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ADAMS OPERATIONS ON $KO(X) \oplus KSp(X)$

by

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B.Sc., Université de Montréal, 1971

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics

We accept this thesis as conforming to the
required standard

THE UNIVERSITY OF BRITISH COLUMBIA

July, 1973

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ABSTRACT

Let $KO(X)$ be the real and $KSP(X)$ be the quaternionic K-theory of a finite CW-complex X . The tensor product and the exterior powers of vector bundles induce on

$$L(X) = KO(X) \oplus KSP(X)$$

the structure of a $\mathbb{Z}_2$ -graded λ -ring.

In this thesis it is shown, that the Adams operations

$$\psi^k : L(X) \rightarrow L(X) \quad , \quad k = 1, 2, 3, \dots,$$

which are associated to this λ -ring, are ring homomorphisms and satisfy the composition law

$$\psi^k \circ \psi^\ell = \psi^{\ell k} = \psi^\ell \circ \psi^k \quad , \quad k, \ell = 1, 2, 3, \dots$$

Finally, the ring $L(X)$ together with its ψ -operations is explicitly determined for the quaternionic and complex projective spaces.

Acknowledgement

The author is pleased to acknowledge the very patient help and guidance of Dr. Ulrich Suter during the preparation of this thesis.

He is also thankful to Dr. Kee Lam for kindly reading the manuscript.

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Introduction

Let X be a finite connected CW-complex and let $KO(X)$, $KU(X)$ and $KSp(X)$ be respectively the real, complex and quaternionic K-theory of X . This thesis is concerned with the functor defined by $L(X) = KO(X) \oplus KSp(X)$.

Real and quaternionic vector bundles can be viewed as complex vector bundles provided with some structure maps. This permits the definition of tensor product and exterior powers in the category of real and quaternionic vector bundles, and, therefore, induces a structure of $(\mathbb{Z}_2\text{-graded})$ λ -ring on $L(X)$. (A λ -ring R is a commutative ring with unit together with a set of maps $\lambda^n : R \rightarrow R$, $n = 0, 1, \dots$, having the formal properties of exterior powers).

To any λ -ring R , one can associate group homomorphisms $\psi^k : R \rightarrow R$, $k = 1, 2, \dots$, given by universal polynomials in λ^n . Our main theorem states that the homomorphisms ψ^k associated to the λ -ring $L(X)$

- (i) are ring homomorphisms
- (ii) satisfy the relation

$$\psi^k \psi^\ell = \psi^{k\ell} = \psi^\ell \psi^k \quad k = 1, 2, \dots$$

Atiyah and Tall [4] have given sufficient conditions on a λ -ring in order that its associated ψ^k 's have properties (i), (ii). A λ -ring satisfying those conditions is called special λ -ring. We prove that $L(X)$ is a special λ -ring. Our proof is based on the

theorem that the representation ring of a compact Lie group is a special λ -ring.

Adams [1] has introduced on the λ -rings $KO(X)$ and $KU(X)$ the homomorphisms:

$$\psi_R^k : KO(X) \rightarrow KO(X)$$

$$\psi_C^k : KU(X) \rightarrow KU(X)$$

The ψ^k 's on $L(X)$ are related to the Adams operations in many ways.

Let $LU(X) = KU(X) \oplus KU(X)$ be given the Z_2 -graded λ -ring structure induced by the λ -ring structure of $KU(X)$ (essentially, define $(a,b) \cdot (a',b') = (aa' + bb', ab' + a'b)$, $\lambda^n(a,0) = (\lambda^n(a),0)$, $\lambda^n(0,b) = (0,\lambda^n(b))$ for n odd, $\lambda^n(0,b) = (\lambda^n(b),0)$ for n even). There is a natural Z_2 -graded ring homomorphism $U : L(X) \rightarrow LU(X)$ commuting with the λ^n 's (and, hence, the ψ^k 's). Moreover, when $L(X)$ is torsion free, U is a monomorphism, and the ψ^k 's on $L(X)$ are induced by the ψ_C^k . This gives an easy proof of our main theorem in the torsion free case, and was done in [8]. In [8] also, one can find an application of the properties of the ψ^k 's on $L(X)$.

The Bott isomorphism $\tilde{K}Sp(X) \cong \tilde{K}O^{-4}(X)$ induces an isomorphism $\tilde{L}(X) \cong \tilde{K}O(X) \oplus \tilde{K}O^{-4}(X)$ ($\tilde{L}(\ )$ defined as usual). This isomorphism is actually a ring isomorphism. By definition, $\tilde{K}O^{-4}(X) = \tilde{K}O(X \wedge S^4)$ ($X \wedge Y$ denotes the smashed product of X and Y). Hence the Adams operations ψ_R^k are defined on $\tilde{K}O(X)$ and on $\tilde{K}O^{-4}(X)$.

The ψ^k 's on $L(X)$ restrict to the ψ_R^k 's on $KO(X)$.

Moreover, they induce the following commutative diagrams:

For k odd:

$$\begin{array}{ccc}
 \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O^{-4}(X) \\
 \downarrow k^2 \cdot \psi^k & & \downarrow \psi_R^k \\
 \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O^{-4}(X)
 \end{array}$$

For k even:

$$\begin{array}{ccc}
 \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O^{-4}(X) \\
 \downarrow \psi^k & & \downarrow \psi_R^k \\
 \tilde{K}O(X) & & \tilde{K}O(X) \\
 \downarrow \frac{k^2}{2} \epsilon & \xrightarrow[\cong]{\text{Bott}} & \downarrow \\
 \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O^{-4}(X)
 \end{array}$$

Hence, in general, the ψ^k 's on $\tilde{L}(X)$ determine the ψ_R^k 's on $\tilde{K}O(X)$ and $\tilde{K}O^{-4}(X)$. Also, $\psi^k : \tilde{L}(X) \rightarrow \tilde{L}(X)$ is a ring homomorphism, whereas $\psi_R^k \oplus \psi_R^k : \tilde{K}O(X) \oplus \tilde{K}O^{-4}(X) \rightarrow \tilde{K}O(X) \oplus \tilde{K}O^{-4}(X)$ is not.

We have actually computed the ring $L(X)$ and the operations ψ^k for the quaternionic and complex projective spaces. The later case as torsion for odd dimensions.

The content of the chapters is as follows:

Chapter 1 gives the definition of the ring $L(X)$ and the maps λ^n on $L(X)$. The main theorem is proved there. The definition of the natural homomorphism U and of the reduced theory $\tilde{L}()$ is given with their main properties.

Chapter 2 has two main theorems. In the first the isomorphism $\tilde{L}(X) \cong \tilde{KO}(X) \oplus \tilde{KO}^{-4}(X)$ is established. In the second, we compare the ψ^k 's on $L(X)$ with Adams' ψ_R^k 's.

Chapter 3 contains the computation of $L(X)$ and the ψ^k 's for the quaternionic and complex projective spaces.

Chapter 1

§1.1. Definitions and preliminaries

For $\Lambda = \mathbb{R}, \mathbb{C},$ or $\mathbb{H}$, and for a finite CW-complex X , let $\text{Vect}_{\Lambda}(X)$ denote the class of Λ -vector bundles over X , and $\text{Vect}_{\Lambda}(X)$ the set of isomorphism classes of Λ -vector bundles over X . The direct (or Whitney) sum of vector bundles induces a structure of abelian semi-group on $\text{Vect}_{\Lambda}(X)$ and the tensor product induces a structure of commutative semi-ring on $\text{Vect}_{\mathbb{R}}(X)$ and $\text{Vect}_{\mathbb{C}}(X)$. Let us write:

$$V(X) = \text{Vect}_{\mathbb{R}}(X) \oplus \text{Vect}_{\mathbb{H}}(X)$$

and let $K_{\Lambda}(X)$ denote the Grothendieck group of $\text{Vect}_{\Lambda}(X)$ and $L(X)$ the Grothendieck group of $V(X)$. We will also use the alternative notations:

$$KO(X) = K_{\mathbb{R}}(X)$$

$$KU(X) = K_{\mathbb{C}}(X)$$

$$KSp(X) = K_{\mathbb{H}}(X) .$$

We have that

$$L(X) \approx KO(X) \oplus KSp(X) .$$

Following [3], §1.5, we recall that a real (resp. quaternionic) vector bundle over X can be viewed as a complex vector bundle over X together with a structure map, i.e. as a pair (v, T) , where

$v \in \text{Vect}_{\mathbb{C}}(X)$ and $T : v \rightarrow \bar{v}$ is an isomorphism of the vector bundle v with its complex conjugate $\bar{v}$, such that

$$T^2 = \text{Id}_v \quad (\text{resp. } T^2 = -\text{Id}_v)$$

(Note, T can be considered as a conjugate-linear map from v to itself).

