THE INDEPENDENCE OF THE WHITEHEAD PROBLEM FROM ZFC

by

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Abstract

An abelian group G is called a W-group if $\mathrm{Ext}(G,Z) = 0$. Whitehead's problem asks which groups are W-groups. Saharon Shelah proved that the answer to Whitehead's problem, for groups of cardinality ω_1, is independent of the axioms of Zermelo-Frankel set theory with the axiom of choice. This thesis gives a complete and detailed proof, based on Shelah's proof, of this independence result.

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Introduction

The following result was proved by Saharon Shelah in (11). "The Whitehead problem, for groups of cardinality ω_1, is independent of and consistent with ZFC." In this thesis we present a proof of this result based on Shelah's proof. Certain alterations had to be made, as well as a good deal of filling in of details. A description of some of the alterations is given in the appendix at the end of the thesis.

A Whitehead group, or simply a W-group, is an Abelian group for which $\text{Ext}(G,Z) = 0$. ($\text{Ext}(G,Z) = 0$ is a mapping property which will be explained in detail in this thesis.) The Whitehead Problem asks:

"Are all W-groups of cardinality ω_1 freely generated."

The axioms of ZFC, Zermelo-Frankel set theory with the axiom of choice, are the axioms on which all current mathematics can be built upon. We will show that within ZFC the Whitehead Problem cannot be solved. We will do this by showing that within one model of ZFC all W-groups are free, and within another model there exists non free W-groups.

Gödel exhibited a construction which produced a model of ZFC. His construction is referred to as $V = L$ and so $\text{ZFC} + V = L$ is consistent. Jensen showed that within such a model of ZFC, a combinatorial property called 'diamond' holds. We will define and use this property diamond to show that within a model of ZFC where $V = L$ holds, all W-groups are freely generated.

Martin and Soloway showed that MA (Martin Axiom) + $2^{\omega} > \omega_1$ is consistent with ZFC. We will define and use MA to show the existence of non free W-groups in a model of ZFC in which MA + $2^{\omega} > \omega_1$ holds.

And so the situation is this. Let X be the following statement: "All W-groups of cardinality ω_1 are freely generated." Then:

(i) ZFC + V = L implies X and so ZFC + X is consistent.

(ii) ZFC + MA + $2^{\omega} > \omega_1$ implies $\neg X$ and so ZFC + $\neg X$ is consistent.

Thus X is consistent with and independent of ZFC.

Whitehead Groups and their Structure

In this section we will give some preliminary facts and definitions about Whitehead groups and ω_1-free groups. We will classify the ω_1-free groups of cardinality ω_1 into three possibilities. In this thesis, by group we will always mean Abelian group.

Definition (1): G is called a Whitehead group or W-group if for every epimorphism $h:H \to G$ such that the kernel of h is isomorphic to Z (the integers) there exists a homomorphism $g:G \to H$ such that $gh:G \to G$ is the identity map on G.

Definition (2): For a group A we say A is the direct sum of subgroups B and C of A if:

(i) $B + C = A$ where $B + C$ is the set of all sums of the form $b + c$ where b is in B and c is in C.

(ii) $B \cap C = 0_A$ where 0_A is the identity element of A.

This is written as $A = B \oplus C$.

Lemma (1): The group H as in the definition of a W-group is a direct sum of a copy of Z and a copy of G.

Proof: Let g and h be as in the definition of a W-group. Since gh is the identity map on G, g is 1-1, so $G^* = \text{image}(g) \simeq G$. By definition $\text{kernel}(h) \simeq Z$. We show that $H = G^* \oplus \text{kernel}(h)$. Clearly G^* and kernel(h) have only 0_H in common, else gh could not be 1-1. Now let h^* be h restricted to G^*. If

$x \in H$, then $x = h^{*^{-1}}h(x) + (x - h^{*^{-1}}h(x))$ and clearly $h^{*^{-1}}h(x) \in G^*$ and $x - h^{*^{-1}}h(x) \in \text{kernel}(h)$.

Some Preliminary Facts about W-groups

Definition (3): A group is called free if it is isomorphic to a direct sum of copies of Z.

Definition (4): A group is ω_1-free if every countable subgroup is free.

If G is a W-group then it is:

(i) Torsion free (2) page 178

(ii) ω_1-free (2) page 178

From now on G will be taken to be torsion free and of cardinality ω_1. So we can assume without loss of generality that the elements of G are all the ordinals $< \omega_1$ where ω_1 is the first uncountable ordinal: $\omega_1 = \{\alpha : \alpha < \omega_1\}$ = the set of all ordinals less than ω_1.

Definition (5): B is a pure subgroup of G if $B \cap zG = zB$ for all $z \in Z$, where $zG = \{g \in G : g = zx$ for some $x \in G\}$. Equivalently B is pure if for any $z \in Z$, $b \in B$, if the equation $zx = b$ is solvable in G then it is solvable in B.

Lemma (2): Let G be a torsion free group of cardinality ω_1. Then G can be well-ordered as $\{g_\alpha : \alpha < \omega_1\}$ in such a way that for any limit ordinal δ, $G_\delta = \{g_\alpha : \alpha < \delta\}$ is a pure subgroup of G.

Proof: We define g_α by transfinite induction. The limit ordinals less than ω_1 are precisely the ordinals of the form $\omega\beta$, where $0 < \beta < \omega_1$. Suppose that for every $\beta < \gamma$, and every $\alpha < \omega\beta$, g_α has been defined, and $G_{\omega\beta}$ is a pure subgroup of G. We will show how to extend the definition so that $G_{\omega\gamma}$ is pure. There are two cases to consider.

Case (i): γ is a limit ordinal. Then of course g_α is already defined for every $\alpha < \omega\gamma$. Since for every $\beta < \gamma$, $G_{\omega\beta}$ is a pure subgroup of G, and $G_{\omega\gamma} = \bigcup_{\beta<\gamma} G_{\omega\beta}$, it is trivial to verify that $G_{\omega\gamma}$ is a pure subgroup of G.

Case (ii): γ is a successor. Take a fixed well-ordering of G in order type ω_1, with the first element in the ordering $\neq 0$. Let g be the first element of G with respect to this fixed order which is not a g_α for any $\alpha < \omega(\gamma-1)$. Let B be the subgroup of G generated by $G_{\omega(\gamma-1)}$ and g (where $G_0 = \phi$). Since G is torsion free B has cardinality ω. So by (1), page 115, B is contained in a countable pure subgroup of G, say B^*. As $zg \notin G_{\omega(\gamma-1)}$ for any z in Z it is clear that $B^* \setminus G_{\omega(\gamma-1)}$ has cardinality ω, and so it may be enumerated as $\{g_\alpha : \omega(\gamma-1) \leq \alpha < \omega\gamma\}$. Thus $G_{\omega\gamma} = B^*$ is a pure subgroup of G.

If G is a (torsion free) group of cardinality ω_1, instead of labelling the elements of G by the ordinals less than ω_1, it is notationally more convenient to assume the elements of G are the ordinals less than ω_1. Whenever such notation is used, it will be understood that for any limit ordinal δ, $G_\delta = \{\alpha : \alpha < \delta\}$ is a pure subgroup of G. Lemma (2) shows this is a harmless assumption. Call any such naming of G admissible.

Classification of ω_1-Free Groups of Cardinality ω_1

We will in this section classify ω_1-free groups of cardinality ω_1 into three possibilities, called unimaginatively Possibility I, Possibility II, and Possibility III. First we need a remark and then some preliminary group theoretic and set theoretic definitions.

Remark (1): For torsion free groups the equation $zx = g$ can have at most one solution, for $zx = g = zy$ implies $zx = zy$ implies $x = y$. So if $zx = g$ is solvable in G, then the unique solution belongs to all pure subgroups containing g and thus the intersection of pure subgroups is again pure. This allows us to make the following definition.

Definition (6): Let $\langle L,G\rangle_*$ be the smallest pure subgroup of Gwhich contains L, $L \subseteq G$.

Remark (2): If S is a pure subgroup of G, then $\langle L,S\rangle_* = \langle L,G\rangle_*$ by Remark (1). We write $\langle L\rangle_* = \langle L,G\rangle_*$ if it is clear which group G we are referring to.

Definition (7): Let S be a subgroup of G, L a finite subset of G, and a an element of G. We say $\Pi(a,L,S,G)$ holds if $\langle S \cup L\rangle_* = \langle S\rangle_* \oplus \langle L\rangle_*$ but for no $b \in \langle S \cup L \cup \{a\}\rangle_*$ is $\langle S \cup L \cup \{a\}\rangle_* = \langle S\rangle_* \oplus \langle L \cup \{b\}\rangle_*$.

Definition (8): A subset C of ω_1 is closed and unbounded if:

(i) For every non-empty subset S of C $\sup S \in C \cup \{\omega_1\}$. This says that C is closed.

(ii) $\sup C = \omega_1$. This says that C is unbounded.

Definition (9): A subset A of ω_1 is stationary if $C \cap A \neq \phi$ for every closed and unbounded subset of ω_1.

Definition (10): Let X be a subset of ω_1. Then X is cofinal in ω_1 if for all α in ω_1, there exists β in X such that $\alpha \leq \beta$.

Remark (3): No countable set is cofinal with ω_1. (3) page 207.

We are now ready to define the three possibilities.

<u>Definition (11):</u> An ω_1-free group G of cardinality ω_1 satisfies Possibility I if for any admissible naming of G there is some limit ordinal $\delta < \omega_1$, and there are elements of G, say $a^{\alpha}_{n(\alpha)}$ for all $\alpha < \omega_1$, (where $n(\alpha)$ is a finite ordinal) and subsets $L_\alpha = \{a^{\alpha}_{\ell}: 0 \leq \ell < n(\alpha)\}$ such that:

(A) $\{a^{\alpha}_{\ell} + G_\delta : \alpha < \omega_1, \ \ell \leq n(\alpha)\}$ is an independent family in G/G_δ .

(B) $\Pi(a^{\alpha}_{n(\alpha)}, L_\alpha, G_\delta, G)$ holds for all $\alpha < \omega_1$.

<u>Remark (4):</u> Since $\delta < \omega_1$, then G_δ is countable and so we can assume without loss of generality that $\delta = \omega$. Rename G_δ by $\{\alpha: 0 \leq \alpha < \omega\}$ which can be done as G_δ is countable. Now rename the rest of G using the technique of Lemma (2).

<u>Definition (12):</u> An ω_1-free group G of cardinality ω_1 satisfies Possibility II if G does not satisfy Possibility I and there is a stationary subset of ω_1, say A, such that for any α in A there are elements of G, say a^{α}_{ℓ}, $\ell \leq n(\alpha)$, (where $n(\alpha)$ is a finite ordinal), and subsets $L_\alpha = \{a^{\alpha}_{\ell}: 0 \leq \ell < n(\alpha)\}$ such that:

(A) $\{a^{\alpha}_{\ell}: 0 \leq \ell < n(\alpha)\}$ is an independent family in G/G_α .

(B) $\Pi(a^{\alpha}_{n(\alpha)}, L_\alpha, G_\alpha, G)$ holds.

<u>Definition (13):</u> An ω_1-free group G of cardinality ω_1 satisfies Possibility III if it doesn't satisfy Possibility I or Possibility II.

Lemma (3): The classification of a given group G to the three possibilities depends on G only up to isomorphism.

Proof: We must show that under any admissible naming of G, it will always satisfy the same possibility. There are three cases to consider.

Case (i): Suppose G satisfies Possibility I. Let $h:G \to G^*$ be an isomorphism. Then G^* can be thought of as a renaming of the elements of G and so by the definition of Possibility I, G^* satisfies Possibility I.

Case (ii): Suppose G satisfies Possibility II. First we show that if $h:G \to G^*$ is an isomorphism, then the set C defined by $C = \{\delta : h|G_\delta$ is an isomorphism from G_δ onto $G^*_\delta\}$ is a closed and unbounded subset of ω_1 where $h|G_\delta$ is the restriction of h to G_δ. G is closed since the union of a chain of isomorphisms is an isomorphism. Suppose C is bounded. Choose $\alpha < \omega_1$ such that α is an upper bound for C. For $n < \omega$ define α_n inductively as follows:

$$\alpha_0 = \alpha$$

$$\alpha_n = \sup\left(\{h(\delta):\delta < \alpha_{n-1}\} \cup \{\beta : h(\beta) < \alpha_{n-1}\}\right)$$

As $\alpha_0 < \omega_1$, then $\alpha_n < \omega_1$ for all n. That is if we assume inductively that $\alpha_{n-1} < \omega_1$, then α_n is the sup of a countable set and since no countable set is cofinal with ω_1, then $\alpha_n < \omega_1$. Let $\alpha^* = \sup_{n<\omega} \alpha_n$. Since $\alpha_n < \omega_1$ for all $n < \omega$, then $\alpha^* < \omega_1$. Let $\beta \in G$ such that $\beta < \alpha^*$, then $\beta < \alpha_n$ for some n, and so $h(\beta) < \alpha_{n+1}$ by the definition of α_{n+1}. Similarly if $\beta \in G^*$ such that $\beta < \alpha^*$ then $\beta < \alpha_n$ for some n and so $\beta = h(\rho)$ where $\rho < \alpha_{n+1}$. Thus $h|G_{\alpha^*}$ is an isomorphism and so $\alpha^* \in C$. Thus α is not an upper bound for C and so C is unbounded.

