THE WEYL FUNCTIONAL CALCULUS

by

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ABSTRACT

The Weyl functional calculus for a family of n self-adjoint operators acting on a Hilbert space provides a map from spaces of functions on R^n into the set of bounded operators. The calculus is not multiplicative under point-wise multiplication of functions unless the self-adjoint operators commute. However, if the operators happen to generate a strongly continuous unitary representation of a Lie group, we can hope to define a "skew product" on the function spaces under which the calculus is multiplicative.

In part I, we show that, for exponential groups, a natural skew product exists by using the exponential map to pull the convolution on the group back to the Lie algebra. Moreover, whenever a skew product is defined in part I, it depends only on the underlying Lie group and not on the particular representation. We then examine when the skew product of two functions is again in the original function space. For compact Lie groups, the theory becomes more complex. A skew product is constucted but by a rather artificial method. The explicit calculations for $SU(2)$ demonstrates the difficulties.

In part II, a unique skew product is developed for the position and momentum operators of one dimensional quantum mechanics. The dynamics of quantum mechanics on phase space can be formulated through this skew product whenever the underlying Hamiltonian corresponds to a tempered distribution on the plane. The resulting evolution operator on phase space is shown to be equivalent to the difference of two "singular" integral operators obtained from the usual configuration space formulation. The evolution and configuration operators are then bounded with appropriate domains for the same set of tempered distributions. The skew product on this set of distributions is used to define noncommutative Banach algebras and to determine the multipliers on these spaces. For real, compactly supported distributions, it is shown that the phase space formulation has a unique solution if and only if there is a unique solution on configuration space. On the other hand, we observe that the symmetries of the evolution operator seem to imply that the two formulations of quantum mechanics are not equivalent for all real tempered distributions.

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INTRODUCTION

The usual von Neumann functional calculus for a single self-adjoint operator assigns, through the spectral resolution, an operator to every Borel-measurable complex-valued function defined on the real line. R. Anderson [1] has extended this to the Weyl functional calculus for certain families of self-adjoint operators and functions on R^n. An n-tuple of self-adjoint operators $A = (A_1, \ldots, A_n)$ on a Hilbert space H is called a self-adjoint n-tuple if; when the operators are restricted to their common domain, any real linear combination of them is essentially self-adjoint. For any self-adjoint n-tuple A, the Weyl functional calculus $T(A)$ maps various subspaces S of functions on R^n into the set of bounded operators $B(H)$ on the Hilbert space.

A multiplicative structure on S will mean a map $S \times S \to S$ given by $(f,g) \to f * g$ that satisfies $T(A) f * g = T(A) f\, T(A) g$. The function $f * g$ is called the skew product of f and g. This multiplicative structure for special n-tuples plays an important role in phase space quantum mechanics as is explained later in the introduction. Our purpose is to define and study the skew product for various self-adjoint n-tuples.

The thesis is divided into two parts. The first examines necessary and sufficient conditions on the self-adjoint n-tuple for a skew product to exist on various spaces of functions. The second studies an evolution equation that arises naturally from the skew product for one of our special n-tuples.

In part I, we will always assume that the self-adjoint n-tuple A comes from the set of generators of a strongly continuous unitary representation of a real, connected Lie group G. In fact, $iA = (iA_1, \ldots, iA_n)$ will be a representation of the Lie algebra Γ of G.

If G is nilpotent, a continuous skew product has already been developed in [2] for these n-tuples on the Schwartz class $S(R^n)$. Unfortunately, this result cannot be extended to other groups for the space of functions $S(R^n)$. Hence, we consider the other three spaces of Fourier transforms that are introduced in Chapter One (namely; $F(L^1(R^n))$, $F(C_o(R^n))$, and $F(C_o^\infty(R^n))$). In Chapter Two, we show that for exponential groups there is a unique continuous skew product on these spaces for the n-tuple associated to the left regular representation. Moreover, this skew product holds for any strongly continuous unitary representation of the exponential group. The example at the end of the chapter demonstrates that G must be nilpotent in order to expect a skew product on $S(R^n)$.

In Chapters Three and Four, we attempt to carry the program of Chapter Two over to compact groups. Again, there is a continuous skew product on $S = F(L^1(R^n))$ that holds for all our representations but it is no longer unique. As the exponential map is no longer a diffeomorphism, the existence of a skew product for general compact groups on the Fourier transform of a space of continuous functions seems particularly difficult to establish. However, with considerable effort, a skew product is produced for $SU(2)$.

In part II, we will be dealing primarily with the pair of self-adjoint operators (Q,cP) on $L^2(R)$ that denote multiplication by x and the differential operator $-ic\frac{d}{dx}$ respectively. The constant c is always a positive number. With c thought of as Planck's constant and h in $S(R^n)$, $T(Q,cP)h$ is nothing but the quantization procedure suggested by Weyl [18 ; page 275] that interprets classical quantities on phase space as quantum mechanical operators (henceforth called <u>Weyl operators $A_c(h)$</u>) on $L^2(R)$. A unique continuous skew product $\underset{c}{*}$ is provided on the Schwartz class $S(R^2)$

in Chapter Six. If h is regarded as a fixed Hamiltonian, the evolution equation on phase space $\frac{df}{dt} = i(h \underset{c}{*} f - f \underset{c}{*} h)$ is equivalent to the Schrodinger equation on configuration space as explained in [2].

However, most interesting Hamiltonians do not come from functions in $S(R^2)$. Fortunately, the above paragraph can be reasonably extended to define a Weyl operator A_c on $S(R)$ and an evolution operator H_c on $S(R^2)$ for any h in the set of tempered distributions $S'(R^2)$. The greater portion of part II studies the equivalence of the resulting evolution equation $\frac{df}{dt} = H_c f$ and the Weyl equation $\frac{d\phi}{dt} = iA_c \phi$. In Chapter Six, our main theorem shows that the evolution operator is equivalent to the difference of two kernel operators on the plane. These kernel operators are intrinsically related to the Weyl operator of the original distribution.

In Chapter Seven, H_c and A_c are regarded as operators on $L^2(R^2)$ and $L^2(R)$ respectively. From this point of view, H_c is bounded with domain $S(R^2)$ if and only if A_c is bounded with domain $S(R)$. As any bounded operator on $L^2(R)$ is an extension of a Weyl operator, the above set of distributions is quite a large subset of $S'(R^2)$ and, in fact, includes $L^2(R^2)$ and $F(L^1(R^2))$.

The multiplicative structure on the Banach spaces $L^2(R^2)$ and $F(L^1(R^2))$ defined through the bounded Weyl operators of Chapter Seven produces two noncommutative Banach algebras. In Chapter Eight, we study the multipliers on these spaces. The multipliers on $L^2(R^2)$ simply correspond to the distributions with bounded Weyl operators. Furthermore, a variation of Wendel's Theorem [6], proves the multipliers on $F(L^1(R^2))$ come from the Fourier transform of finite Radon measures.

In the last chapter, we return to the viewpoint of Chapter Seven to study the relation between H_c and A_c for real tempered distributions. Most Schrodinger operators of a one dimensional particle are extensions of these Weyl operators. One important result of Chapter Nine states that H_c has dense domain if and only if A_c has dense domain. Moreover for distributions with compact support (or whose Fourier transform has compact support), the evolution equation has a unique solution if and only if the Weyl equation has a unique solution. In other words, the two equations are equivalent in this case. Finally, the geometric intuition present in phase space is used to suggest that they are not always equivalent.

PART I THE MULTIPLICATIVE STRUCTURE

CHAPTER ONE

PRELIMINARIES

The functional analysis notation used in this thesis is as in K. Yosida [19] unless otherwise specified. The following four spaces of functions with the given topology will be considered throughout.

DEFINITION 1.1: *Let* F *be the Fourier transform defined on* $L^1(R^n)$ *according to the formula*

$$Ff(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{i\xi\cdot x} f(x)\,dx \qquad \textit{for every} \quad f \in L^1(R^n).$$

This operator induces a topology on the function spaces below.

1. $F(L^1(R^n)) = \{Ff : f \in L^1(R^n)\}$ *is a Banach space with norm* $\|Ff\| = \|f\|_{L^1}$
2. $F(C_o(R^n)) = \{Ff$: f is a continuous function with compact support$\}$. $C_o(R^n)$ *has the topology of uniform convergence on compact subsets.*
3. $F(C_o^\infty(R^n)) = \{Ff$: f is a C^∞ function with compact support$\}$. *The topology is induced from* $C_o^\infty(R^n)$ *considered as the locally convex linear topological space of test functions for distributions on* R^n.
4. $S(R^n)$ *is the space of rapidly decreasing functions with usual topology. The Fourier transform is a homeomorphism on this space* [19 ; chapter VI].

DEFINITION 1.2: (The Weyl Functional Calculus) *Suppose* $A = (A_1, \ldots, A_n)$ *is a self-adjoint n-tuple of operators on H. Let* S *be one of the four above spaces. Let* $e^{-i\xi\cdot A}$ *be the unitary operator associated to the essentially self-adjoint operator* $-(\xi_1 A_1 + \ldots + \xi_n A_n)$ *through Stone's Theorem. Then* $T(A) : S \to B(H)$ *is defined by*

(1.1) $$T(A)f = (2\pi)^{-n/2} \int_{R^n} Ff(\xi)\, e^{-i\xi\cdot A} d\xi \quad \textit{for every} \quad f \in S,$$

where the integral on the right is the Bochner integral [19 ; page 132].

For part I, let us assume that $iA = (iA_1, \ldots, iA_n)$ is a set of generators for a strongly continuous unitary representation U (henceforth called a <u>representation</u> U) on a real, connected Lie group G. Without loss of generality, G is simply connected because the representation can be lifted to the universal covering group of G without affecting the generators. Formula (1.1) may be rewritten in terms of this representation. If exp denotes the exponential map from the Lie algebra Γ to G, then there is a basis for Γ corresponding to A such that $U(\exp\xi) = e^{i\xi\cdot A}$. With this basis

(1.2) $$T(A)f = (2\pi)^{-n/2} \int_{R^n=\Gamma} Ff(\xi)\, U(\exp(-\xi))\, d\xi.$$

We will be particularly interested in the left regular representation R on G. This is constructed through left translation on the group. If μ is a left invariant Haar measure on G, let $R(\sigma) : L^2(G,\mu) \to L^2(G,\mu)$ by

(1.3) $$(R(\sigma)\psi)\,(\rho) = \psi\,(\sigma^{-1}\rho) \quad \text{for} \quad \psi \in L^2(G,\mu) \quad \text{and} \quad \sigma,\rho \in G.$$

This defines a representation on G that is generated by n left invariant vector fields $(iA_1, \ldots, iA_n)$ of skew-adjoint operators on $L^2(G,\mu)$. For all groups considered in this thesis, the skew product for the generators of any representation on G, will depend only on the left regular representation.

CHAPTER TWO

EXPONENTIAL LIE GROUPS

DEFINITION 2.1: *A Lie group* G *is called* exponential *if the exponential map establishes an analytic diffeomorphism of* Γ *onto* G.

Exponential groups have been examined by a number of authors (eg. [4] and [13]). This class of Lie groups may be studied through their Lie algebras as there is a purely algebraic criterion on the Lie algebra to determine if the group is exponential. In particular, all connected, simply connected nilpotent Lie groups are exponential while all exponential groups are solvable - each inclusion being proper.

The following lemma is needed to study the left regular representation on these groups. The result should be obvious for those familiar with the theory of Lie groups [5 ; page 364].

LEMMA 2.2: *If* G *is an exponential group, the measure induced on* G *by Lebesgue measure on* Γ *through the exponential map is of the form* $\Sigma(\sigma)d\mu(\sigma)$ *where* $\Sigma(\sigma)$ *is a positive analytic function. If* G *is nilpotent, then* Σ *is a constant.*

PROOF: Let us construct the left invariant Haar measure μ on G. The left invariant vector fields $iA_1,\ldots,iA_n$ of the left regular representation form a basis for the tangent space at each point of G. Define n 1-forms on G by $\omega^k(iA_j) = \delta_{kj}$ for $1 \le k,j \le n$. These are clearly left invariant and the form $\omega = \omega^1 \wedge \ldots \wedge \omega^n$ is a left invariant n-form that is non-degenerate at each point of G.

By using exponential coordinates with respect to our chosen basis for G, ω can be pulled back to a non-degenerate n-form ν on Γ. But the n-form $dx_1 \wedge \ldots \wedge dx_n$ corresponding to Lebesgue measure is also non-degenerate and so $\nu = \Delta(x_1,\ldots,x_n)dx_1 \wedge \ldots \wedge dx_n$ where $\Delta : R^n \to R$ is non-zero. By the choice of coordinates Δ is obviously analytic and $\Delta(0,\ldots,0) = 1$. Thus Δ is a positive analytic function.

Define the measure μ for $f \in C_o^\infty(G)$ by

$$\mu(f) = \int_\Gamma f(\exp(x))\,\Delta(x)\,dx.$$

It is easy to check μ is a positive measure on G that is left invariant since it corresponds to the n-form ω. The definition of μ states that μ induces the measure $\Delta(x)dx$ on Γ. Alternatively, Lebesgue measure induces the measure $\Sigma(\sigma)d\mu(\sigma)$ on G where $\Sigma(\sigma) = 1/\Delta(\exp^{-1}\sigma)$. Therefore Σ has the desired properties.

If G is nilpotent, an appropriate basis for Γ may be chosen using the Campbell-Baker-Hausdorff Theorem so that

$$\exp^{-1}\{\exp(\xi_1,\ldots,\xi_n)\exp(\eta_1,\ldots,\eta_n)\} = (\xi_1+\eta_1,\ \xi_2+\eta_2+\text{a polynomial in }(\xi_1,\eta_1),\ \ldots,\ \xi_n+\eta_n+\text{a polynomial in }(\xi_1,\eta_1,\ldots,\xi_{n-1},\eta_{n-1})).$$

With these coordinates, Lebesgue measure is clearly left invariant and this insures that Σ is a constant.

<u>THEOREM 2.3</u>: *If* A *is the self-adjoint n-tuple associated to the left regular representation of an exponential group* G, *then there is a unique continuous skew product on* $S = F(L^1(R^n))$, $S = F(C_o(R^n))$ *and* $S = F(C_o^\infty(R^n))$. *In fact, if* $\Gamma = R^n$ *has the basis corresponding to* A, *then* $F(f * g) = K$ *for* $f,g \in S$ *where*

$$K(\xi) = (2\pi)^{-n/2}\int_{R^n} Ff(\exp^{-1}(\exp(-\eta)\exp\xi))Fg(\eta)\,\frac{\Sigma(\exp(-\eta)\exp\xi)}{\Sigma(\exp\xi)}\,d\eta.$$

PROOF: We will determine the skew product on $F(L^1(R^n))$. Define a new function on the group for any $f \in F(L^1(R^n))$ as follows

$$\tilde{f}(\sigma) = Ff(\exp^{-1}\sigma)\cdot\Sigma(\sigma).$$

By formula (1.2), we have for every $\psi \in L^2(G,\mu)$

$$\begin{aligned}(T(A)f)(\psi) &= (2\pi)^{-n/2}\int_{R^n} Ff(\xi)\, R(\exp(-\xi))\psi\, d\xi \\ &= (2\pi)^{-n/2}\int_G Ff(\exp^{-1}\sigma)\, R(\sigma^{-1})\psi\, \Sigma(\sigma)d\mu(\sigma) \\ &= (2\pi)^{-n/2}\int_G \tilde{f}(\sigma)\, R(\sigma^{-1})\psi\, d\mu(\sigma).\end{aligned}$$

Since $\tilde{f} \in L^1(G,\mu)$, the last integral is defined as a convolution on G of an L^1 function and an L^2 function.