The details of this construction are to be found in [3].

However let us recall that two pairs (v, T) and (v', T') are said isomorphic if v and v' are isomorphic complex vector bundles via an isomorphism $\alpha : v \rightarrow v'$ which commutes with the structure maps, i.e.

$$\alpha \circ T = T' \circ \alpha.$$

This definition coincides with the isomorphism of real (resp. quaternionic) vector bundles under the present identification.

Hence

$$1.1.1 \quad V(X) \cong \{([\mu, S], [v, T]) \mid \mu, v, \in \text{Vect}_{\mathbb{C}}(X), S^2 = \text{Id}_{\mu}, T^2 = -\text{Id}_v\}$$

where S, T are structure maps and $[\mu, S]$ and $[v, T]$ denote the isomorphism classes of (μ, S) and (v, T) respectively.

We define a semi-ring structure on $V(X)$ as follows.

Let

$$[\mu, S], [\mu', S'] \in \text{Vect}_{\mathbb{R}}(X)$$

$$[v, T], [v', T'] \in \text{Vect}_{\mathbb{H}}(X)$$

and define:

$$([\mu, S], 0) \cdot ([\mu', S'], 0) = ([\mu \otimes \mu', S \otimes S'], 0)$$

$$([\mu, S], 0) \cdot (0, [\nu, T]) = (0, [\mu \otimes \nu, S \otimes T])$$

$$(0, [\nu, T]) \cdot (0, [\nu', T']) = ([\nu \otimes \nu', S \otimes T'], 0)$$

Here the tensor products are taken in the category of complex vector bundles. The 0 denotes the isomorphism class of the 0-dimensional real or quaternionic vector bundle. It is easy to check that the structure maps $S \otimes S'$, $S \otimes T$, $T \otimes T'$ have the appropriate squares, so that these definitions make sense. We extend the definition of the product for arbitrary elements of $L(X)$ in the obvious way. Hence $V(X)$ becomes commutative semi-ring and $L(X)$ therefore carries the structure of a commutative ring with unit. Notice that $L(X)$ is $\mathbb{Z}_2$ -graded. Actually $L(-)$ is a contravariant functor from the category of finite CW-complexes and continuous maps to the category of $\mathbb{Z}_2$ -graded commutative rings with unit. If Y is another finite CW-complex and $f : X \rightarrow Y$ is a continuous map, we denote by f^* the induced map $L(Y) \rightarrow L(X)$. It is the direct sum of the maps $KO(Y) \rightarrow KO(X)$ and $KSp(Y) \rightarrow KSp(X)$ induced by f .

§1.2. Main theorem:

The ring $L(X)$ admits Adams operations, i.e. there are ring homomorphisms:

$$\psi^k : L(X) \rightarrow L(X) \qquad k = 1, 2, \dots$$

such that

$$(i) \quad \psi^k \circ \psi^\ell = \psi^{k\ell} = \psi^\ell \circ \psi^k \quad k, \ell = 1, 2, \dots$$

$$(ii) \quad \psi^{p^r}(x) = x^{p^r} \pmod{p} \quad p \text{ prime}$$

(iii) for a continuous $f : X \rightarrow Y$,

$$\psi^k \circ f^* = f^* \circ \psi^k.$$

The proof of this theorem will be completed in 1.6. We need to introduce another concept.

§1.3. λ -semi-rings

The following definitions are found in [4] in the case of rings. The extension to semi-rings was without problem.

A λ -semi-ring is a commutative semi-ring R with unit 1 together with a set of maps $\lambda^n : R \rightarrow R$, $n = 0, 1, \dots$, such that $x, y \in R$

$$(1) \quad \lambda^0(x) = 1$$

$$(2) \quad \lambda^1(x) = x$$

$$(3) \quad \lambda^n(x+y) = \sum_{r=0}^n \lambda^r(x) \lambda^{n-r}(y)$$

If the semi-ring is a ring we will talk of a λ -ring.

Let $a_1 \dots a_q$ and $b_1 \dots b_r$ be indeterminates and let s_i

and σ_i be the i^{th} elementary symmetric functions in $a_1 \dots a_q$ and $b_1 \dots b_r$ respectively. Take $q \geq \max(n, mn)$, $r \geq n$ and let

$P_n(s_1 \dots s_n; \sigma_1 \dots \sigma_n)$ be the coefficient of t^n in $\prod_{i,j} (1 + a_i b_j t)$

$P_{m,n}(s_1 \dots s_{mn})$ be the coefficient of t^n in $\prod_{i_1 < \dots < i_m} (1 + a_{i_1} \dots a_{i_m} t)$

P_n and $P_{m,n}$ are uniquely defined polynomials with coefficients in $\mathbb{Z}$; they split in:

$$P_n = P_n^+ - P_n^-$$

$$P_{m,n} = P_{m,n}^+ - P_{m,n}^-$$

where P_n^+ (resp. $P_{m,n}^+$) is the sum of the terms of P_n (resp. $P_{m,n}$) having positive coefficients and P_n^- (resp. $P_{m,n}^-$) is the sum of the terms of P_n (resp. $P_{m,n}$) having negative coefficients, multiplied by -1 . Hence P_n^+ , P_n^- , $P_{m,n}^+$, $P_{m,n}^-$ have only positive coefficients and can be interpreted in any semi-ring with unit.

A λ -semi-ring R is a special λ -semi-ring if the following identities hold $\forall x, y \in R$:

$$\begin{aligned} (4) \quad P_n^- (\lambda^1(x) \dots \lambda^n(x); \lambda^1(y) \dots \lambda^n(y)) + \lambda^n(xy) \\ = P_n^+ (\lambda^1(x) \dots \lambda^n(x); \lambda^1(y) \dots \lambda^n(y)) \end{aligned}$$

$$\begin{aligned}
 (5) \quad P_{m,n}^{-}(\lambda^1(x) \dots \lambda^{mn}(x)) + \lambda^m \lambda^n(x) \\
 = P_{m,n}^{+}(\lambda^1(x) \dots \lambda^{mn}(x)) .
 \end{aligned}$$

(the first identity relates $\lambda^n(xy)$ and the $\lambda^i(x)$, $\lambda^i(y)$ $i = 1, \dots, n$; the second relates $\lambda^m \lambda^n(x)$ and the $\lambda^i(x)$, $i = 1, \dots, mn$) . For convenience we will refer to these identities in the following form:

$$(4') \quad F_n(x, y) = G_n(x, y)$$

$$(5') \quad F_{m,n}(x) = G_{m,n}(x)$$

i.e. we write $F_n(x, y)$ for the left hand side of (4), etc....

Moreover, even though we will consider many λ -semi-rings, we will not use different symbols for the maps λ^n and the identities (1)-(5), (4'), (5'). We hope this will not cause any difficulty.

Let R and S be two λ -semi-rings. A λ -semi-ring homomorphism $g : R \rightarrow S$ is a semi-ring homomorphism commuting with the maps λ^n , i.e. such that $g \circ \lambda^n = \lambda^n \circ g$, $n = 0, 1, \dots$.

The following two propositions motivate the introduction of the previous concepts.

1.3.1 Proposition

Let R_0 be a (special) λ -semi-ring. Then the Grothendieck group R of R_0 is a (special) λ -ring.

