A COMPARISON OF NONPARAMETRIC TESTS OF INDEPENDENCE

by

AMANDA FRANCES NEMEC

B.Sc., University of Victoria, 1975

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

Department of Mathematics

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

August, 1978

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date August 22, 1978.

Abstract

The nonparametric tests of bivariate independence, based on Spearman's rho, Kendall's tau, the Blum-Kiefer-Rosenblatt statistic, the Fisher-Yates normal scores coefficient and the quadrant sum are compared with the parametric test based on the ordinary correlation coefficient. The tests are compared in the bivariate normal case by recording the Pitman and Bahadur efficiencies of each test. The empirical powers resulting from a Monte Carlo study are also given for the tests.

The components of the Blum-Kiefer-Rosenblatt statistic are derived and are related to linear rank statistics. The rank statistic associated with the first component is suggested as a new nonparametric test of independence. This test is included in the comparison and is shown to perform reasonably well.

As one sided tests of independence in bivariate populations the Fisher-Yates coefficient and the sample correlation coefficient are to be preferred over the other tests. As two sided tests the Blum-Kiefer-Rosenblatt statistic is the best test for alternatives near the null hypothesis. When the alternatives are distant from the null hypothesis the Fisher-Yates coefficient or the sample correlation coefficient should be used. The quadrant sum always performs poorly while the other nonparametric tests, including that based on the first component of the Blum-Kiefer-Rosenblatt statistic are acceptable tests.

TABLE OF CONTENTS

LIST OF TABLES

ACKNOWLEDGEMENT

I wish to thank Dr. J. Koziol and Dr. J. Petkau for their help and advice while preparing this thesis. I would also like to thank my husband, Jim, and my family, for their great encouragement and support, Mr. M.E. Linnell for proof-reading the text, and Mrs. F.I. Linnell for typing the thesis.

The research and work for this thesis was supported by a National Research Council of Canada Postgraduate Scholarship.

CHAPTER 1

INTRODUCTION

Given a bivariate population (X,Y) it is often necessary to test for the independence of X and Y using a sample of n observations, (X_i,Y_i), $i=1,2,\ldots,n$. The usual test of independence is to compute the sample correlation coefficient, r_n, directly from the observations and perform a t-test. The nonparametric tests based on Spearman's rho, Kendall's tau, the Fisher-Yates normal scores coefficient, the quadrant sum or the Blum-Kiefer-Rosenblatt statistic depend only on the ranks (R_i,S_i), $i=1,2,\ldots,n$, corresponding to the observations. This thesis compares the performance of these nonparametric tests with the parametric test based on r_n.

The components of the Blum-Kiefer-Rosenblatt statistic are derived. These components may be related to linear rank statistics. The linear rank statistic associated with the first component is suggested as a test of independence and is compared with the other tests.

The tests are compared assuming (X,Y) has a bivariate normal distribution. The Pitman and Bahadur efficiencies of each nonparametric test relative to the test based on r_n are given. The empirical powers of the tests, derived from a Monte Carlo experiment, are also given. These three pieces of information provide a useful comparison of the nonparametric tests of independence.

In the one sided testing situation the tests based on the Fisher-Yates coefficient and on r_n emerge as the best tests of independence. Both tests perform equally well and are to be

preferred over all the other tests. The quadrant test, although simple to execute, lies at the other extreme, being by far the worst test. Slightly inferior to the two best tests are Spearman's and Kendall's tests, which exhibit a similar behavior to each other, and the test based on the Blum-Kiefer-Rosenblatt first component. Kendall's and Spearman's tests have the advantage over the other nonparametric tests, that being well studied, tables of critical values are readily available.

In testing against a two sided alternative, the overall Blum-Kiefer-Rosenblatt test is the best test in regions near the null. Moving away from the null a cross over takes place and an ordering of the tests similar to the one sided situation is assumed, with the Blum-Kiefer-Rosenblatt test slightly poorer than the r_n and Fisher-Yates tests. The Blum-Kiefer-Rosenblatt test should be used when it is known that the alternative is close to the null hypothesis, otherwise the Fisher-Yates test or the correlation coefficient test should be selected.

CHAPTER 2

CRITERIA FOR THE COMPARISON OF TESTS

2.0 Introduction

In comparing two tests of the null hypothesis against the alternative, the obvious approach is an examination of the appropriate power functions. The power function is, however, a function of the sample size, n, the size of the test, α, and the parameter, θ, which "defines" the two hypotheses. Any statement of one test being "better" than the other must be accompanied by a statement of the conditions for which it holds true. For example, for fixed sample size and fixed α, it may be said that one test is better than the other if it has greater power for all θ in the alternative. Or, if the sample size can vary, it may be said that one size α test is better than the other if it requires a smaller sample size to achieve the same power against the same alternative. The ratio of sample sizes is a measure of the relative efficiency of the two tests against that alternative. A complete comparison of two tests would be complicated; it is desirable to eliminate some of the variables (sample size, size, etc.) in order to make a more direct comparison possible. To do this the "relative efficiency" of one test with respect to the other, in the "limiting" case only, is often used. Pitman and Bahadur efficiencies are two such "asymptotic relative efficiencies".

2.1 Pitman Asymptotic Relative Efficiency (or asymptotic relative efficiency or Pitman efficiency, for short)

Given two tests, both of the same size, α, the Pitman efficiency is the limiting relative efficiency of the second test

with respect to the first, as the sample size increases without bound and at the same time the alternative approaches the null hypothesis. The relative efficiency of the second test with respect to the first is the ratio of the sample sizes, n_1/n_2, required by the respective tests, to achieve the same power against the same alternative. Although various authors have generalized the notion, Pitman efficiency will be defined here in the usual way (Noether 1955, Kendall and Stuart 1973). Consider a test based on the statistic T_n which is consistent for testing $H_0:\theta=\theta_0$ against $H_1:\theta>\theta_0$, where T_n is computed from n observations. Suppose there exist functions of θ, ψ_n and σ_n, (usually $ET_n=\psi_n(\theta)$ and $VarT_n=\sigma_n^2(\theta)$) such that:

A(i) $\psi_n'(\theta_0)=\psi_n''(\theta_0)=\ldots=\psi_n^{(m-1)}(\theta_0)=0 \quad \psi_n^{(m)}(\theta_0)>0$

A(ii) $\lim_{n\to\infty} n^{-m\delta}\psi_n^{(m)}(\theta_0)/\sigma_n(\theta_0)=v>0$ for some $\delta>0$

(assuming the existence of these derivatives)

It is now possible to define the simple alternative

$$H_n:\theta=\theta_n=\theta_0+\frac{k}{n^\delta}$$

where k is some positive constant. The following further assumptions are also made:

A(iii) $\lim_{n\to\infty} \psi_n^{(m)}(\theta_n)/\psi_n^{(m)}(\theta_0)=1, \quad \lim_{n\to\infty} \sigma_n(\theta_n)/\sigma_n(\theta_0)=1$

A(iv) $\dfrac{T_n-\psi_n(\theta_n)}{\sigma_n(\theta_n)}$ is asymptotically normal with mean 0 and variance 1, under the null hypothesis $H_0:\theta_n=\theta_0$ and under the alternative hypothesis H_1 (although the asymptotic normality requirement is not necessary for the general definition of Pitman efficiency this is the only case considered as all the statistics to be compared will satisfy the normality assumption).

For large samples, under the above assumptions, the size α test of H_0 versus H_1 may be written in a form:

$$\text{Reject } H_0 \text{ if } T_n > \psi_n(\theta_0) + \lambda_\alpha \sigma_n(\theta_0)$$

where $\Phi(\lambda_\alpha)=\alpha$, Φ is the standard normal cumulative distribution function. The approximate power of this test against the alternative H_n is

$$P_n(\theta_n) = \Phi\left\{\frac{-\psi_n(\theta_0) - \lambda_\alpha\sigma_n(\theta_0) + \psi_n(\theta_n)}{\sigma_n(\theta_n)}\right\}$$

Expanding $\psi_n(\theta_n)-\psi_n(\theta_0)$ in a Taylor series about θ_0 and under the regularity conditions of assumption A(iii) the argument of Φ above, may be approximated by $\frac{k^m v}{m!} - \lambda_\alpha$.

The approximate power of the test is, therefore,

$$P_n(\theta_n) \simeq \Phi\left\{\frac{k^m v}{m!} - \lambda_\alpha\right\}$$

Assume the tests based on T_{1n} and T_{2n} are two consistent tests of H_0 versus H_1, and that T_{in} satisfies A(i)-A(iv) with $\psi_n=\psi_{in}$, $\sigma_n=\sigma_{in}$, $m=m_i$, $\delta=\delta_i$, and $v=v_i$ for $i=1,2$. The tests will then have asymptotic powers

$$\Phi\left\{v_i \frac{k_i^{m_i}}{m_i!} - \lambda_\alpha\right\}, \quad i=1,2$$

against the respective alternatives $H_{in}:\theta=\theta_{in}=\theta_0+\frac{k_i}{n^{\delta i}}$, $i=1,2$.

