DENSITY THEOREMS AND APPLICATIONS

by

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B.Sc., Tel Aviv University, 1976

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the

DEPARTMENT OF MATHEMATICS

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

June, 1977

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ABSTRACT

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One way of getting structure theorems in ring theory is to fix a general class $\sum$ of modules, and to prove Schur's Lemma and the Density Theorem for $\sum$. For example, the Goldie Theorem for prime rings follows from Schur's Lemma and the Density Theorem for the class of rationally uniform, homogeneous modules in a similar way as the Wedderburn-Artin Theorem follows from Schur's Lemma and the Density Theorem for the class of irreducible modules.

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ACKNOWLEDGEMENT

I would like to thank my supervisor, Prof. Tim Anderson for the invaluable help, care and encouragement he gave me in this thesis and during my whole year at the University of British Columbia.

INTRODUCTION

In this paper ring means associative ring which does not necessarily contain a unity element and is not necessarily commutative. Module means left module.

For every ring A let $\sum_A$ denote, for the moment, the class of all irreducible A-modules.

The proof of the classical Wedderburn-Artin Theorem has the following main steps:

(1) If $V \varepsilon \sum_A$, then $\Delta = \mathrm{Hom}_A(V,V)$ is a division ring.
(Schur's Lemma)

(2) If $V \varepsilon \sum_A$ and $\Delta = \mathrm{Hom}_A(V,V)$, then for any Δ-linearly independent elements $v_1,\ldots,v_n \varepsilon V$ and any elements $w_1,\ldots,w_n \varepsilon V$ there exists an element $a \varepsilon A$ such that

$$av_1 = w_1,\ldots,av_n = w_n \ .$$

This means that the ring A_L of left multiplications by elements of A is, in a certain sense, dense in the ring $\mathrm{Hom}_\Delta(V,V)$ of all linear transformations on the vector space V.
(Jacobson Density Theorem)

(3) If $\sum_A$ has a member which is faithful and finite dimensional (over its centralizer), then A is isomorphic to $M_n(D)$ where D is a division ring and $n \geq 1$.

(4) If A is a simple, Artinian ring, then $\sum_A$ contains a member which is faithful and finite dimensional (over its centralizer).

Here, in order to get the desired structure theorem, we concentrate on a certain fixed class, $\sum$, of modules, and prove Schur's Lemma and the Density Theorem for $\sum$. These enable us to describe the structure of those rings A for which $\sum_A$ has a faithful, finite dimensional member. Once we know this, the structure of a ring A of the class that we are interested in is easily obtained by showing that for such A, $\sum_A$ does have a faithful, finite dimensional member. The essential part of the structure theory is, therefore, Schur's Lemma and the Density Theorem for the class $\sum$ of modules.

The aim of this thesis is to emphasize the usefulness of concentrating on a fixed class $\sum$ of modules for which we can prove an analogue of Schur's Lemma and of the Density Theorem. We show that if we take $\sum$ to be the class of rationally uniform, homogeneous modules, then the approach described above gives Goldie's structure theorem for prime rings satisfying the ascending chain condition.

In Chapter 1 we define the notion of a general class of modules, and prove the Theorem of Andrunakievic and Rjabuhin which shows how a general class of modules defines a radical. For example, the general class of irreducible modules defines the Jacobson radical. Then we show that the class $\sum$ of rationally uniform, homogeneous modules is a general class.

In Chapter 2 we give the analogues of Schur's Lemma and the Jacobson Density Theorem for this class $\sum$, and use them to describe the structure of those rings A which have in $\sum_A$ a faithful member satisfying a certain finite dimensionality condition.

In Chapter 3 we deduce from this the Goldie Theorem.

Historically, the generalizations of the Jacobson Density Theorem that will be given here, are due to Faith [2] and to Koh and Mewborn [4]. It was Heinicke [3] who pointed out that they imply the Goldie Theorem on prime rings.

CHAPTER 1

GENERAL CLASSES OF MODULES

In order to study the structure of rings it is very useful to concentrate on a certain fixed class of modules. The classical example is the class of irreducible modules which gives the Jacobson structure theory, and in particular the Wedderburn-Artin Theorem. Another class of modules, as we shall see, gives the Goldie Theorem for prime rings.

In all the paper, a homorphism which is a one-to-one mapping will be called a monomorphism, and a homomorphism which is an onto mapping will be called an epimorphism.

1.1. Radicals

Definition: A class β of rings is called a radical if

(i) for every $A \in \beta$ and epimorphism $A \longrightarrow B$ we have $B \in \beta$.

(ii) for every ring A there exists an ideal I in A such that $I \in \beta$, and if J is an ideal of A and $J \in \beta$ then $J \subset I$.
(This largest β-ideal of A is denoted by $\beta(A)$.)

(iii) for every ring A, $\beta(A/\beta(A)) = 0$.

Example: The class of all nil rings is a radical.

Definition: Let β be a radical. A ring A is called β-semisimple if $\beta(A) = 0$.

Definition: A class M of rings is called regular if for every ring $A \in M$ and nonzero ideal I of A there exists a nonzero ring $B \in M$ and an epimorphism $I \longrightarrow B$.

1.2. Theorem of Kurosh

Let $\mathcal{M}$ be a regular class of rings. Let $\mathcal{U}_{\mathcal{M}}$ be the class of all rings which cannot be homomorphically mapped onto a nonzero member of $\mathcal{M}$. Then

(i) $\mathcal{U}_{\mathcal{M}}$ is a radical.

(ii) For every ring $A \in \mathcal{M}$, $\mathcal{U}_{\mathcal{M}}(A) = 0$.

(iii) If β is a radical such that $\beta(A) = 0$ for every $A \in \mathcal{M}$, then every $\mathcal{U}_{\mathcal{M}}$-semisimple ring is β-semisimple.

For the proof see Divinsky [1].

1.3. General classes

For an A-module V we denote by $(0:V)$ the annihilator of V, i.e. $(0:V) = \{a \in A \mid aV = 0\}$. Of course, $(0:V)$ is an ideal of A. V is called nontrivial if $AV \neq 0$, i.e. $(0:V) \neq A$. V is called faithful if $(0:V) = 0$.

Suppose that to every ring A there is assigned a (possibly empty) class Σ_A of nontrivial A-modules. Such an assignment is called a class of modules.

<u>Definition</u>: A class Σ of modules is said to be a <u>general class</u> provided:

(i) If $f: A \to B$ is an epimorphism and $V \in \Sigma_B$ then V, considered as an A-module in the obvious way, belongs to Σ_A.

(ii) If $f: A \to B$ is an epimorphism, $V \in \Sigma_A$ and $\ker f \subset (0:V)$ then V, considered as a B-module, is in Σ_B.

(iii) If $\bigcap_{V \in \Sigma_A} (0:V) = 0$ then $\Sigma_I \neq \emptyset$ for every nonzero ideal I of A.

(iv) If for every nonzero ideal I of A $\sum_I \neq \emptyset$ then $\bigcap_{V\varepsilon\sum_A} (0:V) = 0$.

Definition: Let $\sum$ be a general class of modules. A ring A is called $\sum$-primitive if $\sum_A$ contains a faithful module.

Definition: Let $\{A_\lambda\}_{\lambda\varepsilon\Lambda}$ be rings. A ring B, isomorphic to a subring of the complete direct sum $\sum_{\lambda}^{c} \oplus A_\lambda$, is called a subdirect sum of the rings $\{A_\lambda\}$ if the restrictions of the canonical projections $\sum_{\lambda}^{c} \oplus A_\lambda \longrightarrow A_\mu$ to B are onto. This is equivalent to the following condition: There exist ideals I_λ in B such that for every λ B/I_λ is isomorphic to A_λ and $\bigcap_\lambda I_\lambda = 0$.

The following theorem shows that every general class of modules determines a radical.

1.4. Theorem of Andrunakievic and Rjabuhin

Let $\sum$ be a general class of modules. Denote $\sigma = \{A \mid \sum_A = \emptyset\}$. Then

(i) σ is a radical.

(ii) For every ring A, $\sigma(A) = \bigcap_{V\varepsilon\sum_A} (0:V)$.

(iii) Every σ-semisimple ring is a subdirect sum of $\sum$-primitive rings.

Proof: Let M be the class of $\sum$-primitive rings. First we observe that

(*) if $V \varepsilon \sum_A$ then $0 \neq A/(0:V) \varepsilon M$.

(As V is nontrivial, $(0:V) \neq A$, so $A/(0:V) \neq 0$. Consider the canonical projection $A \longrightarrow A/(0:V)$. By property (ii) of a general class $V \varepsilon \sum_{A/(0:V)}$ and it is clearly faithful over $A/(0:V)$.) We show now that M is a regular class. Let $A \varepsilon M$ and let I be a nonzero ideal of A . As A is $\sum$-primitive, $\bigcap_{V\varepsilon\sum_A} (0:V) = 0$. So by property (iii) of a general class, $\sum_I \neq \emptyset$. Let $V \varepsilon \sum_I$. By (*), $I/(0:V)$ is a nonzero member of M, and the canonical projection maps I onto $I/(0:V)$. Therefore M is a regular class of rings.

Let U_M be the radical determined by M by the Theorem of Kurosh. We show: $U_M = \sigma$.

We prove this by showing:

$A \notin U_M$ if and only if $A \notin \sigma$.

Suppose $A \notin U_M$. Then there exists a nonzero member B of M and an epimorphism $A \longrightarrow B$. As $B \varepsilon M$, there exists $V \varepsilon \sum_B$. By property (i) of a general class, $V \varepsilon \sum_A$. So $A \notin \sigma$. Suppose $A \notin \sigma$. Then there exists a module $V \varepsilon \sum_A$. By (*), $A/(0:V)$ is a nonzero member of M . The canonical projection maps A onto $A/(0:V)$. Thus $A \notin U_M$. We have proved $U_M = \sigma$, so σ is a radical. Now we want to show that for every ring A, $\sigma(A) = \bigcap_{V\varepsilon\sum_A} (0:V)$.

Suppose $\sigma(A) \not\subset \bigcap_{V\varepsilon\sum_A} (0:V)$. Then there exists $V \varepsilon \sum_A$ such that $\sigma(A) \not\subset (0:V)$. Therefore $\frac{\sigma(A) + (0:V)}{(0:V)}$ is a nonzero ideal of $A/(0:V)$, and by (*), $A/(0:V) \varepsilon M$. As M is a regular class, there exists a nonzero ring $B \varepsilon M$ and an epimorphism $\frac{\sigma(A) + (0:V)}{(0:V)} \longrightarrow B$.
Thus we have epimorphisms

$$\sigma(A) \longrightarrow \frac{\sigma(A)}{\sigma(A)\cap(0:V)} \cong \frac{\sigma(A) + (0:V)}{(0:V)} \longrightarrow B$$

and the composition is an epimorphism from $\sigma(A)$ onto B which is a nonzero member of M . This is a contradiction because $\sigma(A) \varepsilon \sigma = U_M$.

Therefore $\sigma(A) \subset \bigcap_{V \varepsilon \sum_A} (0:V)$.

