APPROXIMATION TECHNIQUES FOR THE STEFAN PROBLEM

by

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ABSTRACT

The macroscopic description of matter undergoing a phase change (the Stefan Problem) can be formulated as a set of coupled, non-linear partial differential equations. For the case of one space dimension, the thesis develops three approximation methods to solve these equations.

(a) Asymptotic Expansions

With the Green's functions to suit the given boundary conditions, the system can be transformed into a set of integral equations. For the case where the initial phase grows without limit as $\tau \to \infty$, the large τ expansions for the integrals are calculated and the large τ behaviour of the interphase boundary found.

(b) Perturbation Expansions

When the latent heat of fusion of a material is large relative to the heat content of that material, a uniformly valid perturbation expansion in a parameter related to the ratio of these heats is possible. The first few terms of the expansions for the temperature distribution and the position of the interphase boundary are calculated and found to be in good agreement with the few known exact solutions and numerically calculated solutions.

(c) Numerical Techniques

Rather than use a traditional finite difference scheme, only time is discretized and an analytic expression for an approximate temperature is found for each time step. This method gives good results for the temperatures over the time intervals required by the physics of the problem. Finally, the method is applied to describe the freezing of a shallow lake.

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INTRODUCTION

There are a number of problems in heat conduction in which matter, in addition to supporting thermal diffusion, also undergoes a phase change. The problem is usually treated macroscopically; no attempt is made to detail the underlying molecular processes which govern the phase change. Further, complications due to mass motion (convection) are also ignored.

Within these restrictions the system can then be characterized by a temperature field and the boundary between the phases. The evolution of the temperature field is governed by the heat equation while the position of the interphase boundary is determined by an energy balance across this interface. The position of this free boundary is not known in advance but must be determined as part of the problem. For these reasons the study has been known as the free boundary problem for the heat equation.

The problem first arose over a century ago in the work of K. Stefan [1] who studied the thickening of polar ice. In fact, the free boundary problem was originally known as, and is still often called, the Stefan problem. The methods he used were extended to describe other phase change problems such as the freezing of a lake, the recrystallization of metals, or the evaporation and condensation of water.

The free boundary problem has been studied analytically in a few important special cases. It is instructive to start with the general description of the problem and see what restrictions lead to these special cases.

In full generality the system can be modelled as follows

(a) Let R be a three dimensional region with boundary ∂R. R^c is considered to be a heat bath; it is a source of heat to R but is unaltered by R.

(b) R contains a medium with temperature dependent conductivity $k(\phi)$ and enthalpy/unit volume $H(\phi)$. It is this temperature dependence which characterizes the presence of the two phases. In the absence of such transitions both phases are considered to have temperature independent properties.

(c) The evolution of the temperature ϕ within R is determined by the partial differential equation

$$\nabla \cdot (k \nabla \phi) = \frac{\partial H}{\partial \tau} = \frac{\partial H}{\partial \phi} \frac{\partial \phi}{\partial \tau} \tag{o.1}$$

(d) If ω is the volume fraction of the initial phase, then

$$\left(\frac{\partial H}{\partial \phi}\right)_p = \left(\frac{\partial H}{\partial \phi}\right)_{p,\omega} + \left(\frac{\partial H}{\partial \omega}\right)_{p,\phi} \left(\frac{\partial \omega}{\partial \phi}\right)_p$$

$$= \rho c_p + \rho L \left(\frac{\partial \omega}{\partial \phi}\right)_p \tag{o.2}$$

where c_p is the specific heat at constant pressure

L is the latent heat; assumed temperature independent.

(e) There is a temperature interval in which the medium undergoes most of its phase change. Let the midpoint of the interval be Φ_M, the length ε. Assume that c_p and k can be written (Figure 1)

$$
\begin{aligned}
c_p &= c_I + (c_{II} - c_I)\,\Lambda_\varepsilon(\Phi_M, \phi) \\
k &= k_I + (k_{II} - k_I)\,\Lambda_\varepsilon(\Phi_M, \phi)
\end{aligned}
\tag{0.3}
$$

where Λ_ε is a phenomenological function such that for $\varepsilon \to 0$

$$\Lambda_\varepsilon(\Phi_M, \phi) \to \Theta(\phi - \Phi_M)$$

where Θ is the Heaviside function.

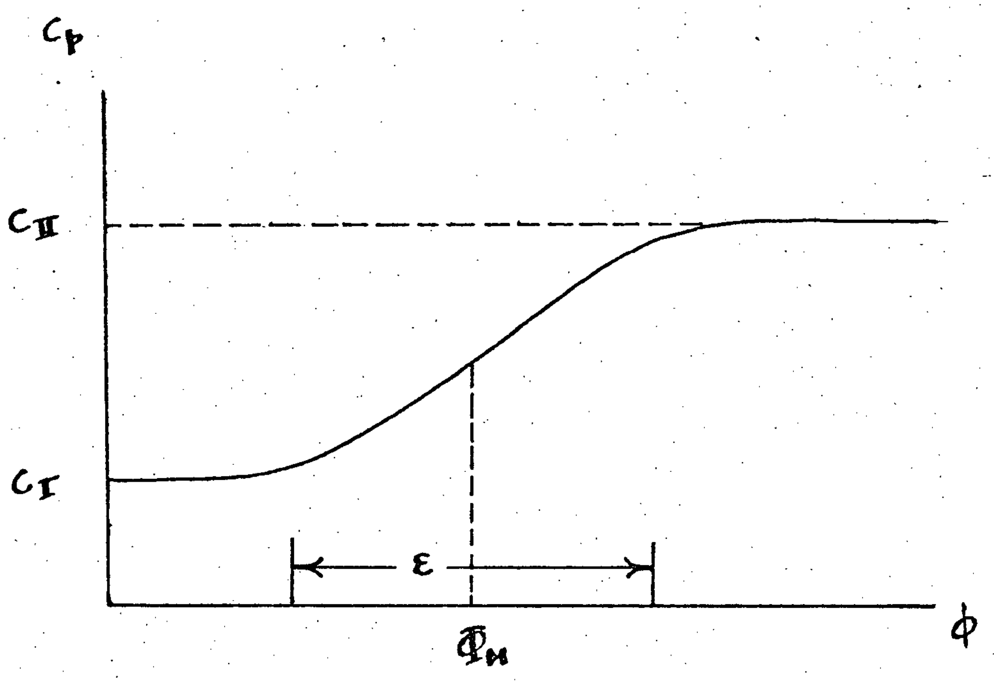

Figure 1: Temperature dependence of c_p .

Here $\{c_I, k_I\}$ are the assumed temperature independent properties of phase I in the absence of phase II; analogously for $\{c_{II}, k_{II}\}$.

Since most of the phase change occurs in the interval $(\Phi_M - \frac{\varepsilon}{2}, \Phi_M + \frac{\varepsilon}{2})$ it follows that

$$\int_{\Phi_M - \frac{\varepsilon}{2}}^{\Phi_M + \frac{\varepsilon}{2}} \frac{\partial \omega}{\partial \phi} \, d\phi - 1 \quad . \tag{0.4}$$

The volume fraction can then be assumed to have the form

$$\omega = \Lambda_\varepsilon(\Phi_M, \phi) \quad . \tag{0.5}$$

The actual form is not important; it is required only that $\frac{\partial \omega}{\partial \phi} \to \delta(\phi - \Phi_M)$ for $\varepsilon \to 0$ so that (0.4) holds for $\varepsilon \to 0$

It then follows from (0.2, 0.3, 0.5) that H can be written in the form

$$H(\phi) = \rho L \Lambda_\varepsilon(\phi, \Phi_M) + H_\varepsilon(\phi)$$

where $\lim_{\varepsilon \to 0} H_\varepsilon(\phi)$ is continuous.

(f) Finally, the boundary conditions that will be considered are

(i) Dirichlet conditions; temperature specified on ∂R

$$\phi = \phi_B \, , \; \underline{x} \in \partial R$$

(ii) Neumann conditions; heat flux specified on ∂R

$$k \nabla \phi \cdot n = J_B \, , \; \underline{x} \in \partial R \, .$$

Consider now the $\varepsilon \rightarrow 0$ limit of the preceding system. At a fixed time t, let the surface $S_o(t)$ be defined by (Figure 2)

$$\phi(\underline{x},t) = \Phi_M \; .$$

In the cases of interest, S_o divides R into two regions: $R_>$ for $\phi > \Phi_M$ and $R_<$ for $\phi < \Phi_M$. Enclose $\underline{x}_o \in S_o$ by a cylindrical pill box V_ε of linear dimensions ε. The position of the pill box will vary in time; for $\underline{v}$ the velocity of $\underline{x} \in \partial V_\varepsilon$ we have

$$\nabla\phi(\underline{x}_o,t)\cdot\underline{v} + \frac{\partial\phi}{\partial t}(\underline{x}_o,t) \sim 0 \; .$$

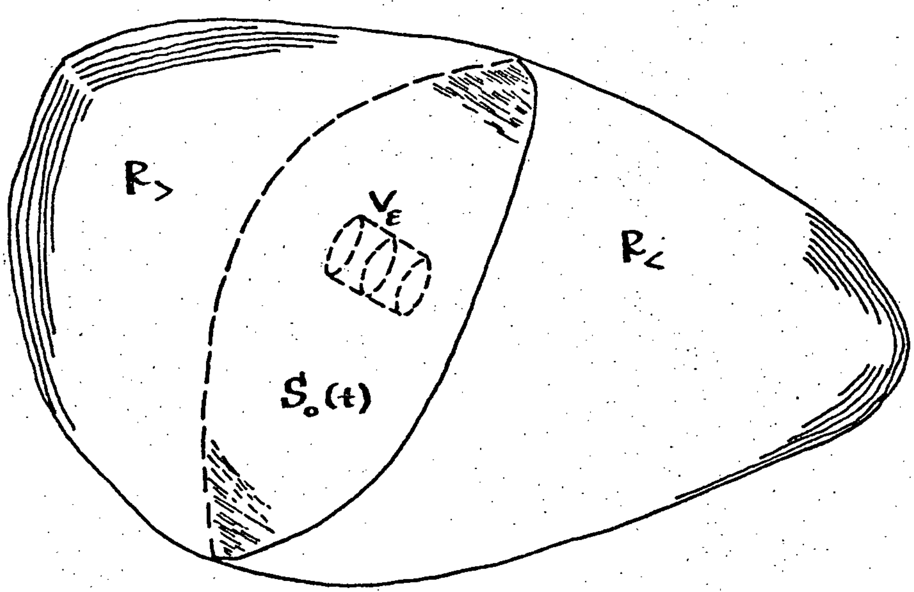

Figure 2. The interphase boundary

Now for the enthalpy H ,

$$\frac{d}{dt}\int_{V_\varepsilon} H\,dV = \int_{V_\varepsilon} \frac{\partial H}{\partial t}\,dV - \int_{\partial V_\varepsilon} H v_n\,dS \qquad (0.6)$$

where $v_n = \underline{v}\cdot\underline{n}$ for $\underline{n}$ the outward normal. Integrate (0.1) over V_ε

$$\int_{V_\varepsilon} \nabla\cdot(k\nabla\phi)\,dV = \int_{V_\varepsilon} \frac{\partial H}{\partial t}\,dV .$$

Apply the divergence theorem to the left-hand side and (0.6) to the right hand side to get

$$\int_{\partial V_\varepsilon} \left(k\frac{\partial\phi}{\partial n} - H v_n\right) dS = \frac{d}{dt}\int_{V_\varepsilon} H\,dV .$$

H is bounded, hence $\int_{V_\varepsilon} H\,dV = O(\varepsilon^3)$ uniformly in t . The left hand side is $O(\varepsilon^2)$.

Now let $\varepsilon \to 0$. The sides of the pill box do not contribute as there, $v_n \to 0$. Only the discontinuities of k and H contribute at $\underline{x}_0$ to give

$$k_{II}\frac{\partial\phi_{II}}{\partial n} - k_I\frac{\partial\phi_I}{\partial n} = \pm \rho L v_n \qquad (0.7)$$

at $\phi(\underline{x}_o, t) = \Phi_H$

where the plus or minus sign is taken if $\underline{v}$ points into $R_>$ or $R_<$ respectively. The sign cannot be determined a priori; that is an additional input to the problem.

The next restriction is imposed by symmetry. Assume translation invariance of the system in two space dimensions; one space co-ordinate survives and the interphase boundary s(t) satisfies $\phi(s(t), t) = 0$. Further, focus on one of the phases. This can be done when the second phase is a heat bath (its evolution would be known and its contribution to the first phase at the interface assumed given) or the second phase is removed as soon as it is formed.

The equations then reduce to

$$k \frac{\partial^2 \phi}{\partial x^2} = \rho c_p \frac{\partial \phi}{\partial t}$$

where at $x = s(t)$

$$\phi(s(t), t) = \Phi_H$$

$$k \phi_x(s(t), t) + H(t) = \pm \rho L \frac{ds}{dt}$$

Here, H(t) can be considered a known heat input from the second phase.

Under these restrictions there are two types of problems

(a) The semi-infinite bar

At $t = 0$ the first phase occupies the semi-infinite interval $0 \leq x < \infty$. The partial differential equation is satisfied on $S(t) < x < \infty$. An initial condition $\phi(x,0) = \phi_0(x)$ on $0 \leq x < \infty$ is needed to render the problem well posed. We consider the cases where $\phi_0(x) \longrightarrow \phi_0$ for $x \longrightarrow \infty$; it follows that $\phi(x,t) \longrightarrow \phi_0$ for $x \longrightarrow \infty$.

Non-dimensionalize as follows:

$$\bar{x} = \frac{x}{\ell} \qquad \sigma = \frac{s}{\ell} \qquad \tau = \frac{\kappa}{\ell^2} t \qquad \bar{\Phi} = \frac{\phi - \phi_0}{\phi_0} \qquad \bar{H} = \frac{\ell}{k\phi_0} H$$

where ℓ is a reference length and $\kappa = \frac{k}{\rho c_p}$. Then, letting $\frac{\Phi_H - \phi_0}{\phi_0} \longrightarrow \Phi_H$ and $\frac{\phi_0(x) - \phi_0}{\phi_0} \longrightarrow \phi_0(x)$ and suppressing bars, we get

$$\phi_{xx} = \phi_\tau \qquad \sigma < x < \infty \tag{0.8a}$$

$$\phi(\sigma(\tau),\tau) = \Phi_H \qquad \phi(\infty,\tau) = 0 \tag{0.8b}$$

$$\varepsilon\left(\phi_x(\sigma(\tau),\tau) + H(\tau)\right) = \pm \frac{d\sigma}{d\tau} \tag{0.8c}$$

$$\sigma(0) = 0 \tag{0.8d}$$

$$\phi(x,0) = \phi_0(x) \tag{0.8e}$$

where $\varepsilon = \frac{c_p \Phi_H}{L}$ is the Stefan number.

For definiteness, assume $\Phi_M > 0$ and take the plus sign in (0.8c). We restrict the study to the cases where $\frac{d\sigma}{d\tau} > 0$; these cases describe the melting of a slab.

For certain boundary conditions, closed solutions can be found after an appropriate change of variables.

With the transformation $y = \frac{x}{\sigma}$, (0.8a) becomes

$$\phi_{yy} = 2r\,\phi_\tau - y\frac{dr}{d\tau}\phi_y \qquad 1 < y < \infty$$

where $r = \frac{1}{2}\sigma^2$, while the boundary conditions transform to

$$\varepsilon\left(\phi_y(1,\tau) + H\sigma\right) = \frac{dr}{d\tau}$$

$$\phi(1,\tau) = \Phi_M \qquad \phi(\infty,\tau) = 0 \; .$$

For $H = \frac{H_o}{\sqrt{\tau}}$ a solution is

$$\frac{dr}{d\tau} = \lambda = \text{const;} \quad \text{hence} \quad \sigma = \sqrt{2\lambda\tau}$$

and

$$\phi = \Phi_M \frac{\int_y^\infty e^{-\frac{\lambda}{2}\xi^2}\,d\xi}{\int_1^\infty e^{-\frac{\lambda}{2}\xi^2}\,d\xi} \; .$$

The temperature distribution satisfies, in the original co-ordinates, the initial condition

$$\phi(x,0) = \begin{cases} \Phi_M & x = 0 \\ 0 & x > 0 \end{cases}$$

and λ satisfies the transcendental equation

$$\lambda = \varepsilon \left(\sqrt{2\lambda}\, H_o - \Phi_M \frac{e^{-\frac{\lambda}{2}}}{\int_1^\infty e^{-\frac{\lambda}{2}x^2} dx} \right) . \tag{0.9}$$

This is a Neumann solution [2] presented by Carslaw and Jaeger [3]. There is a minimal H_o required to support melting i.e. for which (0.9) has a real positive solution. We shall find this H_o in Chapter I.

Another simple solution arises from the transformation $y = x - \sigma$. Then (0.8a) becomes

$$\phi_{yy} = \phi_\tau - \frac{d\sigma}{d\tau}\phi_y \qquad y > 0 \tag{0.10a}$$

while the boundary conditions transform to

$$\phi(0,\tau) = \Phi_M \qquad \phi(\infty,\tau) = 0 \tag{0.10b}$$

$$\varepsilon\,(\phi_y(0,\tau) + H) = \frac{d\sigma}{d\tau} \tag{0.10c}$$

$$\phi(y,0) = \phi_o(y) . \tag{0.10d}$$

For $H(\tau) = H_o$ assume $\sigma = \mu_o \tau$. Then if $\phi_o(y) = \Phi_M e^{-\mu_o y}$ we get

$$\phi = \Phi_M e^{-\mu_o y} .$$

where $\mu_o = \dfrac{\varepsilon H_o}{1 + \varepsilon \Phi_M}$. (0.11)

This is a solution first found by Stefan [1].

(b) The finite bar

For $s(0) = 0$ we look at the problem complimentary to the semi-infinite bar; rather than examine the diminishing semi-infinite phase, we treat the problem of a growing finite phase (which at $t = 0$ is not present).

For $S(0) > 0$ the first phase occupies the interval $0 < x < S(0)$ at t = 0 . Here s may increase or decrease. In both cases the partial differential equation is satisfied on $0 < x < S(t)$.

Non-dimensionalize as follows

$$\bar{x} = \frac{x}{\ell} \quad \sigma = \frac{s}{\ell} \quad \tau = \frac{k}{\ell^2} t \quad \phi = \frac{\phi - \Phi_M}{\Phi_M} \quad \bar{H} = \frac{\ell}{k \Phi_M} H$$

where $\ell = S(0)$ if $S(0) > 0$, otherwise ℓ is a reference length.

Letting $\dfrac{\phi_o(x) - \Phi_M}{\Phi_M} \rightarrow \phi_o(x)$ and suppressing bars, we get

$$\phi_{xx} = \phi_\tau \tag{0.12a}$$

$$\phi(\sigma(\tau), \tau) = 0 \tag{0.12b}$$

$$\varepsilon\,(\,\phi(\sigma(\tau), \tau) + H\,) = \pm \frac{d\sigma}{d\tau} \; . \tag{0.12c}$$

If $\sigma(0)=0$ no initial condition need be specified. If $\sigma(0)>0$ (hence $\sigma(0)=1$) an initial condition is required:

$$\phi(x,0)=\phi_0(x) \qquad 0\leq x\leq 1 \tag{0.12d}$$

For the condition at x = 0 either the temperature

$$\phi(0,\tau)=\Phi_B(\tau) \tag{0.12e}$$

is specified, or the heat flux

$$\Phi_x(0,\tau)=-H_B(\tau) \tag{0.12f}$$

is specified.

For definiteness we choose the minus sign in (0.12c). Let the problem (0.12a-0.12e) be called $B_I(\sigma(0),H)$ and (0.12a-0.12d, 0.12f) be called $B_{II}(\sigma(0),H)$.

To get a closed solution, again let $y=\frac{x}{\sigma}$ to give

$$\phi_{yy}=2r\phi_\tau-y\frac{dr}{d\tau}\phi_y \qquad 0<y<1 \tag{0.13a}$$

with the boundary conditions

$$\phi(1,\tau)=0 \tag{0.13b}$$

$$-\varepsilon\left(\phi_y(1,\tau)+H\sigma\right)=\frac{d\sigma}{d\tau} \qquad (0.13c)$$

$$\phi(0,\tau)=\Phi_B(\tau) \quad \text{or} \quad \phi_y(0,\tau)=-H_B(\tau)\sigma \qquad (0.13d)$$

and if $\sigma(0)=1$

$$\phi(y,0)=\phi_0(y) \qquad 0\leq y\leq 1 .$$

For $\Phi_B(\tau)=\Phi_B=\text{const.}$, the problem $B_I(0,0)$ has a solution of Neumann type:

$$\phi_N(\Phi_B;y)=\Phi_B\,\frac{\int_y^1 e^{-\frac{x^2}{2}\lambda(\Phi_B)}\,dx}{\int_0^1 e^{-\frac{x^2}{2}\lambda(\Phi_B)}\,dx} \qquad (0.14a)$$

$$r_N(\tau)=\lambda(\Phi_B)\tau \qquad (0.14b)$$

hence

$$\sigma_N(\tau)=\sqrt{2\lambda(\Phi_B)\tau} . \qquad (0.14c)$$

Here $\lambda(\Phi_B)$ is a parameter which is the solution to the transcendental equation

$$\varepsilon\,\Phi_B=\lambda\,e^{\frac{\lambda}{2}}\int_0^1 e^{-\frac{\lambda}{2}x^2}\,dx . \qquad (0.14d)$$

The boundary conditions at $x=\sigma(\tau)$ in both (0.8) and (0.12) are non-linear in σ ; hence the free boundary problem as a whole is non-linear. As in most non-linear problems it has been profitable, not to search for more closed solutions, but to develop effective approximation techniques. The principal results of this thesis are conclusions about the effectiveness of these approximation methods.

(1) Asymptotic Methods

In many practical cases only the behaviour of the free boundary $\sigma(\tau)$ is sought. For example, as a lake is freezing over, it is usually more important to know the thickness of the ice rather than the temperature distribution of that ice.

In particular, knowledge of the initial $(\tau \to 0)$ response of σ to the boundary conditions may be sufficient. Such small time expansions have been calculated for σ in system (0.8) by Boley [4], Landau [5] and Evans et al. [6]. In each case the authors transformed (0.8) into an integral equation for σ . The free boundary was then expanded in an asymptotic power series in $\sqrt{\tau}$

$$\sigma \sim \sum_{k=1} \sigma_{\frac{k}{2}} \tau^{\frac{k}{2}}$$

and the coefficients $\sigma_{\frac{k}{2}}$ were found by matching like powers of $\sqrt{\tau}$. Although it has not been done, the same approach will work for system (0.12).

There has been no direct attempt to find the $\tau \to \infty$ response of σ to the boundary conditions. However, providing $\sigma \to \infty$ as $\tau \to \infty$, the work of Cannon and Denson-Hill [7] can be used in an inverse problem where the large τ behaviour of σ is specified and the boundary condition which gives rise to this behaviour is estimated.

They show that for any $\sigma(\tau)$, the temperature distribution

$$\phi_{CDH}(x,\tau) = \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial \tau^n} (x-\sigma)^{2n}$$

satisfies $\phi_{xx} = \phi_\tau$ and the following conditions at $x = \sigma$

$$\phi(\sigma,\tau) = 0 \qquad\qquad -\varepsilon\,\phi_x(\sigma,\tau) = \frac{d\sigma}{d\tau}\,.$$

It follows that

$$\phi(0,\tau) = \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{d^n}{d\tau^n} \sigma^{2n}\,.$$

An asymptotic expansion for σ can be substituted and the behaviour of $\phi(0,\tau)$ estimated. For example, if $\sigma \sim \mu\tau^a$ for $\mu > 0$, $a > 1$, an easy estimate gives the bounds

$$e^{\mu^2\tau^a} < \varepsilon\,\phi(0,\tau) < e^{\mu^2\tau^{2a-1}} \qquad (0.15)$$

It is shown in Chapter I that, providing $\sigma \to \infty$ as $\tau \to \infty$, the leading behavior of σ depends only on the boundary conditions as

$\tau \to \infty$ and not the initial condition. Hence, estimates like (0.15) do not depend on the particular initial condition which ϕ_{CDH} satisfies.

