DENSITY THEOREMS AND APPLICATIONS

by

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One way of getting structure theorems in ring theory is to fix a general class $\sum$ of modules, and to prove Schur's Lemma and the Density Theorem for $\sum$. For example, the Goldie Theorem for prime rings follows from Schur's Lemma and the Density Theorem for the class of rationally uniform, homogeneous modules in a similar way as the Wedderburn-Artin Theorem follows from Schur's Lemma and the Density Theorem for the class of irreducible modules.
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INTRODUCTION

In this paper ring means associative ring which does not necessarily contain a unity element and is not necessarily commutative. Module means left module.

For every ring $A$ let $\Sigma_A$ denote, for the moment, the class of all irreducible $A$-modules.

The proof of the classical Wedderburn-Artin Theorem has the following main steps:

1. If $V \in \Sigma_A$, then $\Delta = \text{Hom}_A(V,V)$ is a division ring.
   (Schur's Lemma)

2. If $V \in \Sigma_A$ and $\Delta = \text{Hom}_A(V,V)$, then for any $\Delta$-linearly independent elements $v_1, \ldots, v_n \in V$ and any elements $w_1, \ldots, w_n \in V$ there exists an element $a \in A$ such that

   \[ av_1 = w_1, \ldots, av_n = w_n. \]

   This means that the ring $A_L$ of left multiplications by elements of $A$ is, in a certain sense, dense in the ring $\text{Hom}_\Delta(V,V)$ of all linear transformations on the vector space $V$.
   (Jacobson Density Theorem)

3. If $\Sigma_A$ has a member which is faithful and finite dimensional (over its centralizer), then $A$ is isomorphic to $M_n(D)$ where $D$ is a division ring and $n \geq 1$.

4. If $A$ is a simple, Artinian ring, then $\Sigma_A$ contains a member which is faithful and finite dimensional (over its centralizer).

Here, in order to get the desired structure theorem, we concentrate on a certain fixed class, $\Sigma$, of modules, and prove Schur's Lemma and
the Density Theorem for $\sum$. These enable us to describe the structure of those rings $A$ for which $\sum_A$ has a faithful, finite dimensional member. Once we know this, the structure of a ring $A$ of the class that we are interested in is easily obtained by showing that for such $A$, $\sum_A$ does have a faithful, finite dimensional member. The essential part of the structure theory is, therefore, Schur's Lemma and the Density Theorem for the class $\sum$ of modules.

The aim of this thesis is to emphasize the usefulness of concentrating on a fixed class $\sum$ of modules for which we can prove an analogue of Schur's Lemma and of the Density Theorem. We show that if we take $\sum$ to be the class of rationally uniform, homogeneous modules, then the approach described above gives Goldie's structure theorem for prime rings satisfying the ascending chain condition.

In Chapter 1 we define the notion of a general class of modules, and prove the Theorem of Andrunakievic and Rjabuhin which shows how a general class of modules defines a radical. For example, the general class of irreducible modules defines the Jacobson radical. Then we show that the class $\sum$ of rationally uniform, homogeneous modules is a general class.

In Chapter 2 we give the analogues of Schur's Lemma and the Jacobson Density Theorem for this class $\sum$, and use them to describe the structure of those rings $A$ which have in $\sum_A$ a faithful member satisfying a certain finite dimensionality condition.

In Chapter 3 we deduce from this the Goldie Theorem.

Historically, the generalizations of the Jacobson Density Theorem that will be given here, are due to Faith [2] and to Koh and Mewborn [4]. It was Heinicke [3] who pointed out that they imply the Goldie Theorem on prime rings.
CHAPTER 1
GENERAL CLASSES OF MODULES

In order to study the structure of rings it is very useful to concentrate on a certain fixed class of modules. The classical example is the class of irreducible modules which gives the Jacobson structure theory, and in particular the Wedderburn-Artin Theorem. Another class of modules, as we shall see, gives the Goldie Theorem for prime rings.

In all the paper, a homorphism which is a one-to-one mapping will be called a monomorphism, and a homomorphism which is an onto mapping will be called an epimorphism.

1.1. Radicals

**Definition:** A class $\mathcal{G}$ of rings is called a radical if

(i) for every $A \in \mathcal{G}$ and epimorphism $A \rightarrow B$ we have $B \in \mathcal{G}$.

(ii) for every ring $A$ there exists an ideal $I$ in $A$ such that $I \in \mathcal{G}$, and if $J$ is an ideal of $A$ and $J \in \mathcal{G}$ then $J \subseteq I$.

(This largest $\mathcal{G}$-ideal of $A$ is denoted by $\mathcal{G}(A)$.)

(iii) for every ring $A$, $\mathcal{G}(A/\mathcal{G}(A)) = 0$.

**Example:** The class of all nil rings is a radical.

**Definition:** Let $\mathcal{G}$ be a radical. A ring $A$ is called $\mathcal{G}$-semisimple if $\mathcal{G}(A) = 0$.

**Definition:** A class $\mathcal{M}$ of rings is called regular if for every ring $A \in \mathcal{M}$ and nonzero ideal $I$ of $A$ there exists a nonzero ring $B \in \mathcal{M}$ and an epimorphism $I \rightarrow B$. 
1.2. Theorem of Kurosh

Let $M$ be a regular class of rings. Let $U_M$ be the class of all rings which cannot be homomorphically mapped onto a nonzero member of $M$.

Then

(i) $U_M$ is a radical.

(ii) For every ring $A \in M$, $U_M(A) = 0$.

(iii) If $\beta$ is a radical such that $\beta(A) = 0$ for every $A \in M$, then every $U_M$-semisimple ring is $\beta$-semisimple.

For the proof see Divinsky [1].

1.3. General classes

For an $A$-module $V$ we denote by $(0:V)$ the annihilator of $V$, i.e.

$(0:V) = \{a \in A | aV = 0\}$. Of course, $(0:V)$ is an ideal of $A$. $V$ is called nontrivial if $AV \neq 0$, i.e. $(0:V) \neq A$. $V$ is called faithful if $(0:V) = 0$.

Suppose that to every ring $A$ there is assigned a (possibly empty) class $\Sigma_A$ of nontrivial $A$-modules. Such an assignment is called a class of modules.

Definition: A class $\Sigma$ of modules is said to be a general class provided:

(i) If $f: A \rightarrow B$ is an epimorphism and $V \in \Sigma_B$ then $V$, considered as an $A$-module in the obvious way, belongs to $\Sigma_A$.

(ii) If $f: A \rightarrow B$ is an epimorphism, $V \in \Sigma_A$ and $\ker f \subseteq (0:V)$ then $V$, considered as a $B$-module, is in $\Sigma_B$.

(iii) If $\bigcap_{V \in \Sigma_A} (0:V) = 0$ then $\sum I \neq \emptyset$ for every nonzero ideal $I$ of $A$. 
Definition: Let $\sum$ be a general class of modules. A ring $A$ is called $\sum$-primitive if $\sum_A$ contains a faithful module.

Definition: Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be rings. A ring $B$, isomorphic to a subring of the complete direct sum $\bigoplus_{\lambda \in \Lambda} A_\lambda$, is called a subdirect sum of the rings $\{A_\lambda\}$ if the restrictions of the canonical projections $\bigoplus_{\lambda \in \Lambda} A_\lambda \to A_\mu$ to $B$ are onto. This is equivalent to the following condition: There exist ideals $I_\lambda$ in $B$ such that for every $\lambda$ $B/I_\lambda$ is isomorphic to $A_\lambda$ and $\bigcap I_\lambda = 0$.

The following theorem shows that every general class of modules determines a radical.

1.4. Theorem of Andrunakievic and Rjabuhin

Let $\sum$ be a general class of modules. Denote $\sigma = \{A|\sum_A = \emptyset\}$. Then

(i) $\sigma$ is a radical.

(ii) For every ring $A$, $\sigma(A) = \bigcap_{V \in \sum_A} (0:V)$.

(iii) Every $\sigma$-semisimple ring is a subdirect sum of $\sum$-primitive rings.

Proof: Let $M$ be the class of $\sum$-primitive rings. First we observe that

(*) if $V \in \sum_A$ then $0 \neq A/(0:V) \in M$.

(As $V$ is nontrivial, $(0:V) \neq A$, so $A/(0:V) \neq 0$. Consider the canonical projection $A \to A/(0:V)$. By property (ii) of a general class $V \in \sum_A/(0:V)$ and it is clearly faithful over $A/(0:V)$. We show now that $M$ is a regular class. Let $A \in M$ and let $I$ be a nonzero ideal
of $A$. As $A$ is $\Sigma$-primitive, $\bigcap_{V \in \Sigma_A} (0:V) = 0$. So by property (iii) of a general class, $\bigcap_I \neq \emptyset$. Let $V \in \sum_I$. By (*), $I/(0:V)$ is a nonzero member of $M$, and the canonical projection maps $I$ onto $I/(0:V)$. Therefore $M$ is a regular class of rings.

Let $U_M$ be the radical determined by $M$ by the Theorem of Kurosh. We show: $\mathcal{U}_M = \sigma$.

We prove this by showing:

$A \notin \mathcal{U}_M$ if and only if $A \notin \sigma$.

Suppose $A \notin \mathcal{U}_M$. Then there exists a nonzero member $B$ of $M$ and an epimorphism $A \rightarrow B$. As $B \in M$, there exists $V \in \Sigma_B$. By property (i) of a general class, $V \in \Sigma_A$. So $A \notin \sigma$. Suppose $A \notin \sigma$. Then there exists a module $V \in \Sigma_A$. By (*), $A/(0:V)$ is a nonzero member of $M$. The canonical projection maps $A$ onto $A/(0:V)$. Thus $A \notin \mathcal{U}_M$. We have proved $\mathcal{U}_M = \sigma$, so $\sigma$ is a radical. Now we want to show that for every ring $A$, $\sigma(A) = \bigcap_{V \in \Sigma_A} (0:V)$.

Suppose $\sigma(A) \neq \bigcap_{V \in \Sigma_A} (0:V)$. Then there exists $V \in \Sigma_A$ such that $\sigma(A) \neq (0:V)$. Therefore $\frac{\sigma(A) + (0:V)}{(0:V)}$ is a nonzero ideal of $A/(0:V)$, and by (*), $A/(0:V) \in M$. As $M$ is a regular class, there exists a nonzero ring $B \in M$ and an epimorphism $\frac{\sigma(A) + (0:V)}{(0:V)} \rightarrow B$.

Thus we have epimorphisms

$$\sigma(A) \rightarrow \frac{\sigma(A)}{\sigma(A) \cap (0:V)} \rightarrow \frac{\sigma(A) + (0:V)}{(0:V)} \rightarrow B$$

and the composition is an epimorphism from $\sigma(A)$ onto $B$ which is a nonzero member of $M$. This is a contradiction because $\sigma(A) \in \sigma = U_M$. 
Therefore \( \sigma(A) \subseteq \bigcap_{V \in \sum_A} (0:V) \).

