A BOOLEAN-VALUED APPROACH TO THE LEBESGUE MEASURE PROBLEM

by

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B.Sc., University of British Columbia, 1973

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES
(Department of Mathematics)

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
December, 1976

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Date January 15 , 1977.

Abstract

We let:

ZF = the Zermelo-Fraenkel axioms of set theory without the Axiom of Choice. (AC) .

ZFC = ZF + AC .

I = " There exists an inaccessible cardinal " .

Ψ = " Every set of reals definable from a countable sequence of ordinals is Lebesgue measurable ".

DC = the Axiom of Dependent Choices.

LM = " Every set of reals is Lebesgue measurable ".

In 1970, Solovay published a proof by forcing of the following relative consistency result:

Theorem If there exists a model $\mathbb{M}$ of ZFC + I, then there exist extensions $\mathbb{M}[G]$ and $\mathbb{N}$ of $\mathbb{M}$ such that:

(a) $\mathbb{M}[G] \models ZFC + \Psi$.

(b) $\mathbb{N} \models ZF + DC + LM$.

Boolean-valued techniques are used here to retrace Solovay's proof on a different foundation and prove the following result:

Theorem Let $\mathbb{K}$ be a non-minimal standard transitive model of ZFC + I. Then:

(a) $\mathbb{K} \models$ there is a model of ZFC + Ψ .

(b) $\mathbb{K} \models$ there is a model of ZF + DC + LM .

Contents

Acknowledgements

My sincere thanks go to Dr. Andrew Adler, not only for his suggestion of the topic of this thesis and willingness to oversee the work, but especially for his encouragement and patience during the long course that has lead to its completion. His advice and insights have had a decisive influence on this presentation of the topic.

I wish also to acknowledge the support the Mathematics Department has given me.

Special thanks are due to Mrs. Anne Rogers for her hard work and diligence in typing the manuscript.

I am grateful to Mr. Ralph Pudritz for his helpful comments on the manuscript, and to Mr. Peter Lomax and Ms. Miriam Silbermann for their help with translations.

And lastly, my acknowledgements go to the many who have shown friendly concern and encouragements, not the least of whom is Mr. Anatole Lebovich.

Introduction

Under what hypothesis can we consistently assume that all sets of reals are Lebesgue measurable? This is the essence of the Lebesgue measure problem. Here we present a recent set theoretical investigation of this problem due to Solovay.

Lebesgue measure is countably additive and translation invariant. Under the hypothesis that the reals can be well-ordered, these two properties allowed early researchers (e.g. Vitali, Bernstein) to construct various sets of reals which are not Lebesgue measurable. Set functions with domain the powerset $\mathbb{P}(\mathbb{R})$ of the reals, which drop one of the above two constraints, have been the focus of some attention. But such would-be measures cannot compete with Lebesgue measure for its central role in modern real analysis. If we must accept the presence of non-measurable sets in ordinary analysis, then it would be useful to know how much their existence depends on the Axiom of Choice (AC).

Solovay's research, published in 1970, shows that it is consistent with the Zermelo-Fraenkel axioms of set theory (ZF) and the Principle of Dependent Choices (DC) to assume that all sets of reals are Lebesgue measurable (LM), given the consistency of ZF+AC together with the statement (I) that a (strongly) inaccessible cardinal exists. This main theorem of Solovay indicates that it is impossible to prove the existence of non-measurable sets from the ZF

axioms with DC, provided that the theory ZF+AC+I is consistent. Hence the existence of non-measurable sets is dependent on some form of AC which is stronger than DC.

To prove this relative consistency result, a model $\mathbb{N}$ of ZF+DC+LM is constructed from a model $\mathbb{M}$ of ZF+AC+I. Solovay's construction uses an unramified form of the forcing method, essentially due to Cohen. His main innovation is the use of Borel sets of positive measure to replace Cohen's finite forcing conditions.

Solovay has conjectured that the hypothesis regarding the consistency of I is dispensable, though no proof has been forthcoming as yet. In 1969, Sacks published an account of another Solovay result, using ramified languages and a measure theoretic forcing argument. He demonstrated that if ZF is consistent, then ZF+DC+ "there exists a countably additive, translation invariant extension of Lebesgue measure on $\mathbb{P}(\mathbb{R})$" is consistent. The statement in quotes is weaker than LM, but its consistency requires no hypothesis regarding I.

In 1965, Solovay and Scott, and independently Vopěnka, noticed that forcing arguments could be translated into constructions involving so-called Boolean-valued models. Our presentation of Solovay's 1970 research begins with a résumé of this method of constructing a generic extension of a given ground model without forcing. A close examination of [23] (e.g. pp. 31 - 8, pp. 49 - 50) leaves no

doubt that Solovay's original conception of his work on the Lebesgue measure problem was in terms of Boolean-valued models, rather than the classical forcing arguments which predominate his final published account. The use of Boolean-valued methods provides a more natural and intuitive development of Solovay's ideas. In our initial section, we have concentrated on those aspects of Boolean-valued models which have a direct bearing on Solovay's 1970 constructions, and have tried to improve and complete some of the standard proofs in this area.

In Section 3 we describe Solovay's notion of a random real, which is his main innovation mentioned above and the notion which motivated the Solovay-Scott development of Boolean-valued models. In this, we have started with Solovay's definition, and translated his development into the language of Section 0. A different development (based on Definition 3.5) in the same Boolean language can be found in [9]. Lemma 3.8 establishes the equivalence of the two approaches.

In this exposition we have endeavored to combine the intuitive clarity of the Boolean-valued approach with a rigorous foundational background. Two points are often glossed over in presentations of this type. It is often not specified whether the models involved are sets or classes. This lack of precision opens any treatment of definable sets to the possibility of set theoretical para-

doxes. To combat this, we consider all our model constructions to take place within a model which is a set, rather than within the "real world" of Solovay. The second stipulation is that our ground model is countable. The reasoning behind this is explained in Section 0. These two points ensure that our model constructions rest on an explicit and correct foundation.

The concluding sections describe the two main models of Solovay, whose origin he attributes to Lévy and McAloon. The latter model is usually defined via the eight fundamental Gödel operations (see [27]), or by an extended form of the Reflection Principle (see [15]). Because of our adherence to models which are sets, we have been able to employ Gödel-numbering to present a simpler construction needing much less background. The Lévy model is constructed from an unpleasant Boolean algebra L which is the subject of Section 4. We have filled in some of the neccessary technical work in this Lévy algebra that is avoided elsewhere. By a classical algebraic argument, a theorem of Jensen ([9], p. 76) indirectly implies that L is homogeneous. In our Lemmas 4.8 and 4.10 we have modified Jensen's theorem considerably, providing a direct proof of the homogeneity of L.

Another often neglected point is the problem of first-order definability of sets. In addressing this topic, we have included relevant material here which is usually only

alluded to. In Section 5, for example, we introduce the notion of uniform definability. This has other names in the literature (e.g. "specifiability", [4]), but there seems to be no standard usage. In this, we differ from Solovay's development which uses his $\mathbb{M}$-$\mathbb{R}$-definability ([23] p. 41). The somewhat informal use of definability in Section 5 is formalized in Section 6. Here we show that the source of our definability problems is Richard's paradox.

While the McAloon model substantiates the main theorem of Solovay quoted above, the Lévy model gives an equally interesting secondary theorem of Solovay: If ZF+AC+I is consistent, then it is consistent with ZF+AC to assume that every set of reals definable from a countable sequence of ordinals is Lebesgue measurable. Using the formulas of set theory, we cannot explicitly define a non-measurable set without an uncountable sequence of ordinals. These notions are made precise in Sections 5 and 6.

Section 2 deals with some absoluteness properties of Lebesgue measure. For this work we have selected a notion of absoluteness due to Shoenfield that is naturally adapted to the model extension process. In most other respects, the development of this section follows Solovay. We differ from the Solovay development by first establishing that the property of being a set of Lebesgue measure zero is absolute. The main lemma of Solovay (Lemma 1.6.4, p. 31,

[23]) follows easily as our Corollary 2.25.

Some prefatory remarks on the use of the Countable Axiom of Choice (AC_ω) in analysis are included in Section 1. Our aim here is to emphasize that the real impact of DC on ordinary analysis is through AC_ω.

Our set theoretic and model theoretic notation and nomenclature is standard for the most part, being consistent with that of [1] and/or [14].

Section 0 : Boolean-valued models and generic extensions

Our approach to the proof of Solovay is based on the concept of a Boolean-valued model of set theory. We start here with a fairly general treatment of this subject, then in later sections select the line of application that leads to the Solovay result.

To begin, we recall some background. As usual, ZF is used to stand for the Zermelo-Fraenkel set theory, i.e. the collection of theorems that follow from the Zermelo-Fraenkel axioms (less the Axiom of Choice). ZFC denotes the full collection of theorems following from ZF and the Axiom of Choice (AC). For this section the terms "set theory" and ZFC will be used synonymously.

A model of set theory is an ordered pair $\mathbb{M} = (M,E)$, where M is a set and E is a binary relation on M (i.e. $E \subseteq M^2$) which satisfies all the axioms of set theory as the interpretation for 'ε'. The symbolism $\mathbb{M} \models \phi$, which says that $\mathbb{M}$ satisfies ϕ, can be defined by induction on the complexity of ϕ (see [1]). M is referred to as the universe or underlying set of $\mathbb{M}$. In stipulating that M is a set rather than a class, we will avoid the danger of set theoretic paradoxes which might otherwise impair the model construction process. However, many popular versions of the theorems we will use assume M to be a class, and care must be taken when we meet such theorems.

Definition 0.1 (a) A binary relation R is well-founded in a set $H \subset \mathrm{dom}(R)$ if there is no sequence $\{x_n\} \subset H$ such that $x_{n+1} R x_n$ holds for each $n \in \omega$.

(b) A model $\mathbb{H} = (H,R)$ is extensional if for all x, y, and z in H,

$$(zRx \leftrightarrow zRy) \rightarrow (x = y) .$$

(c) A model $\mathbb{H} = (H,R)$ is standard if $R \subset H^2 \cap \in$.

(d) A model $\mathbb{H} = (H,R)$ is transitive if H is a transitive set, i.e. for each $x \in H$, $x \subset H$.

In 0.1 (c) above, we assume that there is a "real world" of sets, and that $\in$ is the natural membership relation.

The statement that ZFC has a model implies the statement that ZFC is consistent. The latter of these statements is unprovable in ZFC ([3], p. 56). So in order to proceed very far with the set theoretical manipulation of the models defined above, it becomes convenient and sometimes necessary to add some new axiom to ZFC which asserts their existence. [3], p. 78 discusses this. The following is a relatively strong form of model existence axiom, and will be adequate for our purposes.

Axiom A There is a set H and a binary relation R well-

founded on H, such that $\mathbb{H} = (H,R)$ is an extensional model of ZFC.

Any transitive model is extensional (see [9], p. 21, and note that this proof is valid for our definition of model). For any model $\mathbb{H} = (H,R)$ satisfying Axiom A, the Mostowski Collapsing Theorem ([9], p. 27) guarantees the existence of a unique standard, transitive model $\mathbb{K} = (K, K^2 \cap \in)$ which is isomorphic to $\mathbb{H}$. To see that $\mathbb{K}$ is truly a model, we must keep in mind that the isomorphic copy of a set is also a set (by the Axiom of Replacement). This gives us the following more useful form of Axiom A.

Axiom A' There is a set K such that $\mathbb{K} = (K, K^2 \cap \in)$ is a (standard) transitive model of ZFC.

By the Axiom of Regularity we know that $K^2 \cap \in$ is well-founded ([3], p. 54), and so the two statements A and A' are equivalent.

Axiom A implies the existence of a minimal standard transitive model $\mathbb{M}_0$ of ZFC; one that is countable and is a submodel of all other standard transitive models of ZFC ([3], p. 83; [24], p. 197). $\mathbb{M}_0$ has no standard proper submodels which are transitive. Since $\mathbb{M}_0$ does not satisfy A, Axiom A cannot be proved from the other axioms of ZFC ([4], p. 110; c.f. [26], p. 83 and [9], p. 37).

The upward Löwenheim-Skolem Theorem certainly allows

us to pick K uncountable. We do this to prevent $\mathbb{K} = \mathbb{M}_o$. From this point we fix this $\mathbb{K}$, writing $\in$ for $K^2 \cap \in$.

All further models we will consider are understood to satisfy the condition that their universes belong to $\mathbb{K}$. Because K is a set, the downward Löwenheim-Skolem Theorem allows us to construct, within ZFC, transitive submodels of $\mathbb{K}$ ([3], p. 18, c.f. p. 79, where the problem of models with class universes is discussed). By a suitable argument, one such model $\mathbb{M}$ is countable and has countable rank[1] ([4], p. 110), hence $M \in K$. We fix $\mathbb{M} = (M, M^2 \cap \in)$, and call it the ground model.

By standardness and transitivity, ordinals in $\mathbb{K}$ and $\mathbb{M}$ are ordinals in the real sense. The class of ordinals in $\mathbb{K}$ turns out to be the least ordinal not in $\mathbb{K}$ ([24], p. 197).

We now turn to the topic of Boolean algebra, beginning with the definition.

Definition 0.2 B is a Boolean algebra if $\underline{B} = (B, \cdot, +, -, 0, 1)$ where B is a set, $\cdot$ and $+$ are binary operations on B, $-$ is a unary operation on B, and 0 and 1 are distinguished constants in B, all of which satisfy:

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[1] Transitivity guarantees the existence of a rank function ([3], pp. 68-9).

(a) $x + y = y + x$, $x \cdot y = y \cdot x$.

(b) $x + (y + z) = (x + y) + z$,

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(c) $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$,

$(x \cdot y) + z = (x + z) \cdot (y + z)$.

(d) $x + x = x$, $y \cdot y = y$.

(e) $x + (-x) = 1$, $x \cdot (-x) = 0$.

(f) $- (x + y) = -x \cdot -y$,

$- (x \cdot y) = -x + -y$.

(g) $-(-x) = x$.

We note that $\underline{B}$ is partially ordered by the relation:

$$x \geq y \quad \text{iff} \quad x = y + x .$$

With this partial order, $x + y$ corresponds with $\sup(x,y) = \inf [u : u \geq x, u \geq y]$, and $x \cdot y$ corresponds with $\inf(x,y) = \sup [u : u \leq x, u \leq y]$. Generalizing this notion, we write ΣA for $\sup(A)$ and ΠA for $\inf(A)$, when $A \subset B$. It follows that $\Sigma B = 1$, and $\Pi B = 0$. It is convenient to adopt the convention: $\Sigma \emptyset = 0$ and $\Pi \emptyset = 1$.

Definition 0.3 (a) A Boolean algebra $\underline{B}$ is complete if for each $A \subset B$, ΣA and ΠA exist, and $\Sigma A \in B$ and $\Pi A \in B$.

(b) A Boolean algebra $\underline{B}$ is $\mathbb{M}$-complete if $\underline{B} \in M$, and for each $A \subset B$:

$A \in M$ implies $\Sigma A \in B$ and $\Pi A \in B$.

There is no problem regarding the existence of Boolean algebras in models. Since any field of sets (e.g. the powerset $\mathbb{P}(x)$ of x) is a Boolean algebra, each model abounds in Boolean algebras. If $\underline{B}$ is a Boolean algebra in $\mathbb{M}$, then it is clear that $\underline{B}$ is a Boolean algebra in $\mathbb{K}$. We need not be concerned then, about losing Boolean algebras when we move from a given model to a more inclusive model. However, it is quite conceivable that an incomplete Boolean algebra exists which is complete in a certain model $\mathbb{H}$ simply because the $A \subset B$ for which $\Sigma A \notin B$ do not belong to H. An $\mathbb{M}$-complete Boolean algebra is therefore not necessarily a $\mathbb{K}$-complete Boolean algebra.

For the remainder of this section we will consider $\underline{B}$ to be a fixed $\mathbb{M}$-complete Boolean algebra. Without danger of confusion, we will write B for both $\underline{B}$ and its underlying set B, and write $\mathbb{M}$ in places where its universe M is understood (e.g. $x \in \mathbb{M}$).

Two processes will now be dealt with. The first is the construction of the Boolean-valued model $\mathbb{M}^B$ from $\mathbb{M}$ and B.

The Boolean-valued model $\mathbb{M}^B \in \mathbb{K}$ may be thought of as a generalization of the ground model $\mathbb{M}$, where set theoretic statements may be evaluated for their "degree" of truth. More precisely, where classical logic allows only two truth values, the Boolean-valued model $\mathbb{M}^B$ assigns as truth value a member of the complete (in $\mathbb{M}$) Boolean

algebra B to each statement. This explains the name Boolean valued model. If B is the two-element algebra denoted $\{0,1\}$, the notion of Boolean-valued model reduces to the classical notion of model. We define $\mathbb{M}^B$ from within $\mathbb{M}$, by induction on the ordinals less than $\Theta_{\mathbb{M}}$, the least ordinal not in $\mathbb{M}$.

Definition 0.4

$$\mathbb{M}_0^B = \{0\}$$

$$\mathbb{M}_\alpha^B = \bigcup_{\beta<\alpha} \mathbb{M}_\beta^B \text{ , if } \alpha \text{ is a limit ordinal.}$$

$$\mathbb{M}_{\alpha+1}^B = \{ x : x \text{ is a function, } \mathrm{dom}(x) \subset \mathbb{M}_\alpha^B \text{ , } \mathrm{rng}(x) \subset B \}$$

$$\mathbb{M}^B = \bigcup_{\alpha<\Theta_{\mathbb{M}}} \mathbb{M}_\alpha^B .$$

Notice that each element x of $\mathbb{M}^B$ must by induction belong to $\mathbb{M}_{\alpha+1}^B$ for some least α. This α we may call the rank $\rho(x)$ of that particular object in $\mathbb{M}^B$.

Even though we are working within the ground model, it does not follow that $\mathbb{M}^B \in \mathbb{M}$, and in general this is false. There is, however, a concrete way of envisioning $\mathbb{M}$ as being inside $\mathbb{M}^B$. This is done via the following embedding functor $^\vee : \mathbb{M} \longrightarrow \mathbb{M}^B$, defined by transfinite induction on $\rho(x)$.

Definition 0.5

(a) $\check{0} = 0$

(b) for each $\check{x} \in \mathbb{M}$, we have $x \in \mathbb{M}^B$, with $\mathrm{dom}(\check{x}) = \{ \check{y} : y \in x \}$, and $\check{x}(\check{y}) = 1$ for each $y \in x$.

Notice that $\rho(\check{y}) < \rho(\check{x})$ if $y \in x$, so that $\check{x}$ is indeed defined in terms of elements of lower rank.

The $\check{}$-functor illustrates each set $x \in \mathbb{M}$ as a specialized characteristic function $\check{x} \in \mathbb{M}^B$. Because each $y \in x$ is also a set in $\mathbb{M}$, $\check{y}$ is also a characteristic function of this type, and so $\check{x}$ becomes a composition of characteristic functions on sets of characteristic functions. The rank $\rho(x)$ serves to indicate how long this process has gone on.

It is clear that other function-objects exist in $\mathbb{M}^B$ whose ranges include values other than 1. These of course have no pre-image by $\check{}$ among the sets of $\mathbb{M}$, but they show that the $\mathbb{M}^B$ construction enables the handling of objects which may be set-like to varying degrees. This presents us with the possibility of considering some of these objects set-like enough to combine with the sets of $\mathbb{M}$, thus forming a new, more inclusive model of ZFC. This is the subject of the last portion of this section.

It is possible now to define a <u>Boolean value</u> $[\![\phi(x_1, \ldots, x_n)]\!] \in B$ for each formula ϕ of n free variables, and each $x_1, \ldots, x_n \in \mathbb{M}^B$. These Boolean values behave like the conventional truth values of first order predicate calculus, but since they belong to the $\mathbb{M}$-complete Boolean algebra B, thay extend our notion of semantics beyond the usual duality of truth (1) and falsity (0).

The Boolean values $[\![x \in y]\!]$ and $[\![x = y]\!]$ are defined

for $x, y \in \mathbb{M}^B$ by transfinite induction on the lexicographical ordering $(\rho(x), \rho(y))$ (i.e. the ordering defined by: $(\alpha,\beta) > (\delta,\gamma)$ iff $\alpha > \delta$ or $\alpha = \delta$, $\beta > \gamma$).

Definition 0.6

$$\text{(a)}\quad [\![x \in y]\!] = \sum_{z \in \mathrm{dom}(y)} (y(z) \cdot [\![z = x]\!]) \ .$$

$$\text{(b)}\quad [\![x = y]\!] = \prod_{z \in \mathrm{dom}(x)} (-x(z) + [\![z \in y]\!]) \cdot \prod_{z \in \mathrm{dom}(y)} (-y(z) + [\![z \in x]\!]) \ .$$

For a discussion of the form and efficacy of the above and similar definitions, see [17], pp. 41 - 4, and [25], pp. 121 - 2.

Having defined $[\![\phi]\!]$ for ϕ an atomic formula, we extend the definition to include any set theoretic formula.

Definition 0.7

$$\text{(a)}\quad [\![\neg \phi]\!] = - [\![\phi]\!] \ .$$

$$\text{(b)}\quad [\![\phi \,\&\, \psi]\!] = [\![\phi]\!] \cdot [\![\psi]\!] \ .$$

$$\text{(c)}\quad [\![\phi \vee \psi]\!] = [\![\phi]\!] + [\![\psi]\!] \ .$$

$$\text{(d)}\quad [\![\phi \to \psi]\!] = -[\![\phi]\!] + [\![\psi]\!] \ .$$

$$\text{(e)}\quad [\![(\forall x)\phi]\!] = \prod_{x \in \mathbb{M}^B} [\![\phi(x)]\!] \ .$$

$$\text{(f)}\quad [\![(\exists x)\phi]\!] = \sum_{x \in \mathbb{M}^B} [\![\phi(x)]\!] \ .$$

The $\mathbb{M}$-completeness of B ensures that 0.6 (a), (b) and 0.7 (e), (f) are well-defined. The following is a trivial but useful consequence of the above definitions.

Lemma 0.8 $[\![\phi]\!] \leq [\![\psi]\!]$ iff $[\![\phi \rightarrow \psi]\!] = 1$

A series of lemmas now follows, which give several useful relations concerning the Boolean values of some specific formulas. The proofs are straightforward, and are usually accomplished by transfinite induction on $(\rho(x), \rho(y))$. A sample proof accompanies the first of these lemmas.

Lemma 0.9 (a) $[\![x = x]\!] = 1$.

(b) $x(y) \leq [\![y \in x]\!]$.

(c) $[\![x = y]\!] = [\![y = x]\!]$.