Cannon and Denson-Hill used the above estimates to study the monotone dependence of the free boundary on the boundary conditions, and not primarily to obtain an asymptotic expansion. Thus their method is of limited value in calculating higher order terms in the expansion for σ . In particular, the effect of an arbitrary initial condition cannot be calculated.

Chapter I develops a method for calculating to all orders the large τ behaviour of σ for both systems (0.8) and (0.12). Here the initial conditions present no difficulty in the calculation.

(a) The semi-infinite bar

For the system (0.8) we follow Boley [4] and extend the space domain to $0 \leq x < \infty$ at the cost of introducing another unknown $\phi_x(0,\tau) \equiv F(\tau)$. The equations can then be solved in closed form on this domain. When the boundary conditions at $x = \sigma$ are imposed on the solution a pair of coupled integral equations for σ and F result.

The analysis breaks up into three separate cases according to whether $\Lambda = \dfrac{\sigma^2}{4\tau}$ approaches zero, goes like a constant, or increases without bound as $\tau \to \infty$.

First, the asymptotic expansion of the integral

$$\int_0^\infty \frac{e^{-\Lambda u}}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right) du \tag{0.16}$$

for $\tau \to \infty$ must be calculated.

For $\Lambda \leq O(1)$ the starting point is the work of Handlesman and Lew [8] who use the Parseval theorem for Mellin transforms to derive an asymptotic expansion of

$$\int_0^\infty f(u) e^{-\Lambda u} du$$

for $\Lambda \to 0$. In the case $\Lambda < O(1)$ their method raises the second problem of calculating the analytic continuation in z of the asymptotic expansion of

$$\int_0^\infty \frac{u^{z-1}}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right) du$$

for $\tau \to \infty$. These expansions are carried out in the Appendix to Chapter I.

For $\Lambda > O(1)$ the integral (0.16) can be expanded by standard steepest descent techniques.

It is then straightforward to apply these expansion techniques to the integral equations of Boley. The main results are the following.

[1] To support melting $\left(\frac{d\sigma}{d\tau} > 0\right)$ the heat flux H cannot decrease faster than $H_{min} = \frac{H_o}{\sqrt{\tau}} + o\left(\frac{1}{\sqrt{\tau}}\right)$, $\tau \to \infty$ where $H_o \geq \frac{\Phi_H}{\sqrt{\pi}}$.

[2] When H has the form $H = \frac{\Phi_H}{\sqrt{\pi\tau}} + o\left(\frac{1}{\sqrt{\tau}}\right)$, $\tau \to \infty$ then $\Lambda \to 0$. In particular, when $H \sim \frac{\Phi_H}{\sqrt{\pi\tau}} + \frac{H_1}{\tau^s} + \cdots$ $H_1 > 0$, $\frac{1}{2} < s \leq 1$ then $\sigma = O(\tau^{1-s})$ for $s < 1$ and $\sigma = O(\ln \tau)$ for $s = 1$. The heat flux contribution dominates the initial condition contribution to the order calculated.

[3] When H is of the form $H = \frac{H_o}{\sqrt{\tau}} + o\left(\frac{1}{\sqrt{\tau}}\right)$, $\tau \to \infty$ where $H_o > \frac{\Phi_H}{\sqrt{\pi}}$ then $\Lambda = O(1)$. In the case where $H = \frac{H_o}{\sqrt{\tau}} + o\left(\frac{1}{\tau}\right)$ the initial condition dominates at the second order in the expansion of the integral equations. It follows that $\sigma \sim \sqrt{2\lambda_o\tau} + \frac{2\Lambda_E^o}{\lambda_o} + \cdots$ where λ_o is the solution to (0.9) and Λ_E^o can be expressed explicitly in terms of λ_o, H_o, Φ_H and $\int_0^\infty \phi_o(\bar{x})\, d\bar{x}$.

[4] When the heat flux satisfies $H > O\left(\frac{1}{\sqrt{\tau}}\right)$, then $\Lambda \to \infty$. For the particular case $H = O(\tau^b)$, $b > -\frac{1}{2}$ it follows that $\sigma = O(\tau^{b+1})$. In general, when Λ is unbounded the initial condition contribution is exponentially small relative to the heat flux contribution. For this reason a perturbation procedure applied to the partial differential equation gives the same results as the analysis of the full integral equations.

(b) The finite bar

For the analysis of (0.12) we follow Friedman [9] and Sherman [10] to generate an integro-differential equation for σ. The techniques developed in the study of the semi-infinite bar are used to analyze the dependence of σ on the boundary condition at the fixed [x=0] end.

Here the major integral to be expanded is

$$\int_0^\tau \frac{1}{\sqrt{\tau-\bar{\tau}}}\,\frac{d\sigma}{d\bar{\tau}}\,e^{-\frac{(\sigma(\tau)+\sigma(\bar{\tau}))^2}{4(\tau-\bar{\tau})}}\,d\bar{\tau}\,.$$

Nevertheless, the analysis still depends on the behaviour of $\Lambda = \frac{\sigma^2}{4\tau}$ as $\tau \to \infty$.

The problem $\mathcal{B}_{I}(1,\sigma)$ is analysed in the cases $\Lambda \leq O(1)$, the problem $\mathcal{B}_{II}(1,\sigma)$ when $\Lambda > O(1)$. The main results are the following.

[1] With $\phi(0,\tau) = \Phi_B(\tau) \sim \Phi_B \tau^{-\nu}$, $0 < \nu < 1$, it follows that $\Lambda \to 0$ and the free boundary σ has the leading behaviour $\sigma \sim \sqrt{\frac{2\Phi_B}{1-\nu}}\,\tau^{\frac{1-\nu}{2}}$.

[2] If $\Phi_B(\tau) \sim \Phi_B$, then $\Lambda = O(1)$ and $\sigma \sim \sqrt{2\lambda_0 \tau}$ where λ_0 satisfies (0.14a)

[3] If $\phi_x(0,\tau) = -H_B(\tau) \sim -H_B \tau^{a}$, $H_B > 0$, $a > -\frac{1}{2}$ then $\Lambda \to \infty$. The behaviour of σ for this finite bar case is

compared with the behaviour for the semi-infinite bar undergoing the same heating, but at $x = \sigma$. The free boundary σ grows slower in the finite bar case.

(2) Perturbation Methods

The first suggestion of a perturbation procedure appears in a paper by Landau [5] who studied in a particular case the $\varepsilon \to 0$ and $\varepsilon \to \infty$ limits of the semi-infinite bar. Sherman [11] did a detailed analytic study of the $\varepsilon \to \infty$ case but the results are not suited to numerical calculation.

In Chapter II we study the conditions under which the problem $B_I(1,0)$ admits an expansion of the form

$$\phi = \sum_{k=0} \varepsilon^k T^{(k)} \qquad \sigma = \sum_{k=0} \varepsilon^k \sigma_k \quad . \qquad (0.17)$$

Even though these expansions are in fact asymptotic expansions ordered with respect to the parameter ε , the methods used to derive them are distinct from those of the previous section. To make that distinction, the expansions (0.17) will be called perturbation expansions. The main results are the following.

[1] When $\phi(0,\tau) \leq O(1)$ for $\tau \to \infty$, a perturbation expansion for ϕ and $r = \frac{1}{2}\sigma^2$, carried out in the fixed boundary co-ordinates $(y = \frac{x}{\sigma}, \tau)$ will be uniform on $0 \leq y \leq 1$, $\tau > 0$. In most physical problems two terms of the expansion give good results.

In the case that $\phi(0,\tau)$ is oscillatory the regular procedure produces secular terms. These terms can be summed, however, to produce a uniform expansion. The specific case

$$\phi(0,\tau) = \Phi_0 + \mu \sin \Omega\tau$$

is worked out.

Finally, even if ε is not small, the perturbation expansions can serve as asymptotic expansions for $\tau \rightarrow \infty$.

[2] If $\phi(0,\tau) \geq O(1)$ for $\tau \rightarrow \infty$ the regular perturbation expansion fails to be uniformly valid. In fact, after the independent and dependent variables are rescaled, the first order system incorporates the leading growth for $\sigma(\tau)$ but is as difficult to solve as the original system.

[3] A regular perturbation expansion for $\varepsilon \rightarrow \infty$ also exists. However, by a rescaling of the variables, the first order system can be shown to be of the same difficulty as the original free boundary problem.

(3) <u>Numerical Methods</u>

There have been many types of numerical calculations for the free-boundary problem.

One approach is to solve the general problem (0.1) where the phase change occurs over a finite temperature interval. Although the enthalpy and conductivity are then functions of temperature, the domain for the partial differential equation is fixed. Further, the method can be formulated in any number of dimensions; the two dimensional case has been calculated by Hashemi and Sliepcevich [12], the three dimensional case by Meyer [13].

For the one dimensional case, when only the boundary σ is required, the integral formulation of the free boundary problem can be used. Chuang and Szekeley [14] discretized the time in the integral equations of Friedman and solved for σ at the discrete times.

Finally, in one dimension, when both the temperature and the boundary are to be calculated, finite difference schemes are used. A classic paper on this method is that of Douglas and Gallie [15].

In Chapter III we introduce a semi-analytic approach for the solution of the finite bar. Only the time is discretized and the boundary σ approximated by a straight line in each time step. In each such time interval the solution to system (0.12), satisfying all the boundary conditions except (0.12c), is calculated. This solution, which started at the beginning of the n^{th} time step is then made to satisfy the flux condition (0.12c) at the end of the n^{th} step to calculate the slope of σ for the $(n+1)^{st}$ step.

This method was first suggested by G. W. Bluman [16] who used the theory of Lie Groups to show that such an analytic interpolative function is a similarity solution to the inverse Stefan problem with a linear free boundary. We derive the same solution by a simplified method which has the advantage of more directly including the case of inhomogeneous boundary conditions. The solution itself has two possible series representations; which representation is chosen depends upon the boundary conditions being considered.

The cases where first the heat flux and then the temperature is specified at $x = 0$ are solved. These cases demonstrate where each representation of the analytic solution is more effective.

Finally, the method is applied to the problem of a shallow lake which is freezing. We take advantage of the disparity between the diffusivities of ice and water to reduce the problem to an equivalent one phase problem with an effective latent heat and effective heat source. The results of the numerical calculation are compared with the data for a shallow lake in northern Michigan and are in good agreement for both water temperature and ice thickness.

CHAPTER I: ASYMPTOTIC METHODS

The objective of this chapter is to determine the large τ response of the boundary σ to the initial temperature distribution and the specified boundary temperatures or fluxes. These boundary temperatures or fluxes are to be specified at $x = \sigma$ for the semi-infinite bar and at $x = 0$ for the finite bar.

The analysis is restricted to the cases for which $\sigma \rightarrow \infty$ as $\tau \rightarrow \infty$. It follows that the boundary conditions provide the leading contribution to the growth of σ while the initial condition contribution enters only at higher orders in the large τ expansion for σ.

The cases of the semi-infinite and finite bar are studied separately. We begin with the semi-infinite bar; the equations are easier to analyse and give direction to the discussion of the finite bar.

1. THE SEMI-INFINITE BAR

The system to study is (0.8). Specifically, choose the plus sign in (0.8c) and take $\Phi_M > 0$. This situation describes the melting process.

Following Boley [4], extend the space domain of (0.8a) from (σ, ∞) to $(0, \infty)$. This is done at the cost of introducing another unknown which Boley chooses to be the flux at $x = 0$

$$\frac{\partial \phi}{\partial x}(0,\tau) \equiv F(\tau) \quad .$$

Then solve instead the problem

$$\phi_{xx} = \phi_\tau \qquad 0 < x < \infty \tag{1.1a}$$

$$\phi_x(0,\tau) = F(\tau) \qquad \phi_x(\infty,\tau) = 0 \tag{1.1b}$$

$$\phi(x,0) = \phi_0(x) \quad . \tag{1.1c}$$

The solution to (1.1) can be written in terms of the functions

ϕ_a, ϕ_b both of which satisfy (1.1a) and

$$\frac{\partial \phi_a}{\partial x}(0,\tau) = 1 \qquad \frac{\partial \phi_a}{\partial x}(\infty,\tau) = 0 \qquad \phi_a(x,0) = 0$$

$$\frac{\partial \phi_b}{\partial x}(0,\tau) = 0 \qquad \frac{\partial \phi_b}{\partial x}(\infty,\tau) = 0 \qquad \phi_b(x,0) = \phi_0(x) \quad .$$

This gives

$$\phi_a = -2\sqrt{\tau}\ \text{ierfc}\left(\frac{x}{2\sqrt{\tau}}\right)$$

$$\phi_b = \int_0^\infty \phi_0(\bar{x})\, G^+(x,\tau;\bar{x},0)\, d\bar{x}$$

where $$G^\pm(x,\tau;\bar{x},\bar{\tau}) = \frac{1}{2\sqrt{\pi}}\left\{\frac{e^{-\frac{(x-\bar{x})^2}{4(\tau-\bar{\tau})}} \pm e^{-\frac{(x+\bar{x})^2}{4(\tau-\bar{\tau})}}}{\sqrt{\tau-\bar{\tau}}}\right\}$$

are the even and odd Green's functions for the half-plane. Finally, from the Duhamel theorem [3] the solution to (1.1) can be written

$$\phi = \int_0^{\tau} F(\tau-\bar{\tau})\,\frac{\partial \phi_a}{\partial \bar{\tau}}\,d\bar{\tau} + \phi_b \, . \tag{1.2}$$

Note that the conditions (0.8b) and (0.8c) have yet to be satisfied. It is the application of these conditions that determines the equations for $\sigma(\tau)$.

Before applying these boundary conditions we point out that the bar may have had a heating history before melting commenced. It is this history that determines the temperature distribution at the onset of melting i.e. $\phi_o(x)$ of (1.1c). Temporarily, let $\tau = 0$ mark the beginning of this history and $\hat{\phi}$ the temperature distribution during this initial phase.

Then, since there is no melting, the true space domain is $(0,\infty)$ and system (1.1) describes the physics exactly. In this case $F(\tau)$ can be taken to be a known flux. If, for example, the bar starts at a uniform zero temperature and is subjected to a constant heat flux input H_0 , then set $F(\tau) = -H_o$, $H_o > 0$ and $\phi_o(x) = 0$ in (1.1) to get from (1.2)

$$\hat{\phi} = 2H_o\sqrt{\tau}\ \mathrm{ierfc}\left(\frac{x}{2\sqrt{\tau}}\right) \tag{1.2a}$$

Melting will then commence at a time τ_M such that $\hat{\phi}(0,\tau_M) = \Phi_M$, which can be calculated [4] as

$$\tau_M = \frac{\pi}{4}\left(\frac{\Phi_M}{H_o}\right)^2 .$$

With the onset of melting reinstated as $\tau = 0$ and F no longer known, system (1.1) is used with

$$\phi_o(x) = \hat{\phi}(x, \tau_M)$$

Now, using (1.2), apply the conditions (0.8b) and (0.8c) at $x = \sigma$ to get

$$-\frac{1}{\sqrt{\pi}}\int_0^{\tau} \frac{F(\tau-\bar{\tau})}{\sqrt{\bar{\tau}}}\, e^{-\frac{\sigma^2(\tau)}{4\bar{\tau}}}\, d\bar{\tau} \tag{1.3a}$$

$$+ \int_0^{\infty} \phi_o(\bar{x})\, G^+(\sigma(\tau), \tau; \bar{x}, 0)\, d\bar{x} = \Phi_M$$

$$\frac{\sigma(\tau)}{2\sqrt{\pi}}\int_0^{\tau} \frac{F(\tau-\bar{\tau})}{\bar{\tau}\sqrt{\bar{\tau}}}\, e^{-\frac{\sigma^2(\tau)}{4\bar{\tau}}}\, d\bar{\tau} \tag{1.3b}$$

$$+ \int_0^{\infty} \phi_o(\bar{x})\, G_x^+(\sigma(\tau), \tau; \bar{x}, 0)\, d\bar{x} = \frac{1}{\varepsilon}\frac{d\sigma}{d\tau} - H(\tau) .$$

To expedite the asymptotic analysis, make the transformation

$$1 + u = \frac{1}{\tau - \bar{\tau}}$$

in the first integrals of (1.3a) and (1.3b) to get

$$I F\left(\tfrac{3}{2},\tau,\Lambda\right) = -\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} e^{\Lambda} \int_0^{\infty} \phi_0(\bar{x})\, G^{+}(\sigma(\tau),\tau;\bar{x},0)\, d\bar{x}$$
$$- \Phi_M \left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} e^{\Lambda} \qquad (1.4a)$$

$$I F\left(\tfrac{1}{2},\tau,\Lambda\right) = \left(\frac{\pi}{\Lambda}\right)^{\frac{1}{2}} e^{\Lambda} \int_0^{\infty} \phi_0(\bar{x})\, G_x^{+}(\sigma(\tau),\tau;\bar{x},0)\, d\bar{x}$$
$$- \left(\frac{\pi}{\Lambda}\right)^{\frac{1}{2}} e^{\Lambda} \left\{ \frac{1}{\varepsilon} \frac{d\sigma}{d\tau} - H(\tau) \right\} \qquad (1.4b)$$

where

$$I F(r,\tau,\Lambda) = \int_0^{\infty} \frac{e^{-\Lambda u}}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right) du$$

and $\quad \Lambda(\tau) = \dfrac{\sigma^2(\tau)}{4\tau}$.

This is a set of integral equations for F and σ . Although only the large τ expansion for σ is required, it will be necessary to find the large τ expansion for F in the process.

The asymptotic expansions of F and σ for $\tau \to \infty$ are calculated as follows:

(a) Assume an asymptotic series for F and σ

$$F \sim \sum_{k=0} F_k A_k(\tau) \qquad (1.5a)$$

$$\sigma \sim \sum_{k=0} \sigma_k B_k(\tau) \qquad (1.5b)$$

where $\{A_k\}$ and $\{B_k\}$ are asymptotic sequences for $\tau \to \infty$ and

F_k, σ_k, are coefficients yet to be determined. For the types of heat fluxes being considered, it is sufficient to assume for F the asymptotic expansion

$$F(\tau) \sim e^{\nu\tau^a} \sum_{q=0} F_q \tau^{-r_q} \qquad (1.6)$$

where $r_q \uparrow \infty$ and $\nu \geqslant 0,\ a \geqslant 0$. In practice it is convenient to leave the asymptotic expansion for σ unspecified for the moment.

(b) Substitute the expansions (1.5a,b) into (1.4a,b) and generate large τ asymptotic expansions for the integrals.

(c) When the asymptotic sequences $\{A_k\}$ and $\{B_k\}$ have been chosen correctly it is possible to match to determine the coefficients F_k, σ_k. In practice, once the form (1.6) for F has been assumed, the corresponding asymptotic sequence for σ is determined by the requirement that matching be possible.

Step (b) is the most difficult. The behaviour of Λ as $\tau \to \infty$ is critical. The analysis divides up, accordingly, into three separate cases: $\Lambda < O(1)$, $\Lambda = O(1)$ and $\Lambda > O(1)$ as $\tau \to \infty$.

In this chapter, the Stefan number ε is a parameter that does not enter into the calculations, so without loss of generality, set $\varepsilon = 1$ in (1.4b).

CASE 1: $\Lambda \to 0$ for $\tau \to \infty$

To complete step (a) it is consistent to assume that $\nu = 0$ in (1.6).

In step (b) the first integral to be expanded is $IF(r,\tau,\Lambda)$. This is a complicated calculation and is done in Appendix I. For reference, the result is

$$IF(r,\tau,\Lambda) \sim \sum_{m=0}^{\infty} \Gamma(1-m-r)\, AF(r,\tau,m)\, \Lambda^{m+r-1}$$
$$+ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} DF(r,\tau,m+1)\, \Lambda^m \,. \tag{1.7}$$

The first sum is called the asymptotic contribution and for $\tau \to \infty$,

$$AF(r,\tau,m) \sim \sum_{q=0}^{\infty} (-1)^q F_q \sum_{k=0}^{m} \binom{-k-r}{m-k} \frac{\Gamma(r_q+k)}{\Gamma(r_q)} \frac{(-\tau)^k}{k!} \tau^{-(r_q+k)} \,. \tag{1.7a}$$

The second sum is called the domain contribution and for $\tau \to \infty$,

$$DF(r,\tau,m+1) \sim \sum_{q \in Q^c} \frac{F_q}{\tau^{r_q}} B(m+1-r_q,\, r-m-1) \tag{1.7b}$$
$$+ \ln\tau \sum_{q_k \in Q,\, k \geq m} \binom{r-m-2}{k} \frac{F_{q_k}}{\tau^{k+1}} + \sum_{k=m}^{\infty} \binom{r-m-2}{k} \frac{M_k}{\tau^{k+1}} \,.$$

Here, the q for which $r_q = k+1$ is called q_k, and Q is the sequence $\{q_k\}$. If Q is empty the second sum in (1.7b) is absent. The M_k are unknown constants which can be determined in the matching process to come.

The next integral to consider is, from (1.4a),

$$I\phi_o(\tau,\Lambda) = e^{\Lambda}\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \int_0^\infty \phi_o(\bar{x})\, G^+(\sigma(\tau),\tau;\bar{x},0)\, d\bar{x}$$

$$= \frac{1}{\tau} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\Lambda}{\tau}\right)^n I_n\, \phi_o(\tau) \tag{1.8}$$

where

$$I_n\, \phi_o(\tau) = \int_0^\infty \phi_o(\bar{x})\, \bar{x}^{2n}\, e^{-\frac{\bar{x}^2}{4\tau}}\, d\bar{x} \tag{1.9a}$$

$$= \frac{1}{2} \int_0^\infty \phi_o(\sqrt{u})\, u^{n-\frac{1}{2}}\, e^{-\frac{1}{4\tau}u}\, du \quad . \tag{1.9b}$$

For simplicity, consider the case where $\phi_o(x) \le O(e^{-\mu x^2})$, $\mu > 0$, $x \to \infty$. This is not an unusual condition; for example, it is met by the initial condition (1.2a) which arises when the melting state is approached in a "natural" way as was described in the introduction to this chapter. The exponential in (1.9a) can be expanded to get

$$I_n\, \phi_o(\tau) = \int_0^\infty \phi_o(\bar{x})\, d\bar{x} + O\left(\frac{1}{\tau}\right) \quad . \tag{1.10}$$

The assumption on the initial condition is not critical. For more general initial conditions the method developed in step (a) of Appendix I can be applied to (1.9b) to generate the required expansion for $\tau \to \infty$.

Finally, using (1.10) in (1.8) we have

$$I\phi_o(\tau,\Lambda) = \frac{1}{\tau}\int_0^\infty \phi_o(\bar{x})\,d\bar{x} + O\left(\frac{1}{\tau^2}\right) \quad . \tag{1.11}$$

The last integral to consider is from (1.4b),

$$J\phi_o(\tau,\Lambda) = e^\Lambda \left(\frac{\pi}{\Lambda}\right)^{\frac{1}{2}} \int_0^\infty \phi_o(\bar{x})\, G^+_{\bar{x}}(\sigma(\tau),\tau;\bar{x},0)\,d\bar{x} \quad .$$

The methods just described yield the expansion

$$J\phi_o(\tau,\Lambda) = \frac{1}{\tau}\int_0^\infty \phi_o(\bar{x})\,d\bar{x} + O\left(\frac{1}{\tau^2}\right) \quad . \tag{1.12}$$

The results (1.7), (1.11) and (1.12) can now be used to write the integral equations (1.4a,b) in an expanded form.