The ring \( A/\sigma(A) \) is \( \sigma \)-semisimple, so for every nonzero ideal \( I \) of \( A/\sigma(A) \), \( I \not\subseteq \sigma \), which means \( \sum_I \neq \emptyset \). By property (iv) of a general class we know:

\[
\bigcap_{V \in \sum_A/\sigma(A)} (0:V)_{A/\sigma(A)} = 0.
\]

Let \( a \in \bigcap_{V \in \sum_A} (0:V) \). Let \( V \in \sum_A/\sigma(A) \). By property (i) of a general class, if we consider \( V \) as an \( A \)-module then \( V \in \sum_A \), and therefore \( a \in (0:V)_A \). By the definition of \( V \) as an \( A \)-module,

\[
0 = av = \overline{av} \quad \text{for every } v \in V.
\]

Thus \( a \in (0:V)_{A/\sigma(A)} \). As \( V \) was any member of \( \sum_A/\sigma(A) \), we have:

\[
\overline{a} \in \bigcap_{V \in \sum_A/\sigma(A)} (0:V)_{A/\sigma(A)} = 0.
\]

Thus \( a \in \sigma(A) \). This proves:

\[
\sigma(A) = \bigcap_{V \in \sum_A} (0:V).
\]

Finally, let \( A \) be \( \sigma \)-semisimple. Then \( \sigma(A) = \bigcap_{V \in \sum_A} (0:V) = 0 \). Let \( \{I_\lambda\}_{\lambda \in \Lambda} \) be the set of all ideals of \( A \) which are the annihilator of some module in \( \sum_A \). Then clearly, \( \bigcap_{\lambda} I_\lambda = 0 \), and therefore \( A \) is a subdirect sum of the rings \( \{A/I_\lambda\} \). By (*) the rings \( A/I_\lambda \) are \( \sum \)-primitive. Q.E.D.

1.5. Example

For every ring \( A \) let \( \sum_A \) be the class of irreducible \( A \)-modules. Then \( \sum \) is a general class. (See Heinicke [3].) The radical determined by
\$ \mathfrak{J} \$ is called the Jacobson radical.

The above class of modules gives the classical Jacobson theory. We will work with another general class, the class of rationally uniform, homogeneous modules. First, the definitions and elementary properties.

1.6. Essential extensions

**Definition:** Let \( W \) be a submodule of \( V \). \( V \) is called an essential extension of \( W \) if for every nonzero submodule \( U \) of \( V \), \( W \cap U \neq 0 \).

If \( V \) is an essential extension of \( W \), we also say that \( W \) is an essential submodule of \( V \).

**Proposition**

(i) Let \( V \subseteq V' \subseteq V'' \) be \( A \)-modules. Then \( V \subseteq V'' \) is an essential extension if and only if \( V \subseteq V' \) and \( V' \subseteq V'' \) are essential extensions.

(ii) The intersection of a finite number of essential submodules is an essential submodule.

The proof of this proposition is trivial.

1.7. The singular submodule

A left ideal \( I \) in a ring \( A \) is called essential if it is an essential submodule of \( A \) when we consider \( A \) as a module over itself.

If \( V \) is an \( A \)-module and \( v \in V \) we denote: \( (0:v) = \{ a \in A | av = 0 \} \).

It is clear that \( (0:v) \) is a left ideal in \( A \).

**Proposition**

Let \( V \) be an \( A \)-module, and let \( Z(V) = \{ v \in V | (0:v) \text{ is an essential} \} \).
left ideal of $A$}. Then $Z(V)$ is a submodule of $V$.

**Proof**: If $v, w \in Z(V)$ then $(0:v), (0:w)$ are essential left ideals, so by 1.6 $(0:v) \cap (0:w)$ is essential, and as $(0:v) \cap (0:w) \subseteq (0:v+w)$, we have $v+w \in Z(V)$. It is clear that $0 \in Z(V)$ and that $v \in Z(V)$ implies $-v \in Z(V)$.

Let $a \in A$, $v \in Z(V)$, and we show: $av \in Z(V)$. Let $I$ be a nonzero left ideal of $A$. If $I \subseteq (0:av)$ then $I \cap (0:av) = I \neq 0$. If $I \not\subseteq (0:av)$, take $x \in I$ such that $xav \neq 0$. Consider the left ideal $Ia$. It is nonzero because $0 \neq xa \in Ia$. As $(0:v)$ is essential, $(0:v) \cap Ia \neq 0$. Thus there exists $y \in I$ such that $ya \neq 0$ and $yav = 0$. So we have $0 \neq y \in I \cap (0:av)$. This shows that $(0:av)$ is an essential left ideal of $A$, and $av \in Z(V)$. Q.E.D.

**Definition**: $Z(V)$ is called the singular submodule of $V$.

1.8. Rational extensions

**Definition**: Let $W$ be a submodule of $V$. $V$ is said to be a rational extension of $W$ if for every $v \in V$, $0 \neq v' \in V$ there exists an element $a \in A$ and an integer $n$ such that:

$$av + nv \in W$$

$$av' + nv' \neq 0.$$

**Proposition**

Let $W$ be a submodule of $V$. The following conditions are equivalent:

(i) $V$ is a rational extension of $W$.

(ii) If $T$ is a submodule of $V$ which contains $W$, $f \in \text{Hom}_A(T,V)$ and $f(W) = 0$ then $f = 0$. 

Proof: Let $V$ be a rational extension of $W$, $W \subseteq T \subseteq V$, $f \in \text{Hom}_A(T,V)$, $f(W) = 0$. We want to show: $f = 0$. Suppose $f \neq 0$. Then there exists $v \in T$ such that $f(v) \neq 0$. As $W \subseteq V$ is rational, there exists $a \in A$ and an integer $n$ such that

$$av + nv \in W$$

$$af(v) + nf(v) \neq 0.$$ 

But then $0 = f(av + nv) = af(v) + nf(v)$, a contradiction. Therefore $f = 0$, and (i) implies (ii).

To prove that (ii) implies (i) let $v \in V$, $0 \neq v' \in V$ be given. Suppose that for every $a \in A$ and every integer $n$:

$$av + nv \in W \text{ implies } av' + nv' = 0.$$  

Consider the module $T = W + Av + \mathbb{Z}v$. Clearly, $W \subseteq T \subseteq V$. Define $f : T \rightarrow V$ by $f(w + av + nv) = av' + nv'$ for $w \in W$, $a \in A$, $n \in \mathbb{Z}$.

By (*) $f$ is well defined, and it is clear that $f \in \text{Hom}_A(T,V)$ and $f(W) = 0$. Thus, by (ii), $f = 0$, and in particular $f(v) = 0$. But $f(v) = f(0 + 0v + 1v) = v' \neq 0$, a contradiction. Therefore $V$ is a rational extension of $W$, and (ii) implies (i). Q.E.D.

The next proposition shows the connection between essential and rational extensions.

1.9. Proposition

(i) If $W \subseteq V$ is a rational extension, then it is an essential extension.
(ii) If $W \subseteq V$ is an essential extension and $Z(V) = 0$, then $W \subseteq V$ is an rational extension.

Proof: (i) Let $W \subseteq V$ be rational, and let $U$ be a nonzero submodule of $V$. Take a nonzero element $u \in U$. As $V$ is a rational extension of $W$, there exists $a \in A$ and integer $n$ such that

$$au + nu \in W$$

$$au + nu \neq 0.$$  

Clearly,

$$0 \neq au + nu \in W \cap U.$$  

So $V$ is an essential extension of $W$. (ii) Let $W \subseteq V$ be an essential extension, and $Z(V) = 0$. Let $v \in V$, $0 \neq v' \in V$. Denote

$$I = \{a \in A | av \in W\}.$$  

Then $I$ is a left ideal in $A$. We show that $I$ is essential. Let $J$ be a nonzero left ideal of $A$. Then $Jv$ is a submodule of $V$. If $Jv = 0$, then $J \subseteq I$ and $I \cap J = J \neq 0$. If $Jv \neq 0$, then $Jv \cap W \neq 0$ because $W$ is an essential submodule of $V$. So there exists $j \in J$ such that $jv \neq 0$ and $jv \in W$. Then $0 \neq j \in I \cap J$. So $I$ is an essential left ideal of $A$. As $Z(V) = 0$ and $v' \neq 0$, we know that $(0:v')$ is not an essential left ideal in $A$, and therefore $I \not\subseteq (0:v')$. Take an element $i \in I$ such that $iv' \neq 0$. We have: $iv \in W$, $iv' \neq 0$. Thus, $V$ is a rational extension of $W$. Q.E.D.

1.10. Uniform, rationally uniform and homogeneous modules

Definition: (i) An $A$-module $V$ is said to be uniform if it is an essential extension of each of its nonzero submodules. (ii) An $A$-module $V$ is said to be rationally uniform if it is a rational extension of each
of its nonzero submodules.

It is clear that a module $V$ is uniform if and only if the intersection of any two nonzero submodules of $V$ is nonzero. Every irreducible module is rationally uniform, and by 1.9(i) every rationally uniform module is uniform. By 1.9(ii) a uniform module which has zero singular submodule, is rationally uniform.

**Definition**: An $A$-module $V$ is called **homogeneous** if for every nonzero submodule $W$ of $V$ there exists a monomorphism $f: V \rightarrow W$.

Clearly, every irreducible module is homogeneous. It is also easy to see that $\mathbb{Z}$ is a homogeneous module over itself.

Now we are ready to fix the class of modules we want to work with.

### 1.11. Theorem

For every ring $A$, let $\sum_A$ be the class of all nontrivial, rationally uniform, homogeneous $A$-modules. Then $\sum$ is a general class.

**Proof**: The first two properties in the definition of a general class of modules (see 1.3) are easily verified for $\sum$. To prove property (iii), let $A$ be a ring such that $\bigcap_{V \in \sum_A} (0:V) = 0$, and let $I$ be a nonzero ideal of $A$. We want to show: $\sum_I \neq \emptyset$. There exists an $A$-module $V \in \sum_A$ such that $IV \neq 0$. (Otherwise $I \subseteq \bigcap_{V \in \sum_A} (0:V) = 0$, a contradiction to $I \neq 0$.) Consider $V$ as an $I$-module. We show: $V \in \sum_I$. As $IV \neq 0$, $V$ is a nontrivial $I$-module. Let $N = \{v \in V | Iv = 0\}$. Clearly, $N$ is an $A$-submodule of $V$. Suppose $N \neq 0$. Then, as $V$ is a homogeneous $A$-module, there exists an $A$-monomorphism $f: V \rightarrow N$. For every $i \in I$
and \( v \in V \) we have \( f(iv) = if(v) = 0 \) because \( f(v) \in N \). As \( \ker f = 0 \), we get \( iv = 0 \), for every \( i \in I \) and \( v \in V \), a contradiction to \( IV \neq 0 \). Therefore: \( N = 0 \).

To prove that \( V \) is rationally uniform as an \( I \)-module, let \( W \) be a nonzero \( I \)-submodule of \( V \), \( T \) an \( I \)-submodule of \( V \) containing \( W \), and \( f \in \text{Hom}_I(T,V) \) such that \( f(W) = 0 \). By 1.8 it is enough to show that \( f = 0 \). It is easy to see that \( IW \) and \( IT \) are \( A \)-submodules of \( V \), and \( 0 \neq IW \subseteq IT \subseteq V \). (\( IW \neq 0 \) because \( N = \{ v \in V | Iv = 0 \} = 0 \).) As \( V \) is a rationally uniform \( A \)-module, \( IW \subseteq V \) is a rational extension. The restriction \( f \mid_{IT}: IT \rightarrow V \) is an \( A \)-homomorphism (because \( f(a \sum_k t_k) = f(\sum_k (ai_k)t_k) = \sum_k (ai_k)f(t_k) = a \sum_k f(t_k) = a f(\sum_k t_k) \)), and \( f \mid_{IT}(IW) = f(IW) = 0 \). Therefore \( f \mid_{IT} = 0 \), i.e. \( f(IT) = 0 \).

Let \( t \in T \). Then for every \( i \in I: if(t) = f(it) = 0 \). Thus \( If(t) = 0 \), so \( f(t) \in N = 0 \). Therefore \( f = 0 \), and \( V \) is rationally uniform as an \( I \)-module. To show that \( V \) is a homogeneous \( I \)-module, let \( W \) be a nonzero \( I \)-submodule of \( V \). Then again, \( IW \) is a nonzero \( A \)-submodule of \( V \). As \( V \) is homogeneous as an \( A \)-module, there exists an \( A \)-monomorphism \( f: V \rightarrow IW \).