Proof: Suppose (a) to be true for any x satisfying $\rho(x) < \gamma$. Let $\rho(x) = \gamma$ now. By definition:

(i) $$[\![x = x]\!] = \prod_{y \in \mathrm{dom}(x)} (-x(y) + [\![y \in x]\!]).$$

For each $y \in \mathrm{dom}(x)$:

(ii) $$[\![y \in x]\!] = \sum_{u \in \mathrm{dom}(x)} (x(u) \cdot [\![u = y]\!])$$

$$\geq x(y) \cdot [\![y = y]\!] = x(y) \cdot 1,$$

by hypothesis, as $\rho(y) < \gamma$.

Since x(y) is defined only where $y \in \mathrm{dom}(x)$, (b) follows. Substituting for $[\![y \in x]\!]$ in (i), we have:

$$[\![x = x]\!] \geq \prod_{z \in \mathrm{dom}(x)} (-x(z) + x(z)) = 1 .$$

Therefore: $x = x = 1$ for $\rho(x) = \gamma$.

Calculating directly: $[\![0 = 0]\!] = 1$ (the

Boolean infimum of the empty collection is 1),
so (a) follows by transfinite induction.
(c) is verified directly from the definition.

The next three interdependent statements are proved simultaneously by transfinite induction on ($\rho(x), \rho(y)$) (see [25], p. 123).

Lemma 0.10 (a) $[\![x = y]\!] \cdot [\![x \in z]\!] \leq [\![y \in z]\!]$.

(b) $[\![x \in z]\!] \cdot [\![z = y]\!] \leq [\![x \in y]\!]$.

(c) $[\![x = y]\!] \cdot [\![y = z]\!] \leq [\![x = z]\!]$.

The lemma below follows by induction on the complexity of ϕ, using the fact that the previous lemma establishes the result for atomic formulas.

Lemma 0.11 $[\![x = y]\!] \cdot [\![\phi(x)]\!] \leq [\![\phi(y)]\!]$.

Lemma 0.12 (a) $[\![(\exists y \in x)\phi(y)]\!] = \sum_{y \in dom(x)} (x(y) \cdot [\![\phi(y)]\!])$.

(b) $[\![(\forall y \in x)\phi(y)]\!] = \prod_{y \in dom(x)} (x(y) \cdot [\![\phi(y)]\!])$.

Proof: See [25], p. 125 .

Definition 0.13 Let $x_1, \ldots, x_n \in \mathbb{M}^B$. $\phi(x_1, \ldots, x_n)$ is said to be valid in $\mathbb{M}^B$ if:

$[\![\phi(x_1, \ldots, x_n)]\!] = 1$,

in which case we write:

$\mathbb{M}^B \models \phi(x_1, \ldots, x_n)$.

This notion of validity sets the stage for two of the most important results of this section.

Theorem 0.14 — Every axiom of first order predicate calculus with identity is valid in $\mathbb{M}^B$. Those formulas obtained by rules of inference of first order predicate calculus from formulas valid in $\mathbb{M}^B$, are themselves valid in $\mathbb{M}^B$.

Theorem 0.15 — Every axiom of ZFC is valid in $\mathbb{M}^B$.

Corollary 0.16 — $\mathbb{M}^B$ is a model of ZFC (i.e. every theorem of ZFC is valid in $\mathbb{M}^B$).

No proof will be given here for 0.14 as the usual computational proof (see [25], pp. 60, 124, and [17], pp. 36 - 51) is unaffected by our definition of model.

Theorem 0.15 is also standard, but it is worthwhile to look at some aspects of its proof, particularly those which surface as techniques in later proofs. This selective approach to 0.15 is carried out in the next series of lemmas.

We define the Boolean-valued singleton $\{x\}^B$, for $x \in \mathbb{M}^B$, as follows: $\mathrm{dom}(\{x\}^B) = \{x\}$; $\{x\}^B(x) = 1$. Hence $\{x\}^B \in \mathbb{M}^B$, and for $\rho(x) = \alpha$, we have $\rho(\{x\}^B) = \alpha + 1$. In general, $\{x\}^B$ and $\{x\}$ are distinct, however $[\![\{x\} = \{x\}^B]\!] = 1$ (see Lemma 0.30).

Lemma 0.17 For each $S \subset \mathbb{M}^B$, $S \in \mathbb{M}$, there is a $T \in \mathbb{M}^B$ such that $[\![x \in T]\!] = 1$ for each $x \in S$.

Proof: We take the Boolean sum of functions

$T = \sum_{x \in S} \{x\}^B$, i.e. $\mathrm{dom}(T) = S$;

$T(x) = \{x\}^B(x) = 1$, for each $x \in S$. From 0.9 (b) we have $[\![x \in T]\!] = 1$, for all $x \in S$. Note that $T \in \mathbb{M}^B$ since $\rho(T) = \sup_{x \in S} (\{x\}^B)$, which exists since $S \in \mathbb{M}$.

The verification of 0.15 proceeds one axiom of ZFC at a time. The $\mathbb{M}^B$-validity of some of the axioms of ZFC is a matter of a basic computation. Our previous lemmas reduce the validations of the Axiom of Extensionality (see [17], p. 50 for a proof that can be adapted to our foundations), and the Axiom of Regularity ([25], p. 89, [9], p. 56) to this computational level.

Slightly more sophisticated are those validations which are a consequence of 0.17. These include the validations of the Axiom of Pairing and the Axiom of Unions, whose closely related proofs ([9], p. 55) are immediate. The next few lemmas also use the 0.17 strategy.

A function $F: \Theta_{\mathbb{M}} \longrightarrow B$ is nondecreasing if $F(\alpha) \leq F(\beta)$ whenever $\alpha \leq \beta$, and is eventually constant if there exists an ordinal γ such that for all $\beta > \gamma$, $F(\beta) = F(\gamma)$.

Lemma 0.18 For each formula ϕ of set theory, the function

$$F(\alpha) = \sum_{x \in \mathbb{M}^B_\alpha} [\![\phi(x)]\!]$$

is nondecreasing and eventually constant.

Proof: For increasingly larger ordinals β, $F(\beta)$ is a Boolean supremum taken over increasingly larger sets $\mathbb{M}^B_\beta$, hence F is obviously nondecreasing. We define a function $H: B \longrightarrow \Theta_{\mathbb{M}}$ by: $H(a) = \inf[\ \beta : a \leq F(\beta)\ ]$. Since $B \in \mathbb{M}$, an ordinal γ exists which is the supremum of the image of B by H. For each $\beta > \gamma$, $F(\beta) = F(\gamma)$.

The fact that B is a set is crucial; neither the Axiom of Replacement (0.19), nor the Axiom of Power Set (0.21), nor the Maximum Principle (0.26) hold in some $\mathbb{M}^B$ where B is a proper class ([25], p. 196).

Lemma 0.19 For each formula ϕ of set theory:

$$\mathbb{M}^B \models (\forall x)(\exists y)(\forall u \in x)[\ (\exists v)\phi(u,v) \rightarrow (\exists v \in y)\phi(u,v)\ ]$$

Proof: Let $x \in \mathbb{M}^B$. The function $F_u(\beta) = \sum_{v \in \mathbb{M}^B_\beta} [\![\phi(u,v)]\!]$ is eventually constant, by 0.18, for each $u \in dom(x)$.

This enables us to define the function:

$g(u) = \inf\ [\ \alpha: \forall \beta > \alpha,\ F_u(\beta) = F_u(\alpha)\ ]$ for each $u \in dom(x)$. We may write:

$$\sum_{v \in \mathbb{M}^B} [\![\phi(u,v)]\!] = \sum_{v \in \mathbb{M}^B_{g(u)}} [\![\phi(u,v)]\!].$$

Following 0.17, we let $dom(y) = \bigcup_{u \in dom(x)} \mathbb{M}^B_{g(u)}$

$= \mathbb{M}^B_\nu$, where $\nu = \sup_{u \in dom(x)} g(u)$, and we set

$y(z) = 1$ for all $z \in dom(y)$. For a chosen

$x \in \mathbb{M}^B$, we have constructed a set $y \in \mathbb{M}^B$

such that for each $u \in dom(x)$:

$$[\![(\exists v)\phi(u,v) \rightarrow (\exists v \in y)\phi(u,v)]\!]$$

$$= - \sum_{v \in \mathbb{M}^B} [\![\phi(u,v)]\!] + \sum_{v \in \mathbb{M}^B_\nu} [\![\phi(u,v)]\!]$$

$$= - \sum_{v \in \mathbb{M}^B} [\![\phi(u,v)]\!] + \sum_{v \in \mathbb{M}^B_{g(u)}} [\![\phi(u,v)]\!]$$

$$= 1 .$$

Therefore, for each $x \in \mathbb{M}^B$ there exists $y \in \mathbb{M}^B$

satisfying:

$$[\![(\forall u \in x)[(\exists v)\phi(u,v) \rightarrow (\exists v \in y)\phi(u,v)]]\!]$$

$$= \prod_{u \in dom(x)} [-x(u) + [\![(\exists v)\phi(u,v) \rightarrow (\exists v \in y)\phi(u,v)]\!]]$$

$$= \prod_{u \in dom(x)} (-x(u) + 1) = 1 .$$ The result

now follows.

The above lemma validates one form of the Axiom Schema of Replacement. It is well known that the Axiom Schema of Separation is a logical consequence of the Replacement Axiom. We could infer then, by way of 0.14, that the Axiom of Separation also holds in $\mathbb{M}^B$. The following lemma is a more useful statement of the validity of the Separation

Axiom. Its proof is a basic calculation, so in view of the above discussion we will omit it (see [9], p. 55).

Lemma 0.20 For each $x \in \mathbb{M}^B$ and formula ϕ, there is a set $y \in \mathbb{M}^B$ satisfying: $dom(y) = dom(x)$, and:

$$[\![(\forall z \in y)(z \in x \ \& \ \phi(z))]\!] = 1 \ ,$$

$$[\![(\forall z \in x)(\phi(z) \rightarrow z \in y)]\!] = 1 \ .$$

The main application of the above lemma is in the next result.

Lemma 0.21 $$\mathbb{M}^B \models (\forall x)(\exists y)(\forall u)(u \subset x \rightarrow u \in y)$$

Proof: Let $x, u \in \mathbb{M}^B$. From 0.20, there exists $v \in \mathbb{M}^B$ satisfying: $dom(v) = dom(x)$, $[\![v = u \cap x]\!] = 1$. For each $t \in dom(x)$ we have:

$$[\![t \in v]\!] = [\![t \in u]\!] \cdot [\![t \in x]\!] \ ,$$

thus, $[\![t \in v]\!] \leq [\![t \in x]\!]$.

Following 0.17, we define $y \in \mathbb{M}^B$ as follows:

$$dom(y) = \{ z : dom(z) = dom(x), \ t \in dom(x) \ [\![t \in z]\!] \leq [\![t \in x]\!] \} \ ,$$

and $y(z) = 1$ for $z \in dom(y)$. We know that $y \neq \check{\emptyset}$ when $x \neq \check{\emptyset}$, since $v \in dom(y)$. Furthermore:

$$[\![u \subset x]\!] = [\![u \subset x]\!] \cdot [\![v = u \cap x]\!] \leq [\![u = v]\!] \ ,$$

so that:

$$[\![u \subset x]\!] \leq \sum_{z \in dom(y)} [\![u = z]\!] = [\![u \in y]\!].$$

On the other hand:

$$\llbracket u \in y \rrbracket = \sum_{z \in \mathrm{dom}(y)} \llbracket z = u \rrbracket$$

$$\text{(i)} \qquad = \sum_{z \in \mathrm{dom}(y)} \llbracket z = u \rrbracket \cdot \llbracket z \subset x \rrbracket \ ,$$

since $z \in \mathrm{dom}(y)$ implies $\llbracket z \subset x \rrbracket = 1$.

0.11 gives (i) $\leq \sum_{z \in \mathrm{dom}(y)} \llbracket u \subset x \rrbracket = \llbracket u \subset x \rrbracket$.

Given any $x \in \mathbb{M}^B$, we have produced a $y \in \mathbb{M}^B$ satisfying $\llbracket u \subset x \rrbracket = \llbracket u \in y \rrbracket$, for each $u \in \mathbb{M}^B$. The result now follows.

Lemma 0.21 establishes the validity of the Axiom of Power Set in $\mathbb{M}^B$.

The Axiom of Infinity may be validated by various strategies. Jech sketches a recursive construction of an infinite set in $\mathbb{M}^B$ ([9], p. 56). Requiring slightly more background is Rosser's proof that $\llbracket \check{\omega} \text{ is infinite} \rrbracket = 1$ ([17], p. 77). At this point we quote a general theorem that yields the immediate validation of the Axiom of Infinity, as well as that of the Null Set Axiom.

$\phi(x_1, \ldots, x_n)$ is a <u>bounded formula</u> of set theory if each of the quantified variables of ϕ are restricted to one of the sets $x_1, \ldots, x_n$, e.g. $(\forall x \in a)(\exists y \in b)(x \in y)$.

<u>Theorem 0.22</u> If $\phi(x_1, \ldots, x_n)$ is a bounded formula of set theory, then:

$\mathbb{M}^B \models \phi(\check{x}_1, \ldots, \check{x}_n)$ iff $\mathbb{M} \models \phi(x_1, \ldots, x_n)$.

Proof: This follows from our Corollary 4.14.

For an elementary proof, see [25], p. 127 .

Corollary 0.23

(a) $\mathbb{M}^B \models (\exists x)(x \in \check{\omega}) \ \& \ (\forall x \in \check{\omega})(\exists y \in \check{\omega})(x \in y)$.

(b) $\mathbb{M}^B \models (\forall x \in \check{\emptyset})(x \neq x)$.

The remaining lemmas of this series culminate in the validity proof of the final axiom of ZFC, namely the Axiom of Choice.

Definition 0.24 Let $u \in B$ and $u \neq 0$. $\{ u_\beta : \beta \in I \}$ is a partition of u if $\sum_{\beta \in I} u_\beta = u$, and

$u_\gamma \cdot u_\delta = 0$ for $\gamma \neq \delta$ ($I \subset \Theta_{\mathbb{M}}$, $I \in \mathbb{M}$).

Lemma 0.25 Let $\{ u_\beta : \beta \in I \}$ be a partition of $u \in B$, $u \neq 0$. Let $\{ t_\beta : \beta \in I \} \subset \mathbb{M}^B$, and $I \subset \Theta_{\mathbb{M}}$, $I \in \mathbb{M}$. Then there exists $t \in \mathbb{M}^B$ such that:

$$u_\beta \leq [\![t = t_\beta]\!] \quad \text{for each } \beta \in I .$$

Proof: Letting $\alpha = \sup_{\beta \in I} \rho(t_\beta)$, we define t as follows:

$$\mathrm{dom}(t) = \mathbb{M}^B_{\alpha+1} ,$$

$$t(z) = \sum_{\beta \in I} u_\beta \cdot t_\beta(z) .$$

An immediate consequence is that for each $\beta \in I$ and each $z \in \mathbb{M}^B_{\alpha+1}$: $u_\beta \cdot t_\beta(z) = u_\beta \cdot t(z)$.

This fact gives us two calculations:

(i) $u_\beta \cdot (-t(z) + [\![z \in t_\beta]\!]) \geq [u_\beta \cdot -t(z)] + [u_\beta \cdot t_\beta(z)]$

$= [u_\beta \cdot -t(z)] + [u_\beta \cdot t(z)]$

(ii) $u_\beta \cdot (-t_\beta(z) + [\![z \in t]\!]) \geq [u_\beta \hat{\cdot} -t_\beta(z)] + [u_\beta \cdot t(z)]$

$= [u_\beta \cdot -t_\beta(z)] + [u_\beta \cdot t_\beta(z)]$

$= u_\beta$.

Employing (i) and (ii), we conclude:

$$[\![t = t_\beta]\!] \geq u_\beta \cdot [\![t = t_\beta]\!]$$

$$\geq u_\beta \cdot \prod_{z \in \mathbb{M}^B} u_\beta \cdot (-t(z) + [\![z \in t_\beta]\!])$$

$$\cdot \prod_{z \in dom(t_\beta)} u_\beta \cdot (-t_\beta(z) + [\![z \in t]\!])$$

$$\geq u_\beta \ .$$

In satisfying the above lemma $\mathbb{M}^B$ is said to be a <u>complete</u> Boolean-valued structure (see [25], p. 62). The t in Lemma 0.25 is unique in the sense that if t and t' both satisfy the Lemma, then $[\![t = t']\!] = 1$.

The next lemma is known as the <u>Maximum Principle</u> as it states that Boolean suprema in $\mathbb{M}^B$ are in fact maxima.

<u>Lemma 0.26</u> For each formula $\phi(x)$ of set theory there exists $t \in \mathbb{M}^B$ such that $[\![(\exists x)\phi(x)]\!] = [\![\phi(t)]\!]$.

Proof: Let $u = [\![(\exists x)\phi(x)]\!] = \sum_{x \in \mathbb{M}^B} [\![\phi(x)]\!]$.

Without loss of generality, we assume $u \neq 0$.

It follows that for some t_o we have:

$$u_o = [\![\phi(t_o)]\!] > 0 \ .$$

The sequence $\{ t_\beta : \beta < \alpha \} \subset \mathbb{M}^B$ is constructed inductively. If $u \cdot \prod_{\gamma < \beta} -u > 0$, we may pick $t_\beta \in \mathbb{M}^B$ such that:

$$0 < u_\beta = [\![\phi(t_\beta)]\!] \leq u \cdot \prod_{\gamma < \beta} -u_\gamma ,$$

by virtue of the fact that AC holds in $\mathbb{M}$, and that $B \in \mathbb{M}$. A second consequence of this fact is that B has a cardinality in $\mathbb{M}$, which allows us to conclude that an ordinal $\alpha \in \Theta_{\mathbb{M}}$ exists such that $\sum_{\beta < \alpha} u_\beta = u$. By Lemma 0.25 there is a $t \in \mathbb{M}^B$ such that: $u_\beta \leq [\![t \in t_\beta]\!]$, for $\beta < \alpha$. So $u_\beta = [\![t = t_\beta]\!] \cdot [\![\phi(t_\beta)]\!] \leq [\![\phi(t)]\!]$, for each $\beta < \alpha$; and $u = \sum_{\beta < \alpha} u_\beta \leq [\![\phi(t)]\!]$ $\leq \sum_{x \in \mathbb{M}^B} [\![\phi(x)]\!] = u$. Hence $u = [\![\phi(t)]\!]$.

Definition 0.27 For $x \in \mathbb{M}^B$, we write $\sup(x)$ for $[\![x \neq \emptyset]\!]$ $= \sum_{u \in \mathrm{dom}(x)} x(u)$.

Before entering into the validity proof of the Axiom of Choice, we must briefly review the property of functionhood in $\mathbb{M}^B$. By induction, we know that elements of $\mathbb{M}^B$ are functions in $\mathbb{M}$. What does a function in $\mathbb{M}^B$ look like from the point of view of $\mathbb{M}$? First, a function is a special set of ordered pairs. But the pair (x,y) is foreign to $\mathbb{M}^B$ since it has no domain in $\mathbb{M}^B$ or range in B. Just as we have defined the Boolean-valued singleton (p. 17), we may define Boolean-valued pairs.

Definition 0.28 For each $x, y \in \mathbb{M}^B$:

(a) $\{x,y\}^B = \{x,y\} \times \{1\}$

(b) $(x,y)^B = \{ \{x\}^B, \{x,y\}^B \}^B$

Notice that $\{x\}^B = \{x,x\}^B$.

The definition of a Boolean-valued function parallels the usual notion of functionhood.

Definition 0.29 $f \in \mathbb{M}^B$ is a Boolean-valued function if there exist $u,v \in \mathbb{M}^B$ such that:

(a) $dom(f) \subset \{ (x,y)^B : x \in u, y \in v \}$.

(b) $[\![(\forall x \in u)(\exists y \in v)[(x,y)^B \in f]]\!] = 1$.

(c) if $(x,y)^B, (x,y')^B \in dom(f)$, then: $[\![(x,y)^B \in f]\!] \cdot [\![(x,y')^B \in f]\!] \leq [\![y = y']\!]$

(d) $f(w) \in B$ for each $w \in dom(f)$.

(a) and (d) above provide that $f \in \mathbb{M}^B$. $[\![f$ is a function $]\!] = 1$ iff f satisfies (b) and (c). $[\![f: u \to v]\!] = 1$ iff f satisfies (a) and (b). Definition 0.29 is thus equivalent to the condition:

$$[\![f \text{ is a function } \& \ f: u \to v]\!] = 1 \ .$$

Lemma 0.30 (a) $[\![\{x,y\}^B = \{x,y\}]\!] = 1$

(b) $[\![(x,y)^B = (x,y)]\!] = 1$

(c) $[\![(x,y) = (x',y')]\!] = [\![(x,y)^B = (x',y')^B]\!]$

(d) $[\![(x,y) \in u]\!] = [\![(x,y)^B \in u]\!]$.

Proof: (a) is trivial when we interpret it as:

$$[\![z \in \{x,y\}^B \leftrightarrow (z = x \vee z = y)]\!] = 1 \ .$$

The others follow from (a) via the early lemmas.

Now it is possible to freely exchange (x,y) (unnatural in $\mathbb{M}^B$) with $(x,y)^B$ (natural in $\mathbb{M}^B$) in expressions like (d) above. This will be exploited in the next proof, as ordinary pairs are less cumbersome to use than their Boolean counterparts.

We recall that f is a choice function for a nonempty set x , if dom(f) = x , rng(f) $\subset$ x , and f(z) $\in$ z for each z $\in$ x , z $\neq \emptyset$.

Lemma 0.31 $\mathbb{M}^B \models (\forall x)\ [(\ x \neq \emptyset\ ) \rightarrow (\exists y)(\ y$ is a choice tion for x)]

Proof: Let x $\in \mathbb{M}^B$. For each z $\in$ dom(x) we use the Axiom of Choice in $\mathbb{M}$ and Lemma 0.26 to pick $t_z \in \mathbb{M}^B$ such that: $\sup(z) \leq [\![\ t_z \in z\ ]\!]$.

Let y $\in \mathbb{M}^B$ be defined:

$$\mathrm{dom}(y) = \{\ (z,t)^B : z \in \mathrm{dom}\ (x),\ t = t_z\ \}\ ,\ y(\ (z,t_z)^B) = x(z)\ ,\ \text{for } z \in \mathrm{dom}(x)\ .$$

Then let $\phi(x,y)$ be:

$$(\forall z)[\ (\ z \in x\ \&\ z \neq \emptyset\ ) \rightarrow (\exists t)(\ t \in z\ \&\ (z,t)^B \in y\ )\ ]\ .$$

Using Lemma 0.30 and others, we calculate:

$$[\![\ \phi(x,y)\ ]\!] \geq \prod_{z \in \mathbb{M}^B} [\ -x(z) + (\ -\sup(z)+x(z)\cdot \sup(z))] = 1\ .$$

Now we will show $[\![$ y is a function $]\!] = 1$.

