta_R^+(\zeta) - 2^{4k-1} = 0$ and $\Delta_R^-(\zeta) - 2^{4k-1} = \beta_{8k+1}$, or $\Delta_R^+(\zeta) - 2^{4k=1} = \beta_{8k+1}$ and $\Delta_R^-(\zeta) - 2^{4k-1} = 0$. Without loss of generality, assume the latter. Then, by theorem (2.25), we have that: $\Delta_R^+(\zeta^{8k} \times \tau^{8\ell}) = \Delta_R^+(\zeta) \otimes \Delta_R^+(\tau^{8\ell}) + \Delta_R^-(\zeta) \otimes \Delta_R^-(\tau^{8\ell}) =$

$(\beta_{8k+1} + 2^{4k-1}) \otimes (\beta_{8\ell} + 2^{4\ell-1}) + 2^{4k-1} \otimes (-\beta_{8\ell} + 2^{4\ell-1}) =$
$\beta_{8k+1} \otimes \beta_{8\ell} + 2^{4(k+\ell)-1}$. Applying Bott's periodicity theorem inductively, we have that $0 \neq \beta_{8k+1} \otimes \beta_{8\ell} \in \widetilde{KO}(S^{8k} \times S^{8\ell})$. Therefore, $2^{4(\ell+k)-1} \neq \Delta_R^+(\zeta^{8k} \times \tau^{8\ell}) \in KO(S^{8k+1} \times S^{8\ell})$. The result follows. ∎

Chapter III

Whitney Sums of Stably Trivial Vector Bundles.

This chapter is concerned with the r-fold Whitney sums of the vector bundle $\eta_{n,k}$ defined in Chapter I. The notation is that of that chapter unless otherwise indicated. Moreover, the truncated real projective space RP^n/RP^{m-1} is denoted by P^n_m. Sections 1 and 2 contain auxiliary results.

§1. Sections of $\alpha^*(\tau S^m)$.

Proposition (3.1). Let η_m represent the generator of $\pi_{m+1}(S^m) \cong Z_2$. Assume that m is even and $m \geq 6$. Then

(i) $\eta_m^*(\tau S^m)$ is not trivial;

(ii) if $m \equiv 0(4)$, $\eta_m^*(\tau S^m)$ and $(\eta_m \circ \eta_{m+1})^*(\tau S^m)$ admit exactly one linearly independent section;

(iii) if $m \equiv 2(8)$ and $m \geq 18$, $\eta_m^*(\tau S^m)$ admits at most 5 linearly independent sections.

Proof. (i) If $\eta_m^* \tau S^m$ is trivial, there is a lifting $\bar{\eta}_m$ of η_m as shown in the following diagram:

$$\begin{tikzcd} & V_{m+1,m} = SO(m+1) \arrow[d, "p"] \\ S^{m+1} \arrow[ur, dashed, "\bar{\eta}_m"] \arrow[r, "\eta"] & S^m \end{tikzcd}$$

Consequently, $\partial(\eta_m) = 0$ where ∂ is the boundary homomorphism in the exact homotopy sequence associated to the fibering $SO(m+1)/SO(m) \cong S^m$. However, this is the case only if $m \equiv 3(4)$ [Kervaire 1960, thm. 1].

(ii) Let $\alpha\colon S^{m+i} \longrightarrow S^m$ be a continuous map. The vector bundle $\alpha^*(\tau S^m)$ admits k linearly independent sections if and only if there is a lifting $\bar{\alpha}$ of α as shown in diagram (1).

$$\begin{tikzcd} & V_{m+1,k+1} \arrow[d] \\ S^{m+i} \arrow[ur, dashed, "\bar\alpha"] \arrow[r, "\alpha"] & S^m \end{tikzcd} \qquad\qquad \begin{tikzcd} & P^m_{m-k} \arrow[d, "c"] \\ S^{m+i} \arrow[ur, dashed, "\bar\alpha"] \arrow[r, "\alpha"] & S^m \end{tikzcd}$$

(1) (2)

If $2k < m - i - 1$, by cellular approximation, this is equivalent, to having the homotopy commutative diagram (2), where $c\colon P^m_{m-k} \longrightarrow S^m$ is the map collapsing the (m-1)-skeleton of P^m_{m-k} to a point.

To prove that $\eta_m^*(\tau S^m)$ admits one non-zero section for m even, it suffices to consider the cofibration sequence

$$\begin{tikzcd} S^{m-1} \arrow[r, "2"] & S^{m-1} \arrow[r] & P^m_{m-1} \arrow[r, "c"] & S^m \arrow[r, "2"] & S^m \\ & & & S^{m+1} \arrow[ul, dashed, "\bar\eta_m"] \arrow[u, "\eta_m"] \arrow[ur, dashed, "2\circ\eta_m \simeq *"'] & \end{tikzcd}$$

Since $2\circ\eta_m$ is homotopically trivial, the map η_m factors through c as desired, and the result follows. Of course, that $(\eta_m \circ \eta_{m+1})^*(\tau S^m) = \eta_{m+1}^*(\eta_m^*(\tau S^m))$ admits at least one non-zero section also follows.

To prove that $(\eta_m \circ \eta_{m+1})^*(\tau S^m)$ does not admit 2 linearly independent sections if $m \equiv 0(4)$, consider the cofibration sequence

$$\begin{tikzcd} S^{m-1} \arrow[r, "A"] & P^{m-1}_{m-2} \simeq S^{m-1} \vee S^{m-2} \arrow[r] & P^m_{m-2} \arrow[r, "c"] & S^m \arrow[r, "\Sigma A"] & P^{m-1}_{m-2} \\ & & & S^{m+2} \arrow[u, "\eta_{m+1}\circ\eta_m"'] & \end{tikzcd}$$

The attaching map $A\colon S^{m-1} \longrightarrow P^{m-1}_{m-2} \simeq S^{m-2} \vee S^{m-1}$ is homotopic to the map $\eta_{m-2} \vee 2\colon S^{m-1} \longrightarrow S^{m-2} \vee S^{m-1}$ (consider the action of the Steenrod algebra on $H^*(P^m_{m-2};\ Z^2))$. Consequently, the composition $(\Sigma A)\circ(\eta_m\circ\eta_{m+1}) \simeq ((\Sigma\eta_{m-2})\circ\eta_m\circ\eta_{m+1})\vee *$ is not homotopically trivial, and the map $\eta_m\circ\eta_{m+1}$ does not lift through c. We deduce that $(\eta_m\circ\eta_{m+1})^*(\tau S^m)$ does not admit 2 linearly independent sections for $m \equiv 0(4)$. This implies that $\eta_m^*(\tau S^m)$ does not admit 2 linearly independent sections either.

(iii) (The following proof was suggested by S. Gitler, thus avoiding an unnecessary reference to the literature). Assume that $m \equiv 2(8)$ and suppose that $\eta_m^*(\tau S^m)$ admits 6 linearly independent sections. Since we assume that $m \geq 18$, we deduce that there exists a factorization $\bar\eta_m$ of η_m through the map c as in the following diagram:

$$\begin{tikzcd}
 & P^m_{m-6} \arrow[d, "c"] \arrow[r, dashed] & P^m_{m-6}\cup_{\bar\eta_m} e^{m+2} \arrow[d] \\
S^{m+1} \arrow[ur, dashed, "\bar\eta_m"] \arrow[r, "\eta_m"'] & S^m \arrow[r, dashed] & S^m \cup_{\eta_m} e^{m+2}
\end{tikzcd}$$

Let $X = P^m_{m-6}\cup_{\bar\eta_m} e^{m+2}$. For $i = m-6, m-5, \ldots, m$, denote the generator of $H^i(X;Z_2) \cong Z_2$ by x_i. Denote also the generator of $H^{m+2}(X;Z_2) \cong Z_2$ by y. By naturality, $Sq^2x_m = y$. Since $Sq^3x_{m-3} = x_m$ we deduce that $Sq^2Sq^3x_{m-3} = y$. Using the Adem relation $Sq^2Sq^3 + Sq^4Sq^1 + Sq^1Sq^4$ and the fact that $H^{m+1}(X;Z_2) = 0$, we have that $Sq^4x_{m-2} = y$. Since $Sq^4x_{m-6} = x_{m-2}$, we deduce that $Sq^4Sq^4x_{m-6} = y$. However, $Sq^1x_{m-6} = 0$ and $Sq^2x_{m-6} = 0$, and using the Adem relation $Sq^4Sq^4 + Sq^7Sq^1 + Sq^6Sq^2$, we deduce that $Sq^4Sq^4x_{m-6} = 0$, a contradiction. ■

<u>Remark</u> (3.2). It has been shown that $\eta_m^*(\tau S^m)$ admits exactly 5 linearly independent sections if $m \equiv 2(8)$ [Hoo 1964]. The author does not know the corresponding result for $m \equiv 6(8)$.

§2. The block map.

Let $1 \leq k_i < n_i$ $(i = 1,2,\ldots,t)$ and let $N = \Sigma n_i$ and $K = \Sigma k_i$. We define the map $b\colon V_{n_1,k_1} \times \ldots \times V_{n_t,k_t} \longrightarrow V_{N,K}$ by:

$$b(A_1,\ldots,A_t) = \begin{bmatrix} A_1 & 0 \ldots & 0 \\ 0 & & 0 \\ \ldots & & \\ 0 & 0 & A_t \end{bmatrix},$$

where A_i is a $k_i \times n_i$ matrix such $A_i A_i^t = I_{k_i}$. We call the map b the block map and we will generally assume that the indices (n_i,k_i) are clearly indicated by the context.

Proposition (3.3). Let $b\colon V_{n_1,k_1} \times \ldots \times V_{n_t,k_t} \longrightarrow V_{N,K}$ be the block map. Then $b^* \eta_{N,K} \cong \eta_{n_1,k_1} \times \ldots \times \eta_{n_t,k_t}$.

Proof. Omitted. ■

Proposition (3.4). Assume that for some i_0, $1 \leq i_0 \leq t$, we have $k_{i_0} \leq \sum_{i \neq i_0} (n_i - k_i)$. Then the restriction of the block map b to the subproduct $V_{n_1,k_1} \times \ldots \times V_{n_{i_0-1},k_{i_0-1}} \times * \times V_{n_{i_0+1},k_{i_0+1}} \times \ldots \times V_{n_t,k_t}$ is homotopically trivial.

