THE CRYSTALLOGRAPHY OF THE ROTATION SUBGROUPS OF COXETER GROUPS

by

NORMA BRODERICK

B.Sc., University of British Columbia, 1972

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in the department
of
MATHEMATICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
April 1975
In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representative. It is understood that copying or publication for financial gain shall not be allowed without my written permission.

Department of Mathematics
The University of British Columbia
Vancouver 8, Canada

Date April 25, 1975
ABSTRACT

The theory of space groups has its origins in crystallography and solid state physics. In this thesis, we study those space groups in n-dimensional Euclidean space whose point groups are the rotation subgroups of crystallographic Coxeter groups.
# TABLE OF CONTENTS

## CHAPTER I.

1. Introduction ................................................. 1
2. Some technical results ....................................... 5
3. The crystallography of Coxeter groups ..................... 8

## CHAPTER II.

1. A presentation of $K^+$ ....................................... 12
2. The case $\dim V_K = 2$ ..................................... 14
3. The normaliser of $K^+$ ...................................... 15
4. Lattices invariant under $K^+$ ............................... 18
5. The groups $\mathcal{H}^1(K^+, V/\Lambda)$ ..................... 22
6. Application to classical crystallography ..................... 34

TABLE I .......................................................... 35

BIBLIOGRAPHY .................................................. 37
ACKNOWLEDGMENT

I would like to thank Professors B. Moyls and T. Anderson for reading this thesis and Professor G. Maxwell for suggesting the topic and helpful advice during its preparation.
CHAPTER I

1. Introduction.

The theory of space groups has its origins in crystallography and solid state physics. After introducing the fundamental concepts, we shall give a brief historical outline of its development.

Let $E$ be an affine space with $n$-dimensional real vector space of translations $V$ and affine group $A(E)$. Once an origin has been chosen in $E$, the elements of $A(E)$ can be expressed in the form $(t, g)$, where $t \in V$ and $g \in \text{GL}(V)$. The composition rule in $A(E)$ is then given by $(t, g)(r, h) = (t + gr, gh)$. Let $(x, y)$ be a positive definite symmetric bilinear form on $V$, $\Gamma$ its orthogonal group and $\Gamma(E)$ the subgroup of $A(E)$ consisting of isometries. A subgroup $S$ of the 'Euclidean group' $\Gamma(E)$ is called a space group if $S \cap V$ is a lattice $\Lambda$ in $V$. The projection of $S$ on $\Gamma$ is called the point group $K$ of $S$. Elements of $K$ leave $\Lambda$ invariant, for if $(a, g)$ is a representative in $S$ of $g \in K$ and $t \in \Lambda$, the element $(a, g)(t, 1)(a, g)^{-1} = (gt, 1) \in S$ and therefore $gt \in \Lambda$. We can thus regard $S$ as an extension of $\Lambda$ by $K$.

A unique cocycle $K \rightarrow V/\Lambda$ can be associated with $S$. Let $\{(s(g), g)\}$ be a system of representatives in $S$ of all $g \in K$. If $\{(s'(g), g)\}$ is another such system, then $(s(g), g)(s'(g), g)^{-1} = (s(g) - s'(g), 1) \in S$, so that the function $\beta: K \rightarrow V/\Lambda$ obtained by reducing the values of $s$ mod $\Lambda$ is uniquely determined by $S$. Since $(s(g), g)(s(h), h) = (s(g) + gs(h), gh)$, we have
\( \tilde{s}(gh) = \tilde{s}(g) + \tilde{g}\tilde{s}(h) \), so that \( \tilde{s} \) is a cocycle. Conversely, given a cocycle \( \tilde{s}: K \to V/\Lambda \), the set of elements in \( \Gamma'(E) \) of the form \( (s(g)+t,g) \), where \( g \in K \), \( t \in \Lambda \) and \( s: K \to V \) is some lifting of \( \tilde{s} \), form the space group with point group \( K \) and lattice \( \Lambda \) which induces the given cocycle by the previous construction.

Two space groups \( S \) and \( S' \) are considered equivalent if they are conjugate by an element of \( A(E) \). If \( S' \) is conjugate to \( S \) by a translation \( (r,1) \), then every element in \( S' \) is of the form \( (r,1)(t,g)(r,1)^{-1} = (t+r-gr,g) \), where \( (t,g) \in S \). Thus \( S' \) has the same point group and lattice as \( S \) but its cocycle \( \tilde{s}' : K \to V/\Lambda \) differs from \( s \) by the coboundary \( g \to \Gamma - g\Gamma \). Conversely, cohomologous cocycles \( K \to V/\Lambda \) correspond to space groups which are conjugate by a translation. Secondly, if \( S' \) is conjugate to \( S \) by an element in \( A(E) \) of the form \( (0,h) \), then \( S' \) consists of elements \( (0,h)(t,g)(0,h)^{-1} = (ht,hgh^{-1}) \), where \( (t,g) \in S \).

It follows that in this case \( \Lambda' = h\Lambda, K' = hKh^{-1} \) and \( \tilde{s}'(g) = h\tilde{s}(h^{-1}gh) \).

Consequently, the equivalence classes of space groups can be determined in the following way. Call a subgroup of \( \Gamma \) crystallographic if it leaves invariant a lattice in \( V \) and determine the conjugacy classes of such subgroups in \( \text{GL}(V) \).

For each class \( \{K\} \) determine the set \( L(K) \) of all lattices left invariant by \( K \) and find the orbits in this set under the natural action of the normaliser \( N(K) \) of \( K \) in \( \text{GL}(V) \).

Calculate \( H^1(K,V/\Lambda) \) for each of these orbits \( \{\Lambda\} \). Let \( N(K,\Lambda) \) be the subgroup of \( N(K) \) consisting of those elements which leave \( \Lambda \) invariant. Then the orbits of \( H^1(K,V/\Lambda) \) under the
action of $N(K, \Lambda)$ given by $h^h(g) = hgh^{-1}g^{-1}$ correspond to the
equivalence classes of space groups with point group $K$ and
lattice $\Lambda$.

The determination of the equivalence classes of space
groups dates back to the last century for $n \leq 3$. In three
dimensions, a space group can be considered as the symmetry
group of an infinite crystal and the problem of finding all
inequivalent space groups was attacked by crystallographers
trying to understand possible crystal structures. In 1830,
Hessel found the 32 classes of point groups; every crystal
could now be placed into one of 32 types according to its
external symmetry. However, Hessel's work remained unnoticed
for over 60 years and these groups were independently derived
by Bravais in 1848, who also found two years later the 73
inequivalent pairs ($\Lambda, K$). In 1879 Sohncke determined the
54 inequivalent space groups whose point groups consisted
only of rotations and observed that Jordan had derived them
10 years previously but had not expressed his results in
geometric terms. Finally, the complete list of 219 inequivalent
space groups was derived independently by Fedorov in 1885,
Schoenflies in 1891 and Barlow in 1894. However, it was not
until 1912 that Laue was able to confirm these predictions of
crystal structure through experiments with X-ray diffraction.
In two dimensions, a space group can be considered as the
symmetry group of a plane pattern repeated periodically in
two directions, such as is common, for example, in wallpaper
and floor tiling designs. However, it was not until 1891
that Fedorov gave the first mathematical treatment of these groups and determined the 17 possible inequivalent space groups. Fricke and Klein rediscovered these groups in 1897, as did Pólya and Niggli in 1924. Examples of all 17 symmetry groups can be found among the decorative patterns of the early Egyptians. Indeed, in Weyl's words, the art of ornamentation contains in implicit form the oldest piece of higher mathematics known to us.

The solution of the problem for \( n \leq 3 \) led Hilbert to ask in 1900 (as part of his 18th problem) whether there were only finitely many inequivalent space groups in any dimension. Fundamental results of Minkowski and Hermite implied that there were only finitely many inequivalent pairs \((\Lambda, K)\). Joined to the finiteness of \( H^1(K, V/\Lambda) \), this enabled Bieberbach and Frobenius to answer Hilbert's question in the affirmative.

In 1951, Hurley [11] published a list of 222 point groups for \( n = 4 \); the list was completed to 227 in 1967. In 1965, Dade [9] found the maximal subgroups of \( \text{GL}_4(\mathbb{Z}) \) by considering automorphisms of positive definite integral quadratic forms. With the help of a computer, Bülow, Neubüser and Wondratschek [5] investigated the subgroups of these groups and determined the 710 inequivalent pairs \((\Lambda, K)\). In 1972, Brown, Neubüser and Zassenhaus [2-4] found the 4733 inequivalent space groups. Ryskov [13] devised a method of constructing a finite family of positive definite quadratic forms in \( n \) variables such that all maximal finite subgroups of \( \text{GL}_n(\mathbb{Z}) \) occurred among the
groups of their integral automorphisms, and actually found the maximal subgroups for \( n = 5 \).

In the general \( n \)-dimensional case, Burkhardt [6] treated in 1946 the special case of \( K \) being a symmetric group of degree \( n \) or its alternating subgroup, with \( \Lambda \) being a certain type of lattice. Some progress was also made by Hermann [10] in 1948, while Zassenhaus [15] developed an algorithm for the calculation of \( H^1(K,V/\Lambda) \). However, one cannot reasonably expect a general solution since every finite group will eventually occur as the point group of some space group.

The class of point groups which are generated by reflections, i.e. are crystallographic Coxeter groups, has the advantage of being well understood and yet general enough to account (together with their subgroups) for all cases if \( n \leq 3 \). It was studied by Maxwell [12]. In this thesis, we shall investigate the case when the point group is the rotation subgroup of a crystallographic Coxeter group. Together with Maxwell's results, we can then derive practically all inequivalent space groups for \( n \leq 3 \) from theorems valid in all dimensions. In the remainder of this chapter, we shall summarise some technical results as well as the situation when \( K \) is a crystallographic Coxeter group. The proofs of these facts can be found in [12].

2. Some technical results.

2.1 If \( K \) is a subgroup of \( \Gamma \), let \( V^K \) be the subspace of elements fixed by \( K \) and \( V_K \) the orthogonal complement of \( V^K \) in \( V \). The group \( K \) is called essential if \( V^K = 0 \); in particular, the restriction
$\overline{K}$ of $K$ to $V_K$ is essential. Subgroups $K$ and $K'$ of $\Gamma$ are conjugate in $\Gamma$ iff they are conjugate in $\text{GL}(V)$; it is therefore sufficient to determine the conjugacy classes of point groups within $\Gamma$.

Furthermore, $K$ and $K'$ are conjugate in $\Gamma$ iff $\overline{K}$ and $\overline{K}'$ are conjugate by an isometry $V^K_K \rightarrow V^K_{K'}$. As $V^K_K$ and $V^K_{K'}$ are invariant under $N(K)$, we see that $N(K) = \text{GL}(V^K) \times N(\overline{K})$. One can also prove that $N(\overline{K}) = C(\overline{K})N_{\overline{K}}(K)$, where $C(\overline{K})$ is the centraliser of $\overline{K}$ in $\text{GL}(V^K_K)$ and $N_{\overline{K}}(K)$ is the normaliser of $\overline{K}$ in $\overline{\Gamma}$.

2.2 If $\Lambda$ is invariant under $K$, we define the weight group $\Lambda^*$ to be the subgroup of $V$ consisting of all elements $x$ such that $x - gx \in \Lambda$ for every $g \in K$. The exact sequence

$$0 \rightarrow \Lambda \rightarrow V \rightarrow V/\Lambda \rightarrow 0$$

gives rise to the long exact sequence

$$\ldots \rightarrow H^0(K, V) \rightarrow H^0(K, V/\Lambda) \rightarrow H^1(K, \Lambda) \rightarrow H^1(K, V) \rightarrow \ldots$$

It follows that if $K$ is essential, we have $\Lambda^*/\Lambda = H^1(K, \Lambda)$, so that $\Lambda^*/\Lambda$ is finite and $\Lambda^*$ is a lattice. A substantial part of $H^1(K, V/\Lambda)$ can be obtained as follows. Call a cocycle $\bar{\alpha}: K \rightarrow V/\Lambda$ weightlike if all its values lie in $\Lambda^*/\Lambda$. The exact sequence

$$0 \rightarrow \Lambda^*/\Lambda \rightarrow V/\Lambda \rightarrow V/\Lambda^* \rightarrow 0$$

gives rise to the long exact sequence

$$0 \rightarrow \Lambda^{**}/\Lambda^* \rightarrow \text{Hom}(K, \Lambda^*/\Lambda) \rightarrow H^1(K, V/\Lambda) \rightarrow \ldots \ldots$$

The image of $\omega$ will be called the group of weightlike classes and denoted by $\mathcal{J}(K, \Lambda)$.