The ring $A/\sigma(A)$ is σ-semisimple, so for every nonzero ideal I of $A/\sigma(A)$, $I \notin \sigma$, which means $\sum_I \neq \emptyset$. By property (iv) of a general class we know:

$$\bigcap_{V \varepsilon \sum_{A/\sigma(A)}} (0:V)_{A/\sigma(A)} = \bar{0} \ .$$

Let $a \ \varepsilon \bigcap_{V \varepsilon \sum_A} (0:V)_A$. Let $V \ \varepsilon \ \sum_{A/\sigma(A)}$. By property (i) of a general class, if we consider V as an A-module then $V \ \varepsilon \ \sum_A$, and therefore $a \ \varepsilon \ (0:V)_A$. By the definition of V as an A-module,

$$0 = av = \bar{a}v \quad \text{for every} \quad v \ \varepsilon \ V \ .$$

Thus $\bar{a} \ \varepsilon \ (0:V)_{A/\sigma(A)}$. As V was any member of $\sum_{A/\sigma(A)}$, we have: $\bar{a} \ \varepsilon \bigcap_{V \varepsilon \sum_{A/\sigma(A)}} (0:V)_{A/\sigma(A)} = \bar{0}$. Thus $a \ \varepsilon \ \sigma(A)$. This proves:

$\sigma(A) = \bigcap_{V \varepsilon \sum_A} (0:V)$.

Finally, let A be σ-semisimple. Then $\sigma(A) = \bigcap_{V \varepsilon \sum_A} (0:V) = 0$. Let $\{I_\lambda\}_{\lambda \varepsilon \Lambda}$ be the set of all ideals of A which are the annihilator of some module in $\sum_A$. Then clearly, $\bigcap_\lambda I_\lambda = 0$, and therefore A is a subdirect sum of the rings $\{A/I_\lambda\}$. By (*), the rings A/I_λ are $\sum$-primitive. Q.E.D.

<u>1.5. Example</u>

For every ring A let $\sum_A$ be the class of irreducible A-modules. Then $\sum$ is a general class. (See Heinicke [3].) The radical determined by

$\sum$ is called the Jacobson radical.

The above class of modules gives the classical Jacobson theory. We will work with another general class, the class of rationally uniform, homogeneous modules. First, the definitions and elementary properties.

1.6. Essential extensions

Definition: Let W be a submodule of V . V is called an essential extension of W if for every nonzero submodule U of V, $W \cap U \neq 0$. If V is an essential extension of W, we also say that W is an essential submodule of V .

Proposition

(i) Let $V \subset V' \subset V''$ be A-modules. Then $V \subset V''$ is an essential extension if and only if $V \subset V'$ and $V' \subset V''$ are essential extensions.

(ii) The intersection of a finite number of essential submodules is an essential submodule.

The proof of this proposition is trivial.

1.7. The singular submodule

A left ideal I in a ring A is called essential if it is an essential submodule of A when we consider A as a module over itself.
If V is an A-module and $v \in V$ we denote: $(0:v) = \{a \in A | av = 0\}$.
It is clear that $(0:v)$ is a left ideal in A .

Proposition

Let V be an A-module, and let $Z(V) = \{v \in V | (0:v)$ is an essential left ideal of $A\}$. Then $Z(V)$ is a submodule of V .

<u>Proof</u>: If $v, w \in Z(V)$ then $(0:v)$, $(0:w)$ are essential left ideals, so by 1.6 $(0:v) \cap (0:w)$ is essential, and as $(0:v) \cap (0:w) \subset (0: v+w)$, we have $v+w \in Z(V)$. It is clear that $0 \in Z(V)$ and that $v \in Z(V)$ implies $-v \in Z(V)$.

Let $a \in A$, $v \in Z(V)$, and we show: $av \in Z(V)$. Let I be a nonzero left ideal of A . If $I \subset (0:av)$ then $I \cap (0:av) = I \neq 0$. If $I \not\subset (0:av)$, take $x \in I$ such that $xav \neq 0$. Consider the left ideal Ia . It is nonzero because $0 \neq xa \in Ia$. As $(0:v)$ is essential, $(0:v) \cap Ia \neq 0$. Thus there exists $y \in I$ such that $ya \neq 0$ and $yav = 0$. So we have $0 \neq y \in I \cap (0:av)$. This shows that $(0:av)$ is an essential left ideal of A, and $av \in Z(V)$. Q.E.D.

<u>Definition</u>: $Z(V)$ is called the <u>singular submodule</u> of V .

<u>1.8. Rational extensions</u>

<u>Definition</u>: Let W be a submodule of V . V is said to be a <u>rational extension</u> of W if for every $v \in V$, $0 \neq v' \in V$ there exists an element $a \in A$ and an integer n such that:

$$av + nv \in W$$

$$av' + nv' \neq 0 .$$

<u>Proposition</u>

Let W be a submodule of V . The following conditions are equivalent:

(i) V is a rational extension of W .

(ii) If T is a submodule of V which contains W, $f \in \mathrm{Hom}_A(T,V)$ and $f(W) = 0$ then $f = 0$.

Proof: Let V be a rational extension of W, $W \subset T \subset V$, $f \in \mathrm{Hom}_A(T,V)$, $f(W) = 0$. We want to show: $f = 0$. Suppose $f \neq 0$. Then there exists $v \in T$ such that $f(v) \neq 0$. As $W \subset V$ is rational, there exists $a \in A$ and an integer n such that

$$av + nv \in W$$

$$af(v) + nf(v) \neq 0 .$$

But then $0 = f(av + nv) = af(v) + nf(v)$, a contradiction. Therefore $f = 0$, and (i) implies (ii).

To prove that (ii) implies (i) let $v \in V$, $0 \neq v' \in V$ be given. Suppose that for every $a \in A$ and every integer n:

$$(*) \qquad av + nv \in W \quad \text{implies} \quad av' + nv' = 0 .$$

Consider the module $T = W + Av + \mathbb{Z}v$. Clearly, $W \subset T \subset V$. Define

$$f\colon T \longrightarrow V$$

by $f(w + av + nv) = av' + nv'$ for $w \in W$, $a \in A$, $n \in \mathbb{Z}$.

By (*) f is well defined, and it is clear that $f \in \mathrm{Hom}_A(T,V)$ and $f(W) = 0$. Thus, by (ii), $f = 0$, and in particular $f(v) = 0$. But $f(v) = f(0 + 0v + 1v) = v' \neq 0$, a contradiction. Therefore V is a rational extension of W, and (ii) implies (i). Q.E.D.

The next proposition shows the connection between essential and rational extensions.

1.9. Proposition

(i) If $W \subset V$ is a rational extension, then it is an essential extension.

(ii) If $W \subset V$ is an essential extension and $Z(V) = 0$, then $W \subset V$ is an rational extension.

<u>Proof</u>: (i) Let $W \subset V$ be rational, and let U be a nonzero submodule of V. Take a nonzero element $u \in U$. As V is a rational extension of W, there exists $a \in A$ and integer n such that

$$au + nu \in W$$

$$au + nu \neq 0 .$$

Clearly, $$0 \neq au + nu \in W \cap U .$$

So V is an essential extension of W. (ii) Let $W \subset V$ be an essential extension, and $Z(V) = 0$. Let $v \in V$, $0 \neq v' \in V$. Denote

$$I = \{a \in A | av \in W\} .$$

Then I is a left ideal in A. We show that I is essential. Let J be a nonzero left ideal of A. Then Jv is a submodule of V. If $Jv = 0$, then $J \subset I$ and $I \cap J = J \neq 0$. If $Jv \neq 0$, then $Jv \cap W \neq 0$ because W is an essential submodule of V. So there exists $j \in J$ such that $jv \neq 0$ and $jv \in W$. Then $0 \neq j \in I \cap J$. So I is an essential left ideal of A. As $Z(V) = 0$ and $v' \neq 0$, we know that $(0:v')$ is not an essential left ideal in A, and therefore $I \not\subset (0:v')$. Take an element $i \in I$ such that $iv' \neq 0$. We have: $iv \in W$, $iv' \neq 0$. Thus, V is a rational extension of W. Q.E.D.

<u>1.10. Uniform, rationally uniform and homogeneous modules</u>

<u>Definition</u>: (i) An A-module V is said to be <u>uniform</u> if it is an essential extension of each of its nonzero submodules. (ii) An A-module V is said to be <u>rationally uniform</u> if it is a rational extension of each

of its nonzero submodules.

It is clear that a module V is uniform if and only if the intersection of any two nonzero submodules of V is nonzero. Every irreducible module is rationally uniform, and by 1.9(i) every rationally uniform module is uniform. By 1.9(ii) a uniform module which has zero singular submodule, is rationally uniform.

Definition: An A-module V is called homogeneous if for every nonzero submodule W of V there exists a monomorphism $f: V \longrightarrow W$.
Clearly, every irreducible module is homogeneous. It is also easy to see that $\mathbb{Z}$ is a homogeneous module over itself.

Now we are ready to fix the class of modules we want to work with.

1.11. Theorem

For every ring A, let $\sum_A$ be the class of all nontrivial, rationally uniform, homogeneous A-modules. Then $\sum$ is a general class.

Proof: The first two properties in the definition of a general class of modules (see 1.3) are easily verified for $\sum$. To prove property (iii), let A be a ring such that $\bigcap_{V\varepsilon\sum_A} (0:V) = 0$, and let I be a nonzero ideal of A. We want to show: $\sum_I \neq \emptyset$. There exists an A-module $V \varepsilon \sum_A$ such that $IV \neq 0$. (Otherwise $I \subset \bigcap_{V\varepsilon\sum_A} (0:V) = 0$, a contradiction to $I \neq 0$.) Consider V as an I-module. We show: $V \varepsilon \sum_I$. As $IV \neq 0$, V is a nontrivial I-module. Let $N = \{v \varepsilon V | Iv = 0\}$. Clearly, N is an A-submodule of V. Suppose $N \neq 0$. Then, as V is a homogeneous A-module, there exists an A-monomorphism $f: V \longrightarrow N$. For every $i \varepsilon I$

and $v \in V$ we have $f(iv) = if(v) = 0$ because $f(v) \in N$. As $\ker f = 0$, we get $iv = 0$, for every $i \in I$ and $v \in V$, a contradiction to $IV \neq 0$. Therefore: $N = 0$.

To prove that V is rationally uniform as an I-module, let W be a nonzero I-submodule of V, T an I-submodule of V containing W, and $f \in \mathrm{Hom}_I(T,V)$ such that $f(W) = 0$. By 1.8 it is enough to show that $f = 0$. It is easy to see that IW and IT are A-submodules of V, and $0 \neq IW \subset IT \subset V$. ($IW \neq 0$ because $N = \{v \in V | Iv = 0\} = 0$.) As V is a rationally uniform A-module, $IW \subset V$ is a rational extension. The restriction $f|_{IT}: IT \longrightarrow V$ is an A-homomorphism (because $f(a \sum i_k t_k) = f(\sum(ai_k)t_k) = \sum(ai_k)f(t_k) = a \sum i_k f(t_k) = a\, f(\sum i_k t_k))$, and $f|_{IT}(IW) = f(IW) = 0$. Therefore $f|_{IT} = 0$, i.e. $f(IT) = 0$.