Proof. For the sake of technical clearness, we give the proof for $t = 2$. The reader will see immediately the modifications necessary for the general case. For $t = 2$, assume that $i_0 = 1$. We have to prove that the map $V_{n_1,k_1} \longrightarrow V_{n_1+n_2,k_1+k_2}$ defined by $A \longrightarrow \begin{pmatrix} A & 0 & 0 \\ 0 & I_{k_1} & 0 \end{pmatrix}$ is homotopically trivial. Since $n_2 - k_2 \geq k_1$, the map

$$(A,s) \longrightarrow \begin{pmatrix} \sqrt{1-s^2}\,A & 0 & sI_{k_1} & 0 \\ 0 & I_{k_2} & 0 & 0 \end{pmatrix}$$

$(0 \le s \le 1)$ defines a homotopy between the above map and the constant map. ■

We will only use the map b in cases where the indices (n_i,k_i) are all equal, i.e. $(n_i,k_i) = (n,k)$ $i = 1,\dots,t$. The following two propositions are related to this case. For a space X and a permutation $\sigma \in S_t$, we denote by $T_\sigma\colon X \times \dots \times X \longrightarrow X \times \dots \times X$ the map which permutes the t factors of $X \times \dots \times X$ according to the permutation σ.

<u>Proposition</u> (3.5). (<u>Symmetry property of</u> b). <u>Consider the block map</u> $b\colon V_{n,k} \times \dots \times V_{n,k} \longrightarrow V_{tn,tk}$. <u>Assume that either</u> k <u>is even, or</u> n <u>is even and</u> k <u>and</u> t <u>are odd</u>. <u>Then</u> $b \circ T_\sigma \simeq b$ <u>for any permutation</u> $\sigma \in S_t$.

<u>Proof</u>. There is an obvious map $T'_\sigma\colon V_{tn,tk} \longrightarrow V_{tn,tk}$ permuting the elements of the tk-frames in $V_{tn,tk}$ in such a way that the following diagram (strictly) commutes:

$$\begin{tikzcd} V_{n,k} \times \dots \times V_{n,k} \arrow[r, "b"] \arrow[d, "T_\sigma"] & V_{tn,tk} \arrow[d, "T'_\sigma"] \\ V_{n,k} \times \dots \times V_{n,k} \arrow[r, "b"] & V_{tn,tk} \end{tikzcd}$$

If k is even, T'_σ is a "row operation of even order" [James 1958]. In this case, it is known that T'_σ is homotopic to the identity map on $V_{tn,tk}$ [Ibidem]. The proposition for k even follows immediately.

If n is even and k and t are odd, T'_σ is also a row operation, but possibly of odd order. However, under these assumptions, tn is even and tk is odd, and all row operations on $V_{tn,tk}$ are homotopic to the identity map [Ibidem, cor. 1.2]. The proposition follows easily in this case also. ∎

Remark (3.6). Let $f\colon X^t = X \times \ldots \times X \longrightarrow Y$ be a based point preserving map such that $f \circ T_\sigma \simeq f$ for all $\sigma \in S_t$. Of course, S_t acts on $H^*(X \times \ldots \times X; Z_2)$ (on the right) via the homomorphisms T^*_σ. Let $y \in H^*(Y;Z_2)$. Then $T^*_\sigma f^*(y) = (f \circ T_\sigma)^*(y) = f^*(y)$ for all $\sigma \in S_t$. Thus, the class $f^*(y) \in H^*(X \times \ldots \times X; Z_2)$ is fixed under the S_t-action.

Now assume that Y is highly connected. If $H^*(f;Z_2)$ is the zero homomorphism, the map f lifts to the first stage of the Postnikov system of Y:

$$\begin{tikzcd} & E_1 \arrow[d] \\ X \times \ldots \times X \arrow[ur, dashed, "\bar{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that $\phi \in H^j(E_1;Z_2)$ is a k-invariant defined by a relation $Sq^{j-i}y = 0$ for some $y \in H^{i+1}(Y;Z_2)$. Then, the set $\{\bar{f}^*\phi\}$, where $\bar{f}$ runs through all the possible liftings of f, determines an element Σ in the quotient group $H^j(X \times \ldots \times X; Z_2)/Sq^{j-i}H^i(X \times \ldots \times X; Z_2)$. Using the naturality of the Steenrod operations, we have that $T^*_\sigma Sq^{j-i}H^i(X \times \ldots \times X; Z_2) = Sq^{j-i} T^*_\sigma H^i(X \times \ldots \times X; Z_2)$ for any $\sigma \in S_t$. Consequently, $Sq^{j-i} H^i(X \times \ldots \times X; Z_2)$ is an S_t-invariant subgroup of $H^j(X \times \ldots \times X; Z_2)$, and S_t acts naturally on $H^j(X \times \ldots \times X; Z_2)/Sq^{j-i} H^i(X \times \ldots \times X; Z_2)$. We wish to point out that Σ is fixed under this S_t-action. Indeed, if $X \times \ldots \times X \longrightarrow E_1$ is a lifting of f, then $\bar{f} \circ T_\sigma$ is also a lifting of $f \simeq f \circ T_\sigma$, up to homotopy. Since $(\bar{f} \circ T_\sigma)^*(\phi) = T_\sigma^* \bar{f}^*(\phi)$, this implies that the set $\{\bar{f}^*(\phi)\}$ is invariant under the action of S_t on $H^j(X \times \ldots \times X; Z_2)$ and that Σ is fixed under the action of S_t on $H^j(X \times \ldots \times X; Z_2)/Sq^{j-i}\, H^i(X \times \ldots \times X; Z_2)$.

The reader is asked to notice that these remarks generalize to k-invariants ϕ defined by more complicated relations and to k-invariants in higher stages of the Postnikov system for Y, even though the details are more complicated as usual.

In §§3,4, we will apply these considerations to the map $b\colon V_{n,k} \times \ldots \times V_{n,k} \longrightarrow V_{tn,tk}$ with n,k and t satisfying the conditions of proposition (3.5). ∎

Let $1 \leq k < n$. Recall [Steenrod-Epstein 1962] that the cohomology ring $H^*(V_{n,k};Z_2)$ is a commutative associative algebra over Z_2 with generators $x_{n-k},\ldots,x_{n-1}$ in dimension $n-k,\ldots,n-1$ respectively, and relations $x_i^2 = x_{2i}$ for $n - k \leq i \leq \frac{1}{2}(n-1)$ and $x_i^2 = 0$ for $\frac{1}{2}(n-1) < i \leq n - 1$. Moreover, the natural inclusion $i\colon V_{n-\ell,k-\ell} \to V_{n,k}$ induces the cohomology homomorphism defined by $i^*(x_j) = x_j$ for $n - k \leq j \leq n - \ell - 1$ and $i^* x_j = 0$ for $n - \ell \leq j \leq n - 1$. The natural projection $p\colon V_{n,k} \longrightarrow V_{n,\ell}$ induces the homomorphism defined by $p^*(x_j) = x_j$ for $n - \ell \leq j \leq n - 1$.

<u>Proposition</u> (3.7). <u>Assume that</u> $1 \leq k \leq (t-1)n/t$. <u>Then the block map</u> $b\colon V_{n,k} \times \ldots \times V_{n,k} \longrightarrow V_{tn,tk}$ <u>induces the zero homomorphism in reduced cohomology with</u> Z_2-<u>coefficients</u>.

<u>Proof</u>. Let $B\colon SO(n) \times \ldots \times SO(n) \longrightarrow SO(tn)$ be the map defined by

$$B(A_1,\ldots,A_t) = \begin{bmatrix} A_1 & 0 \ldots & 0 \\ 0 & & \ldots \\ \ldots\ldots & & 0 \\ 0 & \ldots & 0 \quad A_t \end{bmatrix},$$

and let $\mu\colon SO(n) \times \ldots \times SO(n) \to SO(n)$ be the multiplication map $\mu(A_1,\ldots,A_t) = A_1 \ldots A_t$. Let $i\colon SO(n) \to SO(tn)$ denote the natural inclusion. It is well known that $i \circ \mu \simeq B$. Consequently, the following diagram is homotopy commutative:

$$\begin{tikzcd} & SO(n) \arrow[d, "i"] \\ SO(n) \times \ldots \times SO(n) \arrow[ur, "\mu"] \arrow[r, "B"] \arrow[d, "p \times \ldots \times p"] & SO(tn) \arrow[d, "p"] \\ V_{n,k} \times \ldots \times V_{n,k} \arrow[r, "b"] & V_{tn,tk} \end{tikzcd} .$$

Since $k \leq (t-1)n/t$, it follows that $tn - tk \geq n$ and that the map $p \circ i$ induces the trivial homomorphism $(p \circ i)^*$ in reduced cohomology with Z_2-coefficients. Thus $(\mu \circ i \circ p)^* = (b \circ (p \times \ldots \times p))^* = (p \times \ldots \times p)^* \circ b^*$ is also the zero homomorphism. Since $(p \times \ldots \times p)^*$ is a monomorphism, the proposition follows.

§3. <u>The vector bundle $\eta_{n,2} \oplus \eta_{n,2}$</u>.

In this section we prove the following theorem.

<u>Theorem</u> (3.8). <u>Let</u> $n \geq 4$. <u>The vector bundle</u> $\eta_{n,2} \oplus \eta_{n,2}$ <u>over</u> $V_{n,2}$ <u>is trivial if and only if</u> n <u>is even</u>.

The following corollaries are almost immediate.

Corollary (3.9). Let ξ be an even dimensional real vector bundle over a finite CW-complex such that $\xi \oplus 2\varepsilon$ is trivial. Then $\xi \oplus \xi$ is trivial.

Proof. The vector bundle $\eta_{n,2}$ is "weakly universal" for such ξ (cf. chapter I). ■

Corollary (3.10). If $n - k \geq 3$ is odd, then $\eta_{n,k} \oplus \eta_{n,k}$ is not trivial over $V_{n,k}$.

Proof. Consider the inclusion map $i\colon V_{n-k+2,2} \longrightarrow V_{n,k}$. We have that $i^*(\eta_{n,k} \oplus \eta_{n,k}) \cong \eta_{n-k+2,2} \oplus \eta_{n-k+2,2}$. The latter vector bundle is not trivial by Theorem (3.8). Thus $\eta_{n,k} \oplus \eta_{n,k}$ is not trivial. ■

Proof of Theorem (3.8).

For $n \geq 4$, let $d\colon V_{n,2} \longrightarrow V_{n,2} \times V_{n,2}$ be the diagonal map, and $b\colon V_{n,2} \times V_{n,2} \longrightarrow V_{2n,4}$ the block map. By (3.3), we have $b^*(\eta_{2n,4}) \cong \eta_{n,2} \times \eta_{n,2}$. Hence, $d^*b^*(\eta_{2n,4}) \cong \eta_{n,2} \oplus \eta_{n,2}$. The first step of the proof will be to study the map $b \circ d$. If n is even, we show that $b \circ d$ lifts arbitrarily high into the (modified) Postnikov tower over $V_{2n,4}$. Consequently, $b \circ d$ is homotopically trivial, and the theorem follows for this case. If n is odd, we find that $b \circ d$ cannot be lifted past the first stage of the Postnikov tower over $V_{2n,4}$. It follows that $b \circ d$ is not homotopically trivial. We deduce that $\eta_{n,2} \oplus \eta_{n,2} \cong (b \circ d)^* \eta_{2n,4}$ is not trivial by an argument involving proposition (3.1).

