THE EILENBERG-MOORE SPECTRAL SEQUENCE

by

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A B S T R A C T

For any two differential modules $M$ and $N$ over a graded differential $k$-algebra $\wedge(k$ a commutative ring), there is a spectral sequence $E_r$, called the Eilenberg-Moore spectral sequence, having the following properties: $E_r$ converges to $\text{Tor}_{\wedge}(M, N)$ and $E_2 = \text{Tor}_*(\wedge)(H(M), H(N))$. This algebraic set-up gives rise to a "geometric" spectral sequence in algebraic topology. Starting with a commutative diagram of topological spaces

$$
\begin{array}{ccc}
X \times_B Y & \rightarrow & Y \\
\downarrow & & \downarrow \\
X & \rightarrow & B
\end{array}
$$

where $B$ is simply connected, one gets a spectral sequence $E_r$ converging to the cohomology $H^*(X \times_B Y)$ of the space $X \times_B Y$, and for which $E_2 = \text{Tor}_*(B)(H^*(X), H^*(Y))$.

In this thesis we outline a generalization of the above geometric spectral sequence obtained, by first extending the category of topological spaces and then, extending the cohomology theory $H^*$ to this larger category. The convergence of the extended spectral sequence does not depend on any topological conditions of the spaces involved. It follows algebraically from the way the exact couple (from which the spectral sequence is derived) is set up and from the Suspension Axiom of the extended cohomology theory.
CONTENTS

Chapter I - The Algebraic Construction of Eilenberg-Moore Spectral Sequence

1. The Categories LDk and LDA  1
2. The Definition of Proper Projective Resolutions and the Functor Tor  3
3. The Künneth Spectral Theorem  11

Chapter II - The Geometric Spectral Sequence  13

1. More About Tor  13
2. The Geometric Eilenberg-Moore Spectral Sequence  16

Chapter III - The Extension of the Geometric Spectral Sequence  23

1. The Category of Fibered Spaces Over B  23
2. Topological Constructions in (Top/B)*  24
3. The Products in (Top/B)*  33
4. The Forgetful Functors on (Top/B)*  34
5. The Extended Cohomology on (Top/B)*  35
6. The Multiplicative Cohomology  39
7. The Künneth Spectral Sequence  42
8. The Convergence of the Spectral Sequence  52
9. The Künneth Spectral Theorem  60

Bibliography  62
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Chapter I

The Algebraic Construction of Eilenberg-Moore Spectral Sequence

We will give an account of how the Eilenberg-Moore spectral sequence arises in homological algebra. Throughout this first chapter, \( k \) will denote a fixed commutative ring.

1. The Categories \( \text{LD}_k \) and \( \text{LDA} \).

The object of the category of graded, differential \( k \)-modules, denoted \( \text{LD}_k \), are the families of \( k \)-modules, \( \{M^i\}_{i \in \mathbb{Z}} \), together with a \( k \)-module homomorphisms \( d^i : M^i \rightarrow M^{i+1} \) such that \( d^{i+1} \circ d^i = 0 \). Such objects in \( \text{LD}_k \) will be denoted by \( M = \{M^i\}_{i \in \mathbb{Z}} \). The morphism of degree \( p \) between two objects \( M \) and \( N \) in \( \text{LD}_k \) is a family of \( k \)-module homomorphism \( \{f^i : M^i \rightarrow N^{i+p}\} \) of degree \( p \) such that \( f^{i+1} \circ d^i = d^{i+1} \circ f^i \). Note that \( \text{LD}_k \) is an Abelian category.

The tensor product of two objects \( M \) and \( N \) in \( \text{LD}_k \) is defined as the object \( M \otimes N \) in \( \text{LD}_k \), where \( (M \otimes N)^i = \sum_{m+n=i} M^m \otimes N^n \). The differential in the tensor product is the one induced by the differential in \( M \) and \( N \). We have that \( k \otimes M \cong M \otimes k \), and that \( (L \otimes M \otimes N)^i = \sum_{l+m+n=i} L^l \otimes M^m \otimes N^n \).

The graded, differential \( k \)-algebra is an object \( \Lambda \) in \( \text{LD}_k \) with two structure morphisms \( I : k \rightarrow \Lambda \) and \( \varphi : \Lambda \otimes \Lambda \rightarrow \Lambda \) of degree 0 such that the following diagrams commutes:
The graded $k$-algebra is said to be connected if $\Lambda^0=k$ and $\Lambda^1=0$ for $i=0$.

We are now ready to describe the other category of differential $\Lambda$-modules, denoted $LDA$.

Let $\Lambda$ be a graded, differential, connected $k$-algebra. Then, by a left differential $\Lambda$-module, we will mean an object $M$ in $LDk$ with morphism $\psi: \Lambda \otimes M \rightarrow M$ of degree 0 such that the following diagrams are commutative:

The right differential $\Lambda$-module is similarly defined.

A morphism between two objects $M$ and $N$ in $LDA$ is a morphism in $LDk$ which is compatible with the structure morphisms $\psi$, $\varphi$, and $I$ mentioned above. From this point on, we will omit the word differential and call objects in $LDA$ (or in $RDA$, the category of right differential $\Lambda$-module.) as left (or right) $\Lambda$-modules.

Let $M$ and $N$ be left and right $\Lambda$-module, respectively. Then, the tensor product of $M$ and $N$, denoted $N \otimes^L M$, is defined as the cokernel of the morphism

$$\overline{\Delta} = \psi_N \otimes 1 - 1 \otimes \psi_M: N \otimes \Lambda \otimes M \rightarrow N \otimes M,$$

where $\psi_N: N \otimes \Lambda \rightarrow N$ and $\psi_M: \Lambda \otimes M \rightarrow M$. 
The tensor product sign without subscripts will always mean that it is taken over \( k \).

2. The Definition of Proper Projective Resolution and the Functor \( \text{Tor} \).

Let \( M \) be an object in \( \text{LD}_k \). Then, denote by \( Z(M) \) and \( B(M) \), the graded \( k \)-modules \( \operatorname{Ker} d = \{ \operatorname{Ker} d^i \}_{i \in \mathbb{Z}} \) and \( \operatorname{Im} d = \{ \operatorname{Im} d^i \}_{i \in \mathbb{Z}} \), respectively, where \( d = \{ d^i \}_{i \in \mathbb{Z}} \) is the differential in \( M \). The homology of \( M \) is defined in the usual way by \( H(M) = Z(M)/B(M) \).

**Definition 2.1:**

A short sequence \( 0 \to K \to L \to M \to 0 \) of objects and morphisms in \( \text{LD}_A \) is said to be proper exact if the two sequences, \( 0 \to K \to L \to M \to 0 \) and \( 0 \to Z(M) \to Z(L) \to Z(M) \to 0 \) are exact in \( \text{LD}_k \).

The short exact sequences \( 0 \to Z(K) \to K \to B(K) \to 0 \) and \( 0 \to B(K) \to Z(K) \to H(K) \to 0 \) in \( \text{LD}_k \) shows that sequences \( 0 \to B(K) \to B(L) \to B(M) \to 0 \) and \( 0 \to H(K) \to H(L) \to H(M) \to 0 \) are also exact in \( \text{LD}_k \).

From above definition, we will define the notion of long exact sequence in \( \text{LD}_A \). Let \( \ldots \to x^{n-1} \to x^n \to x^{n+1} \to \ldots \) be a sequence of objects and morphisms in \( \text{LD}_A \). Then, as objects and morphisms in \( \text{LD}_k \), the above sequence can be factorized to a sequence of short exact sequences as follows:
Each $K^n$ is a (left) $\Lambda$-module by the morphism $\Psi: \Lambda \otimes K^n \to K^n$ induced in the diagram

$$
\begin{array}{c}
\Lambda \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \\
\downarrow \Psi_{n-1} \downarrow \Psi_n \downarrow \Psi \downarrow \Psi \\
x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \otimes x_n \\
\end{array}
$$

where the rows are exact and $\Psi_{n-1}$, $\Psi_n$ are the structure morphisms.

Therefore, a long sequence in $\text{LDA}$ can be factorized into a sequence of short sequences.

**Definition 2.2:**

A long sequence of objects and morphisms,

$$
\cdots \longrightarrow x_{n-1} \longrightarrow x_n \longrightarrow x_{n+1} \longrightarrow \cdots
$$

in $\text{LDA}$ is proper exact if its factorized short sequences,

$$
0 \longrightarrow K^n \longrightarrow x^n \longrightarrow K^{n+1} \longrightarrow 0,
$$

is proper exact in $\text{LDA}$, for all $n$.

The proper exactness in $\text{LDA}$ allows us to define the properness of the morphisms in $\text{LDA}$. 
Definition 2.3:
A morphism $f$ in $\text{LDA}$ is proper epimorphism if there is a morphism $g$ in $\text{LDA}$ such that the short sequence,
$$0 \rightarrow K \xrightarrow{g} L \xrightarrow{f} M \rightarrow 0$$
is proper exact in $\text{LDA}$.

Definition 2.4:
A $\Lambda$-module $P$ is proper projective if for every morphism $h: P \rightarrow N$ and every proper epimorphism $f: M \rightarrow N$ in $\text{LDA}$, there is a morphism $h^*: P \rightarrow M$ in $\text{LDA}$ such that the triangle
$$\begin{array}{cc}
P & \xrightarrow{h} & N \\
\downarrow{h^*} & & \downarrow{f} \\
M & \xrightarrow{f} & N \\
\end{array}$$
commutes.

Definition 2.5:
A proper projective resolution of $\Lambda$-module $M$ is a long sequence of $\Lambda$-modules,
$$\cdots \rightarrow X^{-n} \rightarrow \cdots \rightarrow X^{-1} \rightarrow X^{0} \rightarrow M \rightarrow 0,$$
which is proper exact and each $X^{-n}$, $n \geq 0$ is proper projective.

The proper projective resolution of $\Lambda$-module $M$ will be denoted by $X \xrightarrow{\varepsilon} M \rightarrow 0$.

Proposition 2.6:
Every object in $\text{LDA}$ has a proper projective resolution.

Proof:
Let $M$ be a $\Lambda$-module. Then as object in $\text{LDk}$, there is
a V in LDk such that V is projective and \( V \rightarrow M \rightarrow 0 \).

Consider the tensor product \( \Lambda \otimes V \) with the morphism \( \varphi \otimes 1 : \Lambda \otimes \Lambda \otimes V \rightarrow \Lambda \otimes V \), where \( \varphi : \Lambda \otimes \Lambda \rightarrow \Lambda \). The morphism \( \varphi \otimes 1 \) makes \( \Lambda \otimes V \) a \( \Lambda \)-module. We will show that \( \Lambda \otimes V \) is proper projective.

Let \( h : \Lambda \otimes V \rightarrow N \) and \( f : K \rightarrow N \) be morphisms in LD\( \Lambda \) with \( f \) a proper epimorphism. Define a morphism, 
\[
p : V \rightarrow \Lambda \otimes V \text{ by } p(v) = 1 \otimes v \text{ for every } v \in V.
\]
Then, since \( V \) is projective, we have the lifting \( h' : V \rightarrow K \) of \( h \circ p : V \rightarrow N \) such that \( h \circ p = f \circ h' \).

The composition of morphisms, 
\[
\Lambda \otimes V \xrightarrow{h'} \Lambda \otimes K \xrightarrow{\psi_K} K
\]
gives the lifting of \( h : \Lambda \otimes V \rightarrow N \) such that the triangle, 
\[
\begin{array}{ccc}
\Lambda \otimes V & \xrightarrow{h} & N \\
\downarrow & & \downarrow \\
K & \xrightarrow{f} & N
\end{array}
\]
commutes.

The composition of morphisms, 
\[
\Lambda \otimes V \xrightarrow{1 \otimes h} \Lambda \otimes M \xrightarrow{\psi_M} M \rightarrow 0
\]
is an epimorphism. Hence, in the usual way, we can construct the proper projective resolution of \( M \). □

The proper projective resolution \( X \rightarrow M \rightarrow 0 \) has two differentials; the internal differential \( d_I \) of each \( \Lambda \)-module \( X^{-n} \) and the external differential of the resolution, \( d_E^{-n+1} : X^{-n+1} \rightarrow X^{-n} \). The differentials are displayed in the following diagram:
Denote by

\[ Z_I(X^{-n}) = \ker d_I^{-n,1}: X^{-n,1} \longrightarrow X^{-n,1+1} \] \] \[ B_I(X^{-n}) = \text{im} d_I^{-n,1}: X^{-n,1} \longrightarrow X^{-n,1+1} \] \]

Then the homology of \( X \) with respect to the internal differential \( d_I \) is \( H_I(X^{-n}) = Z_I(X^{-n}) / B_I(X^{-n}) \). Note that \( H_I(X^{-n}) \) is bigraded with gradings on \( n \) and \( i \).

**Proposition 2.7:**

If \( X \) is a proper projective resolution of \( \Lambda \)-module \( M \), then the sequence,

\[ ... \rightarrow H_I(X^{-n}) \rightarrow H_I(X^{-n+1}) \rightarrow ... \]

is a projective resolution of \( H(M) \) as \( H(\Lambda) \)-module.

**Proof:**

From the proper exactness of the resolution \( X \), the above sequence is also proper exact. The projectiveness of \( H_I(X^{-n}) \) follows from the projectiveness of \( X^{-n} \). \( \square \)
Remark:

We need to comment about the algebraic structure of $H(M)$ and $H(\Lambda)$ in Proposition 2.7. If $\Lambda$ is a graded, differential $k$-algebra, then $H(\Lambda)$ is also a graded, differential $k$-algebra with the two structure morphisms,

$$\phi_{H(\Lambda)}: H(\Lambda) \otimes H(\Lambda) \xrightarrow{\text{p}} H(\Lambda \otimes \Lambda) \xrightarrow{\text{H}(\varphi)} H(\Lambda), \quad \text{and}$$

$$I_{H(\Lambda)}: k = H(k) \xrightarrow{H(I)} H(\Lambda),$$

where $p$ is the external homology product, and $\phi: \Lambda \otimes \Lambda \longrightarrow \Lambda$ and $I: k \longrightarrow \Lambda$.

If $M$ is a $\Lambda$-module, then $H(M)$ is a $H(\Lambda)$-module by the morphism,

$$\psi_{H(M)}: H(\Lambda) \otimes H(M) \xrightarrow{\text{p}} H(\Lambda \otimes M) \xrightarrow{\text{H}(\psi)} H(M),$$

where $p$ is the external homology product and $\psi: \Lambda \otimes M \longrightarrow M$.

The differential in $H(M)$ is the one induced in the homology of the differential in $M$.

Definition 2.8:

A sequence of objects and morphisms in $LDA$,

$$\ldots \longrightarrow \chi^{n-1} \xrightarrow{d^{n-1}} \chi^{n} \longrightarrow \ldots,$$

such that $d^{n} \cdot d^{n-1} = 0$ for all $n$, will be called a complex of $\Lambda$-modules.

A complex of $\Lambda$-modules will be denoted by $X$.

A morphism between two complexes of $\Lambda$-modules $X$ and $Y$ is a family of $\Lambda$-module morphisms $f^{n}: \chi^{n} \longrightarrow \chi^{n}$, such that $f^{n} \cdot d^{n-1} = d^{n-1} \cdot f^{n-1}$, for all $n$. 
Definition 2.9:

Let \( f, g: X \rightarrow Y \) be morphisms of complexes of \( \Lambda \)-modules \( X \) and \( Y \). Then, \( f \) and \( g \) are chain homotopic if there is a family of morphisms \( \{ s^n: X^n \rightarrow Y^{n-1} \} \) such that \( d \cdot s + s \cdot d = g - f \).