Now we show that $C \cap A$, where A is the stationary set required by the definition of Possibility II, is a stationary set, and then G^* will satisfy Possibility II using $C \cap A$ as the required stationary set. $C \cap A$ is stationary because any closed and unbounded set is stationary. That is if C_1 and C_2 are closed and unbounded sets then $C_1 \cap C_2 \neq \phi$, for let $\{\xi_1, \xi_2, \xi_3, \ldots\}$ be an increasing sequence of ordinals such that for n even $\xi_n \in C_1$ and for n odd $\xi_n \in C_2$. Then $\psi = \sup \{\xi_1, \xi_3, \xi_5, \ldots\} = \sup \{\xi_2, \xi_4, \xi_6, \ldots\}$ and ψ is in both C_1 and C_2 since C_1 and C_2 are closed. Actually $C_1 \cap C_2$ is closed and unbounded. Clearly $C_1 \cap C_2$ is closed as C_1 and C_2 are closed. If $C_1 \cap C_2$ was bounded, by say $\alpha < \omega_1$, then define a sequence $\{\xi_1, \xi_2, \xi_3, \ldots\}$ as before with $\xi_1 = \alpha$ to get a contradiction. Thus $C \cap C^*$, for any closed unbounded set C^*, is closed and unbounded. Thus $(C \cap A) \cap C^* = A \cap (C \cap C^*) \neq \phi$. Thus $C \cap A$ is stationary and so G^* satisfies Possibility II.

Case (iii): By definition Possibility III holds if and only if neither Possibility I nor Possibility II holds.

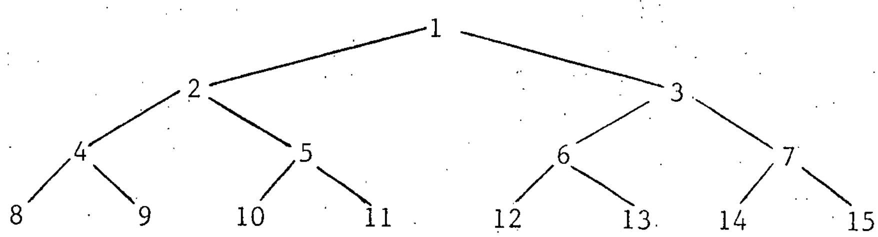
The classification into the three possibilities depends on G only up to isomorphism.

By the definition of the three possibilities, an ω_1-free group G can satisfy only one, and so the three possibilities form a partition. The following lemma shows that each possibility is satisfied by a particular ω_1-free group.

<u>Lemma (4):</u> Each Possibility is satisfied by some ω_1-free group.

<u>Proof:</u> Again there are three cases to consider.

<u>Case (i):</u> We will construct an ω_1-free group satisfying Possibility I. First we define a set C of increasing sequences of natural numbers of length ω such that the cardinality of C is ω_1, and if η and τ are in C, $\eta \neq \tau$, then η and τ have at most finitely many natural numbers in common; that is $\eta \cap \tau$ is finite. To show that such a set C exists we give an example. Consider the following diagram:



The sequences are defined by taking possible paths. For example:

$\{1,2,4,8,\ldots\}$

$\{1,2,4,9,\ldots\}$

$\{1,2,5,10,\ldots\}$

etc.

By the n^{th} row of the diagram 2^n sequences or paths are defined, and in the limit there are $2^\omega \geq \omega_1$ sequences. Choose any ω_1 sequences. The intersection of any two is finite for they can agree only up to the point where their corresponding paths separate.

Let G be generated by:

(i) x_k for $k < \omega$

(ii) $x_\tau^m = \sum_{k=m}^{\infty} (\frac{k!}{m!}) x_{\tau(k)}$ for $m < \omega$ and $\tau \in C$

Using the notation of the definition of Possibility I and $C = \{\tau(\alpha) : \alpha < \omega_1\}$ let:

(i) G_δ be the group freely generated by the x_k's.

(ii) $n(\alpha) = 0$, and so $L_\alpha = \phi$.

(iii) $a_0^\alpha = x_{\tau(\alpha)}^m$ for $\alpha < \omega_1$ and m fixed.

We must show that G satisfies conditions (A) and (B) in the definition of Possibility I. That is:

(A) $\{a_0^\alpha + G_\delta : \alpha < \omega_1\}$ is an independent family in G/G_δ.

(B) $\Pi(a_0^\alpha, \phi, G_\delta, G)$ holds for all $\alpha < \omega_1$.

(A) follows from the finite intersection property of the elements of C. That is if $z_1 x_{\tau(\alpha_1)}^m + \ldots + z_n x_{\tau(\alpha_n)}^m = g$ where $z_i \in Z$, $z_i \neq 0$, $\tau(\alpha_i) \in C$, and $g \in G_\delta$, then $z_1 x_{\tau(\alpha_1)}^m + \ldots + z_n x_{\tau(\alpha_n)}^m$ is a finite linear combination of the x_k's that generate G_δ. As the $x_{\tau(\alpha_i)}^m$'s are infinite linear combinations of the x_k's, then $z_1 x_{\tau(\alpha_1)}^m + \ldots + z_n x_{\tau(\alpha_n)}^m$ must be an infinite linear combination of the x_k's since for $i \neq j$ $x_{\tau(\alpha_i)}^m$ and $x_{\tau(\alpha_j)}^m$ agree at only finitely many x_k's for $\tau(\alpha_i) \cap \tau(\alpha_j)$ is finite. This is a contradiction and so (A) holds.

Now we show condition (B) holds. As $L_\alpha = \phi$, then $\langle G_\delta \cup L_\alpha \rangle_* = \langle G_\delta \rangle_* \oplus \langle L_\alpha \rangle_*$. Choose any $a_0^\alpha = x_{\tau(\alpha)}^m$. Then $(m+1) x_{\tau(\alpha)}^{m+1} = x_{\tau(\alpha)}^m - x_{\tau(\alpha)(m)}^m$ and so by definition of purity $x_{\tau(\alpha)}^{m+1} \in \langle G_\delta \cup \{x_{\tau(\alpha)}^m\} \rangle_*$. Similarly $x_{\tau(\alpha)}^k \in \langle G_\delta \cup \{x_{\tau(\alpha)}^m\} \rangle_*$ for all $k \geq m+1$. Using the finite intersection property for elements of C it is clear that no other elements of G will be thrown into $\langle G_\delta \cup \{x^m_{\tau(\alpha)}\}\rangle_*$. So if for some $x \in \langle G_\delta \cup \{x^m_{\tau(\alpha)}\}\rangle_*$, $\langle G_\delta \cup \{x^m_{\tau(\alpha)}\}\rangle_* = \langle G_\delta\rangle_* \oplus \langle x\rangle_*$, then we can assume that $\langle x\rangle_* = \langle x\rangle$ = the group generated by x for some $x = \sum_{i=1}^{n} z_i y_i$ where $z_i \in Z$ and each y_i is some x_k or $x^k_{\tau(\alpha)}$. Clearly x will cause only finitely many of the $x^k_{\tau(\alpha)}$'s to be in $\langle G_\delta\rangle_* \oplus \langle x\rangle_*$ and so this is impossible. Therefore $\Pi(x^m_{\tau(\alpha)}, \phi, G_\delta, G) = \Pi(a^\alpha_0, L_\alpha, G_\delta, G)$ holds for all $\alpha < \omega_1$ and so condition (B) is satisfied.

Now we show that G is ω_1-free. It is sufficient to show that for any $g_1,\ldots,g_n \in G$, the pure subgroup generated by $g_1,\ldots,g_n$ is free on a finite number of generators, (4) page 25. Without loss of generality $g_1,\ldots,g_n$ are independent and so $\langle\{g_1,\ldots,g_n\}\rangle_*$ has rank, n (1) page 116. So let $b_1,\ldots,b_n$ generate $\langle\{g_1,\ldots,g_n\}\rangle_*$, that is $\langle\{g_1,\ldots,g_n\}\rangle_* = \langle\{b_1,\ldots,b_n\}\rangle$. We do an induction on the number of generators. For $n = 1$ clearly $\langle\{b_1\}\rangle$ is free since G is torsion free. Assume any pure subgroup on $n - 1$ generators is free and let $\langle\{g_1,\ldots g_n\}\rangle_*$ be generated by $b_1,\ldots,b_n$. If $\langle\{b_1,\ldots,b_n\}\rangle$ is not freely generated by $b_1,\ldots,b_n$, then for some $z_i \in Z$, not all zero, $\sum_{i=1}^{n} z_i b_i = 0 \Rightarrow \sum_{i=1}^{n} z_i b_i = -z_n b_n \Rightarrow b_n \in \langle\{b_1,\ldots,b_{n-1}\}\rangle_*$. Thus the pure subgroup generated by $g_1,\ldots,g_n$ has rank less than n, a contradiction. So $b_1,\ldots,b_n$ freely generates $\langle\{g_1,\ldots,g_n\}\rangle$ and by the induction hypothesis any pure subgroup generated by a finite subset of G is free. Thus G is ω_1-free.

Now let G^* be any admissible naming of the elements of G. Choose $\delta < \omega_1$ such that x_n is in G^*_δ for all $n < \omega$. As $\{x^m_{\tau(\alpha)}:\alpha < \omega_1\}$ is uncountable and G_δ is countable, we can find uncountably many $x^m_{\tau(\alpha)}$'s such that $x^k_{\tau(\alpha)} \notin G_\delta$ for $k < \omega$. let $\{x^m_{\tau(\beta)}:\beta < \omega_1\}$ be such a set. By letting:

(i) $G_\delta = G_\delta$

(ii) $n(\beta) = 0$, and so $L_\beta = \phi$

(iii) $a^\beta_0 = x^m_{\tau(\beta)}$ for $\beta < \omega_1$

it follows that G^* satisfies Possibility I in exactly the same way as G was shown to satisfy Possibility I.

Case (ii): We will construct a group satisfying Possibility II. For this example, the stationary set A required by the definition for Possibility II will be the set of all limit ordinals. First we show that this set $A = \{\delta < \omega_1:\delta \text{ is a limit ordinal}\}$ is stationary. This follows from the observation that any closed and unbounded set C contains a limit ordinal. That is if $\{\alpha_1,\alpha_2,\alpha_3,\ldots\}$ is any countably infinite subset of C where $\alpha_i < \alpha_{i+1}$, then $\sup_{i<\omega} \alpha_i = \alpha$ is a limit ordinal for if not then α has a predessor $\alpha - 1$ which would be an upper bound to the sequence.

Now for δ a limit ordinal, let τ_δ be a sequence of ordinals of length ω such that $\sup_{n<\omega} \tau_\delta(n) = \delta$ where $\tau_\delta(n)$ is the n'th ordinal of the sequence τ_δ. Let G be generated by:

(i) x_α for $\alpha < \omega_1$

(ii) $x^m_\delta = \sum_{k=m}^{\infty} (\frac{k!}{m!}) x_{\tau_\delta(k)}$ for $m < \omega$, $\delta < \omega_1$, and δ a limit ordinal

Using the notation of the definition of Possibility II let:

(i) $x_\alpha \in G_\delta$, $x_\delta^m \notin G_\delta$, for $\alpha < \delta$, $m < \omega$, and δ a limit ordinal

(ii) $n(\delta) = 0$, and so $L_\delta = \phi$

(iii) $a_0^\delta = x_\delta^m$, m fixed

Since $x_\delta^m \notin G_\delta$, then $\{x_\delta^m + G_\delta\}$ is an independent family in G/G_δ and so condition (A) in the definition of Possibility II holds. $\Pi(x_\delta^m, L_\delta, G_\delta, G)$ holds using the same arguement as used for Possibility I , and so condition (B) is satisfied. The ω_1-freeness of G is again similar to Possibility I.