Let f and g be two arbitrary functions in $F(L^1(R^n))$. Then

$$\big(T(A)f\ T(A)g\big)\psi$$

$$= (2\pi)^{-n/2}\int_G \tilde{f}(\sigma)\, R(\sigma^{-1})\left\{ (2\pi)^{-n/2}\int_G \tilde{g}(\delta)\, R(\delta^{-1})\psi\, d\mu(\delta)\right\} d\mu(\sigma)$$

$$= (2\pi)^{-n}\int_G\int_G \tilde{f}(\sigma)\,\tilde{g}(\delta)\, R(\sigma^{-1}\delta^{-1})\psi\, d\mu(\delta)\, d\mu(\sigma)$$

Interchange the order of integration (legitimate since these are convolutions) and replace σ by $\delta^{-1}\sigma$ (recall: μ is left invariant).

$$= (2\pi)^{-n}\int_{G\times G} \tilde{f}(\delta^{-1}\sigma)\,\tilde{g}(\delta)\, R(\sigma^{-1})\psi\, d\mu(\sigma)\, d\mu(\delta)$$

$$= (2\pi)^{-n/2}\int_G \left\{ (2\pi)^{-n/2}\int_G \tilde{f}(\delta^{-1}\sigma)\,\tilde{g}(\delta)\, d\mu(\delta)\right\} R(\sigma^{-1})\psi\, d\mu(\sigma)$$

$$= (2\pi)^{-n/2} \int_G \{ \quad (2\pi)^{-n/2} \; \tilde{g} \underset{G}{*} \tilde{f}(\sigma) \quad 1/\Sigma(\sigma) \quad \} \; R(\sigma^{-1})\psi \; \Sigma(\sigma) d\mu(\sigma)$$

where $\underset{G}{*}$ means the usual convolution on a group [6].

$$= (2\pi)^{-n/2} \int_{R^n} K(\xi) \; R(\exp(-\xi))\psi \; d\xi$$

where K is the function

$$K(\xi) = (2\pi)^{-n/2} \; \tilde{g} \underset{G}{*} \tilde{f}(\exp\xi) \; \frac{1}{\Sigma(\exp\xi)}$$

$$= (2\pi)^{-n/2} \int_{R^n} Ff(\exp^{-1}(\exp(-\eta)\exp\xi)) Fg(\eta) \; \frac{\Sigma(\exp(-\eta)\exp\xi)}{\Sigma(\exp\xi)} \; d\eta .$$

K is a function in $L^1(R^n)$ because $\tilde{g} \underset{G}{*} \tilde{f} \in L^1(G,\mu)$ and their norms satisfy the equality $\|K\|_1 = (2\pi)^{-n/2} \|\tilde{g} \underset{G}{*} \tilde{f}\|_1$. This last equation also shows that the skew product is continuous with respect to the induced topology on $F(L^1(R^n))$ when $f \mathbin{\dot{*}} g$ is set equal to $F^{-1}K$. Furthermore, the continuity of the skew product on $F(C_o(R^n))$ and $F(C_o^\infty(R^n))$ is a direct consequence of the continuity of the convolution on G for $C_o(G)$ and $C_o^\infty(G)$.

The skew product for all these spaces is unique since $T(A)f$ is essentially a convolution operator on $L^2(G,\mu)$ with kernel uniquely determined by f.

From the proof, it is seen that the skew product for exponential groups is really convolution on the group pulled back to the Lie algebra. In order to generalize this method to other groups, it seems essential that the exponential map is onto the group. In the next chapter we look at another case of surjective exponentials; namely, the compact groups.

The space $S(R^n)$ is conspicuous by its absence in Theorem 2.3.

We will see in part II that a skew product on $S(R^n)$ should exist to study the multiplicative structure further. For nilpotent Lie groups, this is indeed the case (Corollary 2.4). Unfortunately, even for the simplest exponential group that is not nilpotent we have no skew product on the Schwartz class (Theorem 2.7).

COROLLARY 2.4: *If* G *is nilpotent in addition to the hypothesis of Theorem 2.3, then there is a unique continuous skew product on* $S = S(R^n)$ *for the left regular representation.*

PROOF: See [2 ; section 2].

Remark: In the special case of abelian groups, the skew product is exactly what is expected - it is simply the point-wise product of functions (that is $f \dot{*} g(\xi) = f(\xi)g(\xi)$). The self-adjoint n-tuple is composed of operators whose commuting spectral families enable the von Neumann functional calculus to be immediately extended.

The left regular representation plays such an important role because of the following.

THEOREM 2.5: *Let* G *be an exponential group. The skew product of Theorem 2.3 holds for the Weyl calculus of the generators of any representation of* G.

PROOF: If $iA = (iA_1,\ldots,iA_n)$ are the generators of the representation U, then there is a basis of Γ such that (1.2) holds. As in the proof of Theorem 2.3, we have

$$T(A)f\ T(A)g = (2\pi)^{-n} \int_{R^n \times R^n} Ff(\xi)\ Fg(\eta)\ U(\exp(-\xi)\exp(-\eta))\ d\eta d\xi$$

$$= (2\pi)^{-n} \int_{G\times G} \tilde{f}(\sigma)\ \tilde{g}(\delta)\ U(\sigma^{-1}\delta^{-1})\ d\mu(\delta)d\mu(\sigma)$$

$$= (2\pi)^{-n/2} \int_{G} (2\pi)^{-n/2} \int_{G} \tilde{f}(\delta^{-1}\sigma)\ \tilde{g}(\sigma)\ d\mu(\delta)\quad U(\sigma^{-1})\ d\mu(\sigma)$$

$$= (2\pi)^{-n/2} \int_{R^n} F(f * g)(\xi)\ U(\exp(-\xi))\ d\xi.$$

where $f * g$ is given in Theorem 2.3.

The rest of this chapter calculates the skew product for one specific exponential group in order to demonstrate that there is more to this structure than is first apparent.

<u>EXAMPLE</u>: Suppose G is the Lie group consisting of points on the plane together with the multiplication $(t,x)(s,y) = (t+s,\ x+e^{t}y)$. Let us calculate the exponential map on G.

We must determine the differentiable one parameter subgroup $\gamma : R \to G$ given by $\gamma(s) = (\alpha(s),\beta(s))$ that satisfies the three conditions

i) $\gamma(0) = (0,0)$ ii) $\gamma'(0) = (t_o,x_o)$ iii) $\gamma(s+t) = \gamma(s)\gamma(t)$.

When we find such a curve, then $\exp(t_o,x_o) = \gamma(1)$. Writing these conditions out in terms of α and β, we really have two differential equations whose solutions are

$$\alpha(s) = t_o s \quad \text{and} \quad \beta(s) = \frac{e^{(st_o)}-1}{t_o}x_o \quad \text{where} \quad \frac{e^{t}-1}{t} = 1 \ \text{at}\ t=0.$$

Therefore, $\exp(t,x) = (t, \frac{e^{t}-1}{t}x)$.

The exponential obviously establishes an analytic diffeomorphism between $\Gamma = R^2$ and G. With the coordinate system on the Lie algebra given by the basis $\{(1,0), (0,1)\}$, one can show the multiplication on Γ comes from the bracket $[(1,0), (0,1)] = (0,1)$.

The original coordinates on G are not exponential coordinates. In order to find the skew product according to Theorem 2.3, the following results are required.

LEMMA 2.6: *Let G be the above group and let Γ have the indicated basis. Then* a) $\exp^{-1}(\exp(t,x)\exp(s,y)) = (t+s, \frac{t+s}{e^{t+s}-1}\{\frac{e^t-1}{t}x + e^t(\frac{e^s-1}{s})y\})$

and b) $d\mu(s,y) = \frac{1-e^{-s}}{s}\, ds\, dy$ *is Haar measure in exponential coordinates.* (*that is,* $\Delta(s,y) = \frac{1-e^{-s}}{s}$ *where* Δ *is as in the proof of Lemma 2.2*)

PROOF: a) On the one hand, by definition

$$\exp(t,x)\exp(s,y) = (t, \frac{e^t-1}{t}x)(s, \frac{e^s-1}{s}y)$$

$$= (t+s, \frac{e^t-1}{t}x + e^t(\frac{e^s-1}{s})y).$$

On the other hand,

$$\exp(t+s, \frac{t+s}{e^{t+s}-1}\{\frac{e^t-1}{t}x + e^t(\frac{e^s-1}{s})y\}) = (t+s, \frac{e^t-1}{t}x + e^t(\frac{e^s-1}{s})y).$$

b) To show μ is left invariant, the equation below must be verified when exponential coordinates are used in the integrals.

$$\int_{R^2} f((t,x)(s,y))\, \frac{1-e^{-s}}{s}\, ds\, dy = \int_{R^2} f(s,y)\, \frac{1-e^{-s}}{s}\, ds\, dy \quad \text{for all} \quad f \in C_o(R^2).$$

By part a), we have

$$\int_{R^2} f((t,x)(s,y))\, \frac{1-e^{-s}}{s}\, ds\, dy = \int_{R^2} f(t+s, \frac{t+s}{e^{t+s}-1}\{\frac{e^t-1}{t}x + e^t(\frac{e^s-1}{s})y\})\, \frac{1-e^{-s}}{s}\, dsdy$$

$$= \int_{R^2} f(s, \frac{s}{e^s-1}\{\frac{e^t-1}{t}x + e^t(\frac{e^{s-t}-1}{s-t})y\})\ \frac{1-e^{t-s}}{s-t}\, ds\, dy$$

Interchange the order of integration and change the variable y to obtain

$$= \int_{R^2} f(s,y)\ \frac{e^s-1}{s}\ e^{-t}\ \frac{s-t}{e^{s-t}-1}\ \frac{1-e^{t-s}}{s-t}\, ds\, dy$$

$$= \int_{R^2} f(s,y)\ \frac{e^s-1}{s}\ e^{-t}\ \frac{e^{-s}(e^s-e^t)}{e^{-t}(e^s-e^t)}\, ds\, dy$$

$$= \int_{R^2} f(s,y)\ \frac{1-e^{-s}}{s}\, ds\, dy.$$

<u>THEOREM 2.7</u>: *If* G *is the above group and* A *is the self-adjoint n-tuple associated to the left regular representation with the indicated basis for* Γ, *then* $S = \mathcal{S}(R^2)$ *has* <u>*no*</u> *skew product.*

<u>PROOF</u>: Assume that a skew product does exist. For $f,g \in \mathcal{S}(R^2)$, the function $F(f*g)$ has the following form according to Theorem 2.3 and Lemma 2.6.

$F(f*g)(t,x)$

$$= \frac{1}{2\pi}\int_{R^2} Ff(t-s, \frac{t-s}{e^{t-s}-1}\{\frac{e^{-s}-1}{s}y + e^{-s}(\frac{e^t-1}{t}x)\})\ Fg(s,y)\ \frac{t-s}{1-e^{s-t}}\ \frac{1-e^{-t}}{t}\, ds\, dy\ .$$

It is apparent that $F(f*g)$ is defined everywhere and is in fact a continuous function (since the factor $(t-s)/(1-e^{s-t})$ is a polynomially bounded analytic function of s). As there is a skew product, $F(f*g)$ as defined above is in $\mathcal{S}(R^2)$.

To obtain a contradiction, we have only to exhibit two functions f and g in $\mathcal{S}(R^2)$ such that $F(f*g)$ is not in $\mathcal{S}(R^2)$. To this end, suppose $Ff = \phi\times\psi$ and $Fg = \alpha\times\beta$ (that is, $Ff(u,v) = \phi(u)\psi(v)$ etc.) where ϕ,ψ,α,β are all non-negative functions in $\mathcal{S}(R)$ to be chosen below. With this decomposition

$$F(f*g)(0,x) = \frac{1}{2\pi}\int_{R^2} Ff(-s, \frac{-s}{e^{-s}-1}(\frac{e^{-s}-1}{s}y + e^{-s}x))Fg(s,y)\ \frac{-s}{1-e^s}\ ds\,dy$$

$$= \frac{1}{2\pi}\int_R \phi(-s)\alpha(s)\ \frac{-s}{1-e^s}\left\{\int_R \psi(-y - \frac{s}{1-e^s}x)\ \beta(y)\ dy\right\} ds.$$

Define ψ and β so that they are non-negative functions in $S(R)$ and satisfy

i) $\psi(y) \equiv 1$ for $|y| \le 1$ and ii) $\beta(y) \equiv e^y$ for $y \le 1$.

Since $s/(1-e^s) \le 0$ for all s, we have for any $x \le 0$

$$F(f*g)(0,x) \ge \frac{1}{2\pi}\int_R \phi(-s)\alpha(s)\ \frac{-s}{1-e^s}\int_{(-1-sx/(1-e^s))}^{(1-sx/(1-e^s))} e^y\,dy\,ds$$

$$= \frac{e^1-e^{-1}}{2\pi}\int_R \phi(-s)\alpha(s)\ \frac{-s}{1-e^s}\ e^{-sx/(1-e^s)}\ ds.$$

Now define ϕ and α so that they are non-negative, in $S(R)$, and satisfy

iii) $\phi(-s) \equiv -s/(1-e^s)$ for $s \ge 0$ and

iv) $\alpha(s) \equiv -\frac{d}{ds}\{-s/(1-e^s)\}$ for $s \ge 0$.

Substituting these functions into our equation for $F(f*g)$, we obtain

$$F(f*g)(0,x) \ge \frac{1}{4}\int_0^\infty (\frac{-s}{1-e^s})^2\ (-\frac{d}{ds}\{-s/(1-e^s)\})\ e^{-sx/(1-e^s)}\ ds \quad \text{for any} \quad x \le 0$$

$$= \frac{1}{4}\int_1^0 -s^2\ e^{sx}\ ds$$

$$= \frac{1}{4}\{\ e^x/x - 2e^x/x^2 + 2e^x/x^3 - 2/x^3\ \} \quad \text{for} \quad x < 0.$$

Obviously, $F(f*g)(0,x)$ does not belong to $S(R)$. Therefore, there is no skew product on $S(R^2)$.

CHAPTER THREE

COMPACT LIE GROUPS

If G is a compact group, the technique of Theorem 2.3 must be altered because the exponential map is not a diffeomorphism. Fortunately, the theory of Riemannian geometry is applicable and supplies the necessary facts concerning the exponential in place of Lemma 2.2.

LEMMA 3.1: *Let G be a compact group with n dimensional Lie algebra.*

a) *The set C of singular points of the exponential map is a closed set in R^n with measure zero.*

b) *There is an open, relatively compact neighborhood E of $0 \varepsilon \Gamma$ such that*

   i) $\exp$ *is a diffeomorphism of E onto $\exp E$*

   ii) E *is the largest connected neighborhood with this property*

   iii) $\exp(\bar{E}) = G$ *where $\bar{E}$ is the closure of E.*

c) *With ω the left invariant n-form on G and $\Delta(x)$ defined through the exponential as in the proof of Lemma 2.2,*

   i) *Haar measure is given by*

$$\int_G \phi(\sigma)\, d\mu(\sigma) = \int_E \phi(\exp\xi)\, \Delta(\xi)\, d\xi \quad \textit{for all continuous } \phi \textit{ on } G$$

   ii) *When f is a continuous function on R^n with support where $\exp$ is a diffeomorphism (so $\exp^{-1}$ is defined)*

$$\int_{R^n} f(\xi)\, d\xi = \int_{\exp(\mathrm{supp}\, f)} f(\exp^{-1}\sigma) \frac{1}{|\Delta(\exp^{-1}\sigma)|}\, d\mu(\sigma).$$

PROOF: a) Since $C = \{\xi \varepsilon \Gamma \mid \exp : \Gamma \to G$ is not a local diffeomorphism at $\xi\}$, it is clearly a closed set. If the exponential is composed with an analytic coordinate chart, C is locally the set of points where the determinant of an analytic map is zero. The calculation of the determinant is an analytic operation so, locally, C is the set of zeroes of an analytic real-valued map.

If this analytic map vanished identically on any non-empty open set in Γ, then exp would be singular everywhere by extending the chart to an analytic atlas. But exp is not always singular, hence, C is the set of zeroes of a not identically zero analytic function. It is an easy exercise to show that such a set has measure zero.

b) This is the part in which the Riemannian metric plays a leading role. The details of its construction and certain theorems concerning Riemannian manifolds will not be proved because the Riemannian structure of our groups is overall of secondary importance. For a rigorous presentation of the theory, the reader is referred to [9] - especially chapters IV and VIII.

For our purposes, the most important aspect of the metric is that the one parameter subgroups obtained from the exponential map are the geodesics of the manifold that pass through the identity e of G. Since G is compact, it is a complete Riemannian manifold. As such, any point in G can be joined to e by a geodesic that minimizes arclength [9 ; chapter IV, page 172].

Define the set E of the theorem as follows

$$E = \{\xi \varepsilon \Gamma : \text{the curve } \gamma_\eta(t) = \exp(t\eta) \text{ minimizes arclength from } e \text{ to } \exp(\eta) \text{ for all } \eta \text{ in some neighborhood of } \xi\}.$$

Then E is an open, relatively compact neighborhood of $e = 0$ in Γ and $\exp : E \longrightarrow \exp E$ is a diffeomorphism onto an open set of G. In addition, G is the disjoint union of $\exp E$ and $\exp(\overline{E} \sim E)$. See [9 ; chapter VIII, page 100].

c) The right side of the equation in i) is a left invariant positive measure because $\Delta(\xi)$ is positive on the connected set E and the measure of the boundary of E is zero. It should be noted that $\Delta(\xi)$ is not always positive on the entire Lie algebra.

The second statement is simply the construction of the Haar measure from the left invariant n-form ω as in Lemma 2.2. The absolute value appears on the integration factor because the exponential map need not be orientation preserving at all points where it is a local diffeomorphism.

With the above result, a skew product may be developed for general compact groups.

THEOREM 3.2: *If* A *is the self-adjoint n-tuple associated to the left regular representation of a compact group* G, *then there is a continuous skew product on* $S = F(L^1(R^n))$.