We will denote the chain homotopy by \( s: f \sim g \).

A morphism \( f: X \rightarrow Y \) of complexes of \( \Lambda \)-modules is chain equivalent if there is another morphism \( h: Y \rightarrow X \) such that \( h \cdot f = 1_X \) and \( f \cdot h = 1_Y \).

If \( X \) is a complex of \( \Lambda \)-modules, then we denote by \( D(X) \) the graded object, \( D^n(X) = \sum_{p+q = n} X^p, q \). \( D(X) \) is a \( \Lambda \)-module and has the differential \( d = d_I + d_E \), where \( d_I: X^p, q \rightarrow X^{p, q+1} \) and \( d_E: X^p, q \rightarrow X^{p+1, q} \).

We state the following proposition and theorem without proof.

Proposition 2.10:

A chain homotopy \( s: f \sim g: X \rightarrow Y \) of complexes of \( \Lambda \)-modules induces a chain homotopy \( s^*: f^* \sim g^*: D(X) \rightarrow D(Y) \) of \( \Lambda \)-modules.

Theorem 2.11: (The Comparison Theorem)

Let \( f: M \rightarrow N \) be a morphism in \( \text{LDA} \). If \( X \xrightarrow{\xi} M \rightarrow 0 \) and \( Y \xrightarrow{\xi} N \rightarrow 0 \) are proper projective resolutions of \( M \) and \( N \), respectively, then there is a morphism \( g: X \rightarrow Y \) such that \( \xi \cdot g = f \cdot \xi \).

Moreover, if \( g^*: X \rightarrow Y \) is another morphism such that \( \xi \cdot g = f \cdot \xi \), then \( g \) and \( g^* \) are chain homotopic.
Let $M$ be a right $\Lambda$-module and $N$ a left $\Lambda$-module. Let $X \longrightarrow M \longrightarrow 0$ and $Y \longrightarrow N \longrightarrow 0$ be their respective resolutions. Then, we set $L^p, q = (X^p \otimes_\Lambda N)^q$ and denote this bigraded object by $L$. $L$ has the differential

$$d_1 \otimes_\Lambda 1 : L^p, q \longrightarrow L^{p+1}, q,$$

where $d_1 : X^p, q \longrightarrow X^{p+1}, q$ is the external differential of $X$.

**Definition 2.12:**

With the homology being taken with respect to $d_1 \otimes_\Lambda 1$ given above, define

$$\text{Tor}^p, q(M, N) = H_{p, q}(L).$$

Since $\text{Tor}^n(M, N) = \sum_{p+q=n} H_{p, q}(L) = H_n(D(L)) = H_n(D(X) \otimes_\Lambda N)$, we will also denote by

$$\text{Tor}_\Lambda(M, N) = H(D(X) \otimes_\Lambda N).$$

The two morphisms,

$$\xi \otimes_\Lambda 1 : D(X) \otimes_\Lambda D(Y) \longrightarrow M \otimes_\Lambda D(Y)$$

and

$$1 \otimes_\Lambda \xi : D(X) \otimes_\Lambda D(Y) \longrightarrow D(X) \otimes_\Lambda N,$$

induces the isomorphisms

$$H(D(X) \otimes_\Lambda D(Y)) \cong H(M \otimes_\Lambda D(Y)) \cong H(D(X) \otimes_\Lambda N).$$

Therefore, $\text{Tor}$ could also have been defined using the resolution of $N$ or using both resolutions of $M$ and $N$. The Comparison Theorem makes the definition of $\text{Tor}$ independent of the choice of the resolutions.
Lemma 2.13:

Let \( P \) be a proper projective \( \Lambda \)-module. Then for any \( \Lambda \)-module \( M \),

\[ H(P \otimes_{\Lambda} M) = H(P) \otimes_{H(\Lambda)} H(M). \]

Proof:

As \( k \)-modules, the external homology product,

\[ p : H(P \otimes M) = H(P) \otimes H(M), \]

is an isomorphism. (See Theorem 10.1, Chapter V in (1).)

This isomorphism preserves the structure of \( H(\Lambda) \)-module on both side, i.e., the diagram

\[ \begin{array}{ccc}
H(\Lambda) \otimes (H(P) \otimes H(M)) & \xrightarrow{1 \otimes p} & H(\Lambda) \otimes H(P \otimes M) \\
\downarrow \psi & & \downarrow \psi' \\
H(P) \otimes H(M) & \xrightarrow{p} & H(P \otimes M)
\end{array} \]

commutes, where \( \psi \) and \( \psi' \) are the modular structure morphisms for \( H(P) \otimes H(M) \) and \( H(P \otimes M) \), respectively.

Hence, \( p \) is an isomorphism of \( H(\Lambda) \)-modules. \( \blacksquare \)

3. The Künneth Spectral Theorem.

Let \( X \to M \to 0 \) and \( Y \to N \to 0 \) be the proper projective resolutions of \( \Lambda \)-modules \( M \) and \( N \). We have already mentioned in Sec.2 that the bigraded object \( L^{p,q} = (x^p \otimes_{\Lambda} N)^q \) has an (external) differential \( d_E \otimes 1 \). There is also an (internal) differential \( d_I \otimes 1 : (x^p \otimes_{\Lambda} N)^q \to (x^p \otimes_{\Lambda} N)^{q+1} \), where \( d_I \) is the internal differential of \( X \). Since

\[ d_I \cdot d_E + d_E \cdot d_I = 0, \]

we have \( (d_I \otimes 1) \cdot (d_E \otimes 1) + (d_E \otimes 1) \cdot (d_I \otimes 1) = 0. \)
We filter the graded object $D(L)$ by

$$(F^m D(L))_n = \sum_{p+q=n} I^p, q.$$ 

This filtration determines a spectral sequence $\{E_r, d_r\}$ such that,

$$E_1^{p,q} = H_{p+q}(FPD(L)/FP^{p+1}D(L)).$$

(See Sec.6, Chapter XI of (1), where this is shown for $k$-modules.) But, $(FPD(L)/FP^{p+1}D(L))_{p+q} = I^p, q$. Therefore,

$$E_1^{p,q} = H_{p+q}(L),$$

where the homology is taken with respect to the differential $d_1 \otimes 1$. We singly grade $E_1^{p,q}$ in the usual way by

$$E_1^n = \sum_{p+q=n} E_1^{p,q} = \sum_{p+q=n} H_{p+q}(X^p \otimes \Lambda)$$

$$= \sum_{p+q=n} (H(X^p) \otimes_{H(\Lambda)} H(N)) q$$ from Lemma 2.13.

Hence, we get that $E_1 = D(H(X)) \otimes_{H(\Lambda)} H(N)$.

Since $H(X)$ is a proper projective resolution of $H(M)$ as $H(\Lambda)$-module,

$$E_2 = H(E_1) = Tor_{H(\Lambda)}(H(M), H(N)).$$

We have shown the first part of the following theorem.

**Theorem 3.1:**

Let $M$ and $N$ be a right and left $\Lambda$-modules, respectively.

Then, there is a spectral sequence $\{E_r, d_r\}$ such that

1) $E_2 = Tor_{H(\Lambda)}(H(M), H(N))$, and

11) $E_r$ converges to $Tor_{\Lambda}(M, N)$.

For the proof of the second part, refer to Part I of (4).
Chapter II

The Geometric Spectral Sequence

We will look at an example of the Eilenberg-Moore spectral sequence in geometry. This will be done by applying Theorem 3.1 of the last chapter to modules obtained from the ordinary cohomology of topological spaces. Before we begin, we need to mention a few facts about Tor.

1. More About Tor.

Let $\Lambda$ and $\Gamma$ be a graded, differential $k$-algebras with a morphism $f: \Lambda \rightarrow \Gamma$ in $LD_k$. Let $g: M \rightarrow K$ and $h: N \rightarrow L$ be morphisms in $LD_k$, where $M$ and $N$ are $\Lambda$-modules, and $K$ and $L$ are $\Gamma$-modules. Then, the morphisms $g$ and $h$ are said to be $f$-semilinear if the following diagrams commute:

\[
\begin{align*}
M \otimes \Lambda & \xrightarrow{\psi_M} M \\
g \otimes f & \downarrow \\
K \otimes \Gamma & \xrightarrow{\psi_K} K
\end{align*}
\text{ and }
\begin{align*}
\Lambda \otimes N & \xrightarrow{\psi_N} N \\
f \otimes h & \downarrow \\
\Gamma \otimes L & \xrightarrow{\psi_L} L
\end{align*}
\]

where $\psi_M$, $\psi_N$, $\psi_K$ and $\psi_L$ are the respective modular structure morphisms for $M$, $N$, $K$ and $L$. 
The $f$-semilinear morphisms $g$ and $h$ induces the morphism, 
\[ \text{Tor}_f(g,h) : \text{Tor}_A(M,N) \longrightarrow \text{Tor}_A(K,L) \]
Moreover, with \( \{ E_r, d_r \} \) and \( \{ E'_r, d'_r \} \) being the respective spectral sequences associated with \((M,N)\) and \((K,L)\) from Theorem 3.1 of Chapter I, we have the following lemma.

**Lemma 1.1:**

The $f$-semilinear morphisms $g$ and $h$ induces the morphism,
\[ \text{Tor}_f(g,h)_r : E_r \longrightarrow E'_r. \]
Furthermore, if $g$ and $h$ induces the isomorphisms,
\[ H(g) : H(M) \cong H(K) \quad \text{and} \quad H(h) : H(N) \cong H(L), \]
in the homology, then $\text{Tor}_f(g,h)_r$ is an isomorphism for all $r \geq 2$.

**Proof:**

Let $X$ and $Y$ be the projective resolutions of $H(M)$ and $H(K)$, respectively. Then, by the Comparison Theorem, $g$ induces the morphism $g' : X \longrightarrow Y$ in the resolutions. In the same way, if $R$ and $S$ are the respective resolutions of $H(N)$ and $H(L)$, then $h$ induces the morphism $h' : R \longrightarrow S$.

Consider the diagram,

\[
\begin{array}{ccc}
D(X) \otimes H(\Lambda) \otimes D(R) & \xrightarrow{g' \otimes h} & D(Y) \otimes H(\Gamma) \otimes D(S) \\
D(X) \otimes D(R) & \xrightarrow{g' \otimes h'} & D(Y) \otimes D(S) \\
D(X) \otimes H(\Lambda) \otimes D(R) & \xrightarrow{g' \otimes h'} & D(Y) \otimes H(\Gamma) \otimes D(S) \\
0 & & 0
\end{array}
\]
The top square of the above diagram commutes by the two diagrams in the definition of $f$-semilinearity. Hence, $g \circ \Lambda h'$ is induced on the cokernels. Then, $g \circ \Lambda h'$ induces the morphism,

$$\text{Tor}_f(g,h)_2: \text{Tor}_H(\Lambda)(H(M), H(N)) \to \text{Tor}_H(\Lambda)(H(K), H(L)).$$

If $g$ and $h$ induces the isomorphisms in the homology, then $g \circ \Lambda h'$ is an isomorphism. Therefore,

$$\text{Tor}_f(g,h)_2: E_2 \cong E_2.'$$

Let $M$ be a left $\Lambda$-module. Then the tensor product of $k$-modules, $M \otimes M$, is a left $\Lambda \otimes \Lambda$-module by the morphism

$$\Psi: (\Lambda \otimes \Lambda) \otimes (M \otimes M) \to M \otimes M,$$

defined by

$$\Psi((\lambda \otimes \lambda') \otimes (x \otimes x')) = \Psi_M(\lambda \otimes x) \otimes \Psi_M(\lambda' \otimes x'),$$

for $\lambda, \lambda' \in \Lambda$ and $x, x' \in M$, where $\Psi_M: \Lambda \otimes M \to M$. Similarly, if $N$ is a right $\Lambda$-module, then we have a right $\Lambda \otimes \Lambda$-module, $N \otimes N$.

Let $X \longrightarrow M \longrightarrow 0$ be the proper projective resolution of the $\Lambda$-module $M$. Then the sequence

$$\cdots \longrightarrow (X \otimes X)^{-n} \longrightarrow \cdots \longrightarrow (X \otimes X)^{0} \longrightarrow M \otimes M$$

is a complex of $\Lambda \otimes \Lambda$-module, where $(X \otimes X)^{-n} = \sum_{p+q=-n} x^p \otimes x^q$ is a projective $\Lambda \otimes \Lambda$-module for all $n \neq 0$. If $P$ is a proper projective resolution of $M \otimes M$ as $\Lambda \otimes \Lambda$-module, then the identity morphism induces the morphism $I: X \otimes X \longrightarrow P$. Hence, with $N$ being the left $\Lambda$-module, there is a morphism

$$\Phi: \text{Tor}_\Lambda(M, N) \otimes \text{Tor}_\Lambda(M, N) \longrightarrow \text{Tor}_\Lambda(\Lambda \otimes \Lambda)(M \otimes M, N \otimes N).$$
Explicitly, the morphism $\Phi$ is the composition

$$H(D(X) \otimes N) \otimes H(D(X) \otimes N)$$

where $p'$ is the morphism induced by the external homology product.

The two structure morphisms,

$\Psi: (\Lambda \otimes \Lambda) \otimes (M \otimes M) \longrightarrow M \otimes M$ and

$\Psi': (N \otimes N) \otimes (\Lambda \otimes \Lambda) \longrightarrow N \otimes N,$

defined above, is $\mathcal{Q}$-semilinear, where $\mathcal{Q}: \Lambda \otimes \Lambda \longrightarrow \Lambda.$

Therefore, by Lemma 1.1, there is a morphism

$\text{Tor}_{\mathcal{Q}}(\Psi, \Psi') : \text{Tor}_{\Lambda \otimes \Lambda}(M \otimes M, N \otimes N) \longrightarrow \text{Tor}_{\Lambda}(M, N)$.

The composition of morphisms $\text{Tor}_{\mathcal{Q}}(\Psi, \Psi') \cdot \Phi$ makes $\text{Tor}_{\Lambda}(M, N)$ a $k$-algebra.


In this section, we will assume that all topological spaces have the homotopy type of a countable CW-complex and that their ordinary cohomology is finite. We will also assume that $k$ is a field.

For any topological space $X$, let $C_*(X)$ denote the normalized singular chain complex of $X$ (i.e., the singular chain complex of $X$ with all its degenerate elements taken out). And denote by $C^*(X) = \text{Hom}(C_*(X), k)$. 
Then, the graded object $C^*(X)$ is a differential $k$-algebra by the two morphisms,

$$\phi: C^*(X) \otimes C^*(X) \longrightarrow C^*(X) \quad \text{and} \quad I: k \longrightarrow C^*(X),$$
defined as follows:

The morphism $\phi$ is the composition

$$C^*(X) \otimes C^*(X) \xrightarrow{p} \text{Hom}(C^*(X) \otimes C^*(X), k) \xrightarrow{w} C^*(X),$$

where $p$ is the external product and $w$ is the morphism induced by a map of Eilenberg-Zilber Theorem and the diagonal map $\Delta: C^*(X) \longrightarrow C^*(X) \times C^*(X)$.