2.3 Suppose $\Lambda$ is a lattice invariant under $K$. Let $\Lambda^K = \Lambda \cap V^K$, $\Lambda^K = \Lambda \cap V^K$ and $\Lambda_0 = \Lambda^K + \Lambda_K$. If $x = y + z \in \Lambda$, where $y \in V^K$ and $z \in V_K$, then $x - gx = z - gz \in \Lambda_K$, so that $z \in \Lambda^*_K$.

The mapping $x \rightarrow z$ induces a homomorphism $\Lambda/\Lambda_0 \rightarrow \Lambda/K^*\Lambda_K$ which is injective since if $z \in \Lambda^*_K$, $y = x - z \in \Lambda^K$ and thus $x \in \Lambda_0$. 

$$\begin{array}{c}
\overline{\text{K of K to V}_K \text{ is essential. Subgroups K and K' of } \Gamma \text{ are conjugate in } \Gamma \text{ iff they are conjugate in } \text{GL}(V); \text{ it is therefore sufficient to determine the conjugacy classes of point groups within } \Gamma. \\
\text{Furthermore, K and K' are conjugate in } \Gamma \text{ iff } \overline{K} \text{ and } \overline{K}' \text{ are conjugate by an isometry } V^K_K \rightarrow V^K_{K'}. \text{ As } V^K_K \text{ and } V^K_{K'} \text{ are invariant under } N(K), \text{ we see that } N(K) = \text{GL}(V^K) \times N(\overline{K}). \text{ One can also prove that } N(\overline{K}) = C(\overline{K})N_{\overline{K}}(K), \text{ where } C(\overline{K}) \text{ is the centraliser of } \overline{K} \text{ in } \text{GL}(V^K_K) \text{ and } N_{\overline{K}}(K) \text{ is the normaliser of } \overline{K} \text{ in } \overline{\Gamma}. \\
2.2 \text{ If } \Lambda \text{ is invariant under } K, \text{ we define the weight group } \Lambda^* \text{ to be the subgroup of } V \text{ consisting of all elements } x \text{ such that } x - gx \in \Lambda \text{ for every } g \in K. \text{ The exact sequence } \\
0 \rightarrow \Lambda \rightarrow V \rightarrow V/\Lambda \rightarrow 0 \text{ gives rise to the long exact sequence } \\
\ldots \rightarrow H^0(K, V) \rightarrow H^0(K, V/\Lambda) \rightarrow H^1(K, \Lambda) \rightarrow H^1(K, V) \rightarrow \ldots \ldots \\
\text{It follows that if } K \text{ is essential, we have } \Lambda^*/\Lambda = H^1(K, \Lambda), \text{ so that } \Lambda^*/\Lambda \text{ is finite and } \Lambda^* \text{ is a lattice. A substantial part of } H^1(K, V/\Lambda) \text{ can be obtained as follows. Call a cocycle } \bar{\alpha}: K \rightarrow V/\Lambda \text{ weightlike if all its values lie in } \Lambda^*/\Lambda. \text{ The exact sequence } \\
0 \rightarrow \Lambda^*/\Lambda \rightarrow V/\Lambda \rightarrow V/\Lambda^* \rightarrow 0 \text{ gives rise to the long exact sequence } \\
0 \rightarrow \Lambda^{**}/\Lambda^* \rightarrow \text{Hom}(K, \Lambda^*/\Lambda) \rightarrow H^1(K, V/\Lambda) \rightarrow \ldots \ldots \\
\text{The image of } \omega \text{ will be called the group of weightlike classes and denoted by } \mathcal{J}(K, \Lambda). \\
2.3 \text{ Suppose } \Lambda \text{ is a lattice invariant under } K. \text{ Let } \Lambda^K = \Lambda \cap V^K, \Lambda^*_K = \Lambda \cap V_K \text{ and } \Lambda_0 = \Lambda^K + \Lambda_K. \text{ If } x = y + z \in \Lambda, \text{ where } y \in V^K \text{ and } z \in V_K, \text{ then } x - gx = z - gz \in \Lambda_K, \text{ so that } z \in \Lambda^*_K. \text{ The mapping } x \rightarrow z \text{ induces a homomorphism } \Lambda/\Lambda_0 \rightarrow \Lambda^*_K/\Lambda_K \text{ which is injective since if } z \in \Lambda^*_K, y = x - z \in \Lambda^K \text{ and thus } x \in \Lambda_0. \end{array}$$
It follows that $\Lambda/\Lambda_0$ is finite and $\Lambda_0$ is a lattice. Let $\Theta(\Lambda)$ be the image of $\Lambda/\Lambda_0$ in $\Lambda^K/\Lambda_K$. The mapping $x \mapsto y$ also induces an injective homomorphism $\Lambda/\Lambda_0 \rightarrow V^K/\Lambda^K$ whose image is isomorphic to $\Theta(\Lambda)$. Conversely, given lattices $\Lambda^K \subset V^K$ and $\Lambda_K \subset V_K$ and a subgroup $\Theta$ of $\Lambda^K/\Lambda_K$ which is isomorphic to a subgroup $\Theta_1$ of $V^K/\Lambda^K$, one can construct a lattice $\Lambda \supset \Lambda_0 = \Lambda^K \oplus \Lambda_K$ such that $\Theta(\Lambda) = \Theta$. Namely, let $\kappa: \Theta \rightarrow \Theta_1$ be some isomorphism and define

$$\Lambda = \bigcup_{\Theta \in \Theta} (y_K(\Theta) + z_\Theta + \Lambda_0),$$

where $z_\Theta$ (resp. $y_K(\Theta)$) are representatives of the elements of $\Theta$ (resp. $\Theta_1$) in $V_K$ (resp. $V^K$). One can show that if $\Lambda'$ is another lattice invariant under $K$, then $\Lambda' = g\Lambda$ for some $g \in N(K)$ iff there exists $h \in N(K)$ which maps $\Lambda_K$ to $\Lambda'_K$ and induces an isomorphism $\Theta(\Lambda) \rightarrow \Theta(\Lambda')$. Therefore, in order to obtain a set of representatives of the orbits of $L(K)$ under the action of $N(K)$, it is sufficient to first choose a set of lattices $\Lambda_K$ representative of the orbits of $L(\overline{K})$ under $N(\overline{K})$ and an arbitrary lattice $\Lambda^K \subset V^K$. For each $\Lambda_K$, one then chooses a representative set of subgroups $\Theta$ of $\Lambda^K/\Lambda_K$ from those which are isomorphic to subgroups of $V^K/\Lambda^K$. Finally, one combines $\Lambda^K, \Lambda_K$ and $\Theta$ by (1).

2.4 To calculate $H^1(K, V/\Lambda)$, we shall essentially use the idea of Zassenhaus [15]. Choose a presentation $F/R$ of $K$ with generators $\{e_i\}_{i \in I}$ and relations $\{R_j\}_{j \in J}$. By first reducing the elements of $F$ modulo $R$, one obtains an action of $F$ on $V$.

A cocycle $t:F \rightarrow V$ is uniquely determined by the vector $(t(e_i))$, which may be chosen arbitrarily. Since $t(uR_ju^{-1}) = ut(R_j)$, the values of $t$ on $R$ are determined by the vector $(t(R_j))$. 
The cocycle \( t: F \rightarrow V \) induces a cocycle \( \tilde{t}: K \rightarrow V/\Lambda \) iff \( t(R_j) \in \Lambda \) for all \( j \in J \), and every cocycle \( K \rightarrow V/\Lambda \) can be obtained in this way.

2.5 If the inversion \(-1_V\) is not an element of \( K \), we shall denote by \( +K \) the subgroup generated by \( K \) and \(-1_V\). Clearly \( L(K) = L(+K) \) and \( N(K) \subseteq N(+K) \); however, there may be lattices inequivalent under \( N(K) \) but equivalent under \( N(+K) \). If \( \Lambda \in L(K) \), it follows from the exact sequence

\[
0 \rightarrow H^1(\pm 1_V, (V/\Lambda)^K) \xrightarrow{\text{inf}} H^1(K, V/\Lambda) \xrightarrow{\text{res}} H^1(K, V/\Lambda) \xrightarrow{\pm 1_V} 0
\]

that \( H^1(\pm K, V/\Lambda) \) is isomorphic to the direct sum of the group of elements of order \( \leq 2 \) in \( H^1(K, V/\Lambda) \) and the quotient of \( A^* / \Lambda \) by \( 2(A^* / \Lambda) \).

3. The crystallography of Coxeter groups.

Suppose \( \overline{K} \) is the Weyl group \( W(R) \) of a root system \( R \) in \( V \). Following Bourbaki [1], we denote by \( s_\alpha \) the reflection corresponding to the root \( \alpha \) and by \( Q(R) \) and \( P(R) \) the root and weight lattices of \( R \). Let \( B = \{ \alpha_i \} \) be a basis of \( R \) and \( S = \{ s_i \} \) the corresponding set of generators of \( K \), where we extend the function \( \alpha_i \) from \( V \) to \( V^* \) by letting it vanish on \( V \). Then \( \{ (s_i s_j)^m_{ij} = 1 \} \) is a presentation of \( K \), summarised by the Coxeter matrix \( (m_{ij}) \).

3.1 We define \( \widetilde{A}(R) \) to be the group of those \( g \in GL(V) \) which satisfy \( g_\alpha = a_\alpha h_\alpha \) for all \( \alpha \in B \), where \( h \) is an angle preserving permutation of \( B \) and the \( a_\alpha \) are positive real numbers satisfying \( n(\beta , \alpha) a_\alpha = n(h \beta , h_\alpha) a_\beta \) for all \( \alpha, \beta \in B \). Then \( N(\overline{K}) \) is equal to the semidirect product of \( \overline{K} \) and \( \widetilde{A}(R) \). If \( R \) is irreducible, one can see that \( \widetilde{A}(R) = H \cdot \widetilde{A}_0(R) \), where \( H \) is the
group of positive homotheties and $\tilde{A}_0(R)$ the group of graph automorphisms of $B$ except for the cases of $B_2, G_2$ and $F_4$, when $\tilde{A}_0(R)$ possesses an additional element.

3.2 Every lattice $\Lambda_K \in L(K)$ is such that $Q(R') \subset \Lambda_K \subset P(R')$ and $\Lambda_K^* = P(R')$ for some root system $R'$ satisfying $W(R') = \bar{K}$. If $R$ and $R'$ are two root systems of the same type for which $W(R) = W(R') = \bar{K}$, it follows from 3.1 that there exists $g \in \tilde{A}(R)$ such that $R' = gR$. Therefore lattices $\Lambda_K^*$ satisfying $Q(R') \subset \Lambda_K \subset P(R')$ are equivalent under $N(K)$ to lattices $\Lambda_K$ such that $Q(R) \subset \Lambda_K \subset P(R)$. Consequently, in order to find a representative set of lattices, it is sufficient to consider only different types of root systems $R$ for which $W(R) = \bar{K}$ and for each type $R$ only those lattices $\Lambda_K$ satisfying $Q(R) \subset \Lambda_K \subset P(R)$ and $\Lambda_K^* = P(R)$; the latter condition is equivalent to requiring that $\alpha/2 \not\in \Lambda_K$ for all $\alpha \in B$.