As stated before, Pitman efficiency requires that the tests have the same power against the same alternative. This is achieved if

$$\frac{v_1 k_1^{m_1}}{m_1!} = \frac{v_2 k_2^{m_2}}{m_2!} \quad \text{and} \quad \frac{k_1}{n_1^{\delta_1}} = \frac{k_2}{n_2^{\delta_2}}$$

Equivalently, the condition

$$\frac{n_1^{\delta_1}}{n_2^{\delta_2}} = \left(\frac{v_2\ m_1!\ k_2^{(m_2-m_1)}}{v_1\ m_2!}\right)^{1/m_1}$$

guarantees the same power against the same alternative. The Pitman efficiency of T_{2n} relative to T_{1n}, A_{21}, is defined to be

$$A_{21} = \begin{cases} 0 & \text{if} \quad \delta_1 > \delta_2 \\ \infty & \text{if} \quad \delta_1 < \delta_2 \\ \lim\limits_{n_1, n_2 \to \infty} \dfrac{n_1}{n_2} & \text{if} \quad \delta_1 = \delta_2 \end{cases}$$

In the case $\delta_1 = \delta_2$ the expression for the Pitman efficiency is

$$A_{12} = \lim_{n_1, n_2 \to \infty} \frac{n_1}{n_2} = \left(\frac{v_2\ m_1!\ k_2^{(m_2-m_1)}}{v_1\ m_2!}\right)^{1/m_1\delta}$$

Frequently, $m_1 = m_2 = m$ as well as $\delta_1 = \delta_2$ and A_{21} is simplified to be

$$A_{21} = \left(\frac{v_2}{v_1}\right)^{\frac{1}{m\delta}} = \lim_{n\to\infty}\left\{\frac{\psi_{2n}^{(m)}(\theta_0)/\sigma_{2n}(\theta_0)}{\psi_{1n}^{(m)}(\theta_0)/\sigma_{1n}(\theta_0)}\right\}^{1/m\delta} \qquad 2.1.1$$

The last expression is often given as the definition of asymptotic relative efficiency.

It should be noted that the Pitman efficiency may also be interpreted as a limiting ratio of the derivatives of the power functions of the two tests (Blomqvist 1950, Noether 1955; Kendall and Stuart 1973). If the Pitman efficiency turns out to be independent of α, as it does for asymptotically normal statistics, it is a single summary measure of how well one test does compared with the other, in the region of the null hypothesis.

2.2 Bahadur Relative Efficiency (or Bahadur efficiency, for short)

Given two tests of the null versus the alternative hypothesis, the Bahadur efficiency is the limiting relative efficiency of the two tests as the sample size increases without bound and at the same time the size of each test approaches 0. In order to compute the Bahadur efficiency it is necessary to define the "exact slope" associated with a sequence of test statistics, $\{T_n\}_{n=1}^{\infty}$, as in Bahadur (1967). The notation used here is Bahadur's.

Consider a test statistic, T_n, based on n observations, for testing the null hypothesis, $H_0:\theta=\theta_0$ against the one sided alternative, $H_1:\theta>\theta_0$. The test rejects H_0 when T_n is greater than some constant. Let $L_n(T_n)=1-F_n(T_n)$, where $F_n(t)=P_{\theta_0}(T_n<t)$, be the probability under the null hypothesis of obtaining T_n greater than or equal to the observed value. Now, $L_n \to 0$[1] with probability (w.p.) 1, under the alternative, exponentially fast, typically. If then, there exists a function of θ, $c(\theta)$, defined over the alternative, H_1, such that c is positive and finite and $n^{-1}\log L_n \to -\frac{1}{2}c(\theta)$ as $n\to\infty$, w.p. 1, when $\theta>\theta_0$ is the true parameter value, then $c(\theta)$ is called the exact slope of the sequence $\{T_n\}$.

Bahadur defines for two test sequences $\{T_{in}\}_{n=1}^{\infty}$, i=1,2 with corresponding exact slopes $c_i(\theta)$, i=1,2, $B_{21}=c_2(\theta)/c_1(\theta)$; called the Bahadur efficiency, as a measure of the asymptotic relative efficiency of T_{2n} with respect to T_{1n} when θ is the parameter. (Here the asymptotic efficiency is in the sense described at the beginning of this section). Bahadur efficiency assesses the asymptotic efficiency for all θ while Pitman efficiency has meaning

[1] In the above discussion the convergence is w.p. 1. This is relaxed to convergence in probability under the alternative hypothesis, by Woodworth (1970).

only in the neighbourhood of the null hypothesis.

Bahadur (1967) outlines the following procedure for determining the existence and evaluation of the exact slope.

Suppose $T_n/n^{\frac{1}{2}} \to b(\theta)$ w.p. 1 where $\theta > \theta_0$ and b is a positive and finite function defined on H_1. Further, suppose $n^{-1}\log(1-F_n(n^{\frac{1}{2}}t)) \to -f(t)$ as $n \to \infty$ for every t in an open interval including each value of b, where f is continuous on the interval, finite and positive. Then $c(\theta)$ exists for each $\theta \varepsilon H_1$ and equals $2f(b(\theta))$.

Very often the exact distribution, $F_n(t)$, is not known. In such a case it may still be possible to get an approximation to the exact slope, if the asymptotic distribution is known. Bahadur (1960) defines a "standard sequence", $\{T_n\}$, for which it is possible to compute the "approximate slope".

A sequence of test statistics $\{T_n\}_{n=1}^{\infty}$ is called a standard sequence if:

B(i) There exists a continuous probability distribution, F, such that

$$\lim_{n\to\infty} P_{\theta_0}(T_n < t) = F(t) \qquad \forall t$$

B(ii) There exists a constant, a, $0<a<\infty$, such that

$$\log(1-F(t)) = -at^2/2(1+o(1)) \quad \text{as } t\to\infty$$

B(iii) There exists a function, b, on $\theta > \theta_0$, with $0<b<\infty$, such that $\forall\ \theta > \theta_0$

$$\lim_{n\to\infty} P_\theta(|T_n/n^{\frac{1}{2}} - b(\theta)| > t) = 0 \qquad \forall t>0$$

For a standard sequence, Bahadur (1960) shows that $\log L_n^{(a)}/n \to -\frac{1}{2}c^{(a)}(\theta)$ in probability for $\theta > \theta_0$. The superscript "(a)" refers to approximate values. L_n is approximated by $L_n^{(a)} = 1-F(T_n)$ and $c^{(a)}(\theta) = a(b(\theta))^2$ is the approximate slope (or inexact slope).

Sometimes condition B(iii) above, is replaced by the stronger

condition

B(iii)* $P_\theta(\lim_{n\to\infty} T_n/n^{\frac{1}{2}}=b(\theta))=1$ for $\theta>\theta_0$

in which case, $\log L_n^{(a)}/n \to -\frac{1}{2}c^{(a)}(\theta)$ w.p. 1 $\forall\theta>\theta_0$

The definition of the approximate slope parallels that of the exact slope with the exact distribution replaced by the asymptotic distribution as an approximation. Similarly given two standard sequences, $\{T_n^{(i)}\}_{n=1}^{\infty}$, i=1,2 with approximate slopes, $c_i^{(a)}(\theta)$, i=1,2, $B_{12}^{(a)}=c_1^{(a)}(\theta)/c_2^{(a)}(\theta)$, the approximate Bahadur efficiency, can be interpreted as an asymptotic measure of the relative efficiency of the two sequences.

Bahadur (1960 and 1967) states that the use of the approximate slope is often informative but gives examples where it is a very bad approximation to the exact slope. In general, the inexact slope is reasonable in the neighbourhood of the null hypothesis.

The approximate Bahadur efficiency generally depends on the parameter, θ . The limit, as θ approaches the null value, θ_0, is called the limiting Bahadur efficiency. In the case of one-sided tests, the limiting Bahadur efficiency is equal to the Pitman efficiency. (Bahadur 1960, Wieand 1976)

Pitman and Bahadur efficiencies serve as two means of comparing the selected nonparametric tests of independence with the parametric test based on the correlation coefficient. The main reason for comparing asymptotic behaviour is that the distribution functions, and therefore power functions, of some of the nonparametric statistics are unknown or are difficult to work with in the finite sample case. Where possible the exact slopes will be given to allow for comparison in regions distant from the null. Otherwise approximate slopes will be given.

Pitman efficiencies and approximate Bahadur efficiencies provide comparison criteria in regions near the null.

2.3 A Monte Carlo Study

Pitman and Bahadur efficiencies provide comparison criteria of tests when the sample size is large. The comparison of the tests would not be complete without consideration of the performance of the tests when the sample is small. The easiest way to examine the small sample behaviour is through a Monte Carlo study. A large number of samples of given size are generated under a fixed alternative. The tests are carried out for each sample and the relative frequencies of making the correct decision (reject the null hypothesis) are recorded. These empirical powers provide a way to assess the performance of the tests as a function of sample size and the alternative, when the experiment is repeated for chosen values of n and for different alternatives. The size of the test remains fixed at the same constant value for all tests.

CHAPTER 3

TESTS TO BE COMPARED

3.0 Introduction

Nonparametric tests for the independence of two random variables, like other nonparametric tests, require that only very weak assumptions be made. In all that follows it is assumed that a random sample of n independent bivariate observations is available, denoted by $(X_1,Y_1),(X_2,Y_2)\ldots(X_n,Y_n)$. Further, it is assumed that the sample is drawn from a population with distribution function, $F(x,y)=\text{Prob}(X\leqslant x,Y\leqslant y)$. It is desired to test the null hypothesis, H_0, that X and Y are independent, or $H_0:F(x,y)=F(x,\infty)F(\infty,y)$. The tests to be discussed are all rank tests, that is, the test statistic, T_n, depends on the observations only through their ranks, (R_i,S_i), $i=1,2\ldots n$, where $R_i(S_i)$ is the rank of $X_i(Y_i)$ in the joint ranking of the n X's (Y's). Finally, it is assumed that F is continuous, thereby ruling out the possibility of tied ranks.