The morphism $I$ is given by $I_a: C_n(X) \longrightarrow k$,

$$I_a(x) = a \quad \text{for all} \ x \in C_n(X).$$

Furthermore, the $k$-algebra $C^*(X)$ is connected since $C^0(X) = \text{Hom}(C_0(X), k) \cong k$ and $C^n(X) = 0$ for all $n > 0$.

We will be in a situation where we have a commutative diagram of topological spaces

$$\begin{array}{ccc}
X & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
X_x \times Y & \xrightarrow{g} & Y
\end{array}$$

where $g$ is a Serre fibration and $X_x \times Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$. The space $B$ is assumed to be simply connected.

The maps $f$ and $g$ above give a modular structure to $C^*(X)$ and $C^*(Y)$ in the following way:

Define

$$\psi_X: C^*(B) \otimes C^*(X) \longrightarrow C^*(X) \quad \text{by}$$
\[ \Psi_X(a \otimes x) = (a \cdot \bar{f})x \] for \( a \otimes x \in C^*(B) \otimes C^*(X) \),

where \( \bar{f} : C^*_+(X) \to C^*_-(B) \) is induced by \( f \). This morphism \( \Psi_X \) makes \( C^*(X) \) a left differential \( C^*(B) \)-module. The right \( C^*(B) \)-module structure morphism is analogously defined.

Similarly, \( C^*(Y) \) is a left and right \( C^*(B) \)-module.

**Remark:**

We remark that the modular structure of \( C^*(X) \) is preserved in the homology. Hence, \( H^*(X;k) = H(C^*(X)) \) is a left and right \( H^*(B;k) \)-module.

Let

\[ \phi : C^*(X) \otimes C^*(B) \otimes C^*(Y) \to C^*(X) \otimes C^*(Y) \]

be the morphism given by

\[ \phi(x \otimes a \otimes y) = (a \cdot \bar{f}) \otimes y - x \otimes (a \cdot \bar{g}) \]

for \( x \otimes a \otimes y \in C^*(X) \otimes C^*(B) \otimes C^*(Y) \), i.e.,

\[ \text{Coker} \phi = C^*(X) \otimes C^*(B) \otimes C^*(Y). \]

Then, define the morphism

\[ \varphi : C^*(X) \otimes C^*(Y) \to C^*(X \times B \times Y) \]

as follows:

We have the morphism

\[ \nu : C^*(X \times B \times Y) \otimes C^*(X \times B \times Y) \to \text{Hom}(C^*(X \times B \times Y) \otimes C^*(X \times B \times Y), k) \]

given by

\[ \nu(a \otimes b)(x \otimes y) = (-1)^{\deg a \cdot \deg x} a(x) \otimes b(y), \]

for \( a \otimes b \in C^*(X \times B \times Y) \otimes C^*(X \times B \times Y) \) and \( x \otimes y \in C^*(X \times B \times Y) \otimes C^*(X \times B \times Y) \).

There is also a morphism
Let $w: \text{Hom}(C_*(X \times_B Y) \otimes C_*(X \times_B Y), k) \rightarrow C_*(X \times_B Y)$, which is induced by the composition of the map of the Eilenberg-Zilber Theorem and the diagonal map

$\Delta: X \times_B Y \rightarrow (X \times_B Y) \times (X \times_B Y)$.

We set the composition of morphisms by $U = w \circ v$.

Let $f': X \times_B Y \rightarrow Y$ and $g': X \times_B Y \rightarrow X$ be the induced maps of $f$ and $g$, respectively, in the commutative diagram given above. Then, $f'$ and $g'$ induces the morphisms

$f^*: C_*(X \times_B Y) \rightarrow C_*(Y)$ and $g^*: C_*(X \times_B Y) \rightarrow C_*(X)$.

Define

$V: C_*(X) \otimes C_*(Y) \rightarrow C_*(X \times_B Y) \otimes C_*(X \times_B Y)$

by $V(a \otimes b) = (-1)^{\deg f \cdot \deg g} (a \cdot g^*) \cdot (b \cdot f^*)$ for $a \otimes b \in C_*(X) \otimes C_*(Y)$.

The morphism $\varphi$ is defined as the composition of morphisms

$\varphi = U \circ V$.

Lemma 2.1:

The composition of morphisms $\varphi \cdot \phi = 0$.

Proof:

Let $x \otimes a \otimes y \in C_*(X) \otimes C_*(B) \otimes C_*(Y)$. Then,

$\varphi \cdot \phi (x \otimes a \otimes y) = w((x(a \cdot f)) \cdot g^* \otimes y \cdot f^* - x \cdot g^* \otimes ((a \cdot g)y) \cdot f^*)$.

Considering $C_*(X \times_B Y)$ as a right and left $k$-module, then

$(x(a \cdot f)) \cdot g^* \otimes y \cdot f^* - x \cdot g^* \otimes ((a \cdot g)y) \cdot f^*$ is 0 in $C_*(X \times_B Y) \otimes C_*(X \times_B Y)$.

Hence, $\varphi \cdot \phi (x \otimes a \otimes y) = 0$. ■

By the above lemma, there is an induced morphism

$\varphi^*: C_*(X) \otimes C_*(B) \otimes C_*(Y) \rightarrow C_*(X \times_B Y)$. 
Let $\mathcal{P} \xrightarrow{\mathcal{C}} \mathcal{C}^*(X) \to 0$ be the proper projective resolution of $\mathcal{C}^*(X)$ as $\mathcal{C}^*(B)$-module. Define the morphism $\vartheta$ as the composition of morphisms,

$$D(P) \otimes_{\mathcal{C}(B)} \mathcal{C}^*(Y) \xrightarrow{\vartheta} \mathcal{C}^*(X) \otimes_{\mathcal{C}(B)} \mathcal{C}^*(Y) \xrightarrow{\varphi} \mathcal{C}^*(X \times_B Y).$$

**Lemma 2.2:**

The morphism $\vartheta$ induces an isomorphism

$$\vartheta^* : \text{ Tor } \mathcal{C}^*(B)(\mathcal{C}^*(X), \mathcal{C}^*(Y)) \cong H^*(X \times_B Y).$$

**Proof:**

Let $\{E_r, d_r\}$ be the Serre spectral sequence for the fibration $F \xrightarrow{} X \times_B Y \xrightarrow{} X$, i.e., there is a filtration $F^p \mathcal{C}^*(X \times_B Y)$ of $H^*(X \times_B Y; k)$ which determines a spectral sequence $\bar{E}_r$ such that $\bar{E}_2^{p,q} = H^p(X; H^q(F; k))$ and $\bar{E}_r$ converges to $H^*(X \times_B Y; k)$. Similarly, let $\{E^*_r, d^*_r\}$ be the Serre spectral sequence for the fibration $F \xrightarrow{} Y \xrightarrow{} B$ with the filtration $F^p \mathcal{C}^*(Y)$ of $H^*(Y; k)$.

Now, since $B$ is simply connected and $k$ is a field, the Serre Theorem state that $E_2^* \cong H^*(B; k) \otimes H^*(F; k)$. Since the fibration $F \xrightarrow{} X \times_B Y \xrightarrow{} X$ is induced from the fibration $F \xrightarrow{} Y \xrightarrow{} B$, we also have that $E_2^* \cong H^*(X; k) \otimes H^*(F; k)$.

The filtration $F^j(D(P)) \sum_{m+n=j}^{\infty} \mathcal{P}^{m,n}$ of the graded object $D(P)$ determines a spectral sequence $\{E_r, d_r\}$ such that $E_1 = D(H(P))$, where the homology is taken with respect the external differential $d_E$ of the resolution $P$. The differential $d_1 : E_1 \xrightarrow{} E_1$ corresponds with the internal differential $d_I$.
of P.

Note that both $E'_r$ and $E_r$ are $H^*(B; k)$-modules, and further that $E_r$ is projective since $H(P)$ is projective.

From the definition of the resolutions,

$$E_p^q = H_p, q(P) = \begin{cases} 0 & \text{if } p \neq 0 \\ C^q(X) & \text{if } p = 0 \end{cases}$$

Hence,

$$E_p^q = \begin{cases} 0 & \text{if } p \neq 0 \\ H^q(X; k) & \text{if } p = 0 \end{cases}$$

Define the spectral sequence $\{ E'^r, d'^r \}$ by

$$E'^r = E_r \otimes_{\Lambda_r} E'^r,$$

with the differential $d'^r: E'^r \rightarrow E'^r$ given by

$$d'^r = d_r \otimes_{\Lambda_r} 1 - 1 \otimes d_r.$$ The differential $k$-algebra $\Lambda_r$ is defined inductively by $\Lambda_1 = C^*(B)$ and $\Lambda_{r+1} = H(\Lambda_r) = H^*(B; k)$ for all $r \geq 2$. From Lemma 2.13, we have the isomorphism

$$E'^r = E_{r+1} \otimes_{\Lambda_{r+1}} E'^{r+1} \cong H(E_r) \otimes_{H(\Lambda_r)} H(E'^r) \cong H(E_r \otimes_{\Lambda_r} E'^r) = H(E^r).$$

Therefore, $E'^r$ is a spectral sequence.

Since $E_r$ converges to $H(D(P))$ and $E^r$ converges to $H^*(Y; k) = H(C^*(Y))$, the spectral sequence $E'^r$ converges to the object $H(D(P)) \otimes H^*(B; k) H(C^*(Y)) = H(D(P)) \otimes C^*(B) C^*(Y))$ by Lemma 2.13. Hence, $E'^r$ converges to $\text{Tor}_{C^*}(B) (C^*(X), C^*(Y))$.

We have the isomorphism $\Theta_2: E'^r \cong E_r$ given by

$$E'^2 = E_2 \otimes_{H^*(B; k)} E^2 = H^*(X; k) \otimes_{H^*(B; k)} H^*(B; H^*(F; k))$$

$$\cong H^*(X; k) \otimes_{H^*(B; k)} H^*(B; k) \otimes H^*(F; k) = H^*(X; k) \otimes H^*(F; k)$$

$$\cong H^*(X; H^*(F; k)) = E_2.$$
We are now ready to apply Theorem 3.1 of the last chapter to the following theorem.

Theorem 2.3:

Let

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow g \\
X & \xrightarrow{\epsilon} & B
\end{array} \]

be a commutative diagram of topological spaces, where B is a simply connected space and g is a Serre fibration.

Then, there is a spectral sequence of k-algebras, \( \{E_r, d_r\} \), such that

1) \( E_r \) converges to the k-algebra \( H^*(X \times_BY;k) \), and

2) \( E_2 \cong \text{Tor}_{H^*(B;k)}(H^*(X;k), H^*(Y;k)) \).

Proof:

We know that \( C^*(X) \) and \( C^*(Y) \) are \( C^*(B) \)-modules. Therefore, by Theorem 3.1, there is a spectral sequence \( \{E_r, d_r\} \) such that \( E_2 \cong \text{Tor}_{H^*(B;k)}(H^*(X;k), H^*(Y;k)) \) and that \( E_r \) converges to \( \text{Tor}_{C^*(B)}(C^*(X), C^*(Y)) \) which is isomorphic to \( H^*(X \times_BY;k) \).

The fact that \( \{E_r, d_r\} \) is a sequence of k-algebras follows from the discussion in Sec.1.
Chapter III

The Extension of the Geometric Spectral Sequence

We have seen in Chapter II how the Eilenberg-Moore spectral sequence came about in geometry. Such spectral sequence existed because there were certain algebraic structures in the ordinary cohomology on the category of topological spaces. Our aim in this chapter is to generalize the conditions under which this spectral sequence exists. The first step is to generalize the category of topological spaces. And then, after defining a cohomology on the extended category we will show that we obtain an extended version of the Eilenberg-Moore spectral sequence.

In this chapter, all topological spaces will be assumed to have the homotopy type of a countable CW-complex.

1. The Category of Fibered Spaces Over B.

We will denote the category of (pointed) topological spaces by \((\text{Top})_\ast\). Then, with some space \(B\) in \((\text{Top})_\ast\) fixed, we define the following category:

The object in the category is the sequence of maps

\[
X \xrightarrow{p_X} B \xrightarrow{s_X} X
\]

in \((\text{Top})_\ast\) such that \(p_X \cdot s_X = 1_B\). We will denote the object in the category by \(\overline{X} : X \rightarrow B \rightarrow X\).
The morphism $\phi: \bar{X} \rightarrow \bar{Y}$ between two objects is a map $\phi: X \rightarrow Y$ in $(\text{Top})_\bullet$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{p_X} & B \\
\downarrow & & \downarrow s_X \\
Y & \xrightarrow{p_Y} & B
\end{array}
\begin{array}{ccc}
\xrightarrow{s_Y} & & \xrightarrow{X} \\
\end{array}
$$

commutes.

This category will be called the category of fibered spaces over $B$, and will be denoted by $(\text{Top}/B)_\bullet$. Note that if $B$ is a one-point space, then $(\text{Top}/B)_\bullet$ is just $(\text{Top})_\bullet$.

**Remark:**

For object $\bar{X}$ in $(\text{Top}/B)_\bullet$, each point $b \in B$ determines a subspace $F_b(X) = \{ x \in X | p_X(x) = b \}$ of $X$, which has the basepoint, $s_X(b)$. The subspace $F_b(X)$ is called the fiber of $X$ at $b$.

Thus, the name "fibered spaces".

The initial and terminal object in the category $(\text{Top}/B)_\bullet$ is $\bar{B}: B \rightarrow B \rightarrow B$. This is easily seen from the fact that, given an object $\bar{X}$, $p_X: \bar{X} \rightarrow \bar{B}$ and $s_X: \bar{B} \rightarrow \bar{X}$ are the only morphisms between $\bar{X}$ and $\bar{B}$.

2. **Topological Constructions in $(\text{Top}/B)_\bullet$.**

We have extended the category $(\text{Top})_\bullet$ to $(\text{Top}/B)_\bullet$. We will define a cohomology on $(\text{Top}/B)_\bullet$. But, whenever there is a cohomology (or homology), we need to have such topological notions as homotopies, mapping cylinders, cones, cofibrations,
etc., and since \((\text{Top}/B)\) is an extension of the category \((\text{Top})\), the topological constructions in \((\text{Top}/B)\) should be an extension of the ones in \((\text{Top})\). We, therefore, mimic the topological constructions in \((\text{Top})\).

**Definition 2.1:**

Let \(\overline{X}\) be an object in \((\text{Top}/B)\). Then, \(\overline{X}'\) is a fibered subspace of \(\overline{X}\) over \(B\) if

1) \(\overline{X}'\) is a topological subspace of \(\overline{X}\), and

ii) the inclusion map \(i: \overline{X}' \rightarrow \overline{X}\) is a morphism in \((\text{Top}/B)\).

The pair \((\overline{X}, \overline{X}')\) will be called the pair of fibered spaces over \(B\). From this, one can define the category of pairs of \((\text{Top}/B)\).

If \((\overline{X}, \overline{X}')\) is a pair of fibered spaces over \(B\), then we define the quotient space \(\overline{X}/\overline{X}'\) over \(B\) to be the object in \((\text{Top}/B)\), where the sequence of maps \(\overline{X}/\overline{X}' \xrightarrow{p_X} B \xrightarrow{s_X} \overline{X}/\overline{X}'\) is given by

\[p_X^*(x) = p_X(x) \text{ for } (x) \in \overline{X}/\overline{X}' \quad \text{and} \quad s_X^*(b) = (s_X(b)) \text{ for } b \in B.\]

**Definition 2.2:**

A continuous morphism \(F:\overline{X} \times I \rightarrow \overline{Y}\), where \(I = [0,1]\), is a continuous map \(F:X \times I \rightarrow Y\) with \(F(\cdot,t)\) a morphism in \((\text{Top}/B)\) for all \(t \in I\).
Two morphisms \( \varphi_0, \varphi_1 : X \longrightarrow Y \) in \( \text{Top}/B \) are homotopic if there is a continuous morphism \( F : X \times I \longrightarrow Y \) such that \( F(x,0) = \varphi_0(x) \) and \( F(x,1) = \varphi_1(x) \) for all \( x \in X \). The fact of the homotopy between \( \varphi_0 \) and \( \varphi_1 \) will be denoted by \( \varphi_0 \simeq \varphi_1 \).