Let $t \in T$. Then for every $i \in I$: $if(t) = f(it) = 0$. Thus $If(t) = 0$, so $f(t) \in N = 0$. Therefore $f = 0$, and V is rationally uniform as an I-module. To show that V is a homogeneous I-module, let W be a nonzero I-submodule of V. Then again, IW is a nonzero A-submodule of V. As V is homogeneous as an A-module, there exists an A-monomorphism $f: V \rightarrow IW$. Clearly, f is also an I-monomorphism $f: V \longrightarrow W$. Therefore V is homogeneous as an I-module, and we have $V \in \sum_I$. This proves (iii).

To prove property (iv) of a general class for $\sum$, suppose that A is a ring such that for every nonzero ideal I of A, $\sum_I \neq \emptyset$. We want to show $\bigcap_{V \in \sum_A} (0:V) = 0$. Denote $K = \bigcap_{V \in \sum_A} (0:V)$, and suppose $K \neq 0$. Then K is a nonzero ideal of A, and therefore $\sum_K \neq \emptyset$. Take a K-module $W \in \sum_K$. As $KW \neq 0$, there exists $w_0 \in W$ such that $Kw_0 \neq 0$.

Denote: $U = Kw_0$. Clearly U is an abelian subgroup of W. We define an A-module structure $A \times U \longrightarrow U$ on U in the following way:

For $a \varepsilon A$, $kw_0 \varepsilon Kw_0 = U$ define $a * kw_0 = (ak)w_0$.
To show that $*$ is well-defined, we have to prove: $kw_0 = 0$ implies $(ak)w_0 = 0$, for $k \varepsilon K$ and $a \varepsilon A$.
Let $k \varepsilon K$ be fixed and suppose $kw_0 = 0$. We show $(Ak)w_0 = 0$.
Suppose the contrary. Then $(Ak)w_0$ is a nonzero K-submodule of W, and as W is homogeneous over K, there exists a K-monomorphism $f: W \longrightarrow (Ak)w_0$. For every $x \varepsilon K$, $w \varepsilon W$ we have:
$f(xw) = xf(w) = x[(ak)w_0]$ for some $a \varepsilon A$ because $f(w) \varepsilon (Ak)w_0$.
$f(xw) = x[(ak)w_0)] = [x(ak)]w_0 = [(xa)k]w_0 = (xa)[kw_0] = xa \cdot 0 = 0$.
As $\ker f = 0$, we get $xw = 0$ for every $x \varepsilon K$, $w \varepsilon W$, a contradiction to $KW \neq 0$. Therefore $(Ak)w_0 = 0$, and $*$ is well-defined. It is easy to check that the operation $*$ turns U into an A-module, and that for $k \varepsilon K$, $u \varepsilon U$ we have: $k*u = ku$.
We show now that $U \varepsilon \sum_A$.
Let $M = \{w \varepsilon W | Kw = 0\}$. Then M is a K-submodule of W . If $M \neq 0$, then as W is a homogeneous K-module, there exists a K-monomorphism $f: W \longrightarrow M$. For every $k \varepsilon K$, $w \varepsilon W$, $f(kw) = kf(w) = 0$ because $f(w) \varepsilon M$. As $\ker f = 0$, we get a contradiction to $KW \neq 0$. Thus $M = 0$. As $U = Kw_0 \neq 0$, this implies: $A*U \supset K*U = KU \neq 0$, and so U is a nontrivial A-module.
To show that U is rationally uniform over A, let S be a nonzero A-submodule of U, $u \varepsilon U$, $0 \neq u' \varepsilon U$. Clearly, S is a nonzero K-submodule of W, therefore, as W is a rationally uniform K-module, there exists an element $k \varepsilon K$ and an integer n such that

$$ku + nu \varepsilon S$$

$$ku' + nu' \neq 0 \text{ .}$$

Thus we have $k \in A$ and integer n with

$$k*u + nu \in S$$
$$k*u' + nu' \neq 0 .$$

So U is a rationally uniform A-module. To prove that U is a homogeneous A-module, let S be a nonzero A-submodule of U. Then again, S is a nonzero K-submodule of W. As W is a homogeneous K-module, there exists a K-monomorphism $f: W \longrightarrow S$. Consider the restriction $f|_U: U \longrightarrow S$. For every $a \in A$ and $u = kw_0 \in U$ we have:
$f|_U(a*u) = f(a*kw_0) = f((ak)w_0) = (ak)\ f(w_0) = (ak)*f(w_0) = a*(k*f(w_0)) = a*(kf(w_0)) = a*f(kw_0) = a*f(u) = a*f|_U(u)$. Therefore $f|_U$ is an A-monomorphism, and this shows that U is a homogeneous A-module. We have proved: $U \in \sum_A$. As $K = \bigcap_{V\in\sum_A} (0:V)$, we have $K \subset (0:U)$, thus $K*U = 0$. But $K*U = KU$, so $KU = 0$, in contradiction to $M = \{w \in W | Kw = 0\} = 0$. Therefore $K = \bigcap_{V\in\sum_A} (0:V) = 0$. Q.E.D.

<u>1.12. The weak radical</u>

By the Theorem of Andrunakievic and Rjabuhin, the general class $\sum$ of 1.11 defines a radical. This radical is called the weak radical and is denoted by W. As every irreducible module is nontrivial, rationally uniform and homogeneous, we have $W \subset J$ (where J is the Jacobson radical), and therefore for every ring A, $W(A) \subset J(A)$. But the two radicals are different. In fact, the following example gives a ring A which is weakly primitive (i.e. $\sum$-primitive where $\sum$ is the general class of 1.11), but Jacobson-radical (i.e. $A \in J$).

<u>Example</u>: Let $A = \{\frac{m}{n} | m,n$ integers, m even, n odd$\}$. It is easy to

see that A is a subring of the rational numbers. Consider A as a module over itself, and denote it by ${}_AA$. Then ${}_AA$ is a faithful member of $\sum_A$. Clearly, ${}_AA$ is a nontrivial module. For $0 \neq v \in {}_AA$, $(0:v) = \{a \in A | av = 0\} = 0$ because there are no zero divisors in A, and therefore ${}_AA$ has zero singular submodule. Let V_1, V_2 be nonzero submodules of ${}_AA$. As A is commutative, V_1, V_2 are ideals in A, and so $V_1 \cap V_2 \supset V_1V_2 \neq 0$. Thus ${}_AA$ is uniform, and as $Z({}_AA) = 0$, ${}_AA$ is rationally uniform. To show the homogeneity of ${}_AA$, let V be a nonzero submodule of ${}_AA$. Take a nonzero element $v_0 \in V$ and define $f: {}_AA \longrightarrow V$ by $f(x) = xv_0$ for every $x \in {}_AA$. f is clearly an A-monomorphism. Therefore ${}_AA \in \sum_A$. ${}_AA$ is a faithful module because there are no zero divisors in A. Thus A is a weakly primitive ring. An easy computation shows that every element in A is left-quasi-regular, and therefore A is a Jacobson-radical ring.

CHAPTER 2

ANALOGUES OF SCHUR'S LEMMA AND THE DENSITY THEOREM

From now on $\sum$ will always denote the class of nontrivial, rationally uniform, homogeneous modules. We want to prove the analogues of Schur's Lemma and the Jacobson Density Theorem for $\sum$, and with the help of these to describe the structure of those rings A for which $\sum_A$ contains a faithful module satisfying a certain finite-dimensionality condition. To be able to do this, we need the notion of the quasi-injective hull of a module.

2.1. The injective hull

Definition: A module V is said to be injective if for every module U, submodule W of U and homomorphism $f \in \mathrm{Hom}_A(W,V)$, there exists an extension $\bar{f} \in \mathrm{Hom}_A(U,V)$ of f .

Theorem

Let V be an A-module. Then:

(i) There exists a maximal essential extension of V, i.e. an essential extension $V \subset M$ such that if $V \subset M \subset E$ and $V \subset E$ is essential, then $M = E$.

(ii) If M_1 and M_2 are maximal essential extensions of V, then there is an isomorphism between M_1 and M_2 which fixes every element of V .

Therefore we can speak about the maximal essential extension of V, which will be denoted by V_H .

(iii) V_H is an injective module. Moreover, V_H is a minimal injective module containing V, i.e. if U is injective and $V \subset U \subset V_H$, then $U = V_H$.

For the proof of this well-known result see e.g. Faith [2].

Definition: V_H is called the injective hull of V.

2.2. The quasi-injective hull

The following notion gives a common generalization of irreducibility and injectivity.

Definition: A module V is said to be quasi-injective if for every submodule W of V and homomorphism $f \in \mathrm{Hom}_A(W,V)$ there exists an extension $\bar{f} \in \mathrm{Hom}_A(V,V)$ of f.

Theorem

Let V be an A-module, and let V_H be the injective hull of V. Denote: $\Gamma = \mathrm{Hom}_A(V_H,V_H)$, and let $V_Q = \Gamma V = \{\sum \alpha_i v_i \mid \alpha_i \in \Gamma,\ v_i \in V$, and the sums are finite$\}$. Then:

(i) $V \subset V_Q \subset V_H$, and V_Q is a quasi-injective module.

(ii) V_Q is the smallest quasi-injective module between V and V_H, i.e. if $V \subset Q \subset V_H$ and Q is quasi-injective, then $V_Q \subset Q$.

For the proof see Faith [2].

Definition: V_Q is called the quasi-injective hull of V.

First we prove an analogue of Schur's Lemma for quasi-injective modules.

Definition: A ring A is called Von Neumann-regular if for every element

$a \in A$ there exists $x \in A$ such that $a = axa$.

2.3. Theorem (Schur's Lemma for quasi-injective modules)

Let V be a quasi-injective module over A, and let $\Delta = \mathrm{Hom}_A(V,V)$. Then:

(i) $J(\Delta) = \{\alpha \in \Delta \mid \ker \alpha$ is an essential submodule of $V\}$.

(ii) $\Delta/J(\Delta)$ is Von Neumann-regular.

Proof: Denote $N = \{\alpha \in \Delta \mid \ker \alpha$ is essential in $V\}$. Then N is a left ideal in Δ . For let $\alpha,\beta \in N$. Then $\ker \alpha$, $\ker \beta$ are essential in V, so by 1.6 $\ker \alpha \cap \ker \beta$ is essential, thus $\ker(\alpha+\beta)$ is essential because $\ker \alpha \cap \ker \beta \subset \ker(\alpha+\beta)$, and so $\alpha + \beta \in N$. Clearly $0 \in N$, and $\alpha \in N$ implies $-\alpha \in N$. If $\alpha \in N$ and $\beta \in \Delta$, then $\beta\alpha \in N$ because $\ker \alpha \subset \ker \beta\alpha$. From the Jacobson theory we know that the Jacobson radical of a ring contains every one-sided ideal which is Jacobson-radical. Thus, to show $N \subset J(\Delta)$, it suffices to prove that N is Jacobson-radical, or that every element of N has a left quasi-inverse. Let $\alpha \in N$. Clearly, $\ker \alpha \cap \ker(1-\alpha) = 0$. As $\ker \alpha$ is essential in V, $\ker(1-\alpha) = 0$. So $1 - \alpha: V \longrightarrow (1-\alpha)V$ is an isomorphism. Let $f: (1-\alpha)V \longrightarrow V$ be its inverse. As V is quasi-injective, f can be extended to a homomorphism $\beta \in \Delta$. We have: $\beta(1-\alpha)v = v$ for every $v \in V$, and so $\beta(1-\alpha) = 1$. Take $\gamma = -\alpha\beta$. It is easy to check that $\gamma + \alpha = \gamma\alpha$, and as N is a left ideal of Δ, $\gamma \in N$. So N is Jacobson-radical, and $N \subset J(\Delta)$.