Lastly we must show that G doesn't satisfy Possibility I. It is sufficient to show that a given admissible naming of G does not satisfy it. Let G_δ be generated by:

(i) x_α for $\alpha < \delta$

(ii) x_β^m for $\beta < \delta$, $m < \omega$, β a limit ordinal

Now define the τ_δ's to be increasing sequences for all limit ordinals $\delta < \omega_1$. Then $\Pi(x_\delta^m, \phi, G_\delta, G)$ holds for any $m < \omega$. The "Π" condition cannot hold for any other x_β^m's , $\beta \neq \delta$, since for $\beta < \delta$, x_β^m is in G_δ , and for $\beta > \delta$, $\langle G_\delta \cup \{x_\beta^m\}\rangle_* = G_\delta \oplus \langle x_\beta^k\rangle_*$ where k is the largest element of the increasing sequence τ_β less than δ . As the x_β^m's are the only possibilities for creating the "Π" condition, we can conclude that it is satisfied at only countably many places for each G_δ . Thus Possibility I cannot hold and so G satisfies Possibility II.

<u>Case (iii):</u> Let G be the free group on ω_1 generators. It is sufficient to show that G does not satisfy Possibility I or II for some admissible naming of G. Let G be generated by the elements α_β, $\beta < \omega_1$. Let $G_{\omega\xi}$ be the group generated by α_β, $\beta \leq \xi$, for ξ not a limit ordinal. That is $G_{\omega\xi} = \bigoplus_{\beta \leq \xi} G^\beta$, where G^β is the subgroup of G generated by the element α_β. Let $G_{\omega\xi}$ be the group generated by α_β, $\beta < \xi$, for ξ a limit ordinal. That is $G_{\omega\xi} = \bigoplus_{\beta<\xi} G^\beta$, where G^β is the subgroup of G generated by the element α^β. Clearly this is an admissible naming of G. Claim that $\Pi(a,L,G_\delta,G)$ does not hold for any limit ordinal δ, where L is a finite subset of G and a is an element of G. So suppose for some L and a that $\Pi(a,L,G_\delta,G)$ holds. If we can show that only finitely many elements are in the group W where $W = \langle G_\delta \cup L \cup \{a\}\rangle_* / \langle\langle G_\delta \cup L\rangle_* \cup \{a\}\rangle$, then by the result on page 24 (G_5/G_4 is infinite if "Π" holds) since $W = G_5/G_4$, "Π" must fail. If $L = \{a_1,\ldots a_n\}$ and if $a = a_o$, then each $a_i = \sum_{j=1}^{m_i} z_j \alpha_j$, where $z_j \in Z$, and α_j is the generator of $G^{\alpha j}$. Then the only new elements in W will be linear combinations of the α_j's which make up the a_i's. Clearly there are only finitely many of these in W. Thus the "Π" condition fails in G under this admissible naming, and so Possibility I or II cannot hold. Also G is ω_1-free since any subgroup of a free group is free, (1) page 74. Thus G must satisfy Possibility III.

<u>Lemma (5):</u> Let G be ω_1-free. Then Possibility III is equivalent to G being the direct sum of countable groups.

<u>Proof:</u> Suppose G is the direct sum of countable groups and G is ω_1-free. Then $G = \bigoplus_{\alpha<\omega_1} G^\alpha$ where each G^α is countable. Since G is ω_1-free, each G^α is free, so each G^α is isomorphic to a countable direct sum of copies of Z. Thus G is isomorphic to a direct sum of ω_1 copies of Z and so G is free on ω_1 generators. By Lemma (4) case (iii), G satisfies Possibility III.

Now suppose G satisfies Possibility III. First we show that if C is a closed and unbounded subset of ω_1, then $C^* = \{\delta : \delta \in C$ and δ is a limit ordinal$\}$ is also closed and unbounded. C^* is closed since C is closed and the sup of a sequence of limit ordinals is a limit ordinal. C^* is unbounded since C is unbounded and the sup of an infinite increasing sequence of ordinals of C is a limit ordinal of C. Thus C^* is closed and unbounded and contains only limit ordinals.

Since Possibility I and Possibility II fails, we can find a closed unbounded set C such that if $\delta \in C$, there does not exist $a \in G$ and L, a finite subset of G, such that $\Pi(a,L,G_\delta,G)$ holds. That is, if for every closed and unbounded set such a δ exists, then by taking the set of these δ's we get a stationary set which satisfies condition (B) of Possibility II. By taking $L^* \subseteq L$ such that L^* is a maximal independent family in L, then condition (A) of Possibility II would be satisfied using the L^*'s in place of the L's. Since Possibility I fails, then Possibility II would hold for G, a contradiction. Therefore such a C exists.

From previous remarks in this proof we can assume that C contains only limit ordinals. Since $\sup C = \omega_1$, then the cardinality of C is ω_1 since no countable set is cofinal with ω_1. So let $C = \{\delta_\alpha : \alpha < \omega_1\}$ where each δ_α is a limit ordinal and $\alpha < \beta \Rightarrow \delta_\alpha < \delta_\beta$. Now we rename G as follows:

Rename $\{\beta : \delta_\alpha \leqq \beta < \delta_{\alpha+1}\}$ as $\{\beta : \omega\alpha \leqq \beta < \omega(\alpha + 1)\}$.

Now we can assume that $C = \{\omega\alpha : \alpha < \omega_1\}$, and it is clear that we still have an admissible naming of G.

Now we do an induction to show that $G_{\omega\alpha+\omega} = G_{\omega\alpha} \oplus \langle b_1, b_2, \ldots\rangle_*$ for some b_i's in $G_{\omega\alpha+\omega} \smallsetminus G_{\omega\alpha}$. Suppose $b_1, \ldots b_n$ have been choosen. Let $G^n_{\omega\alpha} = G_{\omega\alpha} \oplus \langle b_1, \ldots b_n\rangle_*$. Now let $L = \{b_1, \ldots b_n\}$ and let $a = \inf\{\delta : \delta \in G_{\omega\alpha+\omega} \smallsetminus G^n_{\omega\alpha}\}$. As $\omega\alpha$ is in C, then $\Pi(a, L, G_{\omega\alpha}, G)$ fails and so there must exist $b_{n+1} \in \langle G_{\omega\alpha} \cup L \cup \{a\}\rangle_*$ such that $\langle G_{\omega\alpha} \cup L \cup \{a\}\rangle_* = G_{\omega\alpha} \oplus \langle L \cup \{b_{n+1}\}\rangle_* = G_{\omega\alpha} \oplus \langle b_1, \ldots, b_n, b_{n+1}\rangle_*$. Since clearly $\bigcup_n G^n_{\omega\alpha} = G_{\omega\alpha+\omega}$ we get that $G_{\omega\alpha+\omega} = G_{\omega\alpha} \oplus \langle b_1, b_2, \ldots\rangle_*$. Let $H_{\omega\alpha} = \langle b_1, b_2, \ldots\rangle_*$. Then $G_{\omega\alpha+\omega} = G_{\omega\alpha} \oplus H_{\omega\alpha}$. Thus $G = G_\omega \bigoplus_{1 \leqq \alpha < \omega_1} H_{\omega\alpha}$ and so G is the direct sum of countable groups.

(G,Z)-Groups

In this section we will define (G,Z)-groups and prove some lemmas about them necessary for the consistency result.

Definition (14): A (G,Z)-group is a group H with underlying set $G \times Z = \{(a,b): a \in G,\ b \in Z\}$ such that:

(i) $(a,b) + (0,c) = (a,b+c)$,

(ii) The map $h: H \to G$ defined by $h(a,b) = a$ is a group homomorphism.

For a given G_i , H_i will denote a (G_i,Z)-group, and the corresponding homomorphism will be denoted by h_i .

Lemma (6): Let G_1 be a countable subgroup of G_2 where G_2 is ω_1-free and the cardinality of G_2 is at most ω_1 . Let H_1 be a (G_1,Z)-group. Then H_1 can be extended to a (G_2,Z)-group.

Proof: First note that G_1 is freely generated since it is countable and G_2 is ω_1-free. Thus from the result noted before, G_1 is a W-group since freely generated groups are W-groups. The result will be proved by transfinite indection. To simplify the induction we will deal with two special cases first. Let $\langle a + G_1\rangle$ be the subgroup of G_2/G_1 generated by the element $a + G_1$, where a is in G_2. Let $G_a = \langle\{a\} \cup G_1\rangle$ be the subgroup of G_2 generated by $\{\{a\} \cup G_1\}$.

Case (i): $\langle a + G_1\rangle$ is isomorphic to Z.

Case (ii): $\langle a + G_1\rangle$ is cyclic of prime order.

We will show that in cases (i) and (ii) H_1 can be extended to a (G_a,Z)-group.

<u>Proof of case (i):</u> Suppose $\langle a + G_1 \rangle$ is isomorphic to Z. Then every $b \in G_a$ has a unique representation as $za + c$ where $z \in Z$ and $c \in G_1$. Now define for b_1, b_2 in G_a and k_1, k_2 in Z, the following:

$$\begin{aligned}
&(b_1,k_1) + (b_2,k_2) \\
&= (z_1a + c_1,k_1) + (z_2a + c_2,k_2) \\
&\overset{\text{def}}{=} ((z_1 + z_2)a + c_3,k_3)
\end{aligned}$$

where $(c_1,k_1) + (c_2,k_2) = (c_3,k_3)$ in H_1. It is easy to check that this natural extension of H_1 forms a group. Call this group H_a.

Then $(b,k) + (0,m) = (za + c,k) + (0,m)$

$$\begin{aligned}
&= (za + c,k + m) \\
&= (b,k + m) \quad \text{, since in } H_1 \ (c,k) + (0,m) = (c,k + m).
\end{aligned}$$

Also the mapping $h_a : H_a \to G_1$ defined by $h_a(b,k) = b$ is clearly a homomorphism, and so H_a is a (G_a,Z)-group.

<u>Proof of case (ii):</u> Suppose $\langle a + G_1 \rangle$ is cyclic of prime order p. Since $h_1 : H_1 \to G_1$ has kernel isomorphic to Z and G_1 is a W-group, then there exists $g_1 : G_1 \to H_1$ such that $h_1 g_1 = 1_{G_1}$. Let $g_1(c) = (c,m(c))$ for $c \in G_1$. Every $b \in G_a$ has a unique representation as $za + c$ where $0 \leqq z < p$, $c \in G_1$. Now for b_1, b_2 in G_a and k_1, k_2 in Z define:

$$\begin{aligned}
&(b_1,k_1) + (b_2,k_2) \\
&= (z_1a + c_1,k_1) + (z_2a + c_2,k_2) \\
&\overset{\text{def}}{=} ((z_1 + z_2)a + c_1 + c_2,k_1 + k_2 - m(c_1) - m(c_2) + m(c_1 + c_2) \\
&\quad + f(z_1 + z_2))
\end{aligned}$$

where $0 \leqq z_1,z_2 < p$, and $f(n) = 0$ when $n < p$ and $f(n) = M \in Z$ otherwise; where M is an arbitrary constant which once chosen remains the same for all such defined sums. We will show that this set, call it H_a, forms a (G_a,Z)-group under this defined operation. To show H_a is a group, the only non trivial thing to show the existence of inverses. Let $(b,k) = (za + c,k)$ be in H_a and $-c$ be the inverse of c in G_1. Then:

$$\begin{aligned}
&(za + c,k) + ((p - z)a - c, -k - M) \\
&= ((z + (p - z))a + c - c, k - k - M - m(c) - m(c) + m(c - c) \\
&\quad + f(z + (p - z))) \\
&= (pa, -M + f(p)) \\
&= (0, -M + M) \\
&= (0,0)
\end{aligned}$$

Thus the inverse of (b,k) is $((p - z)a - c, -k - M)$, and so H_a is a group. Now let $(b,k) = (za + c,k)$ and $(0,t)$ be in H_a. Then:

$$\begin{aligned}
&(b,k) + (0,t) \\
&= (b, k + t - m(c) - m(0) + m(c) + f(z)) \\
&= (b, k + t) \qquad , \text{ as } z < p .
\end{aligned}$$

Also the mapping $h_a : H_a \to G_1$, defined by $h_a(b,k) = b$ is a homomorphism, so H_a is a (G_a,Z)-group.

Since G_2 is ω_1-free and G_1 is countable, then G_a is countable and so it is freely generated. Thus G_a is a W-group.

Now we do the induction. First we find a sequence of elements of G_2, say $A = \{a_\delta : \delta < \sigma, \sigma \text{ an ordinal}\}$, such that $G_1 \cup A$ generates G_2, and such that if $J_\delta = \langle G_1 \cup \{a_\rho : \rho < \delta\}\rangle$ for all $\delta < \sigma$, then $\langle a_\delta + J_\delta \rangle$ is infinite cyclic or cyclic of prime order. The sequence A is defined as follows:

Assume a_β has been defined for all $\beta < \delta$. Let $b = \inf\{\alpha : \alpha \in G_2 \setminus J_\delta\}$. If $\langle b + J_\delta \rangle$ is infinite cyclic or cyclic of prime order, let $a_\delta = b$. If not, then $\langle b + J_\delta \rangle$ is cyclic of non prime order, say of order np where p is prime. Then let $a_\delta = nb$. So $\langle a_\delta + J_\delta \rangle$ has prime order p. It is clear that $\mathrm{card}(A) \leqq \omega_1$, since $G_1 \cup A$ generates G_2 and $\mathrm{card}(G_2) \leqq \omega_1$.