3.3 By 2.4, a map $t:S \rightarrow V$ will induce a cocycle $\bar{t}:K \rightarrow V/\Lambda$ iff

$$t((s_i s_j)^{m_{ij}}) = (1 + s_i s_j + \ldots + (s_i s_j)^{m_{ij}-1})(t(s_i) + s_i t(s_j)) \in \Lambda$$

for all $i, j$. Let $t(s_i) = y_i + \sum_k t_{ki}\alpha_k$, with $y_i \in V_K$ and define $p_{ij} = \langle t(g_i), \alpha_j \rangle$. Then $\tilde{t}$ is a coboundary iff $t(s_i) = t_{ii}\alpha_i \mod \Lambda$ for all $i$. One can deduce from (2) that $2H^1(K, V/\Lambda) = 0$. By subtracting from $\tilde{t}$ the map $b(s_i) = p_{ii}^{-1}/2$, which induces a coboundary, we can assume that $p_{ii} \in \mathbb{Z}$. Equations (2) then amount to the following:

$$2t(s_i) \in \Lambda \quad \text{for all } i$$

$$p_{ij} \alpha_j = p_{ji} \alpha_i \mod \Lambda \quad \text{if } n_{ij} = 0$$

$$t(s_i) - t(s_j) = p_{ji} \alpha_j - p_{ij} \alpha_i \mod \Lambda \quad \text{if } n_{ij} = n_{ji} = -1.$$
(We have abbreviated the Cartan integer \( n(\alpha_i, \alpha_j) \) by \( n_{ij} \).) Since the projection of \( \Lambda \) on \( V_K \) is contained in \( \Lambda_K^* = P(R) \), we can evaluate \( \tilde{\alpha}_j \) at both sides of eqns (3)-(5) to obtain the following necessary conditions for \( t \) to induce a cocycle:

(6) \[ 2p_{ij} \in \mathbb{Z} \quad \text{for all } i, j \]

(7) \[ p_{ij}n_{jk} = p_{ji}n_{ik} \mod \mathbb{Z} \quad \text{for all } k \text{ if } n_{ij} = 0 \]

(8) \[ p_{ik} - p_{jk} = p_{ji}n_{jk} - p_{ij}n_{ik} \mod \mathbb{Z} \quad \text{for all } k \text{ if } n_{ij} = n_{ji} = -1. \]

Call the last root \( \alpha_m \) in a component of \( B \) of type \( C_m \) exceptional; it is characterised by the fact that \( \alpha_m/2 \) is a weight. The equations (6)-(8) imply that by replacing \( \overline{t} \) with a cohomologous cocycle, we can assume that \( p_{ij} \in \mathbb{Z} \) except for the following cases (when \( i \neq j \)):

(i) \( \alpha_i \) and \( \alpha_j \) are exceptional and \( (\alpha_i - \alpha_j)/2 \in \Lambda \), when \( p_{ij} = p_{ji} \mod \mathbb{Z} \).

(ii) \( \alpha_i \) is the last and \( \alpha_j \) the second last root in a component of type \( C_m \), \( m \geq 2 \), when \( \tilde{\alpha}_j(\Lambda) = \mathbb{Z} \).

(iii) \( \alpha_i \) and \( \alpha_j \) belong to a component of type \( A_3, B_3 \) or \( B_4 \) and \( (\alpha_i - \alpha_j)/2 \in \Lambda \), when the only exceptions are \( p_{13} = p_{31} = p_{23} \mod \mathbb{Z} \).

(iv) \( \alpha_i \) and \( \alpha_j \) belong to a component of type \( D_4 \), when the exceptions can be \( p_{13} = p_{31} = p_{23} \mod \mathbb{Z} \), \( p_{14} = p_{41} = p_{24} \mod \mathbb{Z} \) or \( p_{34} = p_{43} = p_{23} + p_{24} \mod \mathbb{Z} \). Correspondingly, the elements \( (\alpha_i - \alpha_j)/2, (\alpha_i - \alpha_4)/2 \) or \( (\alpha_j - \alpha_4)/2 \) must belong to \( \Lambda \).

3.4 When \( p_{ij} \in \mathbb{Z} \) for all \( i, j \), the cocycle \( \overline{t} \) is weightlike in the sense of 2.2. The dimension over \( \mathbb{Z}/2 \mathbb{Z} \) of the subgroup \( \Omega(K, \Lambda) \) can be calculated as follows. Let \( \delta \) be the dimension over \( \mathbb{Z}/2 \mathbb{Z} \) of the set of elements of order \( \leq 2 \) in \( (V \otimes P(R))/\Lambda \), \( \rho \) the number of connected components in the Dynkin diagram of \( B \) after
all double and triple bonds have been removed and \( \chi \) the number of components of type \( C_m \) in \( B \). Then \( \dim \mathbb{Z}/2 \mathbb{Z} \Omega(K, \Lambda) = \delta \rho - \chi \).

3.5 If \( \Pi \) is a component in \( B \) of type \( C_m \), with \( m \) odd and \( \geq 3 \), then \( \omega_{m-1} \in \mathbb{Q}(R) \) so that condition (ii) of 3.3 is satisfied.

Let \( t: \Pi \rightarrow V \) be defined by \( t(s_m) = \omega_{m-1}/2 \) when \( \omega_m \) is the last root of \( \Pi \) and \( t(s_i) = 0 \) otherwise; then \( t \) induces a cocycle \( \tilde{t} : K \rightarrow V/\Lambda \). Let \( \mathcal{C} \) be the subgroup of \( H^1(K, V/\Lambda) \) generated by the cohomology classes of such cocycles. Then \( \mathcal{C} \) has zero intersection with \( \mathcal{N}(K, \Lambda) \) and its dimension over \( \mathbb{Z}/2 \mathbb{Z} \) is equal to the number of components in \( B \) of type \( C_m \), with \( m \) odd and \( \geq 3 \). Frequently, \( \mathcal{N}(K, \Lambda) \oplus \mathcal{C} \) accounts for the entire group \( H^1(K, V/\Lambda) \). More generally, suppose \( \{R_q^q\} \) is the set of components in \( B \) of type \( C_m, A_3, B_3, B_4 \) or \( D_4 \) and \( R' \) the remaining part of \( K \). Suppose that \( \Lambda = \Lambda^K \oplus \Lambda^K_q \oplus (\oplus_q \Lambda^K_q) \), where \( \Lambda^K \) (resp. \( \Lambda^K_q \)) is the intersection of \( \Lambda^K \) with the subspace spanned by the elements of \( R' \) (resp. \( R_q^q \)). Then \( H^1(K, V/\Lambda) = \mathcal{N}(K, \Lambda) \oplus \mathcal{C} \oplus \mathcal{D} \) where the dimension of \( \mathcal{D} \) over \( \mathbb{Z}/2 \mathbb{Z} \) is equal to the number of components \( R^q \) of type \( A_3, B_3 \) or \( B_4 \) plus twice the number of components of type \( D_4 \) for which \( \Lambda^K_q \) equals \( \mathcal{F}(R^q) \). For lattices not covered by these remarks, the existence of additional cocycles can be investigated in each case by starting from the equations (i)-(iii) of 3.3.
CHAPTER II

Let \( K \) be a crystallographic subgroup of \( \Gamma \) generated by reflections and \( K^+ \) its rotation subgroup. In this chapter, we shall investigate the crystallography of \( K^+ \). Suppose \( R \) is a root system in \( V_K \) for which \( W(R) = \overline{K} \). Let \( B = \{ \alpha_i \} \) be a basis of \( R \), \( S = \{ s_i \} \) the set of corresponding generators of \( K \) and \( M = (m_{ij}) \) the Coxeter matrix of \( S \). We shall abbreviate the Cartan integer \( n(\alpha_i, \alpha_j) \) by \( n_{ij} \) and extend the function \( \alpha_i \) from \( V_K \) to \( V \) by letting it vanish on \( V^K \). The case of \( \dim V_K = 1 \) is trivial and will be excluded from discussion. The somewhat atypical case of \( \dim V_K = 2 \) will be discussed in section 2; from then on, we shall assume that \( \dim V_K \geq 3 \).

1. A presentation of \( K^+ \).

The group \( K^+ \) consists of those elements of \( K \) which can be written as a product of an even number of elements of \( S \). Choose any \( \alpha_0 \in B \) and let \( g_i = s_1 s_0 \) for \( i \neq 0 \); then \( g_i \in K^+ \). Since \( g_i g_j^{-1} = s_i s_j \), the \( g_i \)s generate \( K^+ \). They satisfy the following relations:

\[
(1) \quad g_i^{m_{i0}} = 1, \quad (g_i g_j^{-1})^{m_{ij}} = 1.
\]

1.1 Proposition. The relations (1) form a presentation of \( K^+ \).

Proof. Let \( G^+ \) be the abstract group defined by a set of generators \( \{ g_i \} \), subject to the relations (1). Then \( \xi(g_i) = g_i^{-1} \) defines an automorphism of \( G^+ \) such that \( \xi^2 = 1 \), since a word of the form \( g_i^{m_{i0}} \) is mapped to its inverse by \( \xi \), while a word of the form \( (g_i g_j^{-1})^{m_{ij}} \) is mapped to \( g_i^{-1}(g_j g_i^{-1})^{m_{ij}} g_i \). We can obtain an action of the two-element group \( \{ \pm 1 \} \) on \( G^+ \) by
associating the automorphism $\xi$ to the element $-1$. Let $G$ be the semidirect product of $G^+$ and $[\pm 1]$ relative to this action. The function $\phi(g_1) = s_1 s_0$ and $\phi(-1) = s_0$ induces a homomorphism $G \to K$ since the conjugate of $g_1$ by $-1$ in $G$ is mapped to the same element as $\xi(g_1) = g_1^{-1}$, namely $s_0 s_1$. Conversely, the function $\gamma(s^1_i) = (g^1_i, -1)$, for $i \neq 0$, and $\gamma(s^0) = (1, -1)$ induces a homomorphism $K \to G$ since an element of the form $(s^1_i s^1_j)^{m_{ij}}$, for $i, j \neq 0$ is mapped to $((g^1_i, -1)(g^1_j, -1))^{m_{ij}} = ((g^1_i g^1_j)^{-1})^{m_{ij}}, 1)$, while $(s^1_i s^0_j)^{m_{ij}}$, for $i \neq 0$, is mapped to $((g^1_i, -1)(1, -1))^{m_{ij}} = (g^1_i, 1)$. As $\gamma$ is clearly the inverse of $\phi$, the latter must be an isomorphism. The restriction of $\phi$ to $G^+$ is therefore an isomorphism between $G^+$ and $K^+$. \\

The relations (1) are written down by Coxeter and Moser [8, p.124] who do not, however, make any comment as to whether they constitute a presentation. The above method of proof is suggested by an exercise of Bourbaki [1, p.38].

1.2 Proposition. If $K$ is not of type $B_2$, then $K^+/K^+/K^+ \cong (\mathbb{Z}/2\mathbb{Z})^{\rho-1} \oplus (\mathbb{Z}/3\mathbb{Z})^{\mu}$, where $\rho$ is the number of connected components in the Dynkin diagram of $B$ after all double and triple bonds have been deleted and $\mu = 0$, unless $K$ is of type $A_2, G_2, A_3$ or $D_4$, when $\mu = 1$. If $K$ is of type $B_2$, the group $K^+/K^+/K^+ \cong \mathbb{Z}/4\mathbb{Z}$.

Proof. We obtain a presentation for $K^+/K^+/K^+$ by adding the relations $g_1 g_j = g_j g_1$ to (1). Choose $x_0$ to be connected to at most one other root $x_1 \in B$. If $x_2 \neq x_0, x_1$ then $g_1^2 = 1$; if there exists $x_2$ disconnected from $x_1$, then $(g_1 g_1^{-1})^2 = 2$, so that also $g_1^2 = 1$. The relations $(g_1 g_1^{-1})^{m_{ij}} = 1$ then amount to saying that $g_1 = g_j$ if $m_{ij} = 3$ and also that $g_1 = 1$ if $m_{10} = 3$;
the conclusion follows. In the remaining cases, if \( \dim V_K = 2 \), \( K \) is of type \( A_2, B_2 \) or \( G_2 \) and \( K^+ \) is correspondingly cyclic of order 3, 4 or 6. If \( K \) is of type \( B_3 \), then \( g_1^3 = 1, g_2^2 = 1 \) and 
\[
(\bar{g}_1 \bar{g}_2)^4 = 1,
\]
whence \( g_1 = 1 \). If \( K \) is of type \( A_2 \) or \( D_4 \), then 
\[
\begin{align*}
g_1^3 &= 1, \\
g_2^2 &= 1,
\end{align*}
\]
and 
\[
(g_1^{-1} g_1)^3 = 1 \quad \text{for } i > 1,
\]
so that \( g_1 = 1 \).

We shall often need to use the formula

\[
(2) \quad s_i s_j(x) = x - \langle x, \alpha_j \rangle \alpha_j - (\langle x, \alpha_j \rangle - n_j \langle x, \alpha_j \rangle) \alpha_i.
\]

It follows for example that \( V^K = V^K \), since \( s_i s_j(x) = x \) for all \( \alpha_i \neq \alpha_j \) in \( B \) implies \( \langle x, \alpha_j \rangle = 0 \) for all \( \alpha_j \in B \), so that \( x \in V^K \).

2. The case \( \dim V_K = 2 \).

By 1.2.1, \( N(K^+) = GL(V^K)N(K^+) \), where \( K^+ \) is the restriction of \( K^+ \) to \( V_K \); it therefore suffices to determine \( N(K^+) \) in case \( V = V_K \).

2.1 Proposition. If \( K \) is essential, \( N(K^+) \) is the group of similarities of \( V \) if \( R \) is of type \( A_2, B_2 \) or \( G_2 \) and all of \( GL(V) \) if \( R \) is of type \( A_1 \times A_1 \).