Now we show:

(*) for every $\alpha \in \Delta$ there exists $\gamma \in \Delta$ such that $\alpha - \alpha\gamma\alpha \in N$.

Let $\alpha \in \Delta$. Let W be a submodule of V maximal with respect to the property: $W \cap \ker \alpha = 0$. (Such W exists by Zorn's Lemma.) Then V is an essential extension of $W + \ker \alpha$. (For let U be a nonzero submodule of V . If $U \subset W$, then $U \cap (W + \ker \alpha) = U \neq 0$. If $U \not\subset W$, then $W + U \supsetneq W$, so $(W+U) \cap \ker \alpha \neq 0$. So there exist elements $w \in W$, $u \in U$ such that $0 \neq w + u \in \ker \alpha$. $u \neq 0$, otherwise $0 \neq w \in W \cap \ker \alpha = 0$. Thus, $0 \neq u = (-w) + (w+u) \in U \cap (W + \ker \alpha)$.) As $W \cap \ker \alpha = 0$, $\alpha|_W : W \longrightarrow \alpha W$ is an isomorphism. Let $f: \alpha W \longrightarrow W \subset V$ be the inverse of $\alpha|_W$. By the quasi-injectivity of V, f can be extended to a homomorphism $\gamma \in \Delta$. We have: $\gamma\alpha w = w$ for every $w \in W$. $(\alpha - \alpha\gamma\alpha)w = \alpha w - \alpha\gamma\alpha w = \alpha w - \alpha w = 0$ for every $w \in W$. Also: $(\alpha - \alpha\gamma\alpha)v = \alpha v - \alpha\gamma\alpha v = 0$ for every $v \in \ker \alpha$. Thus $W + \ker \alpha \subset \ker(\alpha - \alpha\gamma\alpha)$. As $W + \ker \alpha$ is an essential submodule of V, so is $\ker(\alpha - \alpha\gamma\alpha)$, which means that $\alpha - \alpha\gamma\alpha \in N$. This proves (*). As $N \subset J(\Delta)$, (*) implies that $\Delta/J(\Delta)$ is a Von Neumann-regular ring. It remained to show: $J(\Delta) \subset N$. Let $\alpha \in J(\Delta)$. By (*), there exists $\gamma \in \Delta$ such that $\alpha - \alpha\gamma\alpha \in N$. $J(\Delta)$ is an ideal in Δ, so $\alpha\gamma \in J(\Delta)$. Thus $\alpha\gamma$ has a quasi-inverse $\beta \in J(\Delta)$: $\beta + \alpha\gamma = \beta\alpha\gamma$. We have: $\beta(\alpha - \alpha\gamma\alpha) = \beta\alpha - \beta\alpha\gamma\alpha = \beta\alpha - (\beta + \alpha\gamma)\alpha = \beta\alpha - \beta\alpha - \alpha\gamma\alpha = -\alpha\gamma\alpha$. As $\alpha - \alpha\gamma\alpha \in N$ and N is a left ideal in Δ, we have $-\alpha\gamma\alpha = \beta(\alpha - \alpha\gamma\alpha) \in N$. But also $\alpha - \alpha\gamma\alpha \in N$, and so $\alpha \in N$. Thus $J(\Delta) \subset N$. Q.E.D.

With the help of this theorem, we are able to prove now the analogue of Schur's Lemma for our class $\sum$.

Definition: An element a of a ring A is called regular if it is neither a left, nor a right zero divisor in A, i.e. $x \in A$, $ax = 0$

implies $x = 0$, and also $x \varepsilon A$, $xa = 0$ implies $x = 0$.

Definition: Let B be a ring with unity, A a subring of B. (This of course, does not mean that A contains the unity element of B.) A is said to be a right order in B if:

(i) Every element of A which is regular in A is invertible in B.

(ii) Every element $b \varepsilon B$ has the form $b = a_1 a_2^{-1}$ where $a_1, a_2 \varepsilon A$, a_2 regular.

The definition of a left order is analogous.

2.4. Theorem (Schur's Lemma for rationally uniform, homogeneous modules)

Let A be a ring and $V \varepsilon \sum_A$. Denote:

V_Q = the quasi-injective hull of V.

$\Delta = \mathrm{Hom}_A(V_Q, V_Q)$.

$\Omega = \mathrm{Hom}_A(V, V)$.

Then:

(i) $V_Q = \Delta V$ (where $\Delta V = \{\sum \alpha_i v_i \mid \alpha_i \varepsilon \Delta, \ v_i \varepsilon V\}$).

(ii) Δ is a division ring.

(iii) Ω is a right order in Δ.

(iv) V_Q is a rationally uniform module.

Remark: The homogeneity of V is needed only for the proof of (iii).

Proof: (i) From Theorem 2.2 we know that $V_Q = \Gamma V$ where $\Gamma = \mathrm{Hom}_A(V_H, V_H)$ and V_H is the injective hull of V. Let $\alpha \varepsilon \Gamma$ and $v \varepsilon V_Q$. Then v has the form $v = \sum_{i=1}^{n} \beta_i v_i$, with $\beta_i \varepsilon \Gamma$, $v_i \varepsilon V$. Thus,

$$\alpha v = \alpha(\sum_{i=1}^{n} \beta_i v_i) = \sum_{i=1}^{n} \alpha\beta_i v_i \in \Gamma V = V_Q .$$

Therefore $\alpha|_{V_Q} \in \Delta$ for every $\alpha \in \Gamma$. Let $v \in V_Q$. As we know, v has the form $v = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_i \in \Gamma$ and $v_i \in V$. Thus, $v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i|_{V_Q} v_i \in \Delta V$. So $V_Q \subset \Delta V$. But clearly $\Delta V \subset V_Q$, so $V_Q = \Delta V$.

(ii) First we prove the following:

(*) If $\alpha \in \Delta$ and $\ker \alpha \neq 0$, then $\alpha V = 0$.

Let $\alpha \in \Delta$ and $\ker \alpha \neq 0$. As V_H is an essential extension of V (by 2.1), so is V_Q. Thus, $V \cap \ker \alpha \neq 0$. Let $W = \{v \in V | \alpha v \in V\}$. Then W is a submodule of V, and clearly: $0 \neq V \cap \ker \alpha \subset W \subset V$. V is rationally uniform, therefore V is a rational extension of $V \cap \ker \alpha$. The mapping $\alpha|_W : W \longrightarrow V$ is zero on $V \cap \ker \alpha$, thus by Proposition 1.8, $\alpha W = 0$. This implies $V \cap \alpha V = 0$, and as V_Q is an essential extension of V, we must have $\alpha V = 0$.

As V_Q is quasi-injective, by Theorem 2.3 we know: $J(\Delta) = \{\alpha \in \Delta | \ker \alpha$ is essential in $V_Q\}$, and $\Delta/J(\Delta)$ is Von Neumann-regular. We show now that $J(\Delta) = 0$.

Let $\alpha \in J(\Delta)$. Let $v \in V_Q$. By (i), v has the form $v = \sum_{i=1}^{n} \beta_i v_i$, where $\beta_i \in \Delta$, $v_i \in V$. As $\alpha \in J(\Delta)$ and $\beta_i \in \Delta$, we have $\alpha\beta_i \in J(\Delta)$, for $i = 1,\ldots,n$. Thus $\ker(\alpha\beta_i)$ is an essential submodule of V_Q, and in particular $\ker(\alpha\beta_i) \neq 0$ for $i = 1,\ldots,n$. By (*), $\alpha\beta_i V = 0$ for $i = 1,\ldots,n$. Thus: $\alpha v = \alpha(\sum_{i=1}^{n} \beta_i v_i) = \sum_{i=1}^{n} \alpha\beta_i v_i = 0$. Therefore $\alpha = 0$, and we have $J(\Delta) = 0$. So Δ is a Von Neumann-regular ring. To show that Δ is a division ring, let $0 \neq \alpha \in \Delta$. As Δ is Von Neumann-regular, there exists $\gamma \in \Delta$ such that $\alpha = \alpha\gamma\alpha$. So $\alpha(1 - \gamma\alpha) = 0$. Suppose $\ker \alpha \neq 0$. Then by (*), $V \subset \ker \alpha \subset V_Q$. As V is an essential submodule of V_Q, so is $\ker \alpha$. But then $\alpha \in J(\Delta) = 0$, a contradiction. Therefore $\ker \alpha = 0$. As $\alpha(1 - \gamma\alpha) = 0$, this implies $1 - \gamma\alpha = 0$. Thus $\gamma\alpha = 1$, which proves that Δ is a division ring.

(iii) First we show that Ω can be considered a subring of Δ . Let $\alpha \in \Omega$. By the quasi-injectivity of V_Q, α can be extended to a homomorphism in Δ . We show that this extension is unique. Suppose $f,g \in \Delta$, $f|_V = g|_V$. Then $0 \neq V \subset \ker(f-g)$, and as Δ is a division ring, $f - g = 0$. For every $\alpha \in \Omega$, denote by $\bar{\alpha}$ the (unique) extension of α to V_Q . Then $\alpha \longmapsto \bar{\alpha}$ is a mapping $\Omega \longrightarrow \Delta$. Using the uniqueness of the extension, it is easy to check that this mapping is a monomorphism of rings. Therefore Ω can be considered a subring of Δ . We want to prove that Ω is a right order in Δ . As Δ is a division ring, the first requirement in the definition of a right order is trivially satisfied. Let λ be a nonzero element of Δ . Then $\lambda V \neq 0$ (because λ is invertible), therefore, as V_Q is an essential extension of V, $V \cap \lambda V \neq 0$. Thus $W = \{v \in V | \lambda v \in V\}$ is a nonzero submodule of V . By the homogeneity of V, there exists a monomorphism $f: V \longrightarrow W$. Clearly, $\lambda|_W f \in \Omega$. For every $v \in V$ we have: $\overline{(\lambda|_W f)}v = \lambda|_W(f(v)) = \lambda(f(v))$, and $(\lambda\bar{f})v = \lambda(\bar{f}(v)) = \lambda(f(v))$.

So $\overline{(\lambda|_W f)}$ and $\lambda\bar{f}$ are equal on V, and as Δ is a division ring, this implies $\overline{(\lambda|_W f)} = \lambda\bar{f}$. Clearly, $\bar{f} \neq 0$ because f is one-to-one. Thus $\bar{f}$ is invertible, and we have: $\lambda = \overline{(\lambda|_W f)}(\bar{f})^{-1}$, the required representation of λ . This proves that Ω is a right order in Δ .

(iv) To prove that V_Q is itself rationally uniform, let $0 \neq W \subset T \subset V_Q$ submodules, and let $f \in \mathrm{Hom}_A(T,V_Q)$ such that $fW = 0$. As V_Q is quasi-injective, there exists an extension $\bar{f} \in \Delta$ of f . As Δ is a division ring and $0 \neq W \subset \ker \bar{f}$, we get $\bar{f} = 0$, and so $f = 0$. Thus V_Q is a rationally uniform module. Q.E.D.

The following density theorem for quasi-injective modules is a generalization of the Jacobson Density Theorem.