Let $K_0 = H_1$, a (J_0, Z)-group. We will define K_β to be a (J_β, Z)-group for all $\beta \leqq \sigma$.

<u>(a) β is not a limit ordinal:</u> Since β is not a limit ordinal, it has a predessor. So we can suppose a $(J_{\beta-1}, Z)$-group, $K_{\beta-1}$, has been defined. Then by construction of the sequence A, $\langle a_{\beta-1} + J_{\beta-1} \rangle$ is infinite cyclic, or is cyclic of prime order. If it is infinite cyclic then case (i) can be applied directly to $K_{\beta-1}$ to show it can be extended to a (J_β, Z)-group K_β. If it is cyclic of prime order, then as $J_{\beta-1}$ is countable, it is freely generated and so it is a W-group. Then there exists $g_{\beta-1} : J_{\beta-1} \to K_{\beta-1}$ such that $h_{\beta-1} g_{\beta-1} = 1_{J_{\beta-1}}$ where as usual $h_{\beta-1} : K_{\beta-1} \to J_{\beta-1}$ and $h(a,b) = a$. Thus we can apply case (ii) using $g_{\beta-1}$ as the required map, and so extend $K_{\beta-1}$ to a (J_β, Z)-group K_β.

<u>(b) β is a limit ordinal:</u> Define $K_\beta = \bigcup_{\delta<\beta} K_\delta$. It is easy to check that K_β is a (J_β, Z)-group.

So inductively we can define a (J_σ, Z)-group. Call it H_2. As the set $G_1 \cup A$ generates G_2, then $J_\sigma = G_2$, and so H_2 is a (G_2, Z)-group and the lemma is proved.

<u>Lemma(7):</u> Let H_1 be a (G_1,Z)-group. Let h_1 and g_1 be homomorphisms, $h_1:H_1 \to G_1$ and $h_1(a,b) = a$, $g_1:G_1 \to H_1$, such that $h_1 g_1 = 1_{G_1}$. Let G_2 be ω_1-free and $\operatorname{card}(G_2) \leqq \omega_1$. Suppose $\Pi(a,A,G_1,G_2,)$ holds. Then H_1 can be extended to a (G_2,Z)-group H_2 such that for no homomorphism $g_2:G_2 \to H_2$ does $h_2 g_2 = 1_{G_2}$ where g_2 extends g_1 and as usual $h_2(a,b) = a$.

<u>Proof:</u> Let: (i) $A = \{a_1,\ldots,a_m\}$

(ii) $G_3 = \langle G_1 \cup A\rangle_* = \langle G_1\rangle_* \oplus \langle A\rangle_*$

(iii) $G_4 = \langle G_3 \cup \{a\}\rangle$

(iv) $G_5 = \langle G_1 \cup A \cup \{a\}\rangle_*$

Let H_4 be a (G_4,Z)-group. Consider the homomorphisms $g:G_4 \to H_4$ that extend g_1 and such that $h_4 g = 1_{G_4}$. Any such g is uniquely determined by where g maps $a_1,\ldots,a_m$ and a. That is if $b \in G_4$, then $b = c + za$ where $c \in G_3$ and $z \in Z$. So $b = d + x + za$ where $d \in \langle G_1\rangle_*$ and $x \in \langle A\rangle_*$. $x \in \langle A\rangle_*$ implies nx is a linear combination of the a_i's for $i = 1,\ldots,m$. Thus the $g(a_i)$'s determine $g(nx) = ng(x)$, and so they determine $g(x)$ since there is a unique solution to $ng(x) = y$. As $d \in \langle G_1\rangle_*$, $g(d)$ is already determined by g_1. Let $a_0 = a$. As $h_4 g = 1_{G_4}$ and $h_4(b,z) = b$, then $g(a_i) \in \{(a_i,z):z \in Z\}$ for $i = 0,\ldots,m$. So each $g(a_i)$ can be defined in only countably many ways and since there are only finitely many a_i's, there can be only countably many such g's. Call them $\{g^n:n < \omega\} = R$.

Now we will show that G_5/G_4 must be infinite. Then we will make some observations about the structure of G_5/G_4 and classify it into two possibilities.

$\underline{G_5/G_4 \text{ is infinite:}}$ Since $G_5 = \langle G_1 \cup A \cup \{a\}\rangle_* \neq \langle G_1\rangle_* \oplus \langle A \cup \{a\}\rangle_*$, then there exists $x \in \langle G_1 \cup A \cup \{a\}\rangle_*$, $x \notin \langle G_1\rangle_* \oplus \langle A \cup \{a\}\rangle_*$ such that $nx = g + c + ka$ where for some $n \neq 1$, $g \in \langle G_1\rangle_*$, $c \in \langle A\rangle_*$, and $k \neq 0$. Let n be the smallest positive integer for which there is such a g, k, and c. Then the greatest common divisor of n and k is 1, for if not then say m divides n and k. Then $m((\frac{n}{m})x - (\frac{k}{m})a) = g + c$, and since $\langle G_1 \cup A\rangle_* = \langle G_1\rangle_* \oplus \langle A\rangle_*$, then there exists $g_1 \in \langle G_1\rangle_*$ and $c_1 \in \langle A\rangle_*$ such that $(\frac{n}{m})x - (\frac{k}{m})a = g_1 + c_1$, and so $(\frac{n}{m})x = g_1 + c_1 + (\frac{k}{m})a$, contradicting the minimality of n. Thus there exists integers w and z, such that $nw + kz = 1$. Now consider $wa + zx$:

$$nx = g + c + ka,$$

$$\text{so} \quad wg + wc = wnx - wka,$$

$$\begin{aligned} \text{and so} \quad wg + wc + k(wa + zx) &= wnx - wka + kwa + kzx \\ &= wnx + kzx \\ &= (wn + kz)x \\ &= x \end{aligned}$$

Similarly:

$$\begin{aligned} -zg - zc + n(wa + zx) &= -znx + zka + nwa + nxz \\ &= (zk + nw)a \\ &= a \end{aligned}$$

So if x was the only new element in $\langle G_1 \cup A \cup \{a\}\rangle_*$, then $\langle G_1 \cup A \cup \{a\}\rangle_* = \langle G_1\rangle_* \oplus \langle A \quad \{wa + zx\}\rangle_*$, a contra diction to $\Pi(a, A, G_1, G_2)$, since $wa + zx$ is in $\langle G_1 \cup A \cup \{a\}\rangle_*$. So there must be a y such that $my = g + c + sa + tx$, or $my = g_1 + c_1 + u(wa + zx)$ for some $u \quad Z$. Using the same method we can find an element b in $\langle G_1 \cup A \cup \{a\}\rangle_*$ such that y and $wa + zx$ are in $\langle G_1\rangle_* \oplus \langle A \cup \{b\}\rangle_*$.

Since this process can be repeated for any finite number of such elements, it follows that there must be infinitely many of them else we get a contradiction to the Π condition. Thus G_5/G_4 is a countably infinite torsion group.

Definition (15): A group G is divisible if for every x in G and every integer n, there exists elements in G that satisfy the equation $ny = x$.

From Kaplansky (5), we have the following two results:

(a) Any abelian group G has a unique largest divisible subgroup M, and $G = M \oplus N$ where N has no divisible subgroups. (5) page 9.

(b) Any divisible group is a direct sum of groups, each isomorphic to the additive group of rationals Q, or to $Z(p^{\infty})$, the group of all p^{th} roots of unity for various primes p. (5) page 10.

As G_5/G_4 is a torsion group it cannot have a subgroup isomorphic to Q. So if G_5/G_4 has a non trivial divisible subgroup, then by Kaplansky's two results, G_5/G_4 contains a copy of $Z(p^{\infty})$ for some prime p and the copy of $Z(p^{\infty})$ is a direct summand of the group.

So suppose G_5/G_4 has no non trivial divisible subgroups.

Definition (16): A group G is reduced if it has no non trivial divisible subgroups.

From Kaplansky we have the following result:

(c) If G is a reduced group which is not torsion free, then G has a finite cyclic summand. (5) page 21.

Since G_5/G_4 is a reduced torsion group, then by (c) G has a finite cyclic summand. Now apply (c) to the other summand. Repeated application of (c) to the infinite remaining summand of G_5/G_4 shows that G_5/G_4 contains an infinite direct sum of finite cyclic groups. From each of these choose an element of prime order. Thus we can assume that there exists infinitely many distinct elements, say a_n for $n < \omega$, such that $p_n a_n \in G_4$ where each p_n is prime.

Now we can say that one of the following possibilities occurs in G_5/G_4:

(I) G_5/G_4 contains infinitely many elements of prime order.

or (II) G_5/G_4 contains a copy of $Z(p^\infty)$ for some p.

Let (I) hold in G_5/G_4. Let a_n, $n < \omega$, be the elements of prime order. That is the element $a_n + G_4$ has order p_n in G_5/G_4. Let G_n^* be generated by $G_4 \cup \{a_0,\ldots,a_{n-1}\}$. We will inductively define a (G_{n+1}^*,Z)-group H_{n+1}^* using Lemma (6) so that H_{n+1}^* extends H_n^*. First use Lemma (6) to extend H_1 to a (G_4,Z)-group which we will call H_0^*. Clearly this can be done as G_4 and G_1 meet all the conditions of Lemma (6). That is G_1 is a countable subgroup of G_4, and G_4 is ω_1-free as it is a subgroup of the ω_1-free group G_2. Also H_1 is given to be a (G_1,Z)-group. Thus H_0^* exists. Assume inductively H_n^* is defined. G_n^* and G_{n+1}^* satisfy the conditions of Lemma (6) using H_n^* as the required (G_n^*,Z)-group. So by Lemma (6) H_n^* can be extended to a (G_{n+1}^*,Z)-group, H_{n+1}^*. As $a_n + G_4$ has order p_n, $a_n + G_n^*$ has order p_n. Let M_n be the constant used in Lemma (6) to extend H_n^* to H_{n+1}^*. Inductively we define H_ω^*. Again apply Lemma (6) to extend H_ω^* to a (G_5,Z)-group, say H_5, which extends all the H_n^*'s.

If $g_5 : G_5 \to H_5$ is a homomorphism extending g_1 such that $h_5 g_5 = 1_{G_5}$, then for some n, g_5 extends g^n; that is $g|_{H_0^*} = g^n$ where $g^n \in R$ as defined earlier in the proof. Now we will show using $g_5|_{H_n^*} = g_n^*$ as the required map in extending H_n^* to H_{n+1}^*, (see Lemma (6), case (ii)), that constants M_n can be chosen such that $g_5|_{H_n^*}$ has no extension to G_{n+1}^* for each $n < \omega$. This will show that g_n^* and thus g^n has no extension to G_5 and so such a g_5 does not exist.

As $a_n + G_4$ is of order p_n, then let $p_n a_n = b_n \in G_4$. Let $g^n(b_n) = (b_n, k_n)$ and $g_n^*(a_n) = (a_n, c_n)$. Since b_n is in G_4 and g_n^* extends g^n we have $g^n(b_n) = g_n^*(b_n)$ and so:

$$
\begin{aligned}
g^n(b_n) &= (b_n, k_n) && \text{by definition} \\
&= g_n^*(b_n) && \text{as } b_n \in G_4 \\
&= g_n^*(p_n a_n) && \text{as } p_n a_n = b_n \\
&= p_n g_n^*(a_n) && \text{as } g_n^* \text{ is a homomorphism} \\
&= p_n(a_n, c_n) && \text{by definition} \\
&= (b_n, p_n c_n + M_n) && \text{by definition of "+" in } H_{n+1}^* \text{ as} \\
& && \text{as defined in Lemma (6), case (ii).}
\end{aligned}
$$

So in H_{n+1}^*:

$$
\begin{aligned}
p_n(a_n c_n) &= (a_n, c_n) + \ldots\ldots + (a_n, c_n) \qquad (p_n \text{ times}) \\
&= (a_n + a_n, c_n + c_n + f(1 + 1)) + (a_n, c_n) + \ldots\ldots + (a_n, c_n) \\
&= (2a_n, 2c_n) + (a_n, c_n) + \ldots\ldots + (a_n, c_n) \\
&= \ldots\ldots\ldots\ldots \\
&= ((p_n - 1)a_n, (p_n - 1)c_n) + (a_n, c_n) \\
&= (p_n a_n, p_n c_n + f(p_n)) \\
&= (b_n, p_n c_n + M_n)
\end{aligned}
$$

Recall that the constant M_n as chosen in Lemma (6) case (ii) was arbitrary. By the calculation on the previous page we have that $k_n = p_n c_n + M_n$ and so $k_n \equiv M_n (\text{mod } p_n)$. By choosing $M_n = k_n + 1$, this is impossible, and so g_n^* cannot be extended to G_{n+1}^* and so g^n cannot be extended to G_{n+1}^* and so g^n cannot be extended to G_5 .