Clearly, \( f \) is also an \( I \)-monomorphism \( f: V \rightarrow W \). Therefore \( V \) is homogeneous as an \( I \)-module, and we have \( V \in \sum_I \). This proves (iii).

To prove property (iv) of a general class for \( \sum \), suppose that \( A \) is a ring such that for every nonzero ideal \( I \) of \( A \), \( \sum_I \neq \emptyset \). We want to show \( \bigcap_{V \in \sum_A} (0:V) = 0 \). Denote \( K = \bigcap_{V \in \sum_A} (0:V) \), and suppose \( K \neq 0 \). Then \( K \) is a nonzero ideal of \( A \), and therefore \( \sum_K \neq \emptyset \). Take a \( K \)-module \( W \in \sum_K \). As \( KW \neq 0 \), there exists \( w_0 \in W \) such that \( Kw_0 \neq 0 \).

Denote: \( U = Kw_0 \). Clearly \( U \) is an abelian subgroup of \( W \). We define an \( A \)-module structure \( A \times U \rightarrow U \) on \( U \) in the following way:
For \( a \in A \), let \( kw_0 \in Kw_0 = U \) define \( a \ast kw_0 = (ak)w_0 \).

To show that \( \ast \) is well-defined, we have to prove: \( kw_0 = 0 \) implies \( (ak)w_0 = 0 \), for \( k \in K \) and \( a \in A \).

Let \( k \in K \) be fixed and suppose \( kw_0 = 0 \). We show \( (Ak)w_0 = 0 \).

Suppose the contrary. Then \( (Ak)w_0 \) is a nonzero \( K \)-submodule of \( W \), and as \( W \) is homogeneous over \( K \), there exists a \( K \)-monomorphism \( f: W \rightarrow (Ak)w_0 \). For every \( x \in K \), \( w \in W \) we have:

\[
f(xw) = x[(ak)w_0] = x[(ak)w_0] \quad \text{for some } a \in A \text{ because } f(w) \in (Ak)w_0.
\]

\[
f(xw) = x[(ak)w_0] = x[(ak)w_0] = [(xa)k]w_0 = (xa)[kw_0] = xa \cdot 0 = 0.
\]

As \( \ker f = 0 \), we get \( xw = 0 \) for every \( x \in K \), \( w \in W \), a contradiction to \( Kw_0 \neq 0 \). Therefore \( (Ak)w_0 = 0 \), and \( \ast \) is well-defined. It is easy to check that the operation \( \ast \) turns \( U \) into an \( A \)-module, and that for \( k \in K \), \( u \in U \) we have: \( k \ast u = ku \).

We show now that \( U \in \sum_A \).

Let \( M = \{w \in W| Kw = 0\} \). Then \( M \) is a \( K \)-submodule of \( W \). If \( M \neq 0 \), then as \( W \) is a homogeneous \( K \)-module, there exists a \( K \)-monomorphism \( f: W \rightarrow M \). For every \( k \in K \), \( w \in W \), \( f(kw) = kf(w) = 0 \) because \( f(w) \in M \). As \( \ker f = 0 \), we get a contradiction to \( Kw_0 \neq 0 \). Thus \( M = 0 \). As \( U = Kw_0 \neq 0 \), this implies: \( A \ast U \supseteq K \ast U = KU \neq 0 \), and so \( U \) is a nontrivial \( A \)-module.

To show that \( U \) is rationally uniform over \( A \), let \( S \) be a nonzero \( A \)-submodule of \( U \), \( u \in U \), \( 0 \neq u' \in U \). Clearly, \( S \) is a nonzero \( K \)-submodule of \( W \), therefore, as \( W \) is a rationally uniform \( K \)-module, there exists an element \( k \in K \) and an integer \( n \) such that

\[
ku + nu \in S
\]

\[
ku' + nu' \neq 0.
\]
Thus we have \( k \in A \) and integer \( n \) with
\[
\begin{align*}
k^*u + nu & \in S \\
k^*u' + nu' & \neq 0
\end{align*}
\]
So \( U \) is a rationally uniform \( A \)-module. To prove that \( U \) is a homogeneous \( A \)-module, let \( S \) be a nonzero \( A \)-submodule of \( U \). Then again, \( S \) is a nonzero \( K \)-submodule of \( W \). As \( W \) is a homogeneous \( K \)-module, there exists a \( K \)-monomorphism \( f: W \to S \). Consider the restriction \( f|_U: U \to S \). For every \( a \in A \) and \( u = kw_0 \in U \) we have:
\[
\begin{align*}
f|_U(a^*u) &= f(a^*kw_0) = f((ak)w_0) = (ak)f(w_0) = a^*(k^*f(w_0)) = a^*(kf(w_0)) = a^*f(kw_0) = a^*f(u) = a^*f|_U(u).
\end{align*}
\]
Therefore \( f|_U \) is an \( A \)-monomorphism, and this shows that \( U \) is a homogeneous \( A \)-module. We have proved: \( U \in \Sigma_A \). As \( K = \bigcap_{V \in \Sigma_A} (0:V) \), we have \( K \subseteq (0:U) \), thus \( K^*U = 0 \).

But \( K^*U = KU \), so \( KU = 0 \), in contradiction to \( M = \{w \in W | Kw = 0\} = 0 \).

Therefore \( K = \bigcap_{V \in \Sigma_A} (0:V) = 0 \). Q.E.D.

1.12. The weak radical

By the Theorem of Andrunakievic and Rjabuhin, the general class \( \Sigma \) of 1.11 defines a radical. This radical is called the weak radical and is denoted by \( W \). As every irreducible module is nontrivial, rationally uniform and homogeneous, we have \( W \subseteq J \) (where \( J \) is the Jacobson radical), and therefore for every ring \( A \), \( W(A) \subseteq J(A) \). But the two radicals are different. In fact, the following example gives a ring \( A \) which is weakly primitive (i.e. \( \Sigma \)-primitive where \( \Sigma \) is the general class of 1.11), but Jacobson-radical (i.e. \( A \in J \)).

**Example:** Let \( A = \{\frac{m}{n} | m, n \text{ integers}, m \text{ even}, n \text{ odd}\} \). It is easy to
see that $A$ is a subring of the rational numbers. Consider $A$ as a module over itself, and denote it by $\mathbb{A}_A$. Then $\mathbb{A}_A$ is a faithful member of $\bigoplus A$. Clearly, $\mathbb{A}_A$ is a nontrivial module. For $0 \neq v \in \mathbb{A}_A$, $(0:v) = \{a \in A | av = 0\} = 0$ because there are no zero divisors in $A$, and therefore $\mathbb{A}_A$ has zero singular submodule. Let $V_1, V_2$ be nonzero submodules of $\mathbb{A}_A$. As $A$ is commutative, $V_1, V_2$ are ideals in $A$, and so $V_1 \cap V_2 \supseteq V_1 V_2 \neq 0$. Thus $\mathbb{A}_A$ is uniform, and as $Z(A) = 0$, $\mathbb{A}_A$ is rationally uniform. To show the homogeneity of $\mathbb{A}_A$, let $V$ be a nonzero submodule of $\mathbb{A}_A$. Take a nonzero element $v_0 \in V$ and define $f: \mathbb{A}_A \to V$ by $f(x) = xv_0$ for every $x \in \mathbb{A}_A$. $f$ is clearly an $A$-monomorphism. Therefore $\mathbb{A}_A \in \bigoplus A$. $\mathbb{A}_A$ is a faithful module because there are no zero divisors in $A$. Thus $A$ is a weakly primitive ring. An easy computation shows that every element in $A$ is left-quasi-regular, and therefore $A$ is a Jacobson-radical ring.
CHAPTER 2
ANALOGUES OF SCHUR'S LEMMA AND THE DENSITY THEOREM

From now on \( \mathcal{A} \) will always denote the class of nontrivial, rationally uniform, homogeneous modules. We want to prove the analogues of Schur's Lemma and the Jacobson Density Theorem for \( \mathcal{A} \), and with the help of these to describe the structure of those rings \( A \) for which \( \mathcal{A}_A \) contains a faithful module satisfying a certain finite-dimensionality condition. To be able to do this, we need the notion of the quasi-injective hull of a module.

2.1. The injective hull

Definition: A module \( V \) is said to be injective if for every module \( U \), submodule \( W \) of \( U \) and homomorphism \( f \in \text{Hom}_A(W,V) \), there exists an extension \( \bar{f} \in \text{Hom}_A(U,V) \) of \( f \).

Theorem

Let \( V \) be an \( A \)-module. Then:

(i) There exists a maximal essential extension of \( V \), i.e. an essential extension \( V \subseteq M \) such that if \( V \subseteq M \subseteq E \) and \( V \subseteq E \) is essential, then \( M = E \).

(ii) If \( M_1 \) and \( M_2 \) are maximal essential extensions of \( V \), then there is an isomorphism between \( M_1 \) and \( M_2 \) which fixes every element of \( V \).

Therefore we can speak about the maximal essential extension of \( V \), which will be denoted by \( V_H \).
(iii) $V_H$ is an injective module. Moreover, $V_H$ is a minimal injective module containing $V$, i.e. if $U$ is injective and $V \subseteq U \subseteq V_H$, then $U = V_H$.

For the proof of this well-known result see e.g. Faith [2].

Definition: $V_H$ is called the injective hull of $V$.

2.2. The quasi-injective hull

The following notion gives a common generalization of irreducibility and injectivity.

Definition: A module $V$ is said to be quasi-injective if for every submodule $W$ of $V$ and homomorphism $f \in \text{Hom}_A(W,V)$ there exists an extension $\tilde{f} \in \text{Hom}_A(V,V)$ of $f$.

Theorem

Let $V$ be an $A$-module, and let $V_H$ be the injective hull of $V$. Denote: $\Gamma = \text{Hom}_A(V_H,V_H)$, and let $V_Q = TV = \{ \sum \alpha_i v_i | \alpha_i \in \Gamma, v_i \in V, \text{ and the sums are finite} \}$. Then:

(i) $V \subseteq V_Q \subseteq V_H$, and $V_Q$ is a quasi-injective module.

(ii) $V_Q$ is the smallest quasi-injective module between $V$ and $V_H$, i.e. if $V \subseteq Q \subseteq V_H$ and $Q$ is quasi-injective, then $V_Q \subseteq Q$.

For the proof see Faith [2].

Definition: $V_Q$ is called the quasi-injective hull of $V$.

First we prove an analogue of Schur's Lemma for quasi-injective modules.

Definition: A ring $A$ is called Von Neumann-regular if for every element
2.3. Theorem (Schur's Lemma for quasi-injective modules)

Let \( V \) be a quasi-injective module over \( A \), and let \( \Delta = \text{Hom}_A(V,V) \).

Then:

(i) \( J(\Delta) = \{a \in \Delta \mid \ker a \text{ is an essential submodule of } V\} \).

(ii) \( \Delta/J(\Delta) \) is Von Neumann-regular.

Proof: Denote \( N = \{a \in \Delta \mid \ker a \text{ is essential in } V\} \). Then \( N \) is a left ideal in \( \Delta \). For let \( a, \beta \in N \). Then \( \ker a, \ker \beta \) are essential in \( V \), so by 1.6 \( \ker a \cap \ker \beta \) is essential, thus \( \ker(a+\beta) \) is essential because \( \ker a \cap \ker \beta \subset \ker(a+\beta) \), and so \( a + \beta \in N \). Clearly \( 0 \in N \), and \( a \in N \) implies \( -a \in N \). If \( a \in N \) and \( \beta \in \Delta \), then \( \beta a \in N \) because \( \ker a \subset \ker \beta a \). From the Jacobson theory we know that the Jacobson radical of a ring contains every one-sided ideal which is Jacobson-radical. Thus, to show \( N \subset J(\Delta) \), it suffices to prove that \( N \) is Jacobson-radical, or that every element of \( N \) has a left quasi-inverse.