Proof: It is well known that R is a ring. The maps $\lambda^n : R_0 \rightarrow R_0$ extend to R in a unique way which satisfies (3) (cf. [7], Chap. 12, 1.3). The extension also satisfies (1) and (2), hence R is a λ -ring. If R_0 is special, R is special, since (4) and (5) are compatible with (3) by lemma 1.5 of [4]. /

For $k = 1, 2, \dots$, let Q_k be the unique polynomial with integer coefficients such that:

$$Q_k(s_1 \dots s_k) = a_1^k + \dots + a_q^k$$

(with the notation formerly used and $q \geq k$) . Then we have

1.3.2 Proposition

Let R be a special λ -ring and define

$$\psi^k(x) = Q_k(\lambda^1(x), \dots, \lambda^k(x))$$

for $x \in R$, $k = 1, 2, \dots$. Then ψ^k are ring homomorphisms and

$$\psi^k \psi^\ell = \psi^{\ell k} = \psi^\ell \psi^k \quad \ell, k = 1, 2, \dots$$

$$\psi^{p^r}(x) = x^{p^r} \pmod{p} \quad p \text{ prime, } x \in R$$

Proof: cf [4] prop. 5.1, 5.2. /

Examples:

(a) For $\Lambda = R$ or C , $\text{Vect}_\Lambda(X)$ is a special λ -semi-ring where the λ^n are induced by exterior powers of vector bundles, i.e.:

$$\lambda^n[v] = [\lambda^n(v)] \quad v \in \text{Vect}_\Lambda(X)$$

By 1.3.1 $KO(X)$ and $KU(X)$ are special λ -rings, and the operations:

$$\psi_R^k : KO(X) \rightarrow KO(X)$$

$$\psi_C^k : KU(X) \rightarrow KU(X)$$

obtained via 1.3.2 are the classical Adams operations. The proof of these facts are found in [1] or [4].

(b) Let

$$LU(X) = KU(X) \oplus KU(X)$$

and define a $\mathbb{Z}_2$ -graded ring structure on $LU(X)$ by

$$(a,b) \circ (a',b') = (aa' + bb', ab' + a'b)$$

for $(a,b), (a',b') \in LU(X)$. For $k = 0, 1, \dots$ set

$$\lambda^k(a,0) = (\lambda^k(a), 0)$$

$$\lambda^{2k}(0,b) = (\lambda^{2k}(b), 0)$$

$$\lambda^{2k+1}(0,b) = (0, \lambda^{2k+1}(b)) ,$$

and define

$$\lambda^k(a,b) = \sum_{r=0}^k \lambda^r(a,0) \lambda^{k-r}(0,b) .$$

$LU(X)$ with these maps is a special λ -ring. This is direct using the special λ -ring properties of $KU(X)$.

(c) Other examples can be found in [4].

§1.4. λ -operations on $V(X)$

For $n = 0, 1, \dots$, we define:

$$\lambda^n([\mu, S], 0) = ([\lambda^n(\mu), \lambda^n(S)], 0)$$

$$\lambda^{2n}(0, [\nu, T]) = ([\lambda^{2n}(\nu), \lambda^{2n}(T)], 0)$$

$$\lambda^{2n+1}(0, [\nu, T]) = (0, [\lambda^{2n+1}(\nu), \lambda^{2n+1}(T)])$$

using the characterization of $V(X)$ given by 1.1.1. Checking the squares of the structure maps involved confirms the legitimacy of this definition. We extend it to $V(X)$ by:

$$\lambda^n([\mu, S], [\nu, T]) = \sum_{r=0}^n \lambda^r([\mu, S], 0) \lambda^{n-r}(0, [\nu, T]) .$$

Theorem

$V(X)$ together with the λ -operations just defined, is a special λ -semi-ring.

Proof: Checking the conditions (1), (2), (3) is direct, using the properties of exterior powers and, for (3), verifying a simple identity. Hence $V(X)$ is a λ -semi-ring.

We now have to check (4) and (5). We will do the work for (4) only since a similar treatment deals with (5). Using again the lemma 1.5 of [4], we remark that we need only to check these identities on a set of generators of the semi-group $V(X)$. The convenient set will be:

$$\text{Vect}_R(X) \cup \text{Vect}_H(X) \subset V(X)$$

where $\text{Vect}_R(X)$ and $\text{Vect}_H(X)$ are identified with their image under their canonical embedding in $V(X)$.

In [7], Chap, 5, §6, Husemoller defines the notion of continuous functor, and shows that a continuous functor:

$$\text{Vect}_\Lambda(0) \times \dots \times \text{Vect}_\Lambda(0) \rightarrow \text{Vect}_\Lambda(0)$$

where 0 denotes the 1-point space, induces a functor:

$$\text{Vect}_\Lambda(X) \times \dots \times \text{Vect}_\Lambda(X) \rightarrow \text{Vect}_\Lambda(X).$$

The definitions and theorems of [7] can clearly be extended to the case:

$$\text{Vect}_\Lambda(0) \times \dots \times \text{Vect}_\Lambda(0) \times \text{Vect}_{\Lambda'}(0) \times \dots \times \text{Vect}_{\Lambda'}(0) \rightarrow \text{Vect}_{\Lambda''}(0)$$

where $\Lambda, \Lambda', \Lambda''$ take the values R and H .

For example, we have the following continuous functors:

$$\oplus : \text{Vect}_\Lambda(0) \times \text{Vect}_\Lambda(0) \rightarrow \text{Vect}_\Lambda(0)$$

$$\theta : \text{Vect}_{\Lambda}(0) \times \text{Vect}_{\Lambda'}(0) \rightarrow \text{Vect}_{\Lambda''}(0)$$

for $(\Lambda, \Lambda', \Lambda'') = (R, H, H), (H, R, H), (H, H, R)$ or (R, R, R) .

$$\lambda^n : \text{Vect}_{\Lambda}(0) \rightarrow \text{Vect}_{\Lambda'}(0)$$

for $(\Lambda, \Lambda') = (R, R)$ or (H, H) and n odd, or (H, R) and n even.

These functors respectively induce the direct sum of vector bundles, the tensor product of vector bundles and the exterior powers of vector bundles.

The functors F_n and G_n are compositions of these operations on vector bundles; they are induced by continuous functors

$$F_{n,0}, G_{n,0} : \text{Vect}_{\Lambda}(0) \times \text{Vect}_{\Lambda'}(0) \rightarrow \text{Vect}_{\Lambda''}(0)$$

where $(\Lambda, \Lambda', \Lambda'') = (R, R, R), (R, H, H), (H, R, H)$ or (H, H, H)

for n odd, or (H, H, R) for n even.

By [7], an equivalence of functors:

$$\mu_0 : F_{n,0} \rightarrow G_{n,0}$$

induces an equivalence of functors:

$$\mu : F_n \rightarrow G_n$$

and, passing to isomorphism classes of vector bundles, one gets:

$$F_n = G_n : \text{Vect}_{\Lambda}(X) \times \text{Vect}_{\Lambda'}(X) \rightarrow \text{Vect}_{\Lambda''}(X).$$

However, we are not interested in the functorial properties of F_n and G_n , and an equivalence of the functors $F_{n,0}$ and $G_{n,0}$ for vector bundle isomorphisms will be sufficient to give the result wanted. (This can be seen in greater detail by analyzing the proof of the theorem 6.2 in [7], Chapter 5). Let us find out such an equivalence.

Since Λ -vector bundles over a point are essentially Λ -vector spaces, we can choose bases, and consider, for each $k, k' \in N$:

$$F_{n,0}, G_{n,0} : Gl(k, \Lambda) \times Gl(k', \Lambda') \rightarrow Gl(k'', \Lambda'')$$

with $(\Lambda, \Lambda', \Lambda'')$ as above and $k'' \in N$ depending on $k, k', n, \Lambda, \Lambda'$. (We don't write the more accurate $F_{n,0}^{k,k'}$, $G_{n,0}^{k,k'}$ for fear of heaviness).