Now suppose (II) holds. That is G_5/G_4 contains a copy of $Z(p^\infty)$ for some p. Then from the structure of $Z(p^\infty)$ there are elements, say a_n for $n < \omega$, such that:

(a) $pa_0 = b_0 \in G_4$

(b) $pa_n - a_{n-1} = b_n \in G_4$

That is a_0 is a p^{th} root of unity and a_n is the $(p^n)^{th}$ root of unity such that $(pa_n - a_{n-1}) \equiv 0(\text{mod } G_4)$. Again let G_n^* be generated by $G_4 \cup \{a_0,\ldots,a_{n-1}\}$ and let H_n^* be a (G_n^*,Z)-group constructed inductively as before using the constants M_n . We will again show that by proper choice of the M_n's, that $g^n \in R$ has no extension to G_{n+1}^* and thus no extension to G_5 . As before let $g_n^*(a_n) = (a_n,c_n)$ and $g^n(b_n) = g_n^*(b_n) = (b_n,k_n)$. Then:

$$
\begin{aligned}
g^n(b_0) &= (b_0,k_0) \\
&= g_n^*(b_0) \\
&= pg_n^*(a_0) \\
&= p(a_0,c_0) \\
&= (b_0,pc_0 + M_0)
\end{aligned}
$$

And so $k_0 = pc_0 + M_0$ or $k_0 \equiv M_0(\text{mod } p)$.

Also:

$$
\begin{aligned}
g^n(b_n) &= (b_n,k_n) && \text{by definition} \\
&= g_n^*(b_n) && \text{as } b_n \in G_4
\end{aligned}
$$

$$
\begin{aligned}
&= g_n^*(pa_n - a_{n-1}) && \text{by (b)} \\
&= pg_n^*(a_n) - g_n^*(a_{n-1}) && \text{as } g_n^* \text{ is a homomorphism} \\
&= p(a_n, c_n) - (a_{n-1}, c_{n-1}) && \text{by definition} \\
&= (b_n + a_{n-1}, pc_n + M_n) - (a_{n-1}, c_{n-1}) && \text{"+" in } H_{n+1}^* \\
&= (b_n, pc_n + M_n - c_{n-1}) && \text{"+" in } H_{n+1}^* \text{ using fact} \\
& && \text{that } a_{n-1} \ H_n^*
\end{aligned}
$$

And so $k_n = pc_n + M_n - c_{n-1}$ or $k_n + c_{n-1} \equiv M_n (\text{mod } p)$.

Thus we have: (1) $k_0 \equiv M_0 (\text{mod } p)$

(2) $k_n + c_{n-1} \equiv M_n (\text{mod } p)$

Keeping in mind the M_n's were chosen arbitrarily we can do the following. For (1) choose $M_0 = k_0 + 1$ and for (2) choose $M_n = k_n + c_{n-1} + 1$. Clearly in both cases no such k's exist. Thus g_n^*, and so g^n, cannot be extended to G_5.

Finally use Lemma (6) to extend H_5 to a (G_2,Z)-group, say H_2. Let g_2 be any homomorphism $g_2 : G_2 \to H_2$ such that $h_2 g_2 = 1_{G_2}$ and g_2 extends g_1. Then g_2 extends some $g^n \in R$. We have just shown that g^n cannot be extended to g_{n+1}^* such that $h_{n+1}^* g_{n+1}^* = 1_{G_{n+1}^*}$. As H_2 extends H_{n+1}^* it follows that g^n cannot be extended to g_2. Thus g_2 does not exist. Therefore H_2 satisfies the requirements of the lemma.

$V = L$ and W-groups

In this section we will show that under the assumption $V = L$, groups satisfying Possibility I or Possibility II are not W-groups.

Lemma (8): If G satisfies Possibility I or II then G can be named so that for any limit ordinal δ, there exists an element a^{δ} and a finite subset L_{δ} such that $\Pi(a^{\delta}, L_{\delta}, G_{\delta}, G_{\delta+\omega})$ holds.

Proof: There are two cases to prove.

Case (i): Let G satisfy Possibility I. Thus G is named such that for any limit ordinal $\delta < \omega_1$, G_{δ} is a pure subgroup and for some limit ordinal β, G_{β} is the particular pure subgroup required by conditions (A) and (B) in the definition of Possibility I. That is:

(A) $\{a^{\alpha}_{\ell} + G_{\beta} : \alpha < \omega_1, \ell \leqq n(\alpha)\}$ is an independent family in G/G_{β}.

(B) $(a^{\ell}_{n(\alpha)}, L_{\alpha}, G_{\beta}G)$ holds for all $\alpha < \omega_1$ where $L_{\alpha} = \{a^{\alpha}_{\ell} : \ell < n(\alpha)\}$.

Let $\Phi = \{\delta : \delta$ is a limit ordinal, $\delta < \omega_1$, and G_{δ} does not satisfy conditions (A) and (B)$\}$. That is there do not exist suitable a^{α}_{ℓ}'s, $\alpha < \omega_1$, such that G_{δ} could replace G_{β} in the above. We claim that Φ is bounded in ω_1. If Φ is unbounded then card$(\Phi) = \omega_1$, say $\Phi = \{\delta_{\alpha} : \alpha < \omega_1\}$ where $\delta_{\alpha_1} < \delta_{\alpha_2}$ if $\alpha_1 < \alpha_2$. Rename G so that $\{\theta : \delta_{\alpha} \leqq \theta < \delta_{\alpha+1}\}$ becomes $\{\theta : \omega_{\alpha} \leqq \theta < \omega(\alpha+1)\}$. This renames G so that under the new ordering, if δ is a limit ordinal, then G_{δ} cannot satisfy conditions (A) and (B). As G must satisfy Possibility I under any naming that is admissible, by the definition of Possibility I, this is a contradiction. Thus Φ is bounded, by say $\rho < \omega_1$ where ρ is a limit ordinal. Now rename G so that G_ρ becomes G_ω and $\{\theta : \rho+\omega\alpha \leqq \theta < \rho+\omega(\alpha+1)\}$ becomes $\{\theta : \omega+\omega\alpha \leqq \theta < \omega+\omega(\alpha+1)\}$. Under this admissable naming if δ is a limit ordinal then G_δ will satisfy conditions (A) and (B) for some a_ℓ^α's, $\alpha < \omega_1$.

Now consider G_ω. As G_ω satisfies conditions (A) and (B), then there exists a and L such that $\Pi(a,L,G_\omega,G)$ holds. Choose a limit ordinal $\delta < \omega_1$ so that $<G_\omega \cup L \cup \{a\}>_* \subseteq G_\delta$. Thus $\Pi(a,L,G_\omega,G)$ holds. Now rename G so that $\{\theta : \omega \leqq \theta < \delta\}$ becomes $\{\theta : \omega \leqq \theta < \omega+\omega\}$ and $\{\theta : \delta+\omega\alpha \leqq \theta < \delta+\omega(\alpha+1)\}$ becomes $\{\theta : \omega+\omega+\omega\alpha \leqq \theta < \omega+\omega+\omega(\alpha+1)\}$ for $\alpha < \omega_1$. Under this naming $\Pi(a,L,G_\omega,G_{\omega+\omega})$ holds. Now we do the induction step. Suppose G has been named so that for all $\delta < \beta$ there exists a^δ and L_δ such that $\Pi(a^\delta,L_\delta,G_{\omega\delta},G_{\omega\delta+\omega})$ holds. Since $G_{\omega\beta}$ satisfies conditions (A) and (B), then there exists a and L such that $\Pi(a,L,G_{\omega\beta},G)$ holds. As before choose a limit ordinal $\rho < \omega_1$ such that $<G_{\omega\beta} \cup L \;\; \{a\}>_* \;\; G_\rho$ and so $(a,L,G_{\omega\beta},G_\rho)$ holds. Rename G so that $G_{\omega\beta}$ remains unchanged, $\{\theta : \omega\beta \leqq \theta < \rho\}$ becomes $\{\theta : \omega\beta \leqq \theta < \omega\beta+\omega\}$, and $\{\theta : \rho+\omega\alpha \leqq \theta < \rho+\omega(\alpha+1)\}$ becomes $\{\theta : \omega\beta+\omega+\omega\alpha \leqq \theta < \omega\beta+\omega+\omega(\alpha+1)\}$. Thus $\Pi(a,L,G_{\omega\beta},G_{\omega\beta+\omega})$ holds.

Thus we can assume G can be named so that $\Pi(a^\delta,L_\delta,G_\delta,G_{\delta+\omega})$ holds for any limit ordinal $\delta < \omega_1$ and suitable a^δ's and L_δ's.

<u>Case (ii):</u> Let G satisfy Possibility II. The proof is almost the same. Let A be the required stationary set in the definition of Possibility II. Let $A = \{\delta_\alpha : \alpha < \omega_1\}$, and of course the δ_α's are limit ordinals with $\delta_{\alpha_1} < \delta_{\alpha_2}$ if $\alpha_1 < \alpha_2$. From condition (B) in the definition of Possibility II, there exists a and L such that

$\Pi(a,L,G_{\delta_0},G)$ holds. Choose δ_β so that $\langle G_{\delta_0} \cup L \cup \{a\}\rangle_* \subseteq G_{\delta_\beta}$. Rename G so that G_{δ_0} becomes G_ω, and $\{\theta:\delta_0 \leqq \theta < \delta_\beta\}$ becomes $\{\theta:\omega \leqq \theta<\omega+\omega\}$, and $\{\theta:\delta_\beta+\omega\alpha \leqq \theta < \delta_\beta+\omega(\alpha+1)\}$ becomes $\{\theta:\omega+\omega+\omega\alpha \leqq \theta < \omega+\omega+\omega(\alpha+1)\}$ for all $\alpha < \omega_1$. Then $\Pi(a,L,G_\omega,G_{\omega+\omega})$ holds. The induction step is the same as case (i) except that the limit ordinals will always be chosen from the δ_α's of A.

<u>Definition (17):</u> An infinite cardinal K is regular if no set of cardinality less than K is cofinal in K.

As noted in Remark (3), no countable set is cofinal in ω_1, and so ω_1 is regular.

<u>Definition (18):</u> Let K be an infinite cardinal number and let A be a subset of K. If there is a sequence S_α, α A, such that S_α is a subset of α, and for each subset X of K the set $\{\alpha:X \cap \alpha = S_\alpha\}$ is stationary in K, then we say $\Diamond_K(A)$ holds.

From Jensen (6) page 293, we get the following result:
"Assume V = L and let K be a regular infinite cardinal. Then $\Diamond_K(A)$ holds for every stationary subset A of K."
So this result holds when K is the regular cardinal ω_1.

Now consider a group G of cardinality ω_1. Let A be a stationary set of limit ordinals. As shown before the restriction of any stationary set to its limit ordinals is again stationaty, so there are many such A's. Consider the set $G \times Z = \{(\alpha,z):\alpha \in G, z \in Z\}$.

Name the elements of $G \times Z$ as follows:

(a) Name $J_1 = \{(\alpha,z):\alpha < \omega\}$ as $\{\beta:\beta < \omega\}$ which is easily done as J_1 is countable.

(b) Suppose $\{(\alpha,z):\alpha < \omega\delta\}$ has been named as $\{\beta:\beta < \omega\delta\}$, then name the elements of $J_{\delta+1} = \{(\alpha,z):\omega\delta \leqq \alpha < \omega(\delta+1)\}$ as $\{\beta:\omega\delta \leqq \beta < \omega(\delta+1)\}$ which is easily done as $J_{\delta+1}$ is countable.