Let \( a \in N \). Clearly, \( \ker a \cap \ker(1-a) = 0 \). As \( \ker a \) is essential in \( V \), \( \ker(1-a) = 0 \). So \( 1 - a : V \to (1-a)V \) is an isomorphism. Let \( f : (1-a)V \to V \) be its inverse. As \( V \) is quasi-injective, \( f \) can be extended to a homomorphism \( \beta \in \Delta \). We have: \( \beta(1-a)v = v \) for every \( v \in V \), and so \( \beta(1-a) = 1 \). Take \( \gamma = -\alpha \beta \). It is easy to check that \( \gamma \alpha = \gamma a \), and as \( N \) is a left ideal of \( \Delta \), \( \gamma \in N \). So \( N \) is Jacobson-radical, and \( N \subset J(\Delta) \).

Now we show:

(*) for every \( a \in \Delta \) there exists \( \gamma \in \Delta \) such that \( a - a \gamma a \in N \).
Let \( \alpha \in \Delta \). Let \( W \) be a submodule of \( V \) maximal with respect to the property: \( W \cap \ker \alpha = 0 \). (Such \( W \) exists by Zorn's Lemma.) Then \( V \) is an essential extension of \( W + \ker \alpha \). (For let \( U \) be a nonzero submodule of \( V \). If \( U \subseteq W \), then \( U \cap (W + \ker \alpha) = U \neq 0 \). If \( U \not\subseteq W \), then \( W + U \not\subseteq W \), so \( (W+U) \cap \ker \alpha \neq 0 \). So there exist elements \( w \in W, u \in U \) such that \( 0 \neq w + u \in \ker \alpha \). \( u \neq 0 \), otherwise \( 0 \neq w \in W \cap \ker \alpha = 0 \). Thus, \( 0 \neq u = (-w) + (w+u) \in U \cap (W + \ker \alpha) \).

As \( W \cap \ker \alpha = 0 \), \( \alpha|_W \) is an isomorphism. Let \( f: \alpha W \rightarrow W \subseteq V \) be the inverse of \( \alpha|_W \). By the quasi-injectivity of \( V \), \( f \) can be extended to a homomorphism \( \gamma \in \Delta \). We have: \( \gamma w = w \) for every \( w \in W \). \( (\alpha - \alpha \gamma)w = \alpha w - \alpha \gamma w = \alpha w - \alpha w = 0 \) for every \( w \in W \).

Also: \( (\alpha - \alpha \gamma)v = \alpha v - \alpha \gamma v = 0 \) for every \( v \in \ker \alpha \). Thus \( W + \ker \alpha \subseteq \ker(\alpha - \alpha \gamma) \). As \( W + \ker \alpha \) is an essential submodule of \( V \), so is \( \ker(\alpha - \alpha \gamma) \), which means that \( \alpha - \alpha \gamma \in N \). This proves (*)

As \( N \subseteq J(\Delta) \), (*) implies that \( \Delta/J(\Delta) \) is a Von Neumann-regular ring.

It remained to show: \( J(\Delta) \subseteq N \). Let \( \alpha \in J(\Delta) \). By (*), there exists \( \gamma \in \Delta \) such that \( \alpha - \alpha \gamma \in N \). \( J(\Delta) \) is an ideal in \( \Delta \), so \( \alpha \gamma \in J(\Delta) \).

Thus \( \alpha \gamma \) has a quasi-inverse \( \beta \in J(\Delta) \): \( \beta + \alpha \gamma = \beta \alpha \gamma \). We have:

\[
\beta(\alpha - \alpha \gamma) = \beta \alpha - \beta \alpha \gamma = \beta \alpha - (\beta + \alpha \gamma) \alpha = \beta \alpha - \beta \alpha - \alpha \gamma = -\alpha \gamma.
\]

As \( \alpha - \alpha \gamma \in N \) and \( N \) is a left ideal in \( \Delta \), we have \( -\alpha \gamma = \beta(\alpha - \alpha \gamma) \in N \). But also \( \alpha - \alpha \gamma \in N \), and so \( \alpha \in N \). Thus \( J(\Delta) \subseteq N \). Q.E.D.

With the help of this theorem, we are able to prove now the analogue of Schur's Lemma for our class \( \sum \).

**Definition:** An element \( \alpha \) of a ring \( A \) is called **regular** if it is neither a left, nor a right zero divisor in \( A \), i.e. \( x \in A, \; ax = 0 \)
implies \( x = 0 \), and also \( x \in A, \ xa = 0 \) implies \( x = 0 \).

**Definition:** Let \( B \) be a ring with unity, \( A \) a subring of \( B \). (This of course, does not mean that \( A \) contains the unity element of \( B \).) \( A \) is said to be a **right order** in \( B \) if:

(i) Every element of \( A \) which is regular in \( A \) is invertible in \( B \).

(ii) Every element \( b \in B \) has the form \( b = a_1 a_2^{-1} \) where \( a_1, a_2 \in A \), \( a_2 \) regular.

The definition of a left order is analogous.

2.4. Theorem (Schur's Lemma for rationally uniform, homogeneous modules)

Let \( A \) be a ring and \( V \in \sum_A \). Denote:

\[
V_Q = \text{the quasi-injective hull of } V.
\]
\[
\Delta = \text{Hom}_A(V_Q, V_Q).
\]
\[
\Omega = \text{Hom}_A(V, V).
\]

Then:

(i) \( V_Q = \Delta V \) (where \( \Delta V = \{ \sum a_i v_i | a_i \in \Delta, \ v_i \in V \} \)).

(ii) \( \Delta \) is a division ring.

(iii) \( \Omega \) is a right order in \( \Delta \).

(iv) \( V_Q \) is a rationally uniform module.

**Remark:** The homogeneity of \( V \) is needed only for the proof of (iii).

**Proof:** (i) From Theorem 2.2 we know that \( V_Q = \Gamma V \) where \( \Gamma = \text{Hom}_A(V_H, V_H) \) and \( V_H \) is the injective hull of \( V \). Let \( \alpha \in \Gamma \) and \( v \in V_Q \). Then \( v \) has the form \( v = \sum_{i=1}^{\infty} \beta_i v_i \), with \( \beta_i \in \Gamma, \ v_i \in V \). Thus,
\[ av = \alpha \left( \sum_{i=1}^{n} \beta_i v_i \right) = \sum_{i=1}^{n} \alpha \beta_i v_i \in \Gamma V = V_Q. \]

Therefore \( a_{\mid V_Q} \in \Delta \) for every \( \alpha \in \Gamma \). Let \( v \in V_Q \). As we know, \( v \) has the form \( v = \sum_{i=1}^{n} \alpha_i v_i \) where \( \alpha_i \in \Gamma \) and \( v_i \in V \). Thus,

\[ v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i \mid V_Q v_i \in \Delta V. \]

So \( V_Q \subset \Delta V \). But clearly \( \Delta V \subset V_Q \), so \( V_Q = \Delta V \).

(ii) First we prove the following:

\((*)\) If \( \alpha \in \Delta \) and \( \ker \alpha \neq 0 \), then \( \alpha V = 0 \).

Let \( \alpha \in \Delta \) and \( \ker \alpha \neq 0 \). As \( V_H \) is an essential extension of \( V \) (by 2.1), so is \( V_Q \). Thus, \( V \cap \ker \alpha \neq 0 \). Let \( W = \{ v \in V \mid \alpha v \in V \} \).

Then \( W \) is a submodule of \( V \), and clearly: \( 0 \neq V \cap \ker \alpha \subset W \subset V \).

\( V \) is rationally uniform, therefore \( V \) is a rational extension of \( V \cap \ker \alpha \). The mapping \( \alpha \mid W : W \rightarrow V \) is zero on \( V \cap \ker \alpha \), thus by Proposition 1.8, \( \alpha W = 0 \). This implies \( V \cap \alpha V = 0 \), and as \( V_Q \) is an essential extension of \( V \), we must have \( \alpha V = 0 \).

As \( V_Q \) is quasi-injective, by Theorem 2.3 we know:

\( J(\Delta) = \{ \alpha \in \Delta \mid \ker \alpha \text{ is essential in } V_Q \} \), and \( \Delta / J(\Delta) \) is Von Neumann-regular. We show now that \( J(\Delta) = 0 \).

Let \( \alpha \in J(\Delta) \). Let \( v \in V_Q \). By (i), \( v \) has the form \( v = \sum_{i=1}^{n} \beta_i v_i \), where \( \beta_i \in \Delta \), \( v_i \in V \). As \( \alpha \in J(\Delta) \) and \( \beta_i \in \Delta \), we have \( \alpha \beta_i \in J(\Delta) \), for \( i = 1, \ldots, n \). Thus \( \ker(\alpha \beta_i) \) is an essential submodule of \( V_Q \), and in particular \( \ker(\alpha \beta_i) \neq 0 \) for \( i = 1, \ldots, n \). By \((*)\), \( \alpha \beta_i V = 0 \) for \( i = 1, \ldots, n \). Thus:

\[ \alpha v = \alpha \left( \sum_{i=1}^{n} \beta_i v_i \right) = \sum_{i=1}^{n} \alpha \beta_i v_i = 0. \]

Therefore \( \alpha = 0. \)
and we have $J(\Delta) = 0$. So $\Delta$ is a Von Neumann-regular ring. To show that $\Delta$ is a division ring, let $0 \neq \alpha \in \Delta$. As $\Delta$ is Von Neumann-regular, there exists $\gamma \in \Delta$ such that $\alpha = \alpha \gamma \alpha$. So $\alpha(1 - \gamma) = 0$.

Suppose $\ker \alpha \neq 0$. Then by (*), $V \subseteq \ker \alpha \subseteq V_Q$. As $V$ is an essential submodule of $V_Q$, so is $\ker \alpha$. But then $\alpha \in J(\Delta) = 0$, a contradiction. Therefore $\ker \alpha = 0$. As $\alpha(1 - \gamma) = 0$, this implies $1 - \gamma = 0$. Thus $\gamma = 1$, which proves that $\Delta$ is a division ring.

(iii) First we show that $\Omega$ can be considered a subring of $\Delta$.

Let $\alpha \in \Omega$. By the quasi-injectivity of $V_Q$, $\alpha$ can be extended to a homomorphism in $\Delta$. We show that this extension is unique. Suppose $f, g \in \Delta$, $f|_V = g|_V$. Then $0 \neq V \subseteq \ker(f - g)$, and as $\Delta$ is a division ring, $f - g = 0$. For every $\alpha \in \Omega$, denote by $\bar{\alpha}$ the (unique) extension of $\alpha$ to $V_Q$. Then $\alpha \mapsto \bar{\alpha}$ is a mapping $\Omega \rightarrow \Delta$. Using the uniqueness of the extension, it is easy to check that this mapping is a monomorphism of rings. Therefore $\Omega$ can be considered a subring of $\Delta$.

We want to prove that $\Omega$ is a right order in $\Delta$. As $\Delta$ is a division ring, the first requirement in the definition of a right order is trivially satisfied. Let $\lambda$ be a nonzero element of $\Delta$. Then $\lambda V \neq 0$ (because $\lambda$ is invertible), therefore, as $V_Q$ is an essential extension of $V$, $V \cap \lambda V \neq 0$. Thus $W = \{v \in V | \lambda v \in V\}$ is a nonzero submodule of $V$. By the homogeneity of $V$, there exists a monomorphism $f: V \rightarrow W$. Clearly, $\lambda|_W f \in \Omega$. For every $v \in V$ we have: $(\lambda|_W f)(v) = \lambda|_W(f(v)) = \lambda(f(v))$, and $(\lambda f)|_W = \lambda(f)(v) = \lambda(f(v))$.