PROOF: With C and E as in Lemma 3.1, assume that $Ff \in C_0(R^n \sim C)$ (that is, Ff is a continuous function on R^n with compact support outside C) and define a function on the group analogous to that in Theorem 2.3 by the formula

$$\tilde{f}(\sigma) = \sum_{\substack{\exp\xi = \sigma \\ \xi \in \operatorname{supp}(Ff)}} \left\{ \frac{Ff(\xi)}{|\Delta(\xi)|} \right\} \quad \text{for } \sigma \in G$$

where Δ is taken from the previous lemma. For fixed $\sigma \in G$, $\tilde{f}(\sigma)$ is a finite sum of finite numbers. The numbers are finite since $\Delta(\xi) \neq 0$ for $\xi \in \operatorname{supp} Ff$. If the number of terms were not finite, there would be a sequence ξ_n with a limit point ξ_0 in $\operatorname{supp} Ff$ such that $\exp\xi_n = \sigma$ and thus exp would be singular at a point outside C. This yields a contradiction.

It is easy to see that $\tilde{f}$ is actually a continuous function on the group. Furthermore,

$$\|\tilde{f}\|_1 = \int_G |\tilde{f}(\sigma)| \, d\mu(\sigma)$$

$$= \int_G \left| \sum_{\substack{\exp\xi = \sigma \\ \xi \,\varepsilon\, \text{supp } Ff}} \frac{Ff(\xi)}{|\Delta(\xi)|} \right| d\mu(\sigma)$$

$$\leq \int_G \sum_{\substack{\exp\xi = \sigma \\ \xi \,\varepsilon\, \text{supp } Ff}} \left| \frac{Ff(\xi)}{\Delta(\xi)} \right| d\mu(\sigma)$$

$$= \int_{R^n} |Ff(\xi)|\, d\xi \quad \text{by Lemma 3.1 c) ii)}$$

$$= \|Ff\|_1 .$$

With this definition of $\tilde{f}$, the Weyl calculus has the form (see Theorem 2.3)

$$T(A)f\ (\psi) = (2\pi)^{-n/2} \int_G \tilde{f}(\sigma)\ R(\sigma^{-1})\psi\ d\mu(\sigma) \quad \text{for all} \quad \psi \,\varepsilon\, L^2(G,\mu)$$

and $$T(A)f\ T(A)g\ (\psi) = (2\pi)^{-n} \int_G \tilde{g} \underset{G}{*} \tilde{f}(\sigma)\ R(\sigma^{-1})\psi\ d\mu(\sigma)$$

where $f,g \,\varepsilon\, F(C_0(R^n \sim C))$.

The convolution present in the last integral may be pulled back to a function on R^n as follows

$$(3.1) \qquad K(\xi) = \begin{cases} (2\pi)^{-n/2}\ \Delta(\xi)\ \tilde{g} \underset{G}{*} \tilde{f}(\exp\xi) & \text{for} \quad \xi \,\varepsilon\, E \\ 0 \quad \text{if} \quad \xi \notin E. \end{cases}$$

It is obvious that K satisfies the relation for the product of operators; thus,

$$(2\pi)^{-n/2} \int_{R^n} K(\xi)\ \ R(\exp(-\xi))\psi\ d\xi = T(A)f\ T(A)g\ (\psi).$$

Let us examine this new function more closely. Since $\tilde{g} \underset{G}{*} \tilde{f}$ is a continuous function on G, therefore K is continuous on E. Moreover, $K \,\varepsilon\, L^1(R^n)$ and, in fact, there is an estimate for its norm given below.

$$(2\pi)^{n/2}\ \|K\|_1 = \|\tilde{g} \underset{G}{*} \tilde{f}\|_1 \leq \|\tilde{g}\|_1\ \|\tilde{f}\|_1 \leq \|Ff\|_1\ \|Fg\|_1 .$$

The definition $f * g = F^{-1}K$ provides a multiplicative structure by giving a map $F(C_0(R^n \sim C)) \times F(C_0(R^n \sim C)) \longrightarrow F(L^1(R^n))$ that is continuous, by means of the norm inequality, when $F(C_0(R^n \sim C))$ has the relative topology induced from $F(L^1(R^n))$. As C has measure zero, $F(C_0(R^n \sim C))$ is dense in $F(L^1(R^n))$. Therefore, there is a unique continuous extension of the above map that yields a continuous skew product on $F(L^1(R^n))$.

The analogue of Theorem 2.5 is proved using the Peter-Weyl Theorem.

THEOREM 3.3: *Let* G *be a compact group. The skew product of Theorem 3.2 holds for the Weyl calculus corresponding to any representation of* G.

PROOF: First suppose that $V : \sigma \longrightarrow V(\sigma)$ is a subrepresentation of the left regular representation. Then there is a projection operator P_V on $L^2(G,\mu)$ such that $V(\sigma) = R(\sigma)P_V$. Let A_V denote the generators of the representation corresponding to a fixed basis of Γ. For f in $F(L^1(R^n))$, V is related to R by

$$T(A_V)f\ (\psi) = (2\pi)^{-n/2} \int_{R^n} Ff(\xi)\ V(\exp(-\xi))\psi\ d\xi \quad \text{for} \quad \psi \in P_V(L^2(G,\mu))$$

$$= (2\pi)^{-n/2} \int_{R^n} Ff(\xi)\ R(\exp(-\xi))P_V\psi\ d\xi$$

$$= T(A)f\ (\psi) \quad \text{since} \quad P_V\psi = \psi.$$

From this equation, it is clear that the skew product for the left regular representation is valid for any of its subrepresentations.

Now suppose that $W : \sigma \longrightarrow W(\sigma)$ is an irreducible representation. By the Peter-Weyl Theorem [6 ; page 24], W is equivalent to an irreducible subrepresentation V of the left regular representation. Hence, there is a linear isometry B between the underlying Hilbert spaces such that the equality $W(\sigma) = B^{-1}V(\sigma)B$ is true. If H is the Hilbert space for W, then

$$T(A_W)f\ (\psi) = (2\pi)^{-n/2} \int_{R^n} Ff(\xi)\ W(\exp(-\xi))\psi\ d\xi \quad \text{for all} \quad \psi \in H$$

$$= (2\pi)^{-n/2} \int_{R^n} Ff(\xi)\ B^{-1}V(\exp(-\xi))B\psi\ d\xi$$

$$= (2\pi)^{-n/2}\ B^{-1}\left\{ \int_{R^n} Ff(\xi)\ V(\exp(-\xi))B\psi\ d\xi \right\}$$

$$= B^{-1}(T(A_V)f)\ (B\psi).$$

Writing out the product of two such operators, a trivial cancellation shows that the skew product remains valid for irreducible representations.

Any representation for a compact group is the direct sum of irreducible representations. As the skew product holds for all the summands, it will also hold for the direct sum.

<u>Remark</u>: The skew product of Theorem 3.2 is clearly not unique. A specific set E was chosen in the proof that behaved nicely under the exponential map. There are certainly other sets that would do equally as well.

Since the function K given by (3.1) is usually discontinuous at the boundary of E, our skew product will not suffice for the other three function spaces introduced in Chapter One. However, it is interesting to question the existence of some multiplicative structure on these spaces. The difficulties encountered in Chapter Four for the group $SU(2)$ discourages us from looking at the general case.

CHAPTER FOUR

$SU(2)$

Let G be the real, simply connected Lie group $SU(2)$ of 2×2 unitary matrices of determinant one. Since G is the two-fold universal covering group of the rotation group SO(3), any skew product defined through G will, a priori, establish a skew product for SO(3).

The Lie algebra Γ is the set of 2×2 skew-hermitian matrices of trace zero. Choose the following matrices as our basis for Γ.

$$\left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

As always, we need some facts about the exponential map before producing a skew product.

LEMMA 4.1: a) *With the above basis,* $\exp : \Gamma \to G$ *is given by*

$$\exp(\xi_1,\xi_2,\xi_3) = \begin{pmatrix} \cos|\xi| + \frac{\sin|\xi|}{|\xi|}\xi_1 i & \frac{\sin|\xi|}{|\xi|}(\xi_2 i + \xi_3) \\ \frac{\sin|\xi|}{|\xi|}(\xi_2 i - \xi_3) & \cos|\xi| - \frac{\sin|\xi|}{|\xi|}\xi_1 i \end{pmatrix}$$

where $|\xi| = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2}$ *and* $\frac{\sin|\xi|}{|\xi|} = 1$ *at* $|\xi| = 0$.

b) *The exponential is periodic of period* 2π *along any line through the origin. For any non-negative integer* n, *the set* $\{\xi : n\pi \le |\xi| \le (n+1)\pi\}$ *parameterizes the group* G *and* exp *is a diffeomorphism on the interior of any of these sets. The set* $\overline{E}$ *of Lemma 3.1 is of the above form with* $n = 0$.

c) *If* $\circ$ *denotes group composition on one of these parameterizations, then*

$$\cos|\eta^{-1}\circ\xi| = \cos|\eta|\cos|\xi| + \sin|\eta|\sin|\xi| \frac{\eta\cdot\xi}{|\eta||\xi|}.$$

d) *With the notation of Lemma 3.1,* $\Delta(\xi) = (\sin^2|\xi|)/(|\xi|)^2$.

PROOF: a) and b) are easy exercises using the exponential of a matrix.

c) This is an immediate consequence of matrix mutiplication and the fact that $\cos|\xi| = \frac{1}{2}\,\mathrm{trace}(\exp\xi)$.

d) A slight modification of the Weyl Integration Formula using the roots of G produces this integrating factor. For a more constructive proof, we refer the reader to [12 ; page 220].

THEOREM 4.2: *For* $G = SU(2)$ *, there is a skew product on the space of functions* $S = F(C_0(R^3))$.

PROOF: We have only to consider the left regular representation of G due to Theorem 3.3.

Let $D(\pi,2\pi)$ be the set $\{\xi : \pi \leq |\xi| \leq 2\pi\}$ in 3-space. By Lemma 4.1, $D(\pi,2\pi)$ may be used as a coordinate system for G under the exponential map. The inner and outer boundaries of this set correspond to the matrices $-I$ and I of the group respectively.

Define a function on the group for all $f \in F(C_0(R^3))$, with respect to the above coordinates, by

$$\tilde{f}(\xi) = \sum_{\exp\alpha = \xi} \frac{Ff(\alpha)}{|\Delta(\alpha)|} \qquad \text{for} \quad \pi < |\xi| < 2\pi$$

$$= \frac{1}{\sin^2|\xi|} \sum_{n=-\infty}^{\infty} Ff\left(\xi + \frac{2n\pi\xi}{|\xi|}\right) \left|\xi + \frac{2n\pi\xi}{|\xi|}\right|^2$$

by the periodicity of the exponential (Lemma 4.1). We set $\tilde{f}$ equal to 0 at the boundary of $D(\pi,2\pi)$.

Following Theorem 3.2, the skew product of $f,g \in F(C_0(R^3))$ should be provided through the function K defined on the next page.

$$K(\xi) = (2\pi)^{-3/2} \frac{\sin^2|\xi|}{|\xi|^2} \int_{D(\pi,2\pi)} \tilde{f}(\eta^{-1} \circ \xi)\, \tilde{g}(\eta)\, d\mu(\eta)$$

for ξ in the interior of $D(\pi,2\pi)$. Of course, we do not as yet know if K is defined point-wise because the singular set of the exponential was not avoided as it was in Theorem 3.2. Let us rewrite K in order to examine the last integral more closely. First, for $f \in F(C_0(R^3))$, define

$$f'(\xi) = \frac{1}{|\xi|^2} \sum_{n=-\infty}^{\infty} Ff\left(\xi + \frac{2n\pi\xi}{|\xi|}\right) \left|\xi + \frac{2n\pi\xi}{|\xi|}\right|^2 \quad \text{for all } \xi \in D(\pi,2\pi).$$

With this notation, K becomes

$$(4.1) \quad K(\xi) = (2\pi)^{-3/2} \frac{\sin^2|\xi|}{|\xi|^2} \int_{D(\pi,2\pi)} f'(\eta^{-1} \circ \xi)\, g'(\eta)\, \frac{|\eta^{-1} \circ \xi|^2}{\sin^2|\eta^{-1} \circ \xi|}\, d\eta$$

by changing Haar measure into Lebesgue measure.

It should be evident why $D(\pi,2\pi)$ was chosen for our parameterization instead of the set E as in Lemma 3.1. On $D(\pi,2\pi)$, f' and g' are uniformly continuous and the expressions $|\xi|^2$ and $|\eta^{-1} \circ \xi|^2$ are never zero.

Our proof consists of three steps concerning the properties of K. After these are performed, it will only remain to "round off the edges" of K at the boundary of our coordinate system. The steps are:

Step 1. $K(\xi)$ exists for each ξ in the interior of $D(\pi,2\pi)$.

Step 2. $K(\xi)$ is continuous at each of these points.

Step 3. $K(\xi)$ can be extended to a continuous function on $D(\pi,2\pi)$ that is constant on each boundary.

Step 1. Fix ξ so that $\pi < |\xi| < 2\pi$. Then, for some constant c, depending on ξ

$$c|K(\xi)| \leq \sup_\eta |f'(\eta)| \sup_\eta |g'(\eta)| \int_{D(\pi,2\pi)} \frac{|\eta^{-1} \circ \xi|^2}{\sin^2|\eta^{-1} \circ \xi|}\, d\eta$$

where c is greater than zero and the sup is taken over $D(\pi, 2\pi)$.

The problem in showing the integrand in the last integral is summable is that the denominator is sometimes zero. In fact,

$$\begin{aligned}\{\eta : \sin^2|\eta^{-1} \circ \xi| = 0\} &= \{\eta : |\eta^{-1} \circ \xi| \in \{\pi, 2\pi\}\} \\ &= \{\eta : \eta^{-1} \circ \xi = \pm I\} \\ &= \{\xi , \xi \circ (-I)\}.\end{aligned}$$

Since neither of these points are on the boundary of $D(\pi, 2\pi)$, we can choose balls M_ξ and N_ξ around ξ and $\xi \circ (-I)$ respectively, whose closures stay away from the boundary. Let us split up the integral for the estimate of K into an integral over M_ξ, over N_ξ, and over the remainder. The Haar measure gives a bound for the first and second integral; namely,

$$\begin{aligned}\int_{M_\xi} \frac{|\eta^{-1} \circ \xi|^2}{\sin^2|\eta^{-1} \circ \xi|}\, d\eta &= \int_{M_\xi} \frac{|\eta^{-1} \circ \xi|^2}{\sin^2|\eta^{-1} \circ \xi|} \frac{|\eta|^2}{\sin^2|\eta|}\, d\mu(\eta) \\ &\le c_1 \int_{M_\xi} \frac{|\eta^{-1} \circ \xi|^2}{\sin^2|\eta^{-1} \circ \xi|}\, d\mu(\eta) \\ &\quad \text{where } |\eta|^2/\sin^2|\eta| \text{ is bounded by } c_1 \text{ on } M_\xi \\ &= c_1 \int_{\xi^{-1} \circ M_\xi} d\eta \\ &\le c_1 \cdot \text{volume}(D(\pi, 2\pi)).\end{aligned}$$

Likewise, the integral over N_ξ is bounded by $c_2 \cdot \text{volume}(D(\pi, 2\pi))$.

The integral over the remainder is also bounded since the integrand is a uniformly continuous function on this set. Therefore, formula (4.1) does produce a candidate for the skew product.

Step 2. This is essentially a refinement of the argument used in step 1. As $\sin^2|\xi|/|\xi|^2$ is continuous near a fixed point ξ between π and 2π, we need only show the integral in (4.1) is continuous. To this end, suppose that ξ_1 is close to ξ but is not $\pm I$. The difference of the integrals is

$$\left|\int_{D(\pi,2\pi)} f'(\eta^{-1}\circ\xi)\, g'(\eta)\frac{|\eta^{-1}\circ\xi|^2}{\sin^2|\eta^{-1}\circ\xi|} - f'(\eta^{-1}\circ\xi_1)g'(\eta)\frac{|\eta^{-1}\circ\xi_1|^2}{\sin^2|\eta^{-1}\circ\xi_1|}\,d\eta\right|$$

$$\leq \int_{D(\pi,2\pi)}|g'(\eta)|\left|f'(\eta^{-1}\circ\xi)\frac{|\eta^{-1}\circ\xi|^2}{\sin^2|\eta^{-1}\circ\xi|} - f'(\eta^{-1}\circ\xi_1)\frac{|\eta^{-1}\circ\xi_1|^2}{\sin^2|\eta^{-1}\circ\xi_1|}\right|d\eta.$$

The last integral will again be split up as in step 1 in order to obtain an estimate. Instead of writing the expression inside the long absolute value signs of the last integral each time, it will be denoted by Ω whenever it is used. Let us first calculate a bound over M_ξ.

$$\int_{M_\xi}|g'(\eta)||\Omega|d\eta \leq \sup_\eta|f'(\eta)|\ \sup_\eta|g'(\eta)|\int_{M_\xi}\frac{|\eta^{-1}\circ\xi|^2}{\sin^2|\eta^{-1}\circ\xi|}+\frac{|\eta^{-1}\circ\xi_1|^2}{\sin^2|\eta^{-1}\circ\xi_1|}\,d\eta$$

$$\leq c_3\cdot\{\mathrm{volume}(\xi^{-1}\circ M_\xi)+\mathrm{volume}(\xi_1^{-1}\circ M_\xi)\}$$

if ξ_1 is close enough to ξ. This inequality results by changing to Haar measure and then back again as we did in the calculation on page 25. The constant c_3 is a bound for the expression $\sup|f'|\cdot\sup|g'|\cdot|\eta|^2/\sin^2|\eta|$ when η is restricted to lie in M_ξ.