Let K_i denote the Eilenberg-Maclam space $K(i,Z_2)$ and let ι_i be the fundamental class. τ will denote the cohomology transgression.

Let

$$(3.11) \qquad \longrightarrow E_i \xrightarrow{q_i} \cdots \longrightarrow E_2 \xrightarrow{q_2} E_1 \xrightarrow{q_1} V_{2n,4}$$

be the (modified) Postnikov tower over $V_{2n,4}$ (*).

Case n even.

The following is a list of all k-invariants in dimension $\leq 2n - 3$ occurring in the Postnikov tower over $V_{2n,4}$, $n \equiv 0(2)$:

(0) k-invariants in $H^*(V_{2n,4}; Z_2)$

$$x_{2n-4}\ , \quad x_{2n-3}$$

(1) k-invariants in $H^*(E_1; Z_2)$

$$\phi^{(1)}_{2n-4} \ : \quad Sq^1\, x_{2n-4} = 0$$

$$\psi_{2n-3} \ : \quad Sq^2\, x_{2n-4} = 0$$

(i) k-invariants $H^*(E_i; Z_2)$ $i \geq 2$

$$\phi^{(i)}_{2n-4} \ : \quad Sq^1\, \phi^{(i-1)} = 0\ .$$

Due to proposition (3.7), the map b admits a lifting b_1 to the first stage of (3.11). Proposition (3.5) applies to b. Consequently, we can apply remark (3.6) to the class $b_1^*\, \psi_{2n-3}$. Since $Sq^2 H^{2n-5}(V_{n,2} \times V_{n,2}; Z_2) = 0$, $b_1^*\, \psi_{2n-3}$ does not depend on the choice of the lifting b_1 of b. Thus, by remark (3.6), $b_1^*\, \psi_{2n-3}$ is a class of $H^{2n-3}(V_{n,2} \times V_{n,2}; Z_2)$ invariant under the action of S_2.

(*) We will use the Postnikov towers associated with the path-loop fibrations.

This implies that $b_1^* \psi_{2n-3} = \alpha(x_{n-2}x_{n-1} \otimes 1 + 1 \otimes x_{n-2}x_{n-1}) +$ $\beta(x_{n-2} \otimes x_{n-1} + x_{n-1} \otimes x_{n-2})$ for some $\alpha, \beta \in Z_2$. Hence $d^* b_1^* \psi_{2n-3} = 0$ $H^{2n-3}(V_{n,2}, Z_2)$. Since $H^i(V_{n,2}; Z_2) = 0$ for $i = 2n - 4$ and for $i \geq 2n - 2$, we deduce that $b \circ d$ lifts arbitrarily high in the Postnikov tower (3.11). Therefore $b \circ d$ is homotopically trivial. It follows that $\eta_{n,2} \oplus \eta_{n,2} \cong (b \circ d)^* \eta_{2n,4}$ is trivial.

Case n odd.

The following is a list of all k-invariants in dimension $\leq 2n - 2$ occurring in the Postnikov tower over $V_{2n,4}$, $n \equiv 1(2)$:

(0) k-invariants in $H^*(V_{2n,4}; Z_2)$

$$x_{2n-4}, \quad x_{2n-3}$$

(1) k-invariants in $H^*(E_1; Z_2)$

$$\phi^{(1)}_{2n-4} \quad : \quad Sq^1 x_{2n-4} = 0$$

$$\psi_{2n-3} \quad : \quad Sq^2 x_{2n-4} + Sq^1 x_{2n-3} = 0$$

(i) k-invariants in $H^*(E_i; Z_2)$ $i \geq 2$

$$\phi^{(i)}_{2n-4} \quad : \quad Sq^1 \phi^{(i-1)}_{2n-4} = 0 .$$

As in the case for n even, we obtain a lifting b_1 of b to the first stage of the Postnikov tower (3.11). Again, we study $b_1^* \psi_{2n-3}$ using remark (3.6). In this case, however, we have that

$I = Sq^2\, H^{2n-5}(V_{n,2} \times V_{n,2};\, Z_2) + Sq^1\, H^{2n-4}(V_{n,2} \times V_{n,2};\, Z_2)$ is the subgroup of $H^{2n-3}(V_{n,2} \times V_{n,2};\, Z_2)$ generated by $x_{n-1} \otimes x_{n-2} + x_{n-2} \otimes x_{n-1}$. Consequently, the set $\{b_1^* \psi_{2n-3}\}$, where b_1 runs through all the possible liftings of b to E_1, determines an element Σ in $H^{2n-3}(V_{n,2} \times V_{n,2};\, Z_2)/I$, and, by remark (3.6), Σ is invariant under the induced action of S_2 . After examining the action of S_2 on the group $H^{2n-3}(V_{n,2} \times V_{n,2};\, Z_2)/I$, it is easy to see that $\Sigma = \alpha\, x_{n-2} \otimes x_{n-1} + \beta(x_{n-2}x_{n-1} \otimes 1 + 1 \otimes x_{n-2}x_{n-1}) + I$ for some $\alpha, \beta \in Z_2$. Proposition (3.4) implies immediately that the composition $V_{n,2} \vee V_{n,2} \xrightarrow{j} V_{n,2} \times V_{n,2} \xrightarrow{b} V_{2n,4}$ is homotopically trivial. It is then a routine matter to check that we must have $j^* b_1^* \psi_{2n-3} = 0$ for any lifting b_1 of b . This implies that $\beta = 0$. It is more difficult to determine α . We postpone the proof of the following claim to the end of the proof of theorem (3.8).

<u>Claim</u> (3.12): $\alpha \neq 0$.

Assuming that we have proved this claim, we continue the proof. Thus, $\Sigma = x_{n-2} \otimes x_{n-1} + I$, i.e., $b_1^* \psi_{2n-3} = x_{n-2} \otimes x_{n-1}$ or $b_1^* \psi_{2n-3} = x_{n-1} \otimes x_{n-2}$ depending on the choice of the lifting b_1 of b . It follows that $d^* b_1^* \psi_{2n-3} = x_{n-2}x_{n-1} \neq 0 \in H^{2n-3}(V_{n,2};\, Z_2)$. Since $Sq^2\, H^{2n-5}(V_{n,2};\, Z_2) + Sq^1\, H^{2n-4}(V_{n,2};\, Z_2) = 0$, we deduce that $f^* \psi_{2n-3} = x_{n-2}x_{n-1}$ for any lifting f of the map $d \circ b$ to the first stage of the Postnikov tower (3.11).

Now, recall that the Stiefel manifold $V_{n,2}$ can be obtained by attaching a (2n-3)-cell to the truncated projective space P^{n-1}_{n-2} . Thus, $V_{n,2} \simeq P^{n-1}_{n-2} \cup_A e^{2n-3}$ and we have a map $c\colon V_{n,2} \longrightarrow S^{2n-3}$

collapsing P^{n-1}_{n-2} to a point. By cellular approximation, we have that $c^{\#}\colon [S^{2n-3}, V_{2n,4}] \longrightarrow [V_{n,2}, V_{2n,4}]$ is an isomorphism. Therefore, the map $b \circ d$ factors through the space S^{2n-3} as follows.

$$\begin{tikzcd} V_{n,2} \arrow[rr, "b \circ d"] \arrow[dr, "c"'] & & V_{2n,4} \\ & S^{2n-3} \arrow[ur, "g"'] & \end{tikzcd}$$

Using our previous considerations about the map $b \circ d$, it is easy to deduce that the map g admits a lifting $g_1\colon S^{2n-3} \longrightarrow E_1$ to the first stage of the Postnikov tower (3.11), and that $g_1^* \psi_{2n-3} \neq 0$ $H^{2n-3}(S^{2n-3}; Z_2)$ for any lifting chosen. Comparing the Postnikov tower over $S^{2n-4} \xrightarrow{i} V_{2n,4}$ and the Postnikov tower (3.11), one can deduce easily that the map g is homotopic to the composition $S^{2n-3} \xrightarrow{\eta} S^{2n-4} \xrightarrow{i} V_{2n,4}$ where η represents the generator of $\pi_{2n-3}(S^{2n-4})$. Summing up these considerations, we now have that $b \circ d \simeq i \circ \eta \circ c$. Therefore, $\eta_{n,2} \oplus \eta_{n,2} \cong c^* \ \eta^* \ i^*(\eta_{2n,4})$. Since $i^*\eta_{2n,4} \cong \tau S^{2n-4}$, this means that $\eta_{n,2} \oplus \eta_{n,2} \cong c^* \ \eta^*(\tau S^{2n-4})$.

We wish to deduce that $\eta_{n,2} \oplus \eta_{n,2}$ is not trivial. We suppose that $\eta_{n,2} \oplus \eta_{n,2}$ is trivial and we will get a contradiction. Let Γ_{2n-4} denote the universal vector bundle over $BSO(2n-4)$ and let $t\colon S^{2n-4} \longrightarrow BSO(2n-4)$ be a classifying map for τS^{2n-4}. Consider the following diagram:

$$\begin{tikzcd} S^{2n-4} \arrow[r, "A"] & P^{n-1}_{n-2} \arrow[r] & V_{n,2} \arrow[r, "c"] & S^{2n-3} \arrow[r, "\Sigma A"] \arrow[d, "\eta"] & \Sigma P^{n-1}_{n-2} \arrow[ddl, dashed, "h"] \\ & & & S^{2n-4} \arrow[d, "t"] & \\ & & & BSO(2n-4) & \end{tikzcd}$$

The horizontal sequence of maps is a cofibration sequence. If $\eta_{n,2} \oplus \eta_{n,2}$ is trivial, then the composition $t \circ \eta \circ c$ must be homotopically trivial. Therefore, there is a map $h: \Sigma P^{n-1}_{n-2} \longrightarrow BSO(2n-4)$ such that $h \circ \Sigma A \simeq t \circ \eta$. First let $n \equiv 3(8)$. Notice that the map h factors through $BSO(n) \subset BSO(2n-4)$ for dimensional reasons. It follows that the vector bundle $\eta^*(\tau S^{2n-4}) \cong \eta^* t^*(\Gamma_{2n-4}) \cong (\Sigma A)^* h^*(\Gamma_{2n-4})$ admits at least $n-4$ sections. But $n \equiv 3(8)$ and $n \geq 4$, so that $2n-4 \equiv 2(8)$ and $n-4 \geq 7$. Thus we have obtained a contradiction with proposition (3.1)iii. This proves the theorem for $n \equiv 3(8)$.

