INTERPOLATION THEORY AND LIPSCHITZ CLASSES
ON TOTALLY DISCONNECTED GROUPS

by

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ABSTRACT

This thesis concerns the absolute convergence of the Fourier series of functions belonging to certain Lipschitz classes on totally disconnected groups. The technique used is one of interpolating between certain endpoint results which are proven directly. These results are shown to be best possible and a counterexample in interpolation theory is given.
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INTRODUCTION

This thesis is composed of three chapters. The first is a discussion of some results in interpolation theory. Due to the length and complexity of the proofs involved they have been omitted. The reader is referred to [3] and [9] for details.

A brief discussion of Vilenkin groups is to be found at the beginning of chapter two. The remainder of the chapter concerns relationships between the smoothness of functions and the size of their Fourier coefficients.

Extensions of the results of chapter two, together with a brief discussion of the sharpness of our results, form the bulk of chapter three. A counterexample in interpolation theory is also given.

The technique used in chapter three to extend the results of chapter two is that of interpolating between endpoint results discussed in chapter two. We make no claim that this is the only way to achieve these results; however, it does provide a unified treatment of what would otherwise seem to be an unrelated collection of criteria.

It is suggested that the reader who is unfamiliar with interpolation theory should begin by reading chapters two and three, referring back to one when necessary.

Theorems 2.9 and 2.10 together with all of chapter three are new. All other results can be found in the literature.
CHAPTER ONE

INTERPOLATION THEORY

We begin our discussion of interpolation theory by quoting the theorems of Riesz-Thorin and Marcinkiewicz. The concepts involved are then formally generalized and extended to larger classes of spaces and maps.

To begin, we need the following definitions.

Definition 1.1: Let $L^p(X, \mu)$ and $L^q(Y, \nu)$ be the usual Lebesgue spaces. If $Y$ is a linear operator from $L^p$ to $L^q$ we say $T$ is of (strong) type $(p, q)$ if $||Tf||_q \leq M||f||_p$ for all $f \in L^p$ and we define the $(p, q)$ norm of $T$ to be the infimum of all such $M$'s.

Then

Theorem 1.2: (Riesz-Thorin) Let $T$ be of type $(p_0, q_0)$ and $(p_1, q_1)$ with norms $M_0$ and $M_1$ respectively. Let $0 \leq \theta \leq 1$ and define

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$  Then $T$ is of type $(p, q)$ and the $(p, q)$ norm of $T$ is at most $M_0^{1-\theta}M_1^\theta$.

However powerful we might feel this theorem to be we shall see that somewhat weaker conditions on the "endpoints" $(p_i, q_i)$, $i = 0, 1$, often suffice to insure strong type $(p, q)$ for the "intermediate spaces".

We introduce the following concepts.
Definition 1.3: Let \( f \) be a measurable function. The distribution function of \( f \) is defined by

\[
D_f(x) = \mu\{t : |f(t)| > x\}.
\]

If \( T \) is an operator from some linear space of measurable functions to another space of measurable functions, \( T \) is subadditive if for almost all values of \( x \) we have

\[
|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|.
\]

Definition 1.4: Let the domain of \( T \) contain all finite linear combinations of the characteristic functions of sets of finite measure and all truncations of all its members. When \( 1 \leq p \leq \infty \) and \( 1 \leq q < \infty \) we say that \( T \) is of weak type \((p, q)\) if there exists a constant \( c \) with

\[
D_Tf(y) \leq \left( \frac{c||f||_p}{y} \right)^q.
\]

In the case where \( q = \infty \) we define weak type to be the same as strong type. We define the weak \((p, q)\) norm of \( T \) to be the infimum of all \( c \)'s satisfying the above inequality.

It follows readily that strong type \((p, q)\) implies weak type \((p, q)\) but not conversely (see for example [10]).

We can now state the Marcinkiewicz interpolation theorem.