So $(\lambda|_W f)$ and $\lambda f$ are equal on $V$, and as $\Delta$ is a division ring, this implies $(\lambda|_W f) = \lambda f$. Clearly, $f \neq 0$ because $f$ is one-to-one. Thus $f$ is invertible, and we have: $\lambda = (\lambda|_W f)(f)^{-1}$, the required representation
of $\lambda$. This proves that $\Omega$ is a right order in $\Delta$.

(iv) To prove that $V_Q$ is itself rationally uniform, let $0 \neq W \subseteq T \subseteq V_Q$ submodules, and let $f \in \text{Hom}_A(T,V_Q)$ such that $fW = 0$. As $V_Q$ is quasi-injective, there exists an extension $\tilde{f} \in \Delta$ of $f$. As $\Delta$ is a division ring and $0 \neq W \subseteq \ker \tilde{f}$, we get $\tilde{f} = 0$, and so $f = 0$. Thus $V_Q$ is a rationally uniform module. Q.E.D.

The following density theorem for quasi-injective modules is a generalization of the Jacobson Density Theorem.

**Definition**: Let $\Delta$ be a ring and $V$ a $\Delta$-module. The elements $v_1, \ldots, v_n \in V$ are said to be **linearly independent** over $\Delta$ if none of them is a $\Delta$-linear combination of the others, i.e. if $v_i \notin \sum_{j \neq i} \Delta v_j$ for every $i = 1, \ldots, n$. It is clear that when $\Delta$ is a division ring and $V$ a vector space over $\Delta$, the above definition is equivalent to the usual one.

**2.5. Theorem** (Density theorem for quasi-injective modules)

Let $V$ be a quasi-injective $A$-module which satisfies:

\[ (*) \quad \forall v \in V, \quad Av = 0 \quad \text{implies} \quad v = 0 . \]

Denote: $\Delta = \text{Hom}_A(V,V)$.

Then for every $\Delta$-linearly independent elements $v_1, \ldots, v_n \in V$ there exists an element $a \in A$ such that

$$av_1 \neq 0, \quad av_2 = \ldots = av_n = 0 .$$

**Proof**: The proof is by induction on $n$. For $n = 1$ the result is just the given condition $(*).$
We prove the result for \( n = 2 \). Let \( v_1, v_2 \in V \) be independent over \( \Delta \).

Suppose that there exists no \( a \in A \) such that \( av_1 \neq 0, av_2 = 0 \), i.e. we have:

\[
(**) \quad a \in A, \quad av_2 = 0 \implies av_1 = 0.
\]

Consider the submodule \( Av_2 \) of \( V \) and define \( f: Av_2 \to V \) by \( f(av_2) = av_1 \). Because of \((**)\) \( f \) is well-defined, and it is clearly an \( A \)-homomorphism. As \( V \) is quasi-injective, there exists \( a \in \Delta \) with \( a|_{Av_2} = f \). Then for every \( a \in A \) we have:

\[
\alpha(av_2) = f(av_2) = av_1 \quad \text{and} \quad \alpha(av_2) = aa(v_2) = a(\alpha v_2).
\]

So \( \alpha(v_1 - \alpha v_2) = av_1 - a(\alpha v_2) = 0 \) for every \( a \in A \), i.e. \( A(v_1 - \alpha v_2) = 0 \).

By \((*)\), \( v_1 - \alpha v_2 = 0, v_1 = \alpha v_2 \) in contradiction to the \( \Delta \)-independence of \( v_1, v_2 \). This establishes the result for \( n = 2 \). Now suppose it is true for \( n \), and we will prove it for \( n + 1 \). Let \( v_0, v_1, \ldots, v_n \in V \) be independent over \( \Delta \) (\( n \geq 2 \)). Suppose that there exists no \( a \in A \) such that \( av_0 \neq 0, av_1 = \ldots = av_n = 0 \), i.e. we have:

\[
(***) \quad a \in A, \quad av_1 = \ldots = av_n = 0 \implies av_0 = 0.
\]

Consider \( A \) as a module over itself, and define the maps \( f, g_1, \ldots, g_n : A \to V \) by:

\[
f(a) = av_0, \quad a \in A
\]

\[
g_1(a) = av_1, \quad a \in A
\]

\[
\vdots
\]

\[
g_n(a) = av_n, \quad a \in A.
\]
$f, g_1, \ldots, g_n$ are clearly $A$-homomorphisms. Consider the direct sum of $n$ copies of $V$, and for $i = 1, \ldots, n$ let $e_i : V \to V \oplus \cdots \oplus V$ be the embedding of $V$ on the $i$-th component of $V \oplus \cdots \oplus V$. Let

$$g = \sum_{i=1}^{n} e_i g_i,$$

i.e. $g(a) = (g_1(a), \ldots, g_n(a))$ for every $a \in A$.
From (***), we know that \( \ker g \subseteq \ker f \). Therefore there exists a homomorphism \( h : gA \rightarrow V \) such that \( hg = f \).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & V \\
 & \searrow \downarrow h & \\
 & gA & \\
\end{array}
\]

For \( i = 1, \ldots, n \) let \( K_i = \bigcap_{j \neq i} \ker g_j \), and let \( K = K_1 + \ldots + K_n \).

Clearly, \( K \) is a submodule of \( A \). For \( i = 1, \ldots, n \) we have:

\[
e_{i}g_{i}K \subset gK \quad \text{(For let } a \in K \text{. Then } a = a_1 + \ldots + a_n \text{ where } a_j \in K_j, \ j = 1, \ldots, n, \text{ and so } e_{i}g_{i}(a) = e_{i}(g_{i}(a_1 + \ldots + a_n)) = e_{i}(g_{i}(a_1)) + \ldots + e_{i}(g_{i}(a_n)) = e_{i}(g_{i}(a_1)) = (0, \ldots, g_{i}(a_1), \ldots, 0) = g(a_1) \in gK \text{.) For fixed } i \text{ consider the homomorphisms}
\]

\[
g_{i}K \xrightarrow{e_{i}|g_{i}K} gK \xrightarrow{h|gK} V.
\]

As \( V \) is quasi-injective, the mapping \((h|gK)(e_{i}|g_{i}K) : g_{i}K \rightarrow V\) can be extended to a homomorphism \( \alpha_{i} \in \Delta \). For every \( a \in K \) and \( i = 1, \ldots, n \) we have: \( \alpha_{i}g_{i}(a) = he_{i}g_{i}(a) \), and therefore:

\[
\sum_{i=1}^{n} \alpha_{i}g_{i}(a) = \sum_{i=1}^{n} he_{i}g_{i}(a) = h(\sum_{i=1}^{n} e_{i}g_{i}(a)) = hg(a) = f(a), \text{ for every } a \in K.
\]

In particular, for every \( a \in K_1 = \bigcap_{j=2}^{n} \ker g_j \):

\[
f(a) = \sum_{i=1}^{n} \alpha_{i}g_{i}(a) = \alpha_{1}g_{1}(a), \text{ and using the definition of } f \text{ and } g_1, \text{ we get: } av_0 = \alpha_{1}(av_1) = a(\alpha_1 v_1), \ a(v_0 - \alpha_1 v_1) = 0, \text{ for every } a \in \bigcap_{j=2}^{n} \ker g_j.
\]

This means:
a ∈ A, \( av_1 = \ldots = av_n = 0 \) implies \( a(v_0 - a_1 v_1) = 0 \). Therefore, by the induction hypothesis, the elements \( v_0 - a_1 v_1, v_2, \ldots, v_n \) are linearly dependent over \( A \). But from this it follows easily that the elements \( v_0, v_1, \ldots, v_n \) are also linearly dependent over \( A \), a contradiction. Q.E.D.

The Jacobson Density Theorem follows quickly from the above theorem by observing that an irreducible module \( V \) is quasi-injective, and satisfies the condition (*) (because \( \{ v ∈ V | Av = 0 \} \) is a submodule of \( V \)). (Let \( v_1, \ldots, v_n ∈ V \) independent over \( A \), and \( w_1, \ldots, w_n ∈ V \) arbitrary. By the above theorem there exist \( a_1, \ldots, a_n ∈ A \) such that \( a_i v_i ≠ 0 \) for \( i = 1, \ldots, n \), and \( a_i v_i = 0 \) for \( i ≠ j \). As \( V \) is irreducible, \( A a_i v_i = v_i \) for \( i = 1, \ldots, n \). Therefore there exist \( b_i ∈ A \) such that \( b_i a_i v_i = w_i \), \( i = 1, \ldots, n \). Take \( a = \sum_{j=1}^{n} b_j a_j \). Then clearly: \( a v_i = w_i , i = 1, \ldots, n ) \).

The following density theorem will give, as a corollary, the density theorem for our class of modules, \( \sum \).

2.6. Theorem (Density theorem)

Let \( V \) be an \( A \)-module, \( E \) an extension of \( V \), and suppose that the following properties are satisfied:

(i) \( V ≠ 0 \), \( V \) is homogeneous.

(ii) \( E \) is quasi-injective, uniform, and:

\( (*) \quad v ∈ E , \quad Av = 0 \quad \text{implies} \quad v = 0 . \)

Denote:

\( \Delta = \text{Hom}_A(E,E) \)

\( \Omega = \text{Hom}_A(V,V) \).
Then for any $A$-linearly independent elements $v_1, \ldots, v_n \in E$ and any elements $w_1, \ldots, w_n \in V$ there exists an element $a \in A$ and a nonzero mapping $\lambda \in \Omega$ such that

$$av_1 = \lambda w_1, \ldots, av_n = \lambda w_n.$$ 

Proof: Let the $A$-linearly independent elements $v_1, \ldots, v_n \in E$ and the elements $w_1, \ldots, w_n \in V$ be given. Denote for $i = 1, \ldots, n$:

$$A_i = \{a \in A | av_j = 0 \text{ for every } j \neq i\}.$$ 

As $E$ is quasi-injective and satisfies (*), by 2.5 $A_i v_i \neq 0$ for every $i$. Clearly, $A_1, \ldots, A_n$ are left ideals in $A$, so $A_i v_i, \ldots, A_n v_n$ are nonzero submodules of $E$. As $E$ is uniform, and each of the submodules $V, A_1 v_1, \ldots, A_n v_n$ is nonzero, we have $V \cap (\bigcap_{j=1}^{n} A_j v_j) \neq 0$. As $V$ is homogeneous, there exists a monomorphism

$$\lambda : V \rightarrow V \cap (\bigcap_{j=1}^{n} A_j v_j).$$

Clearly, $0 \neq \lambda \in \Omega$. For $i$ fixed, as $\lambda w_i \in A_i v_i$, there exists $a_i \in A_i$ such that $\lambda w_i = a_i v_i$.

Take $a = a_1 + \ldots + a_n$. Then for every $i = 1, \ldots, n$ we have

$$av_i = (\sum_{j=1}^{n} a_j) v_i = \sum_{j=1}^{n} a_j v_i = a_i v_i = \lambda w_i.$$ 

Thus $av_i = \lambda w_i$, for $i = 1, \ldots, n$. Q.E.D.

Now we are able to state the density theorem for the class $\sum_A$.

2.7. Theorem (Density theorem for rationally uniform, homogeneous modules)

Let $A$ be a ring and $V \in \sum_A$. Denote:
$V_Q$ = the quasi-injective hull of $V$,
\[\Delta = \text{Hom}_A(V_Q,V_Q),\]
\[\Omega = \text{Hom}_A(V,V).\]

Then for any $\Delta$-linearly independent vectors $v_1,\ldots,v_n \in V_Q$ and any vectors $w_1,\ldots,w_n \in V$ there exists an element $a \in A$ and a nonzero mapping $\lambda \in \Omega$ such that:

$$av_1 = \lambda w_1, \ldots, av_n = \lambda w_n.$$

**Remark:** It is justified to use the term "vector" because $\Delta$ is a division ring by Theorem 2.4(ii).

**Proof:** To be able to apply Theorem 2.6, we have to know:

(i) $V \neq 0$, $V$ is homogeneous,

(ii) $V_Q$ is quasi-injective, uniform, and satisfies:

\[(*) \quad v \in V_Q, \ Av = 0 \text{ implies } v = 0.\]

As $V \in \bigcap_A$, we know that $V$ is nonzero and homogeneous. $V_Q$ is, of course, quasi-injective, and by Theorem 2.4(iv) $V_Q$ is rationally uniform, and therefore uniform by 1.10. The only thing to be checked is that $V_Q$ satisfies $(*)$.