Actually, only part (c) of Definition 0.29

needs demonstration.

Let g be the function in $\mathbb{M}$ defined by:

$$\mathrm{dom}(g) = \mathrm{dom}(x) \quad ; \quad g(z) = t_z \quad ,$$

i.e. g is the choice function on dom(x) described above. g is <u>extensional</u>, i.e.

(i) $\forall z \in \mathrm{dom}(x),\ [\![z = z']\!] \leq [\![g(z) = g(z')]\!]$.

The verification of the above involves an elementary applicaton of 0.9 (b) and 0.11 .

For each $z \in \mathrm{dom}(x)$ and $t \in \mathbb{M}^B$, we have:

$$[\![(z,t)^B \in y]\!] = \sum_{z' \in \mathrm{dom}(x)} y((z',g(z'))^B) \cdot [\![(z,t)^B = (z',g(z'))^B]\!]$$

$$= \sum_{z' \in \mathrm{dom}(x)} x(z') \cdot [\![(z,t) = (z',g(z'))]\!] \text{ (by 0.30)}$$

$$= \sum_{z' \in \mathrm{dom}(x)} x(z') \cdot [\![z = z']\!] \cdot [\![t = g(z')]\!]$$

$$\leq \sum_{z' \in \mathrm{dom}(x)} x(z') \cdot [\![g(z) = g(z')]\!] \cdot [\![t = g(z')]\!] \quad \text{(by (i) above)}$$

$$\leq [\![g(z) = t]\!] \ .$$

Applying this calculation, we conclude:

$$[\![(z,t)^B \in y]\!] \cdot [\![(z,t')^B \in y]\!] \leq [\![g(z) = t]\!] \cdot [\![g(z) = t']\!] \leq [\![t = t']\!] \ .$$

This validation of the Axiom of Choice in $\mathbb{M}^B$ concludes our partial proof of Theorem 0.15 .

Our attention now turns to Corollary 0.16 . Is $\mathbb{M}^B$ a model of ZFC, according to our convention on p.7 ? From the beginning we have deliberately confounded the distinction between $\mathbb{M}^B$ and its universe, by neglecting to invent a separate symbol for the latter. Neither have we drawn attention to the membership relation for $\mathbb{M}^B$. Our unconventional

notion of validity tends to further obscure the matter.

$\mathbb{M}^B$ is clearly a set in $\mathbb{K}$: $\mathbb{M}^B_\circ \in \mathbb{K}$; if $\mathbb{M}^B_\alpha \in \mathbb{K}$, then $\mathbb{M}^B_{\alpha+1} \in \mathbb{K}$; if $\mathbb{M}^B_\beta \in \mathbb{K}$ for each $\beta<\alpha$, where α is a limit ordinal in $\mathbb{M}$, then $\bigcup_{\beta<\alpha} \mathbb{M}^B_\beta \in \mathbb{K}$; and since $\theta_{\mathbb{M}} = \mathrm{Ord} \cap M$, we have $\mathbb{M}^B = \bigcup_{\alpha<\theta_{\mathbb{M}}} \mathbb{M}^B \in \mathbb{K}$.

$\mathbb{M}^B$ has a membership relation ε^B, which we may define: $\mathbb{M}^B \models x \,\varepsilon^B\, y$ iff $\sum_{u \in \mathrm{dom}(y)} y(u) \cdot [\![u = x]\!] = 1$, and this relation satisfies (via Theorems 0.14 and 0.15) the theorems of set theory. Of course, we have been refering to ε^B as ε from the beginning, to simplify our notation.

This leads us to another problem: $\mathbb{M}^B \models x = y$ does not necessarily imply $x = y$ in $\mathbb{K}$, i.e. $\mathbb{M}^B$ has a different equality relation than $\mathbb{K}$. This is easily resolved, if we are willing to further complicate our notion of model by relegating the symbol '=' to the status of a predicate constant. In this case, $\mathbb{N} = (N, =_{\mathbb{N}}, \varepsilon_{\mathbb{N}})$ is a model of set theory if N is a set, $=_{\mathbb{N}}$ is a binary relation on N satisfying the axioms and inference rules of first order predicate calculus with identity, etc. , .

This augmented notion of set theoretic model clears up the problem. Both $\mathbb{M}$ and $\mathbb{M}^B$ are easily construed as models of this sort: $\mathbb{M} = (M, =, \varepsilon \cap M^2)$, $\mathbb{M}^B = (\mathbb{M}^B, =^B, \varepsilon^B)$, where $\mathbb{M}^B$ symbolizes both the model and its universe, and $=^B$ and ε^B are defined recursively (as in 0.6). Following this convention, Theorem 0.22 tells us that $\check{}$ is an embed-

ding of $\mathbb{M}$ into $\mathbb{M}^B$. In particular:

(a) $\mathbb{M}^B \models \check{x} =^B \check{y}$ iff $\mathbb{M} \models x = y$.

(b) $\mathbb{M}^B \models \check{x} \in^B \check{y}$ iff $\mathbb{M} \models x \in y$.

Our study of $\mathbb{M}^B$, though not complete, is sufficient for the comming use we are to put it to. We will come to see the $\mathbb{M}^B$ construction as the intermediate stage of a larger process. Moreover, the issue of whether $\mathbb{M}^B$ fits one of several feasible notions of modelhood will have essentially no impact on the work to come.

The remainder of this section deals with the extension of the model $\mathbb{M}$ to a larger model that is related to $\mathbb{M}^B$, but that fits in every way the criteria of modelhood given on p. 7 .

Definition 0.32 (a) A subset G of a Boolean algebra B is an ultrafilter if:

(i) $0 \notin G$.

(ii) $x, y \in G$ implies $x \cdot y \in G$.

(iii) $x \in G$, $y \geq x$ implies $y \in G$.

(iv) $\forall x \in B$, $x \in G$ or $-x \in G$.

(b) $G \subset B$ is an $\mathbb{M}$-generic ultrafilter if, in addition to the above, G satisfies:

(v) $A \subset G$, $A \in \mathbb{M}$ implies $\prod A \in G$.

G is just a filter if it satisfies (i) - (iii) above. G is a proper filter if $G \neq B$. Condition (iv) is equivalent

to saying that G is a maximal proper filter, i.e. one that is not properly included in any other proper filter.

A useful equivalent to 0.32 is the following.

Lemma 0.33 An ultrafilter G on B is $\mathbb{M}$-generic iff for each partition A of $u \in G$ such that $A \in \mathbb{M}$, there exists $b \in B$ such that $A \cap G = \{b\}$.

Proof: Let $A \subset B$, $A \in \mathbb{M}$, then G is $\mathbb{M}$-generic iff:

(i) $\Pi A \notin G$ implies $A \not\subset G$.

We write: $A' = \{ -a: \ a \in A \}$.

Since G is an ultrafilter, we have:

$$\Pi A \notin G \ \text{ iff } \ -(\Pi A) \in G \ \text{ iff } \ \Sigma A' \in G \ .$$

Similarly,

$$A \not\subset G \ \text{ iff } \ (\exists a \in A)(a \notin G) \ \text{ iff}$$

$$(\exists a \in A)(-a \in G) \ \text{ iff } \ (\exists a \in A')(a \in G) \ .$$

Hence (i) is equivalent to:

(ii) $\Sigma A' \in G$ implies $(\exists a \in A')(a \in G)$.

Given A', by simply taking the supremum of the a's which satisfy (ii), we arrive at a unique $b \in A' \cap G$, with no essential change in A'.

Definition 0.34 Let G be an $\mathbb{M}$-generic ultrafilter on B. By transfinite induction on $\rho(x)$, we define the interpretation functor i_G of $\mathbb{M}^B$ by G:

(a) $i_G(0) = 0$.

(b) $i_G(x) = \{ i_G(y) : x(y) \in G \}$.

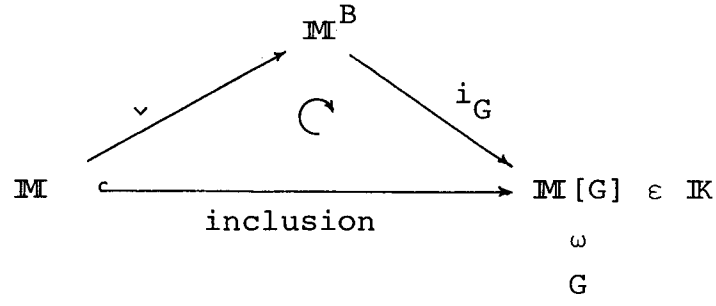
We usually write 'i' for i_G , dropping reference to G when it is understood.

Definition 0.35 $\mathbb{M}[G] = \{ i(x) : x \in \mathbb{M}^B \}$ is called the generic extension of $\mathbb{M}$ by G, where G is an $\mathbb{M}$-generic ultrafilter on B.

As seen below, the notation $\mathbb{M}[G]$ suggests (as in field theory) what it should.

Theorem 0.36 $\mathbb{M}[G]$ is the least model of ZFC extending $\mathbb{M}$ and containing $\{G\}$.

The situation is summarized in the commutative diagram below:



At this point we will only show part of 0.36, i.e. that $\mathbb{M}[G]$ is a model of ZFC extending $\mathbb{M}$ and containing G as an element. This will be done in the next series of lemmas, ending with 0.40 . The minimality of $\mathbb{M}[G]$ is essentially a consequence of our Lemma 5.5 . [9], p. 59 gives another proof using absoluteness, which would be almost unchanged in our system of models.

Lemma 0.37 For each $x \in \mathbb{M}$, $i(\check{x}) = x$.

Proof: Induction on $\rho(\check{x})$:

$i(\check{0}) = i(0) = 0$.

$i(\check{x}) = \{ i(\check{y}) : \check{x}(\check{y}) \in G \}$

$= \{ y : \check{x}(\check{y}) \in G \}$, as $\rho(\check{y}) < \rho(\check{x})$,

$= \{ y : y \in x \}$, as $\check{x}(\check{y}) = 1 \in G$,

$= x$.

We define the canonical generic ultrafilter $\underline{G}$ on $\mathbb{M}^B$:

$dom(\underline{G}) = \{ \check{x} : x \in B \}$; $\underline{G}(\check{x}) = x$, $\forall x \in B$.

$\underline{G}$ belongs to $\mathbb{M}^B$ by definition.

Lemma 0.38 $G \in \mathbb{M}[G]$.

Proof: $i(\underline{G}) = \{ i(\check{x}) : \underline{G}(\check{x}) \in G \}$

$= \{ i(\check{x}) : x \in G \}$

$= G$.

Suppose $\underline{x} \in \mathbb{M}^B$, $x \in \mathbb{M}[G]$, and $i(\underline{x}) = x$. Then we say that $\underline{x}$ is a name for x. For example, $\check{x}$ is a name for x. and $\underline{G}$ is a name for G.

Lemma 0.39 If $\underline{x}$, $\underline{y}$ are names for x, $y \in \mathbb{M}[G]$, respectively

then: $x \in y$ iff $[\![\underline{x} \in \underline{y}]\!] \in G$,

and $x = y$ iff $[\![\underline{x} = \underline{y}]\!] \in G$.

Proof: (a) Given that $x \in y$, we show $[\![\underline{x} \in \underline{y}]\!] \in G$.

If $x \in y$, then there exists $z_o \in dom(\underline{y})$ such

that $\underline{y}(z_o) \in G$ and $i(z_o) = x$. Proceeding by transfinite induction on $(\rho(\underline{x}), \rho(\underline{y}))$, we assume by induction hypothesis that $[\![z_o = \underline{x}]\!] \in G$, as $\rho(z_o)$ $\rho(y)$. Hence $\underline{y}(z_o) \cdot [\![z_o = \underline{x}]\!] \in G$, and since $[\![\underline{x} \in \underline{y}]\!] \geq \underline{y}(z_o) \cdot [\![z_o = \underline{x}]\!]$, we have that $[\![\underline{x} \in \underline{y}]\!] \in G$.

(b) For the converse of (a), see [9], p. 58 .

(c) Given that $x = y$, we show $[\![\underline{x} = \underline{y}]\!] \in G$. Since

$$i(\underline{y}) = \{ i(z) : \underline{y}(z) \in G \}$$
$$= \{ i(z) : \underline{x}(z) \in G \} = i(\underline{x}) ,$$

we have:

$$\forall z \in \mathbb{M}^B, \ \underline{y}(z) \in G \text{ iff } \underline{x}(z) \in G .$$

Hence, for all $z \in \mathrm{dom}(\underline{x})$:

(i) $(\underline{x}(z) \notin G) \rightarrow (-\underline{x}(z) \in G)$
$\rightarrow (-\underline{x}(z) + [\![z \in \underline{y}]\!] \in G)$,

(ii) $(\underline{x}(z) \in G) \rightarrow (i(z) \in y)$
$\rightarrow ([\![z \in \underline{y}]\!] \in G)$
$(\text{as } \rho(z) < \rho(\underline{x}))$
$\rightarrow (-\underline{x}(z) + [\![z \in \underline{y}]\!] \in G)$.

Similarly, for all $z \in \mathrm{dom}(\underline{y})$:

(iii) $(\underline{y}(z) \notin G) \rightarrow (-\underline{y}(z) + [\![z \in \underline{x}]\!] \in G)$,

(iv) $(\underline{y}(z) \in G) \rightarrow (-\underline{y}(z) + [\![z \in \underline{x}]\!] \in G)$.

From the definition of $[\![\underline{x} = \underline{y}]\!]$, (i) - (iv) above, and the genericity of G (0.32), it follows that $[\![\underline{x} = \underline{y}]\!] \in G$.

(d) The converse of (c).

Given that $[\![\underline{x} = \underline{y}]\!] \in G$; for all $z \in \mathrm{dom}(\underline{x})$: $\underline{x}(z) \in G$ implies $[\![z \in \underline{y}]\!] \in G$, because G is an ultrafilter and $-\underline{x}(z) + [\![z \in \underline{y}]\!] \in G$.

So, if $\underline{x}(z) \in G$ (i.e. $i(z) \in x$) then $[\![z \in \underline{y}]\!] \in G$, and by induction hypothesis $i(z) \in y$, as $\rho(z) < \rho(x)$.

For the same reason, for all $z \in \mathrm{dom}(\underline{y})$: $\underline{y}(z) \in G$ (i.e. $i(z) \in y$) implies $[\![z \in \underline{x}]\!] \in G$, which by induction hypothesis implies $i(z) \in x$, as $\rho(z) < \rho(\underline{y})$.

We conclude that for all $z \in \mathrm{dom}(\underline{x}) \cup \mathrm{dom}(y)$ such that $\underline{x}(z) \in G$ and $\underline{y}(z) \in G$:

$$i(z) \in x \quad \text{iff} \quad i(z) \in y \ .$$

Hence, $x = y$.

Lemma 0.40 If $\underline{x}_1, \ldots, \underline{x}_n \in \mathbb{M}^B$ are names for $x_1, \ldots, x_n \in \mathbb{M}[G]$, and ϕ is a formula of set theory, then: $\mathbb{M}[G] \models \phi(x_1, \ldots, x_n)$ iff $[\![\phi(x_1, \ldots, x_n)]\!] \in G$.

Proof: This follows from the previous lemma by induction on the complexity of ϕ .

0.16 and 0.40 prove that $\mathbb{M}[G]$ is a model of ZFC. 0.39 implies that $\mathbb{M}[G]$ is standard, and it is not hard to show (using 0.40) that $\mathbb{M}[G]$ is transitive. Because i_G is defined by transfinite induction over $\Theta_{\mathbb{M}} \in \mathbb{K}$, we have

$i_G \in \mathbb{K}$. The Axiom of Replacement thus implies that $\mathbb{M}[G] \in \mathbb{K}$. The other details of the diagram on p. 33 follow from 0.37 and 0.38 .

Since $\mathbb{K}$ is transitive, our diagram seems to indicate that $G \in \mathbb{K}$. All along however, our tacit assumption has been that $\mathbb{M}$-generic ultrafilters do exist. To make such an assumption is equivalent to adding a very strong axiom to ZFC. Martin's Axiom, which is a weaker and more reasonable form of this assumption, may be invoked for this purpose (see [13]), but we would like to avoid any further additions to our foundations. We shall now show that the foundations laid at the beginning of this section are enough to provide the existence of an $\mathbb{M}$-generic ultrafilter G in $\mathbb{K}$.

An ultrafilter H_a is said to be <u>principal</u> if it is of the form $\{ b \in B : b \geq a \}$. Suppose that H_a is $\mathbb{M}$-generic. Then 0.33 implies that there are no partitions of $a \in B$ in $\mathbb{M}$. Within $\mathbb{M}$, a is an <u>atom</u>, or minimal non-zero member of B (even if B is in reality <u>nonatomic</u>, i.e. having no atoms). A similar argument shows that if an $\mathbb{M}$-generic ultrafilter G belongs to $\mathbb{M}$, then $\cap G$ is an atom of B. Since we do not restrict our attention to Boolean algebras having atoms in the sequel, we cannot rule out the possibility that each $\mathbb{M}$-generic ultrafilter G on B is non-principal and that $G \notin \mathbb{M}$. G, if it exists at all, may be highly non-constructive.

Definition 0.41 Let F be a family of subsets of a Boolean algebra B. A filter U on B is F-complete if for each $E \in F$ such that $\Pi E \in B$:

$$E \subset U \text{ implies } \Pi E \in U .$$

There is an obvious redundancy in the above definition if B is complete.

Two elements $a, b \in B$ are compatible if $a \cdot b \neq 0$. A pairwise compatible subset of B is one whose members are compatible with each other. It is apparent that filters are pairwise compatible, and that each subset of a pairwise compatible set is pairwise compatible. Below, we have a trivial extension lemma which will be helpful in constructing and extending pairwise compatible sets.

Lemma 0.42 If $H \subset B$ is pairwise compatible, then for each $b \in B$, either $H \cup \{b\}$ or $H \cup \{-b\}$ is pairwise compatible.

If H is a pairwise compatible subset of B, then we can enlarge it to the following filter:

$$J = \{ z \in B : z \geq \prod_{k < n} a_k , a_k \in H \} .$$

By maximalization, we may further extend J to an ultrafilter. This is expressed in the well-known Ultrafilter Theorem below.

Theorem 0.43 Each pairwise compatible subset of B is contained in some ultrafilter on B.

Since the above theorem follows from AC, it holds in $\mathbb{K}$, and ultrafilters are plentiful in $\mathbb{K}$. The difficulty of the existence problem we are considering must lie in the property of genericity.

Theorem 0.44 Given a countable family F of subsets of a Boolean algebra B, and an F-complete filter G_o on B, there exists an F-complete ultrafilter G on B extending G_o.

Proof: Let $F^* = \{ A \in F : -\Pi A \notin G_o \}$. Since F is countable, we may enumerate $F^* = \{ A_o, \ldots, A_n, \ldots \}$, and define $P_n = \Pi(A_n)$. By definition, $P_n \neq 0$, for each n. F^* has the following property:

$$\forall n, \forall a \in G_o, \ a \cdot p_n \neq 0 ,$$

since $a \cdot p_n = 0$ implies $a \leq -p_n$, but $-p_n \notin G_o$. Because of this, we know that $H_o = G_o \cup \{p_o\}$ is pairwise compatible. Having defined H_n and assuming that it is pairwise compatible, we use 0.42 to define H_{n+1} as $H_n \cup \{p_n\}$, if this is pairwise compatible; or $H_n \cup \{-p_n\}$ otherwise. $H = \bigcup_n H_n$ is thus a pairwise compatible set containing G_o. We extend H to an ultrafilter G by way of 0.43.

G is clearly G-complete.

Theorem 0.44 is an extended form of the Rasiowa-Sikorski Lemma (c.f. [25], p. 29 et seq.). While the usual form of this result is sufficient for our present purpose, we will need the stronger hypothesis of 0.44 in Section 4. It may seem natural to release F from the countability restriction, but this cannot be done without making further restrictions on B (e.g. Martin's Axiom [9], p. 99). It now becomes apparent just why we have picked a countable ground model $\mathbb{M}$.

Corollary 0.45 If $\mathbb{M} \subset \mathbb{K}$ is a countable model of ZFC, and B is a complete Boolean algebra in $\mathbb{M}$, then $\mathbb{M}$-generic ultrafilters on B exist in $\mathbb{K}$.

Proof: G is $\mathbb{M}$-generic iff G is $\mathbb{P}^{\mathbb{M}}(B)$-complete. Since $\mathbb{P}^{\mathbb{M}}(B)$ is countable, Theorem 0.44 yields the result.

This concludes our study of $\mathbb{M}^B$ and $\mathbb{M}[G]$ as abstract objects. In the sequel we will work with particular examples of these constructions, and virtually all of the material in this section will find application.

By now we are acquainted with the use of a generic ultrafilter as a type of decision process capable of evaluating any set theoretic statement regarding its validity in $\mathbb{M}[G]$.

In practice, many of the properties of $\mathbb{M}[G]$ will be demonstrated by Boolean-valued calculations involving generic ultrafilters. We will also find in the sequel that generic ultrafilters can be used to reflect a number of algebraic and analytical notions. They can be used in certain situations to provide direct answers to purely non-logical problems.

Section 1 : The Lebesgue Measure Problem and the Axiom of Choice

The Axioms of Pairing, Unions, and Infinity ensure that all standard transitive models of ZF contain the set of natural numbers ω (see [24],p.129). Well-known methods (e.g. Dedekind cuts) of generating the rational and real numbers are easily duplicated in these models (see [16],p.271), however the set of Dedekind cuts generated from ω may vary from model to model. It is possible thus to express statements of real analysis as formulas in ZF. As the concept of Lebesgue measure is definable in this context, we may view the Lebesgue Measure Problem from the vantage point of ZF by asking whether models of ZF exist which satisfy the statement:

LM All sets of reals are Lebesgue measureable.

In the event that a model $\mathbb{E}$ does exist such that $\mathbb{E} \models$ LM , we immediately conclude that $\mathbb{E} \not\models$ AC , for AC$\rightarrow\neg$LM. For an analyst, our model $\mathbb{E}$ might be unattractive as there is no guarantee that certain basic prerequisites of analysis hold on $\mathbb{E}$. Not only are the non-constructive principles , such as the Hahn-Banach Theorem, derived from some form of AC, but so are some commonplace facts like the regularity of ω_1 (countable unions of countable sets are countable). Moreover, the presence of AC in some (possibly weaker) form is necessary to provide acceptable properties for

Lebesgue measure in $\mathbb{E}$.