Now, in order to complete the proof, we need to produce

$M \in Gl(k'', \Lambda'')$ such that:

$$M F_{n,0}(A, B) = G_{n,0}(A, B) M$$

for all $(A, B) \in Gl(k, \Lambda) \times Gl(k', \Lambda')$. The next lemma serves that purpose. Hence the proof of the theorem is complete. /

1.4.1 Lemma

There is $M \in Gl(k'', \Lambda'')$ such that:

$$M F_{n,0}(A, B) = G_{n,0}(A, B) M$$

for all $(A, B) \in Gl(k, \Lambda) \times Gl(k', \Lambda')$.

Proof: Similarly as in 1.1, real and quaternionic representations of Lie groups can be viewed as complex representations together with some structure maps. This is carried out in [2], Chapter 3. The following is a direct translation of this treatment in term of matrix representations. (It can also be obtained with the help of §1.1. for the 1-point space) .

Let $E = 1$ or 2 and $T \in \mathcal{G}(E, \mathbb{C})$ such that:

$$T\bar{T} = \text{Id} \quad \text{if } E = 1$$

$$T\bar{T} = -\text{Id} \quad \text{if } E = 2$$

where $\bar{T}$ denotes the complex matrix with as coefficients, the coefficients of T conjugated. Define:

$$\mathcal{G}(E, \mathbb{C})_T = \{A \in \mathcal{G}(E, \mathbb{C}) \mid T\bar{A} = AT\} .$$

Then we have the following isomorphisms:

$$\mathcal{G}(m, \mathbb{R}) \simeq \mathcal{G}(m, \mathbb{C})_S \quad \text{if } S\bar{S} = \text{Id}$$

$$\mathcal{G}(m, \mathbb{H}) \simeq \mathcal{G}(2m, \mathbb{C})_{S'}, \quad \text{if } S'\bar{S}' = -\text{Id} .$$

(The conjugate signs appearing here correspond to the conjugate linearity of the structure maps of [2]).

Examples of S, S' are:

$$S = \text{Id} \quad S' = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & -1 \\ 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

Using these identifications, the functors $F_{n,0}$ and $G_{n,0}$ are as follows:

$$F_{n,0} : Gl(Ek, C)_T \times Gl(E'k', C')_{T'} \rightarrow Gl(E''k'', C)_{F_{n,0}(T, T')}$$

$$G_{n,0} : Gl(Ek, C)_T \times Gl(E'k', C)_{T'} \rightarrow Gl(E''k'', C)_{G_{n,0}(T, T')}$$

where E'' is determined by $F_{n,0}(T, T') \overline{F_{n,0}(T, T')} = G_{n,0}(T, T') \overline{G_{n,0}(T, T')}$

(This equality is easy to check).

The existence of a matrix $M \in Gl(k'', \Lambda'')$ satisfying the lemma is equivalent to the existence of a matrix $N \in Gl(E''k'', C)$ such that:

$$(i) \quad N F_{n,0}(U, V) = G_{n,0}(U, V) N$$

$$\forall (U, V) \in Gl(Ek, C)_T \times Gl(E'k', C)_{T'}$$

$$(ii) \quad N F(T, T') = G(T, T') \overline{N}.$$

One can produce such a N in the following way.

First, the representation ring $R(U(Ek) \times U(E'k'))$ is a special λ -ring, where $U(n)$ is the unitary group of dimension n (cf. [4] th. 1.5); hence the representations

$$F_{n,0}, G_{n,0} : U(Ek) \times U(E'k') \rightarrow Gl(E''k'', C)$$

are equivalent. But two representations of $Gl(Ek, C) \times Gl(E'k', C)$ equivalent on $U(Ek) \times U(E'k')$ are equivalent. (cf. [1], proof 4.5).

Therefore there exist $P \in Gl(E''k'', C)$ such that

$$(iii) \quad P F_{n,0}(U,V) = G_{n,0}(U,V) P$$

$$\forall (U,V) \in Gl(Ek, C) \times Gl(E'k', C).$$

Now, (iii) still holds if one replaces P by $\bar{P}$. First start by writing (iii) with $\bar{U}, \bar{V}$, and note that $F_{n,0}, G_{n,0}$ commute with the conjugation. Now let $\rho \in C$ be such that $-\rho\bar{\rho}^{-1}$ is not an eigenvalue of $P^{-1}\bar{P}$, and define:

$$N = \rho P + \bar{\rho} \bar{P} = P(\rho\bar{\rho}^{-1} I + P^{-1}\bar{P})\bar{\rho}.$$

One sees that $N \in Gl(E''k'', C)$ and N satisfies (iii), hence it satisfies (i). But since $N = \bar{N}$, it satisfies also (ii). This proves the lemma.

1.4.2 Corollary to the theorem

$L(X)$ is a special λ -ring.

Proof: Theorem §1.4 and proposition 1.3.1. /

§1.5. Proof of the main theorem

By 1.4.2 and proposition 1.3.2, we obtain the ring homomorphisms ψ^k with their first two properties. The last one is clear since f^* is a λ -ring homomorphism. /

§1.6. A natural transformation

Recall the canonical homomorphisms

$$\begin{aligned} c : KO(X) &\rightarrow KU(X) & c' : KSp(X) &\rightarrow KU(X) \\ r : KU(X) &\rightarrow KO(X) & q : KU(X) &\rightarrow KSp(X) . \end{aligned}$$

c and c' are induced by the maps $Vect_R(X) \rightarrow Vect_C(X)$ and $Vect_H(X) \rightarrow Vect_C(X)$ forgetting the structure maps (cf 1.1); r is induced by the map $Vect_C(X) \rightarrow Vect_R(X)$ forgetting the complex structure; q is induced by the map $Vect_C(X) \rightarrow Vect_H(X)$ defined by $\eta \rightarrow \eta \otimes_C \varepsilon$ where ε is the trivial 1-dimensional quaternionic vector bundle over X .

Define:

$$\begin{aligned} U &= c \oplus c' : L(X) \rightarrow LU(X) \\ U' &= r \oplus q : LU(X) \rightarrow L(X) . \end{aligned}$$

Proposition

The map $U : L(X) \rightarrow LU(X)$ is a λ -ring homomorphism. If $L(X)$ is torsion free, U is a monomorphism, i.e. $L(X)$ is a sub- λ -ring of $LU(X)$.

Proof: The first part is obvious from the definition of the maps λ^n

on $L(X)$. For the second part we have that

$$U' \circ U = 2$$

where $2 : L(X) \rightarrow L(X)$ is the map $x \rightarrow 2x$ $x \in L(X)$. Cf [5] or [6] for instance. Hence, if $L(X)$ is torsion free, $U' \circ U$ is a monomorphism and so is U . /

§1.7. The reduced theory

From now on we suppose that the spaces considered have base points: by X we mean a pair (X, x_0) s.t. $x_0 \in X$. The inclusion $i : x_0 \rightarrow X$ induces an exact sequence:

$$0 \rightarrow \text{Ker } i^* \rightarrow L(X) \xrightarrow{i^*} L(x_0) \rightarrow 0 .$$

Define

$$\tilde{L}(X) = \text{Ker } i^* .$$

Then we have a natural splitting

$$L(X) \simeq \tilde{L}(X) \oplus L(x_0) .$$

Let ϵ be the 1-dimensional quaternionic vector bundle over x_0 .