Definition: Let Δ be a ring and V a Δ-module. The elements $v_1,\ldots,v_n \in V$ are said to be linearly independent over Δ if none of them is a Δ-linear combination of the others, i.e. if $v_i \notin \sum\limits_{j \neq i} \Delta v_j$ for every $i = 1,\ldots,n$. It is clear that when Δ is a division ring and V a vector space over Δ, the above definition is equivalent to the usual one.

2.5. Theorem (Density theorem for quasi-injective modules)

Let V be a quasi-injective A-module which satisfies:

(*) $v \in V$, $Av = 0$ implies $v = 0$.

Denote: $\Delta = \mathrm{Hom}_A(V,V)$.

Then for every Δ-linearly independent elements $v_1,\ldots,v_n \in V$ there exists an element $a \in A$ such that

$$av_1 \neq 0, \quad av_2 = \ldots = av_n = 0 .$$

Proof: The proof is by induction on n . For $n = 1$ the result is just the given condition (*).

We prove the result for $n = 2$. Let $v_1, v_2 \in V$ be independent over Δ. Suppose that there exists no $a \in A$ such that $av_1 \neq 0$, $av_2 = 0$, i.e. we have:

$$(**) \qquad a \in A, \quad av_2 = 0 \quad \text{implies} \quad av_1 = 0 .$$

Consider the submodule Av_2 of V and define $f: Av_2 \longrightarrow V$ by $f(av_2) = av_1$. Because of (**) f is well-defined, and it is clearly an A-homomorphism. As V is quasi-injective, there exists $\alpha \in \Delta$ with $\alpha|_{Av_2} = f$. Then for every $a \in A$ we have:

$$\alpha(av_2) = f(av_2) = av_1 \text{ and } \alpha(av_2) = a\alpha(v_2) = a(\alpha v_2) .$$

So $a(v_1 - \alpha v_2) = av_1 - a(\alpha v_2) = 0$ for every $a \in A$, i.e. $A(v_1 - \alpha v_2) = 0$. By (*), $v_1 - \alpha v_2 = 0$, $v_1 = \alpha v_2$ in contradiction to the Δ-independence of v_1, v_2. This establishes the result for $n = 2$. Now suppose it is true for n, and we will prove it for $n + 1$. Let $v_0, v_1, \ldots, v_n \in V$ be independent over Δ $(n \geq 2)$. Suppose that there exists no $a \in A$ such that $av_0 \neq 0$, $av_1 = \ldots = av_n = 0$, i.e. we have:

$$(***) \qquad a \in A, \quad av_1 = \ldots = av_n = 0 \text{ implies } av_0 = 0 .$$

Consider A as a module over itself, and define the maps $f, g_1, \ldots, g_n : A \to V$ by:

$$\begin{aligned} f(a) &= av_0 , \qquad a \in A \\ g_1(a) &= av_1 , \qquad a \in A \\ &\vdots \\ g_n(a) &= av_n , \qquad a \in A . \end{aligned}$$

$$\begin{tikzcd}
 & V \\
A \arrow[ur, "f"] \arrow[r, "g_1"] \arrow[ddr, "g_n"] & V \\
 & \vdots \\
 & V
\end{tikzcd}$$

$f, g_1, \ldots, g_n$ are clearly A-homomorphisms. Consider the direct sum of n copies of V, and for $i = 1, \ldots, n$ let $e_i : V \longrightarrow V \oplus \ldots \oplus V$ be the embedding of V on the i-th component of $V \oplus \ldots \oplus V$. Let $g = \sum_{i=1}^{n} e_i g_i$, i.e. $g(a) = (g_1(a), \ldots, g_n(a))$ for every $a \in A$.

$$\begin{tikzcd}
 & V & \\
A \arrow[ur, "f"] \arrow[r, "g_1"] \arrow[ddr, "g_n"'] \arrow[ddrr, bend right=60, "g = \sum_{i=i}^{n} e_i g_i"'] & V \arrow[ddr, "e_1"] & \\
 & \vdots & \vdots \\
 & V \arrow[r, "e_n"] & V \oplus \ldots \oplus V
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
 & V \\
A \arrow[ur, "f"] \arrow[r, "g"] & gA
\end{tikzcd} .$$

From (***) we know that $\ker g \subset \ker f$. Therefore there exists a homomorphism $h\colon gA \longrightarrow V$ such that $hg = f$.

$$\begin{tikzcd} & V \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & gA \arrow[u, "h"'] \end{tikzcd}$$

For $i = 1,\dots,n$ let $K_i = \bigcap\limits_{j \neq i} \ker g_j$, and let $K = K_1 + \dots + K_n$.

Clearly, K is a submodule of A . For $i = 1,\dots,n$ we have:

$e_i g_i K \subset gK$. (For let $a \in K$. Then $a = a_1 + \dots + a_n$ where $a_j \in K_j$, $j = 1,\dots,n$, and so $e_i g_i(a) = e_i(g_i(a_1+\dots+a_n)) =$ $e_i(g_i(a_1)+\dots+g_i(a_i)+\dots+g_i(a_n)) = e_i(g_i(a_i)) = (0,\dots,g_i(a_i),\dots,0) =$ $g(a_i) \in gK$.) For fixed i consider the homomorphisms

$$g_iK \xrightarrow{e_i|_{g_iK}} gK \xrightarrow{h|_{gK}} V .$$

As V is quasi-injective, the mapping $(h|_{gK})(e_i|_{g_iK})\colon g_iK \longrightarrow V$ can be extended to a homomorphism $\alpha_i \in \Delta$. For every $a \in K$ and $i = 1,\dots,n$ we have: $\alpha_i g_i(a) = he_i g_i(a)$, and therefore:

$\sum\limits_{i=1}^{n} \alpha_i g_i(a) = \sum\limits_{i=1}^{n} he_i g_i(a) = h(\sum\limits_{i=1}^{n} e_i g_i(a)) = hg(a) = f(a)$, for every $a \in K$.

In particular, for every $a \in K_1 = \bigcap\limits_{j=2}^{n} \ker g_j$:

$f(a) = \sum\limits_{i=1}^{n} \alpha_i g_i(a) = \alpha_1 g_1(a)$, and using the definition of f and g_1, we get: $av_0 = \alpha_1(av_1) = a(\alpha_1 v_1)$, $a(v_0 - \alpha_1 v_1) = 0$, for every $a \in \bigcap\limits_{j=2}^{n} \ker g_j$.

This means:

$a \in A$, $av_2 = \ldots = av_n = 0$ implies $a(v_0 - \alpha_1 v_1) = 0$. Therefore, by the induction hypothesis, the elements $v_0 - \alpha_1 v_1, v_2, \ldots, v_n$ are linearly dependent over Δ. But from this it follows easily that the elements $v_0, v_1, \ldots, v_n$ are also linearly dependent over Δ, a contradiction. Q.E.D.

The Jacobson Density Theorem follows quickly from the above theorem by observing that an irreducible module V is quasi-injective, and satisfies the condition (*) (because $\{v \in V | Av = 0\}$ is a submodule of V). (Let $v_1, \ldots, v_n \in V$ independent over Δ, and $w_1, \ldots, w_n \in V$ arbitrary. By the above theorem there exist $a_1, \ldots, a_n \in A$ such that $a_i v_i \neq 0$ for $i = 1, \ldots, n$, and $a_i v_j = 0$ for $i \neq j$. As V is irreducible, $Aa_i v_i = V$ for $i = 1, \ldots, n$. Therefore there exist $b_i \in A$ such that $b_i a_i v_i = w_i$, $i = 1, \ldots, n$. Take $a = \sum_{j=1}^{n} b_j a_j$. Then clearly: $av_i = w_i$, $i = 1, \ldots, n$.)

The following density theorem will give, as a corollary, the density theorem for our class of modules, $\sum$.

2.6. Theorem (Density theorem)

Let V be an A-module, E an extension of V, and suppose that the following properties are satisfied:

(i) $V \neq 0$, V is homogeneous.

(ii) E is quasi-injective, uniform, and:

(*) $v \in E$, $Av = 0$ implies $v = 0$.

Denote:
$$\Delta = \mathrm{Hom}_A(E,E)$$
$$\Omega = \mathrm{Hom}_A(V,V).$$

Then for any Δ-linearly independent elements $v_1,\ldots,v_n \in E$ and any elements $w_1,\ldots,w_n \in V$ there exists an element $a \in A$ and a nonzero mapping $\lambda \in \Omega$ such that

$$av_1 = \lambda w_1,\ldots,av_n = \lambda w_n .$$

<u>Proof</u>: Let the Δ-linearly independent elements $v_1,\ldots,v_n \in E$ and the elements $w_1,\ldots,w_n \in V$ be given. Denote for $i = 1,\ldots,n$:
$A_i = \{a \in A | av_j = 0 \text{ for every } j \neq i\}$. As E is quasi-injective and satisfies (*), by 2.5 $A_i v_i \neq 0$ for every i . Clearly, $A_1, \ldots, A_n$ are left ideals in A, so $A_1v_1,\ldots,A_nv_n$ are nonzero submodules of E . As E is uniform, and each of the submodules $V, A_1v_1,\ldots,A_nv_n$ is nonzero, we have $V \cap (\bigcap_{j=1}^{n} A_jv_j) \neq 0$. As V is homogeneous, there exists a monomorphism

$$\lambda: V \longrightarrow V \cap (\bigcap_{j=1}^{n} A_jv_j) .$$

Clearly, $0 \neq \lambda \in \Omega$. For i fixed, as $\lambda w_i \in A_iv_i$, there exists $a_i \in A_i$ such that $\lambda w_i = a_iv_i$.
Take $a = a_1 + \ldots + a_n$. Then for every $i = 1,\ldots,n$ we have

$$av_i = (\sum_{j=1}^{n} a_j)v_i = \sum_{j=1}^{n} a_jv_i = a_iv_i = \lambda w_i .$$

Thus $av_i = \lambda w_i$, for $i = 1,\ldots,n$. Q.E.D.

Now we are able to state the density theorem for the class $\sum$.

<u>2.7. Theorem</u> (Density theorem for rationally uniform, homogeneous modules)

Let A be a ring and $V \in \sum_A$. Denote:

V_Q = the quasi-injective hull of V,

$\Delta = \mathrm{Hom}_A(V_Q, V_Q)$,

$\Omega = \mathrm{Hom}_A(V, V)$.

Then for any Δ-linearly independent vectors $v_1, \ldots, v_n \in V_Q$ and any vectors $w_1, \ldots, w_n \in V$ there exists an element $a \in A$ and a nonzero mapping $\lambda \in \Omega$ such that:

$$av_1 = \lambda w_1, \ldots, av_n = \lambda w_n .$$

Remark: It is justified to use the term "vector" because Δ is a division ring by Theorem 2.4(ii).

Proof: To be able to apply Theorem 2.6, we have to know:

(i) $V \neq 0$, V is homogeneous,

(ii) V_Q is quasi-injective, uniform, and satisfies:

(*) $v \in V_Q$, $Av = 0$ implies $v = 0$.

As $V \in \sum_A$, we know that V is nonzero and homogeneous. V_Q is, of course, quasi-injective, and by Theorem 2.4(iv) V_Q is rationally uniform, and therefore uniform by 1.10. The only thing to be checked is that V_Q satisfies (*).