Theorem 1.5: Let \( T \) be a subadditive operator of weak type \((p_0, q_0)\) and \((p_1, q_1)\). Furthermore suppose \( 1 \leq p_i \leq q_i \leq \infty \) for \( i = 0, 1 \) and \( q_0 \neq q_1 \).
If \( 0 < \theta < 1 \) and \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \), \( \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \), then \( T \) is of strong type \((p, q)\).

Direct proof of theorems 1.2 and 1.5 can be found in [9].

The immediate question would seem to be whether the concepts of these two theorems can be extended in some sense to other larger classes of spaces. The answer is yes. To see this we begin with the following concepts.

Let \( A \) be a linear Hausdorff space and let \( A_0 \) and \( A_1 \) be two Banach subspaces of \( A \) such that the injections of \( A_1 \) \((i = 0, 1)\) are continuous. We call such a pair \((A_0, A_1)\) an interpolation pair and define the intersection \( A_0 \cap A_1 \) and the algebraic sum \( A_0 + A_1 \) in the normal manner.

Much of the following material parallels the discussions in Butzer and Berens [3] and Stein and Weiss [9]. The reader is referred to these works for any of the proofs which are omitted.

**Theorem 1.6**: The spaces \( A_0 \cap A_1 \) and \( A_0 + A_1 \) are Banach spaces under the norms

\[
||f||_{A_0 \cap A_1} = \max \left( ||f||_{A_0}, ||f||_{A_1} \right),
\]

\[
||f||_{A_0 + A_1} = \inf_{f = f_0 + f_1} \left( ||f_0||_{A_0} + ||f_1||_{A_1} \right)
\]

Furthermore \( A_0 \cap A_1 \subset A_1 \subset A_0 + A_1 \) \((i = 0, 1)\).
Definition 1.7: An intermediate space (of $A_0$ and $A_1$) is a Banach space $A \subset A'$ such that $A_0 \cap A_1 \subset A \subset A_0 + A_1$.

The obvious example of an intermediate space is seen in theorems 1.2 and 1.5. If $L^{p_0}$ and $L^{p_1}$ are Lebesgue spaces, then the space $L^p$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ for $0 \leq \theta \leq 1$, are intermediate between $L^{p_0}$ and $L^{p_1}$. Later we will see that a more general class of spaces, namely the Lorentz spaces, are intermediate between the Lebesgue spaces.

There are several methods of generating intermediate spaces. For the purposes of this thesis we will discuss only two of the so-called "real variable" methods; the K-method and the J-method. We will not discuss the "complex-variable" method used by Calderón in [4].

We make the following definitions for $0 < t < \infty$.

Definition 1.8: For $f \in A_0 \cap A_1$ set

$$J(t, f) = \max (\|f\|_{A_0}, t\|f\|_{A_1}).$$

Definition 1.9: For $f \in A_0 + A_1$ set

$$K(t, f) = \inf_{f = f_0 + f_1} (\|f_0\|_{A_0} + t\|f_1\|_{A_1}).$$

Definition 1.10: For any Banach space $X$ we define $L^q_X(X)$ to be the Banach space of all classes of functions $t \rightarrow g(t)$, where $g(t) \in X$ and the map $t \rightarrow g(t)$ is strongly measurable with respect to the measure $\frac{dt}{t}$. 

(Haar measure with respect to the multiplicative group of points \( t \in (0, \infty) \)),
and the norm

\[
\|g(\cdot)\|_{L^q_t(X)} = \left\{ \begin{array}{ll}
\left( \int_0^\infty \|g(t)\|^{q} \frac{dt}{t} \right)^{1/q}, & 1 \leq q < \infty, \\
\text{ess sup} \|g(\cdot)\|, & q = \infty,
\end{array} \right.
\]
is finite.

**Definition 1.11**: The space \((A_o, A_1)_\theta, q; K\) is defined to be the space of all \( f \in A_o + A_1 \) such that \( t^{-\theta} K(t, f) \in L^q_t \).

We will only be interested in the cases \( 0 < \theta < 1, \ 1 \leq q < \infty \)
and/or \( 0 \leq \theta \leq 1, \ q = \infty \) as in all other cases the spaces contain only the zero element.