Let $W = \{v \in V_Q | Av = 0\}$. Then $W$ is clearly a submodule of $V_Q$.

Suppose $W \neq 0$. Then, as $V_Q$ is an essential extension of $V$, $V \cap W \neq 0$. By the homogeneity of $V$ there exists a monomorphism $f: V \rightarrow V \cap W$. For every $a \in A$ and $v \in V$ we have: $f(av) = af(v) = 0$ because $f(v) \in W$. As $\ker f = 0$, we get $av = 0$ for every $a \in A$ and $v \in V$, a contradiction because $V$ is a nontrivial $A$-module.

Therefore $W = 0$, i.e. $V_Q$ satisfies $(*)$. Thus we can apply Theorem 2.6
and get the result. Q.E.D.

Now, as we have in our hands Schur's Lemma and the Density Theorem for the class $\sum$, we are able to describe the structure of those rings $A$ for which $\sum_A$ has a faithful member satisfying a certain condition of finite dimensionality.

2.8. Theorem

Let $A$ be a ring, $V \in \sum_A$ faithful. Denote:

$V_Q = \text{the quasi-injective hull of } V$,

$A = \text{Hom}_A(V_Q, V_Q)$.

Suppose that $V_Q$ is finite dimensional over $A$.

Then $A$ is a left order in the simple, Artinian ring $\text{Hom}_A(V_Q, V_Q)$.

Proof: We know that $A$ is a division ring by Schur's Lemma for the class $\sum$ (Theorem 2.4(ii)). $\text{Hom}_A(V_Q, V_Q)$ is, of course, simple and Artinian, being the ring of linear transformations of a finite dimensional vector space. As $V$ is a faithful $A$-module, $V_Q$ is also faithful. Therefore $A$ can be considered a subring of $\text{Hom}_A(V_Q, V_Q)$, by considering an element $a \in A$ to be the linear transformation $a_L: V_Q \rightarrow V_Q$ defined by $a_L(v) = av$ for $v \in V_Q$. We want to prove that $A$ is a left order in $\text{Hom}_A(V_Q, V_Q)$. Denote by $n$ the dimension of $V_Q$ over $A$. By Theorem 2.4(i) $V_Q = A^\times$, so we can choose a basis $v_1, \ldots, v_n$ of $V_Q$ over $A$ such that $v_1, \ldots, v_n \in V$. (For let $u_1, \ldots, u_n$ be a basis of $V_Q$. As $V_Q = A^\times$, each $u_i$ is a finite linear combination of elements $v_{i1}, \ldots, v_{im_i} \in V$. Clearly the vectors $v_{i1}, \ldots, v_{im_i}, \ldots, v_{nl}, \ldots, v_{nm_n} \in V$ generate $V_Q$, thus we can choose a subset of them which is a basis of $V_Q$.)
Now we show that the singular submodule of $V_Q$ is zero. Suppose the contrary, and let $0 \neq v \in Z(V_Q)$. Let $i$ be fixed. By the Density theorem for $\sum (\text{Theorem 2.7})$, there exists $a_i \in A$ and $0 \neq \lambda_i \in \text{Hom}_A(V,V)$ such that $a_i v = \lambda_i v_i$. As $V_Q$ is quasi-injective, $\lambda_i$ can be extended to a mapping $\alpha_i \in \Delta$. We have:

$$a_i v = \alpha_i v_i \quad \text{for } i = 1, \ldots, n.$$ 

$Z(V_Q)$ is a submodule of $V_Q$, thus $a_1 v, \ldots, a_n v \in Z(V_Q)$, and therefore $I_i = (0:a_i v), \ i = 1, \ldots, n$, are essential left ideals in $A$. We have:

$$0 = I_i a_i v = I_i \alpha_i v_i = a_i I_i v_i \quad \text{for } i = 1, \ldots, n.$$ 

Thus $I_i v_i \in \ker \alpha_i, \ i = 1, \ldots, n$. As $\Delta$ is a division ring, and $0 \neq \alpha_i \in \Delta$, we get $I_i v_i = 0$, for every $i = 1, \ldots, n$. $I_1, \ldots, I_n$ are essential left ideals of $A$, therefore $\bigcap_{i=1}^n I_i$ is essential, and in particular $\bigcap_{i=1}^n I_i \neq 0$. Take $0 \neq a \in \bigcap_{i=1}^n I_i$. Then

$$a v_i = 0 \quad \text{for } i = 1, \ldots, n.$$ 

As $v_1, \ldots, v_n$ is a basis of $V_Q$, this implies $a V_Q = 0$. But then $a = 0$ because $V_Q$ is faithful. This contradiction proves: $Z(V_Q) = 0$.

Next we show that if $I$ is an essential left ideal of $A$, then $I$ contains an element which is invertible in $\text{Hom}_A(V_Q,V_Q)$. Consider $V$ and $V_Q$ as $I$-modules. We will show that they satisfy the conditions of Theorem 2.6 (over the ring $I$). We know that $V \neq 0$. To show that $V$ is homogeneous over $I$, let $W$ be a nonzero $I$-submodule of $V$. Then $I W \subseteq W$, $I W$ is an $A$-submodule of $V$, and $I W \neq 0$. (For let $0 \neq w \in W$. As $Z(V_Q) = 0$, $w \notin Z(V_Q)$, so $(0:w)$ is not essential in
A. Therefore, as I is essential, $I \neq (0; w)$. Thus $Iw \neq 0$, and so $IW \neq 0$. $V$ is a homogeneous A-module, therefore there exists an A-monomorphism $f: V \rightarrow IW$. Clearly, $f$ is also an I-monomorphism from $V$ into $W$. Thus $V$ is a homogeneous I-module.

We show now that $V_Q$ is quasi-injective as an I-module. Let $W$ be an I-submodule of $V_Q$ and $f \in \text{Hom}_I(W, V_Q)$. Then $IW \subseteq W$ and $IW$ is an A-submodule of $V_Q$. Furthermore, $f|_{IW} \in \text{Hom}_A(IW, V_Q)$. (Because for $a \in A$, $\sum_{k=1}^n i_kw_k \in IW$ we have: $f(a \sum_{k=1}^n i_kw_k) = f(\sum (ai_k)w_k) = \sum (ai_k)f(w_k) = a \sum_{k=1}^n i_kf(w_k) = af(\sum_{k=1}^n i_kw_k)$.) As $V_Q$ is quasi-injective as an A-module, there exists $g \in \text{Hom}_A(V_Q, V_Q)$ such that $g|_{IW} = f|_{IW}$. Clearly, $g \in \text{Hom}_I(V_Q, V_Q)$. Let $i \in I$, $w \in W$. Then:

$$i(f(w) - g(w)) = if(w) - ig(w) = f(iw) - g(iw) = 0.$$  

Thus $I(f(w) - g(w)) = 0$. As $I$ is an essential left ideal of $A$, this implies $f(w) - g(w) \in Z(V_Q) = 0$, and so $f(w) = g(w)$, for every $w \in W$, i.e. $g|_W = f$. Therefore $V_Q$ is a quasi-injective I-module.

To prove that $V_Q$ is a uniform I-module, let $W_1, W_2$ be nonzero I-submodules of $V_Q$. Then $IW_1, IW_2$ are nonzero A-submodules of $V_Q$. (Again, they are nonzero because $I$ is essential and $Z(V_Q) = 0$.) By Theorem 2.4(iv) $V_Q$ is rationally uniform as an A-module, so $V_Q$ is a uniform A-module. Thus $IW_1 \cap IW_2 \neq 0$. As $IW_1 \subseteq W_1$ and $IW_2 \subseteq W_2$, we have $W_1 \cap W_2 \neq 0$, and therefore $V_Q$ is a uniform I-module.

To be able to use Theorem 2.6, it remained to check that if $v \in V_Q$ and $Iv = 0$, then $v = 0$. But this is clear, because $Iv = 0$ implies, as $I$ is essential, that $v \in Z(V_Q) = 0$. Thus the I-modules $V \subseteq V_Q$ satisfy the requirements of Theorem 2.6. We show now that
Hom_I(V_Q, V_Q) = \Delta .

Clearly, \( \Delta \subseteq \text{Hom}_I(V_Q, V_Q) \). To show the opposite inclusion, let \( f \in \text{Hom}_I(V_Q, V_Q) \). \( IV_Q \) is an A-submodule of \( V_Q \), and \( f \big|_{IV_Q} : IV_Q \rightarrow V_Q \) is an A-homomorphism. As \( V_Q \) is quasi-injective as an A-module, there exists a mapping \( \alpha \in \Delta \) such that \( \alpha \big|_{IV_Q} = f \big|_{IV_Q} \). For every \( i \in I \) and \( v \in V_Q \) we have: \( i(f(v) - \alpha(v)) = if(v) - i\alpha(v) = f(iv) - \alpha(iv) = 0 \). So for every \( v \in V_Q \), \( I(f(v) - \alpha(v)) = 0 \), and therefore, as \( I \) is essential, \( f(v) - \alpha(v) \in Z(V_Q) = 0 \). Thus \( f = \alpha \in \Delta \). This shows: \( \text{Hom}_I(V_Q, V_Q) = \Delta \). Therefore \( v_1, \ldots, v_n \) are linearly independent over \( \text{Hom}_I(V_Q, V_Q) \). By Theorem 2.6 there exists an element \( e \in I \) and a nonzero mapping \( \lambda \in \text{Hom}_I(V, V) \) such that

\[ ev_1 = \lambda v_1, \ldots, ev_n = \lambda v_n . \]

As \( V_Q \) is quasi-injective as an I-module, \( \lambda \) can be extended to a mapping \( \gamma \in \text{Hom}_I(V_Q, V_Q) = \Delta \). Thus we have \( e \in I \) and \( 0 \neq \gamma \in \Delta \) such that \( ev_1 = \gamma v_1, \ldots, ev_n = \gamma v_n \).

The vectors \( \gamma v_1, \ldots, \gamma v_n \) are, clearly, linearly independent over \( \Delta \), and as \( \dim_{\Delta} V_Q = n \), \( \gamma v_1, \ldots, \gamma v_n \) is a basis of \( V_Q \) over \( \Delta \). Therefore \( e \), considered as a linear transformation on the vector space \( V_Q \), is invertible. So \( I \) contains an element which is invertible in \( \text{Hom}_\Delta(V_Q, V_Q) \).