Let $W = \{v \in V_Q | Av = 0\}$. Then W is clearly a submodule of V_Q . Suppose $W \neq 0$. Then, as V_Q is an essential extension of V, $V \cap W \neq 0$. By the homogeneity of V there exists a monomorphism $f\colon V \longrightarrow V \cap W$. For every $a \in A$ and $v \in V$ we have: $f(av) = af(v) = 0$ because $f(v) \in W$. As $\ker f = 0$, we get $av = 0$ for every $a \in A$ and $v \in V$, a contradiction because V is a nontrivial A-module. Therefore $W = 0$, i.e. V_Q satisfies (*). Thus we can apply Theorem 2.6

and get the result. Q.E.D.

Now, as we have in our hands Schur's Lemma and the Density Theorem for the class $\sum$, we are able to describe the structure of those rings A for which $\sum_A$ has a faithful member satisfying a certain condition of finite dimensionality.

2.8. Theorem

Let A be a ring, $V \varepsilon \sum_A$ faithful. Denote:

V_Q = the quasi-injective hull of V,

$\Delta = \mathrm{Hom}_A(V_Q,V_Q)$.

Suppose that V_Q is finite dimensional over Δ .

Then A is a left order in the simple, Artinian ring $\mathrm{Hom}_\Delta(V_Q,V_Q)$.

Proof: We know that Δ is a division ring by Schur's Lemma for the class $\sum$ (Theorem 2.4(ii)). $\mathrm{Hom}_\Delta(V_Q,V_Q)$ is, of course, simple and Artinian, being the ring of linear transformations of a finite dimensional vector space. As V is a faithful A-module, V_Q is also faithful. Therefore A can be considered a subring of $\mathrm{Hom}_\Delta(V_Q,V_Q)$, by considering an element $a \varepsilon A$ to be the linear transformation $a_L: V_Q \longrightarrow V_Q$ defined by $a_L(v) = av$ for $v \varepsilon V_Q$. We want to prove that A is a left order in $\mathrm{Hom}_\Delta(V_Q,V_Q)$. Denote by n the dimension of V_Q over Δ . By Theorem 2.4(i) $V_Q = \Delta V$, so we can choose a basis $v_1,\ldots,v_n$ of V_Q over Δ such that $v_1,\ldots,v_n \varepsilon V$. (For let $u_1,\ldots,u_n$ be a basis of V_Q . As $V_Q = \Delta V$, each u_i is a finite linear combination of elements $v_{i1},\ldots,v_{im_i} \varepsilon V$. Clearly the vectors $v_{11},\ldots,v_{1m_1},\ldots,v_{n1},\ldots,v_{nm_n} \varepsilon V$ generate V_Q, thus we can choose a subset of them which is a basis of V_Q .)

Now we show that the singular submodule of V_Q is zero. Suppose the contrary, and let $0 \neq v \in Z(V_Q)$. Let i be fixed. By the Density theorem for $\sum$ (Theorem 2.7), there exists $a_i \in A$ and $0 \neq \lambda_i \in \mathrm{Hom}_A(V,V)$ such that $a_i v = \lambda_i v_i$. As V_Q is quasi-injective, λ_i can be extended to a mapping $\alpha_i \in \Delta$. We have:

$$a_i v = \alpha_i v_i \quad \text{for} \quad i = 1,\dots,n \ .$$

$Z(V_Q)$ is a submodule of V_Q, thus $a_1 v,\dots,a_n v \in Z(V_Q)$, and therefore $I_i = (0:a_i v)$, $i = 1,\dots,n$, are essential left ideals in A . We have:

$$0 = I_i a_i v = I_i \alpha_i v_i = \alpha_i I_i v_i \quad \text{for} \quad i = 1,\dots,n \ .$$

Thus $I_i v_i \subset \ker \alpha_i$, $i = 1,\dots,n$. As Δ is a division ring, and $0 \neq \alpha_i \in \Delta$, we get $I_i v_i = 0$, for every $i = 1,\dots,n$. $I_1,\dots,I_n$ are essential left ideals of A, therefore $\bigcap_{i=1}^{n} I_i$ is essential, and in particular $\bigcap_{i=1}^{n} I_i \neq 0$. Take $0 \neq a \in \bigcap_{i=1}^{n} I_i$. Then

$$av_i = 0 \quad \text{for} \quad i = 1,\dots,n \ .$$

As $v_1,\dots,v_n$ is a basis of V_Q, this implies $aV_Q = 0$. But then $a = 0$ because V_Q is faithful. This contradiction proves: $Z(V_Q) = 0$.
Next we show that if I is an essential left ideal of A, then I contains an element which is invertible in $\mathrm{Hom}_\Delta(V_Q,V_Q)$. Consider V and V_Q as I-modules. We will show that they satisfy the conditions of Theorem 2.6 (over the ring I). We know that $V \neq 0$. To show that V is homogeneous over I, let W be a nonzero I-submodule of V . Then $IW \subset W$, IW is an A-submodule of V, and $IW \neq 0$. (For let $0 \neq w \in W$. As $Z(V_Q) = 0$, $w \notin Z(V_Q)$, so $(0:w)$ is not essential in

A . Therefore, as I is essential, $I \not\subset (0:w)$. Thus $Iw \neq 0$, and so $IW \neq 0$.) V is a homogeneous A-module, therefore there exists an A-monomorphism $f: V \longrightarrow IW$. Clearly, f is also an I-monomorphism from V into W . Thus V is a homogeneous I-module.

We show now that V_Q is quasi-injective as an I-module. Let W be an I-submodule of V_Q and $f \varepsilon \mathrm{Hom}_I(W,V_Q)$. Then $IW \subset W$ and IW is an A-submodule of V_Q . Furthermore, $f|_{IW} \varepsilon \mathrm{Hom}_A(IW,V_Q)$. (Because for $a \varepsilon A$, $\sum_{k=1}^{m} i_k w_k \varepsilon IW$ we have: $f(a \sum i_k w_k) = f(\sum (ai_k)w_k) = \sum (ai_k)f(w_k) = a \sum i_k f(w_k) = af(\sum i_k w_k)$.) As V_Q is quasi-injective as an A-module, there exists $g \varepsilon \mathrm{Hom}_A(V_Q,V_Q)$ such that $g|_{IW} = f|_{IW}$. Clearly, $g \varepsilon \mathrm{Hom}_I(V_Q,V_Q)$. Let $i \varepsilon I$, $w \varepsilon W$. Then: $i(f(w) - g(w)) = if(w) - ig(w) = f(iw) - g(iw) = 0$. Thus $I(f(w) - g(w)) = 0$. As I is an essential left ideal of A, this implies $f(w) - g(w) \varepsilon Z(V_Q) = 0$, and so $f(w) = g(w)$, for every $w \varepsilon W$, i.e. $g|_W = f$. Therefore V_Q is a quasi-injective I-module.

To prove that V_Q is a uniform I-module, let W_1,W_2 be nonzero I-submodules of V_Q . Then IW_1, IW_2 are nonzero A-submodules of V_Q . (Again, they are nonzero because I is essential and $Z(V_Q) = 0$.) By Theorem 2.4(iv) V_Q is rationally uniform as an A-module, so V_Q is a uniform A-module. Thus $IW_1 \cap IW_2 \neq 0$. As $IW_1 \subset W_1$ and $IW_2 \subset W_2$, we have $W_1 \cap W_2 \neq 0$, and therefore V_Q is a uniform I-module.

To be able to use Theorem 2.6, it remained to check that if $v \varepsilon V_Q$ and $Iv = 0$, then $v = 0$. But this is clear, because $Iv = 0$ implies, as I is essential, that $v \varepsilon Z(V_Q) = 0$. Thus the I-modules $V \subset V_Q$ satisfy the requirements of Theorem 2.6. We show now that

$$\mathrm{Hom}_I(V_Q,V_Q) = \Delta \ .$$

Clearly, $\Delta \subset \mathrm{Hom}_I(V_Q,V_Q)$. To show the opposite inclusion, let $f \in \mathrm{Hom}_I(V_Q,V_Q)$. IV_Q is an A-submodule of V_Q, and $f|_{IV_Q} : IV_Q \longrightarrow V_Q$ is an A-homomorphism. As V_Q is quasi-injective as an A-module, there exists a mapping $\alpha \in \Delta$ such that $\alpha|_{IV_Q} = f|_{IV_Q}$. For every $i \in I$ and $v \in V_Q$ we have: $i(f(v) - \alpha(v)) = if(v) - i\alpha(v) = f(iv) - \alpha(iv) = 0$. So for every $v \in V_Q$, $I(f(v) - \alpha(v)) = 0$, and therefore, as I is essential, $f(v) - \alpha(v) \in Z(V_Q) = 0$. Thus $f = \alpha \in \Delta$. This shows: $\mathrm{Hom}_I(V_Q,V_Q) = \Delta$. Therefore $v_1,\ldots,v_n$ are linearly independent over $\mathrm{Hom}_I(V_Q,V_Q)$. By Theorem 2.6 there exists an element $e \in I$ and a nonzero mapping $\lambda \in \mathrm{Hom}_I(V,V)$ such that

$$ev_1 = \lambda v_1,\ldots,ev_n = \lambda v_n \ .$$

As V_Q is quasi-injective as an I-module, λ can be extended to a mapping $\gamma \in \mathrm{Hom}_I(V_Q,V_Q) = \Delta$. Thus we have $e \in I$ and $0 \neq \gamma \in \Delta$ such that $ev_1 = \gamma v_1,\ldots,ev_n = \gamma v_n$.

The vectors $\gamma v_1,\ldots,\gamma v_n$ are, clearly, linearly independent over Δ, and as $\dim_\Delta V_Q = n$, $\gamma v_1,\ldots,\gamma v_n$ is a basis of V_Q over Δ . Therefore e, considered as a linear transformation on the vector space V_Q, is invertible. So I contains an element which is invertible in $\mathrm{Hom}_\Delta(V_Q,V_Q)$.