For (1.4a) we have

$$\begin{aligned} &\left\{ DF\left(\tfrac{3}{2},\tau,1\right) - \Lambda\, DF\left(\tfrac{3}{2},\tau,2\right) + O\left(\frac{\Lambda^2}{\tau^{r_o}}\right) \right\} \\ &+ \left\{ \Lambda^{\frac{1}{2}}\,\Gamma\left(-\tfrac{1}{2}\right) AF\left(\tfrac{3}{2},\tau,0\right) + \Lambda^{\frac{3}{2}}\,\Gamma\left(-\tfrac{3}{2}\right) AF\left(\tfrac{3}{2},\tau,1\right) + O\left(\frac{\Lambda^{\frac{5}{2}}}{\tau^{r_o}}\right) \right\} \\ &= -\,\Phi_M \left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \left\{ 1 + \Lambda + O(\Lambda^2) \right\} - \frac{1}{\tau}\int_0^\infty \phi_o(\bar{x})\,d\bar{x} + O\left(\frac{1}{\tau^2}\right) \end{aligned} \tag{1.13}$$

while for (1.4b) we have

$$\{ DF(\tfrac{1}{2},\tau,1) - \Lambda DF(\tfrac{1}{2},\tau,2) + O(\tfrac{\Lambda^2}{\tau^{r_0}}) \} \tag{1.14}$$

$$+ \{ \Lambda^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}) AF(\tfrac{1}{2},\tau,0) + \Lambda^{\frac{1}{2}} \Gamma(-\tfrac{1}{2}) AF(\tfrac{1}{2},\tau,1) + O(\tfrac{\Lambda^{\frac{3}{2}}}{\tau^{r_0}}) \}$$

$$= (\tfrac{\pi}{\Lambda})^{\frac{1}{2}} \{ \tfrac{d\sigma}{d\tau} - H(\tau) \}\{1+\Lambda+O(\Lambda^2)\} + \tfrac{1}{\tau}\int_0^\infty \phi_0(\bar{x})\,d\bar{x} + O(\tfrac{1}{\tau^2})$$

where from (1.6) $F \sim \dfrac{F_0}{\tau^{r_0}}$, $t \to \infty$.

Examine first the leading behaviour of (1.13). The leading order on the R.H.S. is $O(\frac{1}{\sqrt{\tau}})$, on the L.H.S. the domain contribution dominates. To match, set

$$F \sim \frac{F_0}{\sqrt{\tau}} \tag{1.15}$$

then from (1.7b) it follows that $DF(\frac{3}{2},\tau,1) \sim \frac{F_0}{\sqrt{\tau}} B(\frac{1}{2},\frac{1}{2})$. Since $B(\frac{1}{2},\frac{1}{2}) = \pi$, matching then gives

$$F_0 = - \frac{\Phi_M}{\sqrt{\pi}} \quad . \tag{1.16}$$

Next consider the leading order of (1.14). The asymptotic contribution dominates on the L.H.S. From (1.7a) and (1.15) this contribution is $AF(\frac{1}{2},\tau,0) \sim \dfrac{F_0}{\sqrt{\tau}}$ so that matching to first order gives

$$(\tfrac{\pi}{\Lambda})^{\frac{1}{2}} \frac{F_0}{\sqrt{\tau}} = \left(\frac{d\sigma}{d\tau} - H(\tau) \right) (\tfrac{\pi}{\Lambda})^{\frac{1}{2}}$$

or, using (1.16) and $\Lambda = \dfrac{\sigma^2}{4\tau}$,

$$\frac{d\sigma}{d\tau} = \left(H - \frac{\Phi_M}{\sqrt{\pi\tau}}\right) + o\left(\frac{1}{\sqrt{\tau}}\right) \quad .$$

We have assumed that $A \to 0$, hence $\frac{d\sigma}{d\tau} < O(\frac{1}{\sqrt{\tau}})$. For $H = \frac{\Phi_M}{\sqrt{\pi\tau}} + o\left(\frac{1}{\sqrt{\tau}}\right)$ this condition is fulfilled. This behaviour is a turning point: for $H < \frac{\Phi_M}{\sqrt{\pi\tau}}$ melting cannot be supported as then $\frac{d\sigma}{d\tau} < 0$, whereas for $H > \frac{\Phi_M}{\sqrt{\pi\tau}}$ we have $A \geq O(1)$. The minimal H_o described in (0.9) is thus $\frac{\Phi_M}{\sqrt{\pi}}$.

The matching has been carried up to $O(\frac{1}{\sqrt{\tau}})$. The initial condition contribution does not enter until $O(\frac{1}{\tau})$ and thus has not contributed to the leading order in the expansion.

It is yet to be determined how $\sigma \to \infty$. This behaviour can be found by matching the expansion at second order. On physical grounds, the heat flux must be the agent to drive σ to infinity; the initial condition contribution cannot enter even at this second order. This is demonstrated in two cases.

(i) $\sigma \to \infty$ algebraically

Here the heat flux must be of the form

$$H \sim \frac{\Phi_M}{\sqrt{\pi\tau}} + \frac{H_1}{\tau^s} + \cdots$$

where $H_1 > 0$ and $\frac{1}{2} < s < 1$. F has a similar expansion

$$F \sim \frac{F_o}{\sqrt{\tau}} + \frac{F_1}{\tau^s} + \cdots$$

Assume also that $\Lambda \sim \frac{\Lambda_o}{\tau^{m_o}}$, $m_o > 0$, $\Lambda_o > 0$. Then

$$\sigma \sim 2\sqrt{\Lambda_o}\,\tau^{\frac{1}{2}-\frac{m_o}{2}} \tag{1.17}$$

Now from (1.7a,b)

$$DF(\tfrac{3}{2},\tau,1) \sim \frac{F_o}{\sqrt{\tau}} B(\tfrac{1}{2},\tfrac{1}{2}) + \frac{F_1}{\tau^s} B(1-s,\tfrac{1}{2}) + \cdots$$

$$AF(\tfrac{3}{2},\tau,0) \sim \frac{F_o}{\sqrt{\tau}} + \cdots$$

So for (1.13), omitting already matched terms

$$\frac{F_1}{\tau^s} B(1-s,\tfrac{1}{2}) + \cdots \qquad + \frac{F_o}{\tau^{\frac{m_o}{2}+\frac{1}{2}}} \Lambda_o^{\frac{1}{2}} \Gamma(-\tfrac{1}{2}) + \cdots$$

$$= - \Phi_M \left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \frac{\Lambda_o}{\tau^{m_o}} + \cdots$$

The R.H.S. is of higher order. Matching to lowest order gives

$$\frac{m_o}{2} + \frac{1}{2} = s$$

and

$$\frac{F_1}{\sqrt{\Lambda_o}} = - \frac{2\Phi_M}{B(1-s,\tfrac{1}{2})} \tag{1.18}$$

Note that for $\frac{1}{2} < s < 1$ we have $0 < \frac{m_o}{2} < \frac{1}{2}$, so from (1.17) $\sigma \to \infty$ algebraically.

Again from (1.7a,b)

$$DF(\tfrac{1}{2},\tau,1) \sim \frac{F_o}{\sqrt{\tau}} B(\tfrac{1}{2},-\tfrac{1}{2}) + \frac{F_1}{\tau^s} B(1-s,-\tfrac{1}{2}) + \cdots$$

$$AF(\tfrac{1}{2},\tau,0) \sim \frac{F_o}{\sqrt{\tau}} + \frac{F_1}{\tau^s} + \cdots$$

So for (1.14), noting that $B(\frac{1}{2},-\frac{1}{2})=0$ and omitting already matched terms

$$\frac{F_1}{\tau^s} B(1-s,-\tfrac{1}{2}) + \cdots \qquad + \frac{F_1}{\sqrt{A_0}} \frac{\Gamma(\frac{1}{2})}{\tau^{s-\frac{m_0}{2}}} + \cdots$$

$$= (1-m_0)\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} - \left(\frac{\pi}{A_0}\right)^{\frac{1}{2}} \frac{H_1}{\tau^{s-\frac{m_0}{2}}} + \cdots$$

The first term on the L.H.S. is of higher order. Using (1.18), the other terms can be matched to get

$$\frac{H_1}{2\sqrt{A_0}} = (1-s) + \frac{\Phi_M}{B(1-s,\frac{1}{2})}$$

so that finally, for (1.17)

$$\sigma \sim \frac{H_1}{(1-s) + \dfrac{\Phi_M}{B(1-s,\frac{1}{2})}} \tau^{1-s} \qquad \tau \to \infty .$$

Note that the initial condition does not enter into this expression.

(ii) $\sigma \to \infty$ logarithmically

In this case the heat flux must decrease like

$$H \sim \frac{\Phi_M}{\sqrt{\pi\tau}} + \frac{H_1}{\tau} + \cdots ,$$

F must have the same expansion

$$F \sim \frac{F_0}{\sqrt{\tau}} + \frac{F_1}{\tau} + \cdots .$$

For (1.13) use

$$DF(\tfrac{3}{2},\tau,1) \sim \frac{F_0}{\sqrt{\tau}} B(\tfrac{1}{2},\tfrac{1}{2}) + F_1 \frac{\ln\tau}{\tau} + \cdots$$

$$AF(\tfrac{3}{2},\tau,1) \sim \frac{F_0}{\sqrt{\tau}} + \cdots$$

to get, omitting already matched terms

$$F_1 \frac{\ln\tau}{\tau} + \cdots \qquad + \quad F_0 \frac{\Gamma(-\frac{1}{2})}{\sqrt{\tau}} \sqrt{A} + \cdots$$

$$= -\Phi_M \sqrt{\pi} \frac{A}{\sqrt{\tau}} + \cdots .$$

The R.H.S. is of higher order. Assume that

$$A \sim A_0 \frac{(\ln\tau)^2}{\tau} + \cdots \qquad (1.19)$$

and match coefficients to get the ratio

$$\frac{F_1}{2\sqrt{A_0}} = -\Phi_M . \qquad (1.20)$$

In (1.14) use

$$DF(\tfrac{1}{2}, \tau, 1) \sim F_1 \frac{\ln\tau}{\tau} + \cdots$$

$$AF(\tfrac{1}{2}, \tau, 0) \sim \frac{F_0}{\sqrt{\tau}} + \frac{F_1}{\tau} + \cdots$$

so that using (1.19) and omitting already matched terms

$$F_1 \frac{\ln\tau}{\tau} + \cdots \qquad + \frac{F_1}{\sqrt{A_0}} \frac{\Gamma(\frac{1}{2})}{\tau} \frac{\sqrt{\tau}}{\ln\tau} + \cdots$$

$$= \frac{\sqrt{\pi\tau}}{\ln\tau} \frac{1}{\sqrt{A_0}} \left\{ \frac{2\sqrt{A_0}}{\tau} - \frac{H_1}{\tau} \right\} + \cdots .$$

The lowest order is $O(\frac{1}{\sqrt{\tau}\ln\tau})$. Match coefficients and use (1.20) to get

$$F_1 = -H_1$$

Therefore $2\sqrt{A_0} = \frac{H_1}{\Phi_M}$ and finally

$$\sigma \sim \frac{H_1}{\Phi_M} \ln\tau \qquad \tau \to \infty$$

Again the initial condition makes no contribution to this leading behaviour.

CASE II $A = O(1)$ for $\tau \to \infty$

This case arises for a heat flux of the form

$$H = \frac{H_o}{\sqrt{\tau}} + o\left(\frac{1}{\sqrt{\tau}}\right), \quad \tau \to \infty$$

where $H_o > \frac{\Phi_M}{\sqrt{\pi}}$. Set $A = A_o + A_R(\tau)$ where A_o is a positive constant and $A_R(\tau) \to 0$ for $\tau \to \infty$.

To complete step (a) it is again consistent to set $\nu = o$ in (1.6).

For step (6) the first integral to be expanded is $IF(r, \tau, A)$ This calculation is done in Appendix I and the result is

$$IF(r, \tau, A) \sim \sum_{m=o}^{\infty} \frac{(-1)^m}{m!} DF(r, \tau, m+1) A_R^m \tag{1.21}$$

where

$$DF(r, \tau, m+1) \sim \sum_{q \varepsilon Q^c} \frac{F_q}{\tau^{r_q}} \Gamma(m+1-r_q) U(m+1-r_q, m+2-r, A_o) + \ln\tau \sum_{q_k \varepsilon Q, k \geq m} g^r_{m,k} \frac{F_{q_k}}{\tau^{k+1}} + \sum_{k=o}^{\infty} g^r_{m,k} \frac{M_k}{\tau^{k+1}} \, . \tag{1.21a}$$

Here Q is as defined in (1.7b) or Appendix I, $U(a, b, z)$ is the confluent hypogeometric function of the second kind, and

$$g^r_m(u) = (1-u)^{r-m} u^m e^{-A_o \frac{u}{1-u}} = \sum_{k=o}^{\infty} g^r_{m,k} u^k \, .$$

The next integrals are the initial value contributions in (1.4a,b). To first order, these integrals have the same expansions as given by (1.11) and (1.12).

With the results (1.11), (1.12) and (1.21) the integral equations (1.4a,b) can now be written in expanded form. For this calculation it is possible to choose a heat flux such that the initial conditions contribute at the second order. Such a flux satisfies

$$H = \frac{H_0}{\sqrt{\tau}} + o\left(\frac{1}{\tau}\right) \qquad H_0 > \frac{\Phi_H}{\sqrt{\pi}}$$

Assume that $\Lambda_R \sim \frac{\Lambda_R^0}{\sqrt{\tau}} + \cdots$ and put $\Phi_1 = \int_0^\infty \Phi_0(\bar{x})\,d\bar{x}$. Then (1.4a) has the expansion

$$\begin{aligned} &\frac{F_0}{\sqrt{\tau}}\Gamma(\tfrac{1}{2})U(\tfrac{1}{2},\tfrac{1}{2},\Lambda_0) + g_{0,0}^{\frac{3}{2}}\frac{H_0}{\tau} + \cdots \\ &\quad - \frac{\Lambda_R^0}{\sqrt{\tau}}\left\{\frac{F_0}{\sqrt{\tau}}\Gamma(\tfrac{3}{2})U(\tfrac{3}{2},\tfrac{3}{2},\Lambda_0) + \cdots\right\} + \cdots \\ &= -\Phi_H\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} e^{\Lambda_0}\left\{1 + \frac{\Lambda_R^0}{\sqrt{\tau}} + \cdots\right\} - \frac{1}{\tau}\Phi_1 + \cdots \end{aligned} \tag{1.22}$$

while for (1.4b)

$$\begin{aligned} &\frac{F_0}{\sqrt{\tau}}\Gamma(\tfrac{1}{2})U(\tfrac{1}{2},\tfrac{3}{2},\Lambda_0) + g_{0,0}^{\frac{1}{2}}\frac{H_0}{\tau} + \cdots \\ &\quad - \frac{\Lambda_R^0}{\sqrt{\tau}}\left\{\frac{F_0}{\sqrt{\tau}}\Gamma(\tfrac{3}{2})U(\tfrac{3}{2},\tfrac{5}{2},\Lambda_0) + \cdots\right\} + \cdots \\ &= \left(1 - \frac{H_0}{\sqrt{\Lambda_0}}\right)\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} e^{\Lambda_0}\left\{1 + \frac{\Lambda_R^0}{2\sqrt{\tau}} + \cdots\right\} + \frac{1}{\tau}\Phi_1 + \cdots . \end{aligned} \tag{1.23}$$

First, matching to $O\left(\frac{1}{\sqrt{\tau}}\right)$ gives

$$F_0 U(\tfrac{1}{2},\tfrac{1}{2},\Lambda_0) = -\Phi_H e^{\Lambda_0} \tag{1.24a}$$

$$F_0 U(\tfrac{1}{2},\tfrac{3}{2},\Lambda_0) = \left(1 - \frac{H_0}{\sqrt{\Lambda_0}}\right) e^{\Lambda_0}. \tag{1.24b}$$

Next, matching to $O\left(\frac{1}{\tau}\right)$ and using $g_{0,0}^{r} = 1$

$$M_o - \frac{\sqrt{\pi}}{2} \Lambda_R^o \; F_o \, U\left(\tfrac{3}{2}, \tfrac{3}{2}, \Lambda_o\right) = - \Phi_M \sqrt{\pi}\, e^{\Lambda_o} \Lambda_R^o - \Phi_1 \tag{1.25a}$$

$$M_o - \frac{\sqrt{\pi}}{2} \Lambda_R^o \; F_o \, U\left(\tfrac{3}{2}, \tfrac{5}{2}, \Lambda_o\right) = \sqrt{\pi}\, e^{\Lambda_o}\left(1 - \frac{H_o}{\sqrt{\Lambda_o}}\right) \frac{\Lambda_R^o}{2} + \Phi_1 \,. \tag{1.25b}$$

F_o can now be eliminated between (1.24a,b) to derive a transcendental equation for Λ_o,

$$\frac{U\left(\frac{1}{2}, \frac{1}{2}, \Lambda_o\right)}{U\left(\frac{1}{2}, \frac{3}{2}, \Lambda_o\right)} = \Phi_M \frac{\sqrt{\Lambda_o}}{H_o - \sqrt{\Lambda_o}} \,. \tag{1.26}$$

Compare this result with (0.9). With the λ of (0.9) it follows that $\Lambda_o = \frac{\lambda}{2}$ and that equation (0.9) can be rewritten as

$$H_o \sqrt{\Lambda_o} = \Lambda_o + \frac{\Phi_M}{U\left(1, \frac{3}{2}, \Lambda_o\right)}$$

However, with the Kummer transformation $U(a,b,z) = z^{1-b}\, U(1+a-b, 2-b, z)$, this equation can be transformed into (1.26). Thus, to first order, the asymptotic behaviour of Λ with $H \sim \frac{H_o}{\sqrt{\tau}}$ agrees with the exact solution with $H = \frac{H_o}{\sqrt{\tau}}$. This is to be expected on physical grounds and is a check on the validity of the expansion technique.

Using, say (1.24a), F_o can be eliminated from (1.25a,b) and the solution for Λ_R^o found

$$\Lambda_R^o = \frac{2\Phi_1}{G(\Phi_M, A_o, H_o)}$$

where

$$G(\Phi_M, A_o, H_o) = \frac{\sqrt{\pi}}{2} e^{A_o} \left\{ \Phi_H \frac{U(\frac{3}{2},\frac{3}{2},A_o)}{U(\frac{1}{2},\frac{1}{2},A_o)} - \Phi_M \frac{U(\frac{3}{2},\frac{5}{2},A_o)}{U(\frac{1}{2},\frac{1}{2},A_o)} \right.$$
$$\left. + 2\Phi_M + \left(1 - \frac{H_o}{\sqrt{A_o}}\right) \right\} .$$

Note that it was possible to eliminate M_o .

Finally, the expansion for σ is

$$\sigma \sim 2\sqrt{A_o \tau} + \frac{\Lambda_R^o}{\sqrt{A_o}} + \cdots$$

Here the initial conditions enter at second order.

CASE III $A \to \infty$ for $\tau \to \infty$

This case arises for a heat flux satisfying

$$H > O\left(\frac{1}{\sqrt{\tau}}\right) \quad \tau \to \infty .$$

For definiteness, choose a flux of the form

$$H \sim H_o \tau^b \quad \tau \to \infty$$

$b > -\frac{1}{2}$. Then to complete step (a) it will be necessary to choose $\nu > 0$ in (1.6).

To begin step (b), examine the role of the initial condition integrals in (1.4a,b). In (1.4a) we consider

$$I\Phi_o(\tau,\Lambda) = e^{\Lambda}\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \int_o^{\infty} \Phi_o(\bar{x})\, G^+(\sigma(\tau),\tau;\bar{x},o)\, d\bar{x} \, .$$

The first term arising from G^+ is

$$I_1\Phi_o = \frac{e^{\Lambda}}{\tau} \int_o^{\infty} \Phi_o(\bar{x})\, e^{-\frac{(\sigma-\bar{x})^2}{4\tau}}\, d\bar{x}$$
$$= e^{\Lambda}\left(\frac{\sigma}{\tau}\right) \int_o^{\infty} \Phi_o(\sigma u)\, e^{-\Lambda(1-u)^2}\, du \, .$$

Laplace's method then gives the estimate

$$I_1\Phi_o \sim 2\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} e^{\Lambda}\, \Phi_o(\sigma) \qquad \tau \to \infty \, .$$

As in Case I and II consider those Φ_o for which $\Phi_o(x) \leq O(e^{-\mu x^2})$, $\mu > o$, $x \to \infty$. Then

$$I_1\Phi_o \leq 2\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} O(e^{\Lambda-\mu\sigma^2}) \tag{1.27}$$

and since $\sigma^2 > \Lambda$ for $\tau \to \infty$ it follows that $I_1\Phi_o$ is exponentially small for $\tau \to \infty$.

The second term arising from G^+ is

$$I_2\Phi_o = \frac{e^{\Lambda}}{\tau} \int_o^{\infty} \Phi_o(\bar{x})\, e^{-\frac{(\sigma+\bar{x})^2}{4\tau}}\, d\bar{x}$$

Here the boundedness of Φ_o and the Laplace method yield the estimate

$$I_2\Phi_o \leq \|\Phi_o\|_{\infty}\, O\left(\frac{1}{\sqrt{\tau\Lambda}}\right)$$

where

$$\|\Phi_o\|_\infty = \sup\{|\Phi_o(x)| \mid 0 \leq x < \infty\}$$

Together with (1.27) this estimate implies that $I\Phi_o \leq O(\frac{1}{\sqrt{\tau A}})$. Similarly, in (1.4b), for

$$J\Phi_o(\tau, A) = e^{A}\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \int_0^\infty \Phi_o(\bar{x})\, G_x^+(\sigma(\tau), \tau; \bar{x}, 0)\, d\bar{x}$$

we have that $J\Phi_o \leq O(\frac{1}{\sqrt{\tau A}})$.

The leading term to be matched in (1.4a) is $\Phi_H e^{A}\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}}$. The initial condition contribution is therefore exponentially small relative to this leading order and so does not participate in any order of the matching. The same comments apply to (1.4b).

Combine steps (b) and (c) to analyze the final integral $IF(r, \tau, A)$.

With $F \sim F_o e^{\nu\tau^a} \tau^{s_o}$ for $\tau \to \infty$, assume that $A \sim A_o \tau^a$. Then F contributes to the exponential in the integrand of IF to get

$$IF(r, \tau, A) \sim F_o \int_0^\infty \frac{u^{s_o}}{(1+u)^{s_o+r}}\, e^{f(u)\tau^a}\, du \tag{1.28}$$

where $f(u) = \nu\left(\frac{u}{1+u}\right)^a - A_o u$

If f has an interior maximum at u_M then $IF = O(e^{f(u_M)\tau^a} \tau^{s_o - \frac{a}{2}})$, so to match the exponential on the R.H.S. of (1.4a,b) we require

$f(u_M) = A_o$. Using $f'(u_M) = 0$ it follows that $u_M = a$ and $f''(u_M) < 0$ so that $f(u_M)$ is indeed an interior maximum.

The Laplace method can be used to expand (1.28). When the expansions are used in (1.4a,b) and matching carried out to first order the leading behaviour of σ is found to be

$$\sigma \sim \frac{H_o}{(1+b)(1+\Phi_M)} \tau^{b+1} . \tag{1.28a}$$

For the case $b = 0$ the result reduces to (0.11). Again, the asymptotic behaviour of σ for $H \sim H_o$, $\tau \to \infty$ corresponds to the exact solution for $H = H_o$.

Higher order terms can be calculated in a straightforward manner. However, a faster procedure to generate an asymptotic expansion for σ is to use the partial differential equation (0.10) directly.

Take, for example, the case $H(\tau) = H_o + H_R(\tau)$ where $H_R(\tau) \to 0$ for $\tau \to \infty$. Then rewrite (0.10), introducing the artificial parameter λ , as follows

$$\phi_{yy} + \mu \phi_y = \lambda \phi_\tau$$

$$\phi(0,\tau) = \Phi_M \qquad \phi(\infty,\tau) = 0$$

$$\phi_y(0,\tau) + H_o + \lambda H_R = \mu$$

where $\mu = \dfrac{d\sigma}{d\tau}$. Now expand in powers of λ

$$\begin{aligned} \phi &= T_0 + \lambda T_1 + \cdots \\ \mu &= \mu_0 + \lambda \mu_1 + \cdots \quad . \end{aligned} \tag{1.29}$$

If $\mu_{k+1} = o(\mu_k)$ and $T_{k+1} = o(T_k)$ for $\tau \to \infty$, then with $\lambda = 1$ the series (1.29) are asymptotic expansions for the solutions.