For $n \equiv 1,5$ or 7 (8), first recall that the Stiefel manifolds are stably parallelizable. It follows by a standard argument that the attaching map $A: S^{2n-4} \longrightarrow P^{n-1}_{n-2}$ of the top cell e^{2n-3} of $V_{n,2}$ must be stably homotopically trivial. Hence the composition $S^{2n-3} \simeq \Sigma S^{2n-4} \xrightarrow{\Sigma A} \Sigma P^{n-1}_{n-2} \xrightarrow{c'} S^n$ must be stably homotopically trivial also ($c': \Sigma P^{n-1}_{n-2} \longrightarrow S^n$ denotes the map collapsing the bottom cell of ΣP^{n-1}_{n-2} to a point). By the Freudenthal suspension theorem, this implies that the composition $c' \circ (\Sigma A)$ must be homotopically trivial itself. Considering the cofibration $S^{n-1} \xrightarrow{j'} \Sigma P^{n-1}_{n-2} \xrightarrow{c'} S^n$, we deduce that ΣA factors through $j': S^{n-1} \longrightarrow \Sigma P^{n-1}_{n-2}$. Since $n \geq 5$, $[\Sigma P^{n-1}_{n-2}, BSO(2n-4)] \cong [\Sigma P^{n-1}_{n-2}, BSO]$ $(\cong \widetilde{KO}(\Sigma P^{n-1}_{n-2}))$. Using this fact together with Bott's computation of $\pi_i(BSO)$ and the cofibration $S^{n-1} \xrightarrow{2} S^{n-1} \xrightarrow{j'} \Sigma P^{n-1}_{n-2}$, one sees that $\mathrm{Im}\{j'^{\#}: [\Sigma P^{n-1}_{n-2}, BSO(2n-4)] \longrightarrow [S^{n-1}, BSO(2n-4)]\} = 0$. Thus, the composition $h \circ \Sigma A$ must be homotopically trivial. Since $\eta^*(\tau S^{2n-4}) \cong (\Sigma A)^* h^*(\Gamma_{2n-4})$, we obtain that $\eta^*(\tau S^{2n-4})$ is a trivial vector bundle. This is a contradiction with proposition (3.1)i. This proves the theorem for $n \equiv 1,5$ or 7 (8).

This concludes the proof of theorem (3.8), assuming claim (3.12).

Proof of Claim (3.12).

Let us make the supposition that $\alpha = 0$. Recall that n is odd. Then there is a lifting b_1 of b to the first stage of the Postnikov tower (3.11) such that $b_1^* \psi_{2n-3} = 0$. Consider the inclusion $j\colon P_{n-2}^{n-1} \longrightarrow V_{n,2}$ and let $b' = b \circ (j \times j)$. Then $b_1' = b_1 \circ (j \times j)$ is a lifting of b' to E_1 . Our hypothesis implies that $b_1'^* \psi_{2n-3} = 0$. Moreover, since $Sq^1 \phi_{2n-4}^{(1)} = 0$, we must have that $Sq^1 (b_1'^* \phi_{2n-4}^{(1)}) = 0$. This implies that $b_1'^* \phi_{2n-4}^{(1)} = 0$. Thus, b_1' admits a lifting b_2' to E_2 . Again, we must have $b_2'^* (\phi_{2n-4}^{(2)}) = 0$ by the same argument as above. In this way, the map b' lifts arbitrarily high in the Postnikov tower (3.11). Therefore, under the hypotheses that $\alpha = 0$, we have that $b' \simeq *$.

We now obtain a contradiction by showing that, in fact, $b' \not\simeq *$. Specifically, we prove that $(b')^! \colon \widetilde{KU}(V_{2n,4}) \longrightarrow \widetilde{KU}(P_{n-2}^{n-1} \times P_{n-2}^{n-1})$ is not the zero homomorphism. Let ξ denote the restriction of $\eta_{n,2}$ to P_{n-2}^{n-1} . Recall that the composition $P_{n-2}^{n-1} \xrightarrow{j} V_{n,2} \xrightarrow{p} S^{n-1}$ is homotopic to the map $c\colon P_{n-2}^{n-1} \longrightarrow S^{n-1}$ collapsing $S^{n-2} \subset P_{n-2}^{n-1}$ to a point. We can compute $\Delta(\xi)$ (see chapter II) as follows:

$$
\begin{aligned}
\Delta(\xi) &= \Delta(j^*(\eta_{n,2})) \\
&= j^*\,\Delta(\eta_{n,2}) && \text{(by naturality of } \Delta) \\
&= j^*(\Delta(\eta_{n,2})\Delta(\varepsilon)) && (\Delta(\varepsilon) = 1) \\
&= j^*(\Delta^+(\eta_{n,2} \oplus \varepsilon)) && \text{(corollary (2.15))} \\
&= j^*(\Delta^+ p^*(\tau S^{n-1})) && (p^*(\tau S^{n-1}) \cong \eta_{n,2} \oplus \varepsilon) \\
&= j^* p^*(\Delta^+(\tau S^{n-1})) && \text{(by naturality of } \Delta^+) \\
&= c^*(\gamma_{n-1}) + 2^{\frac{1}{2}(n-3)} && \text{(by example (2.11))} \\
&= \nu + 2^{\frac{1}{2}(n-3)}
\end{aligned}
$$

where ν denotes the generator of $\widetilde{KU}(P^{n-1}_{n-2}) \cong Z_2$ and the last equality comes from the fact that $c^!: \widetilde{KU}(S^{n-1}) \twoheadrightarrow \widetilde{KU}(P^{n-1}_{n-2})$. Now, consider $\gamma_{2n-4} = \Delta^+(\eta_{2n,4}) - 2^{n-3} \in \widetilde{KU}(V_{2n,4})$. We have:

$$
\begin{aligned}
b'^!(\gamma_{2n-4}) &= b'^!(\Delta^+(\eta_{2n,4})) - 2^{n-3} \\
&= \Delta^+\, b'^*(\eta_{2n,4}) - 2^{n-3} \\
&= \Delta^+(\xi\times\xi) - 2^{n-3} \\
&= \Delta(\xi) \otimes \Delta(\xi) - 2^{n-3} && \text{(by theorem (2.14))} \\
&= (\nu + 2^{\frac{1}{2}(n-3)}) \otimes (\nu + 2^{\frac{1}{2}(n-3)}) - 2^{n-3} \\
&= \nu \otimes \nu && (\text{since } n \geq 5)\ .
\end{aligned}
$$

Since the exterior tensor product induces a <u>monomorphism</u> $\otimes: KU(X) \otimes KU(X) \longrightarrow KU(X\times X)$ [Atiyah 1962], we deduce that $0 \neq \nu \otimes \nu = b'^!(\gamma_{2n-4}) \in KU(P^{n-1}_{n-2} \times P^{n-1}_{n-2})$. Therefore, $b' \not\simeq *$ and this concludes the proof of the claim (3.12).

This completes the proof of theorem (3.8). ■

§4. The vector bundle $\eta_{n,3} \oplus \eta_{n,3} \oplus \eta_{n,3}$.

We prove the following theorem.

Theorem (3.13). Let $n \geq 4$ be even. Then the vector bundle $\eta_{n,3} \oplus \eta_{n,3} \oplus \eta_{n,3}$ over $V_{n,3}$ is trivial.

The following corollary is immediate.

Corollary (3.14). Let ξ be an odd dimensional vector bundle over a finite CW complex. Assume that $\xi \oplus 3\varepsilon$ is trivial. Then $\xi \oplus \xi \oplus \xi$ is trivial. ■

Proof of theorem (3.13).

Let $d\colon V_{n,3} \longrightarrow V_{n,3} \times V_{n,3} \times V_{n,3}$ be the diagonal map and let $b\colon V_{n,3} \times V_{n,3} \times V_{n,3} \longrightarrow V_{3n,9}$ be the block map. We have that $(b \circ d)^*(\eta_{3n,9}) \cong 3\eta_{n,3}$. We will show that $b \circ d \simeq *$ by lifting this map arbitrarily highly in the Postnikov tower over $V_{3n,9}$. This implies of course that $3\eta_{n,3}$ is trivial. We have to separate the proof in two parts according to the cases $n \equiv 0(4)$ and $n \equiv 2(4)$.

Case $n \equiv 0(4)$.

Since $\eta_{4,3}$ is trivial, we can assume that $n \geq 8$. We first prove that $3\eta_{n,3} \oplus \varepsilon$ is trivial.

We can construct a non-zero section of $3\eta_{n,3}$ as follows. Let (x_1,x_2,x_3) be a 3-frame in R^n , and let $p\colon V_{n,3} \longrightarrow S^{n-1}$ be the natural projection map, $p(x_1,x_2,x_3) = x_1$. Then

$(\eta_{n,3})_{(x_1,x_2,x_3)} = \{u\colon u \in R^n$ and $[u,x_i] = 0, \ i = 1,2,3\}$ and $p^*(\tau S^{n-1})_{(x_1,x_2,x_3)} = \{u\colon u \in R^n$ and $[u,x_1] = 0\}$. There is an obvious vector bundle map $P\colon p^*(\tau S^{n-1}) \longrightarrow \eta_{n,3}$ defined by $P(u) = u - [u,x_2]x_2 - [u,x_3]x_3$. Since $n \equiv 0(4)$, we can find 3 linearly independent sections s_1,s_2,s_3 of τS^{n-1} . Then we define a section s of $3\eta_{n,3}$ by setting $s(x_1,x_2,x_3) = (P(s_1(x_1)), P(s_2(x_1)), P(s_3(x_1)))$. It is easy to check that $s(x_1,x_2,x_3) \neq 0$ for all $(x_1,x_2,x_3) \in V_{n,3}$.

Since $3\eta_{n,3}$ admits a non-zero section, there is a (3n-10)-dimensional vector bundle ζ over $V_{n,3}$ such that $3\eta_{n,3} \cong \zeta \oplus \varepsilon$. We have that $\zeta \oplus 3\varepsilon \cong 3\eta_{n,3} \oplus 2\varepsilon \cong \eta_{n,3} \oplus p'^*(2\eta_{n,2})$ where $p'\colon V_{n,3} \longrightarrow V_{n,2}$. The last vector bundle is trivial since $2\eta_{n,2}$ is trivial by theorem (3.8) and $2(n-2) \geq 3$. Thus $\zeta \oplus 3\varepsilon$ is trivial and by theorem (1.3), there is a map $f\colon V_{n,3} \longrightarrow V_{3n-7,3}$ such that $f^*\eta_{3n-7,3} \cong \zeta$. Therefore, we have that $3\eta_{n,3} \oplus \varepsilon \cong \zeta \oplus 2\varepsilon \cong f^*(\eta_{3n-7,3} \oplus 2\varepsilon) \cong (p'' \circ f)^*(\tau S^{3n-8})$ where $p''\colon V_{3n-7,3} \longrightarrow S^{3n-8}$ is the projection map. We claim that $p'' \circ f \simeq *$. Indeed, by cellular approximation, we have the following commutative diagram where the vertical maps are isomorphisms.

$$\begin{array}{ccc} [V_{n,3}\ ,\ V_{3n-7,3}] & \xrightarrow{p''_\#} & [V_{n,3}\ ,\ S^{3n-8}] \\ \uparrow\cong & & \uparrow\cong \\ [S^{3n-6},\ P^{3n-8}_{3n-10}] & \xrightarrow{c_\#} & [S^{3n-6},\ S^{3n-8}] \end{array}$$

By the <u>proof</u> of proposition (3.1)ii, one sees easily that the arrow forming the bottom side of the square is the zero homomorphism. It follows that the arrow at the top of the square is also the zero homomorphism. This implies that $p'' \circ f = p''_{\#}(f) \simeq *$ as claimed. Since we have shown that $3\eta_{n,3} \oplus \varepsilon \cong (p'' \circ f)^*(\tau S^{3n-8})$, we obtain that

(3.14) $\qquad 3\eta_{n,3} \oplus \varepsilon$ is trivial.