So by this inductive naming process the elements of $G \times Z$ are the ordinals $\{\alpha:\alpha < \omega_1\}$ and each $G_\delta \times Z$ has been named as the ordinals $\{\alpha:\alpha < \delta\}$ for all limit ordinals less than ω_1. Let H be such a naming of the elements of $G \times Z$, and so the set A is stationary in $H = \omega_1 = \{\alpha:\alpha < \omega_1\}$. Assume $V = L$ and apply Jensen's result to H and A. Thus $\diamondsuit_H(A)$ holds. Let $g:G \to G \times Z$ be a function such that $g(\alpha) = (\alpha,z_\alpha)$ where the z_α's are in Z. Then g can be viewed as a set of ordered pairs, say $L = \{(\alpha,z_\alpha):\alpha < \omega_1$ and $z_\alpha \in Z\}$, and so g can be viewed as a subset of H, say Y, where $\delta \in Y$ if and only if δ is the name in H of some (α,z_α) in L. By Jensen's result there exist S_α, for $\alpha \in A$, such that $S_\alpha \subseteq \alpha$ and for any $X \subseteq H$, the set $\{\alpha:X \cap \alpha = S_\alpha\}$ is stationary. In particular, $A^* = \{\alpha:Y \cap \alpha = S_\alpha\}$ is stationary. Since α is a limit ordinal and $\alpha = \{\beta:\beta < \alpha\} = \{(\delta,z):\delta < \alpha,\ z \in Z\} = \{(\delta,z):\delta \in G_\alpha,\ z \in Z\} = G_\alpha \times Z$ and $Y = \{\beta \in H:\beta = (\delta,z_\delta)$ for $\delta \in G_\alpha\}$, then for $\alpha \in A^*$, $Y \cap \alpha = S_\alpha = \{\beta \in H:\beta = (\delta,z_\delta)$ for $\delta \in G_\alpha\}$. Thus $Y \cap \alpha = S_\alpha = g|_{G_\alpha}$ can be viewed as a function $S_\alpha:G_\alpha \to G_\alpha \times Z$. Let $S_\alpha = g_\alpha$ for all $\alpha \in A^*$. As g was arbitrary, then for any function $g:G \to G \times Z$ where $g(\alpha)$ has shape

(α,z), the set $\{\delta < \omega_1 : g|_{G_\delta} = g_\delta = S_\delta\}$ is stationary and since g is a function so is $g_\delta : G_\delta \to G_\delta \times Z$ a function. So let $A^{**} = \{\delta \in A : S_\delta = g_\delta$ is a function from G_δ into $G_\delta \times Z\}$. Thus we can make the following statement:

(J) "If $V = L$, there are functions $g : G_\delta \to G_\delta \times Z$, $\delta \in A^{**} \subseteq A$, such that for any function $g : G \to G \times Z$ where $g(\alpha) = (\alpha, z)$, the set $\{\delta < \omega_1 : g|_{G_\delta} = g_\delta\}$ is stationary."

<u>Theorem (1):</u> Assume $V = L$. Then if G satisfies Possibility I or Possibility II, then G is not a W-group.

<u>Proof:</u> Suppose G sarisfies Possibility I or II. By Lemma (8) G can be named so that for any limit ordinal $\delta < \omega_1$, there exists a^δ and L_δ such that $\Pi(a^\delta, L_\delta, G_\delta, G_{\delta+\omega})$ holds. Let A be the stationary set consisting of all limit ordinals. Thus by Jensen, since $V = L$, we can assume (J) as above. Let $H_\delta = G_\delta \times Z$, and so for $\delta \in A^{**}$, g_δ is a function from G_δ into H_δ and $g(\alpha) = (\alpha, z)$. Let K be the set of these functions, $K = \{g_\delta : \delta \in A^{**}\}$.

We will now construct a (G,Z)-group, H_{ω_1}, such that there does not exist a map $g : G \to H_{\omega_1}$ such that $hg = 1_G$ and as usual $h(\alpha,z) = \alpha$. If we can construct such a H_{ω_1}, then G cannot be a W-group. We do the construction by transfinite induction. Define a (G_ω, Z)-group, H_ω, arbitrarily. Suppose we have defined a $(G_{\omega\alpha}, Z)$-group, $H_{\omega\alpha}$ for all $\alpha < \delta$, such that $H_{\omega\alpha}$ extends $H_{\omega\beta}$ for all $\beta < \alpha$.

Define $H_{\omega\delta}$ as follows:

Case (i): Suppose δ is not a limit ordinal, so we can assume $H_{\omega(\delta-1)}$ is well defined. As before let $H = G \times Z$ and $h:H \to G$ with $h(\alpha,z) = \alpha$.

(a) If $\omega(\delta-1) \notin A^{**}$ or if $\omega(\delta-1) \in A^{**}$ and the corresponding function $g_{\omega(\delta-1)}$ in K is not a homomorphism such that $hg_{\omega(\delta-1)} = 1_{G_{\omega(\delta-1)}}$, then extend $H_{\omega(\delta-1)}$ to $H_{\omega\delta}$ arbitrarily using Lemma (6).

(b) If $\omega(\delta-1) \in A^{**}$ and the corresponding function $g_{\omega(\delta-1)}$ in K is a homomorphism such that $hg_{\omega(\delta-1)} = 1_{G_{\omega(\delta-1)}}$, then $g_{\omega(\delta-1)}$, $a^{\omega(\delta-1)}$, $L_{\omega(\delta-1)}$, $G_{\omega(\delta-1)}$, and $G_{\omega\delta}$ satisfy the conditions of Lemma (7). That is:

(i) $G_{\omega(\delta-1)}$ is a countable subgroup of the ω_1-free group $G_{\omega\delta}$, and $H_{\omega(\delta-1)}$ is a $(G_{\omega(\delta-1)},Z)$-group.

(ii) $\Pi(a^{\omega(\delta-1)},L_{\omega(\delta-1)},G_{\omega(\delta-1)},G_{\omega\delta})$ holds by Lemma (8) as stated in the beginning of the proof.

(iii) $g_{\omega(\delta-1)}:G_{\omega(\delta-1)} \to H_{\omega(\delta-1)}$ is a homomorphism such that $hg_{\omega(\delta-1)} = 1_{G_{\omega(\delta-1)}}$.

So we can apply Lemma (7) to extend $H_{\omega(\delta-1)}$ to $H_{\omega\delta}$ so that $g_{\omega(\delta-1)}$ cannot be extended to a homomorphism $g_{\omega\delta}:G_{\omega\delta} \to H_{\omega\delta}$ such that $hg_{\omega\delta} = 1_{G_{\omega\delta}}$.

Case (ii): Suppose δ is a limit ordinal. Define $H_{\omega\delta} = \bigcup_{\alpha<\delta} H_{\omega\alpha}$, and as before $H_{\omega\delta}$ is a $(G_{\omega\delta},Z)$-group extending $H_{\omega\alpha}$ for all $\alpha < \delta$.

Let H_{ω_1} be the (G,Z)-group constructed by this induction. Suppose $g:G \to H_{\omega_1}$ is a homomorphism such that $hg = 1_G$. Then $A^* = \{\delta : g|_{G_\delta} = g_\delta\}$ is stationary by (J). In particular A^* is non empty, say δ is in A^*. Thus by the construction of H_{ω_1}, g_δ cannot be extended to a homomorphism $g_{\delta+\omega}:G_{\delta+\omega} \to H_{\delta+\omega}$ such that $hg_{\delta+\omega} = 1_{G_{\delta+\omega}}$. Thus g_δ cannot be extended to g such that $hg = 1_G$, and so g is not an extension of g_δ, contradiction. Thus no such g exists and so G is not a W-group.

Martin Axiom and W-groups

In this section we will show that under the assumption of the Martin Axiom and $2^\omega > \omega_1$, any group satisfying Possibility II is a W-group.

Definition (19): Let P be a poset (partially ordered set), and let $a,b \in P$. We say a and b are contradictory if they have no common upper bound in the poset P.

Definition (20): Let P be a poset and let D be a subset of P. We say that D is a dense subset of P if for any a in P there is a b in D such that $a \leqq b$.

Definition (21): Let λ be a cardinal number. Let MA_λ be the following assertion:

"Let P be any poset of cardinality λ. Suppose in P there is no subset of ω_1 pairwise contradictory elements. Also suppose $\{D_\alpha : \alpha < \lambda\}$ are dense subsets of P. Then there exists a subset B of P such that $B \cap D \neq \phi$ for all $\alpha < \lambda$, and such that any two members of B have a common upper bound in B."

Such a set B is called a generic subset of P (with respect to the D_α's). MA (Martin Axiom) says that MA_λ holds for any $\lambda < 2^\omega$.

Theorem (2): Assume the Martin Axiom and $2^{\omega} > \omega_1$. If G has cardinality ω_1, is ω_1-free, and does not satisfy Possibility I. then G is a W-group.

Proof: Suppose G satisfies Possibility III. By Lemma (5) G is the direct sum of countable groups. As G is ω_1-free then each summand is free, and so G is free. Thus G is a W-group. So we can assume G satisfies Possibility II.

Let H be a group whose set of elements is G x Z, and let $h:H \to G$ be defined by $h(a,b) = a$. Now we define a poset P. The elements of P are homomorphisms g from finitely generated pure subgroups I of G into H such that $hg = 1_I$. If g_1 and g_2 belong to P, write $g_1 \leqq g_2$ if g_2 extends g_1. We will now show that the cardinality of P is ω_1.

First we compute the number of finitely generated pure subgroups of G. As each subgroup is countable and G is their union, there must be at least ω_1 different finitely generated pure subgroups. There cannot be more than ω_1 finitely generated pure subgroups of G since there are only ω_1 finite subsets of G. Now each of these pure subgroups is freely generated by a finite set as G is ω_1-free. For a given pure subgroup I, with generators $b_1,\ldots,b_n$, the homomorphisms of I into H are uniquely determined by the images of the generators. If $g:I \to H$ is in P, then for each generator b_i, $g(b_i) \in \{(b_i,z):z \in Z\}$. Thus there are only countably many choices for the image of each b_i, and since there are only finitely many b_i's, there are only countably many distinct mappings of the b_i's. Thus there are only countably many g's for each I. Since the number of finitely generated pure subgroups is ω_1, then the cardinality of P is ω_1.

We now define subsets D_α of P for $\alpha < \omega_1$ as follows:

$$D_\alpha = \{g \in P : \alpha \text{ is in the domain of } g\}.$$

We now show:

(a) Each D_α, $\alpha < \omega_1$, is dense in P.

(b) There do not exist ω_1 pairwise contradictory elements of P.

<u>Proof of (a):</u> Let $\alpha < \omega_1$ and let g_I be in P where $g_I : I \to H$. Then I is pure and freely generated by say $a_1,\ldots,a_n$. We must show that there exists g in D_α such that $g_I \leqq g$. If $\alpha \in \text{Dom } g_I$ (domain of g_I), then $g_I \in D_\alpha$ and $g_I \leqq g_I$, so we can let $g = g_I$. So suppose now $\alpha \notin \text{Dom } g_I$ and consider $I^* = \langle a_1,\ldots,a_n,\alpha\rangle_*$. If I^* is a free group generated by $a_1,\ldots,a_n,a_{n+1}$ then define g as follows:

(i) Let $g(a_i) = g_I(a_i)$ for $i = 1,\ldots,n$.

(ii) Let $g(a_{n+1}) = (a_{n+1}, z)$ for any z Z.

As the a_i, $i = 1,\ldots,n+1$, generate the free group I^*, then this mapping of the generators can be extended to a homomorphism $g : I^* \to H$. Clearly g extends g_I and $\alpha \in \text{Dom } g = I^*$. Thus $g \in D_\alpha$ and $g_I \leqq g$. It remains to show that I^* is freely generated by $a_1,\ldots,a_n,a_{n+1}$ for some a_{n+1}. Since $\alpha \notin I$, then α is independent of $\{a_1,\ldots,a_n\}$ since I is pure. So I^* contains at least $n+1$ independent elements. If $x \in I^* \smallsetminus I$, then $mx \in \langle a_1,\ldots,a_n,\alpha\rangle$ for some m, and so x is not independent of $\{a_1,\ldots,a_n,\alpha\}$. Thus I^* contains exactly $n+1$ independent elements, and so I^* is a free group on $n+1$ generators. Let I^* be freely generated by $b_1,\ldots,b_{n+1}$. As shown before, for any finite number of elements in $\langle a_1,\ldots,a_n,\alpha\rangle_*$ and not in $\langle a_1,\ldots,a_n,\alpha\rangle$, there exists an element, say a_{n+1}, such that the finite set of elements is in $\langle a_1,\ldots,a_n,a_{n+1}\rangle$. (see G_5/G_4 is infinite, page 24). Let B be the set of elements of $\{b_1,\ldots,b_{n+1}\}$ in $\langle a_1,\ldots,a_n,\alpha\rangle_*$ and not in $\langle a_1,\ldots,a_n,\alpha\rangle$. Then for some a_{n+1}, $B \subseteq \langle a_1,\ldots,a_{n+1}\rangle$, and so $\{b_1,\ldots,b_{n+1}\} \subseteq \langle a_1',\ldots,a_{n+1}\rangle$. Since $I^* = \langle b_1,\ldots,b_{n+1}\rangle$, then $I^* = \langle a_1,\ldots,a_{n+1}\rangle$ and the result is proved.