$L(x_0)$ is the free abelian group on two generators $1, \epsilon$, and the product structure is given by the relation $\epsilon^2 = 4$. The maps ψ^k are defined by:

$$\psi^k(\epsilon) = \begin{cases} \epsilon & k = 1, 3, \dots \\ 2 & k = 2, 4, \dots \end{cases} .$$

Moreover,

$$\tilde{L}(X) \simeq \tilde{K}O(X) \oplus \tilde{K}Sp(X)$$

and the ψ^k 's and U pass to the reduced functors (with

$$\tilde{L}U(X) \simeq \tilde{K}U(X) \oplus \tilde{K}U(X) \text{ .}$$

Finally, if X^+ denotes the disjoint union of X with a point which we take as canonical base point, we have that

$$L(X) \simeq \tilde{L}(X^+) \text{ .}$$

Chapter 2

§2.1. Preliminaries

Let S^n denote the n -dimensional sphere and (X, x_0) and (Y, y_0) be finite CW-complexes with base point. Define as usual:

$$X \wedge Y = (X \times Y) / (X \times y_0 \cup x_0 \times Y)$$

Consider the obvious projection maps:

$$\begin{array}{ccccc} & & X \times S^4 & & \\ & \swarrow p & \downarrow q & \searrow f & \\ X & & S^4 & & X \wedge S^4 \end{array}$$

and denote by ν the generator of $\widetilde{KSp}(S^4) \cong \mathbb{Z}$. Define:

$$\phi_1 : \widetilde{KSp}(X) \rightarrow \widetilde{K0}(X \times S^4) \subset \widetilde{L}(X \times S^4)$$

$$\text{by} \quad \phi_1(\beta) = p^*(\beta) \cdot q^*(\nu) \quad \forall \beta \in \widetilde{KSp}(X)$$

$$\phi_2 : \widetilde{K0}(X) \rightarrow \widetilde{KSp}(X \times S^4) \subset \widetilde{L}(X \times S^4)$$

$$\text{by} \quad \phi_2(\alpha) = p^*(\alpha) \cdot q^*(\nu) \quad \forall \alpha \in \widetilde{K0}(X).$$

Since ν is a quaternionic bundle, $p^*(\alpha) \cdot q^*(\nu)$ and $p^*(\beta) \cdot q^*(\nu)$ are respectively quaternionic and real bundles as indicated.

$$f^* : \widetilde{K0}(X \wedge S^4) \rightarrow \widetilde{K0}(X \times S^4) \quad \text{and} \quad f^* : \widetilde{KSp}(X \wedge S^4) \rightarrow \widetilde{KSp}(X \times S^4)$$

are monomorphisms, and we have the

Bott isomorphism theorem:

(i) Consider

$$\widetilde{K}Sp(X) \xrightarrow{\phi} \widetilde{K}O(X \times S^4) \xleftarrow{f^*} \widetilde{K}O(X \wedge S^4)$$

then ϕ_1 is an isomorphism onto $\text{Im } f^*$.

(ii) Consider

$$\widetilde{K}O(X) \xrightarrow{\phi_2} \widetilde{K}Sp(X \times S^4) \xleftarrow{f^*} \widetilde{K}Sp(X \wedge S^4)$$

then ϕ_2 is an isomorphism onto $\text{Im } f^*$.

Proof: cf [5]. /

Next, we recall that one defines for $n = 0, 1, 2, \dots$

$$\widetilde{K}O^{-n}(X) = \widetilde{K}O(X \wedge S^n)$$

and that $\widetilde{K}O(X) \oplus \widetilde{K}O^{-4}(X)$ can be given a ring structure in the following way. Let $\alpha, \alpha' \in \widetilde{K}O(X)$, $\beta, \beta' \in \widetilde{K}O^{-4}(X) = \widetilde{K}O(X \wedge S^4)$, and denote the product in $\widetilde{K}O(X) \oplus \widetilde{K}O^{-4}(X)$ by $\times$.

First define $\alpha \times \alpha' = \alpha \cdot \alpha'$.

Now let

$$X \times S^4 \xrightarrow{\Delta} X \times X \times S^4 \begin{array}{l} \xrightarrow{p_1} X \\ \xrightarrow{p_2} X \times S^4 \end{array}$$

be the maps defined by:

$$\Delta(x, s) = (x, x, s)$$

$$p_1(x, y, s) = x$$

$$p_2(x, y, s) = (y, s) \quad \text{for } x, y \in X, s \in S^4.$$

Then $\gamma = \Delta^*[p_1^*(\nu) \cdot p_2^*p^*(\beta)] \in \text{Im}(f^* : \widetilde{KO}(X \wedge S^4) \rightarrow \widetilde{KO}(X \times S^4))$ and

define $\alpha \times \beta = (f^*)^{-1}(\gamma) \in \widetilde{KO}(X \wedge S^4)$. Similarly, let

$$\begin{array}{ccc} X \times S^4 \times S^4 & \xrightarrow{\Delta'} & X \times S^4 \times X \times S^4 \\ \downarrow g & & \swarrow q_1 \quad \searrow q_2 \\ X \wedge S^4 \wedge S^4 & & X \times S^4 \quad X \times S^4 \end{array}$$

be the obvious projections plus the map defined by $\Delta'(x, s, t) = (x, s, x, t)$ for $x \in X, s, t \in S^4$. Again

$$\gamma' = \Delta'^*[q_1^* p^*(\beta) \cdot q_2^* p^*(\beta')] \in \text{Im}(g^* : \widetilde{KO}(X \wedge S^4 \wedge S^4) \rightarrow KO(X \times S^4 \times S^4))$$

and define $\beta \times \beta' = (\Phi_1 \Phi_2)^{-1} (g^*)^{-1} (\gamma') \in \widetilde{KO}(X)$.

(These definitions are equivalent to those of [3] where the details can be obtained).

§2.2. A ring isomorphism

From the Bott isomorphism theorem, we obtain a group isomorphism:

$$B = \text{Id} \oplus \Phi_1 : \widetilde{L}(X) \rightarrow \widetilde{KO}(X) \oplus \widetilde{KO}^{-4}(X)$$

We have the following

Theorem: $B : \tilde{L}(X) \rightarrow \tilde{K}O(X) \oplus \tilde{K}O^{-4}(X)$

is a ring isomorphism.

Proof: Essentially, the proof consists of checking the commutativity of the two following diagrams:

$$\begin{array}{ccc}
 \tilde{K}O(X) \oplus \tilde{K}Sp(X) & \xrightarrow{\text{Id} \oplus \phi_1} & \tilde{K}O(X) \oplus \tilde{K}O(X \wedge S^4) \\
 \downarrow & & \downarrow \text{Id} \oplus f^* \\
 & & \tilde{K}O(X) \oplus \tilde{K}O(X \times S^4) \\
 & & \downarrow p_1^*(-) \cdot p_2^*(-) \\
 & & \tilde{K}O(X \times X \times S^4) \\
 & & \downarrow \Delta^* \\
 & & \tilde{K}O(X \times S^4) \\
 & & \downarrow f^* \\
 \tilde{K}Sp(X) & \xrightarrow{\phi_1} & \tilde{K}O(X \wedge S^4)
 \end{array}$$

(I)

$$\begin{array}{ccc}
\widetilde{K}Sp(X) \oplus \widetilde{K}Sp(X) & \xrightarrow{\Phi_1 \oplus \Phi_1} & \widetilde{K}O(X \times S^4) \oplus \widetilde{K}O(X \wedge S^4) \\
\downarrow & & \downarrow f^* \oplus f^* \\
& & \widetilde{K}O(X \times S^4) \oplus \widetilde{K}O(X \times S^4) \\
& & \downarrow q_1^*(-) \cdot q_2^*(-) \\
& & \widetilde{K}O(X \times S^4 \times X \times S^4) \\
& & \downarrow \Delta^* \\
& & \widetilde{K}O(X \times S^4 \times S^4) \\
& & \downarrow g^* \\
& & \widetilde{K}O(X \wedge S^4 \wedge S^4) \\
& \xrightarrow{\Phi_1 \oplus \Phi_2} & \\
\widetilde{K}O(X) & &
\end{array}$$

(II)

In each case, the left hand vertical arrow is the product in $\widetilde{L}(X)$ and the right hand vertical sequence is the product in $\widetilde{K}O(X) \oplus \widetilde{K}O^{-4}(X)$. Let us check (I): $\forall \alpha \in \widetilde{K}O(X)$, $\forall \beta \in \widetilde{K}Sp(X)$,

$$\begin{aligned}
& \Delta^* [p_1^*(\alpha) \cdot p_2^*(\beta)] \\
&= \Delta^* [p_1^*(\alpha) \cdot p_2^*(p^*(\beta) \cdot q^*(v))] \\
&= f^* [p^*(\alpha \cdot \beta) \cdot q^*(v)] \\
&= f^* \Phi_1(\alpha \cdot \beta) .
\end{aligned}$$

We used mainly that continuous maps induce ring homomorphisms on

$\widetilde{L}(-)$ and that $p_2 \circ \Delta = \text{Id}_{X \times S^4}$ and $p_1 \circ \Delta = p$.