The first order system has the governing equation

$$\frac{\partial^2 T_0}{\partial y^2} + \mu_0 \frac{\partial T_0}{\partial y} = 0$$

For this equation an arbitrary initial condition cannot be specified. But by the previous discussion the initial condition makes only an exponentially small contribution to the expansion for μ. Consider, for example, the case where $H_R \to 0$ algebraically. Then a simple calculation gives for μ (after setting $\lambda = 1$)

$$\mu \sim \frac{H_0}{1+\Phi_M} + \frac{H_R}{1+\Phi_M} - \frac{H_0 \Phi_M}{(1+\Phi_M)^3} \frac{dH_R}{d\tau} + \cdots$$

+ exponentially small terms .

This is an asymptotic series for $\tau \to \infty$; to first order it agrees with result (1.28a) obtained through the rigorous expansion of the integral equations.

2. THE FINITE BAR

For the finite bar, the large τ response of the boundary $\sigma(\tau)$ to a heat source at $x = 0$ is examined. Here, as opposed to the direct heating case of the semi-infinite bar, the heat source is remote from the free boundary and must diffuse through the material before affecting the phase transition process. As a result, the boundary σ responds differently to the same heating applied in these two different situations.

The relevant system to study is (0.12). Specifically, choose the minus sign in (0.12c) and take $H = 0$. For the analysis an integral equation for σ must first be formulated.

Consider first the problem $\mathcal{B}_I(1,0)$ as defined in the Introduction. Following Friedman [9], suppose that $\phi(x,\tau)$, $\sigma(\tau)$ form a solution to the problem. Then the Green's identity

$$\frac{\partial}{\partial\bar{x}}\left[G^-\phi_{\bar{x}} - G^-_{\bar{x}}\phi\right] - \frac{\partial}{\partial\bar{\tau}}\left[G^-\phi\right] = 0$$

can be integrated over the domain $0 < \bar{x} < \sigma(\bar{\tau})$, $0 < \varepsilon < \bar{\tau} < \tau - \varepsilon$ to get, upon letting $\varepsilon \to 0$

$$\phi(x,\tau) = \int_0^1 \phi_0(\bar{x})\, G^-(x,\tau;\bar{x},0)\, d\bar{x}$$

$$+ \int_0^\tau \Phi_B(\bar{\tau})\, G^-_{\bar{x}}(x,\tau;0,\bar{\tau})\, d\bar{\tau} - \int_0^\tau \frac{d\sigma}{d\bar{\tau}}\, G^-(x,\tau;\sigma(\bar{\tau}),\bar{\tau})\, d\bar{\tau}$$

Now set $x = \sigma(\tau)$ to get an integro-differential equation for $\sigma(\tau)$:

$$\int_0^\tau \frac{d\sigma}{d\bar\tau}\, G^-(\sigma(\tau),\tau;\,\sigma(\bar\tau),\bar\tau)\, d\bar\tau =$$

$$\int_0^\tau \Phi_B(\bar\tau)\, G^-_{\bar x}(\sigma(\tau),\tau;\,0,\bar\tau)\, d\bar\tau \qquad (1.30)$$

$$+ \int_0^1 \phi_0(\bar x)\, G^-(\sigma(\tau),\tau;\,\bar x,0)\, d\bar x \,.$$

For $B_{II}(1,0)$ a similar analysis gives

$$\int_0^\tau \frac{d\sigma}{d\bar\tau}\, G^+(\sigma(\tau),\tau;\,\sigma(\bar\tau),\bar\tau)\, d\bar\tau =$$

$$-\int_0^\tau H_B(\bar\tau)\, G^+(\sigma(\tau),\tau;\,0,\bar\tau)\, d\bar\tau \qquad (1.31)$$

$$+ \int_0^1 \phi_0(\bar x)\, G^+(\sigma(\tau),\tau;\,\bar x,0)\, d\bar x \,.$$

As in the semi-infinite bar, the behaviour of $\Lambda = \frac{\sigma^2}{4\tau}$ for $\tau \to \infty$ breaks the analysis up into three distinct cases.

Case I $\quad \Lambda < O(1)$

For this case analyse the system $B_I(1,0)$ where

$$\phi(0,\tau) = \Phi_B(\tau) \sim \Psi_0 \tau^{-\nu}, \quad \tau \to \infty$$

for $\Psi_0 > 0,\ \nu > 0$. As in the semi-infinite bar, in order to carry out the matching, the integrals in (1.30) must first be expanded.

The first integral to consider is

$$I\sigma(\tau) = \int_0^\tau \frac{d\sigma}{d\bar\tau}\, G^-(\sigma(\tau),\tau;\,\sigma(\bar\tau),\bar\tau)\, d\bar\tau \,.$$

In what follows let

$$\gamma_\sigma^{\pm}(\tau,\bar\tau) = \frac{(\sigma(\tau) \pm \sigma(\bar\tau))^2}{4(\tau-\bar\tau)} .$$

The first term of I_σ ,

$$I_1\sigma(\tau) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{1}{\sqrt{\tau-\bar\tau}} \frac{d\sigma}{d\bar\tau} e^{-\gamma_\sigma^-} d\bar\tau$$

is straightforward; since $\gamma_\sigma^- = O(\Lambda)$ uniformly on $0 \le \bar\tau \le \tau$ the exponential can be expanded to get

$$I_1\sigma(\tau) \sim \frac{1}{2\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\tau \frac{1}{\sqrt{\tau-\bar\tau}} \frac{d\sigma}{d\bar\tau} (\gamma_\sigma^-)^m d\bar\tau . \qquad (1.32)$$

The m^{th} term of this expansion is $O(\Lambda^m)$.

For the second term

$$I_2\sigma(\tau) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{1}{\sqrt{\tau-\bar\tau}} \frac{d\sigma}{d\bar\tau} e^{-\gamma_\sigma^+} d\bar\tau$$

the argument of the exponential has a singularity at $\bar\tau = \tau$.
Consider the transformation

$$\gamma_\sigma^+(\tau,\bar\tau) = (1+u)\,\Lambda\,(1+\frac{1}{\sigma})^2 ; \qquad (1.33)$$

then $u \to 0$ for $\bar\tau \to 0$ and $u \to \infty$ for $\bar\tau \to \tau$. If this transformation is applied to $I_2\sigma$ the analysis in case I of Appendix I can be used.
The result is

$$I_2\sigma \sim \frac{1}{2\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} D\sigma(\tau, m+1) \tag{1.34}$$

$$+ \frac{1}{2\sqrt{\pi}} e^{-\Lambda(1+\frac{1}{\sigma})^2} \sum_{m=0}^{\infty} B_{\tau,m} \Gamma(1-r_m) \Lambda^{r_m-1} (1+\tfrac{1}{\sigma})^{2(r_m-1)} .$$

In the nomenclature of Appendix I, the first sum is the domain contribution. Here $D\sigma(\tau,z)$ is the analytic continuation in z of

$$\int_0^{\tau} \frac{1}{\sqrt{\tau-\bar{\tau}}} \frac{d\sigma}{d\bar{\tau}} (\gamma_\sigma^+)^{z-1} d\bar{\tau} .$$

The second sum is the asymptotic contribution. Here, for $\bar{\tau} = \delta_\sigma^+(\tau,u)$ the inverse transformation of (1.33), the function

$$B_\tau(u) = \frac{\frac{d\delta_\sigma^+(\tau,u)}{du} \frac{d\sigma}{d\bar{\tau}}(\delta_\sigma^+(\tau,u))}{(\tau - \delta_\sigma^+(\tau,u))^{\frac{1}{2}}} .$$

has the expansion

$$B_\tau(u) \sim \sum_{m=0}^{\infty} B_{\tau,m} u^{-r_m} \qquad u \to \infty$$

for $r_m \uparrow \infty$. The $B_{\tau,m}$ and r_m are yet to be determined.

The $m = 0$ term of the domain contribution cancels the $m = 0$ term of (1.32), so consider the next order--the $m = 0$ term of the asymptotic contribution. For $\delta_\sigma^+(\tau,u)$ assume the expansion

$$\bar{\tau} = \delta_\sigma^+(\tau,u) \sim \tau - \frac{a(\tau)}{u}\tau + \dots \qquad u \to \infty .$$

Then expand (1.33) about $\bar{\tau} = \tau$ and substitute in the above expression to get

$$a(\tau) = 4 + O\left(\frac{1}{\sigma}\right)$$

The leading behaviour of $\mathcal{B}_\tau$ is then

$$\mathcal{B}_\tau(u) \sim 2\sqrt{\tau}\,\frac{d\sigma}{d\tau}(\tau)\,u^{-\frac{3}{2}}, \quad u \to \infty$$

so that the asymptotic contribution to (1.34) has the leading behaviour

$$\sigma\,\frac{d\sigma}{d\tau}\,.$$

The results (1.32) and (1.34) can then be combined to give

$$I\sigma \sim \sigma\,\frac{d\sigma}{d\tau}\,. \tag{1.35}$$

The second integral to expand is

$$I\Phi_B(\tau) = \int_0^\tau \Phi_B(\bar{\tau})\,G_{\bar{x}}(\sigma(\tau),\tau;0,\bar{\tau})\,d\bar{\tau}$$

$$= \left(\frac{\Lambda}{\pi}\right)^{\frac{1}{2}} e^{-\Lambda} \int_0^\tau \frac{e^{-\Lambda u}}{\sqrt{1+u}}\,\Phi_B\left(\frac{\tau u}{1+u}\right) du\,.$$

This is in a form covered by Appendix I, and to first order the result is

$$I\Phi_B(\tau) \sim \Psi_0\,\tau^{-\nu}, \quad \tau \to \infty\,. \tag{1.36}$$

The last integral is

$$I\phi_o(\tau) = \int_o^1 \phi_o(\bar{x})\, G^-(\sigma(\tau),\tau;\bar{x},o)\, d\bar{x} \ .$$

Here, since the domain of integration is finite and $\Lambda \to o$, the exponentials in G^- can be expanded to get

$$I\phi_o \sim \frac{1}{2\sqrt{\pi\tau}} \int_o^1 \phi_o(\bar{x}) \Big\{ \Big[1 - \frac{(\sigma-\bar{x})^2}{4\tau} + O(\Lambda^2)\Big] - \Big[1 - \frac{(\sigma+\bar{x})^2}{4\tau} + O(\Lambda^2)\Big] \Big\}$$

$$\sim \frac{1}{\tau}\left(\frac{\Lambda}{\pi}\right)^{\frac{1}{2}} \int_o^1 \phi_o(\bar{x})\bar{x}\, d\bar{x} + O\left(\frac{\Lambda^2}{\sqrt{\tau}}\right) . \tag{1.37}$$

Now assume that $\sigma \sim \mu\tau^a$, $a > o$, $\tau \to \infty$, and collect the results (1.35), (1.36) and (1.37)

$$I\sigma \sim \mu^2 a \tau^{2a-1} \tag{1.35'}$$

$$I\Phi_B \sim \Psi_o \tau^{-\nu} \tag{1.36'}$$

$$I\phi_o \sim \frac{\Phi_1}{2\sqrt{\pi}} \mu\tau^{a-\frac{3}{2}} \tag{1.37'}$$

where $\Phi_1 = \int_o^1 \phi_o(\bar{x})\bar{x}\, d\bar{x}$.

For $a > o$, (1.35') dominates (1.37'). We expect this result. In the phase transition process the initial heat contained within the bar contributes to the latent heat of fusion to cause σ to grow. But then no finite initial amount of heat could provide the infinite amount of heat required to cause $\sigma \to \infty$.

Matching the powers in (1.35') and (1.36') implies that

$a = \frac{1-\nu}{2}$, so $a > 0$ for $\nu < 1$. This is also an intuitively clear result; if Φ_B is not integrable for $\tau \to \infty$, then $\sigma \to \infty$. Finally, matching the coefficients gives $\mu = \sqrt{\frac{2\Psi_0}{1-\nu}}$ so that

$$\sigma \sim \sqrt{\frac{2\Psi_0}{1-\nu}}\ \tau^{\frac{1-\nu}{2}} \ . \tag{1.38}$$

CASE II $\quad \Lambda = O(1)$

For the system $B_I(1,0)$ this case arises when $\Phi_B(\tau) \sim \Phi_B + \Phi_R(\tau)$ where Φ_B is a constant and $\Phi_R \to 0$ for $\tau \to \infty$. Then $\Lambda \sim \Lambda_0 + \Lambda_R(\tau)$ where $\Lambda_R \to 0$ for $\tau \to \infty$.

As has just been shown in Case I, the initial configuration does not affect the leading behaviour of σ for $\tau \to \infty$. In particular, the solution to $B_I(1,0)$ approaches the solution to $B_I(0,0)$ providing $\Phi_B(\tau)$ is the same for both problems. For the same reason, the solution to $B_I(0,0)$ approaches the exact solution (0.14), so it follows that

$$\sigma \sim 2\sqrt{\Lambda_0 \tau}\ , \qquad \tau \to \infty$$

where Λ_0 satisfies the transcendental equation

$$\Phi_B = 2\Lambda_0\, e^{\Lambda_0} \int_0^1 e^{-\Lambda_0 x^2}\, dx \ .$$

CASE III $\quad \Lambda > O(1)$

For this last case examine $B_{II}(1,0)$ with a heat flux satisfying

$$H_B(\tau) \sim -H_0 + H_R(\tau) \quad , \tau \to \infty$$

where $H_0 > 0$ is a constant and $H_R \to 0$ for $\tau \to \infty$.

The integrals in (1.31) must be expanded for $\tau \to \infty$. Consider first the integral

$$I\sigma(\tau) = \int_0^\tau \frac{1}{\sqrt{\tau - \bar{\tau}}} \frac{d\sigma}{d\bar{\tau}} G^+(\sigma(\tau), \tau; \sigma(\bar{\tau}), \bar{\tau})\, d\bar{\tau} \; .$$

In the notation of Case I, for the first term $I_1\sigma$ we have $\gamma_\sigma^- = O(\Lambda)$ for $0 \le \bar{\tau} \le \tau$. Define the transformation

$$\gamma_\sigma^-(\tau, \bar{\tau}) = \Lambda u \left(1 + \frac{1}{\sigma}\right)^2$$

then $u \to 1$ for $\bar{\tau} \to 0$ and $u \to 0$ for $\bar{\tau} \to \tau$. Since $\Lambda \to \infty$, only the $u \to 0$ behaviour of the transformation is needed. A simple calculation gives

$$\tau - \bar{\tau} = \left(\frac{\sigma}{\frac{d\sigma}{d\tau}}\right)^2 \frac{u}{\tau} + O(u^2) \quad , u \to 0 \; .$$

Therefore, the first term has the leading behaviour

$$I_1\sigma \sim \frac{1}{2\sqrt{\pi}} \frac{d\sigma}{d\tau} \left(\frac{\sigma}{\frac{d\sigma}{d\tau}}\right) \frac{1}{\sqrt{\tau}} \int_0^\infty \frac{e^{-\Lambda u}}{\sqrt{u}}\, du = 1 + O\left(\frac{1}{\sqrt{\Lambda}}\right) .$$

The second term $I_2\sigma$ is exponentially small relative to the first so that

$$I\sigma(\tau) = 1 + O\left(\frac{1}{\sqrt{\Lambda}}\right) \; . \tag{1.39}$$

Use the change of variable $\frac{1}{\tau - \bar{\tau}} = \frac{1}{\tau}[1 + u]$ to transform the second integral $IH_B(\tau) = -\int_0^\tau H_B(\bar{\tau})\, G^+(\sigma(\tau), \tau; 0, \bar{\tau})\, d\bar{\tau}$ to give

$$I H_B(\tau) \sim H_o \left(\frac{\tau}{\pi}\right)^{\frac{1}{2}} e^{-A} \int_0^\infty \frac{e^{-Au}}{(1+u)^{\frac{3}{2}}}\, du$$

to get for $\tau \to \infty$ the leading behaviour

$$I H_B(\tau) \sim H_o \left(\frac{\tau}{\pi}\right)^{\frac{1}{2}} \frac{e^{-A}}{A} \quad . \tag{1.40}$$

It is easy to show that the last integral $I\Phi_o(\tau) = \int_0^1 \Phi_o(\bar{x})\, G^+(\sigma(\tau),\tau;\bar{x},0)\, d\bar{x}$ has the leading behaviour

$$I\Phi_o(\tau) \sim \frac{1}{2\sqrt{\pi\tau}}\, e^{-A} \int_0^1 \Phi_o(\bar{x})\, d\bar{x}$$

so that $\quad I\Phi_o(\tau) = O\left(\frac{A}{\tau}\right) I H_B(\tau)$.

Assume that $I H_B$ is of lower order than $I\Phi_o$. Then, matching (1.39) and (1.40) gives

$$A e^{A} \sim H_o \left(\frac{\tau}{\pi}\right)^{\frac{1}{2}} \quad , \tau \to \infty \ .$$

This is a transcendental expression in A and cannot be simplified further. However, it follows that $A = o(\ln \tau)$ so that $\frac{A}{\tau} \to 0$ and $I\Phi_o$ is indeed of higher order.

Finally, this result for the heating of the finite bar can be compared with the results for the same, but direct, heating of the semi-infinite bar. From case III of the semi-infinite bar, $\sigma = O(\tau)$, so that $A = O(\tau)$ as $\tau \to \infty$, whereas for the remote heating of the finite bar we have just shown that $\Lambda = o(\ln\tau)$.

SUMMARY

The behaviour of $\sigma(\tau)$ for $\tau \to \infty$ has been calculated for a variety of boundary conditions. Although, in most cases, only the leading behaviour has been calculated it is clear that the determination of the higher orders is straightforward. Further, the asymptotic forms for the boundary conditions were chosen for demonstration purposes only; the methods presented in this chapter do not depend upon the forms chosen.

CHAPTER II: PERTURBATION METHODS

The free boundary problem encompasses two distinct physical mechanisms:

(a) Heat transfer through diffusion. This mechanism is represented in the equation $\phi_{xx} = \phi_{\tau}$ which is valid throughout the material

(b) Phase transition. This is represented by the flux condition at $x = \sigma(\tau)$

$$-\varepsilon\,[\,\phi_x(\sigma(\tau),\tau) + H\,] = \frac{d\sigma}{d\tau}$$

Here $\varepsilon = \dfrac{c_p \Phi_H}{L}$ is the non-dimensional Stefan number which specifies the coupling between these two physical modes.

For the $\varepsilon \to 0$ case, the problem $B_I(1,0)$ is examined. In this case it is possible to calculate a perturbation expansion of the form

$$\phi = \sum_{n=0}^{\infty} \varepsilon^n T^n \tag{2.1a}$$

$$\sigma(\tau) = \sum_{n=0}^{\infty} \varepsilon^n \sigma_n(\tau) \,. \tag{2.1b}$$

The arguments of T^n have deliberately been left unspecified. In fact, the original co-ordinates (x,τ) are not suitable to generate the expansions.

Problems of uniform validity of the expansions for $0 \le \tau < \infty$ arise only in the particular case $\sigma \to \infty$ for $\tau \to \infty$. In this situation, however, uniform validity does hold if and only if Φ_B is bounded for $\tau \to \infty$. When the approximation is uniform there are the following additional features:

(a) In (0.12) let $M_0 = Max\{\|\Phi_B\|_\infty, \|\phi_0\|_\infty\}$. Then by rescaling the temperature

$$\phi \to M_0 \phi$$

the system remains invariant except that now $|\Phi_B| < 1$, $|\phi_0| < 1$ and in (0.12c) there is a new expansion parameter

$$\varepsilon \to \varepsilon M_0 .$$

For many physical problems εM_0 is small; for example, if ice is being melted by water at 10°C we have $\varepsilon M_0 \approx .14$. This scaling argument then guarantees that for εM_0 small, even if the expansion is carried out in the original non-dimensional co-ordinates, only a few terms of the expansions (2.1) are needed to give good results.

(b) The expansions (2.1) are asymptotic for $\tau \to \infty$. Thus even if εM_0 is not small, the perturbation method is a direct way to generate a large τ expansion.

For the $\varepsilon \to \infty$ case the semi-infinite bar is examined. It is shown that a uniform expansion in powers of $\frac{1}{\varepsilon}$ is possible, but that the equations to determine the first order system are as difficult to solve as the original problem.

1. THE CASE $\varepsilon \to 0$

Consider the problem $\mathcal{B}_I(1,0)$ where $\sigma \to \infty$ for $\tau \to \infty$ Set $H = o$ in (0.12c).

It is easy to show that a perturbation expansion of the form (2.1) in the original co-ordinates will not work. Indeed, from (0.12c) it follows that $\frac{d\sigma_0}{d\tau} = 0$ so that $\sigma_0(\tau) = \sigma(0) = 1$. The boundary condition at $x = \sigma$ can now be expanded:

(a) From $\phi(\sigma,\tau) = 0$ we have

$$0 = T^0(1,\tau) + \varepsilon\{T^1(1,\tau) + \sigma_1 T^0_x(1,\tau)\} + O(\varepsilon^2\sigma^2)$$

(b) From $-\varepsilon\phi_x(\sigma,\tau) = \frac{d\sigma}{d\tau}$ we get

$$\varepsilon\frac{d\sigma_1}{d\tau} + O(\varepsilon^2\sigma) = -\varepsilon T^0_x(1,\tau) + O(\varepsilon^2\sigma) \qquad (2.2)$$

Note that, since $\sigma \to \infty$, these are already non-uniform expansions. Consider, in particular, the case where

$$\phi(0,\tau) = \Phi_B(\tau) \to \Psi_0 = \text{const.} \qquad \tau \to \infty .$$

The $0(1)$ systems is, for $\tau \to \infty$

$$T^0_{xx} - T^0_\tau = 0$$

$$T^0(0,\tau) \sim \Psi_0 \qquad T^0(1,\tau) = 0 .$$

The initial condition need not be considered as it does not contribute to the leading behaviour of the solution as $\tau \to \infty$. Now for large τ

it follows that

$$T^{\circ} \sim \Psi_{o}(1-x) \; .$$

This expansion can be used in (2.2) to give

$$\frac{d\sigma_1}{d\tau} \sim \Psi_o \; .$$

Therefore $\sigma_1 = O(\tau)$. But we have shown in Chapter I, case II of the finite bar that $\sigma = O(\sqrt{\tau})$. The non-uniformity of the expansion is then apparent.

To avoid the non-uniformities presented by the expansions of the boundary conditions at $x = \sigma$, use instead the fixed boundary representation (0.13) to generate the perturbation expansion.

With $r = \frac{1}{2}\sigma^2 = \frac{1}{2} + \varepsilon R$ $\qquad$ (2.3)

system (0.13) can be rewritten as

$$\begin{aligned} &\phi_{yy} - \phi_\tau = \varepsilon\left[2R\phi_\tau - y\frac{dR}{d\tau}\phi_y\right] \qquad 0<y<1 \\ &\phi(0,\tau) = \Phi_B(\tau) \qquad \phi(1,\tau) = 0 \\ &\phi(y,0) = \phi_o(y) \end{aligned} \qquad (2.4)$$

together with the flux condition

$$-\frac{dR}{d\tau} = \phi_y(1,\tau) \qquad (2.4a)$$

The analysis breaks up into three cases according to whether $\Phi_B < O(1)$, $\Phi_B = O(1)$, or $\Phi_B > O(1)$, as $\tau \to \infty$.

CASE I $\Phi_B < O(1)$

Expand ϕ as follows

$$\phi(y,\tau) = \sum_{k=0} \varepsilon^k T^k(y,\tau)$$

and rather than σ , expand R

$$R(\tau) = \sum_{k=0} \varepsilon^k R_k(\tau) \; .$$

To analyse the systems generated by substituting these expansions into (2.4) the following results are needed.