Proof. If \( R \) is of type \( A_1 \times A_1 \), the group \( K^+ \) consists of \( \{\pm 1 \} \) so that \( N(K^+) = GL(V) \). Otherwise, \( K^+ \) is generated by a rotation \( g \) such that \( g^2 \neq 1 \). By 1.2.1, we have \( N(K^+) = C(K^+)N_{\Gamma^+}(K^+) \).

The group \( N_{\Gamma^+}(K^+) \) contains the entire group of rotations \( \Gamma^+ \) since the latter is abelian. If \( s \) is a reflection of \( V \), so is \( gs \) and therefore \( (gs)^2 = 1 \) or \( sgs = g^{-1} \in K^+ \). Thus \( N_{\Gamma^+}(K^+) = \Gamma^+ \).

In general, if \( f = pu \) is the polar decomposition of an arbitrary \( f \in N(K) \) with \( p \) hermitian and \( u \in \Gamma^+ \), we have \( p \in C(K^+) \) and \( u \in N_{\Gamma^+}(K^+) \). Since the minimum polynomial of \( g \) is of maximum possible degree, \( C(K^+) \) is spanned by \( 1_V \) and \( g \). Suppose \( p = a_1 + bg \) for some \( a, b \in R \); then the adjoint \( p^* = a_1 + bg^{-1} = p \), which is
only possible if $b = 0$, since $g^2 \neq 1$. Therefore $f$ is a similarity of $V$.//

When $V = V_K$, an elementary argument (e.g. [6, p. 55]) shows that (i) if $R$ is of type $A_1 \times A_1$, every lattice is invariant under $K^+$ (ii) if $R$ is of type $A_2$ or $G_2$, every hexagonal lattice is invariant and (iii) if $R$ is of type $B_2$, every square lattice is invariant. In each case, prop. 2.1 shows that every lattice is equivalent to the root lattice $Q(R)$. From eqn. (2), one sees that $Q(R)^* = P(R)$. When $V_K \neq 0$, the determination of invariant lattices proceeds according to the general principles of I.2.3.

If $\Lambda$ is a lattice invariant under $K^+$, the group $H^1(K^+, V/\Lambda)$ can be calculated by the general method available for cyclic groups. Let $g$ be a generator of $K^+$, of order $m$. Since $1+g+\ldots+g^{m-1}$ vanishes on $V_K$, while the restriction of $1-g$ to $V_K$ is invertible, one sees that

$$H^1(K^+, V/\Lambda) = \{ y \in V_K^\Lambda | my = 0 \} / \Theta_1(\Lambda).$$

3. The normaliser of $K^+$.

We shall prove in this section that $N(K^+) = N(K)$ if $\dim V_K \geq 3$. Since $N(K^+) = \text{GL}(V_K)N(\overline{K}^+)$ and $N(K) = \text{GL}(V_K)N(\overline{K})$, it is sufficient to prove the assertion in case $V = V_K$.

3.1 Proposition. $C(K^+) = C(K)$.

Proof. If $f \in C(K^+)$, then $fs_is_j(x) = s_is_jf(x)$ for all $\alpha_i \neq \alpha_j$ in $B$ and all $x \in V$. Using (2), this can be written as

$$\langle x, \tilde{\alpha}_j \rangle f(\alpha_j) + (\langle x, \tilde{\alpha}_i \rangle - n_{ij}\langle x, \alpha_j \rangle) f(\alpha_i)$$

$$= \langle f(x), \alpha_j \rangle \alpha_j + (\langle f(x), \alpha_i \rangle - n_{ij}\langle f(x), \alpha_j \rangle) \alpha_i.$$

If we choose $x$ to satisfy $\langle x, \tilde{\alpha}_j \rangle = 0$ and $\langle x, \alpha_i \rangle = 1$, we see
that \( f(\alpha_i) \) belongs to the plane \( P_{ij} \) spanned by \( \alpha_i \) and \( \alpha_j \). If \( \alpha_k \neq \alpha_i, \alpha_j \), the same argument shows that \( f(\alpha_i) \in P_{ik} \); consequently \( f(\alpha_i) \in P_{ij} \cap P_{ik} = \mathbb{R} \alpha_i \). Suppose \( f(\alpha_i) = a_1 \alpha_i \) for some \( a_1 \in \mathbb{R} \). Letting \( x = \alpha_i \) in the above equation and comparing the coefficients of \( \alpha_j \), we see that \( n_{ij} a_j = n_{ij} a_i \). It follows that \( a_i = a_j \) whenever \( \alpha_i \) and \( \alpha_j \) belong to the same component of \( B \). Therefore \( f \in C(K) \).

Since \( N(K^+) = C(K^+)N_\Gamma(K^+) \) and \( N(K) = C(K)N_\Gamma(K) \) by I.2.1, it remains to prove that \( N_\Gamma(K^+) = N_\Gamma(K) \). We need the following well-known result (e.g. [7, p.28]):

3.2 Proposition. Suppose \( w \in K \) and let \( V^w = \{ v \in V \mid w(v) = v \} \). Then \( w \) can be expressed as a product of reflections corresponding to roots in the orthogonal complement of \( V^w \) in \( V \).

For each pair of independent roots \( \alpha, \beta \in \mathbb{R} \), let \( P_{\alpha\beta} \) be the plane spanned by \( \alpha \) and \( \beta \) and let \( P \) be the collection of all such planes in \( V \).

3.3 Proposition. An element \( g \in N_\Gamma(K^+) \) permutes the elements of \( P \).

Proof. Suppose \( \alpha, \beta \in \mathbb{R} \), with \( \alpha \neq \pm \beta \), and consider the element \( w = gs_\alpha s_\beta g^{-1} \in K^+ \). Since \( g \in \Gamma \), the subspace \( V^w = g(H_\alpha \cap H_\beta) \) is of codimension 2 in \( V \) and its orthogonal complement in \( V \) is the plane \( U = g(P_{\alpha\beta}) \). Prop.3.2 implies that \( w = s_\gamma s_\delta \ldots \) for some roots \( \gamma, \delta, \ldots \) in \( U \). There are at least two independent roots among the \( \gamma, \delta, \ldots \) since otherwise \( w \), being in \( K^+ \), would equal 1. Therefore \( U = P_{\lambda\mu} \) for some roots \( \lambda, \mu \in U \).

3.4 Proposition. Suppose \( P_{\alpha\beta} \) and \( P_{\gamma\delta} \) are distinct elements of \( P \) such that the subspace \( L = P_{\alpha\beta} + P_{\gamma\delta} \) is three-dimensional.

Let \( R' \) be the root system \( R \cap L \). Suppose that either (a) \( R' \) is reducible or (b) there exists a rotation \( w \) in \( P_{\alpha\beta} \) of order 3
and a rotation \( w_2 \) in \( P_{\gamma \delta} \) of order 3 or 4 such that \( (w_1 w_2)^2 = 1 \).

Then \( P_{\alpha \beta} \cap P_{\gamma \delta} \) is spanned by a root \( \rho \in R \).

**Proof.** If \( R' \) is reducible, it is easy to see that every intersection \( P_{\alpha \beta} \cap P_{\gamma \delta} \) is spanned by a root. On the other hand, a well-known formula of spherical trigonometry asserts that if \( r_1 \) and \( r_2 \) are rotations of \( L \) through \( \Theta_1 \) and \( \Theta_2 \), and if their planes are inclined at an angle \( \phi \), then the angle \( \Theta \) of the rotation \( r_1 r_2 \) is given by

\[
\cos \Theta/2 = -\cos \Theta_1/2 \cos \Theta_2/2 + \cos \phi \sin \Theta_1/2 \sin \Theta_2/2.
\]

Applying this formula to the rotations \( w_1 \) and \( w_2 \), we conclude that the angle \( \phi \) between \( P_{\alpha \beta} \) and \( P_{\gamma \delta} \) is given by \( \cos \phi = 1/3 \) or \( 1/\sqrt{3} \). An explicit look at root systems of type \( A_3, B_3 \) and \( C_3 \) now reveals that this is only possible if \( P_{\alpha \beta} \cap P_{\gamma \delta} \) is spanned by a root. //

3.5 Proposition. \( N_{\Gamma}(K^+) = N_{\Gamma}(K) \).

**Proof.** We have seen in 3.3 that an element \( g \in N_{\Gamma}(K^+) \) permutes the elements of \( P \). If \( B \) has no components of length \( \geq 3 \), we choose any three elements \( \alpha_1, \alpha_2, \alpha_3 \in B \) and consider the planes \( P_{\alpha \beta} = gP_{12} \) and \( P_{\gamma \delta} = gP_{13} \). Since \( P_{12} + P_{13} \) is three-dimensional and \( R' \) necessarily reducible, prop.3.4 shows that \( P_{\alpha \beta} \cap P_{\gamma \delta} \) is spanned by a root \( \rho \). Since \( P_{12} \cap P_{13} \) is spanned by \( \alpha_1 \), we conclude that \( g \) maps \( \alpha_1 \) into a multiple of \( \rho \) and therefore \( gs_1 g^{-1} = \rho \in K \). As \( gs_1 s_1 g^{-1} \in K^+ \) for all \( i \), we have \( gs_1 g^{-1} \in K \), so that \( g \) belongs to \( N_{\Gamma}(K) \). If \( B \) has a component of length \( \geq 3 \), then we can choose the elements \( \alpha_i \) (\( i = 1, 2, 3 \)) in such a way that \( m_{12} = 3 \) and \( m_{23} = 3 \) or 4. If we let \( w_1 = gs_2 s_1 g^{-1} \) and \( w_2 = gs_1 s_2 g^{-1} \) and apply prop.3.4, the argument proceeds as before. //
4. Lattices invariant under $K^+$.  

We shall assume that $V = V_K^+$; the general case can be treated by the general method of I.2.3. It turns out that a lattice invariant under $K^+$ is also invariant under $K$, apart from the following exceptional cases:

(a) Suppose $R$ is of type $A_1 \times \ldots \times A_j$. Let $\Lambda_e$ be the sublattice of $P(R)$ containing $Q(R)$ and all elements of the form $(\alpha'_i + \alpha'_j)/2$ for $\alpha'_i \neq \alpha'_j$ in $B$. When $\dim V$ is even, define

$$\Lambda_{e1} = \Lambda_e \cup (\Lambda_e + (\sum \alpha_i)/4)$$
$$\Lambda_{e2} = \Lambda_e \cup (\Lambda_e + (\sum \alpha_i)/4 + \alpha_0/2)$$

(b) Suppose $R$ is the direct sum of two root systems $R_1$ and $R_2$ of types $A_1$ and $A_2$ respectively. Let $\Lambda_e = Q(R_1) \oplus P(R_2)$ and define

$$\Lambda_{e1} = \Lambda_e \cup (\Lambda_e + (\alpha_1 + \alpha_2)/3) \cup (\Lambda_e - (\alpha_1 + \alpha_2)/3)$$
$$\Lambda_{e2} = \Lambda_e \cup (\Lambda_e + (\alpha_1 - \alpha_2)/3) \cup (\Lambda_e - (\alpha_1 - \alpha_2)/3),$$

where $\{\alpha_1\}$ and $\{\alpha_2, \alpha_3\}$ are bases of $R_1$ and $R_2$. (Crystallographers call these lattices 'rhombohedric'.)

(c) Suppose $R$ is the direct sum of two root systems $R_1$ and $R_2$, both of type $A_2$. Let $\Lambda_e = P(R)$ and define

$$\Lambda_{e1} = \Lambda_e \cup (\Lambda_e + (\alpha_1 + \alpha_3)/3) \cup (\Lambda_e - (\alpha_1 + \alpha_3)/3)$$
$$\Lambda_{e2} = \Lambda_e \cup (\Lambda_e + (\alpha_1 - \alpha_3)/3) \cup (\Lambda_e - (\alpha_1 - \alpha_3)/3),$$

where $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$ are bases of $R_1$ and $R_2$.

In each of the above cases, the lattices $\Lambda_{e1}$ and $\Lambda_{e2}$ will be called of exceptional type w.r.t. $R$. One can verify that they are invariant under $K^+$, but interchanged by an element of $K$ not in $K^+$ (and thus equivalent under $N(K^+)$).
Suppose $\Lambda$ is a lattice in $V$ invariant under $K^+$. For each $\alpha_i \in B$, let $\Lambda_i = \Lambda \cap R\alpha_i$ and define $\Lambda^# = \oplus_i \Lambda_i$. Recall that $\Lambda^* = \{v \in V \mid v = gv \in \Lambda \text{ for all } g \in K^+\}$ is also a lattice in $V$ which contains $\Lambda$.

4.1 Proposition. Suppose $\alpha_i, \alpha_j$, and $\alpha_k$ are distinct elements of $B$ such that $n_{jk} = 0$. Then if $x \in \Lambda^*$, we have

(a) If $n_{ji} = n_{ki} = 0$, then $2 \langle x, \alpha_i \rangle \alpha_i \in \Lambda$.