We will use this result later. We now proceed to show that the map $b \circ d$ is homotopically trivial. Let

(3.15) $$\cdots \longrightarrow E_i \longrightarrow \cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow V_{3n,9}$$

be the Postnikov tower over $V_{3n,9}$. Following are lists of the k-invariants occurring in this tower in dimension $\leq 3n - 6$.

<u>For $n \equiv 0(8)$</u>:

(0) k-invariant in $H^*(V_{3n,9}; Z_2)$

x_{3n-9}

(1) k-invariant in $H^*(E_1; Z_2)$

ϕ_{3n-7} : $Sq^{2,1} x_{3n-9} = 0$

(2) k-invariant in $H^*(E_2; Z_2)$

ψ'_{3n-6} : $Sq^2\ \phi_{3n-7} = 0$.

<u>For $n \equiv 4(8)$</u>:

(0) k-invariant in $H^*(V_{3n,9}; Z_2)$

x_{3n-9}

(1) k-invariants in $H^*(E_1; Z_2)$

$$\phi_{3n-7} : \quad Sq^{2,1}\, x_{3n-9} = 0$$

$$\psi_{3n-6} : \quad Sq^4\, x_{3n-9} = 0$$

(2) k-invariants in $H^*(E_2; Z_2)$

$$\psi'_{3n-6} : \quad Sq^2\, \phi_{3n-7} = 0 .$$

By proposition (3.7), $H^*(b;Z_2) = 0$. Hence there exists a lifting b_1 of b to E_1 . We now study the composition $b_1 \circ d$. Since $H^{3n-7}(V_{n,3}; Z_2) = 0$, $(b_1 \circ d)^* \phi_{3n-7} = 0$. For $n \equiv 4(8)$, we also want to evaluate $(b_1 \circ d)^* \psi_{3n-6}$. In this case, $Sq^4 H^{3n-10}(V_{n,3} \times V_{n,3} \times V_{n,3}; Z_2) = 0$. Hence the class $(b_1 \circ d)^* \psi_{3n-6}$ is independent of the choice of the lifting b_1 . Notice that proposition (3.5) applies to the map b . Therefore, by remark (3.6), $b_1^* \psi_{3n-6}$ must be an element of $H^{3n-6}(V_{n,3} \times V_{n,3} \times V_{n,3}; Z_2)$ fixed by the S_3-action. It is easy to check any such element is pulled back trivially by d^* . Thus $(b \circ d)^* \psi_{3n-6} = 0$ if $n \equiv 4(8)$. This shows that $H^*(b_1 \circ d; Z_2) = 0$ for $n \equiv 0$ or $4(8)$. Consequently, there is a lifting $g: V_{n,3} \longrightarrow E_2$ of the composition $b_1 \circ d$.

<u>Claim</u> (3.16). $g^* \psi'_{3n-6} = 0$.

Let us assume that we have proved this claim. Then $b_1 \circ d$ and, hence, $b \circ d$, lift to E_3 . However, E_3 is already (3n-5)-connected. Since $\dim V_{n,3} = 3n - 6$, we deduce that $b \circ d \simeq *$. This completes the proof of the theorem for $n \equiv 0(4)$, assuming (3.16).

Proof of claim (3.16).

Suppose that $0 \neq g^* \psi'_{3n-6} \in H^{3n-6}(V_{n,3}; Z_2)$. We will obtain a contradiction.

Let

$$(3.17) \qquad \longrightarrow E'_i \longrightarrow \cdots \longrightarrow E'_2 \longrightarrow E'_1 \longrightarrow V_{3n,8}$$

be a (modified) Postnikov tower over $V_{3n,8}$. The following is a list of the k-invariants occurring in that tower in dimension $\leq 3n - 6$.

(0) k-invariants in $H^*(V_{3n,8}; Z_2)$

$$x_{2n-8} , \quad x_{3n-7}$$

(1) k-invariants in $H^*(E_1;Z_2)$

$$\bar\phi^{(1)}_{3n-8} : \quad Sq^1\, x_{3n-8} = 0$$

$$\bar\psi_{3n-7} : \quad Sq^2\, x_{2n-8} = 0$$

$$\kappa_{3n-6} : \quad Sq^2\, x_{3n-7} = 0$$

(2) k-invariants in $H^*(E_2;Z_2)$

$$\bar\phi^{(2)}_{3n-8} : \quad Sq^1\, \bar\phi^{(1)}_{3n-8} = 0$$

$$\bar\psi'_{3n-6} : \quad Sq^3\, \phi^{(1)}_{3n-8} + Sq^2\, \psi_{3n-7} = 0 .$$

(i) k-invariants in $H^*(E_i;Z_2)$ $\quad i \geq 3$

$$\bar\phi^{(i)}_{3n-8} : \quad Sq^1\, \bar\phi^{(i-1)}_{3n-8} = 0 .$$

The projection map $p\colon V_{3n,9} \Longrightarrow V_{3n,8}$ induces a map between the Postnikov towers. Let $p_i\colon E_i \longrightarrow E'_i$ denote the maps induced.

A routine computation shows that $p_2^* \bar{\psi}'_{3n-6} = \psi'_{3n-6}$. Now, notice that $p_2 \circ g$ is a lifting of the composition $p \circ b \circ d$, and that under our hypothesis, $(p_2 \circ g)^* \psi_{3n-6} \neq 0$. By cellular approximation, we can consider $p \circ b \circ d \colon V_{n,3} \longrightarrow V_{3n,8}$ as mapping into P^{3n-5}_{3n-8} . Since $3n - 8 \equiv 0(4)$, we have $P^{3n-5}_{3n-8} \simeq S^{3n-8} \vee P^{3n-5}_{3n-7}$. Then, it is easy to see that the condition $(p_2 \circ g)^* \bar{\psi}_{3n-6} \neq 0$ implies that the map $p \circ b \circ d$ factors as shown in the following diagram

$$\begin{tikzcd}
V_{n,3} \arrow[rr, "b \circ d"] \arrow[d, "c"] & & V_{3n,9} \arrow[d, "p"] \\
S^{3n-6} \arrow[r, "\eta^2"] & S^{3n-8} \arrow[r, "i"] \arrow[dr] & V_{3n,8} \\
& & P^{3n-5}_{3n-8} \arrow[u]
\end{tikzcd}$$

where $c \colon V_{n,3} \longrightarrow S^{3n-6}$ is the degree 1 map.

We deduce that $c^*(\eta^2)^*(\tau S^{3n-8}) \cong c^*(\eta^2)^* i^*(\eta_{3n-8}) \cong (b \circ d)^* p^*(\eta_{3n,8}) \cong (b \circ d)^*(\eta_{3n,9} \oplus \varepsilon) \cong 3\eta_{n,3} \oplus \varepsilon$. We have already shown (3.14) that the last vector bundle is trivial. Therefore, under the hypothesis that $g^* \psi'_{3n-6} \neq 0$, we deduce that $c^*(\eta^2)^*(\tau S^{3n-8})$ is trivial. Is is easy to see that this implies that $(\eta^2)^*(\tau S^{3n-8})$ admits more than 1 section (cf. proof of theorem (3.8), case $n \equiv 3(8)$). However, this is a contradiction with proposition (3.1)ii. Hence we must deduce that $g^* \psi'_{3n-6} = 0$. This completes the proof of claim (3.16).

This completes the proof of theorem (3.13) for the case $n \equiv 0(4)$.

Case $n \equiv 2(4)$.

In this case it is easy to prove that the map $b \circ d\colon V_{n,3} \longrightarrow V_{3n,9}$ is homotopically trivial.

Again, let

$$(3.18) \qquad \ldots \longrightarrow E_i \longrightarrow \ldots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow V_{3n,9}$$

be the Postnikov tower over $V_{3n,9}$. The k-invariants occurring in dimensions $\leq 3n-6$ are as follows:

For $n \equiv 2(8)$:

(0) k-invariants in $H^*(V_{n,3}; Z_2)$

$$x_{3n-9}, \quad x_{3n-7}$$

(1) k-invariants in $H^*(E_1; Z_2)$

$$\psi_{3n-6} : \quad Sq^4\, x_{3n-9} + Sq^2\, x_{3n-7} = 0 .$$

For $n \equiv 6(8)$:

(0) k-invariants in $H^*(V_{n,3}; Z_2)$

$$x_{3n-9}, \quad x_{3n-7}$$

(1) k-invariants in $H^*(E_1; Z_2)$

$$\psi_{3n-6} : \quad Sq^4\, x_{3n-9} = 0$$

By proposition (3.7), the map b admits a lifting b_1 to E_1. We will show that $d^* b_1^* \psi_{3n-6} = 0$. The theorem follows.

Notice that the map b satisfies the conditions of proposition (3.5). Consequently, we can apply remark (3.6). If $n \equiv 2(8)$, let $I = Sq^4 H^{3n-10}(V_{n,3} \times V_{n,3} \times V_{n,3}; Z_2) + Sq^2 H^{3n-8}(V_{n,3} \times V_{n,3} \times V_{n,3}; Z_2)$. The set $\{b_1^* \psi_{3n-6}\}$, where b_1 runs through all the possible liftings of b, determines an element Σ in $H^{3n-6}(V_{n,3} \times V_{n,3} \times V_{n,3})/I$. As explained in remark (3.6), Σ must be fixed under the S_3-action. Let $a = x_{n-1} \otimes x_{n-2} \otimes x_{n-3} + x_{n-3} \otimes x_{n-2} \otimes x_{n-1} + x_{n-2} \otimes x_{n-2} \otimes x_{n-2}$ and $b = x_{n-2}x_{n-3} \otimes x_{n-1} \otimes 1 + x_{n-2}x_{n-1} \otimes x_{n-3} \otimes 1$. It is easy to compute that I is the subgroup of $H^{3n-6}(V_{n,3} \times V_{n,3} \times V_{n,3}; Z_2)$ generated by the elements in the orbits of a and b under the S_3-action. One computes quickly that the only elements of $H^{3n-6}(V_{n,3} \times V_{n,3} \times V_{n,3}; Z_2)/I$ fixed under the S_3-action are $[0]$ and $[x_{n-2} \otimes x_{n-2} \otimes x_{n-2}]$. It follows that $d^* b_1^* \psi_{3n-6} = 0$ as desired.