Suppose $\alpha>0$ is given. Choose M_ξ so that $\mathrm{volume}(\xi^{-1}\circ M_\xi)<\alpha/8c_3$. Then there is a $\delta_1>0$ such that $|\xi-\xi_1|<\delta_1 \Rightarrow \mathrm{volume}(\xi_1^{-1}\circ M_\xi)<\alpha/6c_3$ and δ_1 is less than half the radius of M_ξ. With this restriction for ξ, we have

$$\int_{M_\xi}|g'(\eta)||\Omega|d\eta < c_3\cdot\{\alpha/8c_3+\alpha/6c_3\} < \alpha/3.$$

Likewise, choose N_ξ and then δ_2 such that δ_2 is less than half the

the distance from ξ to the boundary of $N_\xi o(-I)$ and $|\xi-\xi_1| < \delta_2 \Rightarrow$

$$\int_{N_\xi} |g'(\eta)||\Omega| d\eta < \alpha/3.$$

We have only to estimate the integral over the remainder. Define a function $u : \{D(\pi,2\pi) \sim (M_\xi \cup N_\xi)\} \times \{\xi_1 : |\xi-\xi_1| \leq \text{minimum}(\delta_1,\delta_2)\} \longrightarrow \mathbb{C}$, by

$$u(\eta,\xi_1) = f'(\eta^{-1}o\xi_1)\cdot|\eta^{-1}o\xi_1|^2/\sin^2|\eta^{-1}o\xi_1|$$

where $\mathbb{C}$ denotes the complex numbers. This function is uniformly continuous since $\eta^{-1}o\xi$ remains away from $\pm I$ on this set.

Therefore, there is a $\delta < \text{minimum}(\delta_1,\delta_2)$ such that $|\xi-\xi_1| < \delta \Rightarrow$ $|u(\eta,\xi_1) - u(\eta,\xi)| < \alpha/3\sup|g'|$ for all η where u is defined. Putting all this together, we conclude that $|\xi-\xi_1| < \delta$ implies

$$\int_{D(\pi,2\pi)} |g'(\eta)||\Omega| \eta < \alpha.$$

This completes step 2.

Step 3. It will only be proved here that $K(\xi)$ can be extended continuously to a constant on the sphere $|\xi| = 2\pi$. The analogous proof for the other boundary is left to the reader.

We must show that for a given $\alpha>0$ there is a $\delta>0$ such that $2\pi - \delta < |\xi_j| < 2\pi$ for $j=1,2$ implies $|K(\xi_1) - K(\xi_2)| < \alpha$. The demonstration of this fact rests heavily on the following statement:

$$(4.2)\left\{\begin{array}{l} \text{Given } \alpha>0 \text{ there exists } \delta_1 > \delta_2 > 0 \text{ such that } 2\pi - \delta_2 < |\xi| < 2\pi \Rightarrow \\ \displaystyle\int_{D(\pi,\pi+\delta_1)} \frac{\sin^2|\xi|}{\sin^2|\eta^{-1}o\xi|} d\eta + \int_{D(2\pi-\delta_1,2\pi)} \frac{\sin^2|\xi|}{\sin^2|\eta^{-1}o\xi|} d\eta < \alpha. \end{array}\right.$$

The notation $D(a,b)$ means the set $\{\xi : a \leq |\xi| \leq b\}$. The proof of (4.2) will be postponed until the end of the theorem for fear of losing the flow of step 3 if given at this time. Let us continue assuming (4.2).

By formula (4.1), $(2\pi)^{3/2}\ |K(\xi_1) - K(\xi_2)|$ is bounded by

$$\int_{D(\pi,2\pi)} |g'(\eta)| \left| f'(\eta^{-1}o\xi_1)\frac{|\eta^{-1}o\xi_1|^2\cdot\sin^2|\xi_1|}{|\xi_1|^2\cdot\sin^2|\eta^{-1}o\xi_1|} - f'(\eta^{-1}o\xi_2)\frac{|\eta^{-1}o\xi_2|^2\cdot\sin^2|\xi_2|}{|\xi_2|^2\cdot\sin^2|\eta^{-1}o\xi_2|}\right| d\eta.$$

This integral will be split up again and the expression inside the long absolute value signs will be denoted by $\mathcal{Q}$.

Suppose $\alpha>0$ is given. Notice that $|\eta^{-1}o\xi|^2/|\xi|^2 \leq 4$ for our parameterization. Choose $\delta_1 > \delta_2 > 0$ so that the integral in (4.2) is bounded by $\alpha/8\cdot\sup|f'|\cdot\sup|g'|$. Then, if $2\pi-\delta_2 < |\xi_j| < 2\pi$ for $j=1,2$

$$\int_{D(\pi,\pi+\delta_1)} |g'(\eta)||\mathcal{Q}|d\eta + \int_{D(2\pi-\delta_1,2\pi)} |g'(\eta)||\mathcal{Q}|d\eta < \alpha/2.$$

Let us estimate the integral over the remainder. As in step 2, introduce a function $u : D(\pi+\delta_1, 2\pi-\delta_1) \times D(2\pi-\delta_2, 2\pi) \longrightarrow \mathbb{C}$ through

$$u(\eta,\xi) = f'(\eta^{-1}o\xi)\ \frac{|\eta^{-1}o\xi|^2\cdot\sin^2|\xi|}{|\xi|^2\cdot\sin^2|\eta^{-1}o\xi|}\ .$$

This function is uniformly continuous on its domain and, in fact, satisfies the following statement since the second variable is restricted to lie in a neighbourhood of the identity matrix I in G. Given $\alpha>0$ there is a positive $\delta_3 < \delta_2$ such that $|u(\eta,\xi_1) - u(\eta,\xi_2)| < \alpha$ whenever $2\pi-\delta_3 < |\xi_j| < 2\pi$ for all η where u is defined. Thus, for small enough δ, we have

$$\int_{D(\pi+\delta_1,2\pi-\delta_1)} |g'(\eta)||\mathcal{Q}|d\eta < \alpha/2.$$

This completes step 3.

The above three steps prove that $K(\xi)$ is a uniformly continuous function on $D(\pi, 2\pi)$ that is constant on each boundary. In order to exhibit a skew product of $F(C_0(R^3))$, let us change K into a continuous function with compact support in R^3 while retaining the multiplicative property.

Define $\beta : R \to R$ by the formula

$$\beta(t) = \begin{cases} 1 - \frac{1}{\pi}\left|t - \frac{3\pi}{2}\right| & \text{for } \pi/2 \le t \le 5\pi/2 \\ 0 & \text{otherwise.} \end{cases}$$

$\beta(t)$ is a continuous function with compact support, maximum value of 1 attained at $t=3\pi/2$, and decreases linearly to zero at $t=\pi/2$ and $t=5\pi/2$. In addition, it satisfies the equation

$$\sum_{n=-\infty}^{\infty} \beta(|t+2n\pi|) = 1 \quad \text{for every } t.$$

Finally, the stage is set to introduce the skew product of the two functions $f, g \in F(C_0(R^3))$ by means of

$$F(f * g)(\xi) = \sum_{n=-\infty}^{\infty} \beta(|\xi|) \frac{1}{|\xi|^2}\left|\xi + \frac{2n\pi\xi}{|\xi|}\right|^2 K(\xi + \frac{2n\pi\xi}{|\xi|}) \quad \text{for all } \xi.$$

Then $F(f * g) \in C_0(R^3)$ and has support in $D(\pi/2, 5\pi/2)$. Going back to the constuction of the function $\tilde{f}$ on page 23, we can see immediately that this definition yields a skew product. Moreover, the skew product is a linear map that is continuous with respect to the topology of $F(C_0(R^3))$.

<u>Proof of (4.2)</u>: It will only be shown that the integral near 2π in (4.2) may be made small uniformly for ξ near 2π. The method described easily extends to the other integral and together they imply (4.2).

If δ_1 is a small positive number to be restricted later (always assume that it is less than $\pi/2$), then the integral is estimated as follows:

$$\int_{D(2\pi-\delta_1,2\pi)} \frac{\sin^2|\xi|}{\sin^2|\eta^{-1}\circ\xi|}\,d\eta$$

$$= \int_{D(2\pi-\delta_1,2\pi)} \frac{\sin^2|\xi|\ \ d\eta}{1-\{\cos|\eta|\cos|\xi|+\sin|\eta|\sin|\xi|(\eta\cdot\xi/|\eta||\xi|)\}^2} \quad \text{by Lemma 4.1}$$

Change to polar coordinates with ξ along the positive z-axis and $|\xi|=r$

$$= \int_0^{2\pi}\int_{2\pi-\delta_1}^{2\pi}\int_0^{\pi} \frac{\sin^2 r}{1-\{\cos\rho\cos r + \sin\rho\sin r\cos\phi\}^2}\,\rho^2\sin\phi\,d\phi d\rho d\theta$$

Let $v(\phi) = \cos\rho\cos r + \sin\rho\sin r\cos\phi$

$$= 2\pi\int_{2\pi-\delta_1}^{2\pi}\int_{\cos(\rho-r)}^{\cos(\rho+r)} \frac{-\rho^2\sin^2 r}{\sin\rho\sin r\,(1-v^2)}\,dv d\rho$$

$$= \pi\sin r\int_{2\pi-\delta_1}^{2\pi} \frac{\rho^2}{\sin\rho}\log\left\{\frac{1+\cos(\rho-r)}{1-\cos(\rho-r)}\,\frac{1-\cos(\rho+r)}{1+\cos(\rho+r)}\right\}d\rho\,.$$

The proof depends on an estimate of this integral.

Since $\delta_1<\pi/2$ and ξ is taken such that $2\pi-\delta_1 < |\xi| < 2\pi$, the function $\sin r$ is negative and so is $w(\rho)$ below

$$w(\rho) = \frac{1}{\sin\rho}\log\left\{\frac{1+\cos(\rho-r)}{1-\cos(\rho-r)}\,\frac{1-\cos(\rho+r)}{1+\cos(\rho+r)}\right\} \qquad 2\pi-\delta_1<\rho<2\pi\,.$$

In fact, we claim that w increases from $-\infty$ in the interval $r<\rho<2\pi$. It suffices to show that the derivative of w is non-negative in this interval. Only a sketch of this result will be provided here. The verification of each step is left to the reader. If we set $x(\rho) = (\sin^2\rho/\cos\rho)\frac{dw}{d\rho}$, then $x(2\pi)=0$ and $dx/d\rho = \{\sin\rho\sin r\sin 2\rho\}/\{\sin^2(\rho-r)\sin^2(\rho+r)\}$ which is less than zero in our interval. Thus $x(\rho)$ is positive in the interval and so is $dw/d\rho$.

We are now ready to estimate the integral at the top of the page. It is split into an integral for $2\pi-\delta_1<\rho<\frac{1}{2}(2\pi+r)$ and then for

$\frac{1}{2}(2\pi+r) < \rho < 2\pi$. The second integral is bounded by the expression

$$4\pi^3 \sin r\,(2\pi-\frac{2\pi+r}{2})\ \frac{1}{\sin(\frac{2\pi+r}{2})}\ \log\left\{\frac{1+\cos(\frac{2\pi-r}{2})}{1-\cos(\frac{2\pi-r}{2})}\ \frac{1-\cos(\frac{2\pi+3r}{2})}{1+\cos(\frac{2\pi+3r}{2})}\right\}$$

since w is increasing on this interval. Write the last part of this expression as $\log(\dagger)$. We have the following limits.

$$\lim_{r\to 2\pi} \frac{\sin r}{\sin(\frac{2\pi+r}{2})} = 2$$

$$\lim_{r\to 2\pi} \log(\dagger) = \log 9 .$$

Therefore, if $\alpha>0$ is fixed, there is a $\delta_2 < \delta_1$ such that $2\pi-\delta_2 < |\xi| < 2\pi$ implies that

$$\int_{D(\frac{2\pi+|\xi|}{2},2\pi)} \frac{\sin^2|\xi|}{\sin^2|\eta^{-1}o\xi|}\,d\eta < \alpha/2 .$$

To estimate the integral over the first interval, expand the functions $1-\cos(\rho-r)$ and $1-\cos(\rho+r)$ about $\rho=r$ and $\rho+r=4\pi$ respectively by means of Taylor's Formula. The expansion up to second order provides the following approximations if δ_1 is small enough.

$$1-\cos(\rho-r) \geq (\rho-r)^2/4 \quad \text{and} \quad 1-\cos(\rho+r) < (\rho+r-4\pi)^2$$

As these are the only factors in the integral that cause problems over this interval, there is a constant c such that

$$\int_{D(2\pi-\delta_1,\frac{2\pi+|\xi|}{2})} \frac{\sin^2|\xi|\,d\eta}{\sin^2|\eta^{-1}o\xi|} < c\,\frac{\sin|\xi|}{\sin(\frac{2\pi+|\xi|}{2})}\int_{2\pi-\delta_1}^{\frac{2\pi+|\xi|}{2}} \log\left\{\frac{4(\rho+|\xi|-4\pi)}{\rho-|\xi|}\right\}^2 d\rho .$$

Again, there is a limit for this integral as below

$$\lim_{\delta_1\to 0}\int_{2\pi-\delta_1}^{2\pi} 2\log\left|\frac{4(\rho+|\xi|-4\pi)}{\rho-|\xi|}\right| d\rho = 0 .$$

Of course in this last limit, the ξ was restricted to have absolute value between $2\pi-\delta_1$ and 2π.

Combining all these estimates, we have shown statement (4.2).

PART II THE EVOLUTION EQUATION

CHAPTER FIVE

PRELIMINARIES AND GENERALIZATIONS

The self-adjoint pair considered in this part is provided through a representation of the Heisenberg group. The Heisenberg group $G(1)$ is the subgroup of unitary operators on $L^2(R)$ that have the form

$$U(p,t)\phi(x) = e^{ip(x)}\phi(x+t) \quad \text{for every} \quad \phi \in L^2(R)$$

where $p(x)$ is a real-valued polynomial of degree at most 1. Under the usual product for operators, $G(1)$ becomes a nilpotent Lie group. The generators of this self-representation are the operators (icI, iQ, icP) where c is a positive constant, I is the identity operator, and the other two operators are as in the introduction. The only non-vanishing bracket of this basis for the Lie algebra is

$$[iQ\, , icP] = -icI\, .$$

The skew product $\underset{c}{*}$ of Chapter Six evolves from the above nilpotent group and the given representation.

As explained in the introduction, the Weyl functional calculus applied to the pair (Q,cP) interprets classical quantities on phase space as Hamiltonians on $L^2(R)$. However, quantum mechanics may also be formulated on phase space. This aspect has been studied by a number of authors; notably, J. E. Moyal in his 1949 paper [11] and also J. Jordan and E. Sudarshan [8]. The evolution equations that appear in these papers and the one that is developed here do not seem to be the same because the methods of formulation vary widely. The equivalence of these formulations is revealed most succinctly in [16].

Before studying the evolution equation, it must be emphasized that most of the results of the ensuing four chapters immediately generalize to the phase space formulation of a system with n degrees of freedom. In this case, the operators $(iI, iQ_1, iP_1, \ldots, iQ_n, iP_n)$ form a basis for the nilpotent Lie algebra where Q_j and iP_j are the obvious self-adjoint operators acting on the j^{th} variable of functions in $L^2(R^n)$. The brackets for this system are of the form

$$[iQ_j , iP_j] = -iI .$$

The skew product and evolution equation can be readily defined by comparison with the Heisenberg group. The generalizations of the theorems are left to the interested reader. Reference [10] provides different aspects of this theory.

In passing, it would be negligent not to mention that the important result (Theorem 6.5) can be generalized in yet another direction. Let $G(m)$ be the group similar to $G(1)$ except the polynomial is of degree at most m. The Lie algebra of $G(m)$ has basis $(icI, iQ_1, \ldots, iQ_m, icP)$ where Q_j is multiplication on $L^2(R)$ by x^j. A skew product $\underset{c}{*}$ exists for the self-adjoint $(m+1)$-tuple $(Q_1, \ldots, Q_m, cP)$ on the space $\mathcal{S}(R^{m+1})$ [2 ; page 430]. The generalization states that there is a unitary operator on $L^2(R^{m+1})$ that is a homeomorphism of $\mathcal{S}(R^{m+1})$ and satisfies for $f,g \in \mathcal{S}(R^{m+1})$

$$U(f \underset{c}{*} U^{-1} g)(x_1, \ldots, x_m, y) = (2\pi c)^{-1/2} \int_R Uf(z, x_2, \ldots, x_m, y)\; g(x_1, \ldots, x_m, z)\, dz .$$

In other words, the skew product is equivalent to the point-wise multiplication of functions except in the first and last variables. The author has performed the explicit calculation of this unitary operator for $m \leq 4$ and firmly believes the equation is true for every $G(m)$.