(b) If $n_{ji} = -1$, then $(4 - n_{ij} n_{ji} n_{ik} n_{ki}) \langle x, \alpha_j \rangle \alpha_i \in \Lambda$.

Proof. Since $x \in \Lambda^*$, the elements

\[
x - s_i s_j(x) = \langle x, \alpha_j \rangle \alpha_i + (\langle x, \alpha_i \rangle - n_{ji} \langle x, \alpha_j \rangle) \alpha_i
\]

\[
x - s_i s_k(x) = \langle x, \alpha_k \rangle \alpha_i + (\langle x, \alpha_i \rangle - n_{ki} \langle x, \alpha_k \rangle) \alpha_i
\]

\[
s_j s_k(x) - x = \langle x, \alpha_j \rangle \alpha_j - \langle x, \alpha_k \rangle \alpha_k
\]

all belong to $\Lambda$. Consequently their sum

\[
(2 \langle x, \alpha_i \rangle - n_{ji} \langle x, \alpha_j \rangle - n_{ki} \langle x, \alpha_k \rangle) \alpha_i
\]

also belongs to $\Lambda$. If $n_{ji} = n_{ki} = 0$, we obtain (a). Otherwise, we apply this conclusion to the element

\[
y = (x - s_i s_j(x)) - (x - s_j s_i(x)) = -n_{ji} \langle x, \alpha_j \rangle \alpha_i + n_{ij} \langle x, \alpha_i \rangle \alpha_j
\]

instead of $x$ and deduce (b).//

Note that the integer $4 - n_{ij} n_{ji} n_{ik} n_{ki}$ in part (b) can only assume the values 1, 2 or 3.

If $\Lambda$ is of exceptional type w.r.t. $R$, one can verify that $\Lambda^# = Q(R)$.

4.2 Proposition. Suppose $\Lambda^# = Q(R)$ and $x \in \Lambda^*$ is such that $\langle x, \alpha_i \rangle \notin \mathbb{Z}$ for some $\alpha_i \in B$.

(a) If $2 \langle x, \alpha_i \rangle \in \mathbb{Z}$, then $R$ is of type $A_1 \times \ldots \times A_1$ and either $\Lambda = \Lambda_e$ or $\dim V$ is even and $\Lambda$ equals $\Lambda_{e1}$ or $\Lambda_{e2}$. Furthermore, $x = (\sum \alpha_i)/4 \mod P(R)$. 

(b) If $3 \langle x, \tilde{x}_1 \rangle \in \mathbb{Z}$, then either (i) $R$ is of type $A_1 \times A_2$, 
$\Lambda = \Lambda_{e1} \text{ or } \Lambda_{e2}$ and $tx = (\alpha_1 \pm \alpha_2)/3 \mod P(R)$ or (ii) $R$ is of type $A_2 \times A_2$, $\Lambda = \Lambda_{e1} \text{ or } \Lambda_{e2}$ and $tx = (\alpha_1 \pm \alpha_2)/3 \mod P(R)$.

Proof. For any $\alpha_i \neq \alpha'_j$ in $B$, consider

(3) $x-s_1s_j(x) = \langle x, \tilde{x}_j \rangle \alpha_j + (\langle x, \tilde{x}_1 \rangle - n_{ji} \langle x, \tilde{x}_j \rangle) \alpha_i \in \Lambda$.

If $N \langle x, \tilde{x}_j \rangle \in \mathbb{Z}$ for some $\alpha_j \in B$ and some integer $N$, multiplying
(3) by $N$ shows, in view of $\Lambda^# = Q(R)$, that $N \langle x, \tilde{x}_i \rangle \in \mathbb{Z}$ for all $\alpha_i \in B$. Thus the hypothesis of the proposition implies that $\langle x, \tilde{x}_i \rangle \notin \mathbb{Z}$ for all $\alpha_i \in B$. It now follows from (3) that

(4) $\langle x, \tilde{x}_i \rangle - n_{ji} \langle x, \tilde{x}_j \rangle \notin \mathbb{Z}$,

since otherwise $\langle x, \tilde{x}_j \rangle$ would have to be in $\mathbb{Z}$.

Suppose we are in case (a); then $2 \langle x, \tilde{x}_i \rangle \in \mathbb{Z}$ for all $\alpha_i \in B$. Eqn. (4) implies that $n_{ji}$ is always even, which can only happen if $R$ is of type $A_1 \times \ldots \times A_1$; furthermore, (3) shows that $\Lambda_e \subset \Lambda$. If $y \in (\Lambda \cap P(R)) \setminus \Lambda_e$, then $y = \alpha_{0}/2 \mod \Lambda_e$, contrary to $\Lambda^# = Q(R)$, so that $\Lambda \cap P(R) = \Lambda_e$. Since

$\langle x, \tilde{x}_i \rangle = \frac{1}{2} \mod \mathbb{Z}$ for all $\alpha_i \in B$, we have $x = (\sum \alpha_i)/4 \mod P(R)$.

If dim $V$ is odd, $x$ cannot belong to $\Lambda$ since then $2x = \alpha_{0}/2 \mod \Lambda_e$ would also belong to $\Lambda$. As $R$ is of type $A_1 \times \ldots \times A_1$, prop. 4.1(a) shows that $\Lambda^2 \langle y, \tilde{x}_1 \rangle \in \mathbb{Z}$. If dim $V$ is odd, it follows that $\Lambda \subset P(R)$ so that $\Lambda_e = \Lambda$. If dim $V$ is even and $\Lambda \notin P(R)$, then $\Lambda$ must be one of the lattices $\Lambda_{e1}$ and $\Lambda_{e2}$ since $(\sum \alpha_i)/4$ and $(\sum \alpha_i)/4 + \alpha_{0}/2$ cannot both be in $\Lambda$.

In case (b), we have $3 \langle x, \tilde{x}_i \rangle \in \mathbb{Z}$ for all $\alpha_i \in B$. Eqn. (4) implies that $\langle x, \tilde{x}_i \rangle = \langle x, \tilde{x}_j \rangle \mod \mathbb{Z}$ whenever $n_{ji} = -1$. Furthermore, 4.1 shows that (i) $B$ does not contain three disconnected roots and (ii) $B$ does not contain roots connected by double or triple bonds or a sequence of three roots connected by single
bonds, since otherwise \( \langle x, x' \rangle \) or \( 2 \langle x, x' \rangle \) would belong to \( \mathbb{Z} \) for some \( \alpha_1 \in B \). This means that \( R \) is either of type \( A_1 \times A_2 \) or \( A_2 \times A_2 \). In both of these cases, (3) implies that \( \Lambda \supset \Lambda_e \).

Suppose \( R \) is of type \( A_1 \times A_2 \). Then \( x = x_1 \omega_1 + x_2 (\omega_2 + \omega_3) \mod P(R) \), where \( x_p = \pm 1/3 \) for \( p = 1,2 \) and \( \omega_1 \) is the weight corresponding to \( \alpha_1 \). Since \( \omega_1/3 = \alpha_1/6 = -\alpha_1/3 \mod P(R) \) and \( (\omega_2 + \omega_3)/3 = (\alpha_2 + \alpha_3)/3 = -\alpha_2/3 \mod P(R) \), we have \( t x = (\alpha_1 + \alpha_2)/3 \mod P(R) \). Eqn. (3) applied with \( i = 1 \) and \( j = 2 \) shows that \( (\alpha_1 + \alpha_2)/3 \in \Lambda \) with the same choice of sign as for the value of \( t x \mod P(R) \). Both of these elements cannot belong to \( \Lambda \) since their sum \( 2 \alpha_1/3 \notin \Lambda \). It follows that either \( \Lambda_{e_1} \subset \Lambda \) or \( \Lambda_{e_2} \subset \Lambda \). If \( y \in (\Lambda \cap P(R)) \setminus \Lambda_e \) then \( y = \alpha_1/2 \mod \Lambda_e \) contrary to \( \Lambda^* = Q(R) \). Therefore \( \Lambda \cap P(R) = \Lambda_e \).

If \( y \in \Lambda \setminus P(R) \), choosing \( i = 2 \), \( j = 3 \) and \( k = 1 \) in prop. 4.1(b) shows that \( \exists (y, \alpha_3) \in \mathbb{Z} \). It follows that \( \Lambda_{e_1} = \Lambda \) or \( \Lambda_{e_2} = \Lambda \).

4.3 Corollary. If \( \Lambda \) is a lattice of exceptional type, then \( \Lambda^* \) can be described as follows: in case (a), \( \Lambda^*_{ep} = \Lambda_{ep} \cup (\Lambda_{ep} + \alpha_1/2) \); in case (b), \( \Lambda^*_{ep} = \Lambda_{ep} \cup (\Lambda_{ep} + \alpha_1/2) \) and in case (b) (i), \( \Lambda^*_{ep} = \Lambda_{ep} \), where \( p = 1,2 \).

4.4 Proposition. Suppose \( \Lambda \) is a lattice invariant under \( K^+ \). Then there exists a root system \( R' \) such that \( W(R') = K \) and \( \Lambda \) is either of exceptional type w.r.t. \( R' \) or else \( Q(R') \subset \Lambda \subset P(R') \) and \( \Lambda^* = Q(R') \). In the latter case, \( \Lambda \) is invariant under \( K \) and \( \Lambda^* = P(R') \), unless \( R' \) is of type \( A_1 \times \cdots \times A_1 \) and \( \Lambda = \Lambda_e \), when \( \Lambda^* = P(R') \cup (P(R') + (\sum_1 \alpha_1)/4) \).

Proof. We first show that \( \Lambda_{1} \neq 0 \) for all \( \alpha_1 \in B \), so that \( \Lambda^* \) is a lattice. If \( \dim V > 4 \), there will exist \( \alpha_j, \alpha_k \in B \)
different from \( \alpha_i \) for which \( n_{jk} = 0 \). Since \( \Lambda \) is a lattice, we can find an element \( x \in \Lambda \) for which neither of \( \langle x, \tilde{\alpha_i} \rangle \) and \( \langle x, \tilde{\alpha_j} \rangle \) is zero. Prop. 4.1 then shows that \( \Lambda_1 \neq 0 \). When \( \dim V = 3 \), this argument works for at least one \( \alpha_i \in \mathcal{B} \). If \( \alpha_i \) is a nonzero element of \( \Lambda_1 \) and \( \alpha_j \in \mathcal{B} \) is connected to \( \alpha_i \), then \( (1 + s_j s_i)(a \alpha_i) = n_{ij} a \alpha_j \) is a nonzero element of \( \Lambda_j \). The only remaining case is that of a root \( \alpha_k \in \mathcal{B} \) which is not connected to either of the other two roots \( \alpha_i, \alpha_j \in \mathcal{B} \), but for which we may assume that \( n_{ij} = -1 \). Let \( \langle x, \alpha_k \rangle \) be such that \( \langle x, \alpha_k \rangle \neq 0 \).

Since \( z = (4 - n_{ij} n_{ji}) \langle x, \tilde{\alpha_j} \rangle \alpha_i \in \Lambda \) by prop. 4.1, it follows that \( (1 + s_j s_i)(z) = n_{ij} (4 - n_{ij} n_{ji}) \langle x, \tilde{\alpha_j} \rangle \alpha_i \in \Lambda \). However, since \( x - s_j s_k(x) = \langle x, \tilde{\alpha_j} \rangle \alpha_j + \langle x, \tilde{\alpha_k} \rangle \alpha_k \in \Lambda \), we deduce that \( n_{ij} (4 - n_{ij} n_{ji}) \langle x, \tilde{\alpha_j} \rangle \alpha_k \) also belongs to \( \Lambda \). Therefore \( \Lambda_k \neq 0 \).

Suppose \( \Lambda_1 = \mathbb{Z} m_i \alpha_i \) for some \( m_i > 0 \). Since \( (1 - s_j)(m \alpha_i) = (1 + s_j s_i)(m \alpha_i) = n_{ij} m \alpha_j \in \Lambda_j \), \( \Lambda^# \) is invariant under \( K \) and therefore equal to \( Q(R') \) for some root system \( R' \) such that \( W(R') = K \) by I.3.2. Applying prop. 4.2, we see that either \( \Lambda \subset \Lambda^* = P(R') \) or \( \Lambda \) is of exceptional type w.r.t. \( R' \) or \( R' \) is of type \( A_1 \times \ldots \times A_1 \) and \( \Lambda = \Lambda_e \). In the last case, the value of \( \Lambda_e \) also follows from prop. 4.2.1//

5. The groups \( H^1(K^+, V/\Lambda) \).

Suppose \( \Lambda \) is a lattice in \( V = V^K \oplus V_K \) invariant under \( K^+ \). Then \( \Lambda \) contains the sublattice \( \Lambda_0 = \Lambda^K \oplus \Lambda_K \) and the projection of \( \Lambda \) on \( V_K \) is contained in \( \Lambda_K^* \). We may assume that \( \Lambda_K \) is as described in prop. 4.4, but with \( R' = R \). It will be convenient to introduce the number

\[
e_{ij} = m_{ij} / (4 - n_{ij} n_{ji})
\]
for $\alpha_i, \alpha_j \in \mathcal{B}$. It equals 1, 2, or 6 according to whether $\alpha_i$ is disconnected from $\alpha_j$ or connected to it by a single, double or triple bond.