<u>Proof of (b):</u> Suppose there exists a set of ω_1 pairwise contradictory elements of P, say $\{g_\delta : \delta < \omega_1\}$. We will derive a contradiction. Let the domain of g_δ be freely generated by $a_1^\delta,\ldots,a_{n(\delta)}^\delta$ where $n(\delta)$ is a finite positive integer. We can replace $W = \{g_\delta : \delta < \omega_1\}$ by any subset of W of the same cardinality without loss of generality. As each $n(\delta)$ is finite and $\text{card}(W) = \omega_1$, then some $n(\delta)$ must occur ω_1 times. So we can assume $n(\delta) = n$, for some fixed n, for all the g_δ's. That is without loss of generality the domain of g_δ is generated by $\{a_1^\delta,\ldots,a_n^\delta\}$ for every g_δ in W. Let $K = \{a_1,\ldots,a_m\}$ be a maximal set of elements of G which freely generate a pure subgroup and $\{a_1,\ldots,a_m\} \subseteq \text{Dom } g_\delta$ for ω_1 δ's. Note that K can be empty. For if any uncountable family of the $\text{Dom }(g_\delta)$'s has a trivial intersection, then K is empty, else K is non empty. So again without loss of generality we can

assume $a_1,\ldots,a_m \in \mathrm{Dom}\ g_\delta$ for all $\delta < \omega_1$. Let $a_1 = a_1^\delta, \ldots, a_m = a_m^\delta$. For any δ we can extend $\{a_1,\ldots,a_m\}$ to an n-element generating set for $\mathrm{Dom}\ g_\delta$. Thus we can assume $\mathrm{Dom}\ g_\delta$ is generated by $\{a_1,\ldots,a_m,a_{m+1}^\delta,\ldots,a_n^\delta\}$ for each g_δ in W.

Now consider $\mathrm{Dom}\ g_\delta$. $\mathrm{Dom}\ g_\delta$ is freely generated by n elements. As any homomorphism g from $\mathrm{Dom}\ g_\delta$ is uniquely determined by where the generators are mapped and $g(a) = (a,z)$ for any generator a, there can be only countably many different homomorphisms from $\mathrm{Dom}\ g_\delta$ into H. So if there were only countably many different domains of the g_δ's, there would only be countably many g_δ's, a contradiction. Thus there are ω_1 different domains on which the g_δ's are defined. Choose one g_δ on each domain. So without loss of generality we can assume $\mathrm{Dom}\ g_\delta \neq \mathrm{Dom}\ g_\alpha$ for $\delta \neq \alpha$. In other words, in the set $\{a_1,\ldots,a_m,a_{n+1}^\delta,\ldots,a_n^\delta\}$, $m < n$.

Again without loss of generality we will take a subset of the g_δ's of cardinality ω_1; this time such that the set $\{\{a_1,\ldots,a_m\} \cup \{a_\ell^\delta : m < \ell \leqq n,\ \delta < \omega_1\}\}$ is independent in G. $\mathrm{Dom}\ g_0$ is generated by $\{a_1,\ldots,a_m,a_{m+1}^0,\ldots,a_n^0\}$ which is an independent set in G. Assume for $\alpha < \beta < \omega_1$ we have chosen g_α's such that the set $\{\{a_1,\ldots,a_m\} \cup \{a_\ell^\alpha : m < \ell \leqq n,\ \alpha < \beta\}\}$ is an independent set in G. Now consider $\langle \bigcup_{\alpha<\beta} \mathrm{Dom}\ g_\alpha \rangle_*$. From the remaining g_δ's, toss out any g_δ such that $(\langle \bigcup_{\alpha<\beta} \mathrm{Dom}\ g_\alpha \rangle_* \smallsetminus \langle a_1,\ldots,a_m \rangle) \cap \mathrm{Dom}\ g_\delta$ is non empty. We claim that only countably many g_δ's will be tossed out. This will follow from the fact that $\langle \bigcup_{\alpha<\beta} \mathrm{Dom}\ g_\alpha \rangle_* \smallsetminus \langle a_1,\ldots,a_m \rangle = D$ is countable. If some element in D, say b, was in uncountably many $\mathrm{Dom}\ g_\delta$'s then as b is not an element of $\langle a_1,\ldots,a_m\rangle$, it is independent of the pure subgroup $\langle a_1,\ldots,a_m\rangle$, and so we would get a contradiction to the maximality of the pure subgroup $\langle a_1,\ldots,a_m\rangle$. That is $\langle a_1,\ldots,a_m,b\rangle_*$ would be a pure subgroup of order $m+1$ contained in the domain of ω_1 of the g_δ's. So for any element $b \in D$, we toss out only countably many g_δ's. Since D is countable we toss out only countably many g_δ's and so there are uncountably many g_δ's left. Choose one and call it g_β. We claim that the elements $\{\{a_1,\ldots,a_m\} \cup \{a_\ell^\alpha : m < \ell \leqq n,\ \alpha \leq \beta\}\}$ are independent in G. This is easily seen. We need only check that no linear combination of the elements of $\{a_{m+1}^\beta,\ldots,a_n^\beta\}$ is a linear combination of the elements of $\{\{a_1,\ldots,a_m\} \cup \{a_\ell^\alpha : m < \ell \leqq n,\ \alpha < \beta\}\}$. Any linear combination of the elements of $\{a_{m+1}^\beta,\ldots,a_n^\beta\}$ is in $\mathrm{Dom}\, g_\beta \smallsetminus \langle a_1,\ldots,a_m\rangle$ and any linear combination of the elements of $\{\{a_1,\ldots,a_m\} \cup \{a_\ell^\alpha : m < \ell \leqq n,\ \alpha < \beta\}\}$ is in $\langle \bigcup_{\alpha<\beta} \mathrm{Dom}\, g_\alpha\rangle_*$. By choice of g_β, $(\mathrm{Dom}\, g_\beta \smallsetminus \langle a_1,\ldots,a_m\rangle) \cap \langle \bigcup_{\alpha<\beta} \mathrm{Dom}\, g_\alpha\rangle_*$ is empty and so the set $\{\{a_1,\ldots,a_m\} \cup \{a_\ell^\alpha : m < \ell \leqq n,\ \alpha < \beta\}\}$ is independent. So inductively we can choose ω_1 of the g_δ's in such a way that $V = \{\{a_1,\ldots,a_m\} \cup \{a_\ell^\delta : m < \ell \leqq n,\ \delta < \omega_1\}\}$ is an independent set in G.

Since Possibility I fails, there is an admissible naming of the elements of G such that there does not exist G_δ so that conditions (A) and (B) of the definition of Possibility I hold. So assume G has such a naming. Also make sure G_ω contains $a_1,\ldots,a_m$. This is easily done. Then there is an uncountable subset of V independent over G/G_ω. If an element $g \in G_\omega$ is a finite linear combination of elements of V, then for any a_i^δ in the representation of g, toss out $a_{m+1}^\delta, \ldots, a_n^\delta$. Since G_ω is countable and if $g \in G_\omega$ can be represented as a linear combination of elements of V, then that representation is unique; only countably many subsets $\{a_{m+1}^\delta, \ldots, a_n^\delta\}$ will be tossed out of V. Let V^* be the remaining elements of V. Clearly V^* is uncountable. Now let J^* be the set of all a_{m+1}^δ's in V^*, say $J^* = \{a_{m+1}^\delta : \delta < \omega_1\}$. Then:

(A) J^* is an independent family in G/G_ω.

(B) $\langle G_\omega \rangle_* = G_\omega$.

Since Possibility I fails, then there must exist b_1^δ in $\langle G_\omega \cup \{a_{m+1}^\delta\}\rangle_*$ such that $\langle G_\omega \cup \{a_{m+1}^\delta\}\rangle_* = G_\omega \oplus \langle\{b_1^\delta\}\rangle_*$ for all but countably many δ's. So $\langle G_\omega \cup \{a_{m+1}^\delta\}\rangle_* = \langle G_\omega \cup \{b_1^\delta\}\rangle_* = G_\omega \oplus \langle\{b_1^\delta\}\rangle_*$ for $\{b_1^\delta : \delta < \omega_1\}$. Now let $L_\delta = \{b_1^\delta\}$ for $\delta < \omega_1$ and J^{**} be the set of all a_{m+2}^δ's corresponding to the b_1^δ's. So $J^{**} = \{a_{m+2}^\delta : \delta < \omega_1\}$. Then:

(A) J^{**} is an independent family in G/G_ω.

(B) $\langle G_\omega \cup L_\delta\rangle_* = G_\omega \oplus \langle L_\delta\rangle_*$ for $\delta < \omega_1$.

Again since Possibility I fails there must exist b_2^δ in $\langle G_\omega \cup L_\delta \cup \{a_{m+2}^\delta\}\rangle_*$ such that $\langle G_\omega \cup L_\delta \cup \{a_{m+2}^\delta\}\rangle_* = G_\omega \oplus \langle L_\delta \cup \{b_2^\delta\}\rangle_*$ for uncountably many of the a_{m+2}^δ's. Continuing this finite process we get that $\langle G_\omega \cup \{a_{m+1}^\delta, \ldots, a_n^\delta\}\rangle_* = G_\omega \oplus \langle b_1^\delta, \ldots, b_{n-m}^\delta\rangle_*$ for uncountably many δ's. Choose such a δ, say α, and let $\sigma(1) = \alpha$. Note that $\langle G_\omega \cup \{a_{m+1}^\alpha, \ldots, a_n^\alpha\}\rangle_* = \langle G_\omega \cup \mathrm{Dom}\, g_\alpha\rangle_*$. Also note that $b_1^\alpha, \ldots, b_{n-m}^\alpha$ are elements of $\langle a_1, \ldots, a_m, a_{m+1}^\alpha, \ldots, a_n^\alpha, c_1^\alpha, \ldots, c_{k(\alpha)}^\alpha\rangle_*$ for some $c_1^\alpha, \ldots, c_{k(\alpha)}^\alpha$ in G_ω. This is because $b_i^\alpha \in \langle G_\omega \cup \{b_1^\alpha, \ldots, b_{i-1}^\alpha\} \cup \{a_{m+i}^\alpha\}\rangle_*$.

Now we can repeat this process for any $G_{\omega\alpha}$ and choose $\sigma(\alpha)$ different each time since we are choosing from an uncountable set of which only countably many have been chosen before. In fact we could choose $\sigma(\alpha) > \sigma(\beta)$ for all $\beta < \alpha$ since the $\sigma(\beta)$'s are not cofinal in our uncountable set. Thus we can define a strictly increasing sequence of ordinals $\sigma(\alpha)$, $\alpha < \omega_1$, such that $\langle G_{\omega\alpha} \cup \text{Dom } g_{\sigma(\alpha)}\rangle = G_{\omega\alpha} \oplus \langle b_1^{\sigma(\alpha)},\ldots,b_{n-m}^{\sigma(\alpha)}\rangle_*$ where $\{b_1^{\sigma(\alpha)},\ldots,b_{n-m}^{\sigma(\alpha)}\} \subseteq \langle a_1,\ldots,a_m, a_{m+1}^{\sigma(\alpha)},\ldots,a_n^{\sigma(\alpha)}, c_1^{\sigma(\alpha)},\ldots,c_{k(\alpha)}^{\sigma(\alpha)}\rangle_*$ and $c_1^{\sigma(\alpha)},\ldots,c_{k(\alpha)}^{\sigma(\alpha)} \in G_{\omega\alpha}$. Note that by choosing $k(\alpha)$ minimally we can assume that $\{a_1,\ldots,a_m,a_{m+1}^{\sigma(\alpha)},\ldots,a_n^{\sigma(\alpha)}, c_1^{\sigma(\alpha)},\ldots,c_{k(\alpha)}^{\sigma(\alpha)}\}$ is an independent set in G. To simplify notation assume $\sigma(\alpha) = \alpha$.

At this point we must make the observation that the set of ordinals $C = \{\delta : \delta < \omega_1 \text{ and } \omega\delta = \delta\}$ is a closed and unbounded subset of ω_1.

(a) C is unbounded: By (7), page 108, ordinals of the form ω^{ω^β}, for any β, are in C. That is $\omega^{\omega^\beta} = \omega(\omega^{\omega^\beta})$. Thus if C was bounded above by say α, $\alpha < \omega_1$, then choose β such that $\alpha < \beta < \omega_1$. Then $\omega^{\omega^\beta} \geq \beta$ and so $\omega^{\omega^\beta} > \alpha$. As ω^{ω^β} is in C we get a contradiction.

(b) C is closed: Let β_ν be a countable sequence of members of C. Then for each element of the sequence $\beta_\nu = \omega\beta_\nu$. Let $\alpha = \lim \beta_\nu$. Then:

$$\alpha = \lim \beta_\nu = \lim \omega\beta_\nu = \omega(\lim \beta_\nu) = \omega\alpha .$$

Thus α is in C and so C is closed.

For every $\alpha < \omega_1$, $k(\alpha)$ is finite and so some $k(\alpha)$, say $k(\alpha) = t$, occurs uncountably many times. Let A be this set under the natural ordering of ordinals. $A = \{\alpha : k(\alpha) = t\} = \{\alpha_\delta : k(\alpha_\delta) = t,\ \delta < \omega_1$, and $\alpha_{\delta_1} < \alpha_{\delta_2}$ if and only if $\delta_1 < \delta_2\}$. Now rename G so that $G_{\omega\delta} = G_{\omega\alpha_\delta}$. Thus at every ordinal α our set $\{c_1^\alpha, \ldots, c_{k(\alpha)}^\alpha\} \subseteq G_{\omega\alpha}$ has t elements.