If $n \equiv 6(8)$, the same argument applies, but the computations are easier. This concludes the proof of the case $n \equiv 2(4)$. ■

§5. Odd multiples of $\eta_{n,k}$.

The following theorem is meaningful only if k is approximately equal to or larger than $2n/3$.

Theorem (3.18). The vector bundle $(2s+1)\eta_{n,k}$ is not trivial if $2s + 1 < (n+\delta)/(n-k)$, where $\delta = \delta(n,k)$ is equal to 0 if n and k are even, 1 if n and k are odd, and -2 otherwise.

Before proving theorem (3.12), we establish the following lemma.

Proposition (3.13). Let ξ^{ℓ} be a stably trivial real vector bundle over a finite CW-complex X. Assume that ξ has a unique Spin reduction. Then $\Delta^{\pm}(\xi \oplus \xi) = 2^{\ell-1}$ in $KU(X)$.

Proof. Let us assume that $\ell = 2r + 1$ is odd. The proof for ℓ even is similar. We use the notation of chapter II. Let E be a principal Spin(ℓ)-bundle over X such that $\xi \cong \rho(E)$. Applying (2.15), we have that $\Delta^{\pm}(\xi \oplus \xi) = \Delta(\xi)\Delta(\xi)$. Hence the element $\Delta^{\pm}(\xi \oplus \xi) \in KU(X)$ is represented by the vector bundle $\Delta_{\ell}(E) \otimes \Delta_{\ell}(E)$. Using the property (2.1) of the α-construction, we have that $\Delta_{\ell}(E) \otimes \Delta_{\ell}(E) \cong (\Delta_{\ell} \cdot \Delta_{\ell})(E)$. Since $\Delta_{\ell} \cdot \Delta_{\ell} = \lambda_r + \lambda_{r-1} + \dots + \lambda_1 + 1$ (theorem (2.6)), we deduce that $\Delta^{\pm}(\xi \oplus \xi)$ is represented by the vector bundle $\lambda_r(\xi) + \lambda_{r-1}(\xi) + \dots + \lambda_1(\xi) + \varepsilon$. It is well known that $\lambda_i(\xi)$ is stably trivial for any stably trivial vector bundle ξ. Thus, $\Delta^{\pm}(\xi \oplus \xi)$ is represented in $KU(X)$ by a stably trivial vector bundle. The proposition follows. ■

Proof of theorem (3.18).

We give the proof for n and k even. The other cases are treated similarly. To show that $(2s+1)\eta_{n,k}$ is not trivial, it is sufficient to show that $\Delta^{+}((2s+1)\eta_{n,k}) \notin Z = \mathrm{Im}\{KU(*) \longrightarrow KU(V_{n,k})\}$ (see example (2.9)). Using successively (2.15), (3.13) and (2.12), we have: $\Delta^{+}((2s+1)\eta_{n,k}) = \Delta^{+}(2s\ \eta_{n,k})\Delta^{+}(\eta_{n,k}) + \Delta^{-}(2s\ \eta_{n,k})\Delta^{-}(\eta_{n,k})$ $= 2^{s(n-k)-1}\ \Delta(\eta_{n,k}) = 2^{s(n-k)-1}\ \tau_{n,k} + 2^{(s+\frac{1}{2})(n-k)-1}$. Since the order of $\tau_{n,k}$ is $2^{\frac{1}{2}k-1}$, we deduce that $(2s+1)\eta_{n,k}$ is not trivial if $s(n-k) - 1 < \frac{1}{2}k - 1$, i.e. $(2s+1) < k/(n-k)$. ■

Remark (3.20)(i). The proof of theorem (3.18) actually gives somewhat stronger information. Let $0 < j \leq k - 1$. There is an embedding $P^{n-k+j}_{n-k} \longrightarrow P^{n-1}_{n-k} \subset V_{n,k}$. The element $\tau_{n,k} \in \widetilde{KU}(V_{n,k})$ restricts to the generator of the torsion subgroup of $\widetilde{KU}(P^{n-k+j}_{n-k})$ [Gitler and Lam 1970]. Using the naturality of the Spin operations, it is easy to see that the vector bundle $(2s+1)\eta_{n,k}$ is not trivial over P^{n-k+j}_{n-k} if j is sufficiently large. For instance, if $n - k \equiv j + 1 \equiv 0(2)$, this is the case if $j > 2s(n-k) - 1$.

(ii) It is interesting to notice that if $r\eta_{n,k}$ is trivial, one can deduce that $(r+1)\eta_{n,k}$ is also trivial <u>as long as $r \geq n/(n-k)$</u> . The results of theorem (3.18) are exactly in the complementary range. Thus, theorem (3.18) gives rise to the intriguing possibility that even and odd multiples of $\eta_{n,k}$ behave in very different ways.

(iii) Slightly better results can be obtained by the use of real and quaternionic Spin operations (§II-5). For instance one can show that $7\eta_{65,56}$ is not trivial in this way.

Chapter IV

Cross-sections of $\eta_{n,k} \oplus (k-1)\varepsilon$.

§1. Sectioning of $\eta_{n,k} \oplus (k-1)\varepsilon$.

Theorem (4.1). *If* n,k *are odd and* $1 \leq k \leq n-2$, *or if* n *is odd,* k *is even and* $\min(\rho(n-k-1)+1, \frac{1}{2}(n-1)) \leq k \leq n-3$, *then the vector bundle* $\eta_{n,k} \oplus (k-1)\varepsilon$ *admits exactly* $k-1$ *linearly independent sections.* *If* n *is even and* $1 \leq k \leq \rho(n)$, *then* $\eta_{n,k} \oplus (k-1)\varepsilon$ *admits exactly* $\rho(n)-1$ *linearly independent sections.*

Remark (4.2). The following facts are elementary consequences of (1.5). (A) If $\eta_{n,k} \oplus (k-1)\varepsilon$ admits at most r l.i. sections, then $\eta_{n,k-1} \oplus (k-2)\varepsilon$ admits at most r l.i. sections also. (B) Let $1 \leq s' < s \leq k-1$. If $\eta_{n,k} \oplus s\varepsilon$ admits at most $s+d$ l.i. sections, then $\eta_{n,k} \oplus s'\varepsilon$ admits at most $s'+d$ l.i. sections. (C) If $\eta_{n,k} \oplus s\varepsilon$ admits at most r l.i. sections, then $\eta_{n+\ell,k+\ell} \oplus s\varepsilon$ admits at most r l.i. sections also. Using these facts, the reader will easily convince himself that theorem (4.1) gives extensive information about the sectioning problem for $\eta_{n,k} \oplus s\varepsilon$, $1 \leq s \leq k-1$, and that, in many cases, the results are best possible.

We must prove the following well known lemma before giving the proof of theorem (4.1).

Lemma (4.3). *Let* Y,Z *be CW-complexes and* $f: Z \rightarrow Y$ *a continuous map.* *Assume that we have cohomology classes* $y_i \in H^i(Y;Z_2)$ *and* $z_i \in H^i(Z;Z_2)$ *for* $2a+1 \leq i \leq 2b$ *and that* $Sq^j y_i = \binom{i}{j} y_{i+j}$ *and* $Sq^j z_i = \binom{i}{j} z_{i+j}$ *for* $j \leq \min(2i,2b) - i$. *Then, if* $f^* y_{i_0} = z_{i_0}$ *for some* i_0, $2a+1 \leq i_0 \leq 2b$, *we have that* $f^* y_i = z_i$ *for all* i, $2a+1 \leq i \leq 2b$.

Proof. Use the Steenrod operations Sq^1 and Sq^2 inductively. ■

Proof of theorem (4.1).

If n is even and $k \leq \rho(n)$, then the projection map $p_1\colon V_{n,k} \longrightarrow S^{n-1}$ admits a cross-section, i.e. there is a map $s\colon S^{n-1} \longrightarrow V_{n,k}$ such that $p_1 \circ s \simeq Id_{S^{m-1}}$. Then we have that $p_1^*(\tau S^{n-1}) \cong \eta_{n,k} \oplus (k-1)\varepsilon$ and $\tau S^{n-1} \cong s^*(\eta_{n,k} \oplus (k-1)\varepsilon)$. Since τS^{n-1} admits exactly $\rho(n) - 1$ l.i. sections, we deduce that $\eta_{n,k} \oplus (k-1)\varepsilon$ admits exactly $\rho(n) - 1$ l.i. sections also. This proves the last assertion of the theorem.

To prove the other cases of the theorem, let us suppose that $\eta_{n,k} \oplus (k-1)\varepsilon$ admits k l.i. sections. Then, there is a $(n-k-1)$-dimensional vector bundle ζ over $V_{n,k}$ such that $\zeta \oplus k\varepsilon \cong \eta_{n,k} \oplus (k-1)\varepsilon$. The sum $\zeta \oplus (k+1)\varepsilon \cong \eta_{n,k} \oplus k\varepsilon$ is trivial. Therefore, by theorem (1.3) there is a continuous map $f\colon V_{n,k} \to V_{n,k+1}$ such that $f^*(\eta_{n,k+1}) \cong \zeta$. Moreover, if $p\colon V_{n,k+1} \longrightarrow S^{n-1}$ is the natural projection,

$$\begin{tikzcd} V_{n,k} \arrow[r, "f"] \arrow[dr, "p \circ f"'] & V_{n,k+1} \arrow[d, "p"] \\ & S^{n-1} \end{tikzcd}$$

then we have that $(p \circ f)^*(\tau S^{n-1}) \cong f^* p^* \tau S^{n-1} \cong f^*(\eta_{n,k+1} \oplus k\varepsilon) \cong \eta_{n,k} \oplus (k-1)\varepsilon$.