Choose $\alpha_0$ to be connected to at most one other root $\alpha_i$ and let $G = \{g_i = s_i s_0 \mid i \neq 0\}$ be the corresponding set of generators of $K^+$. Suppose $t$ is a function $G \rightarrow V$. If $t(g_i) = y_i + \sum_k t_{ki} \alpha_k$, with $y_i \in V^K$, define $u_{ij} = \langle t(g_i), \alpha_j \rangle = \sum_k t_{ki} n_{kj}$.

5.1 Proposition. The function $t$ induces a coboundary $\tilde{t} : K^+ \rightarrow V / \Lambda$ iff there exists a constant $c \in R$ such that $t(g_1) = c \alpha_0 + t_{ii} \alpha_i \mod \Lambda$ for all $i \neq 0$.

Proof. If $a = y + \sum_k a_k \alpha_k \in V$, with $y \in V^K$, the coboundary corresponding to a mod $\Lambda$ maps $g_i$ to

$$\tilde{a} - g_i \bar{a} = \left( \sum_k a_k n_{k0} \right) \alpha_0 + \left( \sum_k a_k n_{ki} - n_{0i} \sum_k a_k n_{k0} \right) \alpha_i \mod \Lambda.$$

Conversely, suppose $t(g_i) = c \alpha_0 + t_{ii} \alpha_i \mod \Lambda$. The system of equations

$$\sum_k a_k n_{k0} = c$$
$$\sum_k a_k n_{ki} - n_{0i} \sum_k a_k n_{k0} = t_{ii}$$

can be written in the form $\bar{a}N = \bar{t}$, where $\bar{a} = (a_k)$, $N$ is the Cartan matrix of $R$ and $\bar{t}$ is the vector with 0-th coordinate equal to $c$ and $i$-th coordinate, for $i \neq 0$, equal to $t_{ii} + n_{0i}c$.

Since $N$ is invertible, the system has a unique solution $\bar{a}$ and $\bar{t}$ coincides with the coboundary defined by $a = \sum_k a_k \alpha_k$.

If $\alpha_1$ exists, let $c$ and $d$ be the solutions of the system of equations

$$n_{0i}c + 2d = u_{11}$$
$$2c + n_{10}d = u_{10}$$;
otherwise, let \( c = 0 \). By subtracting from \( t \) the function

\[
\begin{align*}
\sigma_1 &\rightarrow c \alpha_0 + d \alpha_1 \\
\sigma_i &\rightarrow c \alpha_0 + iu_i \alpha_i \quad (i \neq 1)
\end{align*}
\]

we may assume that \( u_{ii} = 0 \) for all \( i \) and also that \( u_{10} = 0 \).

5.2 Proposition. The function \( t \) induces a cocycle \( \overline{t}: K^+ \rightarrow V/\Lambda \) iff the following conditions are satisfied for all \( i, j \neq 0 \):

\[
\begin{align*}
(6) & \quad 2t(g_i) = u_{i0} \alpha_0 \mod \Lambda \quad \text{if } i \neq 1 \\
(7) & \quad m_{10} t(g_1) \in \Lambda \\
(8) & \quad m_{ij} (t(g_i) - t(g_j)) = e_{ij} \left[ -(2u_{ji} + n_{ji}u_{ij}) \alpha_i + (2u_{ij} + n_{ij}u_{ji}) \alpha_j \right] \mod \Lambda \\
\end{align*}
\]

Proof. If \( h \in K^+ \) is of order \( m \), we shall denote \( 1+h+\ldots+h^{m-1} \) by \( N(h) \). Our first aim is to establish the formula

\[
N(s_is_j) t(g_1) = m_{ij} t(g_1) + e_{ij} (n_{ji} u_{ij} \alpha_i - 2u_{ij} \alpha_j)
\]

for \( i \neq 0 \). Indeed, since \( s_is_j t(g_i) = t(g_1) - u_{ij} \alpha_j + n_{ji} u_{ij} \alpha_i \) by \( (2) \), the left side of \( (9) \) equals \( m_{ij} (t(g_1) - v) \), for some \( v \) in \( P_{ij} \).

Since the left side of \( (9) \) is also invariant under \( s_is_j \), the vector \( v \) must in fact be the orthogonal projection of \( t(g_i) \) on this plane. Consequently, if \( v = v_i \alpha_i + v_j \alpha_j \), we have the equations

\[
\begin{align*}
0 & = u_{ii} = \langle t(g_i), \alpha_i \rangle = \langle v, \alpha_i \rangle = 2v_i + n_{ji} v_j \\
u_{ij} & = \langle t(g_i), \alpha_j \rangle = \langle v, \alpha_j \rangle = n_{ij} v_i + 2v_j
\end{align*}
\]

whose solutions are

\[
\begin{align*}
v_i & = -n_{ji} u_{ij} / (4 - n_{ij} n_{ji}) \\
v_j & = 2u_{ij} / (4 - n_{ij} n_{ji})
\end{align*}
\]

this establishes \( (9) \).

According to the presentation \( (1) \) of \( K^+ \), the function \( t \) will induce a cocycle \( \overline{t}: K^+ \rightarrow V/\Lambda \) iff

\[
t(g_1^{m_{ij}}) = N(g_i) t(g_1) = N(s_is_0) t(g_1) \in \Lambda
\]
and
\[ t((g_i^{-1}g_j^{-1})^{m_{ij}}) = N(g_i^{-1}g_j^{-1})(t(g_i) - g_i^{-1}t(g_j)) \in \Lambda. \]
Since \( g_i^{-1}g_j = s_i s_j \) and \( N(s_i s_j) = N(s_j s_i) \), the latter relation can also be expressed as
\[ N(s_i s_j) t(g_i) = N(s_j s_i) t(g_j) \mod \Lambda. \]
Using (9), one sees that a relation of the first type amounts to (6) or (7), while that of the second type amounts to (8).//

We shall postpone until the end of this section the discussion of the cases when \( \Lambda_k \) is of exceptional type or when \( R \) is of type \( A_1 \times \cdots \times A_1 \) and \( \Lambda_k = \Lambda_e \). This enables us to assume that \( \Lambda_k^* = P(R) \), so that \( \alpha_i(\Lambda) \subset \mathbb{Z} \) for all \( \alpha_i \in B \).

5.3 Proposition. For all \( i, j \neq 1 \), we have
(a) \( 2u_{ij} \in \mathbb{Z} \).
(b) \( 2u_{i1} = n_{01}u_{10} \mod \mathbb{Z} ; m_{10}u_{11} \in \mathbb{Z} \).
(c) \( 2u_{i0} = 2u_{j0} \mod \mathbb{Z} \) if \( m_{ij} = 2 \) or \( 4 \); \( 3u_{10} = 3u_{j0} \mod \mathbb{Z} \)
   if \( m_{ij} = 3 \) or \( 6 \).

Proof. Evaluating \( \hat{\chi}_j \) at (6), we obtain (a); to obtain (b), evaluate \( \hat{\chi}_1 \) and \( \hat{\chi}_i \) at (6) and (7) respectively.

If \( m_{ij} = 2 \), evaluating \( \hat{\chi}_0 \) at (8) shows that \( 2(u_{10} - u_{j0}) \in \mathbb{Z} \).
Similarly, if \( m_{ij} = 3 \), evaluating \( \hat{\chi}_0 \) at (8) shows that \( 3(u_{10} - u_{j0}) \in \mathbb{Z} \). If \( m_{ij} = 4 \) or \( 6 \), then \( e_{ij} \) is even and therefore the right side of (8) is in \( \Lambda \) by (a). Using (6), we conclude that \( \frac{1}{2}m_{ij}(u_{10} - u_{j0}) \alpha_0 \in \Lambda \), which can only happen if \( \frac{1}{2}m_{ij}(u_{10} - u_{j0}) \in \mathbb{Z} \).//

If \( R \) is of type \( B_3 \times A_1 \times \cdots \times A_1 \), we define \( \Lambda_e \) to be the sublattice of \( P(R) \) which contains \( Q(R) \) and all elements \( (\alpha_0 + \alpha_2 + \alpha_i)/2 \), where \( \alpha_0 \) and \( \alpha_2 \) are the first and third roots in the component of type \( B_3 \) and \( \alpha_i \) is in a component of type \( A_1 \).
(If there are no factors of type $A_1$, let $\Delta_e = Q(R)$.)

5.4 Proposition. We have $(4-n_{01}n_{10})u_{10} \in \mathbb{Z}$ for all $i \neq 1$, unless

(a) $n_{10} = -2$ and $n_{11} = 0$, when $u_{10} \in \mathbb{Z}$.

(b) $\alpha_0$ belongs to a component of type $A_3, D_4, B_3$ or $B_4$ (in the last two cases, $\alpha_0$ is the first and $\alpha_1$ the third root) and

$\frac{(\alpha_0 + \alpha_1)}{2} \in \Delta$, when $6u_{10} \in \mathbb{Z}$.

(c) $R$ is of type $B_3 \times A_1 \times \ldots \times A_1$ and $\Delta = \Delta_e$, when $6u_{20} \in \mathbb{Z}$.

Proof. Eqn. (8) with $j = 1$ gives

\[ m_{i1}(t(g_i) - t(g_1)) = e_{i1} \left[ -(2u_{11} + n_{11}u_{11})\alpha_1 + (2u_{11} + n_{11}u_{11})\alpha_1 \right] \mod \Delta. \]

Evaluating $\chi_0$ at this equation, we deduce that

\[ (11) \quad m_{i1}u_{10} = e_{i1}n_{10}(2u_{11} + n_{11}u_{11}) \mod \mathbb{Z}. \]

Suppose $n_{11} = 0$. If $n_{10} = -2$, (11) shows that $2u_{10} = -2u_{11} \mod \mathbb{Z}$; since $n_{01} = -1$, 5.3(b) implies that $u_{10} \in \mathbb{Z}$. Otherwise, multiplying (11) by 2 and using 5.3(b), we conclude that $(4-n_{01}n_{10})u_{10} \in \mathbb{Z}$. If $n_{11} \neq 0$ and $n_{10} = -2$, we have $m_{i1} = 3$, so that (11) becomes the equation $3u_{10} = -4u_{11} + 2u_{11} \mod \mathbb{Z}$. Multiplying this by 2 and using 5.3(b), we again conclude that $(4-n_{01}n_{10})u_{10} \in \mathbb{Z}$.

Suppose $n_{11} \neq 0$ and $n_{10} = -1$. Multiplying (11) by $m_{10}$ and using 5.3(b), the conclusion can be expressed as $2(4-n_{01}n_{10})u_{10} \in \mathbb{Z}$. If $n_{01} = -2$, we see from 5.3(b) that both $4u_{11}$ and $4u_{11}$ are in $\mathbb{Z}$. Multiplying (10) by 4 and using (7), we deduce that $12t(g_1) \in \Delta$; it therefore follows from (6) that $6u_{10} \in \mathbb{Z}$ and hence $2u_{10} = (4-n_{01}n_{10})u_{10} \in \mathbb{Z}$. If $n_{01} = -1$ and $m_{i1} = 3$, 5.3(b) implies that $12u_{11}$ and $3u_{11}$ are in $\mathbb{Z}$. Multiplying (10) by 6 and using (7) and 5.3(b), we deduce that $18t(g_1) = -3u_{10} \alpha_1 \mod \Delta$; in view of (6) and the fact that $6u_{10} \in \mathbb{Z}$, this means that $3u_{10}(\alpha_0 + \alpha_1) \in \Delta$. Thus either $3u_{10} = (4-n_{01}n_{10})u_{10} \in \mathbb{Z}$ or
(\alpha_0 + \alpha_1)/2 \in \Lambda$. However, one can easily see that \((\alpha_0 + \alpha_1)/2\) can be a weight only in the cases specified in (b). Finally, suppose \(n_{01} = -1\) and \(n_{11} = 4\). If \(n_{11} = -1\), multiplying (10) by 3 and using 5.3(b), (6) and (7), we see that \(3u_{10} \alpha_1 \in \Lambda\), so that \(3u_{10} = (4 - n_{01} n_{10})u_{10} \in \mathbb{Z}\). The remaining case is when \(\alpha_0\) and \(\alpha_1\) are the first and third roots in a component of type \(B_3\).