Of course, with these two generalizations, one could form direct sums of the Lie algebras considered on the last page and so obtain deeper knowledge of the skew product on many nilpotent Lie groups. This program will not be carried out here.

CHAPTER SIX

THE WEYL CORRESPONDENCE AND EVOLUTION EQUATION

The following operators will be used throughout. It should be noted that the first three operators all extend to unitary operators on $L^2(R^2)$.

DEFINITION 6.1: *Let* $S'(R^2)$ *have the strong dual topology induced by the bounded subsets of* $S(R^2)$. *The operators below are homeomorphisms of* $S(R^2)$ *and* $S'(R^2)$. *Suppose that* f *belongs to* $S(R^2)$.

1. *(Partial Fourier Transform)* $F_2 f(x_1,x_2) = \frac{1}{\sqrt{2\pi}} \int_R e^{(izx_2)} f(x_1,z)\,dz$. F_1 *is defined similarly.*

2. *(Twisting Operator)* $S_c f(x_1,x_2) = \sqrt{c}\, f(x_1 - \frac{1}{2}cx_2, x_1 + \frac{1}{2}cx_2)$.
   $S_c^{-1} f(x_1,x_2) = 1/\sqrt{c}\, f((x_1+x_2)/2, (x_2-x_1)/c)$.

3. *(Translation)* $\tau(x_1,x_2) f(y_1,y_2) = f(x_1+y_1, x_2+y_2)$.

4. *(Rotation and Dilation)* c *may be negative for this operator.*
   $V_c f(x_1,x_2) = f(cx_2/2, -cx_1/2)$.

The Weyl operator has a particularly simple form in terms of these definitions.

PROPOSITION 6.2: *If* $h \in S(R^2)$, $T(Q,cP)h$ *is the bounded integral operator on* $L^2(R)$ *with kernel* $1/\sqrt{2\pi c}\; S_c^{-1} F_2 h$. *That is,*

$$(T(Q,cP)h)u(x) = 1/\sqrt{2\pi c} \int_R S_c^{-1} F_2 h(y,x)\, u(y)\, dy \quad \text{for all } u \in L^2(R).$$

PROOF: See [1 ; page 264].

The skew product of Corollary 2.4 for the self-adjoint triple (cI,Q,cP) can be restricted to a unique continuous skew product $\underset{c}{*}$ on $S(R^2)$. Let $h[f]$ or $h(x_1,x_2)[f(x_1,x_2)]$ denote the action of a tempered distribution h on a function $f \in S(R^2)$. Through duality, $\underset{c}{*}$ is extended to a separately continuous map

$$\underset{c}{*} : S'(R^2) \times S(R^2) \longrightarrow S'(R^2) \quad \text{given by}$$

$$h \underset{c}{*} f[g] = h[f \underset{c}{*} g] \quad \text{for} \quad h \in S'(R^2) \quad \text{and} \quad f,g \in S(R^2) .$$

Let us collect these facts.

<u>PROPOSITION 6.3</u>: a) *The skew product for* (cI,Q,cP) *on* $S(R^3)$ *is*

$$F(f * g)(x_0,x_1,x_2) = (2\pi)^{-3/2}\int_{R^3} Ff\left(x_0-y_0+\left(\frac{y_1x_2-x_1y_2}{2}\right),x_1-y_1,x_2-y_2\right) Fg(y)\,dy \quad \textit{or}$$

$$f * g(x_0,x_1,x_2) = (2\pi)^{-2}\int_{R^4} e^{i(tu+sv)} f(x_0,x_1+sx_0/2,x_2+u)g(x_0,x_1-tx_0/2,x_2+v)\,dudvdsdt$$

where $f,g \in S(R^3)$.

b) *If* f_c *is defined by* $f_c(x_1,x_2) = f(c,x_1,x_2)$ *for* $f \in S(R^3)$, *then* $T(cI,Q,cP)f = T(Q,cP)f_c$. *Thus, the skew product for* $f,g \in S(R^2)$ *is*

(6.1) $f \underset{c}{*} g(x_1,x_2)$

$$= (2\pi)^{-2}\int_{R^4} e^{i(tu+sv)} f(x_1+sc/2,x_2+u)g(x_1-tc/2,x_2+v)\,dudvdsdt$$

$$= \frac{1}{2\pi}\int_{R^2} (V_c\,\tau(x_1,x_2)f)(y_1,y_2)\;(F\tau(x_1,x_2)g)(y_1,y_2)\,dy_1dy_2 .$$

If complex conjugation is denoted by $\overline{}$, *then* $\overline{f \underset{c}{*} g} = \overline{g} \underset{c}{*} \overline{f}$.

c) *For* $h \in S'(R^2)$ *and* $f \in S(R^2)$, *define*

(6.2) $h \underset{c}{*} f(x_1,x_2) = \frac{1}{2\pi}\,(V_c\,\tau(x_1,x_2)h)[F\tau(x_1,x_2)f]$.

Then $h \underset{c}{*} f$ *is a* C^∞ *function with polynomially bounded derivatives of all orders that satisfies for* $g \in S(R^2)$

$$(6.3)\quad \begin{cases} h \underset{c}{*} f[g] = h[f \underset{c}{*} g] & (\textit{duality}) \\ h \underset{c}{*} (f \underset{c}{*} g) = (h \underset{c}{*} f) \underset{c}{*} g & (\textit{associativity}). \end{cases}$$

d) *By the last statement of* b) *and duality, the evolution equation of the introduction extends to* $\frac{df}{dt} = H_c f$ *where*

$$(6.4)\quad H_c f(x) = \frac{i}{2\pi}\{V_c \tau(x) h[F\tau(x) f] - \overline{V_c \tau(x) \bar{h}[F\tau(x) \bar{f}]}\} \quad \textit{for } h \in S'(R^2),\ f \in S(R^2)$$

$$= \frac{i}{2\pi}\{V_c \tau(x) h[F\tau(x) f] - V_{-c} \tau(x) h[F\tau(x) f]\} .$$

Notice that the dependence of H_c *on the distribution is suppressed since* h *is regarded as a fixed Hamiltonian.*

<u>PROOF</u>: See [2 ; section 3].

We are almost ready to express (6.4) in the form of an equivalent singular kernel operator on the plane. But first, a similar result must be proved for the test functions $S(R^2)$.

<u>LEMMA 6.4</u>: *If* $h \in S(R^2)$, *let* $H_c' : S(R^2) \to S(R^2)$ *be the operator* $S_c^{-1} F_2 H_c F_2^{-1} S_c$ *defined through the homeomorphisms at the beginning of the chapter. Writing this in integral form, we obtain for* $f \in S(R^2)$

$$H_c' f(x_1,x_2) = i/\sqrt{2\pi c}\int_R \{S_c^{-1} F_2 h(y,x_2)\ f(x_1,y) - S_c^{-1} F_2 h(x_1,y)\ f(y,x_2)\}\ dy .$$

<u>PROOF</u>: We will show only one half of the formula; namely, that

$$iU(h \underset{c}{*} U^{-1} f)(x_1,x_2) = i/\sqrt{2\pi c}\int_R Uh(y,x_2)\ f(x_1,y)\ dy \quad \text{where } U = S_c^{-1} F_2 .$$

The proof of this is strictly a computation. We have

$$ih \underset{c}{*} f(y_1,y_2) = i/2\pi \int_{R^2} V_c\tau(y_1,y_2)h(z_1,z_2)\ F\tau(y_1,y_2)f(z_1,z_2)\ dz_1dz_2 \quad \text{by (6.1)}$$

$$= i/2\pi \int_{R^2} (F_1V_c\tau(y_1,y_2)h)(z_1,z_2)\ (F_2\tau(y_1,y_2)f)(z_1,z_2)\ dz_1dz_2$$

$$= \frac{i}{2\pi}\frac{2}{c}\int_{R^2} e^{i(2z_1y_2/c)}F_2h(y_1+cz_2/2,-2z_1/c)\ e^{-i(y_2z_2)}F_2f(y_1+z_1,z_2)\ dz_1dz_2$$

$$= \frac{i}{\pi}\int_{R^2}\{e^{iy_2(2z_1/c-z_2)}Uh(y_1+z_1+cz_2/2,y_1-z_1+cz_2/2)\cdot Uf(y_1+z_1-cz_2/2,y_1+z_1+cz_2/2)\}\ dz_1dz_2$$

Let $u_1 = z_1+cz_2/2$ and $u_2 = z_1-cz_2/2$

$$= \frac{i}{\pi c}\int_{R^2} e^{i(2y_2u_2/c)}Uh(y_1+u_1,y_1-u_2)\ Uf(y_1+u_2,y_1+u_1)\ du_1du_2\ .$$

Therefore, $F_2(ih \underset{c}{*} U^{-1}f)(x_1,x_2)$ is equal to

$$i/\pi c\sqrt{2\pi}\int_R e^{i(x_2y_2)}\int_{R^2} e^{i(2y_2u_2/c)}\ Uh(u_1,x_1-u_2)\ f(x_1+u_2,u_1)\ du_1du_2dy_2$$

$$= i/\sqrt{2\pi}\int_R Uh(u_1,x_1+cx_2/2)\ f(x_1-cx_2/2,u_1)\ du_1\ .$$

Thus, $iU(h \underset{c}{*} U^{-1}f)(x_1,x_2) = i/\sqrt{2\pi c}\int_R Uh(s,x_2)\ f(x_1,s)\ ds$.

<u>THEOREM 6.5</u>: *If* $h \in S'(R^2)$, *let* $H_c' : S(R^2) \to S'(R^2)$ *be the operator* $S_c^{-1}F_2H_cF_2^{-1}S_c$. *Then, for* $f,g \in S(R^2)$ *we have the equivalent formulas*

$$H_c'f(x_1,x_2) = i/\sqrt{2\pi c}\ \{S_c^{-1}F_2h(\cdot,x_2)[f(x_1,\cdot)] - S_c^{-1}F_2h(x_1,\cdot)[f(\cdot,x_2)]\} \quad \textit{or}$$

$$H_c'f[g] = i/\sqrt{2\pi c}\ S_c^{-1}F_2h(y_1,y_2)[\int_R f(z,y_1)g(z,y_2)dz - \int_R f(y_2,z)g(y_1,z)dz\ ]\ .$$

PROOF: Define a jointly continuous map $\circ : S(R^2) \times S(R^2) \longrightarrow S(R^2)$ by

$$(6.5)\quad f \circ g(x_1,x_2) = \int_R f(z,x_2)\, g(x_1,z)\, dz .$$

The proof follows immediately from the duality relation (6.3) and the commutativity of the diagram below by Lemma 6.4.

$$\begin{array}{ccccc} S(R^2) & \times & S(R^2) & \xrightarrow{\ \ *_c\ \ } & S(R^2) \\ \Big\downarrow S_c^{-1}F_2 & & \Big\downarrow S_c^{-1}F_2 & & \Big\downarrow S_c^{-1}F_2 \\ S(R^2) & \times & S(R^2) & \xrightarrow{1/\sqrt{2\pi c}\ \circ} & S(R^2) \end{array} .$$

The similarity between the quantum mechanical operator of Proposition 6.2 and the operator in Theorem 6.5 suggests a natural extension of the Weyl quantization procedure to tempered distributions. With this extension, Theorem 6.5 is interpreted as a separation of variables for the evolution equation. Indeed, Weyl operators corresponding to the original tempered distribution h act on each variable separately (see Theorem 6.7).

MAIN DEFINITION 6.6: (The Weyl Correspondence) *Suppose* h *is a tempered distribution on the plane. Define a map* $A_c(h) : S(R) \to S'(R)$ *by*

$$(6.6)\quad \begin{cases} (A_c(h))(\phi)[\psi] = 1/\sqrt{2\pi c}\ S_c^{-1}F_2 h[\phi\times\psi] & \textit{for } \phi,\psi \in S(R) \quad \textit{or equivalently} \\ (A_c(h)\phi)(x) = 1/\sqrt{2\pi c}\ S_c^{-1}F_2 h(\cdot,x)[\phi(\cdot)] . & \end{cases}$$

A_c *is a continuous linear operator called the* *Weyl operator* *corresponding to the Hamiltonian* h *. If the distribution is clear from the context, the dependence of* A_c *on* h *is often suppressed.*

Suppose that $B_j : X_j \longrightarrow Y_j$, j=1,2 are two linear operators with domains $\mathcal{D}(B_j)$ and that X_j, Y_j are complex function spaces. Let the <u>tensor product</u> $B_1 \otimes B_2$ be the operator with domain all finite linear combinations of functions of the form $f_1 \times f_2$, $f_j \in \mathcal{D}(B_j)$ and defined as follows:

$$B_1 \otimes B_2 \left(\sum_{k=1}^{n} \alpha_k \, f_{1_k} \times f_{2_k} \right) = \alpha_k \sum_{k=1}^{n} (B_1 f_{1_k}) \times (B_2 f_{2_k}) \qquad \alpha_k \in \mathbb{C} .$$

Also, define the <u>complex conjugate</u> operator $\overline{B}_1$ by $\overline{B}_1 f = \overline{(B_1 \overline{f})}$ for $\overline{f} \in \mathcal{D}(B_1)$.

Adopting the above notation, we have a reformulation of Theorem 6.5.

<u>THEOREM 6.7</u>: H_c' *is the unique extension of* $i\{I \otimes A_c(h) - \overline{A_c(\overline{h})} \otimes I\}$ *to a continuous linear operator from* $\mathcal{S}(R^2)$ *to* $\mathcal{S}'(R^2)$.

<u>PROOF</u>: It is easy to check the following important equality of distributions.

$$(6.7) \qquad S_c^{-1} F_2 \overline{h}(x_1, x_2) = \overline{S_c^{-1} F_2 h(x_2, x_1)} .$$

Since the linear span of $\mathcal{S}(R) \times \mathcal{S}(R)$ is dense in $\mathcal{S}(R^2)$, we have only to verify the two operators in the theorem are equal for functions of the form $\phi \times \psi$ for $\phi, \psi \in \mathcal{S}(R)$.

$$\begin{aligned} H_c'(\phi\times\psi)(x_1,x_2) &= i/\sqrt{2\pi c}\ \{S_c^{-1}F_2 h(\cdot,x_2)[\phi(x_1)\psi(\cdot)] - S_c^{-1}F_2 h(x_1,\cdot)[\phi(\cdot)\psi(x_2)]\} \\ &= i\phi(x_1)(A_c(h)\psi)(x_2) - i/\sqrt{2\pi c}\ \overline{S_c^{-1}F_2 \overline{h}(\cdot,x_1)}[\phi(\cdot)\psi(x_2)] \text{ by } (6.7) \\ &= i\{\phi\times A_c(h)\psi - \overline{A_c(\overline{h})\overline{\phi}}\times\psi\}(x_1,x_2) \text{ by } (6.6) \\ &= i(\{\ I \otimes A_c(h) - \overline{A_c(\overline{h})} \otimes I\ \}\ \phi\times\psi)(x_1,x_2) . \end{aligned}$$

CHAPTER SEVEN

BOUNDED OPERATORS

One method of solving the evolution equation of (6.4) is to regard H_c and A_c as operators on $L^2(R^2)$ and $L^2(R)$ respectively, with domains $\mathcal{D}(A_c) = \{\phi \in \mathcal{S}(R) : A_c\phi \in L^2(R)\}$ and $\mathcal{D}(H_c) = \{f \in \mathcal{S}(R^2) : H_c f \in L^2(R^2)\}$. In particular, if H_c is a bounded operator with $\mathcal{D}(H_c) = \mathcal{S}(R^2)$, then the usual power series expansion of the exponential provides a group of operators $e^{i(tH_c)}$ that immediately solves the evolution equation.

If we adopt this point of view, Theorem 6.5 establishes a unitary equivalence between H_c and H_c' when $\mathcal{D}(H_c') = \{f \in \mathcal{S}(R^2) : H_c' f \in L^2(R^2)\}$. Although H_c' remains an extension of $i\{I \otimes A_c(h) - \overline{A_c(\overline{h})} \otimes I\}$ which is now defined on linear combinations of functions in $\mathcal{D}(A_c(h)) \times \mathcal{D}(\overline{A_c(\overline{h})})$, we have no guarantee that H_c' is contained in the closure of this operator. Indeed, a priori, H_c could be densely defined while A_c is not (or vice versa).

The relation between H_c and A_c is studied in this chapter and again in Chapter Nine. Here, the case of bounded operators is examined. As a start, we have the following.

<u>THEOREM 7.1</u>: H_c *is bounded with* $\mathcal{D}(H_c) = \mathcal{S}(R^2)$ *if and only if* A_c *is bounded with* $\mathcal{D}(A_c) = \mathcal{S}(R)$.

<u>PROOF</u>: It clearly suffices to prove the statement for H_c' in place of H_c.