The tests to be compared are those based on Kendall's tau, Spearman's rho, the Fisher-Yates normal scores statistic, the quadrant sum, and the Blum-Kiefer-Rosenblatt statistic and its components. All of these test statistics, with the exception of Kendall's tau and the Blum-Kiefer-Rosenblatt statistic, are linear rank statistics. A linear rank statistic in the bivariate case, is a statistic of the form,

$$T_n=n^{-\frac{1}{2}}\sum_{i=1}^{n}a_n(R_i)b_n(S_i) \qquad 3.0.1$$

where a_n and b_n are the scores in the terminology of Hájek and Šidák (1967). Each statistic will now be defined in a way

facilitating efficiency calculations.

3.1 Spearman's Rho

Spearman (1904) introduced a distribution-free statistic which may be used to test H_0. The statistic r_s, is the sample correlation coefficient applied to the ranks, (R_i,S_i), $i=1,2...n$, and is given by

$$r_s=12n^{-1}(n^2-1)^{-1}\sum_{i=1}^{n}(R_i-\tfrac{1}{2}(n+1))(S_i-\tfrac{1}{2}(n+1)).$$

It is clear that r_s is a multiple of

$$T_{sn}=n^{-\frac{1}{2}}\sum_{i=1}^{n}12(R_i/(n+1)-\tfrac{1}{2})(S_i/(n+1)-\tfrac{1}{2})$$

and that T_{sn} is a linear rank statistic. The scores are generated by the function $\zeta(u)=\sqrt{12}(u-\tfrac{1}{2})$, $0\leqslant u\leqslant 1$, in the following way,

$$a_n(i)=b_n(i)=\zeta(i/(n+1)) \qquad i=1,2...n.$$

It may be argued (Kruskal 1958) that r_s is a reasonable estimate of $\rho_s=6\text{Prob}\{(X-X')(Y-Y'')>0\}-3$, where $(X,Y),(X',Y')$, (X'',Y'') are arbitrary, independent observations from the distribution function F. If X and Y are independent ρ_s is 0, although the converse is not true in general.

r_s may also be written as

$$r_s=1-6n^{-1}(n^2-1)^{-1}\sum_{i=1}^{n}(R_i-S_i)^2.$$

3.2 The Fisher-Yates Normal Scores Statistic

The Fisher-Yates normal scores statistic is given by,

$$f=\sum_{i=1}^{n}a_n(R_i)b_n(S_i)$$

where $a_n(i)=b_n(i)=EV_n^{(i)}$ and $V_n^{(i)}$ is the ith largest observation of a sample of size n, drawn from a standard normal population. Apart from some constants, f is the correlation coefficient of the expected normal order statistics corresponding to the

observations. In certain circumstances the statistic arises when a locally most powerful rank test is desired. (Hájek and Šidák 1967) For the purposes of this thesis the statistic,

$$T_{fn}=n^{-\frac{1}{2}}f=n^{-\frac{1}{2}}\sum_{i=1}^{n}a_n(R_i)b_n(S_i)$$

will be used to correspond exactly to the linear rank statistic of 3.0.1. The scores may also be defined in terms of the generating function $\zeta(u)=\Phi^{-1}(u)$, $0\leq u\leq 1$, (Φ is the cumulative distribution function of a standard normal random variable) by noting,

$$a_n(i)=b_n(i)=E\zeta(U_n^{(i)})$$

where $U_n^{(i)}$ is the ith largest observation in a sample of size n from the Uniform (0,1) distribution.

3.3 The Quadrant Sum

Blomqvist (1950) studies the statistic, q, which he attributes to Mosteller (1946). q, as defined by Blomqvist is

$$q=\frac{n_1-n_2}{n_1+n_2}=\frac{n_1-n_2}{n}$$

where n_1=the number of points lying in the first or third quadrants, and n_2=the number of points lying in the second or fourth quadrants. The lines $x=m_x$ and $y=m_y$ (m_x and m_y are the sample medians of the n X values and the n Y values, respectively) divide the (x,y) plane into these four quadrants. For simplicity the sample size, n, is assumed to be even to avoid the situation where a point falls on one of the lines $x=m_x$ and $y=m_y$.

Blomqvist (1950) demonstrates that q is a consistent estimate of $\eta=2\text{Prob}\{(X-\bar{\mu}_x)(Y-\bar{\mu}_y)>0\}-1$, where $\bar{\mu}_x$ and $\bar{\mu}_y$ are the population medians. η may be thought of as a measure of the correlation between X and Y which is 0 when X and Y are independent

(the converse is not necessarily true).

In order to relate q to 3.0.1, n_1-n_2 may be written as

$$n_1-n_2=\sum_{i=1}^{n}\text{sign}(R_i-\tfrac{1}{2}(n+1))\,\text{sign}(S_i-\tfrac{1}{2}(n+1))$$
$$=\sum_{i=1}^{n}\text{sign}((n+1)^{-1}R_i-\tfrac{1}{2})\,\text{sign}((n+1)^{-1}S_i-\tfrac{1}{2})$$

sign(v) is defined in the usual way,

$$\text{sign}(v)=\begin{cases}1 & \text{if } v>0\\ 0 & \text{if } v=0\\ -1 & \text{if } v<0\end{cases}$$

Now, $q=n^{-\frac{1}{2}}T_{qn}$ where

$$T_{qn}=n^{-\frac{1}{2}}\sum_{i=1}^{n}\text{sign}((n+1)^{-1}R_i-\tfrac{1}{2})\,\text{sign}((n+1)^{-1}S_i-\tfrac{1}{2}).$$

The statistic T_{qn} is in the desired form and the score functions, a_n and b_n, of 3.0.1 are given by

$$a_n(i)=b_n(i)=\zeta(i/n+1).$$

The score generating function, ζ , is defined to be

$$\zeta(u)=\text{sign}(u-\tfrac{1}{2}) \qquad 0\leqslant u\leqslant 1.$$

3.4 Kendall's Tau

Although the statistic appeared in the literature before Kendall's extensive work on its properties (Kruskal 1958), it is commonly known as Kendall's tau. Kendall's tau, t_n, is defined as

$$t_n=n^{-1}(n-1)^{-1}\sum\sum_{i\neq j}\text{sign}(R_i-R_j)\,\text{sign}(S_i-S_j).$$

t_n is not a linear rank statistic but Hájek and Šidák (1967) compute the projection, $\hat{t}_n$, of t_n into the family of linear rank statistics, under the null hypothesis. Namely,

$$\hat{t}_n=8n^{-2}(n-1)^{-1}\sum_{i=1}^{n}(R_i-\tfrac{1}{2}(n+1))(S_i-\tfrac{1}{2}(n+1)).$$

Comparing this with Spearman's rho of 3.1, it is apparent that $\hat{t}_n=\frac{2}{3}(n+1)n^{-1}r_s$.

t_n is an estimate of $\tau=2\text{Prob}\{(X-X')(Y-Y')>0\}-1$, where (X,Y) and (X',Y') are independent observations from F(Kruskal 1958).

Under H_0, $\tau=0$, although $\tau=0$ does not necessarily imply that H_0 holds. τ can be interpreted as a measure of "agreement" or correlation between X and Y and therefore a test based on t_n is a reasonable test of independence.

3.5 The Blum-Kiefer-Rosenblatt Statistic And Its Components

Let $F_n(x,\infty), F_n(\infty,y)$ be the marginal distribution functions of the sample joint distribution function $F_n(x,y)$ where,

$$F_n(x,y)=n^{-1}(\text{the number of pairs } (X_i,Y_i) \text{ with } X_i\leq x \text{ and } Y_i\leq y).$$

Define the random process $Q_n(x,y)$ such that,

$$Q_n(x,y)=\sqrt{n}(F_n(x,y)-F_n(x,\infty)F_n(\infty,y)).$$

Blum, Kiefer, and Rosenblatt (1961) propose the statistic

$$B_n=n^{-1}\int Q_n^2(x,y)\ dF_n(x,y)$$

as the basis of a test for the independence of X and Y.

B_n may also be written as the sum,

$$B_n=n^{-1}\sum_{i=1}^{n}(\#(j\ni X_j<X_i, Y_j<Y_i)n^{-1}-n^{-2}\#(j\ni X_j<X_i)\#(j\ni Y_j<Y_i))^2.$$

It may be seen that any test based on B_n is nonparametric by writing B_n in terms of the ranks.

$$B_n=n^{-1}\sum_{i=1}^{n}(n^{-1}\sum_{j=1}^{n}\text{sign}^+(R_i-R_j)\text{sign}^+(S_i-S_j)-n^{-2}\sum_{j=1}^{n}\text{sign}^+(R_i-R_j)\sum_{j=1}^{n}\text{sign}^+(S_i-S_j))^2$$

where $\text{sign}^+(v)=\begin{cases}1 & \text{if } v\geq 0\\ 0 & \text{if } v<0\end{cases}$

Hoeffding (1948) introduced a statistic which is asymptotically equivalent to B_n. Hoeffding's statistic estimates

$$\Delta(F)=\int D^2(x,y)\ dF(x,y)$$

where $D(x,y)=F(x,y)-F(x,\infty)F(\infty,y)$. This parameter has the desirable property that $D(x,y)=0$ for all (x,y) if and only if H_0 is true.