Let ϕ satisfy the diffusion equation with non-homogeneous term $S(y,\tau)$

$$\phi_{yy} - \phi_\tau = S(y,\tau) \qquad 0 < y < 1$$

with the boundary conditions

$$\phi(0,\tau) = \Phi_B(\tau) \qquad \phi(1,\tau) = 0$$

$$\phi(y,0) = \phi_0(y) \; .$$

Then ϕ can be written

$$\phi = \phi_{IC} + \phi_{BV} + \phi_S$$

where ϕ_{IC} arises solely from the initial condition, ϕ_{BV} from the boundary condition, and ϕ_S from the source term. See Appendix II.

One can show that $\phi_{IC}(y,\tau) = O(e^{-\pi^2\tau})$. Since we examine only the large time behaviour of the solutions this term can be ignored. Further, assume that for $\nu_k \uparrow \infty$ and $\mu_k \uparrow \infty$

$$\Phi_B(\tau) \sim \sum_{k=0} \Psi_k \tau^{-\nu_k}$$

$$S(y,\tau) \sim \sum_{k=0} S_k(y) \tau^{-\mu_k} .$$

The Bromwich contour integrals in Appendix II can then be expanded for large τ to give

$$\phi_{BV}(y,\tau) = (1-y) \sum_{k=0} \Psi_k \tau^{-\nu_k} \tag{2.5a}$$

$$+ \frac{y(1-y)(2-y)}{6\tau} \sum_{k=0} \Psi_k \frac{\Gamma(\nu_k+1)\sin\pi(\nu_k+1)}{\Gamma(\nu_k)\sin\pi\nu_k} \tau^{-\nu_k} + O(\tau^{-2-\nu_0})$$

$$\phi_S(y,\tau) = \sum_{k=0} \left\{ \int_y^1 S_k(z) G_0(y,z)\,dz + \int_0^y S_k(z) G_0(z,y)\,dz \right\} \tau^{-\mu_k}$$

$$+ \frac{1}{\tau} \sum_{k=0} \left\{ \int_y^1 S_k(z) G_1(y,z)\,dz + \int_0^y S_k(z) G_1(z,y)\,dz \right\} \tag{2.5b}$$

$$\left\{ \frac{\Gamma(\mu_k+1)\sin\pi(\mu_k+1)}{\Gamma(\mu_k)\sin\pi\mu_k} \tau^{-\mu_k} \right\} + O(\tau^{-2-\mu_0})$$

where
$$G_0(y,z) = y(1-z)$$

$$G_1(y,z) = G_0(y,z)\left\{1 - \frac{y^2}{6} - \frac{(1-z)^2}{6}\right\} .$$

Specifically, to ensure that $\sigma \to \infty$ and yet $\Phi_B \to 0$, choose $0 < \nu_0 < 1$. The first order problem is then

$O(1)$
$$T^0_{yy} - T^0_\tau = 0 \qquad 0 < y < 1$$

$$T^0(0,\tau) = \Phi_B(\tau) \qquad T^0(1,\tau) = 0 \qquad T^0(y,0) = \phi_0(y) .$$

For $\tau \to \infty$ the leading behaviour is then, from (2.5a)

$$T^0(y,\tau) \sim \Psi_0(1-y)\,\tau^{-\nu_0} . \tag{2.6}$$

So from (2.4a) we have $\frac{dR_0}{d\tau} \sim \Psi_0 \tau^{-\nu_0}$, hence

$$R_0(\tau) \sim \frac{\Psi_0}{1-\nu_0}\,\tau^{1-\nu_0} . \tag{2.7}$$

This completes the first order calculation*. The second order can now be calculated as

$$O(\varepsilon) \quad T^1_{yy} - T^1_\tau = 2R_0 T^0_\tau - y\frac{dR_0}{d\tau}T^0_y$$

$$\sim \Psi_0^2\left\{ y\left(\frac{1+\nu_0}{1-\nu_0}\right) - 2\nu_0 \right\}\tau^{-2\nu_0}$$

$$T^1(0,\tau) = T^1(1,\tau) = T^1(y,0) = 0 .$$

---

*The $O(1)$ system can be written in the original co-ordinates as

$$\sigma^2 T^0_{xx} - \left(\frac{x}{\sigma}\right)\frac{d\sigma}{d\tau}T^0_x = T^0_\tau \qquad (A)$$

This is precisely the equation one would write down to describe diffusion in a bar with one end stretching at a rate $\frac{d\sigma}{d\tau}$. An element at x would then experience a velocity $v(x) = \left(\frac{x}{\sigma}\right)\frac{d\sigma}{d\tau}$, giving rise to the convection terms in (A). Further. the factor σ^2 multiplying T^0_{xx} accounts for the change of thermal conductivity of the bar due to the redistribution of matter caused by the stretching.

From (2.5b) the leading behaviour is then

$$T^1(y,\tau) \sim \Psi_o^2 \left\{ y\left\{\frac{1}{6}\frac{1+\nu_o}{1-\nu_o} - \nu_o\right\} + y^2\nu_o - \frac{y^3}{6}\frac{1+\nu_o}{1-\nu_o} \right\} \tau^{-2\nu_o} . \qquad (2.8)$$

So from (2.4a) we have, providing $\nu_o \neq \frac{1}{2}$ *,

$$R_1(\tau) \sim \frac{\Psi_o^2}{1-2\nu_o}\left\{\nu_o - \frac{1}{3}\frac{1+\nu_o}{1-\nu_o}\right\}\tau^{1-2\nu_o} \qquad (2.9)$$

Clearly $R_1 = o(R_o)$ and $T^1 = o(T^o)$ for $\tau \to \infty$

For $k \geq 0$ the general problem is

$O(\varepsilon^{k+1})$

$$T^{k+1}_{yy} - T^{k+1}_{\tau} = \sum_{p=0}^{k}\left\{2R_p T^{k-p}_{\tau} - y\frac{dR_p}{d\tau}T^{k-p}_{y}\right\}$$

$$T^{k+1}(0,\tau) = T^{k+1}(1,\tau) = T^{k+1}(y,0) = 0 . \qquad (2.10)$$

A trivial induction on this system, using (2.5b) repeatedly, demonstrates that $R_{k+1} = o(R_k)$ and $T^{k+1} = o(T^k)$ as $\tau \to \infty$. In fact, it is easy to show that

$$R_k = O(\tau^{1-2k\nu_o})$$

$$T_k = O(\tau^{-2k\nu_o}) .$$

---

*If $\nu_o = \frac{1}{2}$ then (2.4a) can be integrated to give $R_1(\tau) \sim -\frac{1}{2}\Psi_o^2 \ln\tau$

Thus, even if ε is not small, but still 0(1), and the expansions $\phi = \sum_{k=0} \varepsilon^k T^{(k)}$ and $R = \sum_{k=0} \varepsilon^k R_k$ break down as perturbation expansions, they are, for ε fixed, asymptotic expansions for $\tau \to \infty$.

Finally, note that R_0 is the largest term in the expansion for R. Thus from (2.3) and (2.7) the leading behaviour of σ can be calculated as

$$\sigma \sim \sqrt{\frac{2\varepsilon\Psi_0}{1-\nu_0}}\ \tau^{\frac{1-\nu_0}{2}}, \quad \tau \to \infty .$$

This agrees with the asymptotic result (1.38) with $\varepsilon = 1$.

To determine its accuracy, the perturbation solution can be compared with a numerical calculation. Since, when σ is unbounded, asymptotic agreement as $\tau \to \infty$ has already been shown, it is sufficient to compare the solutions in a bounded case. For this case, moreover, the numerical scheme is most accurate and comparison most significant.

The system (2.10) can be used to generate such a perturbation solution. Take for simplicity

$$T^0(y,\tau) = e^{-\gamma^2\tau} \sin\gamma(1-y)$$

for $|\gamma| \neq n\pi,\ n = 0, 1 \ldots$. This satisfies the 0(1) equation and can be used to generate higher order terms. A simple calculation gives for the next order

$$T^1(y,\tau) = e^{-2\gamma^2\tau}\left\{y\cos\gamma(1-y) - \frac{\sin\sqrt{2}\gamma y}{\sin\sqrt{2}\gamma}\right\}$$

$$- e^{-\gamma^2\tau}\left\{y\cos\gamma(1-y) - \frac{\sin\gamma y}{\sin\gamma}\right\}.$$

To second order, therefore, the solution for the temperature is

$$\phi = T^0 + \varepsilon T^1 + O(\varepsilon^2). \qquad (2.11)$$

This solution satisfies the boundary conditions

$$\phi(0,\tau) = e^{-\gamma^2\tau}\sin\gamma \qquad \phi(1,\tau) = 0$$

$$\phi(y,0) = \sin\gamma(1-y) + \varepsilon\left\{\frac{\sin\gamma y}{\sin\gamma} - \frac{\sin\sqrt{2}\gamma y}{\sin\sqrt{2}\gamma}\right\} + O(\varepsilon^2)$$

and has the corresponding free boundary

$$\sigma = \left(1 + 2\varepsilon R_0 + 2\varepsilon^2 R_1 + O(\varepsilon^3)\right)^{\frac{1}{2}} \qquad (2.12)$$

where $R_0 = \frac{1}{\gamma}(1 - e^{-\gamma^2\tau})$

$$R_1 = \frac{1}{\gamma^2}(1 - e^{-2\gamma^2\tau})\left(\frac{\gamma}{\sqrt{2}}\frac{\cos\sqrt{2}\gamma}{\sin\sqrt{2}\gamma} - \frac{1}{2}\right)$$

$$+ \frac{1}{\gamma^2}(1 - e^{-\gamma^2\tau})\left(1 - \gamma\frac{\cos\gamma}{\sin\gamma}\right).$$

The numerical scheme, as outlined in Chapter III, can now be used with the given boundary conditions. The computed temperature and free boundary can then be compared with the perturbation results (2.11) and (2.12) respectively.

Choose $\gamma = 1$. The calculation is then carried out up to $\tau = 3$ at which time the boundary σ has essentially stopped growing. For $\varepsilon = .1$ only the $O(\varepsilon)$ correction is needed to achieve two figure accuracy for both the temperature and the free boundary. For $\varepsilon = .5$ two figure accuracy is maintained when the $O(\varepsilon^2)$ contribution is added. See Tables Ia, and Ib.

CASE II $\Phi_B(\tau) = O(1)$

For the case $\Phi_B(\tau) \sim \Psi_o + \Psi_1 \tau^{-\nu_1} + \ldots$, $\nu_1 > 0$, and $\tau \to \infty$, the results of case I can be applied with $\nu_o = 0$.

From (2.6) and (2.8)

$$\Phi = \{\Psi_o(1-y) + O(\tau^{-\nu_1})\} + \varepsilon\left\{\frac{\Psi_o^2}{3!}(y^3-y) + O(\tau^{-2\nu_1})\right\} + O(\varepsilon^2) \tag{2.13}$$

while from (2.7) and (2.9)

$$\frac{\rho}{\tau} = \{\Psi_o + O(\tau^{-\nu_1})\} + \varepsilon\left\{-\frac{\Psi_o^2}{3} + O(\tau^{-2\nu_1})\right\} + O(\varepsilon^2) . \tag{2.14}$$

For $\tau \to \infty$ these expansions can be compared with the exact solution (0.14) with $\Phi_B = \Psi_o$. First, using (0.14d), expand $\lambda(\Psi_o)$ in powers of ε

$$\lambda(\Psi_o) = \varepsilon\Psi_o - \frac{\varepsilon^2}{3}\Psi_o^2 + O(\varepsilon^3) \tag{2.15}$$

and note from (0.14b) that $r_N(\tau) = \lambda(\Psi_o)\tau$. Now, since $R \to \infty$, it follows from (2.3) that $r \sim \varepsilon R$ for $\tau \to \infty$. Therefore, from (2.14) and (2.15) we have

$$r(\tau) \sim r_N(\tau) \, , \quad \tau \to \infty \; .$$

Next, substitute (2.15) into (0.14a) and expand

$$\phi_N(\Psi_o; y) = \Psi_o(1-y) + \varepsilon \frac{\Psi_o^2}{3!}(y^3 - y) + O(\varepsilon^2) \; . \tag{2.16}$$

Comparing (2.13) and (2.16) we have

$$\phi(y,\tau) \sim \phi_N(\Psi_o; y) \, , \quad \tau \to \infty \; .$$

It is clear that these results are also valid for solutions to $B_I(0,0)$ with the same boundary condition. Thus, for both cases, the time-independent contribution to (2.13) and (2.14) can be summed and the perturbation expansion written as

$$\begin{aligned} \phi &= \phi_N(\Psi_o; y) + \Psi(y,\tau) \\ r &= \frac{\sigma(0)}{2} + \lambda(\Psi_o)\tau + \varepsilon W(\tau) \end{aligned} \tag{2.17}$$

where Ψ , $W \to 0$ for $\tau \to \infty$ and $\sigma(0)$ takes on the values 0 or 1 .

Providing $\Phi_B(\tau)$ is bounded and approaches a constant as $\tau \to \infty$, we have just shown that the perturbation solution has the correct asymptotic behaviour. However, it can happen that $\Phi_B(\tau)$ is bounded but that $\lim_{\tau\to\infty} \Phi_B(\tau)$ does not exist. For these oscillatory boundary conditions the regular perturbation method breaks down.

Consider, for example, the simplest case

$$\Phi_B(\tau) = \Psi_o + \mu \sin \Omega\tau \quad . \tag{2.18}$$

Here, take $\Psi_o > 0$ and $|\mu| < \Psi_o$. If the expansion procedure as outlined in Case I were used, the terms $2R\Phi_\tau$ and $y \frac{dR}{d\tau} \Phi_y$ in (2.4) would generate secular terms of the form $\tau^n \sin\Omega\tau,\ \tau^n \cos\Omega\tau,\ n=1,2$. Therefore, the perturbation expansion would not be uniformly valid for $\tau \to \infty$.

To remedy the problem, the expansions (2.17) for r and Φ can be used. Here, $\Phi_N(\Psi_o; y)$ and $\lambda(\Psi_o)$ are assumed to be derived from the time independent part of $\Phi_B(\tau)$ in (2.18). The important feature is that even though Φ_N and λ depend on ε , they are given by (0.14) and thus can be calculated prior to the perturbation expansion.

Put $\lambda_o = \lambda(\Psi_o)$. Then for either problem $B_I(1,0)$ or $B_I(0,0)$, substitute (2.17) in (0.13) to give

$$\Psi_{yy} + \lambda_o y \Psi_y - \{\sigma(o) + 2\lambda_o\tau\}\Psi_\tau$$
$$= \varepsilon\left\{2W\Psi_\tau - y\frac{dW}{d\tau}\Psi_y - y\frac{dW}{d\tau}\Phi_{N,y}\right\}$$

with the boundary conditions

$$\Psi(0,\tau) = \mu \sin \Omega\tau \qquad \Psi(1,\tau) = 0$$

and, if $\sigma(0) = 1$, the initial condition

$$\Psi(y,0) = \Phi_0(y) - \Phi_N(\Psi_0; y) \, . \tag{2.19a}$$

Finally, the flux condition becomes simply

$$-\Psi_y(1,\tau) = \frac{dW}{d\tau} \, . \tag{2.19b}$$

Now proceed as in case I and assume the expansions

$$\Psi(y,\tau) = \sum_{k=0} \varepsilon^k \Psi^k(y,\tau)$$
$$W(\tau) = \sum_{k=0} \varepsilon^k W_k(\tau) \, .$$

We show that the secular terms are no longer present. In particular, we derive a large τ expansion for W_0 and show that it is bounded for $\tau \to \infty$.

The $O(1)$ system is

$$\begin{aligned} &\Psi^0_{yy} + y\lambda_0 \Psi^0_y - \{\sigma(0) + 2\lambda_0\tau\}\Psi^0_\tau = 0 \\ &\Psi^0(0,\tau) = \mu \sin \Omega\tau \qquad \Psi^0(1,\tau) = 0 \end{aligned} \tag{2.20}$$

The behaviours of $B_I(1,0)$ and $B_I(0,0)$ for $\tau \to \infty$ are the same but the case $\sigma(0)=0$ is simpler to analyse. For this reason, choose $\sigma(0)=0$ in (2.20). Then no initial condition need be specified and the solution can be found first by taking the Mellin transform of (2.20):

$$\bar{\Psi}^{o}_{yy} + \lambda_o y \bar{\Psi}^{o}_{y} + 2\lambda_o p \bar{\Psi}^{o} = 0$$

$$\bar{\Psi}^{o}(0,p) = \mu \Omega^{-p} \Gamma(p) \sin \tfrac{\pi}{2} p$$

$$\bar{\Psi}^{o}(1,p) = 0$$

where $$\bar{\Psi}^{o}(y,p) = \int_0^\infty \tau^{p-1} \Psi^{o}(y,\tau)\, d\tau \ . \qquad (2.21)$$

The solution is

$$\bar{\Psi}^{o}(y,p) = \mu e^{-\frac{\lambda_o}{2} y^2} \Omega^{-p} \Gamma(p) \sin(\tfrac{\pi}{2} p)\, A(y,p)$$

where, with $M(a,b,z)$ the confluent hypergeometric function,

$$A(y,p) = M(\tfrac{1}{2}-p, \tfrac{1}{2}, \tfrac{\lambda_o}{2}) \left\{ \frac{M(\frac{1}{2}-p, \frac{1}{2}, \frac{\lambda_o}{2} y^2)}{M(\frac{1}{2}-p, \frac{1}{2}, \frac{\lambda_o}{2})} - \frac{M(1-p, \frac{3}{2}, \frac{\lambda_o}{2} y^2)}{M(1-p, \frac{3}{2}, \frac{\lambda_o}{2})} \right\} .$$

To get the solution to (2.20), invert (2.21)

$$\Psi^{o}(y,\tau) = \frac{1}{2\pi i} \int_\Delta \tau^{-p} \bar{\Psi}^{o}(y,p)\, dp \qquad (2.22)$$

where Δ is a Bromwich contour in the strip $-1 < \mathcal{R}p < \frac{1}{2}$. The left limit of the strip is dictated by the right most pole of $\Gamma(p) \sin \frac{\pi}{2} p$. The right limit arises from the convergence requirements at $y = 0$ as follows. For $\mathrm{Re}\, p = \Delta_o$, using Stirling's approximation for $\Gamma(p)$ we have

$$|\Gamma(p)\,(\Omega\tau)^{-p} \sin\tfrac{\pi}{2}p| = O\left(|p|^{\Delta_o - \frac{1}{2}}\right) \tag{2.23}$$

for $|\mathrm{Im}\, p| \to \infty$. Because of the oscillations of τ^{-p} along Δ, convergence of (2.21) for $y = 0$ then holds for $\Delta_o < \frac{1}{2}$.

Providing the integral (2.22) continues to converge, the contour Δ can be displaced to the right to derive a large τ expansion for ψ^o. To show this convergence, start with the estimates [17]

$$\begin{aligned} M(\tfrac{1}{2}-p, \tfrac{1}{2}, x) &\sim \frac{e^{\frac{x}{2}}}{\sqrt{\pi}} \cos\sqrt{2px} \\ M(1-p, \tfrac{3}{2}, x) &\sim \frac{e^{\frac{x}{2}}}{\sqrt{\pi}} \frac{\sin\sqrt{2px}}{\sqrt{px}} \qquad |p| \to \infty \end{aligned} \tag{2.24}$$

so that

$$A(y,p) \sim \frac{\sin\sqrt{2p\lambda_o}\,(1-y)}{\sin\sqrt{2p\lambda_o}}\; e^{\frac{\lambda_o}{4}y^2}, \quad |p| \to \infty .$$

Hence with $\phi = \arg p$,

$$|A(y,p)| = O\left(e^{-y\sqrt{2|p|\lambda_o}\,\sin\frac{\phi}{2}}\right) \tag{2.25}$$

for $|\mathrm{Im}\, p| \to \infty$ along any Bromwich contour. Finally, using (2.23), (2.25) and that $|\sin\frac{\phi}{2}| \to \frac{1}{\sqrt{2}}$ for $|\mathrm{Im}\, p| \to \infty$, the integrand of (2.22) damps like

$$|\tau^{-p}\bar{\Psi}^{o}(y,p)| = O(|p|^{\Delta_o - \frac{1}{2}} e^{-y\sqrt{|p|\lambda_o}})$$

for $|\mathrm{Im}\,p| \to \infty$ along $\mathrm{Re}\,p = \Delta_o$. For $y > 0$ this estimate allows the contour Δ to be shifted to the right and the contribution at $|\mathrm{Im}\,p| = \infty$ to be ignored.

From (2.19b), $\frac{dW_o}{d\tau} = -\Psi_y^o(1,\tau)$, so that

$$W_o(\tau) = -\int_0^{\tau} \frac{\partial \psi^o}{\partial y}(1,\bar{\tau})\, d\bar{\tau}$$

and from (2.22) we have, therefore

$$W_o(\tau) = \frac{1}{2\pi i}\int_{\Delta} \frac{\tau^{1-p}}{p-1} \frac{\partial \bar{\Psi}^o}{\partial y}(1,p)\, dp \; .$$

Displace the contour Δ to the right to pick up poles at $p = 1$, and on the set

$$M_{\lambda_o} = \{\alpha_n \mid M(1-\alpha_n, \tfrac{3}{2}, \tfrac{\lambda_o}{2}) = 0\}$$

The set M_{λ_o} is a positive, strictly increasing divergent sequence. Further, for all n, $\alpha_n = O(\frac{1}{\lambda_o})$ [17]. Since $\lambda_o = O(\varepsilon)$, even α_o is large, and the asymptotic estimates (2.24) can be used. Hence we have $\alpha_n \sim \frac{n^2\pi^2}{2\lambda_o}$ and

$$W_0(\tau) = -\mu\lambda_0 e^{-\frac{\lambda_0}{4}}\Big\{ M(\tfrac{1}{2},\tfrac{3}{2},\tfrac{\lambda_0}{2}) + M(-\tfrac{1}{2},\tfrac{1}{2},\tfrac{\lambda_0}{2})$$

$$+ 2\sum_{n=1}^{N} \frac{(-1)^n}{(\Omega\tau)^{\alpha_n}} \frac{\Gamma(\alpha_n)\sin\frac{\pi}{2}\alpha_n}{n\pi - \frac{2\lambda_0}{n\pi}} \Big\} + R_N(\tau) \qquad (2.26)$$

where

$$R_N(\tau) = \frac{1}{2\pi i}\int_{\Delta_N} \frac{\tau^{1-p}}{p-1} \frac{\partial \bar{\psi}^0}{\partial y}(1,p)\,dp$$

for Δ_N a Bromwich contour, in the strip $\alpha_N < \operatorname{Re} p < \alpha_{N+1}$. From the previous estimates it is easy to see that $R_N(\tau) = O(\tau^{-\alpha_0})$ and so (2.26) is an asymptotic expansion for $\tau \to \infty$. It is not convergent as $R_N(\tau) \to \infty$ for τ fixed, $N \to \infty$.

Note that for $\tau \to \infty$, W_0 is bounded and is $O(\varepsilon)$ uniformly. Thus, in the expansion for r

$$r = \lambda_0\tau + \varepsilon W_0 + \cdots$$

the term εW_0 is $O(\varepsilon^2)$ and is uniformly smaller than the first order term. The expansion for σ follows simply from (2.3).

The expansion (2.26) for W_0 holds as well for the problem $B_I(1,0)$. In this case, however, it is worthwhile to compare the perturbation solution with a numerical calculation to determine the accuracy for τ moderate.