There are two main characterizations of the basic topological notions of metric spaces:

(a) ε-δ criteria .

(b) sequential limit criteria .

The definitions of a limit point and the closure of a set, as well as continuity of functions have obvious versions in (a) and (b). If our model under discussion satisfies the Heine-Borel Theorem (which may not be the case; see 1.2), then versions in (a) and (b) exist for the definition of compactness of a set. Using AC we can prove the equivalence of both versions of all the definitions mentioned above. What happens if our model does not satisfy AC ?

Proposition 1.1 For each of the following notions there is a model of ZF not satisfying AC in which versions (a) and (b) of said notion are not equivalent.

(i) limit point of a set.

(ii) closure of a set.

(iii) continuity of a function.

(iv) compactness of a set.

Proposition 1.2 For each of the following statements there is a model of ZF not satisfying AC in which said statement holds:

(i) ω_1 is singular.

(ii) the set of reals $\mathbb{R}$ is the union of countably many countable sets.

(iii) there is an infinite set of reals having no countable subset.

(iv) there is a subspace of the reals which is not separable.

(v) the Heine-Borel Theorem is false.

See [10], pp. 141 - 4 for a demonstration of the above results.

As far as the needs of the analysis and topology of the reals are concerned, the appropriate weakening of AC is the following statement, known as the Countable Axiom of Choice.

$\underline{AC_\omega}$ Every countable collection of nonempty sets has a choice function.

If $\mathbb{E} \models ZF + AC_\omega$ then versions (a) and (b) of each of (i) - (iv) in Proposition 1.1 are equivalent in $\mathbb{E}$. Also, none of the statements (i) - (v) in Proposition 1.2 can hold in any model of AC_ω. In fact, we have the following result.

Proposition 1.3 If $\mathbb{E} \models ZF + AC_\omega$ then each of the following statements must hold in $\mathbb{E}$:

(a) the Heine-Borel Theorem

(b) every subspace of a separable metric space is separable.

(c) Lebesgue measure exists and is countably additive.

(d) the family of First Category sets is countable additive.

Proof: See [10], p. 21 - 2, p. 29 .

The Baire Category Theorem does not depend at all on AC.

AC_ω does not imply the full strength of the general Hahn-Banach Theorem; the McAloon model of Section 6 verifies this (see [23], pp. 2 - 3). However, AC_ω easily yields the Hahn-Banach Theorem for separable Banach spaces.

Since the family of Borel sets will have a special significance in later constructions, it is worthwhile to examine the relation it has with AC. There are two usual definitions for this family, $\mathbb{B}$.

Definition 1.4 (a) $\mathbb{B}$ is the smallest σ-algebra of sets of reals containing the open sets.

(b) $\mathbb{B}$ is the collection of sets hyperarithmetic in some real (see [21], p. 179); or, by a theorem of Souslin:

$\mathbb{B}$ is the collection of Δ^1_1 sets in the Projective Hierarchy (see [21], p. 185, cf. [11], v.1, p. 453, et seq.).

As (b) is the type of definition we will rely on, we need AC_ω at least, in order to show that $\mathbb{B}$ is closed under countable unions (we must pick a code for each of countably many Borel sets). Without AC, (a) and (b) above are not generally equivalent; however AC_ω is strong enough to guarantee the equivalence of (a) and (b), and show (see [10], P. 22):

(i) $\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha$ where $\mathbb{B}_0$ the open sets, and $\mathbb{B}_\alpha$ is the set of all countable unions of elements of $\bigcup_{\gamma<\alpha} \mathbb{B}$ and their complements.

(ii) $\mathbb{B}_\alpha \subsetneq \mathbb{B}_{\alpha+1}$, $\forall\alpha<\omega_1$.

It is evident that AC_ω is a necessary and possibly adequate form of AC as far as elementary analysis and topology are concerned. The question now remains as to whether models of ZF exist which satisfy both LM and AC_ω . Solovay has shown that under a certain hypothesis the construction of a model of ZF satisfying LM + AC_ω is possible, using the techniques of **Section 0** . Solovay's construction is the subject of the sequel.

Section 2 : Some Model Theoretic Properties of Lebesgue Measure

We shall define some types of set theoretic formula from two syntactic hierarchies and develop some of their model theoretic properties. Our first source of notation is Kleene's analytical hierarchy (see [21], p. 173 et seq.). Strictly speaking, this is a classification in recursion theory of formulas of second-order arithmetic. There is a natural translation of these formulas into the first-order language of set theory, however.

Definition 2.1 A formula of set theory ϕ is Π^1_1 if: $\phi \leftrightarrow (\forall x_1, \ldots, x_n \in \omega^\omega) \psi$, where ψ is a formula whose only quantifiers are of the form $\forall y \in \omega$, or $\exists y \in \omega$.

The following is a syntactic classification of the formulas of ZF, known as the Lévy hierarchy (see [12]).

Definition 2.2

(a) A formula is $\Sigma_0 = \Pi_0$ if it is bounded (see p. 21).

(b) ϕ is Σ_{n+1} if $\phi = \exists x \psi$ where ψ is Π_n .

(c) ϕ is Π_{n+1} if $\phi = \forall x \psi$ where ψ is Σ_n .

(d) ϕ is Σ_n^{ZF}, resp. Π_n^{ZF} if $ZF \vdash \phi \leftrightarrow \psi$ where ψ is Σ_n , resp. Π_n .

(e) ϕ is Δ_n , resp. Δ_n^{ZF} if ϕ is both Σ_n and Π_n , resp. Δ_n^{ZF} and Π_n^{ZF} .

Let $\mathbb{E}$, $\mathbb{F}$ be standard models of ZF with universes E, F respectively. It is obvious that $\mathbb{E}$ is a submodel (see [1], p. 21) of $\mathbb{F}$ if $E \subset F$. For such standard models satisfying this condition we write $\mathbb{E} \subset \mathbb{F}$. Let ϕ be a formula of ZF. An assignment f of ϕ in $\mathbb{E}$ is a mapping of the free variables of ϕ into E; we write $\phi[f]$ for the substitution of f(x) for each free variable x occuring in ϕ.

We now define the fundamental model theoretic concept of this section, the notion of absoluteness between standard transitive models of ZF. There are many notions of absoluteness in the literature. Unlike the absoluteness of Goedel (see [7]) or of Cohen (see [3]), the definition we use (due to Shoenfield; see [4], pp. 85, 106) does not employ relativization of formulas to transitive classes.

Definition 2.3 Let $\mathbb{E}$, $\mathbb{F}$ be standard transitive models of ZF, and let $\mathbb{E} \subset \mathbb{F}$.

(a) A formula ϕ is absolute between $\mathbb{E}$ and $\mathbb{F}$ iff for all assignments f of ϕ in $\mathbb{E}$:

$$\mathbb{E} \models \phi[f] \quad \text{iff} \quad \mathbb{F} \models \phi[f] .$$

(b) A term t is absolute between $\mathbb{E}$ and $\mathbb{F}$ iff the formula (x = t) is absolute

between $\mathbb{E}$ and $\mathbb{F}$, and x does not occur in t.

(c) An operation D is absolute between $\mathbb{E}$ and $\mathbb{F}$ iff the term $D(u)$ is absolute between $\mathbb{E}$ and $\mathbb{F}$ for each $u \in dom(D)$.

For the following sequence of lemmas let $\mathbb{E}$, $\mathbb{F}$ be standard transitive models of ZF, and $\mathbb{E} \subset \mathbb{F}$.

Lemma 2.4 (a) Atomic formulas are absolute between $\mathbb{E}$ and $\mathbb{F}$.

(b) If ϕ and ψ are absolute between $\mathbb{E}$ and $\mathbb{F}$ then so are $\neg\phi$, $\phi \vee \psi$.

Proof: (a) and (b) follow from the definitions of submodel and $\models$, respectively.

If every formula is absolute between $\mathbb{E}$ and $\mathbb{F}$, then $\mathbb{F}$ is obviously an elementary extension of $\mathbb{E}$ (see [1], p. 82). Since the models we will build are not elementary extensions of the ground model, the problem of determining whether a formula is absolute in this context is non-trivial. Our purpose is served by a partial solution to the problem.

Lemma 2.5 (a) Bounded (Σ_0) formulas are absolute between $\mathbb{E}$ and $\mathbb{F}$.

(b) Δ_1^{ZF} formulas are absolute between $\mathbb{E}$ and $\mathbb{F}$..

Proof: (a) Either by induction on complexity ([1], p. 478), or by way of Skolem functions ([4], p. 87).

(b) If $\phi(x,y)$ is absolute between $\mathbb{E}$ and $\mathbb{F}$, then $\exists x\ \phi(x,\vec{y})$ is preserved under extension from $\mathbb{E}$ to $\mathbb{F}$, by definition of $\models$.

Hence Σ_1^{ZF} formulas are preserved under the above extension (i.e. they hold in $\mathbb{F}$ if they hold in $\mathbb{E}$).

Similarly, $\forall x\ \phi(x,\vec{y})$ is preserved under restriction from $\mathbb{F}$ to $\mathbb{E}$. Hence Π_1^{ZF} formulas are preserved under the above restriction (i.e. they hold in $\mathbb{E}$ if they hold in $\mathbb{F}$).

It follows from (a) above and Lemma 2.4 that Δ_1^{ZF} formulas are absolute between $\mathbb{E}$ and $\mathbb{F}$.

Most of the fundamental concepts of set theory are expressible as Δ_1^{ZF} formulas, terms, and operators. These include: $x \subset y$; $\{x,y\}$; (x,y) ; (x is an ordinal) ; (successor of x) ; 0 , 1 , 2 , ... ; $(x \in \omega)$; $(x = \omega)$; the ordinal arithmetic operations ; rank of x ; functionhood ; range(x) ; domain(x) ; union(x) (see [4], p. 81, et seq.). These concepts are therefore absolute between $\mathbb{E}$ and $\mathbb{F}$. There are two notable exceptions: neither $\mathbb{P}(\omega)$ nor

$\{ x : \text{rank}(x) = \alpha \}$ are preserved under extensions. Since both notions are Π_1^{ZF}, we cannot expect the higher orders of the Levy hierarchy to add much to our knowledge of absolutness. Lemma 2.5 seems to be the best possible result of its type; fortunately it is enough for our needs.

The following result due to Mostowski and Shoenfield (see [20]) establishes an important connection between the analytic hierarchy and absoluteness.

Theorem 2.6 In any transitive model of ZF: Π_1^1 formulas are equivalent to Δ_1^{ZF} formulas.

Proof: See [4], p. 160.

Corollary 2.7 Π_1^1 formulas are absolute between $\mathbb{E}$ and $\mathbb{F}$.

The assumption is now made that $\mathbb{E} \models AC_\omega$.

Borel sets have a central role in the construction ahead. It is imperative that we have some method of naming and refering to Borel sets within the language of set theory. The method we use is that of Gödel-numbering or 'coding' the $2^{\aleph_0}$ Borel sets with number theoretic functions. Of the many possible recursive coding procedures the following, due to Solovay, is simple and adequate.

Let $\{r_i\}$ be an arithmetic enumeration of $\underline{Q}$ (the rationals). Let J be the following pairing function:

$$J(a,b) = 2^a(2b + 1) .$$

It is easily verified that J is one-to-one from ω^2 onto $\omega \setminus \{0\}$, and is recursive.

Definition 2.8 (a) α codes $[r_i, r_j]$ if:

$$\alpha(0) \equiv 0 \pmod 3 ,$$

$$\alpha(1) = i ,$$

$$\alpha(2) = j .$$

(b) Suppose α_i codes B_i , i = 0, 1, ... then α codes $\bigcup_i B_i$ if:

$$\alpha(0) \equiv 1 \pmod 3 \text{ and}$$

$$\alpha(J(a,b)) = \alpha_a(b) .$$

(c) Suppose β codes B, $\alpha(0) \equiv 2 \pmod 3$, and $\alpha(n+1) = \beta(n)$, then α codes $\mathbb{R} \setminus B$ (the complement of B) .

(d) α codes B only as required by the above cases.

Lemma 2.9 The following holds in $\mathbb{E}$:

(a) Every set coded by $\alpha \in \omega^\omega$ is Borel.

(b) Every Borel set is coded by some $\alpha \in \omega^\omega$.

(c) If α codes A and α codes B, then A = B .

Each code gives a sequential 'recipe' for a Borel set. If a Borel set A with code α is used in the construction of a Borel set B then the resulting code β for B will contain α as a subsequence. The correspondence between Borel sets and their codes is clearly not one-to-one.

The recursive definition of the codes ensures that they are definable by a set theoretic statement. We will continue to use recursiveness for this purpose.

If α codes a Borel set B we will use the notation B_α for B. If, furthermore $\alpha \in \mathbb{E}$, and $B \in \mathbb{E}$ then we write $B_\alpha^{\mathbb{E}}$ for B.

Let $\{s_n\}$ be a non-repetitive recursive enumeration of the finite sequences of positive integers satisfying:

(a) $s_0 = ( )$.

(b) If s_m is an initial segment of s_n, then $m \leq n$.

For $n > 0$, s_n is nonempty and has length k, say. Let s_{n*} be the initial segment of s_n having length k-1. Let the final segment of s_n be n_k. Then $n^*<n$ and $s_n = s_{n*}^\frown (n_k)$[1].

Solovay (in [23]) constructs a code-generating function $\Phi(\alpha,n)$ such that if α is a code, then for each $n\in\omega$ $\Phi(\alpha,n)$ is a code.

<u>Definition 2:10</u> $\Phi(\alpha,n)(i) =$

(a) $\alpha(i)$; $n = 0$.

(b) 0 ; $n > 0$, $\Phi(\alpha,n^*)(0) \equiv 0 \pmod 3$.

(c) $\Phi(\alpha,n^*)(J(n_k,i))$; $n > 0$, $\Phi(\alpha,n^*)(0) \equiv 1 \pmod 3$.

(d) $\Phi(\alpha,n^*)(i+1)$; $n > 0$,

(e) $\Phi(\alpha,n^*)(0) \equiv 2 \pmod 3$.

The purpose of this function is to 'decode' α, yield-

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[1] '$\frown$' denotes the concatenation of finite sequences.

ing the codes of the component Borel sets from which B_α may be constructed. Let $\beta \in \omega^\omega$. We define $\bar{\beta} \in \omega^\omega$ via the finite sequence $s_{\bar{\beta}(n)} = (\beta(0), \ldots, \beta(n-1))$. The following Lemmas are due to Solovay.

Lemma 2.11 Define $\phi_1(\alpha)$ as $(\forall \beta \in \omega^\omega)(\exists n \in \omega)[\Phi(\alpha, \bar{\beta}(n)) = 0]$. Then: $\mathbb{E} \models \phi_1(\alpha) \leftrightarrow$ (α codes a Borel set) .

If α codes a Borel set and $x \in \mathbb{R}$, define:

$$\gamma(i) = \begin{cases} 1 \ ; \ x \in B_{\Phi(\alpha,i)} \ . \\ 0 \ ; \ \text{otherwise} \ . \end{cases}$$

The previous lemma guarantees the existence of $B_{\Phi(\alpha,i)}$.

Lemma 2.12 There is an arithmetic formula (see [22], p. 160) $\phi_4(\alpha,\beta,x)$ such that $\phi_4(\alpha,\beta,x) \leftrightarrow \beta = \gamma$.

Lemma 2.13 Let $x \in \mathbb{R}$. There are Π^1_1 formulas $\phi_2(\alpha,x)$, $\phi_3(\alpha,x)$ such that:

(a) $\mathbb{E} \models \phi_2(\alpha,x) \leftrightarrow$ (α codes a Borel set and $x \in B_\alpha$) .

(b) $\mathbb{E} \models \phi_3(\alpha,x) \leftrightarrow$ (α codes a Borel set and $x \notin B_\alpha$) .

Proof: (a) Define $\phi_2(\alpha,x)$ as the following:

$(\forall \beta \in \omega^\omega)(\phi_4(\alpha,\beta,x) \rightarrow \beta(0) = 1) \ \& \ \phi_1(\alpha)$.

$\phi_2(\alpha,x)$ iff $\gamma(0) = 1$, by Lemma 10, and $\gamma(0) = 1$ iff $x \in B_{\Phi(\alpha,0)}$ iff $x \in B_\alpha$, as $\Phi(\alpha,0) = \alpha$.

(b) Define $\phi_3(\alpha,x)$ as:

$(\forall \beta \varepsilon \omega^{\omega})(\phi_4(\alpha,\beta,x) \rightarrow \beta(0) = 0) \ \& \ \phi_1(\alpha)$.

$\phi_3(\alpha,x)$ iff $\gamma(0) = 0$, by Lemma 10, and $\gamma(0) = 0$ iff $x \notin B_{\Phi(\alpha,0)}$ iff $x \varepsilon B_{\alpha}$.

Both of the above formulas are Π^1_1 .

Corollary 2.14 There are Π^1_1 formulas $\phi_5(\alpha,\beta)$ and $\phi_6(\alpha,\beta)$ such that:

(a) $\mathbb{E} \models \phi_5(\alpha,\beta) \leftrightarrow (B_{\alpha} \subset B_{\beta})$.

(b) $\mathbb{E} \models \phi_6(\alpha,\beta) \leftrightarrow (B_{\alpha} = B_{\beta})$.

Proof: Define $\phi_5(\alpha,\beta)$ as $\phi_1(\alpha) \ \& \ \phi_1(\beta) \ \&$ $(\forall x \varepsilon \mathbb{R})(\phi_3(\alpha,x) \vee \phi_2(\beta,x))$.

Define $\phi_6(\alpha,\beta)$ as $\phi_5(\alpha,\beta) \ \& \ \phi_5(\beta,\alpha)$.

Quantifying over the reals is permissible in the definition of $\phi_5(\alpha,\beta)$. We could code each real by its binary expansion, thus ensuring that $\phi_5(\alpha,\beta)$ is Π^1_1 .

If $\phi(x)$ is Π^1_1 , and α is a code belonging to $\mathbb{E} \subset \mathbb{F}$ then $\phi(\alpha)$ is absolute between $\mathbb{E}$ and $\mathbb{F}$.

Affixing subscripts and superscripts (e.g. $\mathbb{R}_{\mathbb{E}}$) to emphasize that the construction of a defined term is carried out within a specified model, we summarize the above results in the following theorem. From now on we make the additional assumption that $\mathbb{F} \models AC_{\omega}$.

Theorem 2:15 For $\alpha,\beta \varepsilon (\omega^{\omega})_{\mathbb{E}}$ and $x \varepsilon \mathbb{R}_{\mathbb{E}}$ the following notions are equivalent in $\mathbb{E}$ to formulas

absolute between $\mathbb{E}$ and $\mathbb{F}$:

(a) α codes a Borel set.

(b) α codes a Borel set and $x \in B$.

(c) α, β code Borel sets and $B_\alpha \subset B_\beta$.

(d) α, β code Borel sets and $B_\alpha = B_\beta$.

We may define a one-to-one map # by: $\#B_\alpha^{\mathbb{E}} = B_\alpha^{\mathbb{F}}$. Theorem 2.15 indicates that # maps the Borel sets in $\mathbb{E}$ onto a subfamily of the Borel sets in $\mathbb{F}$, which we call the Borel sets rational over $\mathbb{E}$.

Definition 2.16 (a) $B \in \mathbb{F}$ is rational over $\mathbb{E}$ if $B = B_\alpha^{\mathbb{F}}$ for some code $\alpha \in (\omega^\omega)_{\mathbb{E}}$.

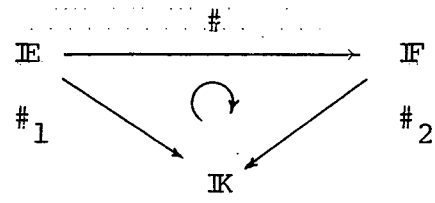
(b) If $\{B_i\}$ is a sequence of Borel sets in $\mathbb{F}$, $\{B_i\}$ is rational over $\mathbb{E}$ if there is a sequence $\{\alpha_i\}$ in $\mathbb{E}$ of codes α_i in $\mathbb{E}$ such that $B_i = B_{\alpha_i}^{\mathbb{F}}$, for each $i \in \omega$.

Solovay points out a redundancy in 2.16(b), namely that if the α_i belong to $\mathbb{E}$, then by AC_ω the sequence of these codes automatically belongs to $\mathbb{E}$.

From Theorem 2.15 we conclude:

Corollary 2.17 : For $B \in \mathbb{F}$ rational over $\mathbb{E}$, $B = \#(B \cap \mathbb{R}_{\mathbb{E}})$

is natural in that the following diagram commutes for Borel sets in $\mathbb{E}$:



where $\#_1$ and $\#_2$ are defined as above, but between $\mathbb{E}$ and $\mathbb{K}$, and $\mathbb{F}$ and $\mathbb{K}$, respectively.

Definition 2.18 $\phi(x,B_\alpha)$ is #-absolute if for all assignments f in $\mathbb{E}$:

$$\mathbb{E} \models \phi[f]\ (B_\alpha) \quad \text{iff} \quad \mathbb{F} \models \phi[f]\ (\#B_\alpha)\ .$$

(Similarly for terms and operations.)

Lemma 2.19 (a) Boolean set operations are #-absolute.

(b) Infinite Boolean set operations are #-absolute.

(c) Let A,B be Borel sets in $\mathbb{E}$, then $A \subset B$ and $A = B$ are #-absolute relations.

Proof: By Boolean set operations, we mean those on the field of sets of reals. Let $\{B_i\}$ be a sequence of Borel sets in $\mathbb{E}$ with codes α_i, then γ defined by: $\gamma(0) \equiv 2 \pmod 3$; $\gamma(1) \equiv 1 \pmod 3$; $\gamma(J(i,0)+1) \equiv 2 \pmod 3$; $\gamma(J(i,j)+1) = \alpha_i(j-1)$, for $j \geq 1$; is a code for both $\bigcap_i B_i$ in $\mathbb{E}$, and $\bigcap_i \#B_i$ in $\mathbb{F}$. By definition: $\# \bigcap_i B_i = \bigcap_i \#B_i$, and (b) follows for the case of infinite intersection. Codes for complementation and infinite unions are covered in Definition 2.8 (b) - (c).

Thus (b). (a) follows from (b). Theorem 2.15 (c) - (d) imply (c).

Of the numerous topological notions that are #-absolute, the two most basic are all we need here.

Lemma 2.20 (a) (B is open) is #-absolute.