Assuming the null hypothesis holds, it is of interest to find the orthogonal representation of nB_n in the asymptotic case, namely $B=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\lambda_{jk}z_{jk}^2$, where the asymptotic distribution of nB_n is the same as that of B. The components z_{jk}, $j,k=1,2\ldots\infty$ are independent, identically distributed normal random variables with mean 0 and variance 1. The procedure used to find the representation is completely analogous to that of Durbin and Knott (1972). For finite n, the corresponding "components" z_{njk}, $j,k=1,2\ldots\infty$ may be computed which are asymptotically equivalent to the components, z_{jk}, $j,k=1,2\ldots\infty$. The z_{njk}'s may be related to linear rank statistics which may themselves be considered for testing for independence. The components are computed in section 3.5(1) and are then related to linear rank statistics in section 3.5(2).

1. The components. Assume that the null hypothesis holds, i.e. $F(x,y)=F(x,\infty)F(\infty,y)$. Let $Q_n(x,y)=\sqrt{n}(F_n(x,y)-F_n(x,\infty)F_n(\infty,y))$ and consider the transformation $(x,y)=(F_X^{-1}(u),F_Y^{-1}(v))$ where $F_X(x)=F(x,\infty)$ and $F_y(y)=F(\infty,y)$. Now $Q_n(x,y)=Q_n(F_X^{-1}(u),F_Y^{-1}(v))$ is asymptotically a Gaussian process, $Q(u,v)$, with mean and covariance given by

$$EQ(u,v)=0=EQ_n(F_X^{-1}(u),F_Y^{-1}(v))$$

$$\begin{aligned}\mathrm{Cov}(Q(u,v),Q(r,s))&=\{\min(u,r-ur\}\{\min(v,s)-vs\}\\&=\lim_{n\to\infty}\mathrm{Cov}(Q_n(F_X^{-1}(u),F_Y^{-1}(v)),\\&\qquad Q_n(F_X^{-1}(r),F_Y^{-1}(s)))\end{aligned}$$

where $\mathrm{Cov}(Q_n(F_X^{-1}(u),F_Y^{-1}(v)),Q_n(F_X^{-1}(r),F_Y^{-1}(s)))$

$$=\{n^{-1}(n-1)\min(u,r)-ur\}\{n^{-1}(n-1)\min(v,s)-vs\}-n^{-2}uvrs.$$

Define

$$z_{njk}=\lambda_{jk}^{-\frac{1}{2}}\iint \ell_{jk}(u,v)Q_n(F_X^{-1}(u),F_Y^{-1}(v))\,dudv \quad j,k=1,2\ldots\infty \qquad 3.5.1.1$$

where $\ell_{jk}(u,v)$ is the eigenfunction corresponding to the eigenvalue λ_{jk} of the equation

$$\int_0^1\int_0^1 \{\min(u,r)-ur\}\{\min(v,s)-vs\}\ell(r,s)\ drds=\lambda\ell(u,v) \qquad 3.5.1.2$$

It has been found (Blum, Kiefer, and Rosenblatt 1961) that the eigenfunctions and eigenvalues of 3.5.1.2 are $2\sin(\pi ju)\sin(\pi kv)$ and $1/\pi^4j^2k^2$, $j,k=1,2\ldots\infty$. Asymptotically, z_{njk} has the same distribution as

$$z_{jk}=\lambda_{jk}^{-\frac{1}{2}}\int_0^1\int_0^1 \ell_{jk}(u,v)Q(u,v)\ dudv \qquad 3.5.1.3$$

Since $Q(u,v)$ is Gaussian, z_{jk} has a normal distribution (Ash and Gardner 1975). It is easy to see, as λ_{jk} and $\ell_{jk}(u,v)$ satisfy 3.5.1.1, that the z_{jk}'s are uncorrelated and have 0 means and variances equal to 1. The z_{njk}'s are, therefore, asymptotically independent, identically distributed Normal (0,1) random variables.

The inverse of the system 3.5.1.3 is

$$Q(u,v)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\lambda_{jk}^{\frac{1}{2}}\ell_{jk}(u,v)z_{jk}.$$

Now,

$$nB_n=\int Q_n^2(x,y)\ dF_n(x,y)=\int Q_n^2(F_X^{-1}(u),F_Y^{-1}(v))\ dF_n(F_X^{-1}(u),F_Y^{-1}(v))$$

and asymptotically, under H_0, nB_n is distributed as

$$B=\int_0^1\int_0^1 Q^2(u,v)\ dudv=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\lambda_{jk}z_{jk}^2=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\pi^{-4}j^{-2}k^{-2}z_{jk}^2$$

Referring to the definition 3.5.1.1, z_{njk} may be written in terms of the original data as follows,

$$z_{njk}=2\pi^2jk\int_0^1\int_0^1 \sin(\pi ju)\sin(\pi kv)Q_n(F_X^{-1}(u),F_Y^{-1}(v))\ dudv$$
$$=2\pi^2jk\int_0^1 \sin(\pi kv)\int_0^1\sin(\pi ju)Q_n(F_X^{-1}(u),F_Y^{-1}(v))du\ dv$$

Integrating the inner integral by parts, with

$$x=Q_n(F_X^{-1}(u),F_Y^{-1}(v))$$
$$dy=\sin(\pi ju)du$$

gives

$$z_{njk}=2\pi k \int_0^1 \sin(\pi kv)\int_0^1 \cos(\pi ju)\frac{\delta}{\delta u}\{Q_n(F_X^{-1}(u),F_Y^{-1}(v))\}du\ dv$$

$$=2\pi k\sqrt{n}\int_0^1 \sin(\pi kv)\{n^{-1}\sum_{\substack{i\\ F_Y(Y_i)\le v}}\cos(\pi jF_X(X_i))-n^{-1}F_n(\infty,F_Y^{-1}(v))\sum_{i=1}^{n}\cos(\pi jF_X(X_i))\}dv$$

Integrating a second time by parts, with

$$x=\{n^{-1}\sum_{\substack{i\\ F_Y(Y_i)\le v}}\cos(\pi jF_X(X_i))-n^{-1}F_n(\infty,F_Y^{-1}(v))\sum_{i=1}^{n}\cos(\pi jF_X(X_i))\}$$

$$dy=\sin(\pi kv)dv$$

gives,

$$z_{njk}=2\sqrt{n}\int_0^1 \cos(\pi kv)\ d\{n^{-1}\sum_{\substack{i\\ F_Y(Y_i)\le v}}\cos(\pi jF_X(X_i))-n^{-1}F_n(\infty,F_Y^{-1}(v))\sum_{i=1}^{n}\cos(\pi jF_X(X_i))\}$$

$$=2\sqrt{n}\{n^{-1}\sum_{i=1}^{n}\cos(\pi jF_X(X_i))\cos(\pi kF_Y(Y_i))-n^{-2}\sum_{i=1}^{n}\cos(\pi jF_X(X_i))\sum_{i=1}^{n}\cos(\pi kF_Y(Y_i))\}$$

z_{njk} may now be related to a linear rank statistic.

2. The related linear rank statistics. It may be necessary to estimate F_X and F_Y by the sample distribution functions $F_n(\cdot,\infty)$ and $F_n(\infty,\cdot)$. Replacing F_X and F_Y by their estimates in the expression for z_{njk} gives

$$\hat{z}_{njk}=2\sqrt{n}\{n^{-1}\sum_{i=1}^{n}\cos(\pi jF_n(X_i,\infty))\cos(\pi kF_n(\infty,Y_i))-n^{-2}\sum_{i=1}^{n}\cos(\pi jF_n(X_i,\infty))\sum_{i=1}^{n}\cos(\pi kF_n(\infty,Y_i))\}$$

Or in terms of the rank statistics, (R_i,S_i)

$$\hat{z}_{njk}=2\sqrt{n}\{n^{-1}\sum_{i=1}^{n}\cos(\pi jn^{-1}R_i)\cos(\pi kn^{-1}S_i)-n^{-2}\sum_{i=1}^{n}\cos(\pi jn^{-1}R_i)\sum_{i=1}^{n}\cos(\pi kn^{-1}S_i)\}$$

The last term of this expression is constant, (i.e. it is equal to $n^{-2}\sum_{i=1}^{n}\cos(\pi jn^{-1}i)\sum_{i=1}^{n}\cos(\pi kn^{-1}i)$). The rank statistic $\hat{z}_{njk}$ suggests the use of the linear rank statistic,

$$T_{njk}=n^{-\frac{1}{2}}\sum_{i=1}^{n}2\cos(\pi j(n+1)^{-1}R_i)\cos(\pi k(n+1)^{-1}S_i)$$

which is of the form 3.0.1. The scores are generated by the functions $\zeta_\ell(u)=2\cos(\pi\ell u)$, $0\le u\le 1$, $\ell=1,2\ldots\infty$ where

$a_n(i)=\zeta_j((n+1)^{-1}i)$, $b_n(i)=\zeta_k((n+1)^{-1}i)$. It will be shown that it is only necessary to consider the linear rank statistics T_{njk}, and not the z_{njk}'s themselves, since T_{njk} is asymptotically equivalent to z_{njk}.

CHAPTER 4

ALTERNATIVES TO INDEPENDENCE

Specification of the alternative to independence is important in making a power comparison of tests of independence. Any conclusions regarding the behaviour of the tests are valid only for the particular alternative under consideration. Usually, only the bivariate normal case is considered. The same will be done in this thesis, but not without mention of some other alternatives which have been investigated.