Assume $A_c(h)$ is bounded by b and $\mathcal{D}(A_c(h)) = \mathcal{S}(R)$. Then, for all $\phi,\psi \in \mathcal{S}(R)$, $|A_c(h)\phi[\psi]| \leq b\,\|\phi\|_2\,\|\psi\|_2$ where $\|\cdot\|_2$ is the L^2-norm. By (6.7), $|A_c(\overline{h})\phi[\psi]| = |\overline{A_c(h)[\overline{\psi\times\phi}]}| \leq b\,\|\psi\|_2\,\|\phi\|_2$. Thus $A_c(\overline{h})$ is bounded by b and $\mathcal{D}(A_c(\overline{h}))=\mathcal{S}(R)$.

For $f\in\mathcal{S}(R^2)$, define two families of functions in $\mathcal{S}(R)$ by $f_{(x_1)}(z)=f(x_1,z)$ and $f^{(x_2)}(z)=f(z,x_2)$. Theorems 6.5 and 6.7 yield

$$H_c'f(x_1,x_2) = i(A_c(h)f_{(x_1)})(x_2) - i\overline{(A_c(\overline{h})f^{(x_2)})(x_1)}\,.$$

Thus, we have the following estimate for the norm:

$$\|H_c'f\|_2 \le \left(\int_{R^2}|(A_c(h)f_{(x_1)})(x_2)|^2dx_2dx_1\right)^{1/2}+\left(\int_{R^2}|(A_c(\overline{h})\overline{f^{(x_2)}})(x_1)|^2dx_1dx_2\right)^{1/2}$$

$$\le \left(\int_R b^2\,\|f_{(x_1)}\|_2^2\,dx_1\right)^{1/2}+\left(\int_R b^2\,\|f^{(x_2)}\|_2^2\,dx_2\right)^{1/2} \quad \text{by hypothesis}$$

$$= 2b\left(\int_{R^2}|f(x_1,x_2)|^2dx_1dx_2\right)^{1/2}$$

$$= 2b\;\|f\|_2\,.$$

Hence, H_c' is bounded by $2b$ and $\mathcal{D}(H_c')=\mathcal{S}(R^2)$.

To prove the converse, assume that H_c' is bounded and $\mathcal{D}(H_c')=\mathcal{S}(R^2)$. Let $\psi\in\mathcal{S}(R)$ be arbitrary and choose $\phi_0,\rho_0\in\mathcal{S}(R)$ such that $\int\phi_0\rho_0\neq 0$. Consider the two inequalities

i) $|H_c'(\phi_0\times\psi)[\rho_0\times\theta]| \le b\,\|\rho_0\|_2\,\|\theta\|_2$ for all $\theta\in\mathcal{S}(R)$ (b is a bound of H_c')

ii) $|H_c'(\phi_0\times\psi)[\rho_0\times\theta]| = |A_c(h)\psi[\theta]\cdot\int\phi_0\rho_0 - \overline{A_c(\overline{h})\overline{\phi_0}}[\rho_0]\cdot\int\psi\theta|$ by Theorem 6.7.

Thus, $|A_c(h)\psi[\theta]| \le \left|\frac{1}{\int\phi_0\rho_0}\right|\{b\,\|\rho_0\|_2\|\theta\|_2+|\overline{A_c(\overline{h})\overline{\phi_0}}[\rho_0]\cdot\int\psi\theta|\}$

$$\le b'\,\|\theta\|_2 \quad \text{for some constant } b' \text{ by Cauchy-Schwarz.}$$

As ψ was arbitrary, the domain of A_c is all of $\mathcal{S}(R)$. By the same argument, $A_c(\overline{h})$ has the same domain. We have only to show that A_c is bounded to complete the proof. •

If A_c were not bounded, there would be a sequence $\phi_n \varepsilon \mathcal{S}(R)$ such that $\|A_c(h)\phi_n\|_2 \longrightarrow \infty$ but $\|\phi_n\| = 1$. For this sequence,

$$\|H_c'(\phi_1 \times \phi_n)\|_2 = \|\phi_1 \times A_c(h)\phi_n - \overline{A_c(\overline{h})\phi_1} \times \phi_n\|_2$$

$$\geq \|\phi_1\| \, \|A_c(h)\phi_n\| - \|A_c(\overline{h})\overline{\phi}_1\| \, \|\phi_n\|$$

$$\longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty .$$

This provides a contradiction and so A_c is bounded.

Remark on the proof: If the operator norm is denoted by $\|\cdot\|_{op}$, the sufficiency part of the proof insures that $\|H_c\|_{op} \leq 2\,\|A_c\|_{op}$. However, there is no such estimate in the other direction. In fact, when $A_c = I$, the evolution operator is the zero operator.

•

Since h can be thought of as the Hamiltonian of the system, we are particularly interested in real tempered distributions (that is, h[f] is real for all real-valued test functions). In this case, the last result may be strengthened to obtain Proposition 7.3 and Theorem 7.4.

DEFINITION 7.2: *Let* K *be a linear operator on a Hilbert space* H.

1. K *is formally skew-adjoint (skew-hermitian) if* $(Ku,v) = -(u,Kv)$ *for all* $u,v \varepsilon \mathcal{D}(K)$.
2. K *is skew-symmetric if* $\mathcal{D}(K)$ *is dense and* K *is formally skew-adjoint.*
3. K *is essentially skew-adjoint if the closure of* K *is skew-adjoint.*
4. K *is skew-adjoint if* K *is skew-symmetric and* $R(K \pm I) = H$. *(* R *means the range of an operator)*

For these definitions without the adjective "skew", see [19 ; chapter XI].

PROPOSITION 7.3: *Let* h *be a real tempered distribution on the plane. Then* H_c *and* iA_c *are formally skew-adjoint. If either one is densely defined, it is skew-symmetric.*

PROOF: Let us show H_c' is formally skew-adjoint. The unitary equivalence then implies that H_c is formally skew-adjoint.

$$(H_c'f,g) = \int_{R^2} H_c'f(x_1,x_2)\,\overline{g}(x_1,x_2)\,dx_1dx_2 \quad \text{for all} \quad f,g \in \mathcal{D}(H_c')$$

$$= H_c'f[\overline{g}]$$

$$= i/\sqrt{2\pi c}\quad S_c^{-1}F_2h(x_1,x_2)\left[\int_R f(z,x_1)\overline{g}(z,x_2)dz - \int_R f(x_2,z)\overline{g}(x_1,z)dz\right]$$

Use (6.7) and the fact h is real

$$= \overline{\left(\frac{-i}{\sqrt{2\pi c}}\, S_c^{-1}F_2h(x_2,x_1)\left[\int_R g(z,x_2)\overline{f}(z,x_1)dz - \int_R g(x_1,z)\overline{f}(x_2,z)dz\right]\right)}$$

$$= -\overline{H_c'g[\overline{f}]}$$

$$= -(f,H_c'g)\ .$$

By a similar argument and formula (6.7), iA_c is formally skew-adjoint.

The last statement in the theorem is true by definition.

THEOREM 7.4: H_c *is a bounded essentially skew-adjoint operator with domain* $\mathcal{D}(H_c) = \mathcal{S}(R^2)$ *if and only if* iA_c *is a bounded essentially skew-adjoint operator with* $\mathcal{D}(A_c) = \mathcal{S}(R)$.

PROOF: One uses the fact that a bounded skew-symmetric operator with dense domain is essentially skew-adjoint (its closure is defined on the whole Hilbert space). The theorem is then a direct consequence of Theorem 7.1 and Proposition 7.3.

As yet, we only know that the Weyl operators of functions in the Schwartz class on phase space are bounded operators on $L^2(R)$. The remainder of this chapter is intended to remove this gap.

PROPOSITION 7.5: *For any bounded (respectively, bounded self-adjoint) operator* A *defined on all of* $L^2(R)$, *there is a unique tempered distribution (respectively, real tempered distribution) whose Weyl operator agrees with* A *on* $S(R)$.

PROOF: If $A : L^2(R) \to L^2(R)$ is bounded, then $A|_{S(R)} : S(R) \to S'(R)$ is a continuous operator. By the Schwartz Kernel Theorem [17 ; page 531], there is a unique $h' \in S'(R^2)$ that satisfies

$$A\phi[\psi] = h'[\phi\times\psi] \quad \text{for every} \quad \phi,\psi \in S(R) .$$

Introduce the tempered distribution h as

$$h = \sqrt{2\pi c}\ \ (S_c^{-1}F_2)^{-1}h' = \sqrt{2\pi c}\ \ F_2^{-1}S_c h' .$$

Trivially, by Definition 6.6, $A_c(h)\phi = A\phi$, for every $\phi \in S(R)$.

If, in addition, A is self-adjoint, then the inner product equality shows that $h'(x_1,x_2) = \overline{h'(x_2,x_1)}$. The converse of (6.7) implies that h is real.

The preceeding proposition asserts that there are many tempered distributions that have a corresponding bounded Weyl operator, but it does not give any indication of the distributions in this set. Theorem 7.6 rectifies the situation.

<u>THEOREM 7.6</u>: *The Weyl operators of the following distributions are continuous linear operators on* $L^2(R)$ *with domain* $S(R)$ *and bound specified below.*

a) *If* h *belongs to* $L^2(R^2)$, A_c *is actually a Hilbert-Schmidt operator on* $L^2(R)$ *with kernel* $1/\sqrt{2\pi c}\ S_c^{-1}F_2h$. *Therefore,* $\|A_c\|_{op} \le 1/\sqrt{2\pi c}\ \|h\|_2$.

b) *If* h *is a finite Radon measure* [17 ; chapter 21] *with total variation* $\|h\|_1$, *then* $\|A_c\|_{op} \le (1/c)(\sqrt{2/\pi})\ \|h\|_1$. *In particular, this is true for* L^1*-functions.*

c) *If* h *is such a distribution, so is* FV_ch *with* $\|A_c(FV_ch)\|_{op} = 2/c\ \ \|A_c\|_{op}$.

<u>PROOF</u>: a) An obvious extension of Proposition 6.2.

b) For $f \in S(R^2)$, it is easy to check $S_c^{-1}F_2h[f] = F_2h[S_cf] = h[F_2S_cf]$. Since h is a finite Radon measure, we have

$$\begin{aligned}|A_c\phi[\psi]| &= |1/\sqrt{2\pi c}\ S_c^{-1}F_2h[\phi\times\psi]| \quad \text{for all } \phi,\psi \in S(R)\\ &= 1/\sqrt{2\pi c}\ |h[F_2S_c\phi\times\psi]|\\ &\le 1/\sqrt{2\pi c}\ \|h\|_1 \sup\{|F_2S_c\phi\times\psi(x_1,x_2)| : (x_1,x_2)\in R^2\}.\end{aligned}$$

But
$$\begin{aligned}\sup_{x_1,x_2\in R}|F_2S_c\phi\times\psi(x_1,x_2)| &= \sup_{x_1,x_2\in R}\left|\sqrt{c}\int_R e^{i(x_2z)}\phi(x_1-\tfrac{cz}{2})\psi(x_1+\tfrac{cz}{2})dz\right|\\ &\le \sup_{x_1\in R}\sqrt{c}\int_R (2/c)|\phi(x_1-z)||\psi(x_1+z)|dz\\ &\le 2/\sqrt{c}\ \|\phi\|_2\|\psi\|_2 \quad \text{by Cauchy-Schwarz.}\end{aligned}$$

By these two inequalities, $|A_c\phi[\psi]| \le (1/c)(\sqrt{2/\pi})\ \|h\|_1\|\phi\|_2\|\psi\|_2$. Hence, $\mathcal{D}(A_c) = S(R)$ and $\|A_c\|_{op} \le (1/c)(\sqrt{2/\pi})\ \|h\|_1$.

c) Let us first demonstrate (7.1) given below. Let δ_0 be the Dirac distribution in $S'(R^2)$ (that is, $\delta_0[f] = f(0)$). By formula (6.2), one calculates that $2\pi\delta_0 \underset{c}{*} f(x) = FV_cf(x)$ for each $f \in S(R^2)$. Through the associativity in (6.3), $FV_c(f \underset{c}{*} g) = (FV_c f) \underset{c}{*} g$ for all $f, g \in \mathcal{S}(R^2)$. The extension through duality yields

$$(7.1) \qquad FV_c(h \underset{c}{*} f) = FV_c h \underset{c}{*} f \quad \text{where} \quad f \in \mathcal{S}(R^2),\ h \in \mathcal{S}'(R^2).$$

Part c) will now be shown. Let $U = S_c^{-1} F_2$. By Theorems 6.5 and 6.7,

$$\begin{aligned} \phi \times A_c(FV_c h)\psi &= U\{(FV_c h) \underset{c}{*} U^{-1}(\phi \times \psi)\} \quad \text{when} \quad \phi, \psi \in \mathcal{S}(R) \\ &= UFV_c\{h \underset{c}{*} U^{-1}(\phi \times \psi)\} \quad \text{by (7.1)} \\ &= UFV_c U^{-1}\{\phi \times A_c(h)\psi\}. \end{aligned}$$

Since U and F are unitary operators while $\|V_c f\|_2 = (2/c)\,\|f\|_2$,

$$\|\phi \times A_c(FV_c h)\psi\|_2 = (2/c)\,\|\phi \times A_c(h)\psi\|_2.$$

Therefore, $\mathcal{D}(A_c(FV_c h)) = \mathcal{S}(R)$ and $\|A_c(FV_c h)\|_{op} = (2/c)\,\|A_c(h)\|_{op}$.

Sections a) and b) of the above theorem were previously known, though in a different setting. They were communicated to the author in the form of an unpublished paper by R. Anderson.

CHAPTER EIGHT

MULTIPLIERS

The bounded Weyl operators of Chapter Seven form an algebra under operator products. Through the Weyl correspondence, a multiplicative structure is established for the set of distributions with bounded Weyl operators. Certain subspaces S of distributions will be invariant under this multiplication and will, perhaps, form noncommutative Banach algebras with respect to some norm.

Following M. Rieffel [15], if S is a Banach space and $\underset{c}{*}$ is a continuous skew product on S, then those bounded operators M on S that satisfy $M(f \underset{c}{*} g) = Mf \underset{c}{*} g$ for all elements in S are called <u>multipliers</u> on S. Theorem 8.1 demonstrates that the multipliers on the Banach spaces we have been considering must be tempered distributions.

<u>THEOREM 8.1</u>: a) *All continuous linear operators* $M : \mathcal{S}(R^2) \to \mathcal{S}'(R^2)$ *that satisfy* $M(f \circ g) = (Mf) \circ g$ *for all functions in* $\mathcal{S}(R^2)$ *are of the form* $Mf = h \circ f$ *for some* $h \in \mathcal{S}'(R^2)$ *and conversely.*
b) *All continuous linear operators* $M : \mathcal{S}(R^2) \to \mathcal{S}'(R^2)$ *that satisfy* $M(f \underset{c}{*} g) = (Mf) \underset{c}{*} g$ *for all functions in* $\mathcal{S}(R^2)$ *are of the form* $Mf = h \underset{c}{*} f$ *for some* $h \in \mathcal{S}'(R^2)$ *and conversely.*

<u>PROOF</u>: b) This follows immediately from a) by letting $M' = UMU^{-1}$ where U is the usual operator $S_c^{-1}F_2$ and using the commutative diagram in Theorem 6.5. The converse statement is just (6.3).

a) If $Mf = h \circ f$, the equality $(h \circ f) \circ g = h \circ (f \circ g)$ is quickly verified by formula (6.5).

Conversely, assume that $M(f \circ g) = (Mf) \circ g$. The Schwartz Kernel Theorem states that there is an $L \varepsilon S'(R^4)$ such that

$$\begin{aligned} Mf[k] &= L[k \times f] \quad \text{for } f,k \varepsilon S(R^2) \\ &= L(x_1,x_2,z_1,z_2)[k(x_1,x_2)f(z_1,z_2)] . \end{aligned}$$

Since M permutes with the multiplicative structure,

$$\begin{aligned} &L(x_1,x_2,z_1,z_2)[k(x_1,x_2)\int_R f(y,z_2)g(z_1,y)dy\ ] \\ &\quad = M(f \circ g)[k] \quad \text{for } f,g,k \varepsilon S(R^2) \text{ by } (6.5) \\ &\quad = (Mf) \circ g[k] \\ &\quad = Mf(x_1,x_2)[\ \int_R g(y,x_1)k(y,x_2)dy\ ] \\ &\quad = L(x_1,x_2,z_1,z_2)[\ \int_R g(y,x_1)k(y,x_2)dy\ f(z_1,z_2)] . \end{aligned}$$

Into the above expression, substitute the functions $f = \alpha \times \beta$, $g = \rho \times \theta$, and $k = \phi \times \psi$ with the assumption that $\alpha,\beta,\rho,\theta,\phi,\psi$ all belong to $S(R)$. Then

$$\begin{aligned} &L(x_1,x_2,z_1,z_2)[\phi(x_1)\psi(x_2)\rho(z_1)\beta(z_2)\ \int\alpha\theta\ ] \\ &\quad = L(x_1,x_2,z_1,z_2)[\theta(x_1)\psi(x_2)\alpha(z_1)\beta(z_2)\ \int\rho\phi\ ] \quad \text{or by extension,} \end{aligned}$$

$$\begin{aligned} &L(x_1,x_2,z_1,z_2)[\phi(x_1)\rho(z_1)q(x_2,z_2)\ \int\alpha\theta\ ] \\ &\quad = L(x_1,x_2.z_1,z_2)[\theta(x_1)\alpha(z_1)q(x_2,z_2)\ \int\rho\phi\ ] \quad \text{for } q \varepsilon S(R^2) . \end{aligned}$$

In order to obtain the distribution h, fix an element $q_0 \varepsilon S(R^2)$ and define $D(q_0) : S(R^2) \rightarrow \mathbb{C}$ by

$$D(q_0)[p] = L(x_1,x_2,z_1,z_2)[p(x_1,z_1)q_0(x_2,z_2)] \quad \text{for } p \varepsilon S(R^2) .$$

Then $D(q_0)$ is a tempered distribution that satisfies, for all test functions on R, the equality $(\int\alpha\theta)D(q_0)[\phi \times \rho] = (\int\rho\phi)D(q_0)[\theta \times \alpha]$.