Similarly for (II). /

§2.3. Relation between ψ^k and ψ_R^k

We need the following

Lemma:

$$(i) \quad \widetilde{L}(S^4) = \varepsilon v \, Z \oplus v \, Z$$

where v is the generator of $\widetilde{KSp}(S^4) \simeq Z$ and ε is the trivial quaternionic 1-dimensional vector bundle;

(ii) the ring homomorphisms ψ^k are given by:

$$\psi^k(v) = \begin{cases} k^2 v & k = 1, 3, 5, \dots \\ \frac{k^2}{2} \varepsilon v & k = 2, 4, 6, \dots \end{cases} .$$

Proof: Let μ be generator of $\widetilde{KO}(S^4) \simeq Z$, and let α be generator of $\widetilde{KU}(S^4)$. By proposition §1.6, $U : \widetilde{L}(S^4) \rightarrow \widetilde{LU}(S^4)$ is a monomorphism. By [1] or [7], $U(\mu) = (2\alpha, 0)$ and $U(v) = (0, \alpha)$. (Elements of $\widetilde{LU}(X) \simeq \widetilde{KU}(X) \oplus \widetilde{KU}(X)$ will be written as pairs (x, y)). Since $U(\varepsilon) \in \widetilde{LU}(S^4)$ is given by $U(\varepsilon) = (0, 2)$, one has that $U(\mu) = U(\varepsilon) \cdot U(v)$, hence $\mu = \varepsilon v$. Using [1], corr. 5.2, one computes that:

$$\psi^k(0, \alpha) = \begin{cases} k^2(0, \alpha) & k = 1, 3, \dots \\ k^2(\alpha, 0) & k = 2, 4, \dots \end{cases} .$$

Hence:

$$\psi^k(v) = \begin{cases} k^2 v & k = 1, 3, \dots \\ \frac{k^2}{2} \epsilon v & k = 2, 4, \dots \end{cases} \quad . /$$

Remark: the ψ^k 's on $L(S^4)$ are now completely determined since we know the value of $\psi^k(\epsilon)$.

Recall (1.3., example (a)) that we have the classical Adams operations $\psi_R^k : \tilde{K}O(X) \rightarrow \tilde{K}O(X)$ and $\psi_R^k : \tilde{K}O(X \wedge S^4) \rightarrow \tilde{K}O(X \wedge S^4)$.

The restriction of $\psi^k : \tilde{L}(X) \rightarrow \tilde{L}(X)$ to $\tilde{K}O(X)$ is ψ_R^k . In the next theorem, we compare the restriction of $\psi^k : \tilde{L}(X) \rightarrow \tilde{L}(X)$ to $\tilde{K}Sp(X) \simeq \tilde{K}O(X \wedge S^4)$, with $\psi_R^k : \tilde{K}O(X \wedge S^4) \rightarrow \tilde{K}O(X \wedge S^4)$.

Theorem

The following diagrams are commutative. For $k = 1, 3, \dots$:

$$\begin{array}{ccc} \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O(X \wedge S^4) \\ \downarrow k^2 \cdot \psi^k & & \downarrow \psi_R^k \\ \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O(X \wedge S^4) \end{array}$$

For $k = 2, 4, \dots$:

$$\begin{array}{ccc} \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O(X \wedge S^4) \\ \downarrow \psi^k & & \downarrow \psi_R^k \\ \tilde{K}O(X) & & \\ \downarrow \frac{k^2}{2} \epsilon & & \\ \tilde{K}Sp(X) & \xrightarrow[\cong]{\text{Bott}} & \tilde{K}O(X \wedge S^4) \end{array}$$

where $\frac{k^2}{2} \varepsilon$ denotes the map $\alpha \rightarrow \frac{k^2}{2} \varepsilon \cdot \alpha$ for $\alpha \in \widetilde{KO}(X)$.

Proof: For $k = 2, 4, \dots$

$$\begin{aligned} \psi_R^k \phi_1(\beta) &= \psi_R^k(p^*(\beta) \cdot q^*(v)) \\ &= \psi^k p^*(\beta) \cdot \psi^k q^*(v) \end{aligned} \quad (*)$$

$$= p^* \psi^k(\beta) \cdot q^* \psi^k(v) \quad (**)$$

$$= p^* \psi^k(\beta) \cdot q^*\left(\frac{k^2}{2} \varepsilon v\right) \quad \text{by the lemma}$$

$$= p^* \psi^k(\beta) \cdot \frac{k^2}{2} \varepsilon q^*(v)$$

$$= \frac{k^2}{2} \phi_1(\varepsilon \cdot \psi^k(\beta)) .$$

(*) because ψ^k is a ring homomorphism on $\widetilde{L}(-)$ and

$$\psi_R^k = \psi^k \Big|_{\widetilde{KO}(-)} \quad \text{for all } k = 1, 2, \dots$$

(**) because ψ^k are natural (Main theorem) .

Similarly for k odd. /

Chapter 3

§3.1. Computation for HP^n

First we recall some results.

Let η be the canonical complex line bundle over CP^n and let ξ be the canonical quaternionic line bundle over HP^n . Also cf. §1.6. for $c' : KSp(X) \rightarrow KU(X)$.

3.1.1 Theorem:

$KU(CP^n)$ is a truncated polynomial ring over Z with one generator $\mu = \eta - 1$ and one relation $\mu^{n+1} = 0$. Moreover:

$$\psi_C^k(\mu) = (1+\mu)^{k-1} \quad k = 1, 2, \dots$$

Proof: cf. [1]. /

For all $k = 1, 2, \dots$, there is a unique polynomial $T_k \in Z[X]$ such that:

$$T_k\left(z + \frac{1}{z} - 2\right) = z^k + \frac{1}{z^k} - 2 \quad .$$

3.1.2 Theorem

$KU(HP^n)$ is a truncated polynomial ring over Z with one generator $v = c' \xi - 2$ and one relation $v^{n+1} = 0$. Moreover:

$$\psi_C^k(v) = T_k(v) \quad k = 1, 2, \dots$$

Proof: The structure of $KU(HP^n)$ is well known. For the Adams operations, the canonical map $f : CP^{2n+1} \rightarrow HP^n$ induces a monomorphism:

$$f^* : KU(HP^n) \rightarrow KU(CP^{2n+1})$$

given by

$$f^*(c'\xi) = \eta + \bar{\eta} . \quad \text{By [1], th. 5.1, } \bar{\eta} = \eta^{-1} .$$

Hence $f^*(v) = f^*(c'\xi - 2) = \eta + \frac{1}{\eta} - 2$, and:

$$\begin{aligned} f^* \psi_C^k(v) &= \psi_C^k f^*(v) = \psi_C^k f^*(c'\xi - 2) \\ &= \psi_C^k(\eta + \eta^{-1} - 2) = \psi_C^k(\eta) + \psi_C^k(\eta^{-1}) - 2 \\ &= \eta^k + \frac{1}{\eta^k} - 2 = T_k(\eta + \frac{1}{\eta} - 2) \end{aligned}$$

Hence by naturality, $\psi_C^k(v) = T_k(v)$. /

We will now compute $L(HP^n)$ and the ψ^k 's on $L(HP^n)$ with the help of the theorem §1.6. We proceed through a series of lemmas.