(b) (B is closed) is #-absolute.

Proof: Let $\{[r_{i_k}, r_{j_k}]\}_k$ comprise the closed rational-endpoint intervals containing B. Define γ as:

$\gamma(0) \equiv 2 \pmod 3$; $\gamma(1) \equiv 1 \pmod 3$;

$\gamma(J(k,0)+1) \equiv 2 \pmod 3$; $\gamma(J(k,1)+1) \equiv 0 \pmod 3$;

$\gamma(J(k,2)+1) = i_k$; $\gamma(J(k,3)+1) = j_k$;

then γ codes the closure of B. B is closed if $B = \bar{B} = B_\gamma$. (b) follows from Lemma 2.19 . B is open if $\mathbb{R} \setminus B$ is closed. (a) follows from (b).

With similar arguments ([23], p. 30) we can show that the intervals are rational over $\mathbb{E}$.

Lemma 2.21 Let $a, b \in \mathbb{R}_{\mathbb{E}}$, then $\#(a,b) = (a,b)$;

$\#[a,b] = [a,b]$; $\#\{a\} = \{a\}$.

We now have all the tools necessary to explore Lebesgue measure in this model theoretic setting. Our concept of Lebesgue measure μ is that of an outer measure

$$\mu^*(E) = \inf \{ \sum_{n=0}^{\infty} (b_n - a_n) : \bigcup_{n=0}^{\infty} (a_n, b_n) \supset E , b_n > a_n \}$$

restricted to the σ-algebra of measurable sets.

Lemma 2.22 If $\mathbb{E} \models AC_\omega$ then for each Lebesgue measurable set E in $\mathbb{E}$ there are sets $G \in \mathcal{G}_\delta^{\mathbb{E}}$ and N such that $\mathbb{E} \models E = G \setminus N \ \& \ \mu^*(N) = 0$.

Proof: We take E measurable to mean that for each $\varepsilon > 0$ there is an open set O_ε and a closed set F_ε such that $F_\varepsilon \subset E \subset O_\varepsilon$,and $\mu^*(O_\varepsilon \setminus F_\varepsilon) < \varepsilon$. By AC_ω we may pick such a pair $(O_{1/n}, F_{1/n})$ for each $n \in \omega$. Let $G = \bigcap_n O_{1/n}$. Then define $N = G \setminus E$. $\forall n \in \omega$, $\mu^*(N) < \mu^*(O_{1/n} \setminus F_{1/n}) < 1/n$.

We now look at various cases of #-absoluteness for Lebesgue measure.

Lemma 2.23 Let B be a $\mathcal{G}_\delta$-set in $\mathbb{E}$, then $\mu_{\mathbb{E}}(B) = \mu_{\mathbb{F}}(\#B)$.

Proof: The equality we wish to prove is expressed in $\mathbb{K}$, so our point of view is that of the diagram following Corollary 2.17 .

(a) Suppose $B = \bigcup_{m=1}^{n} (a_m, b_m)$, then from Lemmas 2.19, 2.20, 2.21, the definition of Lebesgue measure, and the naturality of #,
$\mu(B) = \sum_{m=1}^{n} (b_m - a_m)$ is #-absolute, and
$\mu_{\mathbb{E}}(B) = \mu_{\mathbb{F}}(\#B)$ in $\mathbb{K}$.

(b) Let B be any open set. Enumerate the sets

of form in (a) above: $\{A_n\}$. Then:

$$\mu(B) = \sup_n \{ \mu(A_n) : A_n \subset B \}$$

is sup of a countable collection of #-absolute reals, hence it is #-absolute.

(c) Let $B \in \mathcal{G}_\delta$. We need only look at the represectation: $B = \bigcap_n 0_n$; $0_{n+1} \subset 0_n$. By a well-known property of μ :

$$\mu(B) = \inf_n \mu(0_n)$$

which is #-absolute. The result follows.

The next theorem is the main result of this section.

Theorem 2.24 Let α code a Borel set, then ($\mu(B_\alpha)=0$) is #-absolute.

Proof: Our strategy is reminiscent of Lemma 2.5 .

(a) ($\mu(B_\alpha) = 0$) is preserved under the extension $\mathbb{E} \to \mathbb{F}$. For $\beta \in \omega^\omega$ let $\mathcal{g}_\delta(\beta)$ hold iff (β codes a $\mathcal{G}_\delta$-set). From Lemmas 2.19(b) and 2.20(a), we infer that $\mathcal{g}_\delta(\beta)$ is (#-)absolute. From Lemma 2.23: ($\mathcal{g}_\delta(\beta)$ & $\mu(B_\beta) = 0$) is #-absolute. Using Lemmas 2.4 and 2.19(c) we see that if ($\mu(B_\alpha) = 0$) $\leftrightarrow \exists \beta \in \omega^\omega (\mathcal{g}_\delta(\beta)$ & $B_\alpha \subset B_\beta$ & $\mu(B_\beta) = 0$) holds in $\mathbb{E}$, then by definition of $\Vdash$, it holds in $\mathbb{F}$. Thus $\forall \alpha \in \omega^\omega_{\mathbb{E}}$:

$\mu_{\mathbb{E}}(B_\alpha^{\mathbb{E}}) = 0 \rightarrow \mu_{\mathbb{F}}(B_\alpha^{\mathbb{F}}) = 0$, if α is a code.

(b) $(\mu(B_\alpha) = 0)$ is preserved under the restriction $\mathbb{F} \rightarrow \mathbb{E}$. For $\beta \in \omega^\omega$ let $\mathcal{f}(\beta)$ hold iff (β codes a closed set). Lemma 2.20 says $\mathcal{f}(\beta)$ is (#-)absolute. Since $\mathcal{f}(\beta) \rightarrow \mathcal{g}_\delta(\beta)$, Lemma 2.23 implies that $(\mathcal{f}(\beta) \ \& \ \mu(B_\beta) = 0)$ is #-absolute. If

$(\mu(B_\alpha) = 0)$

$\leftrightarrow \forall \beta \in \omega^\omega ((\mathcal{f}(\beta) \ \& \ B_\beta \subset B_\alpha) \rightarrow \mu(B_\beta) = 0)$

holds in $\mathbb{F}$ then by definition of $\models$, it holds in $\mathbb{E}$. Thus $\forall \alpha \in \omega^\omega$:

$\mu_{\mathbb{F}}(B_\alpha^{\mathbb{F}}) = 0 \rightarrow \mu_{\mathbb{E}}(B_\alpha^{\mathbb{E}}) = 0$, if α is a code.

The following lemma of Solovay is a consequence of the above theorem.

Corollary 2.25 (a) Let B be a Borel set in $\mathbb{E}$, then:

$$\mu_{\mathbb{E}}(B) = \mu_{\mathbb{F}}(\#B) .$$

(b) Let B be a Borel set in $\mathbb{F}$ rational over $\mathbb{E}$, then:

$$\mu_{\mathbb{F}}(B) = \mu_{\mathbb{E}}(B \cap \mathbb{R}_{\mathbb{E}}) .$$

Proof: (a) Consider B as a measurable set; by Lemma 2.22 $B = G \setminus N$, where G is $\mathcal{G}_\delta$ and

N is null (measure zero). Note that in this case N must be Borel also.

$$\mu_{\mathbb{F}}(\#B) = \mu_{\mathbb{F}}(\#G) - \mu_{\mathbb{F}}(\#N) \quad (\text{ by } 2.19)$$
$$= \mu_{\mathbb{E}}(G) - 0 \quad (\text{ by } 2.23,\ 2.24)$$
$$= \mu_{\mathbb{E}}(B) \quad .$$

(b) follows from (a) and Corollary 2.17 .

The proof above might lead us to conjecture that the result holds for B measurable. Our present development offers no ground for this claim, and the reason it doesn't underscores the whole rationale of this section. Because the codes range over the set ω^{ω}, we may express universal statements about Borel sets as Π^1_1 formulas. Even if we could apply codes to the measurable sets, their cardinality would be too large ($2^{2^{\aleph_0}}$) for this treatment. In such a case we have no information regarding absoluteness, even for simple formulas.

Section 3 : The Random Reals

We assume the existence of a standard transitive ground model $\mathbb{M}$ of ZFC which is a submodel of $\mathbb{K}$ countable in $\mathbb{K}$. All our subsequent model constructions will be built from $\mathbb{M}$. Let $\mathbb{R}$ denote the real numbers of $\mathbb{K}$, and $\mathbb{B}$ denote the σ-algebra of Borel sets of reals in $\mathbb{M}$. Let $\tilde{N}$ denote the σ-ideal of Lebesgue measure zero sets in $\mathbb{M}$.

Definition 3.1 $B^* = \mathbb{B}/\tilde{N}$ is the quotient algebra of equivalent classes of Borel sets [A] such that $B \in [A]$ ($B \equiv A \pmod{\tilde{N}}$) iff $\mu(A \Delta B) = 0$ (Δ is symmetric difference).

Proposition 3.2 B^* is a complete Boolean algebra satisfying the countable chain condition.

Proof: See [8], p. 67 . This proof involves AC, but $\mathbb{M} \models$ AC, thus B^* is $\mathbb{M}$-complete.

$\mathbb{R}_{\mathbb{M}} = \mathbb{R} \cap \mathbb{M}$ is countable, so most reals fall outside it. A large portion of these extraneous reals have a special property which is instrumental in the construction of our generic extensions of $\mathbb{M}$.

Definition 3.3 $\mathbb{R}^*_{\mathbb{E}}$ is the set of those reals belonging to no measure zero Borel set which is

rational over $\mathbb{E}$.

For $x \in \mathbb{R}^*_{\mathbb{E}}$ we say that the real number x is random over $\mathbb{E}$. Since $\mathbb{M}$ is fixed we will write $\mathbb{R}^*_{\mathbb{M}}$ as $\mathbb{R}^*$, and call the elements of $\mathbb{R}^*$ random reals. While it is clear that there are no random reals in $\mathbb{M}$ ($\mu(\{x\}) = 0$), the existence of random reals is immediate:

Lemma 3.4 $\mathbb{R}^*$ is a Borel set having measure zero complement.

Proof: As $\mathbb{M}$ is countable we may enumerate the Borel sets of measure zero, rational over $\mathbb{M}$. Their union is a Borel set of measure zero and equals $\mathbb{R} \setminus \mathbb{R}^*$.

Of course $\mathbb{R}^*$ is not rational over $\mathbb{M}$, and since $\underline{Q} \in \mathbb{M}$, each random real is irrational. Solovay ([23], pp. 4, 33) remarks that the random reals are characterized by 'random' binary expansions: for large n, any block of 2n consecutive entries in the expansion contain approximately n zeros and n ones.

Definition 3.5 Let G be an $\mathbb{M}$-generic ultrafilter on B^*: $x_G = \{ r : r \in \underline{Q}, [(r,\infty)] \in G \}$.

It is easy to show that x_G is a (left) Dedekind cut, and as such can be identified with the real: $\sup x_G$. x_G tells us a great deal about the structure of G.

We define a complexity function λ mapping the codes into the ordinals.

Definition 3.6 (a) If a code $\gamma \in \omega^\omega$ satisfies $\gamma(0) \equiv 0 \pmod 3$, define $\lambda(\gamma) = 0$.

(b) If $\gamma(0) \equiv 1 \pmod 3$, let $\gamma_i(j) = \gamma(J(i,j))$ and define $\lambda(\gamma) = \sup_i(\lambda(\gamma_i) + 1)$.

(c) If $\gamma(0) \equiv 2 \pmod 3$, define $\lambda(\gamma) = \lambda(\beta) + 1$, where $\beta(n) = \gamma(n+1)$.

Strictly speaking, the proof below is a transfinite induction on $\lambda_{\mathbb{M}} = \lambda \cap \mathbb{M}$, which maps $(\omega^\omega)_{\mathbb{M}}$ into $\Theta_{\mathbb{M}}$, but we omit the extra notation.

Lemma 3.7 Suppose B is a Borel set in $\mathbb{M}$ and G is an $\mathbb{M}$-generic ultrafilter on B*, then:

$$x_G \in \#B \quad \text{iff} \quad [B] \in G .$$

Proof: We set $\mathbb{E} = \mathbb{M}$, $\mathbb{F} = \mathbb{K}$, and define # as in the last section. Let $B = B_\gamma$ where $\gamma \in (\omega^\omega)_{\mathbb{M}}$.

(a) Suppose $\gamma(0) \equiv 0 \pmod 3$, then $B_\gamma = [r_{\gamma(1)}, r_{\gamma(2)}] = \#[r_{\gamma(1)}, r_{\gamma(2)}] = \#B_\gamma$.

$x_G \in \#B_\gamma \leftrightarrow r_{\gamma(1)} < x_G < r_{\gamma(2)} \leftrightarrow$ $[r_{\gamma(1)}, \infty) \in G \ \& \ (r_{\gamma(2)}, \infty) \notin G \leftrightarrow [B_\gamma] \in G$.

(b) Suppose $\gamma(0) \equiv 1 \pmod 3$, then $B_\gamma = \bigcup_i B_{\gamma_i}$ and $\#B_\gamma = \bigcup_i \#B_{\gamma_i}$ by Lemma 2.19. By induction

hypothesis and filter properties:

$x_G \in \#B_\gamma \leftrightarrow x_G \in \bigcup_i \#B_{\gamma_i} \leftrightarrow \exists i (x_G \in \#B_{\gamma_i})$

$\leftrightarrow \exists i ([B_{\gamma_i}] \in G)$ (as $\lambda(\gamma_i) < \lambda(\gamma)$)

$\leftrightarrow [B_\gamma] \in G$ (as $\sum_i [B_{\gamma_i}] = [B_\gamma]$)

(c) Suppose $\gamma(0) \equiv 2 \pmod 3$, then

$B_\gamma = \mathbb{R} \; B_\beta$, where $\beta(n) = \gamma(n+1)$.

$x_G \in \#B_\gamma \leftrightarrow x_G \notin \#B_\beta$ (by Lemma 2.19)

$\leftrightarrow [B_\beta] \notin G$ (as $\lambda(\beta) < \lambda(\gamma)$)

$\leftrightarrow -[B_\beta] = [B_\gamma] \in G$ (as G is an ultrafilter) .

The result follows by transfinite induction on complexity of codes. (An argument similar to that of Lemma 2.9 proves that every code has a complexity.)

We can use Lemma 3.7 to establish a natural bijection between the generic ultrafilters and the random reals.

Lemma 3.8 $x \in \mathbb{R}^*$ iff $x = x_G$ for some generic ultrafilter G .

Proof: (a) Let $x \in \mathbb{R}^*$. Define $G_x = \{ [B] : x \in \#B \}$. Let $A,B \in \mathbb{B}$; if $A \equiv B \pmod{\tilde{N}}$ then $x \in \#A \leftrightarrow x \in \#B$, as $\mu_{\mathbb{M}}(A \Delta B) = 0$ and $\mu(\#A \Delta \#B) = 0$ (Lemma 2.19 and Theorem 2.24). Thus:

$$[B] \in G_x \leftrightarrow \forall A \in [B],\ x \in \#A \ . \qquad \ldots(1)$$

By Theorem 2.24:

$$[\emptyset] \notin G_x \ . \qquad \ldots(2)$$

If $[A]$, $[B] \in G_x$ then $x \in \#A$, $x \in \#B$, and $x \in \#A \cap \#B = \#(A \cap B)$ by Lemma 2.19 . Thus:

$$[A \cap B] = [A] \cdot [B] \in G_x \ . \qquad \ldots(3)$$

Let $[A] \in G_x$ and $[A] \leq [B]$, then by (1):

$$\forall C \in [B],\ x \in C \text{ and so } [B] \in G_x \ . \qquad \ldots(4)$$

(2) -- (4) imply that G_x is a proper filter. G_x is obviously maximal, and is thus an ultrafilter.

Let $S \quad B^*$, $S \in \mathbb{M}$, $\Sigma S \in G_x$. Since B^* obeys the countable chain condition, there is a $\mathbb{M}$-countable collection of Borel sets:

$$\{ A_{\gamma_i}, \ldots, A_{\gamma_n}, \ldots \} \in \mathbb{M}$$

such that $[A_{\gamma_i}] \in S$, for each $i \in \omega$, and:

$$\sum_i [A_{\gamma_i}] = \Sigma S$$

(see [8], p. 61). Let $\gamma(0) \equiv 1 \pmod 3$, $\gamma(J(i,j)) = \gamma_i(j)$, then $A_\gamma = \bigcup_i A_{\gamma_i}$ and $[A_\gamma] \in G_x$. Hence $x \in \#A_\gamma$ by Lemma 3.7 . Lemma 2.19 implies $\#A_\gamma = \bigcup_i \#A_{\gamma_i}$, so $\exists i (x \in \#A_{\gamma_i})$ and $[A_{\gamma_i}] \in G_x \cap S$ (Lemma 3.7). This establishes the genericity of G_x. Note that even though $|\mathbb{M}| = \aleph_0$, the countable chain condition is required.

Lemma 3.7 now gives: $x = x_G$, where $G = G_x$.

(b) Conversely, let $x = x_G$ for some generic ultrafilter G on B*. For each Borel set B_γ rational over $\mathbb{M}$ and satisfying $\mu(B_\gamma) = 0$, Theorem 2.24 implies $\mu_{\mathbb{M}}(B_\gamma^{\mathbb{M}}) = 0$, and so $[B_\gamma^{\mathbb{M}}] \notin G$. It follows from Lemma 3.7 that $x \notin B_\gamma$.

Corollary 3.9 For each $x \in \mathbb{R}^*$, $G_x = \{ [B] : x \in \#B \}$ is an $\mathbb{M}$-generic ultrafilter on B*.

These results give us some notation and terminology:

(a) Each $x \in \mathbb{R}^*$ has an associated generic ultrafilter G_x on B*.

(b) Each generic ultrafilter G on B* has an associated random real X_G .

Definition 3.10 For $x \in \mathbb{K}$, let $\mathbb{M}[x]$ be the least transitive submodel of $\mathbb{K}$ extending $\mathbb{M}$ and containing $\{x\}$, if this exists; $\mathbb{M}[x] = \mathbb{K}$ otherwise.

We will only use the above notation where it is well defined, primarily by the result below.

Lemma 3.11 For every $x \in \mathbb{R}^*$, $\mathbb{M}[x] = \mathbb{M}[G_x]$.

Proof: $\mathbb{M}[G_x]$ is the least transitive submodel of $\mathbb{K}$ extending $\mathbb{M}$ and containing $\{G_x\}$ (see proof [9], p. 56 via absoluteness, or Lemma 4.8 for direct

proof of a stronger result). $x \in \mathbb{M}[G_x]$, by Definition 3.5 . If $\mathbb{N} \supseteq \mathbb{M}$ is a transitive submodel of $\mathbb{K}$ (hence $\mathbb{N}$ is standard), and $x \in \mathbb{N}$, then $G_x \in \mathbb{N}$ by Corollary 3.9, and so $\mathbb{M}[G_x] \subset \mathbb{N}$, i.e.

$$x \in \mathbb{M}[G_x] \subset \mathbb{N}$$

Definition 3.12 For a given generic ultrafilter G on B^*, define $\underline{x}_G \in \mathbb{M}^{B^*}$ as:

$$\mathrm{dom}(\underline{x}_G) = \{ \check{r} : r \in \underline{Q},\ [(r,\infty)] \in G \}$$

$$\underline{x}_G(\check{r}) = [(r,\infty)]$$

$\underline{x}_G$ is called the canonical random real in $\mathbb{M}^{B^*}$. $\underline{x}_G$ names x_G , i.e. $i_G(\underline{x}_G) = x_G$. For G understood, $\underline{x} = \underline{x}_G$.

We recall this restatement of Lemma 3.8 of Section 0.

Lemma 3.13 For each formula ϕ and $y \in \mathbb{R}^*$: $\mathbb{M}[y] \models \phi(y)$ iff $[\![\phi(\underline{x})]\!] \in G_y$.

This result holds with parameters in $\mathbb{M}$ by making the substitution $\phi_{\vec{p}}(y) = \phi(y,\vec{p})$, $\vec{p} \in \mathbb{M}$. This gives: $\mathbb{M}[y] \models \phi(y,\vec{p})$ iff $[\![\phi(\underline{x},\check{\vec{p}})]\!] \in G_y$, $y \in \mathbb{R}^*$, $\vec{p} \in \mathbb{M}$.

Theorem 3.14 For each formula ϕ, the set $E = \{ y \in \mathbb{R} : \mathbb{M}[y] \models \phi(y) \}$ is Lebesgue measurable.

Proof: Let $E' = \{ y \in \mathbb{R}^* : \mathbb{M}[y] \models \phi(y) \}$.

By Lemma 3.4, $E' \equiv E \pmod{\tilde{N}}$.

Then by Lemma 3.13: $y \in E' \leftrightarrow [\![\phi(\underline{x})]\!] \in G_y$.

Let $\gamma \in (\omega^\omega)_{\mathbb{M}}$ code the Borel B_γ such that: $[B_\gamma] = [\![\phi(\underline{x})]\!]$. Then for all $y \in \mathbb{R}^*$:

$y \in E' \leftrightarrow [B_\gamma] \in G_y \leftrightarrow y \in \#B_\gamma$.

Therefore: $E' \equiv \#B_\gamma \pmod{\tilde{N}}$,

$E \equiv \#B_\gamma \pmod{\tilde{N}}$,

and E is Lebesgue measurable.

The above theorem easily generalizes by adding parameters in $\mathbb{M}$, and it is this form of Theorem 3.14 that finds application in Section 5.

Section 4 : The Lévy Algebra

Every Boolean algebra is a partially ordered set, but seldom does a partially ordered set have the necessary structure to make it a Boolean algebra, much less a complete Boolean algebra. Fortunately, a standard technique exists which transforms any given partially ordered set into a complete Boolean algebra. If P is a partially ordered set we write RO(P) for the regular open algebra of P. This is obtained by imposing the order topology on P (with basic open sets $[p] = \{ q : q \leq p \}$). The elements of RO(P) are those open sets U which are regular (i.e. $U = \bar{U}^{\circ}$). RO(P) is a complete Boolean algebra. Complete details are to be found in [8] (p. 25) and/or [25] (pp. 14 - 17).

As usual, $cf(\alpha)$ denotes the cofinality of an ordinal α.

Definition 4.1 Suppose $\mathbb{M} \models \kappa$ is a cardinal & $cf(\kappa) = \omega$. P_κ is defined by:
$\mathbb{M} \models p \in P_\kappa \leftrightarrow (\exists n \in \omega)(p:n \to \kappa)$, and
is partially ordered by: $p \geq q \leftrightarrow q \supseteq p$.