Konijn (1956) is, perhaps, the first to point out the significance and difficulties arising in defining a class of alternatives to independence. He derives the relative Pitman efficiencies of several nonparametric tests of independence (with respect to the test based on the sample correlation coefficient), when X and Y are given by $X=\lambda_1 Z_1+\lambda_2 Z_2$, $Y=\lambda_3 Z_1+\lambda_4 Z_2$, where Z_1 and Z_2 are independent. The hypothesis of independence is $\lambda_1=\lambda_4=1$, $\lambda_2=\lambda_3=0$. This model incorporates the bivariate normal case and is therefore more general. Bhuchongkul (1964) considers a similar model and a general class of nonparametric tests, computing the Pitman efficiencies of some specific tests. Bhuchongkul's model is $X=(1-\theta)Z_1+\theta Z_2$, $Y=(1-\theta)Z_3+\theta Z_2$, where $0\leqslant\theta\leqslant 1$; Z_1, Z_2 and Z_3 are independent.

Dependence may be specified in terms of the distribution functions. This approach is taken by Farlie (1961) who computes Pitman efficiencies of generalized Daniel's correlation coefficients (which include the ordinary sample correlation coefficient, Spearman's rho and Kendall's tau), when the joint distribution

function of X and Y has the form $F(x,y)=F(x,\infty)F(\infty,y)(1+\Delta A(x)B(y))$. Ghokale (1968) also specifies a general class of bivariate distributions, namely, $F(x,y)=(1-\theta)F(x,\infty)F(\infty,y)+\theta K(x,y), 0\leqslant\theta\leqslant 1$ and considers a subclass of Bhuchongkul's class of rank statistics. Ghokale obtains the Pitman efficiencies of one rank test with respect to another as well as with respect to the test based on the sample correlation coefficient. He shows that there exist alternatives for which these efficiencies are 0 and ∞. This illustrates how crucial knowledge of the alternative is in choosing the best test.

A general notion of dependence, defined by Lehmann (1966), is positive quadrant dependence. The distribution function F is positively quadrant dependent if $F(x,y)\geqslant F(x,\infty)F(\infty,y)$, for every x and y. Behnen (1971) investigates the behaviour of linear rank statistics when testing independence against general contiguous positive quadrant dependence. (Contiguity is as defined by Hájek and Šidák 1967).

CHAPTER 5

THE COMPARISON OF NONPARAMETRIC TESTS OF INDEPENDENCE

5.0 Introduction

The comparison of the tests of independence is restricted to the case of bivariate normality. To be precise, it is assumed that (X,Y) has a bivariate normal distribution with correlation coefficient ρ and that X and Y have means μ_x, μ_y and variances σ_x^2, σ_y^2, respectively. It is desired to test $H_0: \rho=0$ (independence) versus the one sided alternative $H_1: \rho > 0$.

To calculate the Pitman and Bahadur efficiencies of the tests based on the statistics of Chapter 3, relative to the test based on the sample correlation coefficient, two main tools will be used. First, the Pitman efficiencies of tests based on linear rank statistics can be easily derived by applying Behnen (1971), Theorem 1. This result implies that the statistics 3.0.1 are asymptotically Normal(0,1) under the null hypothesis, H_0 and under contiguous alternatives, H_n are asymptotically Normal $(\mu_n, 1)$, where an expression for μ_n is given. The Pitman efficiency is then a straightforward evaluation of the right hand side of 2.1.1.

Second, the Bahadur efficiency of T_n (given by 3.0.1) may be computed by combining Theorem 3 with 4.6, both of Woodworth (1970), to yield the exact slope of the sequence $\{T_n\}$. If the exact slope is difficult to compute, the approximate slope will be given. The approximate slope requires knowledge of the distribution of T_n under H_0, given by Behnen (1971), Theorem 1 and of $\lim_{n\to\infty} T_n/n$ under the

alternative, given by 4.6 of Woodworth (1970).

For the small sample comparison, a Monte Carlo experiment was carried out for samples of sizes, n=10, 24 and 50. One thousand samples were generated from a bivariate normal distribution for each value of n and for several positive ρ values. An estimate of the power of each test is provided by the relative frequency of rejecting the null $H_0:\rho=0$ in favour of $H_1:\rho>0$.

5.1 Pitman Efficiencies

The nonparametric tests are to be compared with the parametric test based on the sample correlation coefficient,

$$r_n = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2 \sum_{i=1}^{n}(Y_i-\bar{Y})^2}}$$

where $\bar{X}$ and $\bar{Y}$ are the sample means of the X's and Y's respectively. The test rejects H_0 in favour of H_1 when r_n is greater than some positive constant. It is known that when (X,Y) has a bivariate normal distribution, with correlation coefficient ρ, the test is consistent against $H_1:\rho\geqslant 0$ and r_n is asymptotically normal with mean ρ and variance $(1-\rho^2)^2/n$. (See Kendall and Stuart 1973, for example). In the notation of Chapter 2, where $\theta=\rho$, $\theta_0=0$

$$\Psi_{r_n}(\rho)=\rho$$

$$\sigma_{r_n}(\rho)=(1-\rho^2)/\sqrt{n}$$

$$\psi'_{r_n}(0)=1,\quad m_r=1$$

$$\lim_{n\to\infty} n^{-\frac{1}{2}}\frac{\psi'_{r_n}(0)}{\sigma_{r_n}(0)} = \lim_{n\to\infty} n^{-\frac{1}{2}}\frac{1}{n^{-\frac{1}{2}}}=1,\quad \delta_r=\tfrac{1}{2},\quad v_r=1.$$

Defining the simple alternative to independence, $H_n:\rho=\rho_n=\frac{b_r}{n^{\frac{1}{2}}}$ for b_r, some positive constant, assumptions A(i)-A(iv) of 2.1 are satisfied.

Now consider a consistent test based on the statistic

$$T_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} a_n(R_i) b_n(S_i)$$

which rejects H_0, in favour of H_1, for large values of T_n. Suppose the score functions are defined by,

$$a_n(i) = Ea(U_n^{(i)})$$
$$b_n(i) = Eb(U_n^{(i)})$$

where $U_n^{(i)}$ is the ith order statistic of a sample of n independent Uniform (0,1) random variables, or by,

$$a_n(i) = a(i/(n+1)) \quad ,$$
$$b_n(i) = b(i/(n+1)).$$

Further, suppose that the score generating functions, a and b, are real valued measurable functions defined on $[0,1]$ which satisfy,

$$\int_0^1 b(z)dz = \int_0^1 a(z)dz = 0$$

and $0<\int_0^1 a^2(z)dz = \sigma_a^2<\infty$, $0<\int_0^1 b^2(z)dz = \sigma_b^2<\infty$.

If these conditions hold, by Behnen (1971), Theorem (1-a,c), T_n is asymptotically normal with mean 0 and variance 1, under $H_0:\rho = 0$ and is asymptotically normal with mean μ_n and variance 1, under the simple alternative, $H_n^T:\rho=\rho_n=b_T/n^{-\frac{1}{2}}$, b_T is any positive constant. The mean μ_n is given by,

$$\mu_n = \frac{n^{\frac{1}{2}}}{\sigma_a\sigma_b}\int a(F(x,\infty))b(F(\infty,y))\, dF_{\rho_n}(x,y)$$

where $F(x,\infty)=\Phi(x)$, $F(\infty,y)=\Phi(y)$ are the standard normal marginals of X and Y and $F_{\rho_n}(x,y)$ is the bivariate normal distribution function with $\rho=\rho_n$. As for r_n, these results can be written

in the notation of Chapter 2, that is

$$\psi_{T_n}(\rho)=\frac{n^{\frac{1}{2}}}{\sigma_a\sigma_b}\int a(F(x,\infty))b(F(\infty,y))\ dF_\rho(x,y).$$

Without loss of generality it may be assumed that $\mu_x=\mu_y=0$ and $\sigma_x=\sigma_y=1$. Substituting for F_ρ gives

$$\psi_{T_n}(\rho)=\frac{n^{\frac{1}{2}}}{\sigma_a\sigma_b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\Phi(x))b(\Phi(y))(2\pi\sqrt{1-\rho^2})^{-1}\exp\{-\tfrac{1}{2}(1-\rho^2)^{-1}[x^2+y^2-2\rho xy]\}\ dxdy$$

$$\sigma_{T_n}(\rho)=1$$

$$\psi'_{T_n}(0)=\frac{n^{\frac{1}{2}}}{\sigma_a\sigma_b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\Phi(x))b(\Phi(y))(2\pi)^{-1}\exp\{-\tfrac{1}{2}(x^2+y^2)\}xy\ dxdy,\quad m_T=1$$

$$\lim_{n\to\infty}\frac{n^{-\frac{1}{2}}\psi'_{T_n}(0)}{\sigma_{T_n}(0)}=(\sigma_a\sigma_b)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\Phi(x))b(\Phi(y))(2\pi)^{-1}\exp\{-\tfrac{1}{2}(x^2+y^2)\}xy\ dxdy=v_T$$

$\delta_T=\frac{1}{2}$.

Provided v_T is positive, A(i)-A(iv) of 2.1 are satisfied.

From 2.1.1 the Pitman efficiency of the test based on T_n, relative to that based on r_n, is given by,

$$A_{Tr}=\lim_{n\to\infty}\left[\frac{\psi'_{T_n}(0)/\sigma_{T_n}(0)}{\psi'_{r_n}(0)/\sigma_{r_n}(0)}\right]^2 \qquad 5.1.1$$

$$=\left[\frac{\partial}{\partial\rho}\left\{\frac{1}{\sigma_a\sigma_b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\Phi(x))b(\Phi(y))(2\pi(1-\rho^2)^{\frac{1}{2}})^{-1}\exp\{-\tfrac{1}{2}(x^2+y^2-2\rho xy)\}dxdy\right\}\right]^2_{\rho=0}$$

Or,

$$A_{Tr}=\left[(\sigma_a\sigma_b)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\Phi(x))b(\Phi(y))(2\pi)^{-1}\exp\{-\tfrac{1}{2}(x^2+y^2)\}xy\ dxdy\right]^2$$

$$=\left[\frac{1}{\sigma_a}\int_{-\infty}^{\infty} a(\Phi(x))\phi(x)x\ dx\right]^2\left[\frac{1}{\sigma_b}\int_{-\infty}^{\infty} b(\Phi(y))\phi(y)y\ dy\right]^2 \qquad 5.1.2$$

where $\phi(x)=\Phi'(x)=(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}x^2)$ (standard normal density).