3.1.3 Lemma

The cofibration $HP^{n-1} \xrightarrow{g_2} HP^n \xrightarrow{g_1} S^{4n}$ induces the exact sequences:

$$(1) \quad 0 \rightarrow \widetilde{L}(S^{4n}) \xrightarrow{g_1^*} \widetilde{L}(HP^n) \xrightarrow{g_2^*} \widetilde{L}(HP^{n-1}) \rightarrow 0$$

$$(2) \quad 0 \rightarrow \widetilde{LU}(S^{4n}) \xrightarrow{g_1^*} \widetilde{LU}(HP^n) \xrightarrow{g_2^*} \widetilde{LU}(HP^{n-1}) \rightarrow 0.$$

Proof: This results from a study of the long exact sequences of the cofibration in KO - and KU -theory. For example, we show that g_1^* is a monomorphism.

One has the following part of exact sequence:

$$\rightarrow \widetilde{KO}(S^{4n+1}) \xrightarrow{\alpha} \widetilde{KO}^{-1}(HP^n) \xrightarrow{\beta} \widetilde{KO}^{-1}(HP^{n-1}) \xrightarrow{\gamma} \widetilde{KO}(S^{4n}) \rightarrow$$

First one shows inductively that $\widetilde{KO}^{-1}(HP^n)$ is torsion or 0.

Indeed $\widetilde{KO}^{-1}(HP^1) \simeq \widetilde{KO}^{-1}(S^4) = 0$. Also $\widetilde{KO}(S^{4n+1}) = 0$ or Z_2 .

Suppose that $\widetilde{KO}^{-1}(HP^{n-1})$ is torsion or 0. Then $\gamma :$

$\widetilde{KO}^{-1}(HP^{n-1}) \rightarrow \widetilde{KO}(S^{4n}) \simeq Z$ is the 0 homomorphism. Hence β is an

epimorphism. But $\text{Ker } \beta = \text{Im } \alpha$ is 0 or Z_2 , so that $\widetilde{KO}^{-1}(HP^n)$

is torsion or 0, as wanted. From that, $\widetilde{KO}^{-1}(HP^{n-1}) \rightarrow \widetilde{KO}(S^{4n}) \simeq Z$

is the 0 homomorphism, hence $g_1^* : \widetilde{KO}(S^{4n}) \rightarrow \widetilde{KO}(HP^n)$ is a mono-

morphism. The same considerations about

$\rightarrow \widetilde{KO}(S^{4n+5}) \rightarrow \widetilde{KO}^{-5}(HP^n) \rightarrow \widetilde{KO}^{-5}(HP^{n-1}) \rightarrow \widetilde{KO}(S^{4n+4}) \rightarrow$ show that $g_1^* : \widetilde{KO}^{-4}(S^{4n}) \rightarrow \widetilde{KO}^{-4}(HP^n)$ is also a monomorphism. Hence

$$g_1^* : \widetilde{L}(S^{4n}) \rightarrow \widetilde{L}(HP^n)$$

is a monomorphism. (We use the natural isomorphism of §2.2). /

3.1.4. Corollary

$L(HP^n)$ is torsion free and $U : L(HP^n) \rightarrow LU(HP^n)$ is a monomorphism.

Proof: $L(HP^1) \simeq L(S^4)$ is torsion free (cf. §2.3 lemma). Then use induction on n with the exact sequence (1) of lemma 3.1.3. The second assertion of the corollary is due to §1.6. /

Let us denote again the elements of $LU(X)$ as pairs (α, β) , $\alpha, \beta \in KU(X)$. Also define:

$$\delta_i = \begin{cases} 1 & i \text{ even} \\ 2 & i \text{ odd} \end{cases}.$$

3.1.5. Lemma

$\text{Im } (U : \tilde{L}(HP^n) \rightarrow \tilde{LU}(HP^n))$ is the free abelian group generated by: $(2v, 0)$, $(v^2, 0)$, ..., $(\delta_n v^n, 0)$, $(0, v)$, $(0, 2v^2)$, ..., $(0, \delta_{n+1} v^n)$.

Proof: Lemma 3.1.3 and the natural transformation U give the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & \tilde{L}(S^{4n}) & \xrightarrow{g_1^*} & \tilde{L}(HP^n) & \xrightarrow{g_2^*} & \tilde{L}(HP^{n-1}) \rightarrow 0 \\ & & \downarrow U_1 & & \downarrow U_2 & & \downarrow U_3 \\ 0 & \rightarrow & \tilde{LU}(S^{4n}) & \xrightarrow{g_1^\#} & \tilde{LU}(HP^n) & \xrightarrow{g_2^\#} & \tilde{LU}(HP^{n-1}) \rightarrow 0 \end{array}$$

Let α be a generator of $\tilde{KU}(S^{4n}) \simeq \mathbb{Z}$. Since:

$$g_2^\#(v^i, 0) = (v^i, 0) \quad g_2^\#(0, v^i) = (0, v^i) \quad i = 1, 2, \dots, n-1$$

$$g_2^\#(v^n, 0) = (0, 0) \quad g_2^\#(0, v^n) = (0, 0)$$

one gets that:

$$g_1^\#(\alpha, 0) = (v^n, 0) \quad g_1^\#(0, \alpha) = (0, v^n) .$$

We also have that $\text{Im } U_1$ is the free abelian group generated by

$(\delta_n \alpha, 0)$ and $(0, \delta_{n+1} \alpha)$. (cf. [1]). By definition,

$$(0, v) = (0, c' \xi - 2) = U(\xi - \varepsilon) \in \text{Im } U_2$$

and also $(2v, 0) = (0, 2) \cdot (0, v) = U(\varepsilon) \cdot U(\xi - \varepsilon) \in \text{Im } U_2$. Hence

$$(2v, 0), (v^2, 0), \dots, (\delta_n v^n, 0), (0, v), (0, 2v^2), \dots, (0, \delta_{n+1} v^n) \in \text{Im } U_2 .$$

Now we can prove the lemma by induction on n . For $HP^1 \approx S^4$,
cf. §2.3, lemma. Assuming the result for HP^{n-1} , look at the split
exact sequence:

$$0 \rightarrow \text{Im } U_1 \rightarrow \text{Im } U_2 \rightarrow \text{Im } U_3 \rightarrow 0 .$$

We can define a splitting map $H : \text{Im } U_3 \rightarrow \text{Im } U_2$ by

$$H(\delta_i v^i, 0) = (\delta_i v^i, 0) \quad \text{and} \quad H(0, \delta_{i+1} v^i) = (0, \delta_{i+1} v^i) \quad \text{for}$$

$i = 1, \dots, n-1$. Then we see that

$$\text{Im } U_2 \simeq g_2^\#(\text{Im } U_1) \oplus H(\text{Im } U_3)$$

is as described in the lemma. /

Theorem: The ring $L(HP^n)$ is generated by $1, \varepsilon$, and $\tau = \xi - \varepsilon$,
with the relations $\varepsilon^2 = 4$ and $\tau^{n+1} = 0$. Moreover:

$$\psi^k(\tau) = \begin{cases} \frac{\varepsilon}{2} T_k\left(\frac{\varepsilon}{2}\tau\right) & k = 1, 3, 5\dots \\ T_k\left(\frac{\varepsilon}{2}\tau\right) & k = 2, 4, 6\dots \end{cases}$$

Proof: By 3.1.5, $\text{Im}(U:L(\text{HP}^n) \rightarrow \text{LU}(\text{HP}^n))$ is generated by $(1,0)$, $(0,2)$, $(0,v)$. Since $U(\tau) = (0,v)$ and $U(\varepsilon) = (0,2)$ we have the structure of $L(\text{HP}^n)$.