Now we can prove that every element $\phi \in \mathrm{Hom}_\Delta(V_Q,V_Q)$ is of the form $\phi = a^{-1}b$ with $a,b \in A$. Let $\phi \in \mathrm{Hom}_\Delta(V_Q,V_Q)$. Denote

$$A_i = \{a \in A | av_j = 0 \ \text{ for every } \ j \neq i\}, \ \text{ for } \ i = 1,\ldots,n \ .$$

Clearly, $A_1,\ldots,A_n$ are left ideals of A . We know that if $v \in V_Q$ and $Av = 0$, then $v = 0$ (because $Z(V_Q) = 0$). Therefore by the Density theorem for quasi-injective modules (Theorem 2.5), $A_i v_i \neq 0$ for every $i = 1,\dots,n$. Thus $A_1 v_1,\dots,A_n v_n$ are nonzero submodules of V_Q, and as V_Q is uniform (since it is rationally uniform by 2.4(iv)), $A_1 v_1,\dots,A_n v_n$ are essential submodules of V_Q. Let $X_i = \{x \in A \mid x\phi(v_i) \in A_i v_i\}$, for $i = 1,\dots,n$. Then $X_1,\dots,X_n$ are essential left ideals of A. (For let J be a nonzero left ideal of A. If $J\phi(v_i) = 0$, then $J \subset X_i$ and so $J \cap X_i = J \neq 0$. If $J\phi(v_i) \neq 0$, then $J\phi(v_i)$ is a nonzero submodule of V_Q, and as $A_i v_i$ is essential in V_Q, $J\phi(v_i) \cap A_i v_i \neq 0$. So there exists $j \in J$ such that $0 \neq j\phi(v_i) \in A_i v_i$. We have $0 \neq j \in J \cap X_i$. Thus X_i is an essential left ideal of A.) Take $X = \bigcap_{i=1}^{n} X_i$. Then X is an essential left ideal of A. Therefore there exists an element $a \in X$ which is invertible in $\mathrm{Hom}_\Delta(V_Q,V_Q)$. As $a \in X = \bigcap_{i=1}^{n} X_i$, we have:

$$a\phi(v_i) \in A_i v_i, \quad \text{for} \quad i = 1,\dots,n .$$

So there exist elements $b_i \in A_i$, $i = 1,\dots,n$, such that

$$a\phi(v_i) = b_i v_i, \quad \text{for} \quad i = 1,\dots,n .$$

Take $b = b_1 + \dots + b_n$. Then for every $i = 1,\dots,n$:

$$bv_i = (\sum_{j=1}^{n} b_j)v_i = \sum_{j=1}^{n} b_j v_i = b_i v_i = a\phi(v_i) .$$

Thus:

$$a\phi(v_1) = bv_1$$
$$\vdots$$
$$a\phi(v_n) = bv_n .$$

So the linear transformations $a\phi$ and b act equally on the basis

$v_1,\ldots,v_n$ of V_Q . Therefore $a\phi = b$, and as a is invertible in $\mathrm{Hom}_\Delta(V_Q,V_Q)$, we have:

$$\phi = a^{-1}b\ , \qquad a,\ b \in A\ .$$

Finally, we have to prove that every element of A which is regular in A, is invertible in $\mathrm{Hom}_\Delta(V_Q,V_Q)$. This will follow from the following elementary property of finite-dimensional vector spaces:

<u>Lemma</u>: Let W be a finite-dimensional vector space over a division ring D . Let $f \in \mathrm{Hom}_D(W,W)$ such that:

$$g \in \mathrm{Hom}_D(W,W),\quad gf = 0 \ \text{ implies } \ g = 0\ .$$

Then f is invertible.

(<u>Proof</u>: Suppose f is not onto, i.e. fW is a proper subspace of W . Then there exists a nonzero linear transformation g on W which is zero on fW . So $gf = 0$ and $g \neq 0$, a contradiction to the given property of f . Therefore f is onto, and because of the finite-dimensionality of W, f is also one-to-one, and the lemma is proved.) Now let $a \in A$ be regular in A . Let $g \in \mathrm{Hom}_\Delta(V_Q,V_Q)$ and suppose $ga = 0$. We know that g has the form $g = b^{-1}c$ with $b,c \in A$. So $b^{-1}ca = 0$, and multiplying by b from the left we get $ca = 0$. As $c \in A$ and a is regular in A, $c = 0$, and therefore $g = b^{-1}c = 0$. As V_Q is finite-dimensional over Δ, by the Lemma a is invertible in $\mathrm{Hom}_\Delta(V_Q,V_Q)$. Thus, A is a left order in $\mathrm{Hom}_\Delta(V_Q,V_Q)$. Q.E.D.

CHAPTER 3

THE GOLDIE THEOREM

Our aim is to prove the theorem of Goldie which gives the structure of prime, Goldie rings. In the previous chapter we obtained, with the help of Schur's Lemma and the Density Theorem for the class $\sum$, the structure of those rings A for which $\sum_A$ contains a faithful member V satisfying the finite-dimensionality condition $\dim_{\Delta} V_Q < \infty$ (where V_Q is the quasi-injective hull of V, and $\Delta = \mathrm{Hom}_A(V_Q, V_Q)$.)

Thus, to get the structure of prime, Goldie rings, all we have to show is the following:

If A is a prime, Goldie ring, then $\sum_A$ contains a faithful module V such that $\dim_{\Delta} V_Q < \infty$.

3.1. Prime rings

The next proposition is elementary.

Proposition: The following conditions on a ring A are equivalent:

(i) If I, J are ideals in A and $IJ = 0$, then $I = 0$ or $J = 0$.

(ii) If I, J are left ideals in A and $IJ = 0$, then $I = 0$ or $J = 0$.

(iii) If I, J are right ideals in A and $IJ = 0$, then $I = 0$ or $J = 0$.

(iv) If $x, y \in A$ and $xAy = 0$, then $x = 0$ or $y = 0$.

Definition: A nonzero ring A is said to be prime if it satisfies one of the above equivalent conditions.

It is clear that every simple ring is prime. But, for example, the ring of integers is prime but not simple.

3.2. Goldie rings

Definition: Let S be a nonempty subset of a ring A. The left annihilator, $\ell(S)$, of S is defined to be

$$\ell(S) = \{a \in A \mid aS = 0\} .$$

It is clear that $\ell(S)$ is a left ideal in A. (If S consists of the single element x, we denote $\ell(S) = \ell(x)$.)

Definition: A left ideal I of a ring A is said to be a left annihilator if there exists a nonempty subset S of A such that $I = \ell(S)$.

Definition: Let I and J be left ideals in A. I is said to be a complement of J if:

(i) $I \cap J = 0$

(ii) For every left ideal I' of A such that $I' \supsetneq I$, we have $I' \cap J \neq 0$.

Definition: A left ideal I of a ring A is called a complement if there exists a left ideal J of A such that I is a complement of J .

Definition: A ring A is said to be Goldie if:

(i) A satisfies the ascending chain condition on left annihilators (i.e. there exists no strictly increasing, infinite sequence of left annihilators in A),

(ii) A satisfies the ascending chain condition on complements.

Of course, every Noetherian ring is Goldie.

3.3. Proposition

The following conditions on a ring A are equivalent:

(i) A satisfies the ascending chain condition on complements.

(ii) There exists no infinite sequence $I_1, I_2, \ldots, I_n, \ldots$ of nonzero left ideals of A such that the sum $\sum_{n=1}^{\infty} I_n$ is direct.

(The sum $\sum_{n=1}^{\infty} I_n$ is direct means that for every $n \geq 1$: $x_1 + \ldots + x_n = 0$, $x_i \in I_i$ implies $x_1 = \ldots = x_n = 0$.)

Proof: (i) implies (ii): Suppose the contrary, i.e. there exists an infinite sequence $I_1, I_2, \ldots, I_n, \ldots$ of nonzero left ideals of A such that the sum $\sum_{n=1}^{\infty} I_n$ is direct. By Zorn's Lemma there exists a left ideal T_0 of A, which is maximal with respect to the property:

$$T_0 \cap (I_1 + I_2 + \ldots) = 0 .$$

Again by Zorn's Lemma there exists a left ideal T_1 of A, which is maximal with respect to the properties:

$$T_1 \cap (I_2 + I_3 + \ldots) = 0$$

$$T_1 \supset T_0 + I_1 .$$

(The set of all left ideals T of A which satisfy $T \cap (I_2 + I_3 + \ldots) = 0$ and $T \supset T_0 + I_1$ is nonempty because $T_0 + I_1$ belongs to it.)

We continue by induction. Suppose we have already found left ideals $T_0, T_1, \ldots, T_{n-1}$ such that for every $i = 1, \ldots, n - 1$, T_i is maximal with respect to the properties:

$$T_i \cap (I_{i+1} + I_{i+2} + \dots) = 0$$

$$T_i \supset T_{i-1} + I_i \ .$$

Then, by Zorn's Lemma, there exists a left ideal T_n of A, which is maximal with respect to the properties:

$$T_n \cap (I_{n+1} + I_{n+2} + \dots) = 0$$

$$T_n \supset T_{n-1} + I_n \ .$$

(The set of all left ideals T of A which satisfy $T \cap (I_{n+1} + I_{n+2} \dots) = 0$ and $T \supset T_{n-1} + I_n$ is nonempty because it contains $T_{n-1} + I_n$. For let $x \in (T_{n-1} + I_n) \cap (I_{n+1} + I_{n+2} + \dots)$. Then $x = t_{n-1} + i_n = i_{n+1} + \dots + i_m$ where $t_{n-1} \in T_{n-1}$, $i_j \in I_j$, $m \geq n+1$.
$t_{n-1} = -i_n + i_{n+1} + \dots + i_m \in T_{n-1} \cap (I_n + I_{n+1} + \dots) = 0$. So $t_{n-1} = 0$, and $-i_n + i_{n+1} + \dots + i_m = 0$, and as $\sum_{n=1}^{\infty} I_n$ is direct, $i_j = 0$ for all j. Thus $x = t_{n-1} + i_n = 0$.)

So we have an infinite sequence $T_0, T_1, \dots, T_n, \dots$ of left ideals such that for every $n \geq 1$ T_n is maximal with respect to the properties

$$T_n \cap (I_{n+1} + I_{n+2} + \dots) = 0$$

$$T_n \supset T_{n-1} + I_n \ .$$

It is clear that T_n is a complement of $I_{n+1} + I_{n+2} + \dots$, and that $T_{n-1} \subset T_n$. Also, $T_{n-1} \cap I_n \subset T_{n-1} \cap (I_n + I_{n+1} + \dots) = 0$, therefore, as $I_n \neq 0$, $T_{n-1} \cap I_n \neq I_n$. Thus $I_n \not\subset T_{n-1}$. But $I_n \subset T_{n-1} + I_n \subset T_n$, and so: $T_{n-1} \subsetneq T_n$. Thus we have a strictly increasing, infinite sequence of complements, $T_0 \subsetneq T_1 \subsetneq \dots \subsetneq T_n \subsetneq \dots$, in contradiction to (i).

(ii) implies (i): Suppose the contrary, i.e. there exists in A a strictly increasing, infinite sequence $I_1 \subsetneqq I_2 \subsetneqq \dots \subsetneqq I_n \subsetneqq \dots$ of complements. For every $n \geq 1$ there exists a left ideal J_n of A such that I_n is a complement of J_n. We have for every $n \geq 1$:

$$I_n \cap J_n = 0$$

$$I_{n+1} \cap J_n \neq 0 \quad \text{(because } I_n \subsetneqq I_{n+1}\text{)} .$$

Denote: $K_n = I_{n+1} \cap J_n$, for $n \geq 1$. Then $K_1, K_2, \dots, K_n, \dots$ are nonzero left ideals of A. We show that the sum $\sum_{n=1}^{\infty} K_n$ is direct.

Let $k_1 + \dots + k_n = 0$, where $k_i \in K_i$, $i = 1, \dots, n$. Then $k_n = (-k_1) + \dots + (-k_{n-1})$, and $-k_1 \in I_2, \dots, -k_{n-1} \in I_n$. As $I_2 \subset \dots \subset I_n$, we get $k_n = (-k_1) + \dots + (-k_{n-1}) \in I_n$. But also $k_n \in J_n$, thus $k_n \in I_n \cap J_n = 0$, $k_n = 0$. Similarly, $k_{n-1} = 0, \dots,$ $k_1 = 0$. So the sum $\sum_{n=1}^{\infty} K_n$ is direct, in contradiction to (ii). Q.E.D.

3.4. Proposition

Let A be a prime ring which satisfies the ascending chain condition on left annihilators. Then, considering A as a module over itself, the singular submodule of A is zero.