Suppose that $D(q_0)$ is not the zero distribution, then choose test functions ϕ_0, ρ_0 such that $D(q_0)[\phi_0 \times \rho_0] \neq 0$. We see immediately that $(\int \rho_0 \phi_0 / D(q_0)[\phi_0 \times \rho_0]) \, D(q_0)$ is a positive tempered distribution and hence a Radon measure. This measure m satisfies

$$\int_R \alpha(y)\theta(y)dy = \int_{R^2} \theta(x_1)\alpha(x_2) \; dm(x_1,x_2) \; .$$

Therefore, m is the measure that assigns each point on the line $x_1 = x_2$ unit mass (in standard notation, $m = \delta(x_1 - x_2)$).

Set $h(q_0) = D(q_0)[\phi_0 \times \rho_0] / \int \rho_0 \phi_0$ for q_0 as in the last paragraph. If $D(q_0) = 0$, set $h(q_0) = 0$. It is obvious that h does not depend on the particular choice of test functions on R and that, in fact, h is a tempered distribution itself. This is a direct consequence of the equation

$$L(x_1,x_2,z_1,z_2)[p(x_1,z_1)q(x_2,z_2)] = h(q) \int_R p(y,y)dy \; .$$

Perhaps we should change notation from $h(q)$ to $h[q]$.

Now choose a function p such that $\int_R p(y,y)dy = 1$. From the last equation, we have $L(x_1,x_2,z_1,z_2) = h(x_2,z_2)\delta(x_1 - z_1)$. Thus

$$\begin{aligned} Mf[k] &= L(x_1,x_2,z_1,z_2)[k(x_1,x_2)f(z_1,z_2)] \\ &= h(x_2,z_2)[\int_R k(y,x_2)f(y,z_2)dy] \\ &= \tilde{h} \circ f[k] \quad \text{where } \tilde{h}(a,b) = h(b,a) \; . \end{aligned}$$

This completes the characterization of the operators M.

If f and g belong to $L^2(R^2)$, then $f \underset{c}{*} g$ is defined as the unique tempered distribution of Proposition 7.5 whose Weyl operator corresponds to the bounded operator $A_c(f)A_c(g)$. Due to the relation between $L^2(R^2)$ and Hilbert-Schmidt operators on $L^2(R)$ (Theorem 7.6 a)), $f \underset{c}{*} g$ is in $L^2(R^2)$ and the skew product is continous with respect to the L^2-norm.

Any multiplier M on this space is clearly in $S'(R^2)$ because M restricted to $S(R^2)$ satisfies the hypothesis of Theorem 8.1 b).
If the operator $f \longrightarrow H_c f$ is replaced by $f \longrightarrow h \underset{c}{*} f$ in Theorem 7.1, the multipliers are seen to correspond to bounded Weyl operators. Recording this, we have:

<u>THEOREM 8.2</u>: *The multipliers on* $(L^2(R^2), \underset{c}{*})$ *are of the form* $Mf = h \underset{c}{*} f$ *for some tempered distribution* h *whose Weyl operator is bounded on* $L^2(R)$ *with domain* $S(R)$ *and conversely.*

The skew product on $F(L^1(R^2))$ may be defined as at the top of the page because Theorem 7.6 confirms that these distributions have bounded Weyl operators. However, it is not entirely obvious that the skew product is again in $F(L^1(R^2))$ and is continous. Lemma 8.3 details a more constructive method of obtaining the skew product. With a little more effort than needed in Theorem 8.2, the multipliers on this space are characterized in Theorem 8.4.

<u>LEMMA 8.3</u>: *There is a continous skew product on the Banach space* $F(L^1(R^2))$ *that has norm induced from* $L^1(R^2)$.

<u>PROOF</u>: Take f, g in $F(L^1(R^2))$. By comparison to formula (6.1), let

$$f \underset{c}{*} g(x_1,x_2) = \frac{1}{2\pi}\int_{R^2} f(x_1+cy_2/2, x_2-cy_1/2)e^{-i(x_1,x_2)\cdot(y_1,y_2)}Fg(y_1,y_2)\,dy_1dy_2 .$$

As f is a uniformly continous, bounded function and $Fg \in L^1(R^2)$, the expression for $f \underset{c}{*} g$ certainly produces an everywhere defined, continous function. In addition, this definition extends the skew product for $S(R^2)$.

Also, formula (7.1) may be rewritten for our situation in the form; $FV_c(f \underset{c}{*} g) = (FV_c f) \underset{c}{*} g$ for all f,g in $F(L^1(R^2))$. Thus,

$$\|f \underset{c}{*} g\|_{(FL^1)} = \|F(f \underset{c}{*} g)\|_1 = \|FV_c(f \underset{c}{*} g)\|_1 = \|(FV_c f) \underset{c}{*} g\|_1$$

$$= \frac{1}{2\pi}\int_{R^2}\left|\int_{R^2} FV_c f(x_1+cy_2/2, x_2-cy_1/2)e^{-i(x_1,x_2)\cdot(y_1,y_2)}Fg(y_1,y_2)dy_1dy_2\right|dx_1dx_2$$

$$\leq \frac{1}{2\pi} \|FV_c f\|_1 \|Fg\|_1 \quad \text{by Fubini's Theorem}$$

$$= \frac{1}{2\pi} \|f\|_{(FL^1)} \|g\|_{(FL^1)} .$$

Therefore, $f \underset{c}{*} g \in F(L^1(R^2))$ and the skew product is continuous.

<u>THEOREM 8.4</u>: *The multipliers on* $(F(L^1(R^2)), \underset{c}{*})$ *are of the form* $Mf = h \underset{c}{*} f$ *for some tempered distribution* h *whose Fourier transform is a finite Radon measure and conversely.*

<u>PROOF</u>: Suppose that $Mf = (Fm) \underset{c}{*} f$ for all functions in $F(L^1(R^2))$ where m is a finite Radon measure with variation $\|m\|$. An easy modification of the argument in Lemma 8.3 shows that $\|F(Fm \underset{c}{*} f)\|_1 \leq \|m\| \; \|f\|_1$. In addition, $(Fm \underset{c}{*} f) \underset{c}{*} g = Fm \underset{c}{*} (f \underset{c}{*} g)$ for all $f,g \in F(L^1(R^2))$ since all these distributions correspond to bounded Weyl operators by Theorem 7.6.

On the other hand, suppose that M is a multiplier on $F(L^1(R^2))$. Theorem 8.1 implies there is an $h \in S'(R^2)$ such that, for $f,g \in S(R^2)$,

i) $Mf = h \underset{c}{*} f$ and

ii) $\|F(h \underset{c}{*} g)\|_1 \leq b \|Fg\|_1$ for some constant b.

Comparing the skew product given in Lemma 8.3 to the usual convolution on the plane, let us choose a sequence $a_n \in C_0^\infty(R^2)$ such that $a_n \to 2\pi\delta_0$ in $S'(R^2)$ and $\|a_n\|_1 = 2\pi$ (see [19 ; page 157]). The Fourier transform sequence has the limit $Fa_n \to 1$ in $S'(R^2)$ and by an argument similar to the usual convolution, Fa_n is an approximate identity for $S(R^2)$. (that is, $Fa_n \underset{c}{*} g \to g$ in $S(R^2)$ as $n \to \infty$).

Using this approximate identity, we have

$$FV_c h[f \underset{c}{*} g] = \lim_{n\to\infty} FV_c h[Fa_n \underset{c}{*} (f \underset{c}{*} g)] \quad \text{for } f,g \in S(R^2)$$

$$= \lim_{n\to\infty} (FV_c h \underset{c}{*} Fa_n)[f \underset{c}{*} g] \quad \text{by the duality in (6.3).}$$

But
$$\|FV_c h \underset{c}{*} Fa_n\|_1 = \|FV_c (h \underset{c}{*} Fa_n)\|_1 \quad \text{by (7.1)}$$

$$= \|F(h \underset{c}{*} Fa_n)\|_1$$

$$\le b\, \|F(Fa_n)\|_1 \quad \text{by property ii) of } h$$

$$= 2\pi b \quad \text{for all } n .$$

Thus, $|FV_c h[f \underset{c}{*} g]| \le 2\pi b\, \|f \underset{c}{*} g\|_\infty$ where $\|k\|_\infty = \sup\{|k(x)| : x \in R^2\}$.

This last statement insures that there is a finite Radon measure m that satisfies $FV_c h[f \underset{c}{*} g] = m[f \underset{c}{*} g]$ for all $f,g \in S(R^2)$. By associativity, $(FV_c h) \underset{c}{*} f[g] = m \underset{c}{*} f[g]$ and so $(FV_c h) \underset{c}{*} f = m \underset{c}{*} f$. It is easy to see by (6.2) that $FV_c h = m$. Hence, $Mf = (Fm') \underset{c}{*} f$ for some finite Radon measure m' .

The reader familiar with harmonic analysis will notice the connection of this theorem and proof with Wendel's Theorem [6 ; page 376].

CHAPTER NINE

REAL TEMPERED DISTRIBUTIONS

In this chapter, we will examine further the equivalence of the evolution and Weyl operators. Only real tempered distributions h will be considered. By Proposition 7.3, H_c and iA_c are formally skew-adjoint on their respective domains. As in Chapter Seven, these operators are regarded as acting on Hilbert space. However, H_c and iA_c are no longer required to be bounded. The cases when the two operators have skew-adjoint extensions will be of particular interest as a means of solving the evolution and Weyl equations.

First, let us demonstrate that most Schrodinger operators associated to a particle moving in one dimension are extensions of Weyl operators. The result is the analogue of Proposition 7.5.

PROPOSITION 9.1: *For any symmetric operator* A *whose domain includes* $S(R)$, *there is a unique real tempered distribution such that* A *agrees with the corresponding Weyl operator on all of* $S(R)$.

PROOF: For $\phi \in S(R)$ with $\|\phi\|_2 \leq 1$, define a map $T_\phi : S(R) \to ¢$ by $T_\phi(\psi) = (A\phi,\psi)$. If ψ is fixed in $S(R)$, then $\{T_\phi(\psi)\}$ is bounded in $¢$ since

$$\sup\{|T_\phi(\psi)|\} = \sup\{|(A\phi,\psi)|\} = \sup\{|(\phi,A\psi)|\} \leq \|A\psi\|_2 ,$$

where the sup is taken over the above ϕ with norm at most one.

By the uniform boundedness theorem [19 ; page 68], $T_\phi(\psi)$ goes to zero uniformly in ϕ as $\psi \to 0$ in $S(R)$. In particular, if B is any bounded set in $S(R)$, then $\sup\{|T_\phi(\psi)| : \psi \varepsilon B, \phi \varepsilon S(R), \|\phi\|_2 \leq 1\}$ is finite.

Thus, for any $\alpha>0$, there exists a $\delta>0$ such that $\|\phi\|_2 < \delta \Rightarrow$

$$p_B(\phi) \equiv \sup\{|A\phi[\bar{\psi}]| : \bar{\psi} \varepsilon B\} < \alpha .$$

Since the family of semi-norms p_B define the strong dual topology on $S(R)$, the map $\phi \longrightarrow A\phi$ is continuous from $S(R)$ to $S'(R)$.

We now continue as in the proof of Proposition 7.5 to obtain the associated real tempered distribution h.

Proposition 7.3 can be improved to show the domains of the evolution and Weyl operators are closely related. As we are now dealing with unbounded operators, the proof of the following theorem is much more delicate than that in Theorem 7.1.

The result is not true for general tempered distributions. For example, when $S_c^{-1}F_2h = u_1 \times u_2$ where u_1 belongs to $L^2(R)$ but u_2 does not, then $\mathcal{D}(A_c(h))$ is all of $S(R)$ while $\mathcal{D}(A_c(\bar{h}))$ is not dense in $L^2(R)$ and $\mathcal{D}(H_c)$ is not dense in $L^2(R^2)$. Of course, h is not a real tempered distribution.

<u>THEOREM 9.2</u>: *Let* h *be a real tempered distribution.* H_c *is densely defined if and only if* A_c *is densely defined. In other words,* H_c *is skew-symmetric if and only if* iA_c *is skew-symmetric.*

<u>PROOF</u>: As always, it suffices to prove the statement with H_c' in place of H_c.

Assume first that A_c is densely defined. Since h is real, Theorem 6.7 states that $H_c'(\overline{\phi}\times\psi) = i(\overline{\phi}\times A_c\psi - \overline{A_c\phi}\times\psi)$ for all $\phi,\psi \in \mathcal{D}(A_c)$. Therefore, the domain of H_c' includes all finite linear combinations of functions in $\overline{\mathcal{D}(A_c)} \times \mathcal{D}(A_c)$. As $\mathcal{D}(A_c)$ is dense in $L^2(R)$, $\mathcal{D}(H_c')$ will be dense in $L^2(R^2)$.

Now assume that H_c' is densely defined. Let $\phi_0 \in \mathcal{S}(R)$ and $f_0 \in \mathcal{D}(H_c')$ be arbitrary. It will be shown that the domain of A_c includes

$$\int_R f_0(y,x)\ \phi_0(y)\ dy .$$

For every $\psi \in \mathcal{S}(R)$, we have

$$iA_c\left(\int_R f_0(y,x)\ \phi_0(y)\ dy \right) [\psi]$$

$$= i/\sqrt{2\pi c}\ \ S_c^{-1}F_2h(x_1,x_2)[\int_R f_0(y,x_1)\ \phi_0(y)\ dy\ \psi(x_2)]$$

$$= H_c'f_0[\phi_0\times\psi] + i/\sqrt{2\pi c}\ \ S_c^{-1}F_2h(x_1,x_2)[\int_R f_0(x_2,y)\ \psi(y)\ dy\ \phi_0(x_1)] .$$

An easy calculation gives the distributional equality:

$$S_c^{-1}F_2h(x_1,x_2)[\int_R f_0(x_2,y)\psi(y)\phi_0(x_1)dy] = \int_R S_c^{-1}F_2h(x_1,x_2)[f_0(x_2,y)\phi_0(x_1)]\psi(y)dy .$$

The expression $S_c^{-1}F_2h(x_1,x_2)[f_0(x_2,y)\phi_0(x_1)]$ is clearly in $\mathcal{S}(R)$ as a function of y. Let the L^2-norm of this function be bounded by b. Then

$$\left| iA_c\left(\int_R f_0(y,x)\phi_0(y)dy \right)[\psi] \right| \le \|H_c'f_0\|_2\|\phi_0\|_2\|\psi\|_2 + b/\sqrt{2\pi c}\ \ \|\psi\|_2$$

$$= b'\ \|\psi\|_2 \quad \text{for some constant } b' .$$

Thus, the domain of A_c contains all such functions.

Suppose that the functions of this form are not dense in $L^2(R)$. There exists some test function $\phi_0 \in \mathcal{S}(R)$ and a positive number α such that, for all $f \in \mathcal{D}(H_c')$,

$$\text{i)}\quad \|\phi_0\|_2 = 1 \quad \text{and} \quad \text{ii)}\quad \left\| \int_R f(y,x)\phi_0(y)dy - \phi_0(x) \right\|^2 > \alpha .$$

$$\text{Hence,}\quad \alpha < \int_R \left| \int_R f(y,x)\phi_0(y)dy - \phi_0(x) \right|^2 dx$$

$$= \int_R \left| \int_R \{f(y,x) - \overline{\phi_0(y)}\phi_0(x)\}\phi_0(y)dy \right|^2 dx \quad \text{by i)}$$

$$\leq \int_R \|\phi_0\|^2 \int_R |f(y,x) - \overline{\phi_0(y)}\phi_0(x)|^2 dydx \quad \text{by Cauchy-Schwarz}$$

$$= \|f - \overline{\phi_0}\times\phi_0\|_2^2$$

This cannot happen because $\mathcal{D}(H_c')$ is dense in $L^2(R^2)$. Therefore, A_c is densely defined.

If h is real, the proof of Theorem 7.1 can be adapted to show that H_c is bounded on a dense set if and only if iA_c is bounded on a dense set. Since a skew-symmetric operator bounded on a dense set is necessarily bounded on its entire domain, an immediate consequence of Theorem 9.2 is the following simpler version of Theorem 7.4.