For each cardinal κ, P_κ is simply a collection of finite functions in $\mathbb{M}$ with range in κ. The ordering of P_κ is reverse of the usual inclusion ordering.

The partially ordered set P_κ gives us a complete Boolean algebra $L_\kappa = RO(P_\kappa)$, called a Collapsing algebra (the reason for this name: for any generic ultrafilter G

on L_κ , the function $\cup G$ maps ω onto κ, and so κ "collapses" onto ω and is countable in $\mathbb{M}[G]$. See [23], p. 8).

For the present we shall assume there is a (strongly) inaccessible cardinal λ. The ramifications of this assumption are discussed in the next section. The family $\{ P_\kappa : cf(\kappa) = \omega, \kappa < \lambda \}$ forms a normal limiting system (see [25], p. 193). It follows that the associated family $\{ L_\kappa : cf(\kappa) = \omega, \kappa < \lambda \}$ is an example of a direct system of complete Boolean algebras. An exhaustive development of this topic is found in [25], pp. 183 - 195 .

Definition 4.2 A Boolean algebra B satisfies the κ-chain condition if each partition of unity in B has cardinality less than κ. In the case where $\kappa = \omega$, we say that B satisfies the countable chain condition.

Two members a, b of a Boolean algebra (or partially ordered set) B are said to be compatible if there exists a nonzero $c \in B$ such that $c \leq a$ and $c \leq b$, otherwise they are said to be incompatible. Since a partition of unity is a maximal family of pairwise incompatible elements of B, the κ-chain condition implies that no family of pairwise incompatible elements of B has cardinality κ, or greater. A partially ordered set satisfies the κ-chain condition if it contains no strictly descending chain (totally ordered set) of cardinality κ.

The proposition below gathers together a number of technical results, mainly from the above reference, which support the work of this section. We quote them without their lengthy but straightforward proofs, some of which derive from the work of Engelking and Karłowicz [5].

Proposition 4.3 (a) For each $\kappa < \lambda$ such that $cf(\kappa) = \omega$,

$$P_\kappa = \bigcup_{\alpha<\lambda} P_\alpha$$

(b) $P = \bigcup_{\kappa<\lambda} P_\kappa$ satisfies the λ-chain condition.

(c) $L = \bigcup_{\kappa<\lambda} L_\kappa$ satisfies the λ-chain condition.

(d) $L \simeq RO(P)$.

(e) For each $\kappa < \lambda$, L_κ is a complete subalgebra of L.

(f) $|L_\kappa| < \lambda$, for each $\kappa < \lambda$.

The Boolean algebra L defined above will be referred to as the Lévy algebra. The significance of Proposition 4.3 lies mainly in two facts. From (c) we have that L satisfies the λ-chain condition. This fact will have applications to situations in both this and the next section. From (c), (e), and (f) on one hand, and (d) on the other, we have two distinct representations of L. The first representation would normally be improper. In general, we cannot say that the union of a family of complete Boolean algebras will be a Boolean algebra, complete or otherwise.

Takeuti and Zaring use Proposition 4.3(a) and the fact that the collapsing algebras L_κ form a direct system to show that this union is equal to the direct limit, or sum, of the L_κ. The usual method of defining the sum of a family of Boolean algebras, and taking the completion of this sum, is thus circumvented. Takeuti and Zaring show further that L thus defined is isomorphic to RO(P), giving us a second representation (within isomorphism) of L.

Our next result takes a closer look at the structure of L by way of this second representation.

Lemma 4.4 (a) P is the collection of finite sets of triples $p = \{ (\alpha_i, n_i, \beta_i) \}_{i \leq k}$ satisfying:

(i) $n_i \in \omega$, $\beta_i < \alpha_i < \lambda$.

(ii) $(\alpha, n, \beta_0), (\alpha, n, \beta_1) \in p \rightarrow \beta_0 = \beta_1$.

(b) Suppose $S \subset L \setminus \{0\}$ and $|S| < \lambda$. For each $\kappa < \lambda$ there is a collection $\{ a_\beta \in L : \beta < \kappa \}$ of pairwise incompatible elements such that:

$$\forall s \in S, \forall \beta < \kappa, a_\beta \cdot s \neq 0 .$$

Proof: (a) is an obvious formal renaming of the elements of P. The proof of (b) is a straightforward calculation using (a) (see [9], p. 76).

Many of the intrinsic properties of L are obtained by looking at P, specifically the representation given in (a) in the lemma above. This habit of calculating in P rather than L carries right over to some of the results concerning

$\mathbb{M}^L$ and $\mathbb{M}[G]$ in the next section, and mirrors the methodology of classical forcing to some extent. In preparation for these calculations we will introduce at this point some indispensable tools.

Definition 4.5 Let P be a partially ordered set and $S \subset P$. S is dense in P if:

$\forall p \in P, \exists s \in S, s \leq p$.

If G is any filter on L (or any other regular open algebra), it is not difficult to induce a related filter G' on the partially ordered set P. Of course, we must describe G' on P in an order language rather than a Boolean language. We say that G' is a filter on P if:

(a) the members of G' are pairwise compatible.

(b) $x \in G', y \geq x \rightarrow y \in G'$.

Complementation and the existence of 0 must not be taken for granted in P; that is why we are forced to simplify the notion of filter from the original Boolean algebraic case.

Definition 4.6 G' is an $\mathbb{M}$-generic filter on P iff for each dense set $S \subset P$, $S \in \mathbb{M}$, we have $S \cap G' \neq \emptyset$.

$G' \subset P$ above is also called a generic set of forcing conditions in the literature.

We will not delve into the method of inducing a generic filter G' on P, given a generic ultrafilter G on L, or that of inducing G from G' on the other hand. The literature contains ample treatment of this (see [25], pp. 25 - 32, especially p. 30; see also [9], pp. 48 - 52).and we shall never need to appeal to the mechanics of it . Suffice it to say that RO induces a one-to-one correspondence between the generic ultrafilters of L and the generic filters of P.

Why have we defined L, and what properties does it have that simpler, more familiar algebras do not? We have already mentioned that by its definition, L satisfies the λ-chain condition, and that this fact is very useful in both this section and the next. L has however, a very strong and unusual property having critical impact on Solovay's application of random reals to the measure problem. It is to this property of homogeneity that the duration of this section is devoted.

All of the prior results we have cited are easily accessible in the literature, and so we have quoted them without proof. The proof that L is homogeneous is not well represented elsewhere, so it deserves a detailed treatment here.

Suppose A is a complete subalgebra of L, and g is an automorphism on A. We say that g lifts from A to L if there is an automorphism g' of L whose restriction to A is g.

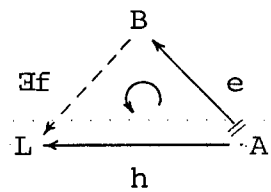
g' is called an extension of g. It is by no means clear what conditions we might impose on A to ensure the lifting of each $g \in Aut(A)$.

Letting Aut(A) denote the set of automorphisms on A, we say that a complete subalgebra A of L has the lifting property if each $g \in Aut(A)$ lifts.

Definition 4.7 L is homogenous if each complete subalgebra A of L satisfying $|A| < \lambda$ has the lifting property.

The term "homogeneous" has various meanings in the literature. For our purposes, the strong notion of homogeneity we use is necessary.

Let us first review some easily obtainable information. L is complete, as it is a regular open algebra. A Hahn-Banach type extension argument can be employed to show that complete Boolean algebras are injective (see [8], pp. 132 - 143), i.e. they satisfy the commutative diagram below for any Boolean algebras A and B:



where e is any monomorphism, and h is any homomorphism. Having fixed all of the above particulars, injectivity

simply means that a homomorphism f exists which completes the diagram.

One of the main consequences of injectivity is the fact that homomorphisms on subalgebras into L extend to homomorphisms on L. This follows directly from the diagram by letting B = L, and e be the inclusion map. We know then, that an automorphism on a subalgebra A of L extends to some homomorphism on L. Using a Hahn-Banach type argument, we can show that embeddings (complete monomorphisms) of subalgebras extend to embeddings of L. Universal techniques show us then, that automorphisms on any subalgebra extend to embeddings of L into L. To show that L is homogeneous however, we must use properties specific to L.

The path we will take involves a novel use of Boolean-valued techniques. Though we are confronted with a non-logical problem concerning L, we will find that much algebraic information about L is reflected in $\mathbb{M}^L$. We recall that $\mathbb{M}$ is our transitive ground model of ZFC.

Lemma 4.8 Let A be a subalgebra of L, and h be an automorphism on A. In $\mathbb{M}^L$ there is an ultrafilter G_h on $\check{A}$ such that for each $a \in A$:

$$h(a) = [\![\check{a} \in G_h]\!] .$$

Proof: $\underline{G}$ is the canonical ultrafilter on $\mathbb{M}^L$. Since $\underline{G}(1) = \check{\check{A}}(1) = 1$, we have:

$$\mathbb{M}^L \models \check{A} \cap \underline{G} \neq \check{\emptyset}$$

and the Maximal Principle (Lemma 0.26) defines G_A such that:

$$\mathbb{M}^L \models G_A = \check{A} \cap \underline{G} \ .$$

Likewise, we use the Maximal Principle to define G_h on $\check{A}$:

$$\mathbb{M}^L \models \check{a} \in G_h \leftrightarrow h(a)\check{} \in G_A \ .$$

An elementary calculation yields:

$$[\![\check{a} \in G_A]\!] = a \ ,$$

for each $a \in A$. So we have:

$$[\![\check{a} \in G_h]\!] = [\![h(a)\check{} \in G_A]\!] = h(a) \ .$$

From this, and the fact that h is a monomorphism, simple calculations give:

(a) $[\![\check{0} \notin G_h]\!] = 1$,

(b) $a \leq b$ implies $[\![\check{a} \in G_h]\!] \leq [\![\check{b} \in G_h]\!]$,

(c) $[\![(a \cdot b)\check{} \in G_h]\!] = [\![\check{a} \in G_h \ \& \ \check{b} \in G_h]\!]$,

(d) $[\![(-a)\check{} \in G_h]\!] = - [\![\check{a} \in G_h]\!]$.

We conclude:

$$\mathbb{M}^L \models G_h \text{ is an ultrafilter on } \check{A}.$$

We refer to G_h as the ultrafilter on $\check{A}$ <u>associated</u> with h. This lemma is understated in the sense that h could just as well have been an embedding of A into L. Even so, we have not extracted all the information about G_h that is reflected in h.

<u>Corollary 4.9</u> $\mathbb{M}^L \models G_h$ is $(\mathbb{P}(A))\check{}$-complete.

Proof: h is complete.

The main work within L is carried out by the following result, which is our modified version of a theorem due to Jensen (see [9], pp. 75 - 76). To prepare for it, we mention the following items that are necessary in the proof:

(a) A subalgebra of a Boolean algebra is said to be <u>regular</u> if suprema (and infima) of subsets common to both algebras correspond to the same value in each algebra. It is easily verified that complete subalgebras of complete Boolean algebras are regular.

(b) The fact that $\check{\lambda} = [\aleph_1]^{\mathbb{M}^L}$, which is proved in the next section (Corollary 5.2) using no information dependent on Lemma 4.10 .

<u>Lemma 4.10</u> Let A, B be complete subalgebras of L having cardinality less than λ, and let A be a complete subalgebra of B. Each automorphism on A lifts to be an automorphism on B.

Proof: Let h be an automorphism on A, and G_h be its associated ultrafilter on $\check{A}$. Using the Maximal Principle again, we induce a filter G* on $\check{B}$:

$\mathbb{M}^L \models (\forall x \in \check{B})\,(\, x \in G^* \leftrightarrow (\exists y \in \check{A})\,(y \check{\leq} x \;\&\; y \in G_h)\,)$.

In $\mathbb{M}^L$, G* is the filter on B generated by the pairwise compatible set G_h. For each $b \in B$,

we define:

$$b^* = \Sigma\{\, a \in A : a \leq b \,\}\,,$$

where the supremum is taken in A, which is complete. Obviously, $b^* \in A$ and $b^* \leq b$. Using completeness of h and Corollary 4.9, for each $b \in B$:

$$[\![\, \check{b} \in G^* \,]\!] = [\![\, (b^*)^{\check{}} \in G_h \,]\!] = h(b^*) \quad (1).$$

For each $E \subset B$, $[\Pi E]^* = \Pi\{\, b^* : b \in E \,\}$, and so we have by (1):

$$\begin{aligned} [\![\, \check{E} \subset G^* \,]\!] &= [\![\, \{\, b^* : b \in E \,\}^{\check{}} \in G_h \,]\!] \\ &= [\![\, (\Pi E)^{*\check{}} \in G_h \,]\!] \\ &= [\![\, (\Pi E)^{\check{}} \in G^* \,]\!] \qquad (2). \end{aligned}$$

This gives:

$\mathbb{M}^L \models G^*$ is a $(\mathbb{P}(B))^{\check{}}$-complete filter on $\check{B}$. Since λ is inaccessible, Theorem 0.44 and item (b) preceding this lemma allow us to extend G^* to an ultrafilter G':

$\mathbb{M}^L \models G'$ is a $(\mathbb{P}(B))^{\check{}}$-complete ultrafilter on B and $G' \supseteq G^*$.

A mapping g may now be defined for each $b \in B$:

$$g(b) = [\![\, \check{b} \in G' \,]\!]\,.$$

The following Boolean suprema, evaluated in their respective algebras, are equal:

$$[\![\, \check{b} \in G' \,]\!]^{(B)} = [\![\, \check{b} \in G' \,]\!]^{(L)}\,,$$

as B is a regular subalgebra of L. Hence g maps B into B.

Using the fact that G' is an ultrafilter and that $\check{}$ is injective, we have for each a, b $\in$ B:

$$g(-a) = [\![(-a)\check{} \in G']\!] = [\![\check{a} \notin G']\!]$$
$$= - [\![\check{a} \in G']\!] = -g(a) ,$$
$$g(a \cdot b) = [\![(a \cdot b)\check{} \in G']\!] = [\![\check{a} \in G' \,\&\, \check{b} \in G']\!]$$
$$= g(a) \cdot g(b) .$$

Thus g is a homomorphism. Similarly, we can use $(\mathbb{P}(B))\check{}$-completeness of G' in $\mathbb{M}^L$ to show that g is a complete homomorphism. For each a $\in$ A, we have: $g(a) = [\![a \in G']\!] = [\![a \in G_h]\!] = h(a)$, so g is an extension of h.

There are many ultrafilters G' for which the above calculations hold. We will select one of these which ensures the injectivity of g. By Lemma 4.4(b), there is a pairwise incompatible family $\{ a_b \in L : b \in B, b \neq 0 \}$ such $a_b \cdot c \neq 0$ for each a_b, and each c $\in$ A where c $\neq$ 0. By Lemma 0.25, there exists t $\in \mathbb{M}^L$ such that $a_b \leq [\![t = \check{b}]\!]$, for each b $\in$ B where b $\neq$ 0. We pick t' $\in \mathbb{M}^L$ such that:

$$[\![(t' \in \check{B}) \,\&\, (t' = -t)]\!] = 1 ,$$

i.e. t' is the complement of t in $\check{B}$. Now we define G' as before, but with the added proviso:

$$[\![t' \notin G^*]\!] \leq [\![t \in G']\!] .$$

This amounts to generating G' from $E = G^* \cup \{t\}$ if E is pairwise compatible, i.e.

$\mathbb{M}^{L} \models$ (G' is a $(\mathbb{P}(B))^{\vee}$-complete ultrafilter on $\check{B}$) & ($G^* \subset G'$) & ($t' \notin G^* \rightarrow t \in G'$) .

Let $b \in B$, and $b \neq 0$.

$$g(b) = [\![\check{b} \in G']\!] \geq [\![\check{b} = t]\!] \cdot [\![t \in G']\!]$$

$$\geq [\![\check{b} = t]\!] \cdot [\![t' \notin G^*]\!] \geq [\![\check{b} = t]\!] \cdot [\![-(\check{b}) \notin G^*]\!]$$

$$= [\![\check{b} = t]\!] \cdot [\![(-b)^{\vee} \notin G^*]\!]$$

$$a_b \cdot -((-b)^*) .$$

Since $-b < 1$; $(-b)^* < 1$, $-((-b)^*) \neq 0$, and so $g(b) \neq 0$, by definition of a_b. Thus $\ker(g) = 0$ and g is injective.

Since B is injective, the diagram indicates that each monomorphism on B (such as g) has a retraction f, i.e. an epimorphism f such that $f \circ g$ is the identity map on B. If f is a retraction for g, then f is complete, and the kernel of f is a complete Boolean ideal. Ideals of this type must necessarily be <u>principal</u>, i.e. there is an element $u \in B$ such that $\ker(f) = [u] = \{ v : v \leq u \}$.

If $\mathrm{rng}(g) \neq B$, then $u \neq 0$ and $u \notin \mathrm{rng}(g)$.

Define: $E = \{ t \in B : [\![\check{t} \in G']\!] \geq u \}$.

Since $\mathrm{dom}(\check{E}) = \{ \check{t} : t \in E \}$, and $\check{E}(\check{t}) = 1$ for each $t \in E$, a simple calculation yields:

$[\![\check{E} \subset G']\!] = \prod_{t \in E} [\![\check{t} \in G']\!] = u$, by definition of of E and ker(f). But (2) above implies:

$u = [\![\check{E} \subset G']\!] = [\![(\Pi E)^{\vee} \in G']\!] = g(\Pi E)$,

i.e. $u \in rng(g)$. We conclude that $ker(f) = [0]$, and $rng(g) = B$.

Theorem 4.11 L is homogeneous.

Proof: Suppose A is a complete subalgebra of L, and that $|A| < \lambda$. Then $A \subset L_\kappa$, where:

$$\kappa = \sup_{a \in A} \inf \{ \gamma < \lambda : a \in L_\gamma \},$$

We show that A is a complete subalgebra of L_κ . Let $E \subset A$. We use superscripts to denote the Boolean operations of various subalgebras. A is a subalgebra of L, and P is a base for the topology of L, so:

$$\Sigma^{(A)}E = \sum_{a \in E}^{(L)} \Sigma\{ p : p \in P, p \leq a \}$$

$$= \sum_{a \in E}^{(L)} \Sigma\{ p : p \in P_\kappa, p \leq a \} .$$

Since P_κ is a base for L_κ, the above equals $\Sigma^{(L_\kappa)}E$.

Now we will show that for each $a \in A$, $-a^{(L)} \in L_\kappa$. P_κ may be represented as the following truncation of P:

$$P_\kappa = \{ p_\kappa : p \in P \}$$

where: $p_\kappa = \{ (\alpha,n,\beta) \in p : \alpha < \kappa \}$.

Then $L_\kappa = \{\Sigma X : X \subset P_\kappa \}$. Recalling Lemma 4.4(a), we see that it is possible for $p, q \in P$ to be incompatible, due to

the functionality constraint (ii). If p and q are incompatible, where $p \in P$, $q \in P_\kappa$; then p_κ and q are also incompatible.

Let $a \in A$, $a = \Sigma X$ where $X \subset P_\kappa$, and $-a^{(L)} = \Sigma Y$, $Y \subset P$. For each $q \in X$ and $p \in Y$, $p \cdot q = 0$; and so $p_\kappa \cdot q = 0$.

Therefore:

$$-a^{(L)} = \sum_{p \in Y} p_\kappa \in L_\kappa \ .$$

A is thus a complete subalgebra of L. If $h \in \mathrm{Aut}(A)$, Lemma 4.10 extends h to an automorphism $\bar{h} \in \mathrm{Aut}(L_\kappa)$. By transfinite induction, we use Lemma 4.10 to define $h_\gamma \in \mathrm{Aut}(L_\gamma)$ for each $\gamma \geq \kappa$:

$$h_\kappa = \bar{h}$$

By Lemma 4.10, if $h_\gamma \in \mathrm{Aut}(L_\gamma)$,

$$h_{\gamma+1} = \bar{h}_\gamma \in \mathrm{Aut}(L_{\gamma+1})$$

$$h_\gamma = \bigcup_{\beta<\gamma} h_\beta \ , \text{ if } \gamma \text{ is a limit ordinal.}$$

Note that for $\gamma < \lambda$ a limit ordinal, h_γ is an embedding and $\mathrm{rng}(h_\gamma) = \bigcup_{\beta<\gamma} L_\beta = L_\gamma$, hence $h_\gamma \in \mathrm{Aut}\ (L_\gamma)$.

$h_\lambda = \bigcup_{\gamma<\lambda} h_\gamma$ is likewise an embedding, and $\mathrm{rng}(h_\lambda) = L$. Thus $h_\lambda \in \mathrm{Aut}(L)$, and we have produced the required lifting of h.

We close this section with an application of Theorem 4.11 to a definability problem to be encountered later. First we will look at a natural method of extending automorphisms of L to automorphisms of $\mathbb{M}^L$. Given $g \in \mathrm{Aut}(L)$, we may induce an automorphism g^* on $\mathbb{M}^L$ by transfinite induction on rank ρ.

Let $g^*(0) = 0$.

Suppose g^* is defined for each $y \in \mathbb{M}^L$ such that $\rho(y) < \rho(x)$, given an $x \in \mathbb{M}^L$. In particular, $g^*(y)$ is defined for each $y \in \mathrm{dom}(x)$. We define:

$$\mathrm{dom}(g^*(x)) = g^*(\mathrm{dom}(x)) ,$$

$$[g^*(x)](g^*(y)) = g(x(y))$$

for each $g^*(y) \in \mathrm{dom}(g^*(x))$.

g^* thus defined is a bijective map of $\mathbb{M}^L$ onto $\mathbb{M}^L$, and $g^*(\check{x}) = \check{x}$ for each $x \in \mathbb{M}_\alpha$.

Lemma 4.12 Let $x, y \in \mathbb{M}^L$, and $g \in \mathrm{Aut}(L)$. Then:

$$g([\![x = y]\!]) = [\![g^*(x) = g^*(y)]\!] ,$$

and $$g([\![x \in y]\!]) = [\![g^*(x) \in g^*(y)]\!] .$$

Proof: The two equations are proved by a simultaneous transfinite induction on $(\rho(x), \rho(y))$. Assume both equations are true for any $z \in \mathbb{M}^L$ satisfying $\rho(z) < \rho(y)$, or $\rho(z) \leq \rho(x)$. Then:

$$g([\![x \in y]\!]) = g[\sum_{z \in \mathrm{dom}(y)} y(z) \cdot [\![z=x]\!]]$$

$$= \sum_{z \in dom(y)} g(y(z)) \cdot g([\![z = x]\!])$$

$$= \sum_{z \in dom(y)} [g^*(y)](g^*(z)) \cdot [\![g^*(z) = g^*(x)]\!]$$

$$= [\![g^*(x) \in g^*(y)]\!] .$$

A similar calculation holds for the other equation, establishing the inductive step for z satisfying $\rho(z) = \rho(y)$.