As in \( \text{Top} \), the homotopy in \( \text{Top}/B \) is an equivalence relation, and the composite of homotopic morphisms are homotopic. Thus, we can form the homotopy category of fibered spaces over \( B \). The morphism \( \varphi : X \longrightarrow Y \) is a homotopy equivalence if there is a morphism \( \psi : Y \longrightarrow X \) such that \( \psi \circ \varphi = 1_X \) and \( \varphi \circ \psi = 1_Y \).

**Remark:**

The equivalence class of homotopic morphisms from \( X \) to \( Y \), denoted \( (X, Y) \), has a basepoint \( (\tilde{B}, \overline{B}) \) = the homotopy class of the trivial morphism \( 0_B : X \longrightarrow \overline{B} \longrightarrow Y \).

**Definition 2.3:**

A pair of fibered spaces \( (X, X') \) is said to have the homotopy extension property with respect to the fibered space \( Y \) if, given morphism \( \varphi : X \longrightarrow Y \) and a continuous morphism \( G : X' \times I \longrightarrow Y \) such that \( G(x',0) = \varphi(x') \) for \( x' \in X' \), then there is a continuous morphism \( F : X \times I \longrightarrow Y \) such that \( F(x,0) = \varphi(x) \) for \( x \in X \) and \( F \big|_{X' \times I} = G \).
**Definition 2.4:**
A morphism \( \phi: \overline{X} \rightarrow \overline{Y} \) is a cofibration if, given a morphism \( \Phi: \overline{Y} \rightarrow \overline{Z} \) and a continuous morphism \( G: \overline{X} \times I \rightarrow \overline{Z} \) such that \( G(\cdot, 0) = \Phi \), there is a continuous morphism \( F: \overline{Y} \times I \rightarrow \overline{Z} \) such that \( F(\cdot, 0) = \Phi \) and \( F(\phi(x), t) = G(x, t) \) for all \( x \in X \) and \( t \in I \).

Considering the diagram
\[
\begin{array}{ccc}
\overline{X}' \times 0 & \xrightarrow{\Phi} & \overline{X}' \times I \\
\downarrow & & \downarrow \\
\overline{X} \times 0 & \xrightarrow{\phi} & \overline{X} \times I \\
\end{array}
\]
the inclusion map \( i: \overline{X}' \rightarrow \overline{X} \) is a cofibration if and only if \((\overline{X}, \overline{X}')\) has the homotopy extension property with respect to any object in \((\text{Top}/B)_{\ast}\).

There are many cofibrations in \((\text{Top}/B)_{\ast}\). We make heavy use of the one we define now.

Let \( \overline{X} \) be an object in \((\text{Top}/B)_{\ast}\). We denote by \( C(X) \) the topological space \( X \times I \) with the identification of the points \( (x, 1) = (x', 1) \) if and only if \( p_X(x) = p_X(x') \) for \( x, x' \in X \), and

\[ \bigvee_{b \in B} s_X(b) \times I \], the disjoint union of \( s_X(b) \times I \).

Note that if \( B \) is a one-point space, \( C(X) \) is just the ordinary reduced cone over \( X \). The space \( C(X) \) is an object in \((\text{Top}/B)_{\ast}\) with the two maps \( p_{C(X)}((x, t)) = p_X(x) \) for \((x, t) \in C(X)\) and \( s_{C(X)}(b) = (s_X(b), 0) \) for \( b \in B \). The fact that \( C(X) \) is a fibered space over \( B \) will be shown by \( \overline{C}(X) \).
If \( \varphi : X \to Y \) is a morphism in \((\text{Top}/B)_*\), then the mapping cone of \( \varphi \), denoted \( \overline{C}(\varphi) \), is defined as the space \( C(\varphi) = C(X) \cup Y \) with the identification of the points \( \{(x, 0) = \varphi(x) \text{ for all } x \in X, \) and \( s_Y(b) = (s_X(b), r) \text{ for all } b \in B \) and \( r \in I \} \).

The two maps \( p_{C(\varphi)} : C(\varphi) \to B \) and \( s_{C(\varphi)} : B \to C(\varphi) \), which makes \( C(\varphi) \) an object in \((\text{Top}/B)_*\), are given by

\[
P_{C(\varphi)}((x)) = p_C(X)(x) \text{ for } (x) \in C(\varphi),
\]
and \( s_{C(\varphi)}(b) = (s_X(b)) \text{ for } b \in B \).

There is a natural inclusion morphism \( \varphi_0 : Y \to \overline{C}(\varphi) \) given by \( \varphi_0(y) = (y) \in C(\varphi) \) for all \( y \in Y \).

**Proposition 2.5:**

The natural inclusion morphism \( \varphi_0 \) given above is a cofibration in \((\text{Top}/B)_*\).

**Proof:**

Consider the diagram

\[
\begin{array}{ccc}
Y \times 0 & \to & Y \times I \\
\downarrow & & \downarrow \\
C(\varphi) \times 0 & \to & C(\varphi) \times I \\
\phi \downarrow & & \downarrow F \\
\overline{Z} & \to & \overline{C}(\varphi)
\end{array}
\]

Given any morphism \( \phi : \overline{C}(\varphi) \to \overline{Z} \) and a continuous morphism \( G : Y \times I \to \overline{Z} \), we need to find the continuous morphism \( F \) which makes the above diagram commutative.

We explicitly define the morphism \( F \) as follows:

\[
F((y), r) = G(y, r) \text{ for } (y) \in \overline{C}(\varphi) \text{ and } r \in I, \text{ and}
\]
\[ F((x,r),r') = \begin{cases} 
((x,(r-r)/(1-r'))) & \text{for } (x,r) \in C(\mathcal{Q}), \ 0 \leq r' < r \leq 1, \\
((x,0)) & \text{for } r = r'. 
\end{cases} \]

Then, \( F((x,r),0) = ((x,r)) \) for all \((x,r) \in C(\mathcal{Q})\), and \( F((y),0) = G(y,0) = \Phi(\mathcal{Q}_0(y)) = \phi((y)) \) for all \((y) \in C(\mathcal{Q})\). Hence, \( F(\cdot,0) = \Phi \).

Now, \( F(\mathcal{Q}_0(y),r) = F(y),r) = G(y,r) \) for all \( y \in Y \) and \( r \in I \).

Therefore, the fibered pair \((\mathcal{C}(\mathcal{Q}), Y)\) has the homotopy extension property. \( \square \)

**Definition 2.6:**

Any sequence of morphisms isomorphic to the sequence

\[ X \xrightarrow{\varphi} Y \xrightarrow{\mathcal{Q}_0} \mathcal{C}(\mathcal{Q}) \]

is called a cofibration sequence.

**Lemma 2.7:**

Let \( X \xrightarrow{\varphi} Y \xrightarrow{\mathcal{Q}_0} \mathcal{Y}_0 \) be a cofibration sequence. Then, \( \mathcal{C}(\mathcal{Q}_0) \simeq \mathcal{Y}_0/\mathcal{Y} \) is a homotopy equivalence in \((\text{Top}/B)^k\).

**Proof:**

We may assume that \( \mathcal{Y}_0 = \mathcal{C}(\mathcal{Q}) \). Then, we have that \( \mathcal{Y}_0/\mathcal{Y} = \mathcal{C}(\mathcal{Q})/\mathcal{Y} = \mathcal{C}(\mathcal{Q}_0)/\mathcal{C}(Y) \). The canonical map \( \mathcal{C}(\mathcal{Q}) \rightarrow \mathcal{C}(\mathcal{Q}_0)/\mathcal{C}(Y) \) is a homotopy equivalence. (An explicit description of the homotopy is made in Corollary 6, Sec. 1 of Chapter 7 in (2). The same description applies to the above canonical map with care taken on \( \mathcal{C}(Y) \), the extension of the ordinary cone over \( Y \).) \( \square \)
Lemma 2.8:

Let $X \overset{p}{\longrightarrow} Y \overset{q_0}{\longrightarrow} Y_0$ be a cofibration sequence. Then, the sequence of homotopy classes

$$(Y_0, Z) \overset{q_0^*}{\longrightarrow} (Y, Z) \overset{q^*}{\longrightarrow} (X, Z)$$

is exact for any object $Z$ in $\text{(Top/B)}_*$. 

Proof:

See Theorem 3, Sec. 1 of Chapter 7 in (2). 

The suspension of an object $\overline{X}$ in $\text{(Top/B)}_*$ is, as expected, a functor on $\text{(Top/B)}_*$. The suspension of $\overline{X}$ will be denoted by $\Sigma \overline{X}$. 

The object $\Sigma \overline{X}$ is the topological space $\Sigma X = X \times I$ with the identification of the points

$$
\begin{cases}
(x, 0) = (y, 0) & \text{if and only if } p_X(x) = p_X(y) \text{ for } x, y \in X, \\
(x, 1) = (y, 1) & \text{for } x, y \in X, \text{ and} \\
(s_X(b), r) = (s_X(b), r') & \text{for } b \in B \text{ and } r, r' \in I,
\end{cases}
$$

together with the two maps $\Sigma p_X: \Sigma \overline{X} \longrightarrow B$ and $\Sigma s_X: B \longrightarrow \Sigma \overline{X}$ given by $\Sigma p_X((x, t)) = p_X(x)$ for all $x \in X$ and $t \in I$, and $\Sigma s_X(b) = (s_X(b), t)$ for all $b \in B$ and $t \in I$.

If $\Phi: \overline{X} \longrightarrow \overline{Y}$ is a morphism in $\text{(Top/B)}_*$, define the morphism $\Sigma \Phi: \Sigma \overline{X} \longrightarrow \Sigma \overline{Y}$ by $\Sigma \Phi((x, r)) = (\Phi(x), r)$ for all $(x, r) \in \Sigma \overline{X}$. Note that $\Sigma \Phi$ is a morphism in $\text{(Top/B)}_*$ since the diagram

$$\begin{array}{ccc}
\Sigma X & \xrightarrow{p_{\Sigma X}} & B \\
\downarrow & & \downarrow \\
\Sigma Y & \xrightarrow{p_{\Sigma Y}} & B
\end{array}
\quad
\begin{array}{ccc}
\Sigma X & \xrightarrow{s_{\Sigma X}} & \Sigma X \\
\downarrow & & \downarrow \\
\Sigma Y & \xrightarrow{s_{\Sigma Y}} & \Sigma Y
\end{array}
$$

commutes.
Furthermore, whenever \( \Phi \) makes sense in \((\text{Top/B})_*\), then 
\[ \sum(\Phi) = \sum \Phi \sum \Phi \] and \( \sum 1_{X} = 1_{\sum X} \). Hence, \( \sum \) is a covariant functor from \((\text{Top/B})_*\) to itself.

**Remark:**

If \( \Phi: \overline{X} \rightarrow \overline{Y} \) is a monomorphism, i.e., \( \Phi: X \rightarrow Y \) is a monic map commuting with \( \{p_X,s_X\} \) and \( \{p_Y,s_Y\} \), then \( \sum \Phi \) is also a monomorphism. This is because we have the identification of the points

\[
\begin{cases}
(Q(x),0) = (Q(x'),0) \text{ if and only if } p_Y \cdot Q(x) = p_Y \cdot Q(x'), \\
(Q(x),1) = (Q(x'),1), \text{ and} \\
(Q(x),r) = (Q(x'),r') \text{ for } Q(x) = Q(x') = s_Y(b),
\end{cases}
\]

in \( \overline{Y} \) from the identification of the points

\[
\begin{cases}
(x,0) = (x',0) \text{ if and only if } p_X(x) = p_X(x'), \\
(x,1) = (x',1), \text{ and} \\
(s_X(b),r) = (s_X(b),r'),
\end{cases}
\]

in \( \overline{X} \).

The suspension of the fibered pair \((\overline{X},\overline{X}')\) is defined as 
\( \sum(\overline{X},\overline{X}') = (\sum \overline{X},\sum \overline{X}') \). The \( n \)-th suspension of \( \overline{X} \) is defined iteratively as, 
\( \overline{X}^0 = \overline{X} \) and 
\( \overline{X}^n = \overline{X}^{n-1} \).

The functor \( \sum \) has the expected properties as exemplified in the following two lemmas.
Lemma 2.9:

Let \( \overline{X} \overline{\phi}_0 \overline{Y} \overline{\phi}_0 \overline{Y}_0 \) be a cofibration sequence. Then, \( \overline{\sum X} = \overline{C}(\phi_0) \).

Moreover, there is a morphism \( \Delta: \overline{Y}_0 \longrightarrow \overline{\sum X} \) which is homotopically equivalent to an inclusion.

Proof:

For the first part, all we need to do is to observe that \( \sum X = \overline{Y}_0 \overline{Y} = \overline{C}(\phi) \overline{Y} \). Then, by Lemma 2.7, \( \sum X = \overline{C}(\phi_0) \).

Now, there is a canonical inclusion \( \phi_1: \overline{Y}_0 \longrightarrow \overline{C}(\phi_0) \) into the mapping cone. The morphism \( \Delta \) is the composition \( \overline{Y}_0 \phi_1 \overline{C}(\phi_0) \phi \longrightarrow \overline{\sum X} \) where \( \phi \) is the homotopy equivalence.

Lemma 2.10:

Let \( \phi: \overline{X} \longrightarrow \overline{Y} \) be a morphism in \((\text{Top}/B)^*\). Then,

\[ \sum \overline{C}(\phi) = \overline{C}(\sum \phi). \]

Proof:

An explicit realization of the objects \( \sum \overline{C}(\phi) \) and \( \overline{C}(\sum \phi) \) shows that the topological spaces \( \sum \overline{C}(\phi) = \overline{C}(\phi) \times I \) (with certain points identified), and \( \overline{C}(\sum \phi) = \overline{C}(\sum X) \lor \overline{\sum Y} \) (with certain points identified) are isomorphic.

Corollary 2.10:

If \( \overline{X} \overline{\phi}_0 \overline{Y} \overline{\phi}_0 \overline{Y}_0 \) is a cofibration sequence, then \( \sum \overline{X} \sum \overline{\phi} \sum \overline{Y} \sum \overline{\phi}_0 \sum \overline{Y}_0 \) is also a cofibration sequence.

Proof:

We may assume that \( \overline{Y}_0 = \overline{C}(\phi) \). Then, \( \sum \overline{C}(\phi) = \overline{C}(\sum \phi) \), i.e.,
is the mapping cone of the morphism $\sum \phi$. 

3. The Products in $(\text{Top}/B)^\ast$.

We will be interested in two products of objects in $(\text{Top}/B)^\ast$. The one is the so-called pullback of the maps $p_X : X \to B$ and $p_Y : Y \to B$ of objects $\overline{X}$ and $\overline{Y}$ in $(\text{Top}/B)^\ast$. The other is the smash product, $\overline{X} \wedge_B \overline{Y}$, of objects $\overline{X}$ and $\overline{Y}$. There are others, such as the cross-product of $\overline{X}$ and $\overline{Y}$, but they are of no interest to us here.