COROLLARY 9.3: H_c *is a bounded essentially skew-adjoint operator if and only if* iA_c *is also.*

The next two results demonstrate that, for certain distributions, the evolution operator is indeed equivalent to the Weyl operator. The condition on the distribution h that produces the Hamiltonian of the system is seen to be physically significant.

PROPOSITION 9.4: *Let* $h = h_1 + Fh_2$ *be a real tempered distribution where* h_1 *and* h_2 *are (not necessarily real) distributions with compact support. Then* a) $\mathcal{D}(H_c) = \mathcal{S}(R^2)$ *and* $\mathcal{D}(A_c) = \mathcal{S}(R)$

and b) H_c' *is in the closure of* $i\{I \otimes A_c - \overline{A}_c \otimes I\}$ *as operators.*

PROOF: The proposition will follow from formula (7.1) and the proof of the statement for h having compact support but perhaps not real.

a) By the local structure of compactly supported distributions [17 ; page 256], there is a positive integer k and an r such that

$$h[f] \leq b \sup\{|D^m f(y)| : 0 \leq |m| \leq k, \ |y| \leq r\} \quad \text{for all } f \text{ in } \mathcal{S}(R^2)$$

where b is some constant depending on h and the notation $m = (m_1, m_2)$, $|m| = m_1 + m_2$ is the usual multiindex notation for derivatives. Therefore,

$$|h \underset{c}{*} f(x_1,x_2)| = |\tfrac{1}{2\pi} h(y_1,y_2)[e^{-i(x_1,x_2)\cdot(-2y_2/c,2y_1/c)} Ff\left(\frac{2(x_2-y_2)}{c}, \frac{2(y_1-x_1)}{c}\right)]|$$

$$\leq b' \sup\left|D^m_{(y)}\left\{e^{-i(x_1,x_2)\cdot(-2y_2/c,2y_1/c)} Ff(2(x_2-y_2)/c, 2(y_1-x_1)/c)\right\}\right|$$

where $D^m_{(y)}$ means the derivative with respect to the y variables and the sup is taken over the same set as in the local structure of h. The first line in the expansion of $h \underset{c}{*} f$ is just (6.2) with the various operators removed.

The derivatives with respect to y will produce polynomials in x multiplied by derivatives of Ff evaluated at a translated point. Because $Ff \in \mathcal{S}(R^2)$, $|h \underset{c}{*} f(x)|$ will decrease faster than any polynomial as $|x| \to \infty$. In fact, as f approaches 0 in $\mathcal{S}(R^2)$, $h \underset{c}{*} f$ will approach 0 in $L^2(R^2)$. The same argument applied to $\overline{h} \underset{c}{*} \overline{f}$ demonstrates that $\mathcal{D}(H_c) = \mathcal{S}(R^2)$. A slight variation of Theorem 9.2 implies $\mathcal{D}(A_c(h)) = \mathcal{D}(A_c(\overline{h})) = \mathcal{S}(R)$.

b) Let f be an arbitrary element in the domain of H'_c. Since the domains of $A_c(h)$ and $A_c(\bar{h})$ are all of $\mathcal{S}(R)$, there is a sequence f_n in $\mathcal{D}(I \otimes A_c(h) - \overline{A_c(\bar{h})} \otimes I)$ that converges to f in the topology of $\mathcal{S}(R^2)$. This being the case, we know that $H'_c f_n$ converges to $H'_c f$ in $L^2(R^2)$ by the proof of a). As $H'_c f_n = i\{I \otimes A_c(h) - \overline{A_c(\bar{h})} \otimes I\} f_n$, section b) is shown.

Remark: It should be pointed out at this time that there is a product rule for the derivative of the skew product. The rule is (compare, [19 ; page 156])

$$\frac{\partial}{\partial x_j}(h \underset{c}{*} f) = \frac{\partial h}{\partial x_j} \underset{c}{*} f + h \underset{c}{*} \frac{\partial f}{\partial x_j} \quad \text{for} \quad f \in \mathcal{S}(R^2),\ h \in \mathcal{S}'(R^2),\ j = 1,2 .$$

With this derivation, one can show that $H_c : \mathcal{S}(R^2) \to \mathcal{S}(R^2)$ is a continuous map when h has compact support.

THEOREM 9.5: *Let $h = h_1 + Fh_2 + h_3$ be a real tempered distribution where h_1 and h_2 have compact support and h_3 corresponds to a bounded Weyl operator with domain $\mathcal{S}(R)$. Then*

a) H_c is essentially skew-adjoint if and only if iA_c is essentially skew-adjoint.

b) The closure of iA_c generates a strongly continuous unitary group $V(t)$ on $L^2(R)$ that solves the Weyl equation $\frac{d\phi}{dt} = iA_c\phi$ if and only if the closure of H_c generates a strongly continuous unitary group $W(t)$ on $L^2(R^2)$ that solves the evolution equation $\frac{df}{dt} = H_c f$. Moreover, $W(t)$ is the closure of $F_2^{-1} S_c(\overline{V(t)} \otimes V(t)) S_c^{-1} F_2$.

PROOF: Proposition 9.4 remains valid with the addition of the distribution that has an associated bounded Weyl operator with domain $\mathcal{S}(R)$.

a) Suppose iA_c is essentially skew-adjoint. Obviously, $\overline{iA_c}$ is also essentially skew-adjoint. By using the resolution of the identity for the operators iA_c and $\overline{iA_c}$, Ju. Berezanskii [3 ; VI §4 of the English translation] proves that this separation of variables produces an essentially skew-adjoint operator $I \otimes iA_c + \overline{iA_c} \otimes I$. This proof is not reproduced here as many new concepts would have to be introduced. Since H_c' is an extension of this tensor product operator, it is essentially skew-adjoint.

For further results along this line that apply to tempered distributions which are not real, the reader is encouraged to see [7] or [14].

For the converse, assume H_c is essentially skew-adjoint while iA_c is not. By the theory of deficiency indices, one of the subspaces

$$\mathcal{R}(iA_c \pm I)^{\perp} \equiv \{u \in L^2(R) \ : \ ((iA_c \pm I)\phi, u) = 0 \quad \text{for all} \quad \phi \in \mathcal{D}(iA_c)\}$$

is non-trivial. Without loss of generality, assume that u_0 is a non-zero element in $\mathcal{R}(iA_c + I)^{\perp}$. Let $\phi, \psi \in \mathcal{D}(iA_c)$ be arbitrary. Then

$$\begin{aligned} ((H_c' + 2I)\overline{\phi}\times\psi, \overline{u_0}\times u_0) &= (\{I \times (iA_c+I) + (\overline{iA_c+I}) \times I\}\overline{\phi}\times\psi, \overline{u_0}\times u_0) \\ &= \overline{(\phi,u_0)}((iA_c+I)\psi,u_0) + (u_0,(iA_c+I)\phi)(\psi,u_0) \\ &= 0 \ . \end{aligned}$$

Since H_c' is in the closure of $I \otimes iA_c + \overline{iA_c} \otimes I$, $\overline{u_0}\times u_0$ is in the set $\mathcal{R}(H_c' + 2I)^{\perp}$. This contradicts the essential skew-adjointness of H_c.

b) If h is a distribution as in the statement of the theorem, then, by Stone's Theorem, the closures of the two operators generate stongly continuous unitary groups for the same set of distributions. Only the relation between $W(t)$ and $V(t)$ is in doubt.

Let $W'(t)$ be the closure of the operator $\overline{V(t)} \otimes V(t)$ defined on $L^2(R^2)$. $W'(t)$ is a strongly continuous unitary group with domain all of $L^2(R^2)$. The generator of $W'(t)$ is an extension of the tensor product $i\{I \otimes A_c - \overline{A_c} \otimes I\}$ because

$$\lim_{t\to 0} \frac{W'(t)(\overline{\phi}\times\psi) - \overline{\phi}\times\psi}{t} = \lim_{t\to 0} \frac{\overline{V(t)\phi}\times V(t)\psi - \overline{\phi}\times\psi}{t} \quad \text{for all} \quad \phi,\psi \in \mathcal{D}(A_c)$$

$$= \lim_{t\to 0} \frac{\overline{(V(t)\phi - \phi)}\times V(t)\psi + \overline{\phi}\times(V(t)\psi - \psi)}{t}$$

$$= \lim_{t\to 0} \frac{\overline{V(t)\phi - \phi}}{t} \times \lim_{t\to 0} V(t)\psi + \overline{\phi} \times \lim_{t\to 0} \frac{V(t)\psi - \psi}{t}$$

$$= \overline{iA_c\phi}\times\psi + \overline{\phi}\times iA_c\psi .$$

Thus, the closure of H_c' generates $W'(t)$. The unitary equivalence established in Theorem 6.5 implies the closure of H_c generates $F_2^{-1}S_cW'(t)S_c^{-1}F_2$.

So far, we have seen only those results that support the equivalence of H_c and iA_c. These last few pages are an attempt to justify the study of the evolution equation on its own. The geometry of the plane is used to point out how the equivalence may break down. No specific counterexample is provided and, indeed, our effort to produce one has been unsuccessful to this time.

The unpublished paper of R. Anderson referred to on page 48 must be acknowledged at this moment. In it, the evolution operator of a real, compactly supported distribution was shown to have dense domain (compare to Proposition 9.4 a)) and, in the case of odd distributions, to have a skew-adjoint extension (Theorem 9.7 a)). The statement of Lemma 9.6 on the next page is reproduced almost verbatim from the paper.

LEMMA 9.6: *If a skew-symmetric operator* K *is unitarily equivalent to* $-K$, *then* K *has a skew-adjoint extension.*

PROOF: It suffices to prove that $\dim\{\mathcal{R}(K+I)^{\perp}\} = \dim\{\mathcal{R}(K-I)^{\perp}\}$ by the theory of deficiency indices [19 ; page 349]. Let $UKU^{-1} = -K$ be the assumed unitary equivalence. Then, with f_0 an arbitrary element of $L^2(R^2)$,

$$\begin{aligned}((K-I)g, U^{-1}f_0) &= (U(K-I)g, f_0) \quad \text{for all } g \in \mathcal{D}(K) \\ &= ((UKU^{-1} - I)Ug, f_0) \\ &= ((-K-I)Ug, f_0) \\ &= -((K+I)Ug, f_0)\,.\end{aligned}$$

Therefore, $U^{-1}f_0$ belongs to $\mathcal{R}(K-I)^{\perp}$ if and only if f_0 is in $\mathcal{R}(K+I)^{\perp}$. As U is unitary, the result is proved.

THEOREM 9.7: *Assume that* H_c *is densely defined and that any of the following conditions are true for the real tempered distribution* h. *Then* H_c *has a skew-adjoint extension.*

a) h *is odd (that is,* $h(x_1,x_2)[f(x_1,x_2)] = -h(x_1,x_2)[f(-x_1,-x_2)]$, $f \in S(R^2)$).

b) $M_\theta h = -h$ *where* M_θ *is a rotation in the plane through an angle* θ.

c) h *is reflexive in any line through the origin. For example,* h *is reflexive in the* x_1*-axis if* $h(x_1,x_2) = h(x_1,-x_2)$.

PROOF: a) This follows from section b) with the angle of rotation π radians.

b) By Lemma 9.6, it suffices to find a unitary operator that effects the equivalence of H_c to $-H_c$. To this end, let $Uf(y) = f(M_\theta y)$. Then, for all $f \in \mathcal{D}(H_cU)$, we have by (6.4)

$$U^{-1}H_cUf(x) = H_cUf(M_{-\theta}x)$$

$$= \frac{i}{2\pi}\{V_c\tau(M_{-\theta}x)h[F\tau(M_{-\theta}x)Uf] - V_{-c}\tau(M_{-\theta}x)h[F\tau(M_{-\theta}x)Uf]\} .$$

To write this as an evolution operator, use the relations below that permute the operators in the last expression and are readily checked.

i) $\tau(M_{-\theta}x)Uf = U\tau(x)f$ ii) $UFf = FUf$

iii) $h[Uf] = U^{-1}h[f]$ iv) $U^{-1}V_{\pm c}h = V_{\pm c}U^{-1}h$

v) $U^{-1}\tau(M_{-\theta}x)h = \tau(x)U^{-1}h$.

Thus, $U^{-1}H_cUf(x) = \frac{i}{2\pi}\{V_c\tau(x)U^{-1}h[F\tau(x)f] - V_{-c}\tau(x)U^{-1}h[F\tau(x)f]\}$.

Since $U^{-1}h = -h$, we have $U^{-1}H_cUf = -H_cf$.

c) The same method as above is used but with $Uf(y) = f(\tilde{y})$ where $\tilde{y}$ is the reflection of y in the given line through the origin. Again the operators are permuted after calculating the various relations. These relations are basically the same as i) to v) except that $UV_ch(x) = V_cUh(-x) = V_{-c}Uh(x)$ and $UV_{-c}h = V_cUh$. Combining these, we obtain

$$U^{-1}H_cUf(x) = \frac{i}{2\pi}\{V_{-c}\tau(x)Uh[F\tau(x)f] - V_c\tau(x)Uh[F\tau(x)f]\}$$

$$= -H_cf \quad \text{since} \quad Uh = h .$$

Remark 1: The origin appears to play a central role in the theorem. This is rather misleading because the symmetries about the origin are considered due to computational convenience as opposed to any intrinsic property associated to this point. Any other point would do equally as well since the evolution operators corresponding to h and $\tau(x_0)h$ are unitarily equivalent under $Uf(x) = f(x+x_0)$.

<u>Remark 2</u>: The purpose of the theorem is to suggest cases in which the two operators H_c and iA_c are not equivalent. Of course, they will not be equivalent when one has a skew-adjoint extension but the other does not. From section c) of the proof, we are struck by the fact that both terms $h \underset{c}{*} f$ and $\overline{h} \underset{c}{*} \overline{f}$ in the evolution operator are needed to insure a skew-adjoint extension. However, the Weyl operator is defined through only the first term and so there is no immediate reason why iA_c should have a skew-adjoint extension.

Therefore, it is plausible that the phase space formulation of quantum mechanics is not equivalent to the usual theory on configuration space. At the very least, the symmetries of phase space contribute to the study of the evolution equation and, ultimately, of the Weyl equation.

BIBLIOGRAPHY

1. Anderson, R.F.V., *The Weyl Functional Calculus*, Journal of Functional Analysis, 4 (1969), 240-269.

2. Anderson, R.F.V., *The Multiplicative Weyl Functional Calculus*, Journal of Functional Analysis, 9 (1972), 423-440.

3. Berezanskii, Ju.M., "Expansions in Eigenfunctions of Selfadjoint Operators", Naukova Dumka, Kiev 1965; English translation, Translations of Mathematical Monographs, volume 17, American Mathematical Society, Providence, R.I., 1968.

4. Dixmier, J., *L'Application Exponentielle dans les Groupes de Lie Résolubles*, Bulletin Societé Mathématique de France, 85 (1957), 113-121.

5. Helgason, S., "Differential Geometry and Symmetric Spaces", Academic Press, New York 1962.

6. Hewitt, E., and Ross, K.A., "Abstract Harmonic Analysis II", Springer-Verlag, Berlin 1970.

7. Ichinose, T., *Operators on Tensor Products of Banach Spaces*, Transactions of the American Mathematical Society, 170 (1972), 197-219.

8. Jordan, J.F., and Sudarshan, E.C.G., *Lie Group Dynamical Formalism*, Reviews of Modern Physics, 33 (1961), 515-524.

9. Kobayashi, S., and Nomizu, K., "Foundations of Differential Geometry, Volumes I and II", Wiley (Interscience), New York 1963.

10. Loupias, G., and Miracle-Sole, S., *C*-Algèbres des Systèmes Canoniques, II*, Annals Institute Henri Poincaré, Section A 6 (1967), 39-58.

11. Moyal, J.E., *Quantum Mechanics as a Statistical Theory*, Proceedings of the Cambridge Philosophy Society, 45 (1949), 99-124.

12. Murnaghan, F.D., "The Theory of Group Representations", Dover Publications, New York 1963.

13. Pukansky, L., *On the Unitary Representations of Exponential Groups*, Journal of Functional Analysis, 2 (1968), 73-113.

14. Reed, M., and Simon, B., *Tensor Products of Closed Operators on Banach Spaces*, Journal of Functional Analysis, 13 (1973), 107-124.

15. Rieffel, M., *Multipliers and Tensor Products of* L^p*-spaces of Locally Compact Groups*, Studia Mathematica, 23 (1969), 71-82.

16. Rubin, H., Proceedings of the International Symposium on the Axiomatic Method at Berkeley, North-Holland Publishing Company, Amsterdam 1959.

17. Treves, F., "Topological Vector Spaces, Distributions and Kernels", Academic Press, New York 1967.

18. Weyl, H., "The Theory of Groups and Quantum Mechanics", Metheuen and Company, London 1931.

19. Yosida, K., "Functional Analysis, Third Edition", Springer-Verlag, New York 1971.