In the case where $a(\cdot)=b(\cdot)$, 5.1.2 reduces to

$$A_{Tr}=\left[\frac{1}{\sigma_a}\int_{-\infty}^{\infty} a(\Phi(x))\phi(x)x\,dx\right]^4 . \qquad 5.1.3$$

It is now a simple matter to calculate the Pitman efficiencies of the tests based on the linear rank statistics of Chapter 3, relative to the test based on r_n. Although these efficiencies have already been derived in various ways by several authors (with the exception of those related to the components of the Blum-Kiefer-Rosenblatt statistic), they are once more computed here for completeness and convenience. The tests reject $H_0:\rho=0$ when the statistics are too large and are consistent for testing against $H_1:\rho>0$. From now on Pitman efficiency will be assumed to be taken relative to the test based on r_n.

1. Spearman's rho (T_{s_n}). In this case $a(u)=b(u)=\sqrt{12}(u-\frac{1}{2})$ and $\sigma_a=\sigma_b=1$. From 5.1.3,

$$A_{sr}=\left[\sqrt{12}\int_{-\infty}^{\infty} x(\Phi(x)-\tfrac{1}{2})\phi(x)dx\right]^4.$$

Evaluating the integral by parts with $v=\Phi(x)-\frac{1}{2}$ and $du=x\phi(x)$ gives the well known result,

$$A_{sr}=\left[\sqrt{12}\int_{-\infty}^{\infty} (2\pi)^{-1}\exp(-x^2)\,dx\right]^4=\left[(12)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}(2)^{-\frac{1}{2}}\right]^4=9/\pi^2\simeq .91.$$

2. Fisher-Yates normal scores statistic (T_{f_n}). Here $a(u)=b(u)=\Phi^{-1}(u)$ and $\sigma_a=\sigma_b=1$. According to 5.1.3 the Pitman efficiency is $A_{fr}=\left[\int_{-\infty}^{\infty} x^2\phi(x)\,dx\right]^4=1$.

3. Quadrant sum (T_{qn}). The statistic T_{qn} of 3.3 corresponds to letting $a(u)=b(u)=\text{sign}(u-\frac{1}{2})$. Obviously, σ_a and σ_b are both 1. From 5.1.3 the Pitman efficiency is,

$$\begin{aligned}A_{qr}&=\left[\int_{-\infty}^{\infty} x\,\text{sign}(\Phi(x)-\tfrac{1}{2})\phi(x)\,dx\right]^4\\&=\left[\int_{-\infty}^{0} -x\phi(x)\,dx+\int_{0}^{\infty} x\phi(x)\,dx\right]^4\\&=\left[(2\pi)^{-\frac{1}{2}}\int_{0}^{\infty} x\exp(-\tfrac{1}{2}x^2)\,dx+(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{0} -x\exp(-\tfrac{1}{2}x^2)\,dx\right]^4\\&=\left[2(2\pi)^{-\frac{1}{2}}\right]^4=4(\pi)^{-2}\simeq .41.\end{aligned}$$

4. The components and related linear rank statistics of the Blum-Kiefer-Rosenblatt statistic (z_{njk} and T_{njk}).

It has already been shown that z_{njk} is asymptotically distributed as z_{jk} when X and Y are independent. Since T_{njk} is of the correct form, Behnen's theorem applies and T_{njk} is also asymptotically Normal (0,1) under independence. T_{njk} and z_{njk} are, therefore, asymptotically equivalent when $H_0:\rho=0$ is true.

Consider the alternative $H_n^T:\rho=\rho_n=n^{-\frac{1}{2}}b_T$. Under H_n^T, letting $(x,y)=(F_X^{-1}(u),F_Y^{-1}(v))=(\Phi^{-1}(u),\Phi^{-1}(v))$ the expected value of Q_n is

$$E(Q_n(\Phi^{-1}(u),\Phi^{-1}(v)))=\sqrt{n}\left[F_{\rho_n}(\Phi^{-1}(u),\Phi^{-1}(v))-n^{-1}F_{\rho_n}(\Phi^{-1}(u),\Phi^{-1}(v)) -(1-n^{-1})uv\right],$$

where F_{ρ_n} is the bivariate normal distribution function with $\rho=\rho_n$. If $F_{\rho_n}(\Phi^{-1}(u),\Phi^{-1}(v))$ is expanded about $\rho=0$, for large n

$$F_{\rho_n}(\Phi^{-1}(u),\Phi^{-1}(v))\simeq \int_{-\infty}^{\Phi^{-1}(v)}\int_{-\infty}^{\Phi^{-1}(u)}(2\pi)^{-1}\exp(-\tfrac{1}{2}(x^2+y^2))(1+\rho_n xy)\,dxdy.$$

Asymptotically, under the sequence of alternatives, $\{H_n^T\}$

$$E(Q_n(\Phi^{-1}(u),\Phi^{-1}(v)))\to \sqrt{n}\left[\int_{-\infty}^{\Phi^{-1}(v)}\int_{-\infty}^{\Phi^{-1}(u)}(2\pi)^{-1}\exp(-\tfrac{1}{2}(x^2+y^2))(1+\rho_n xy)\,dxdy-uv\right]$$

$$=\sqrt{n}\rho_n\int_{-\infty}^{\Phi^{-1}(v)}\int_{-\infty}^{\Phi^{-1}(u)}(2\pi)^{-1}\exp(-\tfrac{1}{2}(x^2+y^2))xy\,dxdy.$$

Also, $\mathrm{Cov}(Q_n(\Phi^{-1}(u),\Phi^{-1}(v)),Q_n(\Phi^{-1}(r),\Phi^{-1}(s)))\to\{\min(u,r)-ur\}\{\min(v,s)-vs\}$

as $n\to\infty$, as well as in the case of independence. Under the sequence $\{H_n^T\}$, $Q_n(\Phi^{-1}(u),\Phi^{-1}(v))$ is asymptotically a Gaussian process, $Q(u,v)$, with

$$E(Q(u,v))=\sqrt{n}\rho_n\int_{-\infty}^{\Phi^{-1}(v)}\int_{-\infty}^{\Phi^{-1}(u)}(2\pi)^{-1}\exp(-\tfrac{1}{2}(x^2+y^2))xy\,dxdy$$

and $\mathrm{Cov}(Q(u,v),Q(r,s))=\{\min(u,r)-ur\}\{\min(v,s)-vs\}$.

From this it may be concluded that under the sequence of alternatives, $\{H_n^T\}$, the component z_{njk} is asymptotically normal

with variance 1 and mean given by,

$$\sqrt{n}\rho_n\pi^2 jk \int_0^1 \int_0^1 2\sin(\pi ju)\sin(\pi kv) \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\Phi^{-1}(u)} (2\pi)^{-1}\exp(-\tfrac{1}{2}(x^2+y^2)) \, xy \, dxdy \, dudv.$$

Integrating by parts, twice, this is equal to,

$$\sqrt{n}\rho_n \int_0^1 \int_0^1 2\cos(\pi ju)\cos(\pi kv)\Phi^{-1}(u)\Phi^{-1}(v) \, dudv.$$

From Behnen's theorem, under the sequence $\{H_n^T\}$, T_{njk} is asymptotically normal with variance 1 and mean given by,

$$\mu_n=\sqrt{n} \int 2\cos(\pi j\Phi(x))\cos(\pi k\Phi(y)) \, dF_{\rho_n}(x,y).$$

Again, approximating $F_{\rho_n}(x,y)$ in the region of $\rho=0$

$$\mu_n=\sqrt{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\cos(\pi j\Phi(x))\cos(\pi k\Phi(y))(2\pi)^{-1}\exp(-\tfrac{1}{2}(x^2+y^2))(1+\rho_n xy) \, dxdy.$$

Making the substitution, $u=\Phi(x)$ and $v=\Phi(y)$

$$\mu_n=\sqrt{n} \int_0^1 \int_0^1 2\cos(\pi ju)\cos(\pi kv)(1+\rho_n\Phi^{-1}(u)\Phi^{-1}(v)) \, dudv$$

$$=\sqrt{n}\rho_n \int_0^1 \int_0^1 2\cos(\pi ju)\cos(\pi kv)\Phi^{-1}(u)\Phi^{-1}(v) \, dudv.$$

T_{njk} and z_{njk} are, therefore, asymptotically equivalent under both the null, H_0, and the sequence of alternatives $\{H_n^T\}$. Because of this equivalence the Pitman and approximate Bahadur efficiencies computed for T_{njk} will be the same as those for z_{njk}.