We define, for objects $\overline{X}$ and $\overline{Y}$ in $(\text{Top}/B)^\ast$, $\overline{X} \times_B \overline{Y}$ to be the topological space $X \times_B Y = \{(x,y) \in X \times Y | p_X(x) = p_Y(y)\}$ together with the two maps $p_X \times_B p_Y : X \times_B Y \to B$ and $s_X \times_B s_Y : B \to X \times_B Y$ given by

$$p_X \times_B p_Y (x,y) = p_X(x) = p_Y(y) \text{ for } (x,y) \in X \times_B Y \text{ and}$$

$$s_X \times_B p_Y (b) = (s_X(b), s_Y(b)) \text{ for } b \in B.$$  

Note that if $B$ is a one-point space, then $X \times_B Y$ is the usual cross-product in $(\text{Top})^\ast$.

The smash product of $\overline{X}$ and $\overline{Y}$ is the quotient space of $X \times_B Y$ with the points $\{(x,s_Y(b)) = (s_X(b), y) \text{ for } (x,y) \in X \times_B Y \text{ and } b \in B\}$ identified. We denote this quotient space by $\overline{X} \wedge_B \overline{Y}$. The two maps $p_X \wedge_B p_Y : \overline{X} \wedge_B \overline{Y} \to B$ and $s_X \wedge_B s_Y : B \to \overline{X} \wedge_B \overline{Y}$ are given by
\[ p_X \wedge_B p_Y((x,y)) = p_X(x) = p_Y(y) \text{ for } (x,y) \in X \wedge_B Y, \text{ and} \]
\[ s_X \wedge_B s_Y(b) = (s_X(b), s_Y(b)) \text{ for } b \in B. \]
The above maps make the smash product, \( X \wedge_B Y \), an object in \((\text{Top}/B)^\#\).

For the fibered pair \((X, X')\), we define \( Y \wedge_B (X, X') \) to be the fibered pair \((Y \wedge_B X, Y \wedge_B X')\).

The following lemma gives a relationship between the suspension functor \( \Sigma \) and the smash product.

**Lemma 3.1:**
\[ (X \wedge_B Y) \wedge_B \Sigma Y \text{ is a homotopy equivalence.} \]

**Proof:**

The two maps
\[ \Phi : \Sigma (X \wedge_B Y) \rightarrow X \wedge_B \Sigma Y \quad \text{and} \quad \Phi : X \wedge_B \Sigma Y \rightarrow \Sigma (X \wedge_B Y) \]
defined by
\[ \Phi (((x,y), t)) = (x, (y, t)) \text{ for } ((x,y), t) \in \Sigma (X \wedge_B Y) \]
\[ \Phi ((x, (y,t))) = ((x,y), t) \text{ for } (x, (y, t)) \in X \wedge_B \Sigma Y \]
gives the homotopy equivalence between \( \Sigma (X \wedge_B Y) \) and \( X \wedge_B \Sigma Y \).

4. **The Forgetful Functors on \((\text{Top}/B)^\#\).**

We will be interested in two certain forgetful functors on \((\text{Top}/B)^\#\) and their respective adjoints. The first functor \( \mathcal{F} \) forgets the fibered structure of the object \( X \), and the other functor \( F \) forgets the basepoints of each fiber space of \( X \).
For describing the second functor $F$, we need another category called the category of spaces over $B$, denoted $\text{Top}/B$. The objects of $\text{Top}/B$ are the maps $p_Z:X \to B$. The morphism between two objects $p_Z:X \to B$ and $p_Y:Y \to B$ is a map $\varphi:X \to Y$ commuting with $p_Z$ and $p_Y$. The forgetful functor $F$ is from $(\text{Top}/B)^\ast$ to $\text{Top}/B$.

The functor $\overline{\mathcal{F}}$ from $(\text{Top}/B)^\ast$ to $(\text{Top})^\ast$ is defined by $\overline{\mathcal{F}(X)} = X/s_X(B)$, for object $X$ in $(\text{Top}/B)^\ast$.

The functor $\overline{\mathcal{G}}$, given by $\overline{\mathcal{G}(X)} = X \times B$ for the topological space $X$, is adjoint to $\overline{\mathcal{F}}$. Note that $\overline{\mathcal{G}(X)}$ is an object in $(\text{Top}/B)^\ast$ by two maps $\overline{\mathcal{G}(X)} \to B$ and $1:B \to X \times B$.

The second functor $F$ is the functor which drops the map $s_X:B \to X$ in the object $X$. Its adjoint functor $G$ is defined by $G(X) = X \vee B$ for the object $p_Z:X \to B$ in $\text{Top}/B$. The maps $p_X \vee 1_B:X \vee B \to B$ and $1:B \to X \vee B$ make $G(X)$ an object in $(\text{Top}/B)^\ast$.

5. The Extended Cohomology on $(\text{Top}/B)^\ast$.

We will use the notation of Chapter II for the ordinary cohomology $H^\ast(\cdot;k)$ with coefficient $k$. Throughout the following sections, $k$ will be assumed a field.

We extend $H^\ast(\cdot;k)$ to a cohomology on $(\text{Top}/B)^\ast$ by first defining the cohomology theory on $(\text{Top}/B)^\ast$ and then show that the extended cohomology satisfies the axioms of the theory.
**Definition 5.1:**

A cohomology theory on \((\text{Top}/B)^*\) is a family of contravariant functors \(\{h^n; n \in \mathbb{Z}\}\) from \((\text{Top}/B)^*\) to a category of \(\mathbb{Z}\)-graded objects from an Abelian category, such that

1) *(The Homotopy Axiom)*

If morphisms \(\varphi\) and \(\phi\) are homotopic, then

\[ h^n(\varphi) = h^n(\phi) \]

for all \(n \in \mathbb{Z}\)

ii) *(The Suspension Axiom)*

For all \(n \in \mathbb{Z}\), there is a natural equivalence

\[ \sigma : h^n \rightarrow h^{n+1} \]

iii) *(The Exactness Axiom)*

Given a cofibration sequence \(\varnothing \rightarrow X \rightarrow Y \rightarrow Y_0\), there is a long exact sequence

\[ \ldots \rightarrow h^{n-1}(X) \rightarrow h^n(Y_0) \rightarrow h^n(Y) \rightarrow h^n(Y) \rightarrow h^n(X) \rightarrow \ldots \]

where \(\delta\) is the connecting homomorphism.

**Remarks:**

1) We define the cohomology of the pair \((\overline{X}, \overline{X}')\) as the cohomology of the mapping cone of the map \(i : \overline{X}' \rightarrow \overline{X}\),

\[ h^n(\overline{X}, \overline{X}') = h^n(\overline{c}(1)) \]

for all \(n \in \mathbb{Z}\).

From the Exactness Axiom, we have the long exact sequence for the pair \((\overline{X}, \overline{X}')\).

2) In the Exactness Axiom, it is enough to know that

\[ h^n(Y_0) \rightarrow h^n(Y) \rightarrow h^n(X) \]

is exact for all \(n \in \mathbb{Z}\). Since \(Y_0 = \overline{c}(\varphi)\) and \(\sum X \circ \overline{c}(\varphi_0)\) by Lemma 2.9, the composition of the morphisms \(h^n(\varphi_1) \circ \sigma\) is the connecting homomorphism \(\delta\), where \(\sigma\)
is the morphism from the Suspension Axiom and \( \varphi_1: \bar{Y}_0 \to \overline{C}(\varphi_0) \) is the canonical inclusion into the mapping cone. In other words,

\[
\cdots \to h^{n-1}(\bar{X}) \to h^n(\bar{Y}_0) \to h^n(\bar{Y}) \to h^n(\overline{C}(\varphi_0)) \to h^n(\bar{X}) \to \cdots
\]

is an exact sequence.

Using the forgetful functor \( \mathcal{F} \), we extend the ordinary cohomology on \((\text{Top})_*\) to a cohomology theory \( H_B^*(\cdot) \) on \((\text{Top}/B)_*\) by

\[
H_B^*(\bar{X}) = H^*(\mathcal{F}(\bar{X}); k) = H^*(X, s_X(B); k)
\]

for object \( \bar{X} \).

**Theorem 5.2:**

\( H_B^*(\cdot) \) is a cohomology theory on \((\text{Top}/B)_*\) with the Abelian category being the category of \( k \)-modules.

**Proof:**

If \( \varphi, \Phi: X \to Y \) are two homotopic morphisms in \((\text{Top}/B)_*\), then, \( \varphi, \Phi: X \to Y \) and \( \varphi|_{s_X(B)}, \Phi|_{s_X(B)}: s_X(B) \to s_Y(B) \) are also homotopic in the category \((\text{Top})_*\). Therefore, the maps \( H^*(\varphi) = H^*(\Phi) \) and \( H^*(\varphi|_{s_X(B)}) = H^*(\Phi|_{s_X(B)}) \) are identical. From the long exact sequences for the topological pairs \((X, s_X(B))\) and \((Y, s_Y(B))\) in the category \((\text{Top})_*\), we have that \( H_B^*(\varphi) = H_B^*(\Phi) \).

Let \( \bar{X} \to \bar{Y} \to \bar{Y}_0 \) be a cofibration sequence in \((\text{Top}/B)_*\) and consider the diagram
The first two rows and all the columns are exact for all \( n \) in the above diagram. Then, the third row is also exact for all \( n \). Hence, \( H^*(\cdot) \) satisfies the Exactness Axiom.

The Suspension Axiom follows from the five lemma applied on the diagram

\[
\begin{array}{c}
\cdots \\
H^n(B) \\
H^n(\Sigma(B)) \\
\end{array}
\begin{array}{c}
\cdots \\
H^n(B) \\
H^n(\Sigma(B)) \\
\end{array}
\begin{array}{c}
\cdots \\
H^n(B) \\
H^n(\Sigma(B)) \\
\end{array}
\begin{array}{c}
\cdots \\
H^n(B) \\
H^n(\Sigma(B)) \\
\end{array}
\begin{array}{c}
\cdots \\
H^n(B) \\
H^n(\Sigma(B)) \\
\end{array}
\begin{array}{c}
\cdots \\
H^n(B) \\
H^n(\Sigma(B)) \\
\end{array}
\]

where the rows are exact by the Exactness Axiom.

Remark:

For the ordinary cohomology \( H^n \) on the category \((\text{Top})_*\),
\( H^n \) is zero for \( n \leq 0 \). Therefore, we make a note that the extended cohomology \( H^*_B(\cdot) = 0 \) for \( n \leq 0 \).

There is a modular structure in \( H^*_B(X) \) induced from the
one in \( C^*(X) = \text{Hom}(C_*(X), k) \) of Chapter II. We have already remarked in Chapter II that \( H^*(X; k) \) is a right and left \( H^*(B; k) \)-module if there is a map \( f: X \to B \). Therefore, for any object \( X \) in \( (\text{Top}/B)_\# \), \( H^*_B(X) = H^*(X, S^X(B); k) \) is also a \( H^*(B; k) \)-module.

**Definition 5.3:**

The n-sphere \( S^n_B \) in \( (\text{Top}/B)_\# \) will be defined by \( S^n_B = \bigcap(S^n) \), where \( S^n \) is the n-sphere in \( \mathbb{R}^{n+1} \).

The coefficient module for the cohomology \( H^*_B(\cdot) \) is \( H^*_B(S^0_B) = H^*(S^0 \times B, B; k) \cong H^*(B; k) \). Hence, \( H^*_B(X) \) is also a \( H^*(B; k) \)-module.

6. **The Multiplicative Cohomology on \( (\text{Top}/B)_\# \).**

The cross product of the ordinary cohomology
\[
\mu: H^*(X/A; k) \otimes H^*(Y/B; k) \longrightarrow H^*(X \times Y/(A \times Y) \cup (B \times X); k)
\]
for the topological pairs \((X, A)\) and \((Y, B)\), gives a multiplicative structure to the extended cohomology \( H^*_B(\cdot) \).

**Definition 6.1:**

A cohomology theory \( \{h^n; n \in \mathbb{Z}\} \) on \( (\text{Top}/B)_\# \) is multiplicative if there is a natural (external) product
\[
h^*(X) \otimes h^*(Y) \longrightarrow h^*(X \wedge_B Y)
\]
for objects \( X \) and \( Y \) in \( (\text{Top}/B)_\# \).
For the extended cohomology $H_B(\ast)$ on $(\text{Top}/B)_\ast$, the composition of the morphisms,

$$
\begin{align*}
H_B^*(X) \otimes H_B^*(Y) \\
\mu^* \\
H_B^*(X \times Y/(s_X(B) \times Y) \vee (s_Y(B) \times X); k) \\
H_B^*(1) \\
H_B^*(X \wedge_B Y)
\end{align*}
$$

where $\mu^*$ is given above and $i$ is the inclusion map, is the external product.

Let $\overline{X}$ and $\overline{Y}$ be objects in $(\text{Top}/B)_\ast$. Consider the pullback of the maps $p_X: X \rightarrow B$ and $p_Y: Y \rightarrow B$,

$$
\begin{array}{ccc}
X & \times_B Y & \rightarrow & X \\
\downarrow & & \downarrow & \\
Y & \rightarrow & P_X \\
\end{array}
$$

Then, the maps $s_X: B \rightarrow X$ and $s_Y: B \rightarrow Y$ induces the maps $r: Y \rightarrow X \times_B Y$ and $u: X \rightarrow X \times_B Y$. Explicitly, $r$ and $u$ are defined by

$$
r(y) = (s_X p_Y(y), y) \text{ for } y \in Y \text{ and } u(x) = (x, s_Y p_X(x)) \text{ for } x \in X.
$$

**Lemma 6.2:**

$$
X \times_B Y/(r(Y) \vee u(X)) \cong X \wedge_B Y/s_X \wedge_B s_Y(B)
$$

**Proof:**

Both of the above spaces are the quotients of the space $X \times_B Y$. The identified points $r(Y) \vee u(X)$ is same as the identified points $\{(x, s_Y(b) = (s_X(b), y) \text{ for all } (x, y) \in X \times_B Y$
and all \( b \in B \), and \((s_x(b), s_y(b)) = (s_x(b'), s_y(b'))\) for all \( b \) and \( b' \in B \). Hence, the lemma. □

**Theorem 6.3:**

Let \( B \) be a simply connected space. Let \( H_B^\ast(\mathbb{X}) \) be finite and projective as \( H_B^\ast(S_B) \)-module and let \( p_Y \) in \( \mathbb{Y} \) be a Serre fibration with \( H_B^\ast(\mathbb{Y}) \) also finite. Then,
\[
H_B^\ast(\mathbb{X}) \otimes_{H_B^\ast(S_B)} H_B^\ast(\mathbb{Y}) \cong H_B^\ast(\mathbb{X} \wedge_B \mathbb{Y}).
\]

**Proof:**

Let \( \{E_r, d_r^\ast\} \) be the Serre spectral sequence for the fibration \( p_Y : (Y, s_Y(B)) \longrightarrow (B, *) \). Then, \( E_r \) converges to \( H^\ast(Y, s_Y(B); k) \) and \( E_2 \cong H^\ast(B; k) \otimes H^\ast(F, s_Y(b); k) \), where \((F, s_Y(b))\) is the fiber of the map \( p_Y \).