The generalization below follows from Lemma 4.12 by induction on the complexity of ϕ. For simplicity, we drop parentheses where convenient.

Corollary 4.13 Let ϕ be a formula with n free variables. For each $x_1, \ldots, x_n \in \mathbb{M}^L$:

$$g[\![\phi(x_1, \ldots, x_n)]\!] = [\![\phi(g^*x_1, \ldots, g^*x_n)]\!].$$

It is apparent that if ϕ is a sentence (i.e. having no free variables), then $[\![\phi]\!]$ is a fixed point for any $g \in Aut(L)$. Hence $[\![\phi]\!]$ is either 0 or 1. The same reasoning gives us a 0-1 law (see [19], p. 408, and [25], p. 171) for formulas whose variables range over $\mathbb{M}$.

Corollary 4.14 Let ϕ be a formula with n free variables. For each $x_1, \ldots, x_n \in \mathbb{M}$:

$$[\![\phi(\check{x}_1, \ldots, \check{x}_n)]\!] \in \{0,1\} .$$

Proof: For each $b \in L \setminus \{0,1\}$, we may use Theorem 4.11 to construct $g_b \in Aut(L)$ such that

$g_b(b) \neq b$. For example, define the automorphism e on the subalgebra $\{0,b,-b,1\}$ by $e(b) = -b$, and lift e to g_b on L.

Let $b = [\![\phi(\check{x}_1, \ldots, \check{x}_n)]\!]$. If $b \notin \{0,1\}$, Corollary 4.13 implies:

$$b = [\![\phi(g_b^*\check{x}_1, \ldots, g_b^*\check{x}_n)]\!]$$
$$= g_b[\![\phi(\check{x}_1, \ldots, \check{x}_n)]\!]$$
$$\neq b .$$

Hence $b \in \{0,1\}$.

It is possible to prove Theorem 0.22 from the above, since bounded formulas are of this type, and their Boolean values are non-zero when they are valid in $\mathbb{M}$.

In the next section, we define L_t as the complete subalgebra of L generated by rng(t), where $t \in \mathbb{M}^L$.

Lemma 4.15 If $|\mathrm{rng}(t)| < \lambda$, then $|L_t| < \lambda$.

Proof: Since P corresponds to the basic open sets in the topology of L, P is dense in L. Using the λ-chain condition, we can thus find for each $a \in L$, a subset $S_a \subset P$ such that $|S_a| < \lambda$, and $a = \Sigma S_a$. Let $S = \cup\{ S_a : a \in \mathrm{rng}(t) \}$. Since $|S_a| < \lambda$, there is a $\kappa < \lambda$ such that $S \subset P_\kappa$. $L_\kappa = RO(P_\kappa)$ and $S \subset L$, so $L_t \subset L_\kappa$, and $|L_t| \leq |L_\kappa| < \lambda$.

A generalized form of Corollary 4.14 now follows.

Theorem 4.16 Let $t \in \mathbb{M}^L$ be such that $dom(t) \subset \{ \check{x} : x \in \mathbb{M} \}$, and $|rng(t)| < \lambda$. Then: $[\![\phi(t)]\!] \in L_t$.

Proof: For each $u \notin L_t$ let $L_t(u)$ be the subalgebra of L generated by $L_t \cup \{u\}$, i.e. $L_t(u) = \{ (a \cdot u) + (b \cdot -u) : a, b \in L_t \}$. Define the following automorphism on $L_t(u)$:

$$e(a \cdot u + b \cdot -u) = (b \cdot u) + (a \cdot -u).$$

$e(a) = a$, for each $a \in L_t$, but $e(u) = -u$. By Theorem 4.11, e lifts to $g_u \in Aut(L)$. For each $u \notin L_t$ there exists $g_u \in Aut(L)$ such that $g_u(a) = a$, for $a \in L_t$, and $g_u(u) \neq u$. For each such u, $g_u^*(t) = t$, by definition of dom(t). Hence $g_u(b) = b$, where $b = [\![\phi(t)]\!]$. Since $u = b$ yields a contradiction, we conclude that $b \in L_t$.

All of the constructions in this section were carried out within $\mathbb{M}$, and so our many uses of AC in various forms have been proper. In spite of the forbidding number of technical results in this section, only two of them have any application in the sequel. These are: Proposition 4.3 (c), used in some cardinality calculations, and Theorem 4.16

above, which is our sole application of the homogeneity of L.

Section 5 : The Model of Lévy

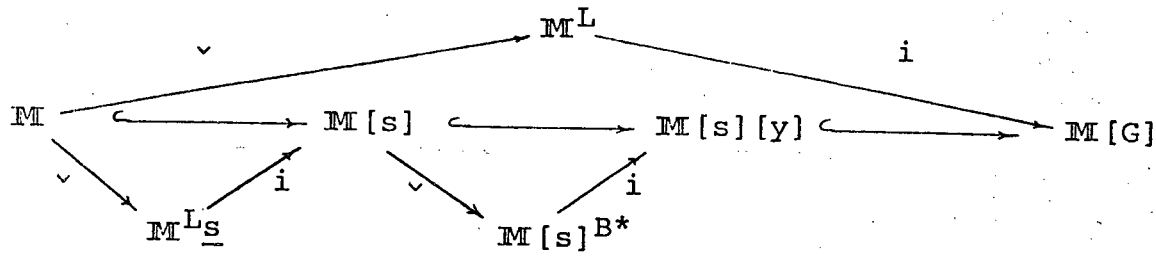
In previous sections we have imposed a number of plausible restrictions on the ground model $\mathbb{M}$. To begin this section, we add one more restriction to the list: namely that $\mathbb{M}$ satisfy the following axiom.

Axiom I There exists an inaccessible cardinal.

The above statement is much stronger than the axiom A we used in section 0 to justify the existence of $\mathbb{K}$. Tarski [26] shows that a model of ZFC can be constructed if there exists an inaccessible cardinal (see also [14], pp. 159-63; [4], p. 109-10). Similar arguments show that such a model may not be a model of I (e.g. [4], p. 110; [9], p. 37). Hence model existence axioms such as A cannot imply I.

The assumption $\mathbb{M} \models$ I is, indirectly, an added constraint on our assumptions regarding $\mathbb{K}$. Though axiom A in its present form is not sufficient to provide such a $\mathbb{K}$, we shall bypass this problem for the present.

Taking $\mathbb{M}$ to be the ground model containing an inaccessible λ, we may construct within $\mathbb{M}$ the Lévy algebra L of Section 4. We fix an $\mathbb{M}$-generic ultrafilter G on L, and in the course of this section construct the following 'tower' of generic extensions:



Since $\mathbb{M} \models$ AC, each generic extension also satisfies AC, and therefore $\neg$LM as well. However, the results of the last section will provide that a large family of sets of reals in $\mathbb{M}[G]$ are Lebesgue measurable.

To begin, we check the behaviour of cardinals with respect to the above tower.

Lemma 5.1 Let $\mathbb{E}$ be a model of ZFC, B be a Boolean algebra in $\mathbb{E}$, satisfying the countable chain condition in $\mathbb{E}$, and let H be an $\mathbb{E}$-generic ultrafilter on B. Then:

$$\Phi = (\ \kappa \text{ is a cardinal }\ )$$

is absolute between $\mathbb{E}$ and $\mathbb{E}[H]$.

Proof: Finite cardinals are absolute, so we assume $\kappa \geq \aleph_0$. Suppose $\mathbb{E} \models \Phi$, and $\mathbb{E}[H] \models \neg\Phi$. Then there exists a Boolean-valued function $f \in \mathbb{E}^B$ such that for $\delta < \kappa$

$$b = [\![\ \underline{\text{dom}}(f) = \check{\delta}\ \&\ \underline{\text{rng}}(f) = \check{\kappa}\ ]\!] \neq 0\ .$$

Since we have not defined the Boolean-valued notions of $\underline{\text{dom}}(f)$, or $\underline{\text{rng}}(f)$, we will assume f satisfies Definition 0.29 and define:

$$b = \prod_{\alpha<\delta} \sum_{\gamma<\kappa} [\![(\check{\alpha},\check{\gamma})^{B} \in \underset{\sim}{f}]\!] \cdot \prod_{\gamma<\kappa} \sum_{\alpha<\delta} [\![(\check{\alpha},\check{\gamma})^{B} \in \underset{\sim}{f}]\!] \neq 0 .$$

We let $b(\alpha,\beta) = b \cdot [\![(\check{\alpha},\check{\beta})^{B} \in f]\!]$. Two facts emerge:

(a) From Definition 0.29(c) and Theorem 0.22, $\beta \neq \gamma \rightarrow b(\alpha,\beta)\cdot b(\alpha,\gamma) \leq b \cdot [\![\check{\beta} = \check{\gamma}]\!] = 0$, as f is a Boolean-valued function.

(b) Since $b \neq 0$, $\forall \beta < \kappa$, $\exists \gamma > \beta$, $\exists \alpha$ ($b(\alpha,\gamma) \neq 0$) . From (b) we have:

$$|\{ \beta < \kappa : \exists \alpha,\ b(\alpha,\beta) \neq 0 \}| = \kappa .$$

Since $\delta < \kappa$, $\exists \alpha_o$ such that:

$$|\{ \beta < \kappa : b(\alpha_o,\beta) \neq 0 \}| = \kappa .$$

By (a), $\{ b(\alpha_o,\beta) : \beta < \kappa \}$ is a set of pairwise incompatible elements of B having cardinality κ . This violates the countable chain condition.

Φ is therefore preserved under the extension $\mathbb{E} \rightarrow \mathbb{E}[H]$. A routine argument shows that Φ is preserved under restriction.

The corollary below recalls some terminology from Section 4. P is the collection of finite sets of triples $p = \{(\alpha_i,n_i,\beta_i)\}_{i \leq k}$ satisfying:

(a) $n_i \in \omega$, $\beta_i < \alpha_i < \lambda$

(b) $(\alpha,n,\beta_o), (\alpha,n,\beta_1) \in p \rightarrow \beta_o = \beta_1$.

P is partially ordered by $p \leq q$ iff $p \supseteq q$. $L \simeq RO(P)$.

If G is an $\mathbb{M}$-generic ultrafilter on L, let G' be its induced $\mathbb{M}$-generic filter on P. We think of $p \in P$ as a function having finite domain $\subset \lambda\times\omega$, with $p[(\alpha_i,n_i)] = \beta_i < \alpha_i$.

Corollary 5.2 $\quad \lambda = (\aleph_1)^{\mathbb{M}[G]}$.

Proof: Let $\delta = \omega$ and $\kappa = \lambda$ in the proof of the preceeding lemma. Using the fact that L obeys the λ-chain condition, we infer $\lambda \geq (\aleph_1)^{\mathbb{M}[G]}$. For each $\alpha < \lambda$, we define:

$$f_\alpha = \{(n,\beta) : \{(\alpha,n,\beta)\} \in G'\} .$$

From (b) above, each f_α is a function in $\mathbb{M}[G]$, and $f_\alpha : \omega \to \alpha$.

For $\alpha \neq 0$, $\forall n \in \omega$, $n \in \mathrm{dom}(f_\alpha)$:

Let $A_o = \{ h \in P : (\alpha,n) \in \mathrm{dom}(h) \}$.

Since $\alpha > 0$, A_o is dense in P. By genericity there is an $h \in G' \cap A_o$ such that $\exists\beta$, $h[(\alpha,n)] = \beta$. Then $\{(\alpha,n,\beta)\} \in G'$, as G' is a filter. $n \in \mathrm{dom}(f_\alpha)$. $\forall\beta < \alpha$, $\beta \in \mathrm{rng}(f_\alpha)$:

Let $A_1 = \{ h \in P : \exists n \in \omega, h[(\alpha,n)] = \beta \}$.

A_1 is dense in P. Therefore there is an $h \in G' \cap A_1$ such that $h[(\alpha,n)] = \beta$, so $\{(\alpha,n,\beta)\} \in G'$ and $\beta \in \mathrm{rng}(f_\alpha)$.

We conclude that for all $\alpha < \lambda$, f_α maps ω onto α. Thus $\lambda \leq (\aleph_1)^{\mathbb{M}[G]}$.

Definition 5.3 Let $s \in \mathbb{M}[G]$, $s \subset \mathbb{M}$, and $\underline{s} \in \mathbb{M}^L$ be a name for s. $L_{\underline{s}}$ is the complete subalgebra of L generated by $rng(\underline{s})$.

Lemma 5.4 For each $x \in \mathbb{M}$, and $s \in \mathbb{M}[G]$, $[\![\check{x} \in \underline{s}]\!] \in L_{\underline{s}}$, where $\underline{s}$ names s.

Proof: Let A be a complete subalgebra of L containing $rng(\underline{s})$. For each $x \in \mathbb{M}$, $\check{x} \in \mathbb{M}^A$ as $\check{x} \in \mathbb{M}^{\{0,1\}}$ and $\{0,1\}$ is a subalgebra of A. Since $dom(\underline{s}) \subset \mathbb{M}^L$, $[\![z = \check{x}]\!] \in A$ for each $z \in dom(\underline{s})$. Thus:

$$[\![\check{x} \in \underline{s}]\!] = \sum_{z \in dom(\underline{s})} \underline{s}(z) \cdot [\![z = \check{x}]\!] \in A ,$$

by completeness of A.

Our attention turns now to the first extension of the tower. The role of this extension concerns the definability of those sets of reals in $\mathbb{M}[G]$ we wish to be Lebesgue measurable. We say that a set E is definable (in $\mathbb{M}[G]$) from $s \in \mathbb{M}[G]$, if there is a formula ϕ having free variables only x, s, such that $E = \{ x : \mathbb{M}[G] \models \phi(x,s) \}$.

For the rest of this section, our interest will focus on those sets of reals in $\mathbb{M}[G]$ definable from a sequence of ordinals. Thus $s \in \mathbb{M}[G]$ is a function with domain ω, ranging over $\Theta_{\mathbb{M}}$, the ordinals of $\mathbb{M}$ ($\mathbb{M}$ and $\mathbb{M}[G]$ have the same ordinals; see [25], p. 128).

In Section 3 we found a connection between the generic

ultrafilters over B* and the random reals. The next result gives us a general connection between the generic ultrafilters induced by G on $L_{\underline{s}}$ and the extensions $\mathbb{M}[s]$.

Lemma 5.5 Let $\underline{s} \in \mathbb{M}^L$ name $s \in \mathbb{M}[G]$. Then:

$$\mathbb{M}[s] = \mathbb{M}[G \cap L_{\underline{s}}] \quad .$$

Proof: Since G is an $\mathbb{M}$-generic ultrafilter on L, $G \cap L_{\underline{s}}$ inherits all the properties necessary to make it an $\mathbb{M}$-generic ultrafilter on $L_{\underline{s}}$. $\mathbb{M}[G \cap L_{\underline{s}}]$ is therefore a true generic extension.

Since for all $x \in \mathbb{M}$:

$\mathbb{M}[G] \models x \in s$

iff $[\![x \in \underline{s}]\!] \in G$ (by 0.40)

iff $[\![x \in \underline{s}]\!] \in G \cap L_{\underline{s}}$ (by 5.4)

iff $\mathbb{M}[G \cap L_{\underline{s}}] \models x \in s$ (by 0.40),

we have: $s \in \mathbb{M}[G \cap L_{\underline{s}}]$.

Let $\mathbb{N} \supset \mathbb{M}$, and $s \in \mathbb{N}$. For each $\alpha \in \Theta_{\mathbb{M}}$ we define:

$\phi_0(\underline{s},b) \leftrightarrow (\exists x \in \mathrm{dom}(\underline{s}))(b = \underline{s}(x) \ \& \ b \in G)$

$\phi^0_\alpha(b) \leftrightarrow (\alpha \text{ is even ordinal }) \ \&$

$(\exists \beta<\alpha)(\exists c \in A_\beta)(b = -c \ \& \ c \notin G)$

$\phi^1_\alpha(b) \leftrightarrow (\alpha \text{ is odd ordinal }) \ \&$

$(\exists \Gamma \subset \bigcup_{\beta<\alpha} A_\beta)(b = \Sigma\Gamma \ \& \ b \in G)$

$$\psi(s,\underline{s},b) \leftrightarrow (i(\underline{s}) = s) \ \&$$
$$[\phi_0(\underline{s},b) \veebar (\exists\alpha \in \Theta_{\mathbb{M}})(\alpha < |L|^+ \ \& \ (\phi^0_\alpha(b) \vee \phi^1_\alpha(b))] .$$

These formulas refer to the following sets:

$A_0 = rng(\underline{s})$,

$A_\alpha = \{-c : c \in \bigcup_{\beta<\alpha} A_\beta\}$; for α even,

$A_\alpha = \{\Sigma\Gamma : \Gamma \subset \bigcup_{\beta<\alpha} A_\beta\}$; for α odd.

Let $G_\alpha = G \cap A_\alpha$.

Since $L \in \mathbb{M} \subset \mathbb{N}$, and $G_0 \subset L$, the separation axiom implies $G_0 \in \mathbb{N}$.

If $\alpha > 0$ is even, $G_\alpha = \{ b : \phi^0_\alpha(b) \}$, and if α is odd, $G_\alpha = \{ b : \phi^1_\alpha(b) \}$. So if $G_\beta \in \mathbb{N}$ for each $\beta < \alpha$, then $G_\alpha \in \mathbb{N}$.

We conclude that $G \cap L_{\underline{s}} = \{ b : \psi(s,\underline{s},b) \}$ $= \bigcup_{\alpha \in \Theta_{\mathbb{M}}} G_\alpha \in \mathbb{N}$, and that $\mathbb{M}[G \cap L_{\underline{s}}]$ is the least model of ZFC extending $\mathbb{M}$ and containing $\{s\}$.

We say that $b = b(\vec{p}) \in \mathbb{E}$ is <u>uniformly $\mathbb{E}$-definable in $\vec{p}$</u> if there is a formula ϕ with free variables among $\vec{p}$, x, such that for any set of values $\vec{p}_0$ of the parameters $\vec{p}$:

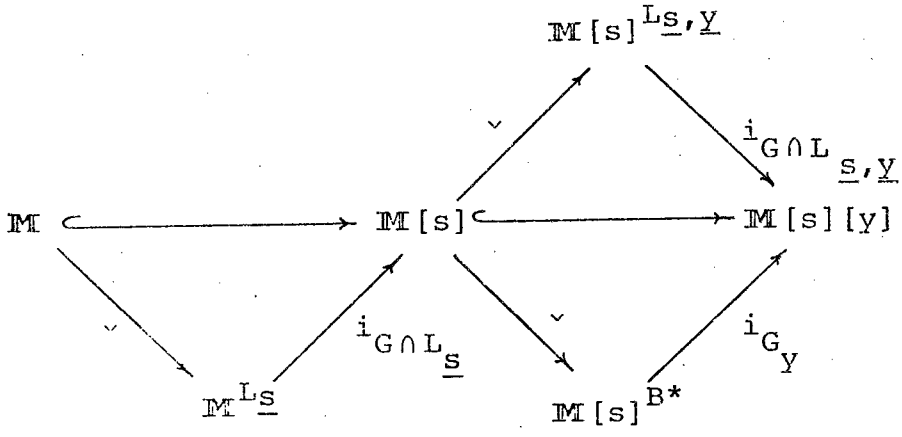
$$b(\vec{p}_0) = \{ x : \mathbb{E} \models \phi(\vec{p}_0,x) \} .$$

It is usually not convenient to mention all the parameters in $\vec{p}$, particularly those which are always fixed. In the corollary below, we make no reference to such parameters as L or G for this reason. Where no parameters are needed,

we will drop the adverb "uniformly".

Corollary 5.6 $G \cap L_{\underline{s}}$ is uniformly $\mathbb{M}[s]$-definable in s, $\underline{s}$.

Our first diagram left out an unusual architectural detail of tower. The second extension is really a simultaneous generic extension, revealed below:



As we shall see, the upper extension apparatus is a a natural repetition of the first extension, using the real number y in the place of s. The lower extension uses the results of Section 3, on the hypothesis that y is "almost always" a random real. Roughly speaking, definability aspects are handled by the upper extension and measure theoretic aspects are handled by the lower extension. How do we know that these extensions agree? Lemmas 3.7 and 5.5 ensure this.

The specific properties of s may now be used to our advantage.

Lemma 5.7 If s is a countable sequence of ordinals in $\mathbb{M}[G]$, it has a name $\underline{s} \in \mathbb{M}^L$ such that:

$$|\mathrm{rng}(\underline{s})| < \lambda .$$

Proof: By Lemma 0.40, we pick $\underline{s}$ so that $[\![\underline{s} \in ({}^{\omega}\theta_{\mathbb{M}} \cap u)^{\check{}}]\!] \in G$, where u is some set in $\mathbb{M}$. For each $n \in \omega$,

$$\kappa_n = |\{ \alpha : [\![(n,\alpha)^{\check{}} \in \underline{s}]\!] \neq 0 \}| < \lambda ,$$

since L obeys the λ-chain condition. By regularity of λ we have:

$$|\mathrm{rng}(\underline{s})| \leq \lim \kappa_n < \lambda .$$

Corollary 5.8 With $\underline{s}$ as above, $|L_{\underline{s}}| < \lambda$.

Proof: By Lemma 4.15 .

The next lemma is the culmination of all our work on the Lévy algebra L and its associated model $\mathbb{M}[G]$. The existence of λ and the resulting homogeneity of L are crucial factors in its proof. The lemma introduces a reduction in the definability of sets of reals in $\mathbb{M}[G]$ that places them within reach of the upper second extension of the tower.

Lemma 5.9 Let E be a set of reals in $\mathbb{M}[G]$ which is definable from a countable sequence of ordinals $s \in \mathbb{M}[G]$. E is uniformly $\mathbb{M}[s][y]$-definable in s, $\underline{s}$, for each $y \in E$.

Proof: We represent $y \in E$ by its Dedekind cut. There is a formula ϕ whose only free variables are s, y, such that the following are equivalent:

(a) $y \in E$,

(b) $\mathbb{M}[G] \models \phi(y,s)$,

(c) $(\exists \underline{y} \in \mathbb{M}^L)([\![\phi(\underline{y},\underline{s})]\!] \in G \ \&\ \mathrm{dom}(\underline{y}) = \{ \check{r} : r \in \underline{Q} \} \ \&\ \mathrm{rng}(\underline{y}) \subset L_{\underline{s},\underline{y}} \ \&\ (\forall r \in \underline{Q})(r \in y \leftrightarrow \underline{y}(\check{r}) \in G))$.