For the statistic T_{njk} $a(u)=\sqrt{2}\cos(\pi ju)$ and $b(u)=\sqrt{2}\cos(\pi ku)$, and $\sigma_a=\sigma_b=1$. The Pitman efficiency of the T_{njk} test is by 5.1.2,

$$A_{(jk)r}=\left[\int_{-\infty}^{\infty} \sqrt{2}\cos(\pi j\Phi(x))\phi(x)x \, dx\right]^2\left[\int_{-\infty}^{\infty} \sqrt{2}\cos(\pi k\Phi(y))\phi(y)y \, dy\right]^2.$$

Substituting $u=\Phi(x)$, $v=\Phi(y)$

$$A_{(jk)r}=4\left[\int_0^1 \cos(\pi ju)\Phi^{-1}(u) \, du\right]^2\left[\int_0^1 \cos(\pi kv)\Phi^{-1}(v) \, dv\right]^2.$$

Consider the integral $I_\ell= \int_0^1 \cos(\pi\ell u)\Phi^{-1}(u) \, du$. I_ℓ is equal to 0 when ℓ is an even integer. The evaluation of I_ℓ, when ℓ is an odd integer, is not straightforward. The work of Beran (1975 a,b), suggests that the behaviour of the first component z_{n11} determines the Pitman efficiency of the overall Blum-Kiefer-Rosenblatt statistic. For this reason the Pitman efficiency of the first

component (or equivalently T_{n11}) is of interest. In order to compare this efficiency with the other efficiencies, I_1 (as well as I_3) has been numerically evaluated, using 8 point Gaussian quadrature. The results are reported here,

$$I_1=.67$$

$$I_3=.17$$

The corresponding efficiencies are,

$$A_{(11)r}=.80$$

$$A_{(13)r}=.051$$

$$A_{(33)r}=.0033$$

These are seen to decrease rapidly with increasing j or k.

5. Kendall's tau. Kendall's tau is asymptotically equivalent to Spearman's rho under both the null hypothesis (Hájek and Šidák 1967) and under the alternative where $\rho>0$ is close to 0 (Farlie 1961). This implies that the Pitman efficiency of Kendall's test is the same as that of Spearman's test. The Pitman efficiency is, therefore, $9/\pi^2 \simeq .91$.

5.2 Bahadur Efficiencies

As another means of comparing tests, the Bahadur efficiencies (exact or approximate) of the nonparametric tests relative to the test based on sample correlation coefficient, will be given. To begin with, it is necessary to know the exact or approximate slope associated with r_n when testing $H_0:\rho=0$ against $H_1:\rho>0$, in the bivariate normal case. The exact slope, of $\left\{r_n/\sqrt{n-2/1-r_n^2}\right\}_{n=1}^{\infty}$, is $c_r(\rho)=-\log(1-\rho^2)$, (Woodworth 1970). The approximate slope is $c_r^{(a)}(\rho)=\rho^2/1-\rho^2$, (Abrahamson 1965).

Now consider the linear rank statistics of 3.0.1,

$T_n=n^{-\frac{1}{2}}\sum_{i=1}^{n}a_n(R_i)b_n(S_i)$. The procedure for calculating the exact or approximate slope of the sequence $\{T_n\}_{n=1}^{\infty}$ requires evaluation of $\lim_{n\to\infty} T_n/n^{\frac{1}{2}}$ when $\rho>0$. From 4.6 of Woodworth (1970)

$$T_n/n^{\frac{1}{2}}\to\iint a(F)b(G)\ dF_\rho=b(\rho) \qquad 5.2.1$$

in probability as $n\to\infty$, where F and G are the Normal (0,1) marginals, Φ , of the bivariate normal distribution F_ρ and a_n and b_n converge in quadratic mean to the square integrable functions a and b, respectively. In addition to this limit, it is necessary to find the function f(t) which satisfies, $n^{-1}\log(1-F_n(n^{\frac{1}{2}}t))\to -f(t)$. Employing theorem 3, Woodworth (1970), f(t) may be found as follows. If there exists a constant $\lambda>0$ and a function s(u), such that

$$\int\frac{\exp[\lambda(a(u)b(v)-s(v))]}{\int\exp[\lambda(a(u)b(v')-s(v'))]\ dv'}\,du=1 \qquad 0<v<1 \qquad 5.2.2$$

and

$$\int\frac{\int a(u)b(v)\exp[\lambda(a(u)b(v)-s(v))]\,dv}{\int\exp[\lambda(a(u)b(v')-s(v'))]\,dv'}\,du=t \qquad 5.2.3$$

then $f(t)=\lambda(t-\int s(v)dv)-\int\log\{\int\exp[\lambda(a(u)b(v)-s(v))]\,dv\}du$. 5.2.4

It is not always simple to find the solution (λ,s) of 5.2.2 and 5.2.3. The approximate slope will have to suffice when it is difficult to solve for f(t) by this method. If f(t) can be found, the exact slope is $2f(b(\rho))$.

Under the null hypothesis $H_0:\rho=0$, T_n is asymptotically Normal (0,1), as stated in section 5.2 and a=1 in B(ii) of 2.2. The approximate slope of $\{T_n\}_{n=1}^{\infty}$ is, therefore, $b^2(\rho)$ where $b(\rho)$ is given by 5.2.1. The above results will now be used to compute the exact (where possible) or approximate slopes of the statistics given in Chapter 3.

1. Fisher-Yates normal scores statistic (T_{fn}). Woodworth (1970) solves 5.2.2 and 5.2.3 for the Fisher-Yates statistic and consequently finds $f(t)=-\frac{1}{2}\log(1-t^2)$. The exact slope of $\{T_{fn}\}_{n=1}^{\infty}$ is, therefore, $c_f(\rho)=-\log(1-\rho^2)$ since $\lim_{n\to\infty} T_{fn}/n^{\frac{1}{2}}$ is

$$\iint xy(2\pi)^{-1}(1-\rho^2)^{-\frac{1}{2}}\exp[-\tfrac{1}{2}(1-\rho^2)^{-1}(x^2+y^2-2\rho xy)]\,dxdy=\rho.$$

2. Quadrant sum (T_{qn}). Applying 5.2.1 to the quadrant sum,

$$\lim_{n\to\infty} T_{qn}/n^{\frac{1}{2}}=\iint \operatorname{sign}(\Phi(x)-\tfrac{1}{2})\operatorname{sign}(\Phi(y)-\tfrac{1}{2})(2\pi)^{-1}(1-\rho^2)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}(1-\rho^2)^{-\frac{1}{2}}(x^2+y^2-2\rho xy))\,dxdy.$$

The right hand side of this equation has already been evaluated as $(2\pi^{-1})\arcsin\rho$. (Woodworth 1970)

The solution to 5.2.2 and 5.2.3 may be found by letting $s(u)=0$. 5.2.2 is then identically equal to 1 for all $\lambda>0$ and $0<v<1$. It remains to find λ by solving 5.2.3. When $s(u)=0$, 5.2.3 becomes

$$t=\int_0^1 \frac{\int_0^1 \operatorname{sign}(u-\tfrac{1}{2})\operatorname{sign}(v-\tfrac{1}{2})\exp[\lambda(\operatorname{sign}(u-\tfrac{1}{2})\operatorname{sign}(v-\tfrac{1}{2}))]\,dv}{\int_0^1 \exp[\lambda(\operatorname{sign}(u-\tfrac{1}{2})\operatorname{sign}(v'-\tfrac{1}{2}))]\,dv'}\,du$$

$$=(e^{\lambda}-e^{-\lambda})/(e^{\lambda}+e^{-\lambda})=\tanh\lambda$$

This gives $\lambda=\tanh^{-1}(t)=\frac{1}{2}\log(1+t/1-t)$. Finally, $f(t)=\frac{1}{2}t\log(1+t/1-t)-\log[\frac{1}{2}(1+t/1-t)^{-\frac{1}{2}}+\frac{1}{2}(1+t/1-t)^{\frac{1}{2}}]$ from 5.2.4.

Combining results, the exact slope for the quadrant sum is, $c_q(\rho)=2\pi^{-1}\arcsin\rho\log(\xi(\rho))-2\log[\frac{1}{2}\xi^{-\frac{1}{2}}(\rho)+\frac{1}{2}\xi^{\frac{1}{2}}(\rho)]$, where

$$\xi(\rho)=\frac{1+2\pi^{-1}\arcsin\rho}{1-2\pi^{-1}\arcsin\rho}.$$

3. Spearman's rho (T_{sn}). Consider Spearman's test.

$$\lim_{n\to\infty} T_{sn}/n^{\frac{1}{2}}=\iint 12(u-\tfrac{1}{2})(v-\tfrac{1}{2})(2\pi)^{-1}(1-\rho^2)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}(1-\rho^2)^{-\frac{1}{2}}(x^2+y^2-2\rho xy))\,dxdy$$

$$=6\pi^{-1}\arctan(\rho(4-\rho^2)^{-\frac{1}{2}}).$$ (Woodworth 1970)

Although, 5.2.2 and 5.2.3 are intractable for Spearman's case, $12(u-\frac{1}{2})(v-\frac{1}{2})$ may be approximated by a function for which the

equations can be solved, providing an approximation to f(t) (Woodworth 1970). For t close to 1, f(t) has a series expansion (Woodworth 1970) and may be approximated by the nonnegligible terms in this expansion. Woodworth (1970) presents a table of f(t) for various values of t, making use of these approximations. Using this table the exact slope, $c_s(\rho)=2f(6\pi^{-1}\arctan(\rho(4-\rho^2)^{-\frac{1}{2}}))$, may be estimated for given values of ρ .

4. Kendall's tau (t_n). Since Kendall's tau is not linear, the methods of this section are not applicable. The exact slope, $c_t(\rho)$, can be found, however, as $c_t(\rho)=2f(b_t(\rho))$ (Woodworth 1970) Where f(t) is the solution of

$$f(t)=\tfrac{1}{4}\lambda t+\tfrac{1}{2}\lambda+\log(\lambda/(e^{\lambda}-1))$$

$$t=1+4\left[\int_0^{\lambda} x(e^x-1)^{-1}\,dx-\lambda\right]/\lambda^2$$

and $b_t(\rho)=4\iint F_\rho\, dF_\rho -1=2\pi^{-1}\arctan(\rho(1-\rho^2)^{-\frac{1}{2}})$.