The induced map \( p_Y^\ast : (X \times \mathbb{Y}, r(Y) \vee u(X)) \longrightarrow (X, s_X(B)) \) is also a Serre fibration. Let \( \{E_r^\ast, d_r^\ast\} \) be the spectral sequence for this map \( p_Y^\ast \). Then, \( E_r^\ast \) converges to \( H^\ast(X \times \mathbb{Y}, r(Y) \vee u(X); k) \) and \( E_2^\ast \cong H^\ast(X, s_X(B); k) \otimes H^\ast(F, s_Y(b); k) \), where \((F, s_Y(b))\) is the fiber of the map \( p_Y^\ast \).

Let the spectral sequence \( \{\overline{E}_r, \overline{d}_r\} \) be defined by
\[
\overline{E}_r = H^\ast(X, s_X(B); k) \otimes_{H^\ast(B; k)} E_r \quad \text{with} \quad \overline{d}_r = d_r \otimes 1, \quad \text{where} \quad \overline{d}_r : \overline{E}_r \longrightarrow \overline{E}_r.
\]

By the projectiveness of \( H^\ast(X, s_X(B); k) \) and the Lemma 2.13 of Chapter I, we have that
\[
\overline{H}(\overline{E}_r) = H(H^\ast(X, s_X(B); k) \otimes_{H^\ast(B; k)} E_r) \cong H^\ast(X, s_X(B); k) \otimes_{H^\ast(B; k)} H(E_r) = H^\ast(X, s_X(B); k) \otimes_{H^\ast(B; k)} \overline{E}_{r+1} = \overline{E}_{r+1}.
\]

Therefore, \( \overline{E}_r \) is a spectral sequence.
Since $E_r$ converges to $H^*(Y, s_Y(B); k)$, the spectral sequence $E_r$ converges to $H^*(X, s_X(B); k) \otimes_{H^*(B; k)} H^*(Y, s_Y(B); k)$. We have the isomorphism

$$E_2 = H^*(X, s_X(B); k) \otimes_{H^*(B; k)} H^*(Y, s_Y(B); k) \cong H^*(X, s_X(B); k) \otimes H^*(F, s_Y(b); k) \cong H^*(X, s_X(B); k) \otimes H^*(F, s_Y(b); k) = E_2.$$  

Hence, $E_0 = E_\infty$ or

$$H^*(X, s_X(B); k) \otimes_{H^*(B; k)} H^*(Y, s_Y(B); k) \cong H^*(X \times_B Y, r(Y) \nu(Y); k).$$

By Lemma 6.2, we have the isomorphism of this theorem.

**Corollary 6.3:**

With the same conditions as in the above theorem, we have the isomorphism for the pair $(X, X^*)$,

$$H^*_B(Y) \otimes_{H^*_B(S_B)} H^*_B(X^*) \cong H^*_B(Y \wedge_B (X, X^*).$$

7. **The Künneth Spectral Sequence.**

The following definition and theorem form the basis from which we build the Künneth spectral sequence.

**Definition 7.1:**

The cohomology theory $\{h^n; n \in \mathbb{Z}\}$ on $(\text{Top}/B)_\bullet$ is said to satisfy the Atiyah property if for all object $\overline{X}$ in $(\text{Top}/B)_\bullet$, there is an object $\overline{Y}$ and a morphism $\chi_{\overline{X}}: \overline{X} \longrightarrow \overline{Y}$ such that

1) $h^n(\chi_{\overline{X}})$ is an epimorphism for all $n \in \mathbb{Z}$, and

2) $h^*(\overline{Y})$ is a projective $h^*(S_B)$-module.
Theorem 7.2:
The extended cohomology $H^*_B(\cdot)$ satisfies the Atiyah property.

Proof:
Let $\alpha : X \rightarrow (X/s_X(B)) \times B$ be a map defined by
$$\alpha(x) = ((x), p_X(x)) \text{ for all } x \in X.$$ 
Note that $\alpha$ is a morphism in $(\text{Top}/B)^*$. We set $Y = \Xi = \bigcap(X/s_X(B))$.

The natural map $\beta : (X/s_X(B)) \times B \rightarrow X$ given by
$$\beta((x), b) = x \text{ for all } ((x), b) \in (X/s_X(B)) \times B,$$
is well-defined and is a morphism in $(\text{Top}/B)^*$. Now, the composition of the maps $\beta \circ \alpha = 1_X$. Therefore, $H^*_B(\alpha) \cdot H^*_B(\beta) = 1_{H^*_B(X)}$. Hence, $H^*_B(\alpha)$ is an epimorphism for all $n \in \mathbb{Z}$.

Since $H^*_B(X)$ is finite and $k$ is a field, from the Künneth theorem for ordinary cohomology,
$$H^*_B(Y) = H^*((X/s_X(B)) \times B, B; k) = H^*((X/s_X(B)); k) \otimes H^*(B; k).$$
Therefore, $H^*_B(Y)$ is a free $H^*(B; k)$-module.

The morphism $\alpha_X$ of Theorem 7.2 is "natural" in the following sense.

Theorem 7.3:
Let $\varphi : X \rightarrow \overline{V}$ be a morphism in $(\text{Top}/B)^*$. Then, with $\alpha_X : X \rightarrow Y$ and $\alpha_Y : Y \rightarrow \overline{W}$ being the morphisms and objects obtained from Theorem 7.2, there is a $Y' \in (\text{Top}/B)^*$ and a morphism $\alpha_X' : X \rightarrow Y'$ such that

1) $H^*_B(\alpha_X')$ is an epimorphism for all $n \in \mathbb{Z}$,
2) $H^*_B(Y')$ is a projective $H^*_B(S^0_B)$-module, and
iii) the diagram

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\phi_X} & X \xrightarrow{\phi_Y} \bar{Y} \\
\downarrow & & \downarrow \\
\bar{Y} & \xrightarrow{\phi_{\bar{Y}}} & Y \\
\end{array}
\]
commutes.

**Proof:**

Let the topological space \( Y^\prime \) be defined by

\[ Y^\prime = (\bar{X}/s_X(B)) \times (\bar{W}/s_W(B)) \times Y. \]

Then, \( Y^\prime \) is an object in \((\text{Top}/B)^*_\) by the two maps \( p_{Y^\prime}: Y^\prime \to B \)

\[ s_{Y^\prime}: B \to Y^\prime \]

given by

\[ p_{Y^\prime}((x),(w),y) = p_Y(y) \quad \text{for all } ((x),(w),y) \in Y^\prime, \]

and

\[ s_{Y^\prime}(b) = (s_X(B), s_W(B), s_Y(b)) \quad \text{for all } b \in B. \]

The above maps are well-defined since \( s_X(B) \) and \( s_W(B) \) are the identified points in \( Y^\prime \).

Now, there is a morphism \( \phi_X^\prime: \bar{X} \to Y^\prime \) defined by

\[ \phi_X^\prime(x) = ((x), (\phi_Y(x)), \phi_X(x)) \quad \text{for all } x \in X. \]

Then, as in the proof of Theorem 7.2, \( H_B(\phi_X^\prime) \) is an epimorphism and \( H_B^\prime(Y^\prime) \) is a projective \( H_B(S_B^0) \)-module.

The projection map \( \pi: Y^\prime \to \bar{W} \) defined by

\[ \pi((x),(w),y) = w \quad \text{for all } ((x),(w),y) \in Y^\prime, \]

and the map \( \gamma: Y \to Y^\prime \) given by

\[ \gamma(y) = ((s_X^*p_Y(y)), (s_W^*s_Y(y)), y) \quad \text{for all } y \in Y \]

make the diagram of this theorem commutative. \( \square \)

Now, we begin the complicated process of constructing a filtration on a certain space, which will give us the Künneth spectral sequence.
Let the sequence of cofibration sequences be defined as follows:

Set \( \overline{X}_0 = \overline{X} \) for some object \( \overline{X} \) in \((\text{Top}/B)\)\(*\). Then, from Theorem 7.2, there is an object \( \overline{Y}_0 \) and a morphism \( \alpha'_0 : \overline{X}_0 \to \overline{Y}_0 \) in \((\text{Top}/B)\)\(*\) such that \( H_B(\alpha'_0) \) is an epimorphism and \( H^*_B(\overline{Y}_0) \) is a projective \( H^*_B(S^0)\)-module.

The inductive step is made by setting \( \overline{X}_{-n} = \overline{X}_{-n+1} \) for \( n > 0 \) and obtaining the objects \( \overline{Y}_{-n} \) and the morphism \( \alpha_{-n} : \overline{X}_{-n} \to \overline{Y}_{-n} \) from Theorem 7.2.

Then, with \( \beta_{-n} : \overline{Y}_{-n+1} \to \overline{X}_{-n} \) being the natural inclusion into the mapping cone, we have the sequence of cofibration sequences,

\[
\overline{X}_0 \overset{\alpha'_0}{\to} \overline{Y}_0 \overset{\beta_{-1}}{\to} \overline{X}_{-1} \to \cdots \to \overline{X}_{-n+1} \overset{\alpha_{-n+1}}{\to} \overline{Y}_{-n+1} \overset{\beta_{-n}}{\to} \overline{X}_{-n} \to \cdots
\]

Recall that for each cofibration sequence

\[
\overline{X}_{-n+1} \to \overline{Y}_{-n+1} \to \overline{X}_{-n}, \text{ for } n > 0,
\]

there is a morphism \( \Delta_{n-1} : \overline{X}_{-n} \to \bigcup X_{-n+1} \) from Lemma 2.9 of this chapter. Applying the suspension functor on each of the cofibration sequences and collapsing them, we get the sequence

\[
\overline{X}_{-n} \Delta_{n-1} \to \cdots \to \overline{X}_{-n-p-1} \overset{n-p-1 \Delta_{n-1}}{\to} \overline{X}_{-p-1} \overset{n-p-1 \Delta p}{\to} \overline{X}_{-p} \overset{n-p \Delta_0}{\to} \cdots \overset{n-1 \Delta_0}{\to} \overline{Y}_0.
\]

for \( 0 \leq p \leq n \).

Since \( \Delta_{n-1} \) is homotopically equivalent to an inclusion we have the (partial) filtration of the smash product \( \overline{Y} \wedge_{B^\infty} \overline{X}_0 \) by taking the smash product of \( \overline{Y} \) with the above sequence.
Theorem 7.4:

There is an exact sequence

\[ H_B^*(\sum \overline{X}_{-n+1}, \overline{X}_{-n}) \longrightarrow \ldots \longrightarrow H_B^*(\sum \overline{X}_{-p}, \sum \overline{X}_{-p-1}) \longrightarrow \ldots \]

\[ H_B^*(\sum \overline{X}_{-n}, \sum \overline{X}_{-n-1}) \longrightarrow H_B^*(\sum \overline{X}) \longrightarrow 0, \]

where each \( H_B^*(\sum \overline{X}_{-p}, \sum \overline{X}_{-p-1}) \) is a projective \( H_B^*(S_B) \)-module for all \( 0 \leq p \leq n-1 \).

Proof:

By the Exactness Axiom of the cohomology and the fact that \( H_B(\overline{X}_{-p}) \) is an epimorphism, there is a short exact sequence of \( H_B^*(S_B) \)-modules,

\[ 0 \longrightarrow H_B^*(\overline{X}_{-p-1}) \longrightarrow H_B^*(\overline{Y}_p) \longrightarrow H_B^*(\overline{X}_p) \longrightarrow 0 \]

for \( 0 \leq p \leq n-1 \).

By the Suspension Axiom of the cohomology, the sequence

\[ 0 \longrightarrow H_B^*(\sum \overline{X}_{-p-1}) \longrightarrow H_B^*(\sum \overline{Y}_{-p}) \longrightarrow H_B^*(\sum \overline{X}_{-p}) \longrightarrow 0 \]

is also exact for \( 0 \leq p \leq n-1 \).

Now, we splice the appropriate sequences from above together and get the long exact sequence,

\[ H_B^*(\sum \overline{Y}_{-n+1}) \longrightarrow \ldots \longrightarrow H_B^*(\sum \overline{Y}_{-p}) \longrightarrow \ldots \longrightarrow H_B^*(\sum \overline{X}) \longrightarrow 0. \]

Consider the sequence

\[ \overline{X}_{-p} \overset{\alpha_p}{\longrightarrow} \overline{Y}_{-p} \overset{\beta_{p-1}}{\longrightarrow} \overline{X}_{-p-1} \overset{k}{\longrightarrow} \overline{C}(\beta_{p-1}) \longrightarrow \overline{C}(k), \]

where \( k \) is the inclusion into the mapping cone. Then, by Lemma 2.9, we have the isomorphism

\[ H_B^*(\sum \overline{Y}_{-p}) \cong H_B^*(\overline{C}k) = H_B^*(\overline{C}\beta_{p-1}, \overline{X}_{-p-1}) \cong H_B^*(\sum \overline{X}_{-p}, \overline{X}_{-p-1}) \]

for \( 0 \leq p \leq n-1 \). Therefore, by the Suspension Axiom,

\[ H_B^*(\sum \overline{Y}_{-p}) \cong H_B^*(\sum \overline{X}_{-p}, \sum \overline{X}_{-p-1}). \]

Hence, we have the long exact sequence of this theorem.
The above isomorphism also shows that $\mathbb{H}^*_B(\sum_{-p}x_{-p}, \sum_{-p}x_{-p-1})$ is a projective $\mathbb{H}^*_B(S^0_B)$-module. ■

We are now ready to describe the Künneth spectral sequence for the cohomology $\mathbb{H}^*_B(\cdot)$.

In order to simplify the notations slightly (and get more confused!), we denote by

$$\mathbb{Z}^n_{n-p} = \mathbb{Y} \wedge B \sum_{-p} x_{-p}$$

with the inclusion morphism $1 \wedge B \sum_{-p} \Delta p$.

Note that $\mathbb{Z}^n_{n-p}$ is only defined for $0 \leq p \leq n$. Therefore, we let

$$\mathbb{Z}^n_{n-p} = \begin{cases} \mathbb{Z}^n_0 & \text{for } p = n, \\ \mathbb{Z}^n_n & \text{for } p = 0. \end{cases}$$

Then, for each $n \geq 0$, we have the filtration of the object $\mathbb{Z}^n_n$.

Let the exact couple of bigraded $k$-modules,

$$D(n) \xrightarrow{\alpha} D(n) \xrightarrow{\beta} E(n)$$

dependent on $n$, be defined by

$$D(n)^{-p,q} = \mathbb{H}^{-p+q-1}(\mathbb{Z}^n_{n-p}) \quad \text{and} \quad E(n)^{-p,q} = \mathbb{H}^{-p+q}(\mathbb{Z}^n_{n-p}, \mathbb{Z}^n_{n-p-1}).$$

for all $p$ and $q$. The three maps $\alpha$, $\beta$ and $\gamma$ come from the long exact sequence of the fibered pair $(\mathbb{Z}^n_{n-p}, \mathbb{Z}^n_{n-p-1})$.

We make a note that $\alpha$ has the bidegree $(-1,1)$, $\beta$ has the bidegree $(1,-1)$ and $\gamma$ has the bidegree $(0,1)$. 
The above exact couple determines a spectral sequence \( \{ E_r(n), d_r(n) \} \), dependent on \( n \neq 0 \).

**Theorem 7.5:**

There is a natural isomorphism

\[ E_r^{-p,q}(n) \cong E_r^{-p,q+1}(n+1) \] for \( 0 \leq p \leq n-1 \).