If \(R\) itself is of type \(B_3\) and \((\alpha_0 + \alpha_1)/2 \notin \Lambda\), \(\Lambda_K\) must be the lattice \(Q(R) = \Lambda_e\). Otherwise, suppose \(\alpha_j\) is not in this component; eqns. (6) and (8) show that

\[(u_{10} - u_{j0}) \alpha_0 = -u_{ji} \alpha_i + u_{ij} \alpha_j \mod \Lambda\.

Multiplying this equation by 3 and using 5.3(a) and the fact that \(3u_{j0} \in \mathbb{Z}\), we deduce that \(3u_{10} \alpha_0 = u_{ji} \alpha_i + u_{ij} \alpha_j \mod \Lambda\). Evaluating \(\alpha_1\) at this equation shows that \(u_{ji} = 3u_{10}\). If \(3u_{10} \notin \mathbb{Z}\) and \((\alpha_0 + \alpha_1)/2 \notin \Lambda\), then we must have \((\alpha_0 + \alpha_1 + \alpha_j)/2 \in \Lambda\).

However, such an element can only be a weight for every \(j\) only if \(\{\alpha_j\}\) is of type \(A_1\); this produces the exception noted in (c), since \(\Lambda_e\) is a maximal sublattice of \(P(R)\) w.r.t. the property \((\Lambda_e)^\# = Q(R)\).

5.5 Proposition. \(H^1(K^+, V/\Lambda) = 0\), unless \(R\) is of type \(A_1 \times \ldots \times A_1, B_2 \times A_1 \times \ldots \times A_1\) or \(B_3 \times A_1 \times \ldots \times A_1\) and, in the last case \(\Lambda_K = \Lambda_e\). In the last two cases, \(6H^1(K^+, V/\Lambda)\) is of order \(\leq 2\).

Proof. Suppose \(B\) contains a component of type \(G_2\). If we choose \(\alpha_0\) to be the first root of such a component, 5.4 says that \(u_{i0} \in \mathbb{Z}\) for all \(i \neq 1\), so that \(2t(g_1) \in \Lambda\) by (6). In view of (7), we have \(6t = 0\). Secondly, suppose \(B\) contains a component of type
other than $A_1$ and $B_2$ and choose $\alpha_0$ to be the first root of such a component; then $m_{01} = 3$. We see from 5.4 that $3u_{10} \in \mathbb{Z}$ for $i \neq 1$, and thus by (6), $6t(g_1) \in \Lambda$, apart from certain exceptional cases, when $6t(g_1) = \alpha_0/2 \mod \Lambda$. If $(\alpha_0 + \alpha_1)/2 \in \Lambda$, then $6t(g_1) = \alpha_1/2 \mod \Lambda$; since $3t(g_1) \in \Lambda$ by (6), prop.5.1 shows that $6t$ is a coboundary. The only remaining exception occurs when $R$ is of type $B_3 \times A_1 \times \ldots \times A_1$ and $\Lambda_K = \Lambda_0$.

Finally, suppose $B$ contains a component of type $B_2$. Choosing $\alpha_0$ to be the second root of such a component, so that $n_{10} = -2$, we see from 5.4 that $u_{10} \in \mathbb{Z}$ for all $i \neq 1$, and thus $2t(g_1) \in \Lambda$. Eqn.(8) with $j = 1$ then says that

$$2t(g_1) = u_{11} \alpha_1 - u_{11} \alpha_1 \mod \Lambda.$$  

Since $4t(g_1) \in \Delta$ by (7) and $2u_{11} \in \mathbb{Z}$ by 5.3(b), we must also have $2u_{11} \in \mathbb{Z}$. If there exist $i,j \neq 1$ such that $n_{ij} = -1$, applying $\alpha_j$ to (12) shows that $u_{11} \in \mathbb{Z}$; then $2t(g_1) = -u_{11} \alpha_1$ and $2t$ is a coboundary. This fails only if $R$ is of type $B_2 \times A_1 \times \ldots \times A_1$, when we may assume that $2t(g_1) = \alpha_1/2 \mod \Lambda$ for any $i \neq 1$.

The only remaining case is when $R$ is of type $A_1 \times \ldots \times A_1$.//

In each of the excluded cases, one can find examples of lattices $\Lambda$ for which $6\mathcal{H}(K^+,V/\Lambda) \neq 0$. For instance, suppose $R$ is of type $A_1 \times \ldots \times A_1$ with 5 factors and that $\Delta$ contains $(\alpha_2 + \alpha_3)/2$, $(\alpha_4 + \alpha_5)/2$ and $(\alpha_0 + \alpha_2 + \alpha_4)/2$. Define $t(g_2) = t(g_3) = (\alpha_0 + \alpha_4 + \alpha_5)/4$ and $t(g_4) = t(g_5) = (\alpha_2 + \alpha_3)/4$; then $t$ is a cocycle but $6t$ is not a coboundary.

5.6 Proposition. $4\mathcal{H}(K^+,V/\Lambda) = 0$, unless $R$ is of type $A_1 \times A_2$, $A_1 \times G_2$, $A_3$, $A_1 \times A_3$, $A_2 \times A_2$, $A_2 \times G_2$, $A_4$, $D_4$, $A_2 \times A_3$, $A_1 \times D_4$, $A_5$, $D_5$ or $A_2 \times D_4$. 
Proof. Suppose \( R \) contains a component of type \( B_m \) or \( C_m \) for some \( m \geq 2 \). Let \( \alpha_0 \) be the last root in such a component. By 5.4, we have \( 2u_{i0} \in \mathbb{Z} \) for all \( i \neq 1 \), so that \( 4t(g_1) \in \Lambda \). From (7), we also have \( 4t(g_1) \in \Lambda \), and thus \( 4t = 0 \).

Secondly, suppose \( R \) contains a component \( \Pi \) of type \( G_2 \). If \( \alpha_0 \) is the first root in \( \Pi \), 5.4 shows that \( u_{i0} \in \mathbb{Z} \) for all \( i \neq 1 \), so that \( 2t(g_1) \in \Lambda \). Eqns. (6) and (8) together imply that for all \( k \neq 1 \),

\[
2t(g_1) = -u_{k1} \alpha_1 + u_{1k} \alpha_k \text{ mod } \Lambda.
\]

Suppose there exist two disconnected roots \( \alpha_i, \alpha_j \in B \setminus \Pi \). Letting \( k = j \) in (13) and applying \( \alpha_1 \), we conclude that \( 2u_{i1} \in \mathbb{Z} \). Since \( 2u_{i1} \in \mathbb{Z} \) by 5.3(b), we see from (13) with \( k = i \) that \( 4t = 0 \).

If \( B \setminus \Pi = \{ \alpha_i, \alpha_j \} \) is of type \( G_2 \), this method of argument shows that \( 2u_{i1} = -3u_{1j} \text{ mod } \mathbb{Z} \) and \( 2u_{1j} = -u_{i1} \text{ mod } \mathbb{Z} \); together, these equations again imply that \( 2u_{i1} \in \mathbb{Z} \). The only other possibilities are \( A_1 \times G_2 \) and \( A_2 \times G_2 \).

Otherwise, if \( \alpha_1 \) exists, we have \( m_{01} = 3 \). Exclude for the moment the cases \( F_4 \) and \( D_6 \) in addition to the listed exceptions. Let \( B' = \{ \alpha_i \in B \mid i \neq 0,1 \} \), and choose some \( \alpha_i \in B' \) which is connected to at most one other \( \alpha_j \in B' \). If \( \alpha_k \in B' \) is different from \( \alpha_j \), 5.3(c) says that \( 2u_{i0} = 2u_{k0} \text{ mod } \mathbb{Z} \). On the other hand, our assumption implies that there exists \( \alpha_k \in B' \) disconnected from \( \alpha_j \), so that \( 2u_{j0} = 2u_{k0} = 2u_{i0} \text{ mod } \mathbb{Z} \). Thus the numbers \( 2u_{i0} \) have, mod \( \mathbb{Z} \), a common value \( c \) for \( i \neq 1 \). If \( \alpha_m \) is disconnected from \( \alpha_1 \), eqns. (6) and (8) show that

\[
2t(g_1) = u_{m0} \alpha_0 - u_{m1} \alpha_1 + u_{1m} \alpha_m \text{ mod } \Lambda.
\]
Again, our assumption implies the existence of two elements $\alpha_i$, $\alpha_j$ in $B'$ disconnected from $\alpha_1$ and from each other. Applying $\beta_1$ to (14) with $m = j$ shows that $2u_{1j} \in D$. Then, on multiplying (14) by 2, with $m = 1$, we deduce that $4t(g_1) = c \alpha_0 - 2u_{11} \alpha_1 \mod \Lambda$. Since $4t(g_1) = c \alpha_0 \mod \Lambda$ for $i \neq 1$, 5.1 shows that $4t$ is a coboundary. The cases $F_4$ and $D_6$ will be handled during the next argument. \\

Since there is no overlap in the exceptional cases of the preceding two propositions, we conclude that either 4 or 6 always annihilates $H^1(K^+, V/\Lambda)$. The group $4H^1(K^+, V/\Lambda)$ is therefore annihilated by 3 in exceptional cases of prop. 5.6. The exact sequence $0 \rightarrow \Lambda^*/\Lambda \rightarrow V/\Lambda \rightarrow V/\Lambda^* \rightarrow 0$, where $\Lambda^* = V^K \oplus P(R)$, gives rise to the exact sequence

$$\text{Hom}(K^+, \Lambda^*/\Lambda) \rightarrow H^1(K^+, V/\Lambda) \rightarrow H^1(K^+, V_K/P(R))$$

1.2 shows that unless $R$ is of type $A_2$ or $D_4$, the first group in this sequence has no 3-torsion. Thus the 3-torsion component of $H^1(K^+, V/\Lambda)$ maps injectively into that of

$H^1(K^+, V_K/P(R))$. When $R$ is of type $F_4$ or $D_6$, $-1V_K \in K^+$ and therefore $H^1(K^+, V_K/P(R))$ is annihilated by 2. Otherwise, one finds by explicit computation that the 3-torsion component of $H^1(K^+, V_K/P(R))$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ in all cases except that of $A_2 \times A_2$, when it is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$. When $R$ is of type $A_2$ or $D_4$, the 3-torsion component of $H^1(K^+, V_K/P(R))$ vanishes, but the first group in the sequence is isomorphic to the group of elements in $(V^K \oplus P(R))/\Lambda$ annihilated by 3. We thus have in all cases a bound for the dimension of $4H^4(K^+, V/\Lambda)$ over $\mathbb{Z}/3\mathbb{Z}$.
As $K^+$ is normal in $K$ and $\Lambda$ is (by assumption) invariant under $K$, we have the exact sequence

$$0 \to H^1(K/K^+, (V/\Lambda)^{K^+}) \xrightarrow{\text{inf}} H^1(K, V/\Lambda) \xrightarrow{\text{res}} H^1(K^+, V/\Lambda) \xrightarrow{\partial}. $$

Since $K/K^+$ acts trivially on $(V/\Lambda)^{K^+} = (V^K \oplus P(R))/\Lambda$, the first group can be identified with the group of elements in $(V^K + P(R))/\Lambda$ annihilated by 2. On the other hand, since $2H^1(K, V/\Lambda) = 0$ by 1.3.3, the image of res is contained in the subgroup $H^1(K^+, V/\Lambda)_2$ of elements annihilated by 2.

5.7 Proposition. The image of res is of index 1 or 2 in $H^1(K^+, V/\Lambda)_2$. In the latter case, the components of $R$ can only be of type $A_1$ or $A_2$. Furthermore, if $R_0$ is a component of type $A_3$, then $Q(R_0) \subseteq \Lambda \cap P(R_0) \subseteq P(R_0)$.

Proof. Suppose $t$ is a cocycle $K^+ \to V/\Lambda$ such that $2t$ is a coboundary, i.e.

$$(15) \quad 2t(g_i) = c \alpha_0 + d_i \alpha_i \mod \Lambda$$

for some $c, d_i \in R$. We shall first show that one can assume $c = 0$. If $R$ is of type $A_1 \times \ldots \times A_1$, we can subtract from $t$ the coboundary inducing function $g_i \to 2c \alpha_0$, which does not affect the assumption that $u_{ii} \in \mathbb{Z}$. Otherwise, we can choose $\alpha_0$ such that $\alpha_1$ exists and $n_{01} = -1$. Evaluating $\alpha_0$ and $\alpha_1$ at (15) with $i = 1$, we conclude that $2c + n_{10} d_1$ and $-c + 2d_1$ both belong to $\mathbb{Z}$. If $n_{10} = -2$ or $-3$, it follows that $c \leq \mathbb{Z}$.