By Lemma 5.7, we may pick $\underline{s}$ so that $|\mathrm{rng}(\underline{s})| < \lambda$. For $y \in E$ we may pick a name $\underline{y} \in \mathbb{M}^L$ so that $|\mathrm{rng}(\underline{y})| < \lambda$ (as a Dedekind cut, y is definable from a countable sequence of ordinals). We define $L_{\underline{s},\underline{y}}$ to be the complete subalgebra of L generated by $\mathrm{rng}(\underline{s}) \cup \mathrm{rng}(\underline{y})$. Corollary 5.8 provides that $|L_{\underline{s},\underline{y}}| < \lambda$. We note that $\underline{y}(\check{r}) \in G \leftrightarrow \underline{y}(\check{r}) \in G \cap L_{\underline{s},\underline{y}}$. From Theorem 4.16, $[\![\phi(\underline{y},\underline{s})]\!] \in G \leftrightarrow [\![\phi(\underline{y},\underline{s})]\!] \in G \cap L_{\underline{s},\underline{y}}$. This is the principal use of the homogeneity of L. By Lemma 5.5, we have $\mathbb{M}[G \cap L_{\underline{s},\underline{y}}] = \mathbb{M}[s][y]$. From Corollary 5.6, we obtain:

$$y \in E \quad \text{iff} \quad \mathbb{M}[s][y] \models \exists \underline{y}\ \phi'(y,\underline{y},s,\underline{s}) \ .$$

We have used the upper second extension to effect a reduction in the criterion of membership in E, from $\mathbb{M}[G]$ to $\mathbb{M}[s][y]$. We will apply the tools of Section 3 to this

reduced criterion by way of the lower extension and obtain the first main theorem of Solovay.

Lemma 5.9 renders E in the parametric form of Theorem 3.14 . The only fact we need check is whether or not $\mathbb{M}[s]$ is an appropriate ground model from the standpoint of Section 3. $\mathbb{M}[s]$ can, in fact, be shown to be countable (by induction on rank, as in [22], p. 361). Instead, we will use a more specific argument about cardinals that re-establishes Lemma 3.4 for the lower second extension.

Theorem 5.10 Let E be a set of reals in $\mathbb{M}[G]$ which is $\mathbb{M}[G]$-definable from a countable sequence of ordinals $s \in \mathbb{M}[G]$. E is Lebesgue measurable.

Proof: Each subset t of ω in $\mathbb{M}[s]$ has a name $\underline{t} \in \mathbb{M}^{L_{\underline{s}}}$ with $\mathrm{dom}(\underline{t}) = \{ \check{n} : n \in \omega \}$, and so determines a function $f_{\underline{t}}:\omega \to L_{\underline{s}}$ defined by:

$$f_{\underline{t}}(n) = [\![\check{n} \in \underline{t}]\!] \quad (\text{see } 5.4).$$

Note that $f_{\underline{t}} \in \mathbb{M}$. $f_{\underline{t}} = f_{\underline{u}} \to$ $\forall n: [\![\check{n} \in \underline{t}]\!] = [\![\check{n} \in \underline{u}]\!] \to [\![\underline{t} = \underline{u}]\!] = 1$ $\to \mathbb{M}[s] \models t = u$.

Thus the number of subsets of ω in $\mathbb{M}[s]$ cannot exceed the number of functions in $\mathbb{M}$ from ω into $L_{\underline{s}}$. Using the inaccessibility of λ in $\mathbb{M}$ and Corollary 5.2:

$$(2^{\aleph_0})^{\mathbb{M}[s]} \leq (|L_{\underline{S}}|^{\aleph_0})^{\mathbb{M}} < \lambda = (\aleph_1)^{\mathbb{M}[G]} .$$

The cardinality of the family of Borel codes in $\mathbb{M}[s]$ is countable in $\mathbb{M}[G]$, so there can only be countably many Borel sets of measure zero in $\mathbb{M}[G]$ which are rational over $\mathbb{M}[s]$. Lemma 3.4 and Theorem 3.14 yield the result.

Section 6 : The Model of McAloon

We cannot expect any generic extension of a ground model of ZFC to be a model of ZF + LM. However, the Lévy model is an example of a generic extension which comes very close to satisfying LM, in that a certain large family of sets of reals are Lebesgue measurable. This fact suggests a new approach. Can we find a suitable submodel of $\mathbb{M}[G]$ whose sets of reals fall within the above family? In this section we follow the McAloon-Solovay construction of one such internal model $\mathbb{N} \subset \mathbb{M}[G]$. All the work of this section is carried out within $\mathbb{M}[G]$.

If $\mathbb{N} \models$ LM, the problems of Section 1 regarding the needs of analysis and measure theory become pertinent. The model $\mathbb{N}$ which we construct will satisfy the axiom below, known as the Principle of Dependent Choices:

DC : If R is a binary relation on a nonempty set A such that for every $x \in A$ there exists $y \in A$ so that xRy, then there is a sequence $\{x_n\}$ of elements in A satisfying:

$$(\forall n \in \omega)(x_n R x_{n+1}) .$$

It is easily shown that DC implies AC_ω (see [10], p. 23), and so the real analysis of $\mathbb{N}$ is "normal". It is also true that AC implies DC (see proof of Theorem 6.12). As an interesting aside, our construction of $\mathbb{N}$

will establish two independence results previously shown by other methods:

(1) The Axiom of Choice is independent of the other axioms of ZF (Cohen, [2]).

(2) AC is independent of DC (Feferman, [6]).

Definability is the central concept in our construction of $\mathbb{N}$. Our ultimate interest is with a family of sets which are the values of abstraction terms having special parameters. These notions are subject to hidden difficulties of which mention is now made. We first look at the simplest type of definability. A set x is <u>definable without parameters</u> (we write: Dwp(x)) if:

$$x = \{ y : \phi(y) \} ,$$

where ϕ is some formula with one free variable. For the class of such sets we write: DWP.

Our previous uses of definability have been informal in the sense that no formula ϕ_o of ZF has been exhibited for which: $\phi_o(x) \leftrightarrow \mathrm{Dwp}(x)$. In general, the following version of Richard's paradox prevents this.

<u>Proposition 6.1</u> $\neg \mathrm{Dwp}(\mathrm{DWP})$.

Proof: Since DWP is countable for the first order language of set theory, undefinable ordinals exist. If $\phi(x)$ has only x free, note that $\gamma = \{ \alpha : \forall \beta < \alpha, \phi(\beta) \}$ is the least ordinal not definable by ϕ.

We have $Dwp(\gamma)$, but $\neg\phi(\gamma)$.

Thus, no formula having one free variable can "define" DWP. Let $DWP^{\mathbb{E}}$ be the class of sets $\mathbb{E}$-definable without parameters, i.e. $Dwp^{\mathbb{E}}(x) \leftrightarrow x = \{ y : \mathbb{E} \models \phi(y) \}$. Because we are working with models having set-universes, the next result is true.

Proposition 6.2 (a) $Dwp^{\mathbb{E}}(a)$ iff there is a formula $\psi(x)$ with free variable x only, such that:

$$\mathbb{E} \models \forall x(\psi(x) \leftrightarrow x = a) .$$

(b) $Dwp(DWP^{\mathbb{E}})$.

Proof: (a) Let $a \in DWP^{\mathbb{E}}$. Then there is a formula ϕ such that $a = \{y : \mathbb{E} \models \phi(y)\}$. Define $\psi_o(y,x) \leftrightarrow (y \in x \leftrightarrow \phi(y))$. Then $\mathbb{E} \models \forall x(\forall y(\psi_o(y,x)) \leftrightarrow x = a)$. Conversely, suppose there is a formula $\psi(x)$, with only x free, and $\mathbb{E} \models \forall x(\psi(x) \leftrightarrow x = a)$. Then $a = \{y : \mathbb{E} \models \forall x\ \phi_o(y,x)\}$, where: $\phi_o(y,x) \leftrightarrow (y \in x \leftrightarrow \psi(x))$.

(b) We arithmetize the set theory of $\mathbb{E}$, and apply Gödel numbers $\ulcorner\phi\urcorner$ to each formula ϕ ([24], pp. 175 - 95). Since $\mathbb{E}$ has a set as universe, there

is a Δ_1^{ZF} formula ([25], p. 193; cf. [4], p. 91):

$Sat(\alpha, \mathbb{E}, a) \leftrightarrow \alpha = \ulcorner \phi \urcorner \ \& \ \mathbb{E} \models \phi(a)$.

From (a) we have:

$Dwp^{\mathbb{E}}(a) \leftrightarrow \mathbb{E} \models \forall x(\psi(x) \rightarrow x = a)$

$\leftrightarrow Sat(\alpha_o, \mathbb{E}, a)$, where $\alpha_o = \ulcorner \forall x(\psi(x) \rightarrow x = a) \urcorner$ is dependent on a, and $\mathbb{E}$ is fixed.

Next, we look at the class of sets which are the values of abstraction terms whose only parameters are ordinals. This class is known in the literature as the ordinal-definable sets, denoted OD. Because of Proposition 6.1, the definability of this class within ZF must be shown with an extended form of the reflection principle ([15], p. 273). Since we are working within $\mathbb{M}[G]$ which has a set as universe, we can use the simpler device of Proposition 6.2(b). We first arithmetize the formulas of ZF, assigning them unique Gödel numbers as in [24], pp. 175 - 95. We let $\mathcal{Y}$ denote the set of Gödel numbers of formulas of ZF. $\mathcal{Y}$ may be thought of as a countable set of ordinals in $\mathbb{M}[G]$. Our analogue to OD must include an additional parameter, namely a countable sequence of ordinals, so we give it a different symbol.

Definition 6.3 $\quad x \in OD' \leftrightarrow \mathbb{M}[G] \models (\exists \ulcorner \phi \urcorner \in \mathcal{Y})(\exists t \in {}^{\omega}\theta_{\mathbb{M}})$

$$(\exists \alpha_1, \ldots, \alpha_n \in {}^{\Theta}\mathbb{M})(\forall y)(y \in x \leftrightarrow \phi(y, t, \alpha_1, \ldots, \alpha_n)) .$$

OD' is the family of sets ordinal-definable in $\mathbb{M}[G]$ from a sequence of ordinals. For each set of reals $E \in OD'$, Theorem 5.10 asserts that E is Lebesgue measurable.

Neither OD nor OD' are necessarily transitive, as elements of OD (resp. OD') sets may not be OD (resp. OD'). OD does contain, however, a transitive submodel known in the literature as the family of hereditarily ordinal-definable sets (HOD). The sets of HOD and all their ancestors via the membership relation are HOD.

The transitive closure TC(x) of a set x is the least set containing x that is transitive. Existence of TC(x) presents no problem:

Proposition 6.4 Let $\mathbb{E}$ be a standard transitive model of ZF. $\forall x \in \mathbb{E}$, $TC(x) \in \mathbb{E}$.

Proof: Let the sequence $\{x_n\}$ be defined: $x_0 = x$, $x_{n+1} = \cup x_n$. By standardness and transitivity, along with the axioms of Infinity and Replacement[1], $\{x_n\} \in \mathbb{E}$. By the axiom of Regularity, $\bigcup_n x_n = TC(x)$. By the axiom of Unions, $TC(x) \in \mathbb{E}$.

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[1] These conditions are essential; see [4], p. 111 .

Just as OD' is our analogue to OD, our definition of $\mathbb{N}$ is analogous to that of $HOD = \{ x \in OD : TC(x) \in OD \}$.

Definition 6.5 $\quad \mathbb{N} = \{ x \in OD' : TC(x) \in OD' \}$.

Lemma 6.6 $\quad \mathbb{N} = \{ x \in OD' : x \subset \mathbb{N} \}$.

Proof: By 6.4, $TC(x) = \{x\} \cup \{ TC(y) : y \in x \}$,
so $x \in \mathbb{N}$ iff $x \in OD' \ \& \ (\forall y \in x)(y \in \mathbb{N})$.

Corollary 6.7 $\quad \mathbb{N}$ is transitive.

Despite the close definitional analogy between the class HOD and the set $\mathbb{N}$, there is one important difference. HOD satisfies AC (see [15], p. 276), but we shall see that $\mathbb{N}$ does not.

The Myhill-Scott proof for $HOD \models ZF$ adapts easily to $\mathbb{N}$.

Theorem 6.8 $\quad \mathbb{N} \models ZF$.

Proof: (a) Since $\mathbb{N}$ is transitive, the following hold (see [9], pp. 21, 23):

(i) $\mathbb{N}$ is extensional.

(ii) $(x,y)^{\mathbb{N}} = (x,y)$.

(iii) $\cup^{\mathbb{N}} x = \cup x$

(iv) $\mathbb{P}^{\mathbb{N}}(x) = \mathbb{P}(x) \cap \mathbb{N}$.

These verify the axioms of Extensionality, Pairs, Unions, and Powerset in $\mathbb{N}$.

(b) $\mathbb{N}$ is standard and $\in \cap \mathbb{N}^2$ is well-

founded, so the axiom of Regularity holds in $\mathbb{N}$.

(c) $\emptyset \in \mathbb{N}$ and $\omega \in \mathbb{N}$, so the axioms of Null set and Infinity hold in $\mathbb{N}$.

(d) The axiom schema of Separation holds in $\mathbb{N}$:

(i) If x is definable from parameters $b_1, \ldots, b_n$ which are in OD', then $x \in OD'$. This is obvious from 6.3; the ordinal parameters for each b_i parametrize x, and the ordinal sequence parameters t_i for each b_i can be amalgamated into a single ordinal sequence parameter t for x, in which the t_i appear as subsequences:

$$t(k) = \begin{cases} t_i(j) & ; \ k = 2^i 3^j \\ 0 & ; \text{ otherwise} \end{cases}$$

(ii) Let ϕ be a formula and let $a, b_1, \ldots, b_n \in \mathbb{N} \subset OD'$. Then: $c = \{ x \in a : \mathbb{N} \models \phi(x, b_1, \ldots, b_n) \} \in OD'$ from (i). $c \in \mathbb{N}$ by Lemma 6.6.

(e) The axiom schema of Replacement holds in $\mathbb{N}$:

Suppose: (i) $f = \{(x,y) \in \mathbb{N}^2 : \mathbb{N} \models \phi(x,y,b_1, \ldots, b_n)\}$ where $b_1, \ldots, b_n$

ε $\mathbb{N} \subset OD'$.

(ii) $\mathbb{N} \models$ f is a function.

Then for a ε $\mathbb{N}$,

$c = \{y : \mathbb{N} \models (\forall x \varepsilon a)\phi(x,y,b_1, \ldots ,b_n)\}$

belongs to OD', as in the argument for (d). Therefore c ε $\mathbb{N}$ by Lemma 6.6 .

From Proposition 6.2(b) and Definitions 6.3 and 6.5, we see that Dwp($\mathbb{N}$). The following uniformity result presents this fact in a more significant and useful form.

Lemma 6.9 There is a formula ϕ_0 free in s, y only, such that:

$\mathbb{M}[G] \models x \varepsilon \mathbb{N} \leftrightarrow (\exists s \varepsilon {}^{\omega}\theta_{\mathbb{M}})(\forall y)(\phi_0(s,y) \leftrightarrow y = x)$.

Proof: Let:

$\phi_0(s,y) \leftrightarrow (\exists \ulcorner\phi\urcorner \varepsilon \mathcal{V})(\exists \alpha_1, \ldots ,\alpha_n \varepsilon \theta_{\mathbb{M}})(\forall z)$ $([z \varepsilon y \leftrightarrow \phi(z,s,\alpha_1, \ldots ,\alpha_n)] \ \& \ \phi_0(y))$,

where $\phi_0(y) \leftrightarrow TC(y) \subset OD'$, which is free in y only (when fully written out in ZF using 6.3 and 6.4).

For any x ε $\mathbb{N}$, Lemma 6.9 tells us that x is uniquely determined by a sequence s ε ${}^{\omega}\theta_{\mathbb{M}}$. The next lemma exploits this to show that a large family of mathematical objects of $\mathbb{M}[G]$ exist in $\mathbb{N}$.

Lemma 6.10 Let $h:\omega \to \mathbb{N}$ and $h \in \mathbb{M}[G]$, then $h \in \mathbb{N}$.

Proof: Working in $\mathbb{M}[G]$, we define an ordinal $\gamma(x)$ as the least ordinal α such that there is an $s:\omega \to \alpha$ such that x is the unique y satisfying $\phi_0(s,y)$ (Lemma 6.9).

Let $\gamma = \sup_n \gamma(h(n))$. Using AC, we define a well-ordering L on $\{ s : s:\omega \to \gamma \}$. Let $s_n:\omega \to \gamma$ be the least of these s such that h(n) is the unique y satisfying $\phi_0(s,y)$. We amalgamate the sequences s_n into a single sequence $g:\omega \to \Theta_{\mathbb{M}}$:

$$g(k) = \begin{cases} s_m(n) & ; \ k = 2^m 3^n \\ 0 & ; \text{ otherwise} \end{cases}$$

h is definable from $\{s_n\}$ via the well-ordering $<$:

$y = h(n) \leftrightarrow (\forall s < s_n)(\neg \phi_0(s,y)) \ \& \ \phi_0(s_n,y)$,

and $\{s_n\}$ is clearly definable from g. Thus, $h \in OD'$.

Since $\omega \in \mathbb{N}$ and the axiom of Pairs holds in $\mathbb{N}$, $h \subset \mathbb{N}$ by definition. $h \in \mathbb{N}$ by Lemma 6.6 .

Corollary 6.11 (a) $\mathbb{R}^{\mathbb{M}[G]} \in \mathbb{N}$.

(b) ${}^{\omega}\Theta_{\mathbb{M}} \subset \mathbb{N}$.

The above techniques are all we need to prove the last two main theorems.

Theorem 6.12 $\mathbb{N} \models DC$.

Proof: Let $A, R \in \mathbb{N}$ satisfy the hypothesis of DC. Suppose $\{x_i\}_i \leq n$ is a finite sequence of elements of A such that $x_i R x_{i+1}$, $\forall i < n$. Since $\mathbb{M}[G] \models AC$, we may pick $x_{n+1} \in A$ such that $x_n R x_{n+1}$ for any value of n. By induction, there is a map $h:\omega \to A$ such that $h(n) = x_n$. From Lemma 6.10, $h \in \mathbb{N}$.

Theorem 6.13 $\mathbb{N} \models LM$.

Proof: By Corollary 6.11, each real in $\mathbb{M}[G]$ belongs to $\mathbb{N}$, each closed interval with rational end-points in $\mathbb{M}[G]$ belongs to $\mathbb{N}$, and each Borel code in $\mathbb{M}[G]$ belongs to $\mathbb{N}$. Thus $\mathbb{M}[G]$ and $\mathbb{N}$ have the same Borel sets. Let $E \subset \mathbb{R}^{\mathbb{N}}$, $E \in \mathbb{N}$, then by definition of $\mathbb{N}$ and Theorem 5.10 and Lemma 2.22, $\mathbb{M}[G] \models \phi(E)$, where: $\phi(E) \leftrightarrow (\exists G \in \mathcal{G}_\delta)(\exists N)(\mu(N) = 0 \ \& \ E = G \setminus N)$. From Theorem 2.24 and the above:

$$\mathbb{N} \models \phi(E) ,$$

i.e., E is Lebesgue measurable in $\mathbb{N}$.

From the assumption that there exists a model $\mathbb{M} \in \mathbb{K}$

such that $\mathbb{M} \models ZFC + I$, we have arrived through these last two sections at a model $\mathbb{N} \in \mathbb{K}$ satisfying $\mathbb{N} \models ZF + DC + LM$.

Conclusion

To summarize the development of the past seven sections, we will present the main results in the form of a theorem.

Theorem Let $\mathbb{K}$ be a non-minimal[1] standard transitive model of ZFC + I . Then:

(a) $\mathbb{K} \models$ there is a model of ZFC + " Every set of reals definable from a countable sequence of ordinals is Lebesgue measurable " .

(b) $\mathbb{K} \models$ there is a model of ZF + DC + LM .

We note that for $\mathbb{E} = (E, E^2 \cap \in)$, where E is a set, and ϕ a sentence, the statements $\mathbb{E} \models \phi$, and $\mathbb{E} \models$ ZF, are expressible as Δ_1^{ZF}-formulas ([4], pp. 94, 96, 97). Hence the statements " there exists a model of ... " are abbreviations of formulas in the language of set theory.

Our point of view is clearly different from that of Solovay [23]. His model construction takes place within the intuitive but ambiguous "real world" of set theory, while our constructions are relativized to a fixed model $\mathbb{K}$ of set theory. For this reason, Solovay's construction appears in the form of a model extension. Our actual constuction takes the same form of model extension, but our models $\mathbb{M}$, $\mathbb{M}[G]$, and $\mathbb{N}$ are internal models with respect to $\mathbb{K}$. This explains the format of our main theorem above.

Theorems 0.36, 5.10, and 6.8, 6.12, 6.13 provide the

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[1] i.e. $\mathbb{K}$ is not the least such model. See pp. 9 - 10 .

main verification of this theorem, except for the foundational aspects to which we now return.

Axiom A, which worked so well as a foundation for Sections 0 - 3, proved not to be powerful enough (p.91) to ensure the existence of an inaccessible in the ground model. Even the adoption of the much stronger Axiom I as a part of our metatheory would fail to do this. Perhaps the best compromise would be to introduce an axiom that guarantees the existence of at least two inaccessibles. Standard techniques (e.g. [4], p.110) then produce a set K such that $\mathbb{K} \models I$.

Now, supposing that K does exist such that $\mathbb{K} \models I$, we must still produce a countable ground model $\mathbb{M}$ satisfying I and belonging to $\mathbb{K}$.

A standard modification of the Löwenheim-Skolem-Tarski technique exists by which we can constuct a countable elementary submodel belonging to $\mathbb{K}$ ($\mathbb{K}$ is non-minimal), which we outline (see [12], c.f. [4]). First we take the closure of $\{\emptyset\}$ under a set of Skolem functions for $\mathbb{K}$ and Mostowski-collapse this to a transitive set M. $\mathbb{M}$ will be a countable elementary submodel of $\mathbb{K}$, thus $\mathbb{M} \models I$. By a theorem of Lévy ([4], p.104) M is <u>hereditarily countable</u> ($|TC(M)| < \omega_1$) which implies that M has countable rank ([4], p.103). Hence $M \subset \{ x \in \mathbb{K} : \text{rank}(x) < \omega_1 \} \in \mathbb{K}$. The power set axiom then tells us that $M \in \mathbb{K}$.

This completes the proof of our theorem.

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