Now we can prove that every element \( \phi \in \text{Hom}_\Delta(V_Q, V_Q) \) is of the form \( \phi = a^{-1}b \) with \( a, b \in A \). Let \( \phi \in \text{Hom}_\Delta(V_Q, V_Q) \). Denote

\[ A_i = \{ a \in A | av_j = 0 \text{ for every } j \neq i \} \text{ for } i = 1, \ldots, n . \]

Clearly, \( A_1, \ldots, A_n \) are left ideals of \( A \). We know that if \( v \in V_Q \) and \( Av = 0 \), then \( v = 0 \) (because \( Z(V_Q) = 0 \)). Therefore by the Density
theorem for quasi-injective modules (Theorem 2.5), $A_i v_i \neq 0$ for every $i = 1, \ldots, n$. Thus $A_1 v_1, \ldots, A_n v_n$ are nonzero submodules of $V_Q$, and as $V_Q$ is uniform (since it is rationally uniform by 2.4(iv)), $A_1 v_1, \ldots, A_n v_n$ are essential submodules of $V_Q$. Let $X_i = \{ x \in A | x \phi(v_i) \in A_i v_i \}$, for $i = 1, \ldots, n$. Then $X_1, \ldots, X_n$ are essential left ideals of $A$. (For let $J$ be a nonzero left ideal of $A$. If $J \phi(v_i) = 0$, then $J \subseteq X_i$ and so $J \cap X_i = J \neq 0$. If $J \phi(v_i) \neq 0$, then $J \phi(v_i)$ is a nonzero submodule of $V_Q$, and as $A_i v_i$ is essential in $V_Q$, $J \phi(v_i) \cap A_i v_i \neq 0$. So there exists $j \in J$ such that $0 \neq j \phi(v_i) \in A_i v_i$. We have $0 \neq j \in J \cap X_i$. Thus $X_i$ is an essential left ideal of $A$.) Take $X = \bigcap_{i=1}^{n} X_i$. Then $X$ is an essential left ideal of $A$. Therefore there exists an element $a \in X$ which is invertible in $\text{Hom}_A(V_Q, V_Q)$. As $a \in X = \bigcap_{i=1}^{n} X_i$, we have:

$$a \phi(v_i) \in A_i v_i, \quad \text{for } i = 1, \ldots, n.$$ 

So there exist elements $b_i \in A_i$, $i = 1, \ldots, n$, such that

$$a \phi(v_i) = b_i v_i, \quad \text{for } i = 1, \ldots, n.$$ 

Take $b = b_1 + \ldots + b_n$. Then for every $i = 1, \ldots, n$:

$$b v_i = \left( \sum_{j=1}^{n} b_j \right) v_i = \sum_{j=1}^{n} b_j v_i = b_i v_i = a \phi(v_i).$$

Thus:

$$a \phi(v_1) = b v_1$$
$$\vdots$$
$$a \phi(v_n) = b v_n.$$ 

So the linear transformations $a \phi$ and $b$ act equally on the basis
v_1, \ldots, v_n \text{ of } V_Q. \text{ Therefore } a\varphi = b, \text{ and as } a \text{ is invertible in } \text{Hom}_\Delta(V_Q, V_Q), \text{ we have:}
\varphi = a^{-1}b, \quad a, b \in A.

Finally, we have to prove that every element of A which is regular in A, is invertible in \text{Hom}_\Delta(V_Q, V_Q). \text{ This will follow from the following elementary property of finite-dimensional vector spaces:}

\textbf{Lemma:} Let W be a finite-dimensional vector space over a division ring D. Let } f \in \text{Hom}_D(W, W) \text{ such that:}
g \in \text{Hom}_D(W, W), \quad gf = 0 \text{ implies } g = 0.

Then f is invertible.

(Proof: Suppose f is not onto, i.e. fW is a proper subspace of W. Then there exists a nonzero linear transformation g on W which is zero on fW. So gf = 0 and g \neq 0, a contradiction to the given property of f. Therefore f is onto, and because of the finite-dimensionality of W, f is also one-to-one, and the lemma is proved.)

Now let a \in A be regular in A. Let } g \in \text{Hom}_\Delta(V_Q, V_Q) \text{ and suppose } ga = 0. \text{ We know that } g \text{ has the form } g = b^{-1}c \text{ with } b, c \in A. \text{ So } b^{-1}ca = 0, \text{ and multiplying by b from the left we get } ca = 0. \text{ As } c \in A \text{ and } a \text{ is regular in } A, \text{ c = 0, and therefore } g = b^{-1}c = 0. \text{ As } V_Q \text{ is finite-dimensional over } \Delta, \text{ by the Lemma a is invertible in } \text{Hom}_\Delta(V_Q, V_Q). \text{ Thus, A is a left order in } \text{Hom}_\Delta(V_Q, V_Q). \text{ Q.E.D.
CHAPTER 3
THE GOLDIE THEOREM

Our aim is to prove the theorem of Goldie which gives the structure of prime, Goldie rings. In the previous chapter we obtained, with the help of Schur's Lemma and the Density Theorem for the class $\sum$, the structure of those rings $A$ for which $\sum_A$ contains a faithful member $V$ satisfying the finite-dimensionality condition $\dim^A V < \infty$ (where $V_Q$ is the quasi-injective hull of $V$, and $\Delta = \text{Hom}_A(V_Q, V_Q)$).

Thus, to get the structure of prime, Goldie rings, all we have to show is the following:
If $A$ is a prime, Goldie ring, then $\sum_A$ contains a faithful module $V$ such that $\dim^A V < \infty$.

3.1. Prime rings

The next proposition is elementary.

Proposition: The following conditions on a ring $A$ are equivalent:

(i) If $I, J$ are ideals in $A$ and $IJ = 0$, then $I = 0$ or $J = 0$.
(ii) If $I, J$ are left ideals in $A$ and $IJ = 0$, then $I = 0$ or $J = 0$.
(iii) If $I, J$ are right ideals in $A$ and $IJ = 0$, then $I = 0$ or $J = 0$.
(iv) If $x, y \in A$ and $xAy = 0$, then $x = 0$ or $y = 0$.

Definition: A nonzero ring $A$ is said to be prime if it satisfies one
of the above equivalent conditions.

It is clear that every simple ring is prime. But, for example, the ring of integers is prime but not simple.

3.2. Goldie rings

**Definition**: Let $S$ be a nonempty subset of a ring $A$. The left annihilator, $\mathcal{E}(S)$, of $S$ is defined to be

$$\mathcal{E}(S) = \{a \in A|aS = 0\}.$$  

It is clear that $\mathcal{E}(S)$ is a left ideal in $A$. (If $S$ consists of the single element $x$, we denote $\mathcal{E}(S) = \mathcal{E}(x)$.)

**Definition**: A left ideal $I$ of a ring $A$ is said to be a left annihilator if there exists a nonempty subset $S$ of $A$ such that $I = \mathcal{E}(S)$.

**Definition**: Let $I$ and $J$ be left ideals in $A$. $I$ is said to be a complement of $J$ if:

(i) $I \cap J = 0$

(ii) For every left ideal $I'$ of $A$ such that $I' \not= I$, we have $I' \cap J \neq 0$.

**Definition**: A left ideal $I$ of a ring $A$ is called a complement if there exists a left ideal $J$ of $A$ such that $I$ is a complement of $J$.

**Definition**: A ring $A$ is said to be Goldie if:

(i) $A$ satisfies the ascending chain condition on left annihilators (i.e. there exists no strictly increasing, infinite sequence of left annihilators in $A$),
(ii) A satisfies the ascending chain condition on complements.

Of course, every Noetherian ring is Goldie.

3.3. Proposition

The following conditions on a ring A are equivalent:

(i) A satisfies the ascending chain condition on complements.

(ii) There exists no infinite sequence \( I_1, I_2, \ldots, I_n, \ldots \) of nonzero left ideals of A such that the sum \( \sum_{n=1}^{\infty} I_n \) is direct.

(The sum \( \sum_{n=1}^{\infty} I_n \) is direct means that for every \( n \geq 1 \): \( x_1 + \ldots + x_n = 0 \), \( x_1 \in I_1 \) implies \( x_1 = \ldots = x_n = 0 \).)

Proof: (i) implies (ii): Suppose the contrary, i.e. there exists an infinite sequence \( I_1, I_2, \ldots, I_n, \ldots \) of nonzero left ideals of A such that the sum \( \sum_{n=1}^{\infty} I_n \) is direct. By Zorn's Lemma there exists a left ideal \( T_0 \) of A, which is maximal with respect to the property:

\[ T_0 \cap (I_1 + I_2 + \ldots) = 0. \]

Again by Zorn's Lemma there exists a left ideal \( T_1 \) of A, which is maximal with respect to the properties:

\[ T_1 \cap (I_2 + I_3 + \ldots) = 0, \]

\[ T_1 \supset T_0 + I_1. \]

(The set of all left ideals T of A which satisfy \( T \cap (I_2 + I_3 + \ldots) = 0 \) and \( T \supset T_0 + I_1 \) is nonempty because \( T_0 + I_1 \) belongs to it.)

We continue by induction. Suppose we have already found left ideals \( T_0, T_1, \ldots, T_{n-1} \) such that for every \( i = 1, \ldots, n - 1 \), \( T_i \) is maximal with
respect to the properties:

\[ T_i \bigcap (I_{i+1} + I_{i+2} + \ldots) = 0 \]

\[ T_i \supseteq T_{i-1} + I_i . \]

Then, by Zorn's Lemma, there exists a left ideal \( T_n \) of \( A \), which is maximal with respect to the properties:

\[ T_n \bigcap (I_{n+1} + I_{n+2} + \ldots) = 0 \]

\[ T_n \supseteq T_{n-1} + I_n . \]

(The set of all left ideals \( T \) of \( A \) which satisfy \( T \bigcap (I_{n+1} + I_{n+2} + \ldots) = 0 \) and \( T \supseteq T_{n-1} + I_n \) is nonempty because it contains \( T_{n-1} + I_n \). For let \( x \in (T_{n-1} + I_n) \bigcap (I_{n+1} + I_{n+2} + \ldots) \). Then \( x = t_{n-1} + i_n = i_{n+1} + \ldots + i_m \) where \( t_{n-1} \in T_{n-1}, \ i_j \in I_j, \ m \geq n + 1 \).

\[ t_{n-1} = -i_n + i_{n+1} + \ldots + i_m \in T_{n-1} \bigcap (I_{n+1} + I_{n+2} + \ldots) = 0 . \] So \( t_{n-1} = 0 \), and \( -i_n + i_{n+1} + \ldots + i_m = 0 \), and as \( \bigcap_{n=1}^\infty I_n \) is direct, \( i_j = 0 \) for all \( j \). Thus \( x = t_{n-1} + i_n = 0 \).

So we have an infinite sequence \( T_0, T_1, \ldots, T_n, \ldots \) of left ideals such that for every \( n \geq 1 \) \( T_n \) is maximal with respect to the properties

\[ T_n \bigcap (I_{n+1} + I_{n+2} + \ldots) = 0 \]

\[ T_n \supseteq T_{n-1} + I_n . \]

It is clear that \( T_n \) is a complement of \( I_{n+1} + I_{n+2} + \ldots, \) and that \( T_{n-1} \subseteq T_n \). Also, \( T_{n-1} \bigcap I_n \subseteq T_{n-1} \bigcap (I_n + I_{n+1} + \ldots) = 0 \), therefore, as \( I_n \neq 0 \), \( T_{n-1} \bigcap I_n \neq I_n \). Thus \( I_n \notin T_{n-1} \). But \( I_n \subseteq T_{n-1} + I_n \subseteq T_n \), and so: \( T_{n-1} \neq T_n \). Thus we have a strictly increasing, infinite sequence of complements, \( T_0 \subsetneq T_1 \subsetneq \ldots \subsetneq T_n \subsetneq \ldots \), in contradiction to (i).
(ii) implies (i): Suppose the contrary, i.e. there exists in $A$ a strictly increasing, infinite sequence $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_n \subsetneq \ldots$ of complements. For every $n \geq 1$ there exists a left ideal $J_n$ of $A$ such that $I_n$ is a complement of $J_n$. We have for every $n \geq 1$:

$$I_n \cap J_n = 0$$

$$I_{n+1} \cap J_n \neq 0 \quad \text{(because } I_n \subsetneq I_{n+1})$$

Denote: $K_n = I_{n+1} \cap J_n$, for $n \geq 1$. Then $K_1, K_2, \ldots, K_n, \ldots$ are nonzero left ideals of $A$. We show that the sum $\sum_{n=1}^{\infty} K_n$ is direct.