If $n_{10} = -1$, then $3c \in \mathbb{Z}$ and $d_1 = 2c \mod \mathbb{Z}$. Therefore, adding to $t$ the coboundary inducing function $g_i \to c \alpha_0$ (i $\neq 1$), $g_i \to c \alpha_0 + 2c \alpha_1$ does not affect the assumption that $u_{ii}$ and $u_{10}$ are in $\mathbb{Z}$, but enables us to assume that $c = 0$. The argument also permits us to assume that $d_1 = 0$, unless $n_{10} = -2$,
when $d_i = 0$ or $\frac{1}{2}$.

Evaluating $\tilde{\chi}_i$ at (15), we conclude that $d_i$ can be assumed to be either 0 or $\frac{1}{2}$. Eqn. (6) also shows that for $i \neq 1$,

$$u_{i0} = d_i \mod Z$$

and

$$d_i = \frac{1}{2} \implies (\alpha_0 + \alpha_i)/2 \in \Lambda.$$  

If $n_{i0} = -3$, then $u_{i0} \in Z$ by 5.4 and thus $d_i = 0$. If $n_{i0} = -1$ and we are not in case (b) of 5.4, it again follows that $d_i = 0$.

If $\alpha_0$ is the last root in a component of type $B_2$, then $d_i = 0$ for $i \neq 1$ since $(\alpha_0 + \alpha_i)/2$ cannot be a weight.

5.8 Lemma. Suppose there exists an $x \in V_K$ such that $\langle x, \tilde{\chi}_0 \rangle \in Z$, $2 \langle x, \tilde{\chi}_1 \rangle \in Z$, $2 \langle x, \tilde{\chi}_2 \rangle = d_1 \mod Z$ for $i \neq 1$ and $2x \in \Lambda$. Then the function $T(s_0) = x$, $T(s_i) = t(g_i) - x - i\frac{1}{2}d_i \alpha_i + a\alpha_0$, where $a = d_1 - \langle x, \tilde{\chi}_1 \rangle$, induces a cocycle $T: K \to V/\Lambda$ such that the restriction of $T$ to $K^*$ is cohomologous to $T$.

Proof. Note that $p_{00} = \langle x, \tilde{\chi}_0 \rangle \in Z$ and $p_{ii} = u_{ii} - \langle x, \tilde{\chi}_i \rangle + d_i + a\alpha_0 \in Z$ for all $i$; therefore, we have only to verify the conditions (3)-(5) of I.3.3. It is clear that $2T(s_0) = 2x \in \Lambda$ and $2T(s_i) \in \Lambda$ for all $i$. If $i \neq 1$, $p_{0i} = d_i \mod Z$ and $p_{i0} = u_{i0} \mod Z$, so that $p_{0i} \alpha_i = p_{i0} \alpha_0 \mod \Lambda$. If $\alpha_i$ and $\alpha_j$ are disconnected roots in $B$ different from $\alpha_0$, we have $p_{ij} = u_{ij} + d_j$, $p_{ji} = u_{ji} + d_i$ and $d_i \alpha_i - d_j \alpha_j = u_{ij} \alpha_j - u_{ji} \alpha_i \mod \Lambda$ by (8), so that $p_{ij} \alpha_j = p_{ji} \alpha_i \mod \Lambda$.

If $m_{01} = 3$, then $3t(g_1) \in \Lambda$ by (7) and $2t(g_1) \in \Lambda$ by (15), so that $t(g_1) \in \Lambda$. Therefore $p_{01} = a \mod Z$, $p_{10} \in Z$ and $T(s_1) - T(s_0) = p_{01} \alpha_0 - p_{10} \alpha_1 \mod \Lambda$. If $\alpha_i$ and $\alpha_j$ are different from $\alpha_0$ and are such that $m_{ij} = 3$, then $p_{ij} = u_{ij} - d_j + \frac{1}{2}d_i \mod Z$, $p_{ji} = u_{ji} - d_i + \frac{1}{2}d_j \mod Z$. Applying
\( \alpha_j \) to (15), we have \( 2u_{ij} = -d_i \) and substituting this in (8), we obtain \( t(g_1) - t(g_j) = (d_j - u_{ij}) \alpha_i - (d_i - u_{ji}) \alpha_j \mod \Lambda \), so that \( T(s_i) - T(s_j) = p_{ji} \alpha_j - p_{ij} \alpha_i \mod \Lambda \). Therefore \( T \) induces a cocycles \( \overline{T}:K \rightarrow V/\Lambda \); since \( \overline{T}(g_1) = \overline{T}(s_i) + s_i \overline{T}(s_0) = \overline{T}(g_1) + \alpha_0 - \langle d_i, a \rangle \overline{T} = (2x_1 + \langle x, \alpha^* \rangle) \alpha_i \mod \Lambda \), the restriction of \( \overline{T} \) to \( K^+ \) is cohomologous to \( \overline{\tau} \).

Returning to the proof of the proposition, suppose \( d_i = 0 \) for \( i \neq 1 \); we can apply 5.8 with \( x = 0 \) to reach the conclusion. If \( \alpha_0 \) and \( \alpha_2 \) are the first and third roots in a component of type \( B_3 \) or \( B_4 \) and if \( d_1 = \frac{1}{2} \), we can let \( x = \frac{1}{2} \omega_2 \), since \( \omega_2 \in \Lambda \) by (16). The same kind of argument works for a component \( R_0 \) of type \( A_3 \) or \( D_4 \) if \( \Lambda \cap P(R_0) = P(R_0) \). In case \( R_0 \) is of type \( D_4 \) and \( \Lambda \cap P(R_0) = Q(R_0) \cup (P(R_0) + \omega_j) \) for some \( j \), we can choose \( \alpha_0 = \alpha_j \) and again have \( d_i = 0 \).

There remains the case when all components of \( R \) are of type \( A_1 \) or \( A_2 \). Define a homomorphism \( \overline{\tau}: H^1(K^+, V/\Lambda)_2 \rightarrow \frac{1}{2} \mathbb{Z}/\mathbb{Z} \) by \( \overline{\tau}(t) = \sum_{i \neq 1} d_i \mod \mathbb{Z} \). If \( \overline{\tau}(t) = 0 \), then \( d_2 = 0 \) in case there is a component of type \( A_2 \), so that the previous argument applies, or the number of \( d_i = \frac{1}{2} \) is even in case \( R \) is of type \( A_1 \times \ldots \times A_1 \). However, we can then let \( x = \sum d_i \omega_i \) in 5.8 since \( 2x = \sum d_i \alpha_i \in \Lambda \) in view of (16).

One can give examples in which the image of \( \text{res} \) is actually of index 2 in \( H^1(K^+, V/\Lambda)_2 \). Suppose first that \( R \) is of type \( A_3 \) and \( \Lambda \) is the lattice \( \Lambda_2 \); then the function \( t(g_1) = 0 \), \( t(g_2) = \frac{\alpha_1}{2} + \frac{\alpha_2}{4} \) provides a counterexample. Secondly, suppose \( R \) is of type \( A_1 \times \ldots \times A_1 \), with 4 factors, and \( \Lambda \) contains \( (\alpha_0 + \alpha_3)/2 \).
and \((\alpha_0 + \alpha_4)/2\); then we can define \(t(g_2) = (\alpha_0 + \alpha_3 + \alpha_4)/4\)
and \(t(g_3) = t(g_4) = 0\).

If we exclude all the exceptional cases of the preceding 3 propositions, we see that \(H^1(K^+, V/\Lambda)\) is annihilated by 2 and that the restriction homomorphism \(H^1(K, V/\Lambda) \to H^1(K^+, V/\Lambda)\) is surjective. Given a cocycle \(\overline{t} : K^+ \to V/\Lambda\), one can therefore find a cocycle \(\overline{T} : K \to V/\Lambda\) such that \(\overline{T}|_{K^+} = \overline{t}\). A space group with point group \(K^+\) corresponding to \(\overline{t}\) can then be viewed as the subgroup of direct isometries of a space group with point group \(K\) corresponding to \(\overline{T}\).

For lattices of exceptional type, explicit computation shows that in case (b), \(H^1(K^+, V/\Lambda) \cong (\mathbb{Z}/2\mathbb{Z})^{\dim V^K}\) if \(\Theta(\Lambda) = 0\) and \((\mathbb{Z}/2\mathbb{Z})^{\dim V^K - 1}\) if \(\Theta(\Lambda) \cong \mathbb{Z}/2\mathbb{Z}\), while in case (c), \(H^1(K^+, V/\Lambda) \cong (\mathbb{Z}/2\mathbb{Z})^{\dim V^K}\). We shall not deal with the case when \(R\) is of type \(A_1 \times \ldots \times A_1\) since the situation is not clear even in the ordinary case.

6. Application to classical crystallography.

In the two and three dimensional cases, the above results enable one to give a 'theoretical' derivation of some space groups well-known in classical crystallography. In two dimensions, there is always only one equivalence class of lattices and the group \(H^1(K^+, V/\Lambda)\) is always zero. The three dimensional situation is presented in Table I, using the Schoenflies notation for ease of comparison with [6]. The entries for groups of the form \(+K^+\) were deduced from I.2.5.
<table>
<thead>
<tr>
<th>$K^+$</th>
<th>$\Lambda$</th>
<th>$\varpi(\Lambda)$</th>
<th>Classical</th>
<th>$H^1(K^+, V/\Lambda)$</th>
<th>#orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_3^+$</td>
<td>$Q(R)$</td>
<td>$T_\alpha$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$A_3^+$</td>
<td>$\Lambda_2$</td>
<td>$T_\alpha$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$B_3^+$</td>
<td>$P(R)$</td>
<td>$O_\alpha$</td>
<td>$Z/4Z$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$C_3^+$</td>
<td>$P(R)$</td>
<td>$O_\gamma$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$(A_1 \times A_2)^+$</td>
<td>$Q(R)$</td>
<td>$\varpi_{\Lambda_{e_1}}$</td>
<td>$Z/3Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$(A_1 \times B_2)^+$</td>
<td>$Q(R)$</td>
<td>$D_3 \delta$</td>
<td>$Z/4Z \oplus Z/2Z$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$(A_1 \times G_2)^+$</td>
<td>$Q(R)$</td>
<td>$D_3 \varepsilon$</td>
<td>$Z/3Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$(A_1 \times A_1 \times A_1)^+$</td>
<td>$Q(R)$</td>
<td>$D_3 \alpha$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$A_2^+$</td>
<td>$Q(R)$</td>
<td>$D_4 \alpha$</td>
<td>$Z/4Z \oplus Z/2Z$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$B_2^+$</td>
<td>$Q(R)$</td>
<td>$D_4 \beta$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$G_2^+$</td>
<td>$Q(R)$</td>
<td>$D_6$</td>
<td>$Z/2Z \oplus Z/3Z$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$(A_1 \times A_1)^+$</td>
<td>$Q(R)$</td>
<td>$D_2 \alpha$</td>
<td>$Z/2Z \oplus Z/2Z \oplus Z/2Z$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$D_2 \gamma$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$D_2 \delta$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$D_2 \beta$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$C_3 \delta$</td>
<td>$Z/3Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$C_3 \alpha$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$C_4 \alpha$</td>
<td>$Z/4Z$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$C_4 \beta$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$C_6$</td>
<td>$Z/2Z \oplus Z/3Z$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$C_2 \alpha$</td>
<td>$Z/2Z$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>$Q(R)$</td>
<td>$C_2 \beta$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$A_1^+$</td>
<td>any</td>
<td>$C_1$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
In the above table, we have abandoned our convention regarding $\alpha_0$ and $\alpha_1$ and simply numbered the elements of $B$ consecutively starting with $\alpha_1$. Notation such as $\{\omega\}$ denotes the lattice $Q(R) \cup (Q(R)+\omega)$. 

| $\pm A_3^+$ | as for $A_3^+$ | $T_h$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 |
| $\pm A_2^+$ | as for $A_2^+$ | $C_{31} \delta$ | 0 | 1 |
| $\pm B_2^+$ | as for $B_2^+$ | $C_{4h} \alpha$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | 4 |
| $\pm G_2^+$ | as for $G_2^+$ | $C_{6h}$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 |
| $\pm A_1^+$ | as for $A_1^+$ | $C_1$ | 0 | 1 |
BIBLIOGRAPHY