Now let $J_0 = C = \{\delta : \delta < \omega_1$ and $\omega\delta = \delta\}$. J_0 is closed and unbounded and hence stationary. Since $k(\alpha) = t$ for all $\alpha < \omega_1$, then $k(\alpha) = t$ for all δ in J_0. Note also that for $1 \leqq \ell \leqq t$, $c_\ell^\delta < \omega\delta$ as $c_\rho^\delta \in G_{\omega\delta}$. Now we can apply a result of Fodor to define a sequence of stationary sets $J_0 \supseteq J_1 \supseteq \ldots \supseteq J_t$ such that for all $\delta \in J_\ell$, $c_\ell^\delta = c_\ell$ for some fixed c_ℓ. We proceed as follows.

<u>Definition (22):</u> A function $f: J \to \Lambda$, where J and Λ are sets of ordinals, is called regressive if $f(\alpha) < \alpha$ for all $\alpha \in J \setminus \{0\}$ and $f(0) = 0$ if $0 \in J$.

Fodor's result (8), page 141, says that for a regular cardinal $\lambda > \omega$, and J a stationary subset of λ, there exists for each defined regressive function f on J a stationaty subset J^* of J such that $f(\alpha) = \beta$ for all α in J^*, and some fixed β in λ.

Now consider $J_0 = C$, a stationary set, and ω_1, a regular cardinal. As noted for $1 \leqq \ell \leqq t$, $c_\ell^\delta < \omega\delta$ and so for $\delta \in J_0$, $c_\ell^\delta < \delta$ as $\delta = \omega\delta$. Thus we can define a regressive function $f: J_0 \to \omega_1$ by $f(\delta) = c_1^\delta$ since $c_1^\delta < \delta$ for all $\delta \in J_0$. By Fodor's result there exists J_1, a stationary subset of J, such that $f(\delta) = c_1$ for all δ in J_1. Now repeat this for J_1 by defining $f: J_1 \to \omega_1$

by $f(\delta) = c_2^\delta < \delta$. This will produce a $J_2 \subseteq J_1$ such that J_2 is stationary and $f(\delta) = c_2$ for all δ in J_2. By repeating this t times we can define a nest of stationary sets $J_t \subseteq J_{t-1} \subseteq \ldots \subseteq J_0$ such that for all $\delta \in J_\ell$, $\ell = 1,\ldots,t$, $c_\ell^\delta = c_\ell$ for some fixed c_ℓ. In particular for $\delta \in J_t$, $c_\ell^\delta = c_\ell$ for all $\ell = 1,\ldots,t$. That is the set $\{c_1^\delta,\ldots,c_t^\delta\} = \{c_1,\ldots,c_t\}$ for all $\delta \in J_t$.

We will now make one more observation. If A and B are pure subgroups of G and if $A \oplus B$ is a pure subgroup then for A^* and B^*, pure subgroups of A and B respectively, $A^* \oplus B^*$ is pure in G. This is easily verified. Let $nx = a^* + b^*$ where a^* is in A^* and b^* is in B^*. As $A \oplus B$ is pure then $x = a + b$ for some a in A and b in B. Then:

$$nx = a^* + b^* = na + nb\,,$$

so $na = a^*$ and $nb = b^*$,

so $a \in A^*$ and $b \in B^*$ as A^* and B^* are pure,

so $x \in A^* \oplus B^*$,

so $A^* \oplus B^*$ is pure in G.

Now consider the pure subgroups $\langle a_1,\ldots,{}_m,a_{m+1}^\delta,\ldots,a_n^\delta,c_1,\ldots,c_t\rangle_*$ where δ is in J_t. We can extend g_δ to this domain. Call these new homomorphisms g^δ. By the above observation $\langle a_1,\ldots,a_m,c_1,\ldots,c_t\rangle_*$ $\oplus$ $\langle b_1^\delta,\ldots,b_{n-m}^\delta\rangle_* = B$ is pure and contained in Dom g^δ since each b_i^δ is an element of Dom g^δ. By construction each a_{m+i}^δ is an element of B and so Dom g^δ is contained in B. Thus Dom $g^\delta = B$. So Dom g^δ is freely generated by $\{a_1,\ldots,a_{m+t}a_{m+t+1}^\delta,\ldots,a_{n+t}^\delta\}$ where $\{a_1,\ldots,a_{m+t}\}$ freely generates $\langle a_1,\ldots,a_m,c_1,\ldots,c_t\rangle_*$ and

$\{a^\delta_{m+t+1},\ldots,a^\delta_{n+t}\}$ freely generates $\langle b^\delta_1,\ldots,b^\delta_{n-m}\rangle_*$.

Since $g^\delta(a_\ell) \in \{(a_\ell,z): z \in Z\}$ for $\ell = 1,\ldots,m+t$, there are only countably many different images of the $g^\delta(a_\ell)$'s, and so must appear uncountably many times. So we can assume that $g^\delta(a_\ell)$ is fixed for $\ell = 1,\ldots,m+t$ where δ is in J^*_t , an uncountable subset of J_t .

Now choose $\alpha,\beta \in J^*_t$ such that $\alpha < \beta$ and so the generators of $g^\alpha \in G_{\omega\beta}$. Then Dom g^α is a pure subgroup of $G_{\omega\beta}$. Also $\langle a^\beta_{m+t+1},\ldots,a^\beta_{n+t}\rangle$ is pure and equal to $\langle b^\beta_{m+1},\ldots,b^\beta_n\rangle_*$. As $G_{\omega\beta} \oplus \langle b^\beta_{m+1},\ldots,b^\beta_n\rangle_*$ is pure then by the observation on the last page Dom $g^\alpha \oplus \langle a^\beta_{m+t+1},\ldots,a^\beta_{m+t}\rangle = \langle \text{Dom } g^\alpha \cup \text{Dom } g^\beta\rangle$ is pure in G.

Finally we have the extension needed to produce the contradiction. $\langle \text{Dom } g^\alpha \cup \text{Dom } g^\beta\rangle$ is freely generated by $\{a_1,\ldots,a_{m+t},a^\alpha_{m+t+1},\ldots,a^\alpha_{n+t},a^\beta_{m+t+1},\ldots,a^\beta_{n+t}\}$ and $g^\alpha(a_\ell) = g^\beta(a_\ell)$ for $\ell = 1,\ldots,m+t$. Thus $g:\langle \text{Dom } g^\alpha \cup \text{Dom } g^\beta\rangle \to H$ defined by $g(a+b) = g^\alpha(a) + g^\beta(b)$ where $a \in \text{Dom } g^\alpha$, $b \in \text{Dom } g^\beta$ is a common extension of g^α and g^β . Now g^α and g^β are themselves extensions of some g_λ and g_δ respectively in our original set assumed to be pairwise contradictory. So g is a common extension of some g_λ and g_δ and $g \in P$ since $\langle \text{Dom } g^\alpha \cup \text{Dom } g^\beta\rangle$ is a finitely generated pure subgroup. This contradicts our original assumption that g_λ and g_δ have no upper bound in P. Thus there doesn't exist any subset of ω_1 pairwise contradictory elements of P.

We can now complete the proof of Theorem (2). Since under our assumptions, $\omega_1 < 2^{\omega_0}$, by Martin's Axiom there exists a (generic) subset B of P such that $B \cap D_\alpha = \phi$ for all $\alpha < \omega_1$, and such that any two members of B have a common upper bound in B. Let $g^* = \bigcup_B g$. Since B is generic it is easy to verify that g^* is a function from G to H. Since each $g \in B$ is a homomorphism, so is g^*. Since hg is the identity map on the domain of g, we have $hg^* = 1_G$. Thus there exists a homomorphism $g^*:G \to H$ such that $hg^* = 1_G$, and so G is a W-group.

The Independence Result

Theorem (3): The statement: "Every W-group of cardinality ω_1 is free" is independent of ZFC (Zermelo-Frankel set theory plus the axiom of choice).

Proof: Since W-groups are ω_1-free, then by Theorem (1) if $V = L$, any W-group must satisfy Possibility III. By Lemma (5) $G = \bigoplus_{\alpha<\omega_1} G_\alpha$ where each G_α is countable. As G is ω_1-free, each G_α is free and so G is free. Thus ZFC + $V = L$ implies that every W-group of cardinality ω_1 is free. But by Gödel (8), ZFC + $V = L$ is consistent if ZFC is consistent.

By Martin and Solovay (10), ZFC + MA + $2^{\omega_0} > \omega_1$ is consistent if ZFC is. But by Theorem (2), in the presence of MA + $2^{\omega_0} > \omega_1$, any group satisfying Possibility II is a W-group. By Lemma (4) there are groups satisfying Possibility II. So it is consistent with ZFC to assume that there are W-groups of cardinality ω_1 which are not free.

Thus the statement: "Every W-group of cardinality ω_1 is free" is independent of ZFC.

Bibliography

1. L. Fuchs, Infinite Abelian Groups, Vol.I, Academic Press, N.Y. and London, 1970.
2. L. Fuchs, Infinite Abelian Groups, Vol.II, Academic Press, N.Y. and London, 1973.
3. Pinter, Set Theory, Addison-Wesley Publishing Company, London, 1971.
4. Jon Barwise, Back and Forth Thru Infinitary Logic, in Studies in Model Theory, M.A.A., 1973,pp. 5-34.
5. I. Kaplansky, Infinite Abelian Groups, University of Michigan Press, Ann Arbor, 1954.
6. R. B. Jensen, The Fine Structure of the Constructible Hierarchy, Annals of Mathematical Logic (1972), Vol.4, pp.229-308.
7. J. Monk, Introduction to Set Theory, McGraw-Hill Book Company, Toronto, 1969.
8. G. Fodor, Eine Bemerdeeng zur Theorie der Regressiven Funktionen, Acta. Sci. Math. (1956), Vol.17, pp. 139-142.
9. K. Gödel, The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory, Princeton University Press, Princeton N.J., 1940.
10. D. M. Martin, R. M. Solovay, Internal Cohen Extensions, Annals of Mathematical Logic (1970), Vol.2, 143-178.
11. S. Shelah, Infinite Abelian Groups, Whitehead Problem, and Some Constructions, Israel J. Math.(1974), Vol.18, pp. 243-256.

Appendix

We will describe the significant alterations that we have made to Shelah's paper (11). The definition of Possibility I was changed from the existence of a G_δ, under some admissible ordering, which satisfies conditions (A) and (B), to the existence of a G_δ, under every admissible ordering, which satisfies conditions (A) and (B). Under the original definition there appeared to be no way of classifying, up to isomorphism, ω_1-free groups into the three possibilities. This altered definition of Possibility I allowed us to simplify some of the proofs. In particular we were able to come up with a lemma (Lemma (8)) which allowed us to deal with both Possibility I and II in a uniform way via Theorem (1). Shelah had used a complicated group theoretic argument in dealing with Possibility I. Thus our set theoretic Lemma (8) eliminated the more difficult group theoretic theorem of Shelah's, (see (11) 3.3).

Shelah used a rather complicated combinatorial argument, (see (11) 3.1(2) and 3.1(3)), to show every Possibility III group is a direct sum of countable groups. Our Lemma (5) gives a simpler and more direct proof of this fact witout using the complicated combinatorial technique of (11) 3.1(2).

In the proof of Theorem (2), the method indicated by Shelah for producing the elements $c_1^\delta, \ldots, c_{k(\delta)}^\delta$ appeared to be incorrect. So a completely different arguement had to be used, see pages 41-44.

In general the set theoretic and group theoretic details were filled in to the point where someone with only a limited knowledge of set theory and group theory could read the thesis. This involved much work in places for Shelah assumed a knowledge of group theory at a level of Fuch's books (1) and (2). In many cases only the broad outline of an argument was given, and so there had to be a significant filling in of detail. As an example, in Lemma (7) G_5/G_4 has to be shown to be infinite and then it has to be shown that this implies either G_5/G_4 contains a copy of $Z(p^\infty)$ or it contains infinitely many elements of prime order, see pages 24-26. Another example was working out all the details in showing that the examples in Lemma (4) actually satisfy the respective possibilities, see pages 11-16. The main difficulties with the set theory, other than redefining Possibility I and the subsequent classification into the three possibilities, was in showing how the results of Jensen (6) and Fodor (8) applied to our problem. So again here there had to be substantial filling in of detail, see pages 32-34 and 44-46. Also we had to show that G could be well-ordered such that for all limit ordinals δ, G_δ is pure, see Lemma (2) on page 5.