Put $\sigma(0) = 1$ in (2.20); then an initial condition is required. For simplicity, choose

$$\phi_0(y) = \phi_N(\Psi_0 ; y) \,. \tag{2.27}$$

So that from (2.19a), $\psi^0(y,0) = 0$. The problem (2.20) can then be solved by an integral transform in y [18]. The solution is

$$\psi^0(y,\tau) = \sum_{n=1}^{\infty} \frac{c_n(\tau)\, E_n(y\sqrt{\lambda_0})}{N_n} \tag{2.28}$$

where for $\alpha_n \varepsilon M_{\lambda_0}$

$$c_n(\tau) = \frac{2\lambda_0 \mu}{(1+2\lambda_0\tau)^{\alpha_n}} \int_0^{\tau} (1+2\lambda_0\bar{\tau})^{\alpha_n - 1} \sin\Omega\bar{\tau}\, d\bar{\tau}$$

$$E_n(z) = z e^{-\frac{z^2}{4}} M(1-\alpha_n, \tfrac{3}{2}, \tfrac{z^2}{2})$$

$$N_n = 2\int_0^{\sqrt{\lambda_0}} e^{\frac{z^2}{2}} E_n^2(z)\, dz \,.$$

For λ_0 small and $1+2\lambda_0\tau \ll \alpha_0$, the following approximations are valid:

$$c_n(\tau) = \mu \frac{\sin\Omega\tau}{\alpha_n} + O\left(\frac{1+2\lambda_0\tau}{\alpha_n^2}\right)$$

$$E_n(z) = \frac{\sin z\sqrt{2\alpha_n}}{\sqrt{2\alpha_n}} + O\left(\frac{1}{\alpha_n}\right)$$

$$N_n \sim \frac{\sqrt{\lambda_0}}{2\alpha_n} \hat{N}_0 \,; \quad \hat{N}_0 = \int_0^1 e^{\frac{\lambda_0}{2}x^2} dx$$

and $\alpha_n \sim \dfrac{n^2\pi^2}{2\lambda_o}$. These approximations can be used in (2.28) to give

$$\psi^o(y,\tau) \sim \frac{2\mu}{\hat{N}_o} \sin\Omega\tau \sum_{n=1}^{\infty} \frac{\sin n\pi y}{n\pi} \quad .$$

This representation is valid away from $y = 0$ and can be summed to

$$\psi^o(y,\tau) \sim \mu \frac{\sin\Omega\tau}{\hat{N}_o}(1-y) \quad .$$

Hence, from (2.19b), $\dfrac{dW_o}{d\tau} \sim \mu \dfrac{\sin\Omega\tau}{\hat{N}_o}$, so that

$$W_o(\tau) \sim \frac{\mu}{\Omega\hat{N}_o}(1 - \cos\Omega\tau) \quad .$$

Finally, from (2.17)

$$\sigma(\tau) = \left(1 + 2\lambda_o\tau + \frac{2\varepsilon\mu}{\Omega\hat{N}_o}\{1 - \cos\Omega\tau\} + O(\varepsilon^2)\right)^{\frac{1}{2}} . \qquad (2.29)$$

The solution for the free boundary σ of $B_\gamma(1,0)$ with boundary condition (2.18) and initial condition (2.27) can now be calculated numerically and compared with (2.24). Rather than fix ε and solve the transcendental equation (0.14b) for λ_o , choose

λ_o and calculate the corresponding ε . For $\lambda_o = .1$ we get $\varepsilon = .1034$. Then with the correction term W_0 in (2.2a) two figure accuracy is achieved over the range $0 \le \tau \le 10$. See Table II.

CASE III $\Phi_B(\tau) > O(1)$

For this case, the regular perturbation procedure will not yield a uniformly valid series. Attempts at multi-scale expansions will also fail. A set of scale transformations can verify these statements.

Assume, for definiteness, that $\Phi_B(\tau) \sim \Psi_o \tau^\nu$, $\nu > 0, \tau \to \infty$
Make the following transformations:

$$\bar{\Phi} = \varepsilon \Phi$$
$$\bar{\tau}^\nu = \varepsilon \tau^\nu .$$

Hence $\bar{\tau} = \varepsilon^{\frac{1}{\nu}} \tau$ and $\frac{d\bar{\tau}}{d\tau} = \varepsilon^{\frac{1}{\nu}}$. Finally, put

$$\bar{r} = \varepsilon^{\frac{1}{\nu}} r .$$

Then $\frac{dr}{d\tau} = \frac{d\bar{r}}{d\bar{\tau}}$ and $r \frac{\partial}{\partial \tau} = \bar{r} \frac{\partial}{\partial \bar{\tau}}$. The system (0.13) becomes

$$\bar{\Phi}_{yy} = 2\bar{r}\,\bar{\Phi}_{\bar{\tau}} - y \frac{d\bar{r}}{d\bar{\tau}} \bar{\Phi}_y \qquad 0 < y < 1$$
$$\bar{\Phi}(0,\bar{\tau}) = \varepsilon\, \Phi_B(\varepsilon^{-\frac{1}{\nu}} \bar{\tau}) \qquad \bar{\Phi}(1,\bar{\tau}) = 0$$
$$\bar{\Phi}(y,0) = \varepsilon\, \Phi_o(y)$$

and $-\bar{\Phi}_y(1,\bar{\tau}) = \frac{d\bar{r}}{d\bar{\tau}}$.

Now we can write $\Phi_B(\tau) = \Psi_o \tau^\nu + \Phi_R(\tau)$ where for some

$\delta > 0$, $\Phi_B(\tau) \leq O(\tau^{\nu-\delta})$. It follows that

$$\varepsilon \Phi_B(\varepsilon^{-\frac{1}{\nu}}\bar{\tau}) = \Psi_0 \bar{\tau}^{\nu} + O(\varepsilon^{\frac{\delta}{\nu}} \bar{\tau}^{\nu-\delta}) .$$

Assume a regular expansion of the form

$$\bar{\Phi} = \sum_{k=0} \varepsilon^k T^k(y,\bar{\tau})$$

$$\bar{r} = \sum_{k=0} \varepsilon^k r_k(\bar{\tau}) .$$

The first order system is therefore (dropping the bars)

$$T^0_{yy} = 2r_0 T^0_\tau - \frac{dr_0}{d\tau} T^0_y \qquad 0 < y < 1$$

$$T^0(0,\tau) = \Psi_0 \tau^\nu \qquad T^0(1,\tau) = 0$$

$$T^0(y,0) = 0$$

and $$-T^0_y(1,\tau) = \frac{dr_0}{d\tau} .$$

Thus the first order system incorporates the leading growth behaviour. Any multiscale expansion procedure would have this system as its first order problem. But this system is as difficult to solve as the original one; hence a perturbation expansion is not useful here.

2. THE CASE $\varepsilon \rightarrow \infty$

Consider the semi-infinite bar. First note in the special case (0.10) and (0.11) that $\frac{d\sigma}{d\tau}$ admits a regular perturbation expansion in powers of ε^{-1}

$$\frac{d\sigma}{d\tau} = \frac{H_o}{\Phi_M}\left(1 - \frac{1}{\Phi_M}\varepsilon^{-1} + O(\varepsilon^{-2})\right) .$$

Hence, we do not expect the expansion in the general case to be of a singular nature.

Let $\varepsilon \to \frac{1}{\varepsilon}$ in (0.10), to give the system

$$\begin{aligned}
&\phi_{yy} = \phi_\tau - \frac{d\sigma}{d\tau}\phi_y \\
&\phi(0,\tau) = \Phi_M \quad \phi_y(0,\tau) + H(\tau) = \varepsilon\frac{d\sigma}{d\tau} \\
&\phi(y,0) = \phi_o(y) \quad \phi(\infty,\tau) = 0 .
\end{aligned} \tag{2.30}$$

Examine the melting case where $\Phi_M > 0$. Then, to support melting, $H(\tau) > 0$.

Now assume a regular expansion for ϕ and σ .

$$\phi = \sum_{k=0}^{\infty} \varepsilon^k T^k \qquad \sigma = \sum_{k=0}^{\infty} \varepsilon^k \sigma_k$$

Then the 0(1) system is

$$\begin{aligned}
&T^o_{yy} = T^o_\tau - \frac{d\sigma_o}{d\tau} T^o_y \qquad y > 0 \\
&T^o(0,\tau) = \Phi_M \quad T^o(\infty,\tau) = 0 \\
&T^o(y,0) = \phi_o(y) .
\end{aligned} \tag{2.31}$$

The flux condition at y = 0 no longer contains σ explicitly. The dependence can be made explicit, however, by using (2.31) at y = 0 to get

$$\frac{1}{H(\tau)} T^{o}_{yy}(o,\tau) = \frac{d\sigma_o}{d\tau} .$$

This system is no simpler than the original one, hence again a perturbation expansion is not useful. Similar comments hold for the finite bar.

SUMMARY

For the one-dimensional free boundary problem we have shown, for a selected set of boundary conditions, when a feasible perturbation expansion is possible. For other situations, eg. $H \neq 0$ in (0.12c), or the flux rather than the temperature specified at $x = 0$ in (0.12b), the approach is the same and the details can be worked out.

Finally, for the case $\varepsilon \to 0$, the expansion techniques of this chapter can be applied to an n-dimensional free-boundary problem providing the system is rotationally invariant in n-dimensions. Then the equations are, with $H(\tau) \equiv 0$.

$$\phi_{rr} + \frac{n}{r}\phi_r = \phi_\tau \qquad 0 < r < \sigma$$

$$\phi(\sigma,\tau) = \Phi_M \qquad \pm\, \phi_r(\sigma,\tau) = \frac{d\sigma}{d\tau}$$

$$\phi(r,o) = \phi_o(r)$$

With the change of variables $y = \frac{r}{\sigma}$ the equations become

$$\phi_{yy} + \frac{n}{y}\phi_y = 2R\phi_\tau - y\frac{dR}{d\tau}\phi_y$$

$$\phi(1,\tau) = \Phi_H \qquad \pm\ \varepsilon\,\phi_y(1,\tau) = \frac{dR}{d\tau}$$

$$\phi(y,0) = \phi_0(y)$$

where $R = \frac{1}{2}\sigma^2$ and $\sigma(0) = 1$. This problem has the same structure as those already studied and a similar analysis can be carried out.

CHAPTER III: NUMERICAL METHODS

The objective of any numerical calculation for the free boundary problem is first to locate the boundary $\sigma(\tau)$ at time τ and then to represent the solution $\phi(x,\tau)$ at that time. For the method presented in this chapter we accomplish this objective by the following steps. (See Figure 3)

(a) Given the requirement to solve the problem on $[0,T]$, choose a partition $\tau_0 < \tau_1 < \cdots < \tau_N$ where $\tau_0 = 0$ and $\tau_N = T$. Let $I_n = [\tau_{n-1}, \tau_n]$.

(b) Pick an R-parameter family of $C^\infty(0,\infty)$ curves $\sigma(a_1, \cdots a_R; \tau)$. Then represent σ on $[0,T]$ by

$$\sigma^{(N)} = \sum_{n=1}^{N} \chi_n \, \sigma(a_1^n \cdots a_R^n ; \tau - \tau_{n-1})$$

where χ_n is the characteristic function with support on I_n and a_i^j is the value of the i^{th} parameter in the j^{th} interval. The method for determining these values is yet to be specified. Let

$\sigma_n(\tau) = \sigma(a_1^n \cdots a_R^n ; \tau - \tau_{n-1})$.

(c) For each I_n calculate a solution $\phi^n(x, \tau - \tau_{n-1})$ to the heat equation on the domain

$$D_n = \{ (x,\tau) \mid \tau \in I_n ; x \in (0, \sigma_n(\tau)) \}$$

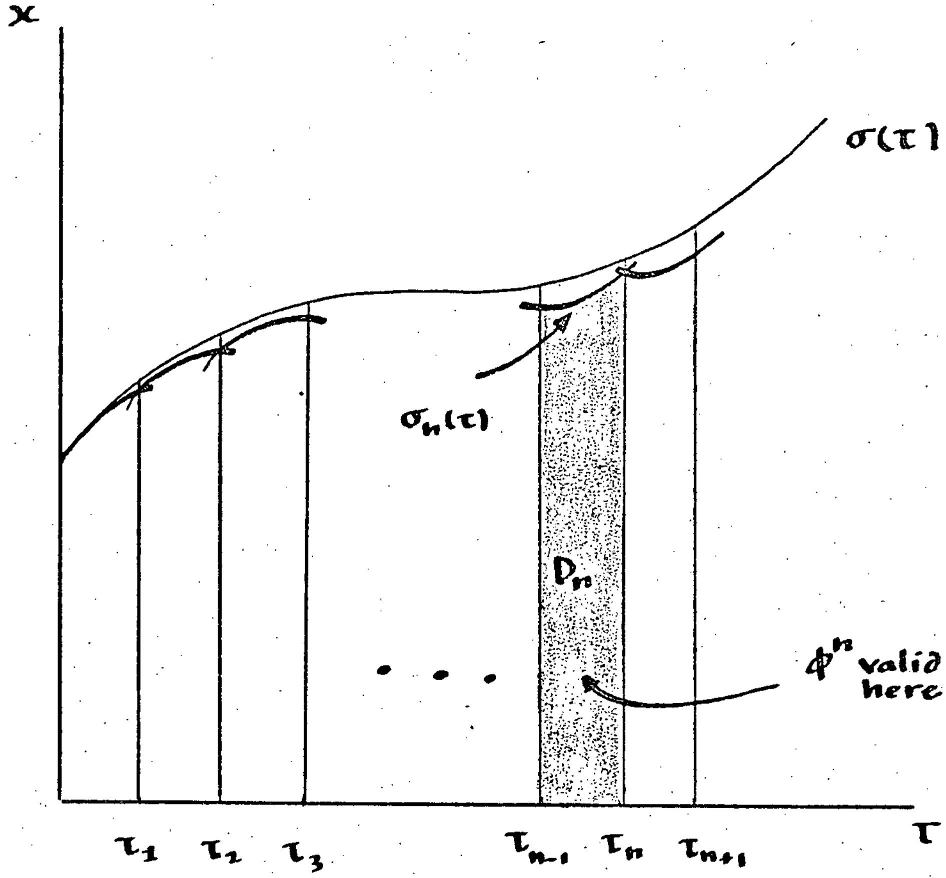

Figure 3. A Numerical Scheme for the Stefan Problem

Subject to the conditions

$$\phi^n_x(0, \tau-\tau_{n-1}) = H_S(\tau) \qquad \phi^n(\sigma_n(\tau), \tau-\tau_{n-1}) = 0$$

$$\phi^n(x, 0) = \phi^{n-1}(x, \tau_{n-1}-\tau_{n-2}) \quad 0 \leq x \leq \sigma_n(0)$$

This task is a particular example of the so-called inverse Stefan problem. The solution ϕ^n is the analytic interpolation function to be used on the interval I_n. Note that the function has been defined on the interval $[0, \tau_n - \tau_{n-1}]$; thus for numerical purposes, ϕ^n need be calculated only for small times.

(d) Determine the parameters $\{a_j^n\}_{j=1}^{R}$ by requiring that $\sigma^{(N)}$ be $C^R[0,T]$. Here, use the results of (c) together with the flux condition at $x = \sigma$ to determine these values. (More details later.)

To demonstrate the method we do numerical calculations for both $B_I(1,0)$ and $B_{II}(1,0)$.

1. HEAT FLUX SPECIFIED AT $x = 0$

The difficult step in this scheme is step (c) since , in general, no analytic solutions to the inverse problem have been found. However, such interpolating solutions can be calculated for a special three parameter family of curves. To find these special solutions it is convenient to use the fixed boundary representation (0.13) and then solve the system subject to all the boundary conditions except (0.13c). Even though the domain is rectangular, the equation cannot be solved by integral transforms in y or τ because the coefficient of ϕ_y is a function of both y and τ. To eliminate this derivative put $\phi = v e^{w}$. Then the equation becomes

$$\{v_{yy} - v_\tau\} + v_y\{2w_y + \frac{dr}{d\tau} y\}$$
$$+ v\{w_{yy} - 2r w_\tau + \frac{dr}{d\tau} y w_y + w_y^2\} = 0$$

where $r = \frac{1}{2}\sigma^2$. The derivative v_y can be eliminated by setting $w_y = -\frac{\dot{r}}{2} y$, so that $w = -\frac{\dot{r}}{4} y^2 + F(\tau)$ where F is an arbitrary function. The coefficient of v then becomes

$$\{\tfrac{1}{2} r\ddot{r} - \tfrac{1}{4}\dot{r}^2\} y^2 - \{\tfrac{1}{2}\dot{r} + 2r\dot{F}\} .$$

First set $\dot{F} = -\frac{1}{4}\frac{\dot{r}}{r}$. The τ-dependence can then be eliminated provided

$$r\ddot{r} - \tfrac{1}{2}\dot{r}^2 = 2c_0 = \text{const.} \qquad (3.1)$$

Differentiate this expression with respect to τ to get $\dddot{r} = 0$. Integrating three times, we have then

$$\sigma = (a\tau^2 + b\tau + c)^{\frac{1}{2}} .$$

The transformation of the dependent variable can now be written

$$\phi = \frac{v}{\sqrt{\sigma}}\, e^{-\frac{\dot{r}}{4} y^2} \tag{3.2}$$

where v satisfies

$$v_{yy} - c_0 y^2 v = 2r v_\tau$$

$$v_y(0,\tau) = -H_B(\tau)\,\sigma(\tau)^{\frac{3}{2}} \equiv G(\tau) \qquad v(1,\tau) = 0$$

$$v(y,0) = \phi_0(y)\,\sigma(0)^{\frac{1}{2}} \quad .$$

For the numerical calculation we choose a linear boundary $\sigma = a + b\tau$, which is contained in the three parameter family just derived. For this case it follows from (3.1) that $c_0 = 0$ and the equation for v reduces to

$$v_{yy} = 2r v_\tau \quad . \tag{3.3}$$

In the k^{th} time step v then satisfies the boundary conditions

$$v_y^k(0,\tau) = -H_B(\tau + \tau_{k-1})\,\sigma_k(\tau)^{\frac{3}{2}} \equiv G_k(\tau)$$

$$v^k(1,\tau) = 0 \tag{3.4}$$

$$v^k(y,0) = \sigma_k(0)^{\frac{1}{2}}\,\phi^{k-1}(y\sigma_k(0), \tau_{k-1} - \tau_{k-2})\, e^{\frac{\dot{r}_k(0)}{4} y^2} \equiv \Phi_0^k(y)$$

where

$$\sigma_k(\tau) = a_k + b_k(\tau - \tau_{k-1}) \quad , \qquad r_k = \frac{1}{2}\sigma_k^2 \quad .$$

Here a_k is determined by continuity

$$a_k = \sigma_k(\tau_{k-1}) = \sigma_{k-1}(\tau_{k-1})$$

and b_k is determined by matching the flux condition at τ_{k-1} :

$$b_k = -\varepsilon\,\phi_x^{k-1}(\sigma_{k-1}(\tau_{k-1}),\tau_{k-1}) . \qquad (3.5)$$

Since ϕ^{k-1} has already been calculated from the previous step and does not involve b_k , the equation is as explicit as it appears.

Now, omit reference to the time step. Return to (3.3) and put

$$\mu = \int_0^\tau \frac{d\bar{\tau}}{\sigma^2(\bar{\tau})} = \frac{\tau}{a(a+b\tau)} .$$

Then (3.3) becomes simply $v_{yy} = v_\mu$. This can be solved by a Fourier transform in y . With $\nu_n = (n-\frac{1}{2})\pi$, $n = 1, 2, \cdots$ and

$v_n = \int_0^1 v \cos\nu_n y \, dy$ we have

$$\frac{d}{d\mu} v_n + \nu_n^2 v_n = G$$

which has the solution

$$v_n = e^{-\nu_n^2 \mu} v_n(0) + I_n(\mu) \qquad (3.6)$$

where $v_n(0) = \int_0^1 \Phi_0(y) \cos\nu_n y \, dy$ (3.7)

and $I_n(\mu) = \int_0^\mu G(\bar{\mu})\, e^{-\nu_n^2(\mu-\bar{\mu})} \, d\bar{\mu}$. (3.8)

The initial value integral (3.7) can be approximated by the simple quadrature

$$v_n(0) \sim \sum_{k=0}^{N-1} \left\{ \Phi_k \int_0^h \cos\nu_n(x_k+x)\,dx + \frac{\Phi_{k+1}-\Phi_k}{h} \int_0^h x\cos\nu_n(x_k+x)\,dx \right.$$

where $x_k = (k-1)h$, $h = \frac{1}{N-1}$ and $\Phi_k = \Phi_0(x_k)$. This is a simplified Filon quadrature which was found to be adequate for the present calculations. For $N = 10$ the quadrature was accurate to four significant figures over the range $0 \le \nu_n \le 25$.

As for the boundary integral (3.8), note that the calculated solution is to be used only for small time increments, hence small μ. Since σ has already been approximated by a linear function in these time steps, there is no significant added error in linearizing the boundary condition at $x = 0$ as well. Hence, represent $G(\bar{\mu})$ by the linear approximation

$$G(\bar{\mu}) \sim G(0) + \frac{G(\mu)-G(0)}{\mu}\,\bar{\mu}, \quad 0 \le \bar{\mu} \le \mu .$$

The integration can then be carried out to give

$$I_n(\mu) \sim \frac{G(0)}{\nu_n^2}\left\{1-e^{-\nu_n^2\mu}\right\} + \frac{G(\mu)-G(0)}{\nu_n^2\mu}\left\{\mu - \frac{1}{\nu_n^2}\left\{1-e^{-\nu_n^2\mu}\right\}\right\}.$$

Finally, the solution to (3.3) has the expansion

$$v(y,\mu) = 2\sum_{n=1}^{\infty} v_n(\mu)\cos\nu_n y .$$

So from (3.2) the interpolating function ϕ can be written in the original co-ordinates as

$$\phi(x,\tau) = \frac{2}{\sqrt{a+b\tau}} e^{-\frac{b}{4}\frac{x^2}{a+b\tau}} \sum_{n=1}^{\infty} v_n\left(\frac{\tau}{a(a+b\tau)}\right) \cos\left(\nu_n \frac{x}{a+b\tau}\right) .$$

This expression can be differentiated term by term to find $\phi_x(\sigma,\tau)$ as required in (3.5).

With the above representation for the ϕ^k , the method can be checked against a Neumann solution similar to (0.14)

$$\phi_N(H_o; y) = H_o \int_y^1 e^{-\frac{\lambda}{2}\xi^2} d\xi$$

where $y = \frac{x}{\sqrt{2\lambda\tau}}$ and λ satisfies the transcendental equation

$$\varepsilon H_o = \lambda e^{\frac{\lambda}{2}} .$$

Initialize time at $\tau = 1$ and choose $\lambda = .5$ and $H_o = 1$. Then $\varepsilon = .6420$ and ϕ_N is a solution of the free boundary problem with the heat flux

$$H_B(\tau) = (1+\tau)^{-\frac{1}{2}}$$

and the initial condition

$$\phi_o(x) = \int_x^1 e^{-\frac{\xi^2}{4}} d\xi , \quad 0 \le x \le 1 .$$

The corresponding free boundary is

$$\sigma(\tau) = (1+\tau)^{\frac{1}{2}} .$$

For the numerical calculation equal time steps are chosen and the temperature is computed at fixed $y = \frac{x}{\sigma}$ intervals. In this co-ordinate the exact solution is time independent and comparison with the exact solution is easier. See Table III.

With a step size of $\Delta\tau = .1$ the calculated temperature was accurate to within 1% except near $x = 0$. The error is largest at this boundary because the Fourier series representation for the temperature converges most slowly. The error in the free boundary σ is larger; it grows as $.03\tau$.

2. TEMPERATURE SPECIFIED AT $x = 0$

For this case, to determine the interpolating function we must again solve (3.3) for $\mathcal{V}$. The boundary conditions (3.4) are the same except that at $y = 0$

$$\mathcal{V}^k(o,\tau) = \Phi_B(\tau + \tau_{k-1})\,\sigma_k(\tau)^{\frac{1}{2}} \equiv P_k(\tau) \;.$$

Hereafter, omit reference to the time step. Solving the equation by Fourier transforming in y leads to a sine series representation for ϕ. This series, however, is not suitable for numerical calculation. At $x = 0$ it does not converge to $\phi(o,\tau)$ and converges only very slowly for x near zero. Further, the series representation for Φ_x at $x = a + b\tau$ does not converge at all.

These problems can be eliminated when (3.3) is solved instead by a Laplace transform in the variable

$$\mu = \frac{\tau}{a(a+b\tau)} \; .$$

The solution can then be written

$$v = v_{Ic} + v_{Bv}$$

where v_{Ic} and v_{Bv} are defined in Appendix II. Here identify $\Phi_B(\tau)$ of the appendix with $P(\tau)$.