Now, let us assume that n and k are odd. Let $\gamma_{n-1} \in \widetilde{KU}(S^{n-1})$ and $\tau_{n,k} \in \widetilde{KU}(V_{n,k})$ be as in example 2.11 and 2.12 respectively. Recall that we can choose γ_{n-1} such that $\gamma_{n-1} = \Delta^+(\tau S^{n-1}) - 2^{(n-3)/2}$, and $\tau_{n,k} = \Delta(\eta_{n,k}) - 2^{(n-k)/2}$. Using corollary 2.15 and the naturality of the Spin operations, we have that $(p \circ f)^!(\gamma_{n-1}) = \Delta^+(f^* p^*(\tau S^{n-1})) - 2^{(n-3)/2} = \Delta^+(\eta_{n,k} \oplus (k-1)\varepsilon) - 2^{(n-3)/2} = 2^{(k-3)/2}\,\Delta(\eta_{n,k}) - 2^{(n-3)/2} = 2^{(k-3)/2}\,\tau_{n,k}$. On the other hand, we have that $p^!(\gamma_{n-1}) = 2^{(k-1)/2}\,\tau_{n,k+1}$ (by computing as above, for instance). Therefore, we must have $2^{(k-3)/2}\,\tau_{n,k} = f^! p^!(\gamma_{n-1}) = 2^{(k-1)/2}\,\tau_{n,k+1}$. However, this is impossible because the order of $\tau_{n,k}$ is $2^{(k-1)/2}$. This is a contradiction, and we must deduce that $\eta_{n,k} \oplus (k-1)\varepsilon$ admits only $(k-1)$ l.i. sections for n and k odd, as stated.

If n is odd and k is even, we find that $(p \circ f)^!(\gamma_{n-1}) = 2^{(k-2)/2}\,\tau_{n,k}$ by the same computation as above. Let $j\colon P^{n-1}_{n-k} \to V_{n,k}$ be the usual embedding. Since $j^!(\tau_{n,k})$ generates $\widetilde{KU}(P^{n-1}_{n-k}) \cong Z_{2^t}$ and $t = (k-1)/2$ [Gitler and Lam 1970], we deduce that $(p \circ f \circ j)^!(\gamma_{n-1}) \neq 0$. Therefore, the map $p \circ f \circ j$ is not homotopically trivial. By the Hopf classification theorem, we deduce that $p \circ f \circ j$ is homotopic to the map collapsing $P^{n-2}_{n-k} \subset P^{n-1}_{n-k}$ to a point. Therefore, $(p \circ f \circ j)^*\, x_{n-1} = x_{n-1}$ where x_i is as in §III-2. Since $p^* x_{n-1} = x_{n-1}$, we deduce that $(f \circ j)^*\, x_{n-1} = x_{n-1}$. Using lemma (4.3), it follows that $(f \circ j)^*\, x_i = x_i$ for $n - k \leq i \leq n - 1$. Firstly, assume that $\frac{1}{2}(n-1) \leq k \leq n - 3$. Then $Sq^{n-k-1}\, x_{n-k-1} = x_{2n-2k-2}$ in $H^*(V_{n,k+1}; Z_2)$. Since $H^{n-k-1}(P^{n-1}_{n-k}; Z_2) = 0$, $(f \circ j)^*(x_{n-k-1}) = 0$. Therefore $(f \circ j)^*\, x_{2n-2k-2} = Sq^{n-k-1}(f \circ j)^*(x_{n-k-1}) = 0$. However, $n - k \leq 2n - 2k - 2 \leq n - 1$, and this is a contradiction with our previous statement. Secondly, assume that $\rho(n-k-1) \leq k - 1$. We can also assume that $k < \frac{1}{2}(n-1)$ in view of the previous case. Since the $(n-1)$-skeleton of $V_{n,k+1}$ is P^{n-1}_{n-k-1}, we can homotope the map $f \circ j$

so that $f \circ j(P^{n-1}_{n-k}) \subset P^{n-1}_{n-k-1}$. Let $q\colon P^{n-1}_{n-k-1} \to P^{n-1}_{n-k}$ be the map collapsing the bottom cell of P^{n-1}_{n-k-1} to a point. Then, of course, $(q \circ f \circ j)^* x_{n-1} = x_{n-1}$ again. Therefore, by applying lemma (4.3), we obtain that $(q \circ f \circ j)^*\colon H^*(P^{n-1}_{n-k}; Z_2) \to H^*(P^{n-1}_{n-k}; Z_2)$ is an isomorphism. By the universal coefficient theorem, we deduce that $(q \circ f \circ j)^* H^*(P^{n-1}_{n-k}; Z) \to H^*(P^{n-1}_{n-k}; Z)$ is an isomorphism. It follows, by Whitehead's theorem, that $q \circ f \circ j$ is a homotopy equivalence, i.e. we have the following homotopy commutative diagram:

$$\begin{tikzcd} P^{n-1}_{n-k} \arrow[r, "f\,j"] \arrow[rr, bend right, "\mathrm{Id}"'] & P^{n-1}_{n-k-1} \arrow[r, "q"] & P^{n-1}_{n-k} \end{tikzcd}$$

Therefore, P^{n-1}_{n-k-1} is co-reducible. However, this is impossible by [Adams 1962, Theorem 1.2]. This contradiction implies the theorem in this case.

This completes the proof of (4.1). ■

§2. Non-triviality of $\eta_{n,k} \oplus (k-1)\varepsilon$.

Theorem (4.4). If n is odd and $2 \le k \le n-2$, or if $n \ne 4,8$ is even and $2 \le k \le n-3$, then $\eta_{n,k} \oplus (k-1)\varepsilon$ is not trivial.

Proof. Under the projection map $V_{n,k+\ell} \longrightarrow V_{n,k}$, the vector bundle $\eta_{n,k} \oplus (k-1)\varepsilon$ pulls back to $\eta_{n,k+\ell} \oplus (k+\ell-1)\varepsilon$. Therefore, it is sufficient to prove theorem (4.4) for large values of k . If n is odd, $\eta_{n,n-2} \oplus (n-3)\varepsilon$ is not trivial by theorem (4.1). We deduce that $\eta_{n,k} \oplus (k-1)\varepsilon$ is not trivial for $1 \le k \le n-2$ in this case.

For n even, $n \neq 4,8$ $\eta_{n,k} \oplus (k-1)\varepsilon$ is not trivial for $k \leq \rho(n)$ by theorem (4.1) also.

The only case remaining is n even and $\rho(n) < k \leq n-3$. Assume this. By the remark at the beginning of the proof, we can also assume that k is odd. By theorem (1.3), if $\eta_{n,k} \oplus (k-1)\varepsilon \cong (n-1)\varepsilon$, we have a map $f\colon V_{n,k} \longrightarrow V_{n-1,k-1}$ such that $f^* \eta_{n-1,k-1} \cong \eta_{n,k}$. Let $i\colon V_{n-1,k-1} \longrightarrow V_{n,k}$ be the inclusion map. Then $(i \circ f)^*(\eta_{n,k}) \cong \eta_{n,k}$.

$$\begin{tikzcd} V_{n,k} \arrow[r, "f"] \arrow[dr, "i \circ f"'] & V_{n-1,k-1} \arrow[d, "i"] \\ & V_{n,k} \end{tikzcd}$$

Since $\tau_{n,k} = \Delta(\eta_{n,k}) - 2^{(n-k-1)/2}$, by naturality of the Spin operations, we have that $(i \circ f)^!(\tau_{n,k}) = \tau_{n,k}$. Because $k \leq n-3$, P_{n-k}^{n-k+2} is the $(n-k+2)$-skeleton of $V_{n,k}$. Let $j\colon P_{n-k}^{n-k-2} \longrightarrow V_{n,k}$ denote the inclusion map and $g\colon P_{n-k}^{n-k+2} \longrightarrow P_{n-k}^{n-k+2}$ the restriction of $i \circ f$ to P_{n-k}^{n-k-2}. Using [Gitler and Lam 1970] again, we have that $j^!(\tau_{n,k})$ is the generator of $\widetilde{KU}(P_{n-k}^{n-k+2}) \cong Z_2$. It follows easily that $g^!$ is an isomorphism. After studying the homotopy commutative diagram

$$\begin{tikzcd} P_{n-k}^{n-k+2} \arrow[r, "g"] \arrow[d, "c"] & P_{n-k}^{n-k+2} \arrow[d, "c"] \\ S^{n-k+2} \arrow[r] & S^{n-k+2} \end{tikzcd}$$

first in KU-theory, and then in Z_2-cohomology, one deduces that $H^*(g;Z_2)$ must be an isomorphism. Since $H^{n-k}(V_{n,k};Z_2) \cong H^{n-k}(V_{n-1,k-1}) \cong Z_2$, it follows that $H^{n-k}(i \circ f;\, Z_2)$ and $H^{n-k}(f;\, Z_2)$ are isomorphisms. Now let $j'\colon P_{n-k}^{n-1} \longrightarrow V_{n,k}$ and $j''\colon P_{n-k}^{n-2} \longrightarrow V_{n-1,k-1}$ be the usual inclusions.

Let N be a very large power of 2. Recall from [James 1959, thm. 2.5], that there is a map $h\colon \Sigma^N V_{n-1,k-1} \longrightarrow \Sigma^N P_{n-k}^{n-2}$ such that the

$$\begin{tikzcd}
\Sigma^N P_{n-k}^{n-2} \arrow[rrr, "F"] \arrow[d, "\Sigma^N j_1"'] & & & \Sigma^N P_{n-k}^{n-2} \arrow[d, shift right, "\Sigma^N j''"'] \\
\Sigma^N P_{n-k}^{n-1} \arrow[r, "\Sigma^N j'"] & \Sigma^N V_{n,k} \arrow[rr, "\Sigma^N f"] & & \Sigma^N V_{n-1,k-1} \arrow[u, shift right, "h"']
\end{tikzcd}$$

composition $h \circ (\Sigma^N j'')$ is homotopic to the identity map on $\Sigma^N P_{n-k}^{n-2}$. Since $H^{N+n-k}(\Sigma^N V_{n-1,k-1}; Z_2) \cong Z_2$, $H^{N+n-k}(h;Z_2)$ must be an isomorphism. Let $j_1\colon P_{n-k}^{n-2} \longrightarrow P_{n-k}^{n-1}$ be the inclusion and $F = h \circ \Sigma^N(f \circ j' \circ j_1)$. $H^{N+n-k}(F;Z_2)$ is an isomorphism. By lemma (4.3) it follows that $H^*(F;Z_2)$ is an isomorphism also. Using the universal coefficients theorem and Whitehead's theorem, we deduce that F is a homotopy equivalence

$$F\colon \Sigma^N P_{n-k}^{n-2} \xrightarrow{\Sigma^N j_1} \Sigma^N P_{n-k}^{n-1} \xrightarrow{h\, \Sigma^N(f\; j')} \Sigma^N P_{n-k}^{n-2}$$

Because N is large, this implies that $\Sigma^N P_{n-k}^{n-1}$ is reducible. However, since N is also a power of 2, $\Sigma^N P_{n-k}^{n-1} \simeq P_{N+n-k}^{N+n-1}$ [Atiyah 1961]. Therefore, we must have $k \leq \rho(N+n)$. But this is a contradiction with our hypothesis. Thus, we have proven the theorem for n even and $\rho(n) < k \leq n-3$.

This completes the proof of theorem (4.4). ■

Chapter V

Examples of Stably Free Modules and Unimodular Matrices.