Woodworth (1970) has tabled f(t) for $.020<t<.996$ and $c_t(\rho)$ may be computed for given values of ρ.

5. The components and related linear rank statistics of the Blum-Kiefer-Rosenblatt statistic (z_{njk} and T_{njk}). It has not been possible to find the solution to 5.2.2 and 5.2.3 in this case and, therefore, only the approximate slope will be given. From 5.2.1

$$\lim_{n\to\infty} T_{njk}/n^{\frac{1}{2}}=\iint 2\cos(\pi j\Phi(x))\cos(\pi k\Phi(y))\, dF_\rho$$

Expanding F_ρ about $\rho=0$, to get an approximation to F_ρ for small ρ,

$$\lim_{n\to\infty} T_{njk}/n^{\frac{1}{2}}=\iint 2\cos(\pi j\Phi(x))\cos(\pi k\Phi(y))(2\pi)^{-1}\exp(-\tfrac{1}{2}(x^2+y^2))(1+\rho xy)\, dxdy.$$

If the substitution $u=\Phi(x)$, $v=\Phi(y)$ is made in the expression above, $\lim_{n\to\infty} T_{njk}/n^{\frac{1}{2}}$ reduces to

$$2\rho\int_0^1 \cos(\pi ju)\Phi^{-1}(u)\ du \int_0^1 \cos(\pi kv)\Phi^{-1}(v)\ dv.$$

The approximate slope, $c_{jk}^{(a)}(\rho)$, is

$$c_{jk}^{(a)}(\rho)=4\rho^2\left[\int^1 \cos(\pi ju)\Phi^{-1}(u)\ du\right]^2\left[\int^1 \cos(\pi kv)\Phi^{-1}(v)\ dv\right]^2$$
$$=4\rho^2 I_j^2\ I_k^2$$

where I_ℓ was defined in section 5.1.

Abrahamson (1965) gives the approximate slope of the overall Blum-Kiefer-Rosenblatt statistic which is noted here.

$$c_B^{(a)}(\rho)=12^{-1}\pi^2\rho^2 \qquad \text{for } \rho \text{ near } 0.$$

It should be pointed out that the test based on B_n is two sided in contrast to the other tests previously discussed. A direct comparison of the approximate Bahadur efficiency of B_n cannot be made with the other Bahadur efficiencies.

The Bahadur efficiencies of the nonparametric tests relative to the test based on r_n, may now be computed by dividing the slopes by $c_r(\rho)$. These are summarized below,

Fisher-Yates normal scores statistic	$B_{fr}(\rho)=1$
Quadrant sum	$B_{qr}(\rho)=-\log^{-1}(1-\rho^2)\left[2\pi^{-1}\arcsin\rho\right.$ [2] $\left.\log(\xi(\rho))-2\log\left[\tfrac{1}{2}\xi^{-\frac{1}{2}}(\rho)+\tfrac{1}{2}\xi^{\frac{1}{2}}(\rho)\right]\right]$
Spearman's rho	[3] $B_{sr}(\rho)=-\log^{-1}(1-\rho^2)\ c_s(\rho)$
Kendall's tau	[3] $B_{tr}(\rho)=-\log^{-1}(1-\rho^2)\ c_t(\rho)$
T_{njk}	$B_{(jk)r}^{(a)}(\rho)=4(1-\rho^2)I_j^2\ I_k^2$
Blum-Kiefer-Rosenblatt statistic (overall)	$B_{Br}^{(a)}(\rho)=12^{-1}\pi^2(1-\rho^2)\simeq .82(1-\rho^2)$

[2] $\xi(\rho)=\dfrac{1+2\pi^{-1}\arcsin\rho}{1-2\pi^{-1}\arcsin\rho}$.

[3] c_s, c_t are not given explicitly but are tabled by Woodworth (1970).

The Pitman and approximate Bahadur efficiencies of T_{njk} rapidly decrease with j and k. For this reason it is suggested that T_{n11} be used in testing for independence.

5.3 A Monte Carlo Comparison

The size of each test, was fixed throughout the study, at $\alpha=.05$. The critical values were first estimated for each statistic by generating 1000 independent samples of size n. In each sample, the n observations (X_i, Y_i), $i=1,2...n$, were drawn from the null distribution, that is the X's and Y's were independent Normal (0,1). The statistics of Chapter 3: Spearman's rho, Kendall's tau, the Fisher-Yates Normal scores coefficient, the quadrant sum, the Blum-Kiefer-Rosenblatt statistic and T_{n11} (related to its first component), as well as the ordinary sample correlation coefficient were all computed from the same n observations. The resulting 1000 values of each statistic were ordered. The critical value was then taken to be the smallest observed number such that the relative frequency of values greater than or equal to it, was less than or equal to α. Where tables are available the critical values obtained numerically can be compared with exact values.

For the Fisher-Yates normal scores statistic of section 3.2 the approximate scores, $\Phi^{-1}(i/(n+1))$, were used instead of the exact scores, $E(\Phi^{-1}(U_n^{(i)}))$. This eliminated the need for a table of normal scores. The approximation improves as $n\to\infty$. (Hájek and Šidák 1967)

The Blum-Kiefer-Rosenblatt test is inherently two sided. To make a fair comparison with the other tests it is necessary that they also be two sided. Since the alternative of interest is the one sided $H_1:\rho>0$, and, only positive ρ's were used in the power calculations, it will be assumed that the appropriate one sided tests with $\alpha=.025$ are good approximations to the corresponding two sided tests with $\alpha=.05$.

The Monte Carlo critical values as well as some exact values are given in Table I.

After the critical values were computed, a power study was carried out. Again, 1000 samples of each size, n=10, 24 and 50, were generated. This time the samples were generated from the bivariate normal distribution (with 0 means and unit variances) with ρ=.1,.25 and .5. The additional values ρ=.75 and .9 for n=10, ρ=.4 and .6 for n=24 and ρ=.15, .3 and .4 for n=50, were included to give a better picture of the power curve. All tests were performed for each sample and a running count of the number of times H_0 was rejected was kept for each test. The final count divided by 1000 estimates the power. The results of these computations are given in Table II. The powers for both one sided and two sided tests, with α=.05 are recorded for all statistics except the Blum-Kiefer-Rosenblatt statistic. Only the .05 level two sided test was performed for the latter.

Computations were done at the University of British Columbia Computing Centre on an IBM 360 computer. The normal observations were generated by a built-in routine which transforms independent normal observations, produced by Marsaglia's rectangle-wedge-tail-method, into observations from the desired bivariate normal distribution. A different random starting value was used to generate the 1000 samples for each n and ρ combination.

Table I

Critical values derived from a Monte Carlo experiment

n=10

Test	Critical value	Exact critical value[4]	α
Correlation coefficient	2.2660	2.3060	.025
	1.8939	1.8595	.05
Kendall's tau	0.5111	0.5111	.025
	0.4222	0.4667	.05
Spearman's rho	0.6242	0.6485	.025
	0.5273	0.5636	.05
Quadrant sum[5]	1.0000	1.0000	.025
	1.0000	1.0000	.05
Fisher-Yates coefficient	1.2069	-	.025
	1.0661	-	.05
T_{nll}	1.7544	-	.025
	1.5039	-	.05
Blum-Kiefer-Rosenblatt statistic	0.00803	-	.05

n=24

Test	Critical value	Exact critical value	α
Correlation coefficient	2.0350	2.0739	.025
	1.6942	1.7171	.05
Kendall's tau	0.2826	0.2899	.025
	0.2319	0.2464	.05
Spearman's rho	0.3948	-	.025
	0.3313	-	.05
Quadrant sum[5]	0.5000	0.5000	.025
	0.5000	0.5000	.05
Fisher-Yates coefficient	1.4778	-	.025
	1.2831	-	.05
T_{nll}	1.9306	-	.025
	1.5563	-	.05
Blum-Kiefer-Rosenblatt statistic	0.00278	-	.05

[4] References for exact values: Blomquist (1950), Ödeh et al (1977).

[5] Because of the discrete nature of the statistic, the .05 and .025 critical values of the quadrant sum are the same. This is also reflected in equal powers for the corresponding entries in Table II.

Table I (continued)

n=50

Test	Critical value	Exact critical value	α
Correlation	2.3116	2.0106	.025
coefficient	1.7523	1.6772	.05
Kendall's tau	0.2033	-	.025
	0.1657	-	.05
Spearman's rho	0.3007	-	.025
	0.2400	-	.05
Quadrant sum	0.3600	0.3600	.025
	0.2800	0.2800	.05
Fisher-Yates	1.8486	-	.025
coefficient	1.4817	-	.05
T_{n11}	1.9738	-	.025
	1.7134	-	.05
Blum-Kiefer-Rosenblatt statistic	0.00133	-	.05

Power derived from

ρ=.1	ρ=.25	ρ=.1	ρ=.25
0.048	0.098	0.068	0.226
0.086	0.168	0.120	0.332
0.040	0.068	0.072	0.213
0.086	0.165	0.133	0.323
0.051	0.101	0.073	0.219
0.102	0.170	0.130	0.314
0.005	0.020	0.042	0.104
0.005	0.020	0.042	0.104
0.053	0.094		
0.098	0.163		
0.054	0.101		
0.092	0.160		
0.070	0.127		