Consider first the boundary term v_{Bv} . Expand the integrand of (II.3)

$$\frac{\sinh\sqrt{p}\,(1-y)}{\sinh\sqrt{p}} = \sum_{k=0} \left\{ e^{-\sqrt{p}\,(2k+y)} - e^{-\sqrt{p}\,(2k+2-y)} \right\}$$

and use

$$\frac{1}{2\pi i}\int_{\Gamma} \overline{P}(p)\, e^{-\sqrt{p}\,\Omega}\, e^{p\mu}\, dp$$

$$= \frac{\Omega}{2\sqrt{\pi}} \int_0^{\mu} \frac{P(\mu-\bar{\mu})}{\bar{\mu}^{\frac{3}{2}}}\, e^{-\frac{\Omega^2}{4\bar{\mu}}}\, d\bar{\mu}$$

to get the representation

$$v_{Bv} = \frac{1}{\sqrt{\pi}} \sum_{k=0} \left\{ \frac{2k+y}{2\sqrt{\mu}}\, e^{-\frac{(2k+y)^2}{4\mu}} \int_0^{\infty} \frac{e^{-\frac{(2k+y)^2}{4\mu}\,e}}{\sqrt{1+e}}\, P\!\left(\frac{\mu e}{1+e}\right) de \right.$$

$$\left. - \frac{2k+2-y}{2\sqrt{\mu}}\, e^{-\frac{(2k+2-y)^2}{4\mu}} \int_0^{\infty} \frac{e^{-\frac{(2k+2-y)^2}{4\mu}\,e}}{\sqrt{1+e}}\, P\!\left(\frac{\mu e}{1+e}\right) de \right\} . \tag{3.9}$$

As in the previous problem the boundary condition at $x = 0$ can be linearized. Here, using the approximation

$$P\left(\frac{\mu e}{1+e}\right) \sim P(0) + \{P(\mu) - P(0)\}\frac{e}{1+e}$$

leads to the problem of evaluating integrals of the form

$$I_{ab}(\lambda) = \int_0^\infty e^a (1+e)^b e^{-\lambda e} \, de$$

for $a = 0, 1$; $b = -\frac{1}{2}, \frac{3}{2}$. Rather than evaluate the integrals directly, use the identity

$$I_{ab}(\lambda) = \Gamma(1+a) \, U(1+a, 2+a+b, \lambda) .$$

A power series representation for U can then be used when $\lambda \leq 5$. For $\lambda > 5$ a one term asymptotic expansion for $\lambda \to \infty$ gives an accurate result. (See [17].)

For the initial value term a similar calculation yields the representation

$$v_{IC} = \frac{1}{2\sqrt{\pi\mu}} \sum_{k=0} \int_0^1 \Phi_0(z) \qquad (3.10)$$

$$\Big\{ \theta(z-y) \Big\{ e^{-\frac{(z+2k-y)^2}{4\mu}} + e^{-\frac{(z+2k-2-y)^2}{4\mu}} \Big\}$$

$$+ \theta(y-z) \Big\{ e^{-\frac{(z-2k-y)^2}{4\mu}} + e^{-\frac{(z+2k+2-y)^2}{4\mu}} \Big\}$$

$$- \Big\{ e^{-\frac{(z+2k+y)^2}{4\mu}} + e^{-\frac{(z-2k-2+y)^2}{4\mu}} \Big\} \Big\} dz .$$

To calculate v_{Ic} , integrals of the type

$$I = \int_0^1 \phi(z)\, e^{-\beta(z+a)^2}\, dz \qquad \beta > 0$$

must to evaluated. For a large range of β the Filon quadrature

$$I \sim \sum_{k=0}^{N-1} \phi_k \int_0^h e^{-\beta(z+z_k+a)^2}\, dz + \sum_{k=0}^{N-1} \frac{\phi_{k+1}-\phi_k}{h} \int_0^h e^{-\beta(z+z_k+a)^2}\, z\, dz$$

gives good results. Here $h = \frac{1}{N-1}$, $z_k = kh$ and $\phi_k = \Phi(z_k)$.

The expansions (3.9) and (3.10) together give an expansion for v . The expansion for ϕ can then be found from (3.2) .

With this method of representation the numerical calculation was checked against two exact solutions. See Tables IVa, and IVb.

(a) A Stefan solution with $\frac{d\sigma}{d\tau} < 0$.

$$\phi(x,\tau) = 1 - e^{x+\tau-1}$$

$$\sigma(\tau) = 1 - \tau$$

(b) A Neumann solution with $\frac{d\sigma}{d\tau} > 0$.

In (0.14) initialize time at $\tau = 1$ and take the particular case

$$\phi(x,\tau) = \int_{\frac{x}{\sqrt{1+\tau}}}^{1} e^{-\frac{\xi^2}{4}} d\xi$$

$$\sigma(\tau) = \sqrt{1+\tau} \quad .$$

The calculation was done with step sizes $\Delta\tau$ between .1 and .5 . Then with at most two terms in the series (3.9) and (3.10) the accuracy for the temperature was .5% while for the free boundary σ an even higher accuracy of .1% was achieved.

3. DESCRIPTION OF A FREEZING LAKE

The numerical technique can be applied to a special two phase problem - the freezing of a shallow lake. (Fig. 4) A lake is considered shallow when during the period of freezing the effect of the lake bottom must be taken into account.

Let T_o be the period in which freezing takes place. Then there are two possibilities:

(a) If $s(T_o) \sim o$, clearly the bottom must be included in the description. For the lake we study, this is not the case.

(b) Even if $s(T_o) \sim H$ the heat flux from the bottom of the lake may affect, through diffusion, the water temperature near the top of the lake.

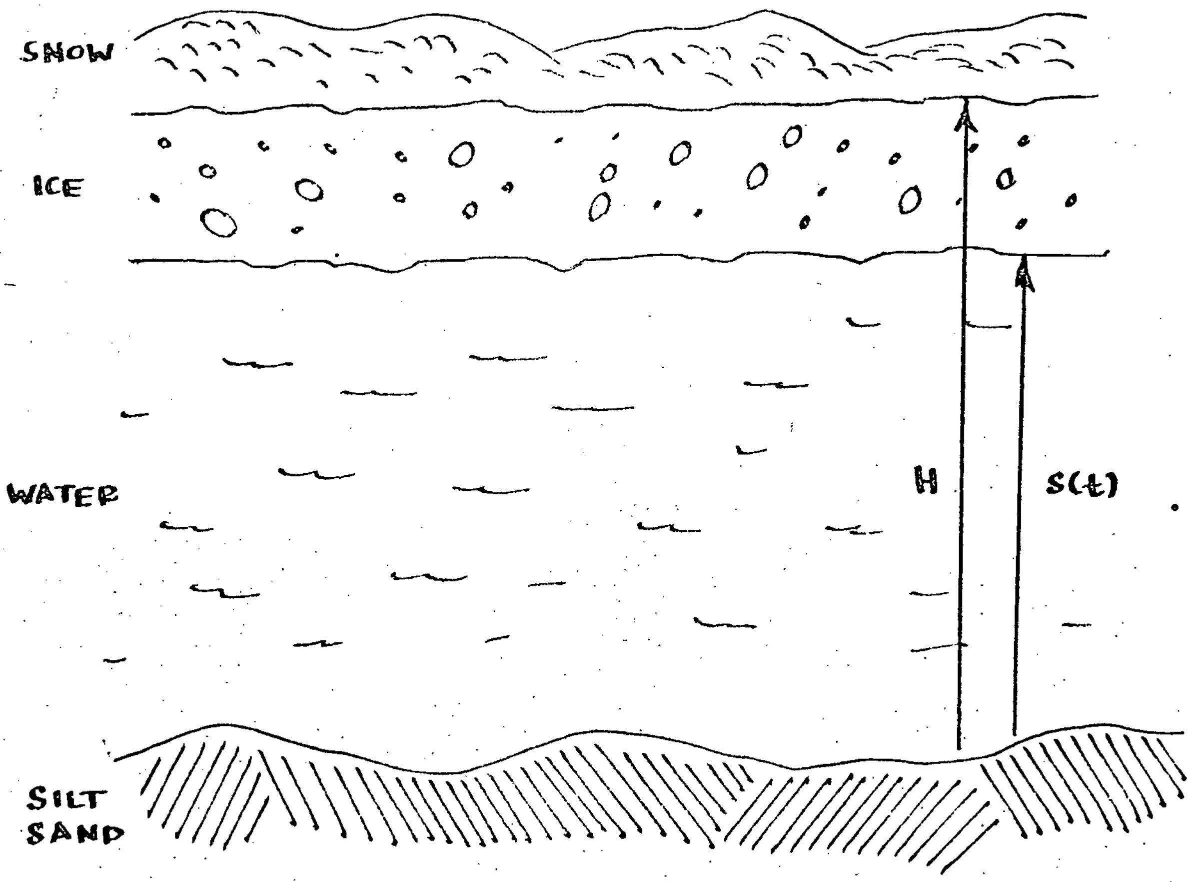

Figure 4 Cross-section of a shallow lake

Consider the second effect. In the limit $S(T_o) \sim H$ the water phase can be approximated by the fixed boundary problem

$$K_w \frac{\partial^2 \phi_w}{\partial x^2} = \frac{\partial \phi_w}{\partial t} \qquad 0 < x < H$$

with the boundary conditions

$$-k_w \frac{\partial \phi_w}{\partial x}(H,t) = H_{TOP}$$

$$-k_w \frac{\partial \phi_w}{\partial x}(0,t) = H_{BOT}$$

where K_w is the diffusivity and k_w is the conductivity of water. The physical heat fluxes are approximated by their time averages over the freezing period. Let $R = \frac{H_{TOP}}{H_{BOT}} = const.$

The solution can be written as

$$\begin{aligned}\phi_w &= \frac{1}{k_w}\{H_{BOT}\,\Phi(H-x,t) + H_{TOP}\,\Phi(x,t)\} \\ &= \phi_{BOT}(x,t) + \phi_{TOP}(x,t)\end{aligned}$$

where

$$\Phi(x,t) = \frac{1}{2}\left(\frac{K_w}{\pi}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \int_0^t \left\{ e^{-\frac{[(2k+1)H+x]^2}{4(t-\bar{t})}} + e^{-\frac{[(2k+1)H-x]^2}{4(t-\bar{t})}} \right\} \frac{d\bar{t}}{\sqrt{t-\bar{t}}} .$$

The effect of the bottom will become significant for a depth H such that

$$k_w \left| \frac{\partial \phi_{BOT}}{\partial x}(H,T_o) \right| \sim k_w \left| \frac{\partial \phi_{TOP}}{\partial x}(H,T_o) \right| .$$

This will hold only if it holds for the $k = 0$ terms in the sums for ϕ_{TOP} and ϕ_{BOT}. For $\epsilon = \frac{H^2}{4K_w T_o}$ it follows that

$$\sqrt{e}\, e^{-e} \int_{\sigma}^{\infty} \frac{e^{-eu}}{\sqrt{1+u}}\, du \sim R$$

which can be rewritten as

$$1 - 2\left(\frac{e}{\pi}\right)^{\frac{1}{2}} M\left(1, \tfrac{3}{2}, e\right) \sim R \tag{3.11}$$

For a given lake this equation can be solved for e and the critical H determined. If the lake is shallower than this H, it can be considered a "shallow" lake.

We apply the numerical method to an analytic model of Seneca Lake in northern Michigan [19]. For this lake, $R \sim .1$ so from (3.11), $e \sim 1.5$ which implies a critical depth of 10 meters. Seneca Lake is 2 meters deep and so can be considered shallow.

To set up the model, first non-dimensionalize with the water parameters

$$y = \frac{x}{H} \qquad \tau = \frac{k_W}{H^2} t \qquad \sigma = \frac{s}{H}$$

$$T = \frac{\phi - \Phi_M}{\Phi_M}$$ for both water and ice temperatures.

For the ice, the equations are

$$\frac{\partial^2 T_I}{\partial y^2} = \mu \frac{\partial T_I}{\partial \tau} \qquad \sigma < y < 1 \qquad \mu = \frac{k_W}{k_I} = .1252$$

$$T_I(\sigma, \tau) = 0 \tag{3.12}$$

$$T_I(1,\tau) = \frac{\Phi_B(\tau) - \Phi_M}{\Phi_M} \equiv T_B(\tau) \quad .$$

When the data for Seneca Lake was accumulated the ice was covered by 20 cm. of snow. As a result the temperature Φ_B at the top of the ice remained constant throughout the freezing period.

Since $\sigma(0) = 1$, no initial condition need be specified for the ice.

For the water the equations are:

$$\frac{\partial^2 T_W}{\partial y^2} = \frac{\partial T_W}{\partial \tau} \quad 0 < y < \sigma$$

$$T_W(\sigma,\tau) = 0$$

$$\frac{\partial T_W}{\partial y}(0,\tau) = H_B(\tau)\frac{H}{k_W \Phi_M} \equiv J_B(\tau) \quad .$$

For Seneca Lake the stored summer heat provided a heat flux $H_B(\tau) \simeq 10^{-4}\ cal/cm^2\text{-}sec$. This flux overshadows the ever-present geothermal gradient of $10^{-6}\ cal/cm^2\text{-}sec$.

An initial condition is required for the water

$$T_W(y,0) = T_W^0(y) \, , \quad 0 \leq y \leq 1 \quad .$$

Finally, the flux condition at the interphase boundary $\sigma(\tau)$ is

$$\frac{d\sigma}{d\tau} = \gamma_I \frac{\partial T_I}{\partial y}(\sigma,\tau) - \gamma_W \frac{\partial T_W}{\partial y}(\sigma,\tau) \tag{3.13}$$

where $$\gamma_I = \left(\frac{\Phi_M}{\rho L}\right)\frac{k_I}{K_W} = 13.75$$

$$\gamma_W = \left(\frac{\Phi_M}{\rho L}\right)\frac{k_W}{K_W} = 3.73 .$$

A two-phase calculation where both the water and ice temperatures are calculated numerically leads to difficulties. With $\mu \approx .1$ in (3.12) the ice temperature propagates almost ten times faster than the water temperatures and so makes the calculation awkward.

This difficulty can be turned to an advantage, however, for the ice equillibrates ten times faster as well. Thus, it is a good approximation to represent the ice solution by a perturbation series in μ and match it to the numerical solution for the water at every time step. This results in an equivalent one phase problem for the water with a modified latent heat and extra heating terms.

Assume a perturbation expansion for (3.12)

$$T_I = T_I^0 + \mu T_I^1 + \ldots$$

to get

$$T_I = T_B\frac{y-\sigma}{1-\sigma} + \frac{\mu}{3!}\left\{\frac{\dot{T}_B}{1-\sigma}(y-\sigma)^3 - \frac{T_B\dot{\sigma}}{(1-\sigma)^2}(1-y)^3 - (\dot{T}_B + T_B\dot{\sigma})y + (T_B\dot{\sigma} + \dot{T}_B\sigma)\right\}$$
$$+ O(\mu^2) .$$

Therefore

$$\frac{\partial T_I}{\partial y}(\sigma,\tau) = \frac{T_B}{1-\sigma} - \mu\left\{\tfrac{2}{3}T_B\dot{\sigma} + \tfrac{1}{6}\dot{T}_B\right\} + O(\mu^2) .$$

This expression can be used in (3.13) to give the flux condition

$$L_\mu \frac{d\sigma}{d\tau} = -\gamma_W \frac{\partial T_W}{\partial y}(\sigma,\tau) + H_\mu$$

where $$L_\mu = 1 + \frac{2}{3}\mu T_B + O(\mu^2)$$

is the effective latent heat and

$$H_\mu = \gamma_I \left\{ \frac{T_B}{1-\sigma} - \frac{\mu}{6}\dot{T}_B + O(\mu^2) \right\}$$

is the added heat source. This flux condition can now be used in the numerical solution for the water temperature.*

For the freezing case, $T_B < 0$, hence $L_\mu < 1$ and $H_\mu < 0$ for small μ. Both of these effects enter the flux condition so as to make $\frac{d\sigma}{d\tau}$ more negative as T_B becomes more negative. In physical terms, the lower the temperature above the ice, the faster the lake freezes over.

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* To generate higher orders in μ , note that the expressions are to be used in conjunction with a linear boundary for a given time step. Thus in the perturbation calculation there is no significant added error in setting

$$\frac{d^k\sigma}{d\tau^k} = \frac{d^k T_B}{d\tau^k} = 0 \quad k \geq 2.$$

APPLICATION TO SENECA LAKE [19]

On December 2nd, after 25 days of alternate freezing and thawing, a permanent ice cover of approximately 15 cm. formed on Seneca Lake. The numerical calculation was initialized to this day. As the ice grew during the month of December, the temperature of the water near the bottom rose from 1°C (Dec. 2) to near 4°C (Dec 26). At 4°C, water is most dense so that any further heating from the bottom of the lake results in convection rather than an increase in water temperature. This convection is beyond the scope of the model; hence the calculation was terminated on December 26th.

Further aspects of the calculation are the following:

(a) The heat flux at the bottom was determined by soil temperature profiles beneath the lake. In the period Dec 2 - 26 the total heat released to the water was 303 cal/cm^2 to give an average heat flux of 1.45×10^{-4} cal/cm^2-sec. In dimensionless units, $J_B \approx .0735$

(b) Despite the large fluctuations in air temperature, a 20 cm. snow cover kept the temperature at the top of the ice very nearly a constant -1°C. In dimensionless units $T_B \approx -.018$

(c) The calculation was started when the ice was 15cm. thick or in dimensionless units $\sigma(0) \approx .925$

At this time the water temperature was linear in the depth; for the calculation, the initial temperature was taken to be

$$T_w(y,0) \approx .018\,(.925 - y)\,.$$

(d) For the temperatures entering the calculation, the $O(\mu)$ term in L_μ represented less than a 1% correction and so was dropped.

In Fig. 5 the numerical calculation is compared with the experimental obervations.

At depths of 50 cm. and 150 cm. the difference between the calculated temperature and the observed temperature was within 20% of the observed temperature. Only this rough agreement can be expected because the instantaneous heat flux was not available and a heat flux averaged over the freezing period was used.

Finally, Bilello reported that by Dec. 26 the ice was 20-25 cm. thick. The calculated value of 22 cm. is within this range.

APPENDIX I

Expansion of $IF(r,\tau,A) = \int_0^\infty \frac{e^{-Au}}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right) du$ for $r = \frac{1}{2}, \frac{3}{2}$; $\tau \to \infty$

The function $F(\tau)$ is assumed to be locally integrable on $(0,\infty)$ and to have an asymptotic expansion of the form

$$F(\tau) \sim \sum_{q=0} F_q \tau^{-r_q} \qquad (I.1)$$

for $\tau \to \infty$, $r_q \uparrow \infty$

CASE 1 $\quad A < O(1)$ for $\tau \to \infty$

The expansion is carried out in two steps.

(a) First expand the integral for fixed τ, $A \to 0$

$$IF(r,\tau,A) \sim \sum_{k=0} IF_k(r,\tau)\, G_k(A)$$

where $\{G_k(A)\}$ is an asymptotic sequence for $A \to 0$.

(b) Then expand the $IF_k(r,\tau)$ for $\tau \to \infty$

$$IF_k(r,\tau) \sim \sum_{p=0} IF_{kp}(r)\, H_p(\tau)$$

where $\{H_p(\tau)\}$ is an asymptotic sequence for $\tau \to \infty$.

Step (a)

With τ fixed at this step, put $f(u) = \frac{1}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right)$.

It is then required to calculate the expansion of the Laplace transform

$$Lf(A) = \int_0^\infty f(u) e^{-Au} \, du \tag{I.2}$$

for small A . We indicate how the work of Handlesman and Lew [17] can be used to complete this step.

Break up f at $u = 1$ and put

$$f(u) = f(u)\,\Theta(1-u) + f(u)\,\Theta(u-1)$$
$$= f_1(u) + f_2(u) \, .$$

Then $Mf_1[z] = \int_0^1 u^{z-1} f(u) \, du$ defines a function analytic for $\mathrm{Re}\, z > a$ for some $0 < a \leq 1$. Even in the case $a = 1$, the $\lim_{z \to 1} Mf_1[z]$ exists : i.e. $Mf_1[z]$ does not have a pole at $z = 1$.

Now for $u \to \infty$, $f(u)$ has the asymptotic expansion

$$f(u) \sim \sum_{m=0}^{\infty} c_m u^{-s_m}$$

where
$$c_m = \sum_{k=0}^{m} \frac{(-\tau)^k}{k!} \binom{-k-r}{r-k} F^{(k)}(\tau) \tag{I.3}$$

and $s_m = m + r$; note that no s_m is an integer.

It follows that $Mf_2[z] = \int_1^\infty u^{z-1} f(u) \, du$ defines a function analytic for $\mathrm{Re}\, z < s_0$ which can be analytically continued to a meromorphic function with simple poles at $z = s_m$ and residues c_m .

Define $M_0 f[z] = Mf_1[z] + Mf_2[z]$.
If $a < s_0$, then the integral representations of Mf_1 and Mf_2 are defined on the common vertical strip S for which $a < \mathrm{Re}\, z < s_0$. The Mellin transform

$$Mf[z] = \int_0^\infty u^{z-1} f(u)\, du$$

then exists in this strip S and

$$M_0 f[z] = Mf[z] \ , \quad z \in S \ ,$$

so that $M_0 f$. is the analytic continuation of Mf .
If $s_0 < a$, so that S is empty , the function $M_0 f$ still plays the role of the Mellin transform. Hereafter, the distinction between $M_0 f$ and Mf is suppressed and the notation Mf is used.

With this definition of the Mellin transform, the Parseval theorem for Mellin transforms is valid:

$$\int_0^\infty f(u)\, g(u)\, du = \frac{1}{2\pi i} \int_\Delta Mf[z]\, Mg[1-z]\, dz \ .$$

Choose $g(u) = e^{-\Lambda u}$; then $Mg[1-z] = \Gamma(1-z)\Lambda^{z-1}$ and the Bromwich contour Δ is to be taken in the strip $a < \mathrm{Re}\, z < 1$. The contour Δ can then be displaced to the right to pick up the poles of $Mf[z]$ and $\Gamma(1-z)$; since no s_m is an integer these poles are simple. This gives an asymptotic expansion for (I.2) :

$$Lf(\Lambda) \sim \sum_{m=0}^{\infty} c_m \Gamma(1-s_m)\Lambda^{s_m-1} + \sum_{m=0}^{\infty} Mf[m+1]\frac{(-\Lambda)^m}{m!}$$

(I.4)

where the C_m and S_m are given by (I.3). Even in the limit $a \to 1$ the result is valid.

The first sum of (I.4), which depends only on the asymptotic behavior of f will be called the ASYMPTOTIC contribution: the second sum will be called the DOMAIN contribution.

Finally, using the definition of $f(u)$, the results (I.3) and (I.4) can be collected to give

$$IF(r,\tau,A) \sim \sum_{m=0}^{\infty} \Gamma(1-m-r)\, AF(r,\tau,m)\, A^{m+r-1} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} DF(r,\tau,m+1)\, A^m \qquad \text{(I.5a)}$$

where $$AF(r,\tau,m) = \sum_{k=0}^{m} \frac{(-\tau)^k}{k!} \binom{-k-r}{m-k} F^{(k)}(\tau) \qquad \text{(I.5b)}$$

and $DF(r,\tau,z)$ is the analytic continuation in z of

$$\int_0^\infty \frac{u^{z-1}}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right) du \quad . \qquad \text{(I.5c)}$$

This completes step (a).