Now use 3.1.2 to compute the ψ^k 's on $\text{LU}(\text{HP}^n)$. The obvious result is:

$$\psi^k(0,v) = \begin{cases} (0, T_k(v)) & k = 1, 3, 5\dots \\ (T_k(v), 0) & k = 2, 4, 6\dots \end{cases}.$$

Noting that $(v^i, 0)$ and $(0, v^i)$ can be written respectively as

$$\frac{U(\varepsilon) \cdot U(\tau)^i}{2} \quad \text{and} \quad U(\varepsilon) \cdot \frac{U(\varepsilon) \cdot U(\tau)^i}{2} \quad \text{for } i = 1, 2, \dots, n,$$

we use the fact that U commutes with the ψ^k 's, we get the result stated in the theorem. /

Examples

$$\psi^2(\tau) = \tau^2 + 2\varepsilon\tau$$

$$\psi^3(\tau) = \tau^3 + 3\varepsilon\tau^2 + 9\tau$$

Proof: $T_2(x) = x^2 + 4x$

$$T_3(x) = x^3 + 6x^2 + 9x.$$

§3.2. Computation for $\mathbb{CP}^n$

We still denote by μ a generator of $\widetilde{KU}(\mathbb{CP}^n)$. Let g be a generator of $\widetilde{KU}(S^2) \simeq \mathbb{Z}$. Let $\mu_0 = r(\mu) \in \widetilde{KO}(\mathbb{CP}^n)$ and

$\mu_2 = r(\mu g^2) \in \widetilde{KO}^{-4}(\mathbb{CP}^n)$, where $r : \widetilde{KU}(X) \rightarrow \widetilde{KO}(X)$ is as in §1.6.

Let $v_2 = \phi_1^{-1}(\mu_2) \in \widetilde{KSp}(\mathbb{CP}^n)$.

3.2.1. Theorem

Let n be even. $\widetilde{L}(\mathbb{CP}^n)$ is the free abelian group generated by $\mu_0, \mu_0^2, \dots, \mu_0^{\frac{n}{2}-1}, v_2, v_2\mu_0, \dots, v_2\mu_0^{\frac{n}{2}-1}$. The multiplicative

structure is completed by the relation $v_2^2 = \mu_0^2$. Moreover:

$$\psi^k(\mu_0) = T_k(\mu_0) \quad k = 1, 2, \dots$$

$$\psi^k(v_2) = \begin{cases} T_k(\mu_0) & k = 2, 4, \dots \\ \frac{v_2}{\mu_0} T_k(\mu_0) & k = 1, 3, \dots \end{cases}$$

Proof: [6] gives the structure of $\widetilde{KO}(\mathbb{CP}^n) \oplus \widetilde{KO}^{-4}(\mathbb{CP}^n)$. Hence we have the structure of $\widetilde{L}(\mathbb{CP}^n)$ by §2.2. In order to compute the ψ^k 's, we remark that $\widetilde{L}(\mathbb{CP}^n)$ is torsion-free (for n even). Therefore, $U : \widetilde{L}(\mathbb{CP}^n) \rightarrow \widetilde{LU}(\mathbb{CP}^n)$ is a monomorphism. We first determine this map.

Let $\Phi : \widetilde{KU}(X) \rightarrow \widetilde{KU}^{-2}(X)$ be the complex Bott isomorphism as described in [3] §2.2 for instance. The following diagram is

commutative (compare [5])

$$\begin{array}{ccc} \widetilde{KO}^{-4}(X) & \xrightarrow{c} & \widetilde{KU}^{-4}(X) \\ \uparrow \Phi_1 & & \uparrow \Phi^2 \\ \widetilde{KSp}(X) & \xrightarrow{c'} & \widetilde{KU}(X) \end{array}$$

where c, c' as in §1.6. Moreover $c \circ r = 1 + (\overline{}) : KU(X) \rightarrow KU(X)$ (cf [6] for instance).

With the help of these remarks, we find out that:

$$U(\mu_0) = (c \circ r(\mu), 0) = (\mu + \overline{\mu}, 0)$$

$$U(\nu_2) = (0, \Phi^{-2} c'(\Phi_1(\nu_2)))$$

$$= (0, \Phi^{-2} c'(\mu_2))$$

$$= (0, \Phi^{-2}(g^2 \mu + \overline{g^2 \mu}))$$

$$= (0, \Phi^{-2}(g^2(\mu + \overline{\mu}))) \quad \text{since } \overline{g^2} = g^2 \text{ (cf [6])}$$

$$= (0, \mu + \overline{\mu}) .$$

By theorem 3.1.1 we can compute $\psi^k : LU(CP^n) \rightarrow LU(CP^n) :$

$$\psi^k(\mu, 0) = ((1+\mu)^{k-1}, 0) \quad k = 1, 2, \dots$$

$$\psi^k(0, \mu) = \begin{cases} (0, (1+\mu)^{k-1}) & k = 1, 3, \dots \\ ((1+\mu)^{k-1}, 0) & k = 2, 4, \dots \end{cases} .$$

From this, we get:

$$\psi^k(\mu+\bar{\mu}, 0) = (T_k(\mu+\bar{\mu}), 0) \quad k = 1, 2, \dots$$

$$\psi^k(0, \mu+\bar{\mu}) = \begin{cases} (0, T_k(\mu+\bar{\mu})) & k = 1, 3, \dots \\ (T_k(\mu+\bar{\mu}), 0) & k = 2, 4, \dots \end{cases}$$

(Details as in proof of 3.1.2). Finally, since

$$((\mu+\bar{\mu})^i, 0) = U(\mu_0)^i$$

$$(0, (\mu+\bar{\mu})^i) = U(\mu_0)^{i-1} U(v_2)$$

for $i = 1, 2, \dots, \frac{n}{2}$, we get the result as stated in the theorem. /

3.2.2. Theorem

(i) For $n = 4t + 1$, $\widetilde{L}(\mathbb{CP}^n)$ is the direct sum of the free abelian group generated by $\mu_0, \mu_0^2, \dots, \mu_0^{2t}, v_2, v_2\mu_0, \dots, v_2\mu_0^{2t-1}$,

and the cyclic group of order two generated by μ_0^{2t+1} . The multiplicative structure is completed by the relation $v_2^2 = \mu_0^2$.

(ii) For $n = 4t + 3$, $\widetilde{L}(\mathbb{CP}^n)$ is the direct sum of the free abelian group generated by $\mu_0, \mu_0^2, \dots, \mu_0^{2t+1}, v_2, v_2\mu_0, \dots, v_2\mu_0^{2t}$ and the cyclic group of order two generated by $v_2\mu_0^{2t+1}$. The multiplicative structure is completed by $v_2^2 = \mu_0^2$.

Moreover

$$\begin{aligned}\psi^k(\mu_0) &= T_k(\mu_0) & k &= 1, 2, \dots \\ \psi^k(\nu_2) &= \begin{cases} T_k(\mu_0) & k = 2, 4, \dots \\ \frac{\nu_2}{\mu_0} T_k(\mu_0) & k = 1, 3, \dots \end{cases}\end{aligned}$$

where the suitable coefficient has to be taken modulo 2 in each case.

Proof: [6] and our §2.2 give again the structure of $\widetilde{L}(\mathbb{CP}^n)$.

For the ψ^k 's, the natural inclusion $\mathbb{CP}^n \rightarrow \mathbb{CP}^{n+1}$ induces an epimorphism:

$$\widetilde{L}(\mathbb{CP}^{n+1}) \twoheadrightarrow \widetilde{L}(\mathbb{CP}^n).$$

$n+1$ is even. Hence, using the result just found for $\widetilde{L}(\mathbb{CP}^{n+1})$, it is clear out to get the result for $\widetilde{L}(\mathbb{CP}^n)$, by naturality of the ψ^k 's. /

3.2.3. Remark

The result for the truncated complex projective spaces are now also obvious. Looking at the cofibration $\mathbb{CP}^\ell \rightarrow \mathbb{CP}^{n+\ell} \xrightarrow{\pi} \mathbb{CP}^{n+\ell}/\mathbb{CP}^\ell$, one observes that π induces an embedding:

$$\pi^* : \widetilde{L}(\mathbb{CP}^{n+\ell}/\mathbb{CP}^\ell) \rightarrow \widetilde{L}(\mathbb{CP}^{n+\ell})$$

since $\tilde{K}O^{-5}(CP^{\ell}) = \tilde{K}O^{-1}(CP^{\ell}) = 0$. It is then clear how to get the results for $CP^{n+\ell}/CP^{\ell}$.

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