Therefore, \( d_r^{-p,q}(n) = d_r^{-p,q+1}(n+1) \).

**Proof:**

Applying Lemma 3.1, a calculation shows that for \( 0 \leq p \leq n-1 \),

\[ H_B^{-p+q+1}(Z^{n+1}_p, Z^{n+1}_p) = H_B^{-p+q+1} \sum (Z^{n}_{n-p}, Z^{n}_{n-p-1}) \].

By the Suspension Axiom,

\[ E_1^{-p,q}(n) \cong E_1^{-p,q+1}(n+1) \].

The above theorem makes the following definition of the Kunneth spectral sequence independent of \( n \).

**Definition 7.6:**

The Kunneth spectral sequence is defined by

\[ E_r^{-p,q}(X, Y) = E_r^{-p,q+n}(n) \]

for objects \( X \) and \( Y \) in \( \text{Top}/B \). Its differential \( d_r^{-p,q}(X, Y) \) is \( d_r^{-p,q+n}(n) \).

**Remark:**

If \( m \equiv n \neq 0 \), then by Theorem 7.5, \( E_r^{-p,q+m}(m) \cong \ldots \cong E_r^{-p,q+n}(n) \), for \( 0 \leq p \leq n-1 \). In other words, for \( p \neq 0 \),

\[ E_r^{-p,q}(X, Y) = E_r^{-p,q+n}(n) \cong E_r^{-p,q+p+1}(p+1) \], i.e.,

independent of \( n \). Note that \( E_r^{-p,q+p}(p) \not\cong E_r^{-p,q+p+1}(p+1) \).
Lemma 7.7:

The Künneth spectral sequence is a second quadrant spectral sequence.

Proof:

Consider the morphism $\alpha'$ in the diagram

\[
\begin{array}{ccc}
D^{-p},q^{+n}(n) & \xrightarrow{\alpha'} & D^{-p-1},q^{+n+1}(n) \\
H^{-p+q+n-1}(\mathbb{Z}^n_{n-p}) & \xrightarrow{\alpha'} & H^{-p+q+n-1}(\mathbb{Z}^n_{n-p-1}).
\end{array}
\]

Now, for $p<0$, $\mathbb{Z}^n_{n-p}=\mathbb{Z}^n_{n-p-1}=\mathbb{Z}^n_n$. Hence, for fixed $q$, $\alpha'^{-p},q^{+n}$ is an isomorphism. From the long exact sequence,

\[
\ldots \xrightarrow{D^{-p},q^{+n}(n)} \xrightarrow{\alpha'} D^{-p-1},q^{+n+1}(n) \xrightarrow{\alpha'} E^{-p},q^{+n}(n) \xrightarrow{D^{-p},q^{+n+1}(n)} \ldots
\]

we get that $E^{-p},q(\bar{x},\bar{y})=E^{-p},q^{+n}(n)=0$, for $p<0$.

Consider the cofibration sequence,

\[
\mathbb{Z}^n_{n-p-1} \xrightarrow{J} \mathbb{Z}^n_{n-p} \xrightarrow{\alpha'} \bar{c}(j).
\]

Now, $E^{-p},q(\bar{x},\bar{y})=H_B^{-p+q+n}(\bar{c}(j))$. By Lemma 3.1 and the Suspension Axiom,

\[
H_B^{-p+q+n}(\mathbb{Z}^n_{n-p}) \cong H_B^q(\bar{X} \wedge \bar{X}_{n-p}) \text{ and } H_B^{-p+q+n}(\mathbb{Z}^n_{n-p-1}) \cong H_B^{q+1}(\bar{X} \wedge \bar{X}_{n-p-1}).
\]

From the remark made after Theorem 5.2,

\[
H_B^q(\bar{X} \wedge \bar{X}_{n-p})=0 \text{ for } q<0 \text{ and } H_B^{q+1}(\bar{X} \wedge \bar{X}_{n-p-1})=0 \text{ for } q+1<0.
\]

Hence, from the long exact sequence of the above cofibration sequence, $H_B^{-p+q+n}(\bar{c}(j))=0$ for $q<0$.

Since we will be using Theorem 6.3 and its Corollary to make the identification on the $E_2(n)$-terms, we need to assume, at this point, the conditions of Theorem 6.3, i.e., that $B$ is simply connected, that $H_B^*(\bar{X})$ is a finite, projective $H_B^*(S^0)$-
module, and that \( p : Y \to B \) in \( Y \) is a Serre fibration with \( H^*_B(Y) \) also finite.

For each spectral sequence \( \{ E_r(n), d_r(n) \} \), \( E_2(n) \) is the homology of the complex

\[
\cdots \to E^{-n+1},q(n) \to \cdots \to E^0,q(n) \to 0 \to \cdots
\]

where \( E^{-n+1},q(n) = H_{E^n}^{-n+1+q}(\mathbb{Z}_1, Z_{1-1}) \), for all \( 1 \leq i \leq n \).

Then, from Corollary 6.3, we have the isomorphisms

\[
E^{-n+1},q(n) = H_{E^n}^{-n+1+q}(\mathbb{Z}_1, Z_{1-1}) = H_{E^n}^{-n+1+q}(Y \wedge_B(\sum \mathbb{X}^{n+1}, \sum \mathbb{X}^{n+1-1}))
\]

\[
\cong (H^*_B(Y) \otimes H^*_B(S_B, H^*_B(\sum \mathbb{X}^{n+1}, \sum \mathbb{X}^{n+1-1})))^{-n+1+q},
\]

for all \( 1 \leq i \leq n \).

Hence, the two complex of \( H^*_B(S_B^0) \)-modules,

\[
\cdots 0 \to H^*_B(\mathbb{Z}_1^n, \mathbb{Z}_0^n) \to \cdots \to H^*_B(\mathbb{Z}_1^n, \mathbb{Z}_1^n) \to \cdots
\]

\[
H^*_B(\mathbb{Z}_n^n, \mathbb{Z}_n^{n-1}) \to 0 \to \cdots
\]

and

\[
\cdots 0 \to H^*_B(Y) \otimes H^*_B(S_B, H^*_B(\sum \mathbb{X}^{n+1}, \sum \mathbb{X}^{n+1-1})) \to \cdots \to \cdots
\]

\[
H^*_B(Y) \otimes H^*_B(S_B, H^*_B(\sum \mathbb{X}^{n+1}, \sum \mathbb{X}^{n+1-1})) \to \cdots \to \cdots
\]

\[
H^*_B(Y) \otimes H^*_B(S_B, H^*_B(\sum \mathbb{X}^{n+1}, \sum \mathbb{X}^{n+1-1})) \to 0 \to \cdots
\]

are chain equivalent.

Now, by Theorem 7.4, the above complexes are a (partial) projective resolutions of \( H^*_B(\sum X) \). Therefore, by the definition of the Tor functor

\[
E_2^{-p,q}(n) = \text{Tor}_{H^*_B(S_B)}^{-p,q}(H^*_B(Y), H^*_B(\sum X))
\]

for \( 0 \leq p \leq n-1 \).

By the Suspension Axiom,

\[
E_2^{-p,q}(n) = \text{Tor}_{H^*_B(S_B)}^{-p,q}(H^*_B(Y), H^*_B(\sum X)).
\]

Hence, for all \( p \) and \( q \),
We have shown above that the spectral sequence \( \{ E_r(X,Y), d_r(X,Y) \} \) is independent of the choice of the morphism \( \alpha_X \) from Theorem 7.2. We will now show that \( E_r(X,Y) \) is functorial with respect to the object \( X \) and \( Y \).

**Theorem 7.8:**

Let \( \varphi: X \rightarrow V \) and \( \phi: Y \rightarrow W \) be morphisms in \((\text{Top}/\mathcal{B})_\mathcal{B}\) with both pairs of objects, \( \{ X,Y \} \) and \( \{ V,W \} \) satisfying the conditions of Theorem 6.3.

Then, there is a natural morphism

\[
E_r(\varphi,\phi):E_r(V,W) \rightarrow E_r(X,Y)
\]

with \( E_2(\varphi,\phi)=\text{Tor}_{H^*_B(S_B)}(H^*_B(\varphi),H^*_B(\phi)) \).

**Proof:**

Consider the two diagrams obtained from Theorem 7.2,

\[
\begin{align*}
\begin{array}{ccc}
X & \xrightarrow{\varphi} & V \\
\downarrow{\alpha_X} & & \downarrow{\alpha_V} \\
U & \xrightarrow{\beta} & \tilde{U} & \xrightarrow{\hat{\beta}} & \tilde{U}' & \xrightarrow{\hat{\beta}} & \tilde{U}'' & \xrightarrow{\hat{\beta}} & \tilde{U}'''
\end{array}
& \quad & 
\begin{array}{ccc}
Y & \xrightarrow{\phi} & W \\
\downarrow{\alpha_Y} & & \downarrow{\alpha_W} \\
T & \xrightarrow{\gamma} & \tilde{T} & \xrightarrow{\hat{\gamma}} & \tilde{T}' & \xrightarrow{\hat{\gamma}} & \tilde{T}'' & \xrightarrow{\hat{\gamma}} & \tilde{T}'''
\end{array}
\end{align*}
\]

Now, the morphisms \( \{ \varphi, \tilde{\varphi} \} \) and \( \{ \phi, \tilde{\phi} \} \) induces a morphism

\[
E_r^*(\varphi,\phi):E_r^*(\tilde{V},\tilde{W}) \rightarrow E_r^*(\tilde{X},\tilde{Y})
\]

through the construction of the Künnneth spectral sequences, \( E_r^*(\tilde{V},\tilde{W}) \) and \( E_r^*(\tilde{X},\tilde{Y}) \), starting with the morphisms \( \alpha_X^* \) and \( \alpha_Y^* \), respectively. Let \( E_r(\tilde{V},\tilde{W}) \) and \( E_r(\tilde{X},\tilde{Y}) \) be the Künnneth spectral sequence using the morphisms \( \alpha_X^* \) and \( \alpha_Y^* \), respectively. Then,
since $E_r(\overline{V},\overline{W})$ and $E_r(\overline{X},\overline{Y})$ are independent of the choice of the morphisms $\alpha_\overline{X}$ and $\alpha_\overline{Y}$, we have the morphism

$$E_r(\phi, \phi): E_r(\overline{V}, \overline{W}) \cong E_r(\overline{V}, \overline{W}) \to E_r(\overline{X}, \overline{Y}) \cong E_r(\overline{X}, \overline{Y}).$$

On the $E_2$-level, we have the commutative diagram

$\begin{array}{ccc}
E_2(\overline{V}, \overline{W}) & \xrightarrow{E_2(\phi, \phi)} & \text{Tor}_{H^*_B(S_B)}(H^*_B(\overline{W}), H^*_B(\overline{V})) \\
\downarrow & & \downarrow \\
E_2(\overline{X}, \overline{Y}) & \xrightarrow{E_2(\phi, \phi)} & \text{Tor}_{H^*_B(S_B)}(H^*_B(\overline{Y}), H^*_B(\overline{X})).
\end{array}$

8. **The Convergence of the Spectral Sequence.**

The convergence of the Künneth spectral sequence will be discussed from the point of view of the convergence of each spectral sequence $\{E_r(n), d_r(n)\}$ defined in the last section.

The convergence of $E_r(n)$ is a consequence of the following lemma.

**Lemma 8.1:**

For each spectral sequence $\{E_r(n), d_r(n)\}$,

$$E_{n+p, q}^r(n) \cong E_{n+p, q}^r(n) \cong \ldots \cong E_{\infty, q}^r(n).$$

**Proof:**

$E_2(n)$ is the homology of the complex

$$\begin{array}{ccc}
0 & \xrightarrow{E_{1, q}^r(n)} & E_1^0, q(n) \xrightarrow{d_1} E_1^0, q(n) \xrightarrow{d_1} 0.
\end{array}$$

Hence, the lemma. \]

With the notations of the last section, we have the
following theorem.

Theorem 8.2:

For all \( n \geq 0 \), \( E_r(n) \) converges to \( H_B(\bar{Y} \wedge_B \overline{X_0}) \).

Proof:

The filtration

\[
\cdots \rightarrow \bar{Y} \wedge_B \sum_{-p}^{n-p-1} X_{p-1} \rightarrow \bar{Y} \wedge_B \sum_{-p}^{n-p} X_p \rightarrow \cdots \rightarrow \bar{Y} \wedge_B \sum_{0}^{n} X_0
\]

of \( \bar{Y} \wedge_B \sum_{0}^{n} X_0 \) induces the filtration \( \mathcal{F}_p^B(\bar{Y} \wedge_B \sum_{0}^{n} X_0) \) of \( H_B(\bar{Y} \wedge_B \sum_{0}^{n} X_0) \) through the singular chain complex of the topological pair \( (\bar{Y} \wedge_B \sum_{0}^{n} X_0, s_{Y_B} \sum_{0}^{n} X_0(B)) \).

Hence, \( E_r(n) \) converges to \( H_B(\bar{Y} \wedge_B \sum_{0}^{n} X_0) \).

We will be taking the inverse limits of the families of spectral sequence \( \{E_r(n)\}_{n \geq 0} \) and of objects \( \{H(n)\}_{n \geq 0} \), both indexed on the positive integers. The definition of inverse systems and facts used in this section are the ones in Sec.1 of the Introduction in (2).

Although the next theorem is slightly more general than what we will need, we will apply the theorem to show the convergence of the Künneth spectral sequence.

Theorem 8.3:

Let \( \{E_r(n)\}_{n \geq 0} \) be a family of spectral sequence indexed on the non-negative integers. Let \( \{H(n)\}_{n \geq 0} \) be a family of graded object, indexed on the non-negative integers, with morphisms \( \{g_{n,m}^{m}: H(m) \rightarrow H(n)\}_{m \geq n} \) of degree \( r(n,m) \) such that the inverse limit, \( H^q = \lim_n \{H^q(n)\} \), exists, for each \( q \).
Then, if

1) $E_r(n)$ converges to $H(n)$ for each $n$, and

2) each filtration $FP^pH(n)$ of $H(n)$ in 1) is such that

$$g^m_n(F^pH^q(m)) \subseteq F^pH^q+r(n,m)(n)$$

for $n \leq m$, there exists

1) a family of morphisms $\{f^m_n: E^p_{\omega}(m) \to E^p_{\omega}(m+r(n,m))(n)\}$

such that the inverse limit $E^p_{\omega} = \lim \{E^p_{\omega}(0,n)(n)\}$

exists, and

2) a filtration $FP^pH^q$ of $H^q$ such that $E^p_{\omega}F^pH^q/F^p-1H^q$, for every $p$ and $q$.

Remark:

The inverse limit $H^q = \lim H^q(n)$ is not well-defined.

We make a convention that $H^q$ is the inverse limit of the system, $\{\ldots \to H^q-r(o,n)(n) \to \ldots \to H^q-r(o,1)(1) \to H^q(0)\}$.

Proof:

Let $h^m_n = g^m_n| F^pH^q(m) \to F^pH^q+r(n,m)(n)$ for $n \leq m$.

Then, we can take the inverse limit of the system

$$\ldots \to F^pH^q-r(0,n)(n) \to \ldots \to F^pH^q-r(0,1)(1) \to F^pH^q(0)$$

and denote by $F^pH^q = \lim \{F^pH^q-r(0,n)(n)\}$. Since by definition

$$F^pH^q = \{\{(x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} F^pH^q-r(0,n)(n) | x_n = h_n^m(x_m), n \leq m\}$$

and

$$H^q = \{\{(x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} H^q-r(0,n)(n) | x_n = g_n^m(x_m), n \leq m\},$$

$F^pH^q$ is a filtration of $H^q$, for each $q$.