Step (b)

The asymptotic expansion of $AF(r,\tau,m)$ for $\tau \to \infty$ follows simply by substituting (I.1) into (I.5b)

$$AF(r,\tau,m) \sim \sum_{p=0}^{\infty} (-1)^p F_p \sum_{k=0}^{m} \binom{-k-r}{n-k} \frac{\Gamma(r_p+k)}{\Gamma(r_p)} \frac{(-\tau)^k}{k!} \tau^{-(r_p+k)} . \tag{I.6}$$

The expansion of $DF(r,\tau,m+1)$ for $\tau \to \infty$ is more difficult. For convenience, transform the integral (I.5c) to the interval (0,1) with the transformation $\frac{u}{1+u} \mapsto u$

$$DF(r,\tau,z) = \int_0^1 (1-u)^{r-z-1} u^{z-1} F(\tau u)\, du \tag{I.7}$$

This integral is of the general form

$$J(\tau) = \int_0^1 g(u) F(\tau u)\, du \tag{I.8}$$

where g has a convergent power series expansion for $|u|<1$

$$g(u) = \sum_{k=0}^{\infty} g_k u^k \qquad |u|<1 . \tag{I.9}$$

Now the Mellin fransform of g,

$$M_g[z] = \int_0^1 u^{z-1} g(u)\, du$$

is a meromorphic function with poles at $z = -k$, $k = 0, 1, 2 \cdots$. In fact, use the series representation (I.9) in the transform; since the expansion is absolutely convergent, integration and summation can be interchanged to get

$$M_g[z] = \sum_{k=0}^{\infty} \frac{g_k}{z+k} \tag{I.10}$$

Define

$$MF(\tau, z) = \int_0^1 u^{z-1} F(\tau u)\, du \tag{I.11}$$

then, by substituting (I.9) into (I.8) we have

$$J(\tau) = \sum_{k=0}^{\infty} g_k\, MF(\tau, k+1) \tag{I.12}$$

while from (I.11) we get

$$\frac{d}{d\tau}\left\{ \tau^{k+1} MF(\tau, k+1) \right\} = \tau^k F(\tau) \,. \tag{I.13}$$

Now use the expansion (I.1). If there is a q such that $r_q = k+1$ denote it by q_k; then from (I.13)

$$MF(\tau, k+1) \sim \sum_{q \neq q_k} \frac{F_q}{\tau^{r_q}} \frac{1}{k - r_q + 1} \tag{I.14}$$

$$+ F_{q_k} \frac{\ln \tau}{\tau^{k+1}} + \frac{M_k}{\tau^{k+1}}$$

with the understanding that if there is no such q_k the term $F_{q_k} \frac{\ln \tau}{\tau^{k+1}}$ is absent. Here M_k is an integration constant.

Let $Q = \{q_k\}$. Then from (I.10),(I.12) and (I.14) we get the final form

$$J(\tau) \sim \sum_{q \in Q^c} \frac{F_q}{\tau^{r_q}} M_g[1-r_q] + \ln\tau \sum_{q_k \in Q} g_k \frac{F_{q_k}}{\tau^{k+1}} + \sum_{k=0}^{\infty} g_k \frac{M_k}{\tau^{k+1}}$$

where if no r_q is an integer then Q is empty and the second sum is absent.

This result can now be applied to (I.7) with the identification

$$g(u) \longleftrightarrow g_z^r(u) = (1-u)^{r-z-1} u^{z-1}$$

so that

$$M g_z^r[1-r_q] = B(z-r_q, r-z)$$

where $B(z,w) = \dfrac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ is the Beta function.

The g_k of (I.9) can be read off from the expansion

$$g_{m+1}^r(u) = u^m \sum_{k=0}^{\infty} \binom{r-m-2}{k} u^k$$

to give finally

$$DF(r,\tau,m+1) \sim \sum_{q \in Q^c} \frac{F_q}{\tau^{r_q}} B(m+1-r_q, r-m-1)$$

$$+ \ln\tau \sum_{q_k \in Q,\, k \geq m} \binom{r-m-2}{k} \frac{F_{q_k}}{\tau^{k+1}}$$

$$+ \sum_{k=m} \binom{r-m-2}{k} \frac{M_k}{\tau^{k+1}}. \qquad \text{(I.15)}$$

The expressions (I.6) and (I.15) are the required expansions to complete step (b).

These expansions can now be used in (I.5a) to complete case 1.

CASE 2 $\Lambda = O(1)$ for $\tau \to \infty$.

Write $\Lambda \sim \Lambda_o + \Lambda_R(\tau)$ where Λ_o is a positive constant and $\Lambda_R(\tau) \to 0$ for $\tau \to \infty$.

As in case 1. the expansion is carried out in two steps:

Step (a)

With τ fixed, put $f(u) = \frac{e^{-\Lambda_o u}}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right)$.

It is then required to calculate the expansion of the Laplace transform

$$Lf[\Lambda_R] = \int_0^\infty f(u)\, e^{-\Lambda_R u} \qquad \text{(I.16)}$$

for $\Lambda_R \to 0$. As in case 1, Mf_1 and Mf_2 can be defined.

Then, because $\ddot{f}$ if exponentially decreasing for $u \longrightarrow \infty$,
Mf_2 is entire and $Mf = Mf_1 + Mf_2$ is analytic for
$\mathrm{Re}\, z > 1$. The contour Δ in the Parseval theorem can
then be shifted to the right. There is no asymptotic contribution;
the result is simply

$$Lf[A_E] \sim \sum_{m=0}^{\infty} Mf[m+1] \frac{(-A_E)^m}{m!} . \tag{I.17}$$

Note that the same result could have been obtained simply by
expanding the exponential in (I.16). Finally, from the definition
of f and (I.17) it follows that

$$IF(r,\tau,A) \sim \sum_{m=0}^{\infty} DF(r,\tau,m+1) \frac{(-A_E)^m}{m!} \tag{I.18a}$$

where $$DF(r,\tau,m+1) = \int_0^\infty \frac{u^m}{(1+u)^r} F\left(\frac{\tau u}{1+u}\right) e^{-A_o u} du , \tag{I.18b}$$

<u>Step (b)</u>

The expansion of $DF(r,\tau,m+1)$ for $\tau \rightarrow \infty$
is calculated as in case 1, step (b).

Here, after transforming the integral (I.18b) to the
interval $(0,1)$ we get

$$g_m^r(u) = (1-u)^{r-m} u^m e^{-A_o \frac{u}{1-u}}$$

so that

$$Mg_m^r[1-r_q] = \Gamma(m+1-r_q)\, U(m+1-r_q, m+2-r, A_o)$$

where $U(a,b,z)$ is the second confluent hypergeometric function.

The calculation of the power series expansion

$$g_m^r(u) = \sum_{k=0}^{\infty} g_{m,k}^r u^k$$

is straight forward, but tedious. To the order which the matching of the integral equations is carried out, we need only that $g_{0,r}^r = 1$.

Finally, the result analogous to (I.15) is

$$DF(r,\tau,m+1) \sim \sum_{q \in Q^c} \frac{F_q}{\tau^{r_q}} \Gamma(m+1-r_q)\, U(m+1-r_q, m+2-r, A_0)$$
$$+ \ln\tau \sum_{q_k \in Q,\, k \geq m} g_{m,k}^r \frac{F_{q_k}}{\tau^{k+1}} + \sum_{k=0}^{\infty} g_{m,k}^r \frac{M_k}{\tau^{k+1}} .$$

This completes step (b).

When (I.19) is substituted into (I.18a) case 2 is completed.

APPENDIX II

The solution to the system

$$\phi_{yy} - \phi_\tau = S(y,\tau) \qquad 0<y<1 \qquad \tau>0$$
$$\phi(0,\tau) = \Phi_B(\tau) \qquad \phi(1,\tau)=0$$
$$\phi(y,0) = \Phi_0(y)$$

can be found by using the Laplace transform. The solution has three contributions:

(a) The initial value term which satisfies

$$\phi_{yy} - \phi_\tau = 0$$
$$\phi(0,\tau) = \phi(1,\tau) = 0$$
$$\phi(y,0) = \Phi_0(y) .$$

Transform in τ and invert to give

$$\phi_{IV}(y,\tau) = \int_0^1 \Phi_0(z)\left\{\frac{1}{2\pi i}\int_\Gamma G(p,y,z)\, e^{p\tau}\, dp\right\} dz \tag{II.1}$$

where

$$G(p,y,z) = \frac{\sinh\sqrt{p}\,y\ \sinh\sqrt{p}\,(1-z) + \sinh\sqrt{p}\,z\ +\ \sinh\sqrt{p}\,(1-y)}{\sqrt{p}\ \sinh\sqrt{p}}$$

and Γ is a Bromwich contour for which $\mathrm{Re}\, p > 0$.

(b) The source term which satisfies

$$\phi_{yy} - \phi_\tau = S(y,\tau) \qquad 0<y<1$$
$$\phi(0,\tau) = \phi(1,\tau) = 0$$
$$\phi(y,0) = 0$$

Then with $\bar{S}(y,p) = \int_0^\infty S(y,\tau)\, e^{-p\tau}\, d\tau$ it follows

that

$$\phi_S(y,\tau) = -\int_0^1 \left\{\frac{1}{2\pi i}\int_\Gamma \bar{S}(z,p)\, G(p,y,z)\, e^{p\tau}\, dp\right\} dz . \tag{II.2}$$

(c) The boundary term which satisfies

$$\Phi_{yy} - \Phi_\tau = 0 \qquad 0 < y < 1$$

$$\Phi(0,\tau) = \Phi_B(\tau) \quad \Phi(1,\tau) = 0$$

$$\Phi(y,0) = 0$$

with $\bar{\Phi}_B(p) = \int_0^\infty \Phi_B(\tau) e^{-p\tau} d\tau$ the solution is

$$\Phi_{BY}(y,\tau) = \frac{1}{2\pi i} \int_\Gamma \bar{\Phi}(p) \frac{\sinh\sqrt{p}(1-y)}{\sinh\sqrt{p}} e^{p\tau} dp \quad , \qquad \text{(II.3)}$$

With the solution given in integral form by (II.1, II.2, II.3) both the large τ and small τ expansions for ϕ can be found.

TABLE I.a Perturbation Solution to a Stefan Problem with boundary conditions

$$\phi(0,\tau) = e^{-\tau} \sin(1)$$

$$\phi(x,0) = \sin(1-x) + \varepsilon\left[\frac{\sin(x)}{\sin(1)} - \frac{\sin(\sqrt{2}\,x)}{\sin(\sqrt{2}\,)}\right]$$

$$\varepsilon = .1$$

Comparison with a numerical solution; temperatures compared at fixed $y=x/\sigma$ intervals

	Perturbation Soln.	Numerical Soln.	
	Free Boundary		
τ=.25	1.0219	1.0218	
	Temperature		
	.0864	.0811	y=.22
	.3349	.3294	y=.55
	.4816	.4779	y=.77
	Free Boundary		
τ=.50	1.0386	1.0384	
	Temperature		
	.1337	.1254	y=.22
	.2608	.2579	y=.55
	.4256	.4248	y=.77
	Free Boundary		
τ=.75	1.0514	1.0513	
	Temperature		
	.1041	.0990	y=.22
	.2491	.2487	y=.55
	.3315	.3318	y=.77

TABLE I.a

(CONTINUED)

	Perturbation Soln.	Numerical Soln.	
	Free Boundary		
$\tau=1.0$	1.0613	1.0613	
	Temperature		
	.0811	.0779	y=.22
	.1940	.1945	y=.55
	.2581	.2590	y=.77
	Free Boundary		
$\tau=2.0$	1.0830	1.0832	
	Temperature		
	.0298	.0285	y=.22
	.0714	.0713	y=.55
	.0950	.0952	y=.77
	Free Boundary		
$\tau=3.0$	1.0907	1.0910	
	Temperature		
	.0171	.0161	y=.22
	.0276	.0277	y=.55
	.0367	.0369	y=.77

TABLE I.b Perturbation Solution to a Stefan Problem with boundary conditions

$$\phi(0,\tau) = e^{-\tau} \sin(1)$$

$$\phi(x,0) = \sin(1-x) + \varepsilon\left[\frac{\sin(x)}{\sin(1)} - \frac{\sin(\sqrt{2}\,x)}{\sin(\sqrt{2}\,)} \right]$$

$$\varepsilon = .5$$

Comparison with a numerical solution; temperatures compared at fixed $y=x/\sigma$ intervals

	Perturbation Soln.	Numerical Soln.	
	Free Boundary		
τ=.25	1.0883	1.0914	
	Temperature		
	.1574	.1496	y=.22
	.3967	.3927	y=.55
	.5390	.5376	y=.77
	Free Boundary		
τ=.5	1.1581	1.1617	
	Temperature		
	.1301	.1227	y=.22
	.3203	.3131	y=.55
	.4286	.4253	y=.77
	Free Boundary		
τ=.75	1.2129	1.2165	
	.1059	.1004	y=.22
	.2563	.2505	y=.55
	.3387	.3357	y=.77

TABLE I.b

(CONTINUED)

	Perturbation Soln.	Numerical Soln.	
	Free Boundary		
τ=1.	1.2559	1.2596	
	Temperature		
	.0852	.0815	y=.22
	.2038	.1998	y=.55
	.2667	.2646	y=.77
	Free Boundary		
τ=2.	1.3523	1.3562	
	Temperature		
	.0336	.0321	y=.22
	.0784	.0774	y=.55
	.1006	.0996	y=.77
	Free Boundary		
τ=3.	1.3869	1.3915	
	Temperature		
	.0133	.0132	y=.22
	.0308	.0311	y=.55
	.0392	.0392	y=.77

TABLE II Perturbation Solution to a Stefan Problem with boundary conditions

$$\phi(0,\tau) = 1. + .25 \sin(\tau)$$

$$\phi(x,0) = 1. - \frac{\mathrm{erf}(\sqrt{(\lambda/2)}\ x)}{\mathrm{erf}(\sqrt{(\lambda/2)}\ )}$$

$$\varepsilon = .1034 \qquad \lambda = .1$$

Comparison with numerical solution for $\sigma(\tau)$

Time τ	Free Boundary $\sigma(\tau)$ Perturbation Soln.	Free Boundary $\sigma(\tau)$ Numerical Soln.
1.0	1.1060	1.0995
2.0	1.2132	1.2035
3.0	1.3042	1.3000
4.0	1.3726	1.3777
5.0	1.4270	1.4375
6.0	1.4839	1.4912
7.0	1.5532	1.5523
8.0	1.6304	1.6242
9.0	1.7020	1.6991
10.0	1.7588	1.7656

TABLE III Numerical Solution to a Stefan Problem with Boundary Conditions

$$-\phi_x(0,\tau) = \frac{1}{\sqrt{1+\tau}} \qquad \phi(x,0) = \int_x^1 e^{-\xi^2/4}\, d\xi$$

Interpolating temperature found by FOURIER TRANSFORMATION

Time Step	.1
Number of terms in Fourier Representation	4
Number of nodes in Filon Quadrature	10

Temperature compared with exact solution at fixed $y=x/\sigma$ intervals

	Numerical Soln.	Exact Soln.	
	Free Boundary		
$\tau=.5$	1.239	1.225	
	Temperature		
	.178	.182	y=.22
	.483	.485	y=.55
	.697	.701	y=.77
	Free Boundary		
$\tau=1.0$	1.438	1.414	
	Temperature		
	.178	.182	y=.22
	.483	.485	y=.55
	.697	.701	y=.77
	Free Boundary		
$\tau=1.5$	1.613	1.581	
	.178	.182	y=.22
	.482	.485	y=.55
	.696	.701	y=.77

TABLE IVa Numerical Solution to a Stefan Problem with Boundary Conditions

$$\phi(0,\tau) = 1.- e^{\tau-1}$$

$$\phi(x,0) = 1.- e^{x-1}$$

Interpolating temperature found by LAPLACE TRASFORM

Time step	.25
Number of terms in Laplace Representation	2
Number of nodes in Filon Quadrature	10

Temperature compared with exact solution at fixed y=x/σ intervals

	Numerical Soln.	Exact Soln.	
	Free Boundary		
τ=.25	.7500	.7500	
	Temperature		
	.1554	.1476	y=.22
	.2905	.2835	y=.55
	.3623	.3588	y=.77
	Free Boundary		
τ=.50	.4958	.5000	
	Temperature		
	.1016	.1052	y=.22
	.2502	.2425	y=.55
	.3279	.3222	y=.77
	Free Boundary		
τ=.75	.2506	.2500	
	Temperature		
	.0517	.0540	y=.22
	.1223	.1297	y=.55
	.1636	.1767	y=.77

TABLE IVb Numerical Solution to a Stefan Problem with Boundary Conditions

$$\phi(0,\tau) = \int_0^1 e^{-\xi^2/4}\,d\xi \quad \Big| \quad \phi(x,0) = \int_x^1 e^{-\xi^2/4}\,d\xi$$

Interpolating Temperature found by LAPLACE TRANSFORM

Time Step	.5
Number of Terms in Laplace Representation	2
Number of Nodes in Filon Quadrature	10

Time τ	Free Boundary $\sigma(\tau)$ Numerical Soln.	Free Boundary $\sigma(\tau)$ Exact Soln.
1.0	1.0958	1.0954
2.0	1.1829	1.1832
3.0	1.2647	1.2649
4.0	1.3418	1.3416
5.0	1.4150	1.4142
6.0	1.4848	1.4832
7.0	1.5516	1.5492
8.0	1.6157	1.6125
9.0	1.6775	1.6733

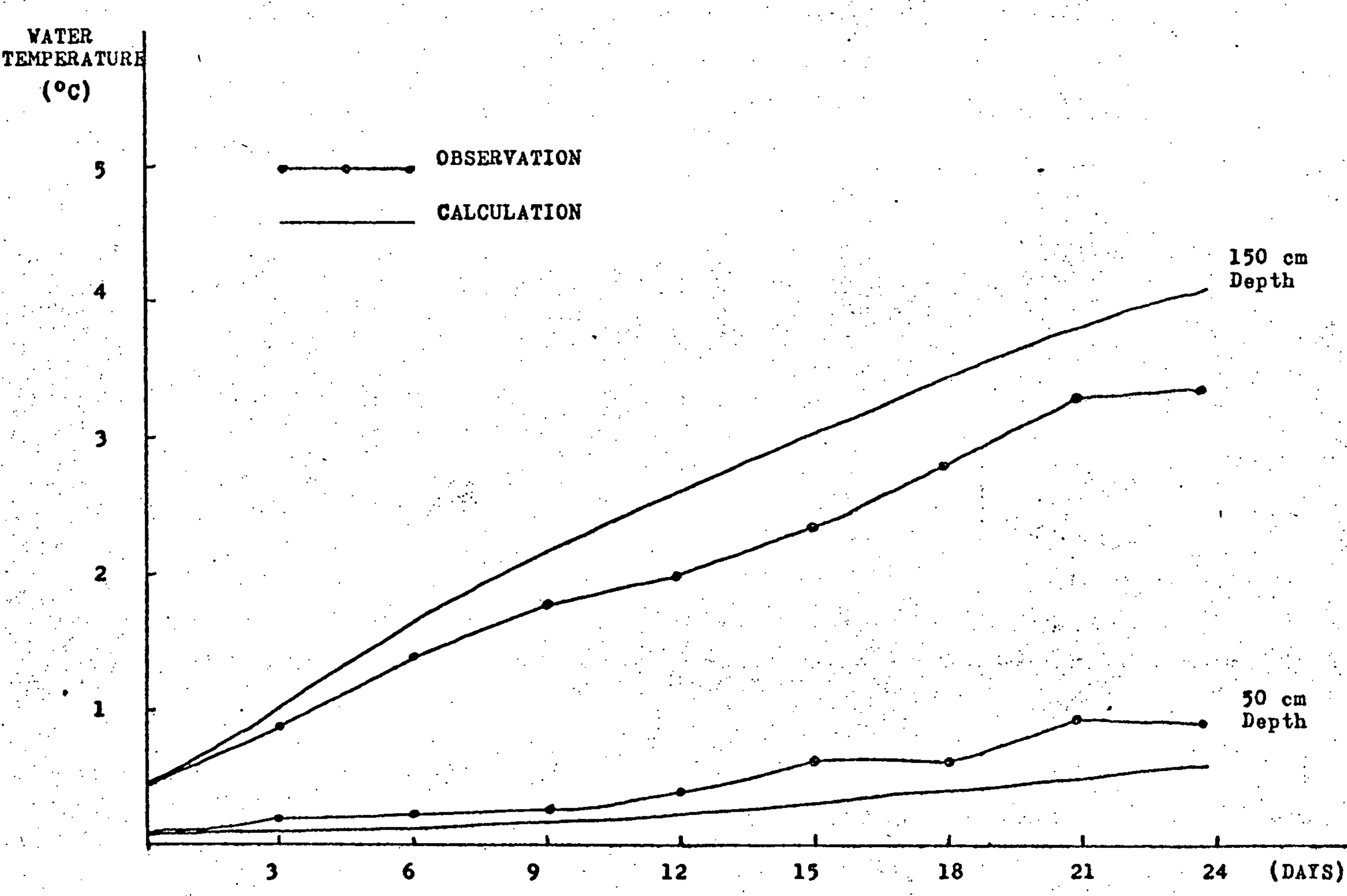

Figure 5 Calculated and Observed Temperature Profiles for Seneca Lake; December 2 to December 26

BIBLIOGRAPHY

[1] K. STEFAN, Ann. Phys. u. Chem. 42 (1891) 269

[2] F. NEUMANN, Lectures c. 1860. See Die Partiellen Diff. Der Math. Physik 2 (1912) 121

[3] H. CARSLAW & J. JAEGER, Conduction of Heat in Solids; Clarendon Press, Oxford 1969

[4] B. BOLEY, Heat Conduction Problems, J. of Math & Phys., 40 (1961) 301

[5] H. LANDAU, Heat Conduction in a Melting Solid, Quart. of Applied Math., 8 (1950) 81

[6] G. EVANS, H. ISAACSON & T. MACDONALD, Stefan Like Problems, Quart. of Applied Math., 8 (1950) 312

[7] J. CANNON & C. DENSON HILL, Existence, Uniqueness, Stability and Monotone Dependence in a Stefan Problem for the Heat Equation, J. of Math. & Mech., 17 (1967) 1

[8] R. HANDELSMAN & J. LEW, Laplace Transforms in the Neighbourhood of the Origin, Siam J. of Appl. Math., 1 (1970) 118

[9] A. FRIEDMAN, Free Boundary Problems for Parabolic Equations, J. of Math. & Mech., 8 (1959) 499

[10] B. SHERMAN, General One Phase Stefan Problems for the Heat Equation, Siam J. of Appl. Math., 20 (1971) 555 (June)

[11] B. SHERMAN, Limiting Behavior in Some Stefan Problems as the Latent Heat Goes to Zero, Siam J. of Appl. Math., 20 (1971) 319 (March)

[12] H. HASHEMI & C. SLIEPCEVICH, A Numerical Method for Solving Two Dimensional Problems of Conduction With Change of Phase, Chem. Eng. Symp. Series, 63 (1967) 34

[13] G.H. MEYER, Multidimensional Stefan Problems, Siam J. Num. Anal. 10 (1973) 520 (June)

[14] Y.K. CHUANG & J. SZEKELEY, Use of Green's Functions for Solving Melting or Solidification Problems, Int. J. Heat & Mass Trans., 14 (1971) 1285

[15] J. DOUGLAS & T. GALLIE, On the Numerical Integration of a Parabolic Differential Equation Subject to a Moving Boundary Condition, Duke Math. J., 22 (1955) 557

[16] G. BLUMAN, Applications of the General Similarity Solution of the Heat Equation to Boundary-Value Problems, Quart.Applied Math. 31 (1974) 403

[17] M. ABRAMOWITZ & I. STEGUN, Handbook of Mathematical Functions, Nat. Bureau of Standards - Appl. Math. Series, (1964)

[18] G. DUFF & R. NAYLOR, Differential Equations of Applied Mathematics, N.Y. (1966)

[19] M. BILELLO, Water Temperatures in a Shallow Lake During Ice Formation, Growth and Decay, U.S. Army Cold Regions Research and Engineering Laboratory, Dec. 1967

[20] L. SIROVICH, Techniques of Asymptotic Analysis, Springer Verlag (1971)

[21] S. CHURCHILL, Modern Operational Mathematics in Engineering, N.Y. (1944)