Proof: Suppose the contrary, i.e. $Z(A) \neq 0$. Let $X = \{\ell(x) \mid 0 \neq x \in Z(A)\}$. Then X is a nonempty set of left annihilators in A, and therefore, as A satisfies the ascending chain condition on left annihilators, X contains a maximal element, $\ell(x_0)$, where $0 \neq x_0 \in Z(A)$. Let $L = \{y \in A \mid yZ(A) = 0\}$. Then L is a left ideal in A, and $LZ(A) = 0$. $Z(A)$ is also a left ideal in A (by 1.7 it is a submodule of A as a module over itself), therefore, as A is a prime ring and $Z(A) \neq 0$, we have $L = 0$.

Therefore $x_0 \notin L$, and so $x_0 Z(A) \neq 0$. Thus there exists $z \in Z(A)$ such that $x_0 z \neq 0$. We show now: $\ell(x_0) \subsetneqq \ell(x_0 z)$. It is clear that $\ell(x_0) \subset \ell(x_0 z)$. Ax_0 is a nonzero left ideal in A. ($Ax_0 \neq 0$ because if $Ax_0 = 0$ then $x_0 A x_0 = 0$ and by the primeness of A, $x_0 = 0$, a contradiction.) As $z \in Z(A)$, $\ell(z)$ is an essential left ideal of A. Therefore $Ax_0 \cap \ell(z) \neq 0$. Thus there exists $a \in A$ such that $ax_0 \neq 0$, $ax_0 z = 0$. We have $a \in \ell(x_0 z)$, $a \notin \ell(x_0)$, and so $\ell(x_0) \subsetneqq \ell(x_0 z)$. As $0 \neq x_0 z \in Z(A)$, $\ell(x_0 z) \in X$, a contradiction to the maximality of $\ell(x_0)$ in X. Thus: $Z(A) = 0$. Q.E.D.

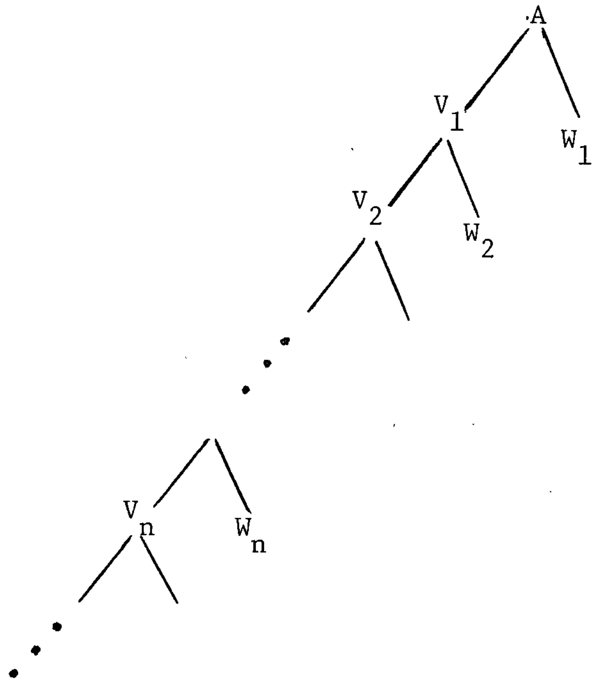
Now we are ready to prove the theorem which was the aim of this chapter, and which gives, as an immediate corollary, the Goldie Theorem.

3.5. Theorem

Let A be a prime, Goldie ring. Then $\sum_A$ contains a faithful module V such that $\dim_\Delta V_Q < \infty$. (V_Q denotes the quasi-injective hull of V, and $\Delta = \mathrm{Hom}_A(V_Q, V_Q)$.)

Proof: First we show that A contains a nonzero left ideal V which is uniform as an A-module.

Suppose that it does not. Then A is not uniform as an A-module. Therefore there exist nonzero left ideals V_1, W_1 in A such that $V_1 \cap W_1 = 0$. As V_1 is not a uniform A-module, there exist nonzero left ideals V_2, W_2 of A such that $V_2 \subset V_1$, $W_2 \subset V_1$ and $V_2 \cap W_2 = 0$. Continuing by induction we obtain two infinite sequences of nonzero left ideals, $V_1, V_2, \ldots, V_n, \ldots$ and $W_1, W_2, \ldots, W_n, \ldots$, with the properties: $V_n \supset V_{n+1}$, $V_n \supset W_{n+1}$ and $V_n \cap W_n = 0$ for every $n \geq 1$.



We show that the sum $\sum_{n=1}^{\infty} W_n$ is direct. Suppose $w_1 + \ldots + w_n = 0$, where $w_i \in W_i$, $i = 1,\ldots,n$. Then $w_1 = (-w_2) + \ldots + (-w_n)$, and $-w_2 \in W_2 \subset V_1, \ldots, -w_n \in W_n \subset V_{n-1} \subset V_1$. Thus
$w_1 = (-w_2) + \ldots + (-w_n) \in V_1 \cap W_1 = 0$, $w_1 = 0$. Similarly, $w_2 = 0, \ldots, w_n = 0$, and the sum $\sum_{n=1}^{\infty} W_n$ is direct. By Proposition 3.3 this is a contradiction to the fact that A satisfies the ascending chain condition on complements. Therefore A contains a nonzero left ideal V which is a uniform A-module.

We will prove that $V \in \sum_A$, V is faithful, and $\dim_\Delta V_Q < \infty$.
First, as A and V are nonzero left ideals of A, and A is a prime ring, $AV \neq 0$ so V is a nontrivial A-module. Clearly, $Z(V) \subset Z(A)$. By Proposition 3.4, $Z(A) = 0$, therefore $Z(V) = 0$. Thus V is

rationally uniform. To prove that V is homogeneous, Let W be a nonzero submodule of V. As V and W are nonzero left ideals in A, and A is a prime ring, $VW \neq 0$. Thus there exists $w_0 \varepsilon W$ such that $Vw_0 \neq 0$. Define $f: V \longrightarrow W$ by $f(v) = vw_0$ for every $v \varepsilon V$. It is clear that f is an A-homomorphism. Suppose $\ker f \neq 0$. Then $\ker f$ is a nonzero submodule of V, and as V is rationally uniform, V is a rational extension of $\ker f$. $f \varepsilon \mathrm{Hom}_A(V,V)$ and $f|_{\ker f} = 0$, therefore, by 1.8, $f = 0$. But then $fV = Vw_0 = 0$, a contradiction. So f is an A-monomorphism, and this proves the homogeneity of V. Thus: $V \varepsilon \sum_A$.

For the annihilator $(0:V)$ of V we clearly have $(0:V)V = 0$, and as $V \neq 0$ and A is a prime ring, this implies $(0:V) = 0$. So V is a faithful module. Denote:

$$V_Q = \text{the quasi-injective hull of } V,$$
$$\Delta = \mathrm{Hom}_A(V_Q,V_Q).$$

It remained to prove that $\dim_\Delta V_Q < \infty$. Suppose the contrary, i.e. that V_Q is infinite dimensional over Δ. Then there exists an infinite sequence $v_1,v_2,\ldots,v_n,\ldots$ of Δ-linearly independent vectors in V_Q. Let

$$I_1 = (0:v_1)$$
$$I_2 = (0:v_1) \cap (0:v_2)$$
$$\vdots$$
$$I_n = (0:v_1) \cap \ldots \cap (0:v_n)$$
$$\vdots$$

Consider the decreasing sequence of left ideals

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots \quad .$$

We show that for every $n \geq 1$ there exists a nonzero left ideal J_n of A such that:

$$J_n \subset I_n \quad \text{but} \quad J_n \cap I_{n+1} = 0 \ .$$

Let n be fixed. Take a nonzero vector $w \in V$. As $v_1,\ldots,v_n,v_{n+1}$ are linearly independent over Δ, and as $w \in V$, by the Density theorem for the class $\sum$ (Theorem 2.7), there exists an element $a \in A$ and a nonzero mapping $\lambda \in Hom_A(V,V)$ such that

$$av_1 = 0,\ldots,av_n = 0, \quad av_{n+1} = \lambda w \ .$$

We claim that $0 \neq av_{n+1} \in V$. As $\lambda\colon V \longrightarrow V$ and $w \in V$, $av_{n+1} = \lambda w \in V$. By the quasi-injectivity of V_Q, λ can be extended to a mapping $\alpha \in \Delta$. Clearly $\alpha \neq 0$, and as Δ is a division ring, α is invertible. Thus, as $0 \neq w \in V$,

$$av_{n+1} = \lambda w = \alpha w \neq 0 \ .$$

We know that $Z(V) = 0$, therefore $av_{n+1} \notin Z(V)$, and $(0:av_{n+1})$ is not an essential left ideal of A. So there exists a nonzero left ideal K of A such that $(0:av_{n+1}) \cap K = 0$. Take: $J_n = Ka$. Clearly, J_n is a left ideal of A. As $K \neq 0$, there exists $0 \neq k \in K$. As $(0:av_{n+1}) \cap K = 0$, $k \notin (0:av_{n+1})$, and therefore $kav_{n+1} \neq 0$. Thus $0 \neq ka \in Ka = J_n$. So J_n is a nonzero left ideal of A. As $av_1 = 0,\ldots,av_n = 0$, we have $J_nv_1 = Kav_1 = 0,\ldots,J_nv_n = Kav_n = 0$. Thus $J_n \subset I_n$.

Let $x \in J_n \cap I_{n+1}$. Then $x \in Ka \cap (0:v_{n+1})$. So $x = ka$ for some $k \in K$, and $kav_{n+1} = 0$. Therefore $k \in (0:av_{n+1}) \cap K = 0$, and $x = ka = 0$. Thus $J_n \cap I_{n+1} = 0$.

We have an infinite sequence $J_1, J_2, \ldots, J_n, \ldots$ of nonzero left ideals of A such that: $J_n \subset I_n$, $J_n \cap I_{n+1} = 0$, for every $n \geq 1$. We show that the sum $\sum_{n=1}^{\infty} J_n$ is direct. Suppose $j_1 + \ldots + j_n = 0$, where $j_i \in J_i$ for $i = 1, \ldots, n$. Then $j_1 = (-j_2) + \ldots + (-j_n)$, $-j_2 \in J_2 \subsetneq I_2, \ldots, -j_n \in J_n \subset I_n \subset I_2$. So $j_1 = (-j_2) + \ldots + (-j_n) \in J_1 \cap I_2 = 0$, $j_1 = 0$.

Similarly, $j_2 = 0, \ldots, j_n = 0$, and the sum $\sum_{n=1}^{\infty} J_n$ is direct. By Proposition 3.3 this contradicts the fact that A satisfies the ascending chain condition on complements. Therefore V_Q is finite-dimensional over Δ . Q.E.D.

<u>3.6. Corollary</u> (Theorem of Goldie)

Let A be a prime, Goldie ring. Then A is a left order in a simple, Artinian ring.

<u>Proof</u>: By the previous theorem, there exist a module $V \in \sum_A$, which is faithful, and satisfies $\dim_\Delta V_Q < \infty$ (where V_Q is the quasi-injective hull of V, and $\Delta = \mathrm{Hom}_A(V_Q, V_Q)$.) By Theorem 2.8, A is a left order in the simple, Artinian ring $\mathrm{Hom}_\Delta(V_Q, V_Q)$. Q.E.D.

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