Let $k_1 + \ldots + k_n = 0$, where $k_i \in K_i$, $i = 1, \ldots, n$. Then $k_n = (-k_1) + \ldots + (-k_{n-1})$, and $-k_1 \in I_2, \ldots, -k_{n-1} \in I_n$. As $I_2 \subseteq \ldots \subseteq I_n$, we get $k_n = (-k_1) + \ldots + (-k_{n-1}) \in I_n$. But also $k_n \in J_n$, thus $k_n \in I_n \cap J_n = 0$, $k_n = 0$. Similarly, $k_{n-1} = 0, \ldots, k_1 = 0$. So the sum $\sum_{n=1}^{\infty} K_n$ is direct, in contradiction to (ii). Q.E.D.

3.4. Proposition

Let $A$ be a prime ring which satisfies the ascending chain condition on left annihilators. Then, considering $A$ as a module over itself, the singular submodule of $A$ is zero.

Proof: Suppose the contrary, i.e. $Z(A) \neq 0$. Let $X = \{\ell(x) | 0 \neq x \in Z(A)\}$. Then $X$ is a nonempty set of left annihilators in $A$, and therefore, as $A$ satisfies the ascending chain condition on left annihilators, $X$ contains a maximal element, $\ell(x_0)$, where $0 \neq x_0 \in Z(A)$. Let $L = \{y \in A | yZ(A) = 0\}$. Then $L$ is a left ideal in $A$, and $LZ(A) = 0$. $Z(A)$ is also a left ideal in $A$ (by 1.7 it is a submodule of $A$ as a module over itself), therefore, as $A$ is a prime ring and $Z(A) \neq 0$, we have $L = 0$. 

Therefore $x_0 \notin L$, and so $x_0 Z(A) \neq 0$. Thus there exists $z \in Z(A)$ such that $x_0 z \neq 0$. We show now: $\ell(x_0) \nsubseteq \ell(x_0 z)$. It is clear that $\ell(x_0) \subseteq \ell(x_0 z)$. $Ax_0$ is a nonzero left ideal in $A$. ($Ax_0 \neq 0$ because if $Ax_0 = 0$ then $x_0 Ax_0 = 0$ and by the primeness of $A$, $x_0 = 0$, a contradiction.) As $z \in Z(A)$, $\ell(z)$ is an essential left ideal of $A$.

Therefore $Ax_0 \cap \ell(z) \neq 0$. Thus there exists $a \in A$ such that $ax_0 \neq 0$, $ax_0 z = 0$. We have $a \in \ell(x_0 z)$, $a \notin \ell(x_0)$, and so $\ell(x_0) \nsubseteq \ell(x_0 z)$.

As $0 \neq x_0 z \in Z(A)$, $\ell(x_0 z) \subseteq X$, a contradiction to the maximality of $\ell(x_0)$ in $X$. Thus: $Z(A) = 0$. Q.E.D.

Now we are ready to prove the theorem which was the aim of this chapter, and which gives, as an immediate corollary, the Goldie Theorem.

3.5. Theorem

Let $A$ be a prime, Goldie ring. Then $\sum_A$ contains a faithful module $V$ such that $\dim_A V_Q < \infty$. ($V_Q$ denotes the quasi-injective hull of $V$, and $\Delta = \text{Hom}_A(V_Q, V_Q)$.)

Proof: First we show that $A$ contains a nonzero left ideal $V$ which is uniform as an $A$-module.

Suppose that it does not. Then $A$ is not uniform as an $A$-module.

Therefore there exist nonzero left ideals $V_1, W_1$ in $A$ such that $V_1 \cap W_1 = 0$. As $V_1$ is not a uniform $A$-module, there exist nonzero left ideals $V_2, W_2$ of $A$ such that $V_2 \subseteq V_1$, $W_2 \subseteq V_1$ and $V_2 \cap W_2 = 0$.

Continuing by induction we obtain two infinite sequences of nonzero left ideals, $V_1, V_2, \ldots, V_n, \ldots$ and $W_1, W_2, \ldots, W_n, \ldots$, with the properties: $V_n \supset V_{n+1}$, $V_n \supset W_{n+1}$ and $V_n \cap W_n = 0$ for every $n \geq 1$. 
We show that the sum $\sum_{n=1}^{\infty} W_n$ is direct. Suppose $w_1 + \ldots + w_n = 0$, where $w_i \in W_i$, $i = 1, \ldots, n$. Then $w_1 = (-w_2) + \ldots + (-w_n)$, and $-w_2 \in W_2 \subseteq V_1, \ldots, -w_n \in W_n \subseteq V_{n-1} \subseteq V_1$. Thus $w_1 = (-w_2) + \ldots + (-w_n) \in V_1 \cap \bigcap_{i=1}^{\infty} W_i = 0$, $w_1 = 0$. Similarly, $w_2 = 0, \ldots, w_n = 0$, and the sum $\sum_{n=1}^{\infty} W_n$ is direct. By Proposition 3.3 this is a contradiction to the fact that $A$ satisfies the ascending chain condition on complements. Therefore $A$ contains a nonzero left ideal $V$ which is a uniform $A$-module.

We will prove that $V \in \sum_A$, $V$ is faithful, and $\operatorname{dim}_A V Q < \infty$.

First, as $A$ and $V$ are nonzero left ideals of $A$, and $A$ is a prime ring, $AV \neq 0$ so $V$ is a nontrivial $A$-module. Clearly, $Z(V) \subseteq Z(A)$.

By Proposition 3.4, $Z(A) = 0$, therefore $Z(V) = 0$. Thus $V$ is
rationally uniform. To prove that $V$ is homogeneous, let $W$ be a nonzero submodule of $V$. As $V$ and $W$ are nonzero left ideals in $A$, and $A$ is a prime ring, $VW \neq 0$. Thus there exists $w_0 \in W$ such that $Vw_0 \neq 0$. Define $f: V \rightarrow W$ by $f(v) = vw_0$ for every $v \in V$. It is clear that $f$ is an $A$-homomorphism. Suppose $\ker f \neq 0$. Then $\ker f$ is a nonzero submodule of $V$, and as $V$ is rationally uniform, $V$ is a rational extension of $\ker f$. $f \in \text{Hom}_A(V, V)$ and $f|_{\ker f} = 0$, therefore, by 1.8, $f = 0$. But then $fV = Vw_0 = 0$, a contradiction. So $f$ is an $A$-monomorphism, and this proves the homogeneity of $V$.
Thus: $V \in \sum_A$.

For the annihilator $(0:V)$ of $V$ we clearly have $(0:V)V = 0$, and as $V \neq 0$ and $A$ is a prime ring, this implies $(0:V) = 0$. So $V$ is a faithful module. Denote:

$$V_Q = \text{the quasi-injective hull of } V,$$

$$\Delta = \text{Hom}_A(V_Q, V_Q).$$

It remained to prove that $\dim_{\Delta} V_Q < \infty$. Suppose the contrary, i.e. that $V_Q$ is infinite dimensional over $\Delta$. Then there exists an infinite sequence $v_1, v_2, \ldots, v_n, \ldots$ of $\Delta$-linearly independent vectors in $V_Q$.

Let

$$I_1 = (0:v_1)$$

$$I_2 = (0:v_1) \cap (0:v_2)$$

$$\vdots$$

$$I_n = (0:v_1) \cap \ldots \cap (0:v_n)$$

$$\vdots$$
Consider the decreasing sequence of left ideals
\[ I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots \]
We show that for every \( n \geq 1 \) there exists a nonzero left ideal \( J_n \) of \( A \) such that:
\[ J_n \subseteq I_n \quad \text{but} \quad J_n \cap I_{n+1} = 0. \]
Let \( n \) be fixed. Take a nonzero vector \( w \in V \). As \( v_1, \ldots, v_n, v_{n+1} \)
are linearly independent over \( A \), and as \( w \in V \), by the Density theorem
for the class \( \sum \) (Theorem 2.7), there exists an element \( a \in A \) and
a nonzero mapping \( \lambda \in \text{Hom}_A(V, V) \) such that
\[ av_1 = 0, \ldots, av_n = 0, \quad av_{n+1} = \lambda w. \]
We claim that \( 0 \neq av_{n+1} \in V \). As \( \lambda: V \rightarrow V \) and \( w \in V \), \( av_{n+1} = \lambda w \in V \).
By the quasi-injectivity of \( V_Q \), \( \lambda \) can be extended to a mapping \( a \in A \).
Clearly \( a \neq 0 \), and as \( A \) is a division ring, \( a \) is invertible. Thus,
as \( 0 \neq w \in V ,
\[ av_{n+1} = \lambda w = aw \neq 0 . \]
We know that \( Z(V) = 0 \), therefore \( av_{n+1} \notin Z(V) \), and \( (0:av_{n+1}) \) is not
an essential left ideal of \( A \). So there exists a nonzero left ideal \( K \)
of \( A \) such that \( (0:av_{n+1}) \cap K = 0 \). Take: \( J_n = Ka \). Clearly, \( J_n \)
is a left ideal of \( A \). As \( K \neq 0 \), there exists \( 0 \neq k \in K \). As
\( (0:av_{n+1}) \cap K = 0 \), \( k \notin (0:av_{n+1}) \), and therefore \( kav_{n+1} \neq 0 \). Thus
\( 0 \neq ka \in Ka = J_n \). So \( J_n \) is a nonzero left ideal of \( A \). As
\( av_1 = 0, \ldots, av_n = 0 \), we have \( J_n v_1 = Kav_1 = 0, \ldots, J_n v_n = Kav_n = 0 \).
Thus \( J_n \subseteq I_n \).
Let \( x \in J_n \cap I_{n+1} \). Then \( x \in Ka \cap (0:v_{n+1}) \). So \( x = ka \) for some \( k \in K \), and \( kav_{n+1} = 0 \). Therefore \( k \in (0:av_{n+1}) \cap K = 0 \), and \( x = ka = 0 \). Thus \( J_n \cap I_{n+1} = 0 \).

We have an infinite sequence \( J_1, J_2, \ldots, J_n, \ldots \) of nonzero left ideals of \( A \) such that: \( J_n \subseteq I_n \), \( J_n \cap I_{n+1} = 0 \), for every \( n \geq 1 \). We show that the sum \( \sum_{n=1}^{\infty} J_n \) is direct. Suppose \( j_1 + \ldots + j_n = 0 \), where \( j_i \in J_i \) for \( i = 1, \ldots, n \). Then \( j_1 = (-j_2) + \ldots + (-j_n) \), \( -j_2 \in J_2 \subseteq I_2, \ldots, -j_n \in J_n \subseteq I_n \subseteq I_2 \). So \( j_1 = (-j_2) + \ldots + (-j_n) \in J_1 \cap I_2 = 0 \), \( j_1 = 0 \).

Similarly, \( j_2 = 0, \ldots, j_n = 0 \), and the sum \( \sum_{n=1}^{\infty} J_n \) is direct. By Proposition 3.3 this contradicts the fact that \( A \) satisfies the ascending chain condition on complements. Therefore \( V_Q \) is finite-dimensional over \( \Delta \). Q.E.D.

3.6. Corollary (Theorem of Goldie)

Let \( A \) be a prime, Goldie ring. Then \( A \) is a left order in a simple, Artinian ring.

**Proof:** By the previous theorem, there exist a module \( V \in \sum_A \), which is faithful, and satisfies \( \dim_{\Delta} V_Q < \infty \) (where \( V_Q \) is the quasi-injective hull of \( V \), and \( \Delta = \text{Hom}_A(V, V_Q) \)). By Theorem 2.8, \( A \) is a left order in the simple, Artinian ring \( \text{Hom}_{\Delta}(V_Q, V_Q) \). Q.E.D.
REFERENCES