The morphism $h^m_n$ induces the morphism

$$E^p_{\omega}F^pH^q(m)/F^p-1H^q(m)$$

for

$$E^p_{\omega}F^pH^q+r(n,m)(n)/F^p-1H^q+r(n,m)(n)$$
The commutativity $f^k_n \cdot f^m_k = f^m_n$ for $n \leq k \leq m$ follows from the commutativity $g^k_n \cdot g^m_k = g^m_n$. Hence, we can take the inverse limit of the system,

$$\ldots \rightarrow E^p,q-r(0,n)(n) \rightarrow E^p,q-r(0,1)(1) \rightarrow E^p,q(0)$$

and let $E^p,q = \lim_n \{E^p,q-r(0,n)(n)\}$.

Let the morphism $h_n : F^p_H q \rightarrow F^p_H q-r(0,n)(n)$ be defined by $h_n(\{x_n\}_{n=0}^\infty) = x_n$ for $\{x_n\}_{n=0}^\infty \in F^p_H q$. Then, for $n \leq m$, we have

the commutativity $h^m_n \cdot h^m_k(\{x_n\}_{n=0}^\infty) = h^m_n(x_m) = x_n = h_n(\{x_n\}_{n=0}^\infty)$.

We will show, subsequently, the isomorphism

$$F^p_H q/F^p-1_H q = \lim_n \{F^p_H q-r(0,n)(n)/F^p-1_H q-r(0,n)(n)\}$$

$$= \lim_n \{E^p,q-r(0,n)(n)\}.$$ Then, $E^p,q \cong F^p_H q/F^p-1_H q$.

The morphism $h_n$ defined above induces

$$f^m_n : F^p_H q/F^p-1_H q \rightarrow F^p_H q-r(0,n)(n)/F^p-1_H q-r(0,n)(n)$$
as follows:

$$f^m_n(\{x_n\}_{n=0}^\infty) = h^m_n(\{x_n\}_{n=0}^\infty) = f^m_{n-1}(x_n)$$

for all $\{x_n\}_{n=0}^\infty \in F^p_H q/F^p-1_H q$.

Then, for $n \leq m$, we have the commutativity $f^m_n \cdot f^m_{n-1} = f^m_{n-1} \cdot f^m_n$.

Now, if $f^m_n(\{x_n\}_{n=0}^\infty) = f^m_{n-1}(\{x_n'\}_{n=0}^\infty)$ for all $n$ with $\{x_n\}_{n=0}^\infty,\{x_n'\}_{n=0}^\infty \in F^p_H q$, then

$$x_n \cdot F^p-1_H q-r(0,n)(n) = x_n' \cdot F^p-1_H q-r(0,n)(n) \text{ for all } n,$$

$$x_n = x_n' \mod F^p-1_H q-r(0,n)(n) \text{ for all } n.$$

Hence, $\{x_n\}_{n=0}^\infty = \{x_n'\}_{n=0}^\infty \mod F^p-1_H q$.

Moreover, if $\{x_n F^p-1_H q-r(0,n)(n)\}_{n=0}^\infty$ is in the inverse limit, $\lim_n \{F^p_H q-r(0,n)(n)/F^p-1_H q-r(0,n)(n)\}$, then by definition

$x_n \cdot F^p-1_H q-r(0,n)(n) = h^m_n(x_m) F^p-1_H q-r(0,n)(n)$ for $n \leq m$.
Therefore, \( \{x_n\}_{n=0}^{\infty} \) in Eq. \( \frac{P^{-1}H^q}{F^pH^q} \), and we have that \( f_n(\{x_n\}_{n=0}^{\infty}) = x_n F^{-1}H^q r(0,n)(n) \) for all \( n \). Hence, the isomorphism above.

We will apply the above theorem to the family of spectral sequence \( \{E_r(n), d_r(n)\} \) defined in Sec. 7. We will denote by \( H_B^*(n) = H_B^*(Y \wedge B \wedge X) \) and by \( F^pH_B^*(n) \), the filtration of \( H_B^*(n) \) such that \( E_\infty^p, q(n) = F^pH_B^*(n)/F^{p-1}H_B^*(n) \).

Let \( g^m_H : H_B^q(m) \rightarrow H_B^{q+n-m}(n) \), for \( n=m \), be the isomorphism given by the Suspension Axiom of the cohomology. Then, we can take the inverse limit of the system

\[
\cdots \xrightarrow{\partial} H_B^{q+n}(n) \xrightarrow{\partial} \cdots \xrightarrow{\partial} H_B^{q+1}(1) \xrightarrow{\partial} H_B^q(0),
\]

and let \( H_B = \lim_{\rightarrow n}[H_B^{q+n}(n)] \).

Lemma 8.3:

With the above notations,

\( g^m_n(F^pH_B^*(m)) \subseteq F^pH_B^{q+n-m}(n) \) for \( n=m \).

Proof:

We will prove the lemma using induction on \( n \). For \( n=0 \),

\( E_r^p, q(n) = 0 \) for all \( p \) and \( q \). Therefore, the filtration of \( H_B^q(n) \) is \( F^pH_B^q(n) = H_B^q(n) \) for all \( p \). Then, for any \( m \geq n \), we have

\( g^m_n(F^pH_B^q(m)) \subseteq H_B^{q+n-m}(n) = F^pH_B^{q+n-m}(n) \).

Assume that \( g_{n-1}^m(F^pH_B^q(m)) \subseteq F^pH_B^{q+n-m-1}(n-1) \) for all \( m \geq n-1 \), and consider the following isomorphisms,

\[ g^m_n(F^pH_B^q(m))/g^m_n(F^{p-1}H_B^q(m)) \cong F^pH_B^q(m)/F^{p-1}H_B^q(m) \]
for \(-n+1 \leq p \leq 0\), from Theorem 7.5. We also have, from Theorem 7.5, the isomorphism

\[
\mathbb{F}_p H^n_B(n) / \mathbb{F}_p H^{n-1}_B(n) \cong \mathbb{F}_p H^n_B(n) / \mathbb{F}_p H^{n-1}_B(n-1).
\]

By induction hypothesis, \(g_{n-1}(\mathbb{F}_p H^n_B(n)) \subseteq \mathbb{F}_p H^{n-1}_B(n-1)\) and

\[
g_n(\mathbb{F}_p H^q_B(m)) \cong \mathbb{F}_p H^{n-1}_B(n-1).
\]

Hence, for \(-n+1 \leq p \leq 0\), we have that

\[
g_n(\mathbb{F}_p H^q_B(m)) \cong \mathbb{F}_p H^{n-1}_B(n) \cong \mathbb{F}_p H^{n-1}_B(n-1).
\]

For \(p = -n\), we, again, have the isomorphisms

\[
\mathbb{F}^{-n+1}_B H^n_B(n) / \mathbb{F}^{-n}_B H^{n-1}_B(n) \cong \mathbb{F}^{-n+1}_B H^q_B(m) / \mathbb{F}^{-n}_B H^q_B(m)
\]

\[
\cong g_n(\mathbb{F}^{-n+1}_B H^q_B(m) / g_n(\mathbb{F}^{-n}_B H^q_B(m))).
\]

and \(g_n(\mathbb{F}^{-n+1}_B H^q_B(m)) \subseteq \mathbb{F}^{-n}_B H^{n-1}_B(n)\) from Theorem 7.5. Hence,

\[
g_n(\mathbb{F}^{-n}_B H^q_B(m)) \subseteq \mathbb{F}^{-n}_B H^{n-1}_B(n).
\]

Finally, for \(p < -n\), \(E^p, q(n) = 0\), or that

\[
\mathbb{F}^{p-1}_B H^q_B(n) = \mathbb{F}^p H^q_B(n) \text{ for all } p < -n.
\]

Hence, \(g_n(\mathbb{F}^p H^q_B(m)) \subseteq \mathbb{F}^{-n}_B H^{n-1}_B(n)\) for all \(p < -n\).

By Theorem 8.2, there exists an inverse limit \(E^\infty, q = \lim_n \mathbb{F}^p H^q_B(n) = \mathbb{F}^p H^q_B / \mathbb{F}^{p-1}_B H^q_B\) and a filtration \(F^p B^n q = \lim_n \mathbb{F}^{p, q-r(0, n)} B^n\) (In this case, \(r(0, n) = -n\)) such that

\(E^\infty, q = \mathbb{F}^p H^q_B / \mathbb{F}^{p-1}_B H^q_B\).
Lemma 8.4:

With the above notations, we have the isomorphisms,
\[ E^p_q = E^p_q(\overline{X}, \overline{Y}), \]
where \( E(\overline{X}, \overline{Y}) \) is the Künneth spectral sequence,
and \( H^q_B(\overline{Y} \wedge_B \overline{X}) \).

Proof:

Let the morphism \( g_n : H^q_B(\overline{Y} \wedge_B \overline{X}) \to H^{q+n}_B(\overline{Y} \wedge_B \overline{X}) \) be defined as the isomorphism given by the Suspension Axiom.

Then, \( g_n = (g_n^0)^{-1} \). The commutativity \( g^m_n \cdot g_m = g_n \), for \( n \leq m \), follows from the commutativity \( g^0_n \cdot g^m_n = g^m_0 \).

The condition that \( g_n(x) = g_n(x') \) for all \( n \) imply \( x = x' \) for \( x \) and \( x' \) in \( H^q_B(\overline{Y} \wedge_B \overline{X}) \) is trivial since \( g_n \) is an isomorphism.

Let \( \{x_n\}_n = \lim_{\to n} \{H^{q-n}_B(n)\} \). Then, \( x_0 = g^0_0(x_0) \) for all \( n \geq 0 \), by definition of the inverse limit. Hence,

\[ g_n(x_0) = g_n \cdot g^0_n(x_0) = x_n \text{ for all } n \geq 0. \]

Therefore, \( H_B^q(\overline{Y} \wedge_B \overline{X}) = \lim_{\to n} \{H^{q+n}_B(n)\} \).

We will show that for each \( p \) and \( q \), \( E^{-p,q}_\infty(\overline{X}, \overline{Y}) \) is the inverse limit of the system

\[ \cdots \xrightarrow{f_n} E^{-p,q+n}_\infty(n) \xrightarrow{f_{n-1}} \cdots \to E^{-p,q+p+1}(p+1) \to \cdots \]

Note that, for \( 0 \leq n \leq p+1 \), \( E^{-p,q+n}_\infty(n) = 0 \).

Let the morphism \( f_n : E^{-p,q}_\infty(\overline{X}, \overline{Y}) \to E^{-p,q+n}_\infty(n) \) be defined by

\[ f_n = \begin{cases} (f^p_{n+1})^{-1} & \text{for } n = p+1, \\ 0 & \text{for } 0 \leq n \leq p+1. \end{cases} \]

For \( m \geq n \geq p+1 \), the commutativity \( f^m_n \cdot f_m = f_n \) follows from the commutativity \( f_n^{p+1} \cdot f^m_n = f^m_{n+1} \).

For \( m = p+1 \) and \( n \leq p+1 \), we have the commutativity...
The final case, $n=m^p+1$, is trivial since $E_{-p, q+n}(m)=0$.

Hence, $f_m^m f_m = f_m$ for all $0 \leq n \leq m$.

If for all $n \geq 0$ $f_n(x) = f_n(x')$ with $x$ and $x'$ in $E_{-p, q(X, \overline{Y})}$, then, since $f_n$ is an isomorphism for $n=p+1$, $x = x'$.

Moreover, if $\{x_n\}_{n=0}^{\infty} \subseteq E_{-p, q+n}(n)$ such that $x_n = f_m^m(x_m)$, then we have $y = x_{p+1} \in E_{-p, q+p+1}(p+1) = E_{-p, q(X, \overline{Y})}$ such that $f_n(y) = x_n$ for all $n \geq 0$. Hence, $E_{-p, q(X, \overline{Y})} = \lim_{n \to \infty} E_{-p, q+n}(n)$.

It is evident from Theorem 8.3 and the previous two lemmas that the filtration $F^n H_B^q(n)$ of $H_B^q(n)$ induces a filtration $F^n H_B^q(\overline{Y} \wedge_B \overline{X})$ of $H_B^q(\overline{Y} \wedge_B \overline{X})$ such that

$$E_{-p, q}^r(X, \overline{Y}) = F^n H_B^q(\overline{Y} \wedge_B \overline{X}) / F^{n-1} H_B^q(\overline{Y} \wedge_B \overline{X}).$$


We have shown, in the last two sections, the following theorem. We will use the notations of the previous sections. Theorem 9.1:

Let $\overline{X}$ and $\overline{Y}$ be the objects in $(\text{Top/B})_*$ such that $H_B^*(\overline{X})$ and $H_B^*(\overline{Y})$ are finite as $H_B^*(S_B)_*$-modules. Then, if $H_B^*(\overline{X})$ is projective and $p_X: \overline{Y} \to B$ in $\overline{Y}$ is a Serre fiberation, there is a natural, second quadrant spectral sequence $\{E_r^r(X, \overline{Y}), d_r(X, \overline{Y})\}$ such that
1) $E_1(x, y)$ converges to the object $H^*_B(y \land_B x)$, and
2) $E_2^{p, q}(x, y) = \text{Tor}^p_{H^*_B(S_B)}(H^*_B(y), H^*_B(x))$ for all $p$ and $q$.

In order to see how Theorem 9.1 generalizes the geometric
Theorem 2.3 of Chapter II, we note two properties of the
adjoint functor $G: \text{Top}/B \rightarrow \text{(Top}/B)^*$ defined in Sec. 4.

**Proposition 9.2:**

The adjoint functor $G$ has the following two properties:

1) for $p_x: X \rightarrow B$ in $\text{Top}/B$, $H^*_B(G(X)) = H^*(X; k)$, and
2) for $X \times_B Y \rightarrow B$ in $\text{Top}/B$ with $p_x: X \rightarrow B$ and
   $p_y: Y \rightarrow B$, $G(X \times_B Y) \cong G(X) \land_B G(Y)$.

**Proof:**

The first property follows from the fact that $\overline{x}G(x) = X \lor B/B = X$, where $\overline{x}$ is the functor used in Sec. 5 to extend the
ordinary cohomology.

For the second property, we have that $(X \lor B) \land_B (Y \lor B) = (X \times_B Y) \lor \{(b, b) | b \in B\}$ from the definition of the smash product.
Hence, the isomorphism above. \[qed\]

Consider the commutative diagram of topological spaces

\[
\begin{array}{ccc}
X \times_B Y & \longrightarrow & Y \\
\downarrow & & \downarrow p_y \\
X & \longrightarrow & B
\end{array}
\]

where, as in Chapter II, $B$ is simply connected and the map
$p_y$ is a Serre fibration. Then, from Theorem 9.1, we have
a spectral sequence \( \{E_r, d_r\} \) such that

1) \( E_r \) converges to \( H^*_B(G(X) \land_B G(Y)) = H^*(X \times_B Y; k) \), and

11) \( E_2^{-p,q} = \text{Tor}^{H_B(S_B)}_{H_B(S_B)}(H^*_B(G(X)), H^*_B(G(Y))) = \text{Tor}^{H_B(S_B)}_{H_B(S_B)}(H^*(X), H^*(Y)) \) for all \( p \) and \( q \).
BIBLIOGRAPHY