In this chapter, we use the results of chapter III and IV to construct examples of stably free modules and of unimodular matrices over commutative noetherian rings.

§1. Stably free modules.

For a compact Hausdorff space X and a real vector bundle ξ over X, recall that $C(X)$ denotes the ring of real valued continuous functions on X and $\Gamma(X)$ denotes the $C(X)$-module of continuous sections of ξ. The basic properties of the functor $\Gamma(-)$ are well explained in [Swan 1962]. The most important one is given in the following theorem.

Theorem (5.1). (Swan). The correspondence $\xi \longrightarrow \Gamma(\xi)$ defines a bijection between the set of isomorphism classes of real vector bundles over X and the set of isomorphism classes of finitely generated projective $C(X)$-modules.

Proof. [Swan 1962]. ■

We now wish to use the vector bundle $\eta_{n,k}$ over $V_{n,k}$ to construct examples of stably free modules. However, the ring $C(V_{n,k})$ is generally too large to be algebraically interesting. Therefore, following [Swan 1962, example 1], we proceed as follows.

Denote the points of R^n by column vectors and let $\{e_i;\ i = 1,\ldots,n\}$ be the standard basis. Recall that the manifold $V_{n,k}$ consists of the $k \times n$ matrices $x = (x_{ij})$ with real entries satisfying the equation $xx^t = I_k$. The trivial n-dimensional vector bundle $\varepsilon^n = n\varepsilon$ over $V_{n,k}$ consists of the pairs (x,u) where $x \in V_{n,k}$ and $u \in R^n$. Let μ^k be the subbundle of ε^n consisting of the pairs (x,u) where u is a linear combination of the vectors in the k-frame x, i.e. $u = x^t\lambda$ for some $\lambda \in R^k$. The k column vectors of the matrix x^t define a trivialization of μ^k. Let $f\colon \varepsilon^n \longrightarrow \varepsilon^n$ be the vector bundle map defined by $f(x,u) = (x, x^t x\, u)$. Then, $\operatorname{Im} f = \mu^k$ and $\operatorname{Ker} f = \eta_{n,k}$.

The sections $s_i\colon x \longrightarrow (x,e_i)$, $i = 1,\ldots,n$, form a basis of the free $C(V_{n,k})$-module $\Gamma(\varepsilon^n)$. Relatively to the basis $\{s_i\}$, the $C(V_{n,k})$-module homomorphism $\Gamma(f)\colon \Gamma(\varepsilon^n) \longrightarrow \Gamma(\varepsilon^n)$ is represented by the matrix $x^t x$ where $x = (x_{ij})$ now stands for the matrix of coordinate functions on $V_{n,k}$.

For $1 \leq k \leq n-1$, define the ring $\Lambda = \Lambda(n,k)$ to be the quotient of the polynomial ring on nk unknown $\mathbb{R}[x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},\ldots,x_{kn}]$ by the ideal generated by the polynomials $\sum_{\ell} x_{i\ell}x_{j\ell} - \delta_{ij}$, $1 \leq i \leq j \leq k$ (δ_{ij} is the Kronecker symbol). Notice that $\Lambda(n,k)$ is a commutative noetherian ring with unit. Moreover, $\Lambda(n,k) \subset C(V_{n,k})$. Let F be the free $\Lambda(n,k)$-module over the set $\{s_i\colon i = 1,\ldots,n\}$. There is an obvious isomorphism $F \otimes_\Lambda C(V_{n,k}) \cong \Gamma(\varepsilon^n)$. Assume that Λ^k is given its standard basis, and let $h\colon F \longrightarrow \Lambda^k$ and $h'\colon \Lambda^k \longrightarrow F$ be the module homomorphisms corresponding to the matrices $x = (x_{ij})$ and x^t respectively. Since hh' is represented by the matrix xx^t, $hh' = I_k$. Let $g = h'h\colon F \longrightarrow F$. The homomorphism g is represented by the matrix $x^t x$. Thus $\operatorname{Ker} g \otimes_\Lambda C(V_{n,k}) \cong \Gamma(\eta_{n,k})$. Obviously, $\operatorname{Ker} g \supset \operatorname{Ker} h$. Since $h = (hh')h = h(h'h) = hg$, $\operatorname{Ker} h \supset \operatorname{Ker} g$, i.e. $\operatorname{Ker} g = \operatorname{Ker} h$. Therefore, we have an exact sequence $0 \longrightarrow \operatorname{Ker} g \longrightarrow F \longrightarrow \Lambda^k \longrightarrow 0$. Hence $\operatorname{Ker} g$ is a projective module and $\ker g \oplus \Lambda^k \approx \Lambda^n$. Let $P = P(n,k) = \operatorname{Ker} g$. Notice that $P \approx F/\operatorname{Im} h'$. These results are included in the following theorem.

Theorem (5.2). Let $1 \le k \le n-1$ and let R be the field of real numbers. Let $\Lambda = \Lambda(n,k)$ be the quotient ring of the polynomial ring in nk variables $R[x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},\ldots,x_{kn}]$ by the ideal generated by the polynomials $\sum_{\ell} x_{i\ell}x_{j\ell} - \delta_{ij}$, $1 \le i \le j \le k$. Let $P = P(n,k)$ be the $\Lambda(n,k)$-module with generators s_i, $i = 1,\ldots,n$ and relations $\sum_j x_{ij}s_j$, $i = 1,\ldots,k$. Then:

(i) $P \oplus \Lambda^k$ is free of rank n;

(ii) If n is odd and $1 \le k \le n-2$, then $P \oplus \Lambda^{k-1}$ is not free; if k is also odd, then $P \oplus \Lambda^{k-1}$ does not contain a free submodule of rank $> k-1$;

(iii) If $n \ne 2,4,8$ is even and $1 \le k \le n-3$, then $P \oplus \Lambda^{k-1}$ is not free.

Proof. (i) was proven above. To prove (ii), notice that $(P \oplus \Lambda^{k-1}) \otimes_\Lambda C(V_{n,k}) \cong (P \otimes_\Lambda C(V_{n,k})) \oplus C(V_{n,k})^{k-1} \cong \Gamma(\eta_{n,k} \oplus (k-1)\varepsilon)$. The vector bundle $\eta_{n,k} \oplus (k-1)\varepsilon$ is not trivial by theorem (4.4). Therefore, by theorem (5.1), $\Gamma(\eta_{n,k} \oplus (k-1)\varepsilon)$ is not free. It follows that $P \oplus \Lambda^{k-1}$ is not free. If k is odd, Theorem (4.1) implies that the vector bundle $\eta_{n,k} \oplus (k-1)\varepsilon$ does not contain a trivial subbundle of dimension $> k-1$. Therefore, $\Gamma(\eta_{n,k} \oplus (k-1)\varepsilon)$ does not contain free submodule of rank $> k-1$. Hence $P \oplus \Lambda^{k-1}$ does not contain a free submodule of rank $> k-1$ either. (iii) is proved in a similar way. ■

Remark (5.3). For $k = 1$, theorem (5.2) was proven in [Swan 1962].

Theorem (5.4). *Let* $\Lambda = \Lambda(n,k)$ *and* $P = P(n,k)$ *be as in theorem (5.2). Then:*

(i) *Let* $2 \leq k \leq n - 3$. *If* $n - k$ *is odd then the module* $P \oplus P$ *is not free;*

(ii) *Let* δ *be as in theorem (3.18) and let* r *be odd. Then the module* $rP = P \oplus \dots \oplus P$ *is not free if* $r < (n+\delta)/(n-k)$.

Proof. Same as for theorem (5.2), using corollary (3.10) and theorem (3.18) instead. ■

Remark (5.6). If $n \geq 5$ is odd and $k = 2$, theorem (5.4) shows that T.Y. Lam's theorem (1.7) is best possible. For large values of k, notice that theorem (5.4) states that rP is not trivial if r is odd and $r \leq k/(n-k)$ approximately. Thus, there is a gap of approximately k units between the "positive result" (theorem (1.7)) and the "negative result" (theorem (5.4)).

§2. Unimodular matrices.

Let R be a commutative ring with unit. Recall that a $k \times n$ matrix α with entries in R is said to be unimodular if the map $\alpha: R^n \longrightarrow R^k$ is an onto mapping. The following definition is adapted from [Gabel-Geramita 1974, def. 2.1].

Definition (5.7). *Let* α *be a* $k \times n$ *unimodular matrix over* R *and let* $1 \leq \ell \leq n$. *We say that* α *is* ℓ*-stable if there exists a*

$\ell \times (n-\ell)$ matrix γ with entries in R such that the matrix $\alpha \begin{pmatrix} I_{n-\ell} \\ \gamma \end{pmatrix}$ is also unimodular.

The following theorem relates unimodular matrices to stably free modules. It is adapted from [Gabel-Geramita 1974, thm. 2.3].

Theorem (5.8). Let R be a commutative ring with unit and let α be a $k \times n$ unimodular matrix over R. Assume that α is ℓ-stable. Then:

(1) $\operatorname{Ker} \alpha \cong R^{\ell} \oplus Q$, for some projective R-module Q.

(2) If $1 \leq \ell \leq k$, then $\operatorname{Ker} \alpha \oplus R^{k-\ell} \cong R^{n-\ell}$.

(3) If $\ell \geq k$, then $\operatorname{Ker} \alpha$ is free.

Proof. See [Gabel-Geramita 1974, p. 101]. ■

We apply theorem (5.8) to prove the following theorem.

Theorem (5.9). Let $\Lambda = \Lambda(n,k)$ be as in theorem (5.2). Assume that $n \neq 2,4,8$, and that $0 \leq s < k \leq n-3$ if n is even and $0 \leq s < k \leq n-2$ if n is odd. Let $\alpha(n,k,s)$ be the $k \times (n+s)$ matrix $((x_{ij}),0_s)$ where 0_s denotes the $k \times s$ 0-matrix. Then the matrix $\alpha(n,k,s)$ is ℓ-stable if and only if $1 \leq \ell \leq s$.

Proof. The matrix $\alpha(n,k,s)$ is obviously ℓ-stable for $1 \leq \ell \leq s$. Notice that $\operatorname{Ker} \alpha(n,k,s) \cong P(n,k) \oplus \Lambda^s$. By theorem (5.2), this module is not free for $0 \leq s \leq k-1$. Hence $\alpha(n,k,s)$ is not ℓ-stable for $\ell \geq k$ by theorem (5.8)(3). Therefore, assume that $1 \leq \ell \leq k$ and suppose that $\alpha(n,k,s)$ is ℓ-stable. Then

Ker $\alpha(n,k,s) \oplus \Lambda^{k-\ell}$ is free by theorem (5.8)(2). By theorem (5.2), it follows that $k - s \leq k - \ell$, i.e., $\ell \leq s$. This concludes the proof of the theorem. ■

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