**Proposition 1.12**: The spaces \((A_o, A_1)_\theta, q; K\) for \( 0 < \theta < 1, \ 1 \leq q < \infty \)
and/or \( 0 \leq \theta \leq 1, \ q = \infty \) are Banach spaces under the norm

\[
\|f\|_{\theta, q; K} = \|t^{-\theta} K(t, f)\|_{L^q_t}.
\]

Furthermore

\[
A_o \cap A_1 \subset (A_o, A_1)_\theta, q; K \subset A_o + A_1.
\]

Thus we observe that in the nontrivial cases the spaces

\((A_o, A_1)_\theta, q; K\)

are intermediate spaces of \( A_o \) and \( A_1 \). It turns out that

\[
A_o \subset (A_o, A_1)_{0, \infty; K} \quad \text{and} \quad A_1 \subset (A_o, A_1)_{1, \infty; K}.
\]
Definition 1.13: For $-\infty < \theta < \infty$ and $1 \leq q \leq \infty$ we define

$$(A_0, A_1)_{\theta, q; J}$$

to be the space of all $f \in A_0 + A_1$ such that there exists a strongly measurable (with respect to $\frac{dt}{t}$) function $u : (0, \infty) \rightarrow A_0 + A_1$ such that

$$f = \int_0^\infty u(t) \frac{dt}{t}, \quad (u(\cdot) \in L^q_t(A_0 + A_1)), \quad t^{-\theta}J(t, u(t)) \in L^q.$$ 

Theorem 1.14: The spaces $(A_0, A_1)_{\theta, q; J}$ are meaningful for $0 \leq \theta \leq 1$, $q = 1$ and/or $0 < \theta < 1$, $1 < q < \infty$. Under the norms

$$||f||_{\theta, q; J} = \inf_{f = \int_0^\infty u(t) \frac{dt}{t}} (||t^{-\theta}J(t, u(t))||_{L^q_t})$$

they are Banach spaces with $A_0 \cap A_1 \subseteq (A_0, A_1)_{\theta, q; J} \subseteq A_0 + A_1$.

Thus we see that we have again generated an intermediate space of $A_0$ and $A_1$. An alternate criterion for $f \in A_0 + A_1$ to belong to this intermediate space can be shown.

Theorem 1.15: A function $f$ belongs to $(A_0, A_1)_{\theta, q; J}$ for $0 < \theta < 1$, $q = 1$ and/or $0 < \theta < 1$, $1 < q < \infty$ iff there exists an infinitely often, strongly, differentiable function $u = u(t)$, $0 < t < \infty$ in $A_0 \cap A_1$ such that

$$f = \int_0^\infty u(t) \frac{dt}{t}, \quad (u(\cdot) \in L^1_t(A_0 \cap A_1)),$$

and $t^{-\theta}J(t, u(t)) \in L^q$. 

In the course of proving this one sees

**Theorem 1.16**: $A_0 \cap A_1$ is a dense subspace of $(A_0, A_1)_{\theta, q; J}$ for $1 \leq q \leq 0$ if $0 < \theta < 1$, and for $q = 1$ if $\theta = 0$ or 1.

Under certain conditions the $K$- and $J$- methods are equivalent. In particular we have

**Theorem 1.17**: For $0 < \theta < 1$, $1 \leq q \leq \infty$, $(A_0, A_1)_{\theta, q; J} = (A_0, A_1)_{\theta, q; K}$ with equivalent norms. Also

$$(A_o, A_1)_{0,1; J} \subset A_o \subset (A_o, A_1)_{0,\infty; K}$$

and

$$(A_o, A_1)_{1,1; J} \subset A_1 \subset (A_o, A_1)_{1,\infty; K}.$$ 

Further for $1 \leq p \leq q \leq \infty$, $0 < \theta < 1$, we have

$$(A_o, A_1)_{\theta, p; K} \subset (A_o, A_1)_{\theta, q; K}.$$ 

In particular for $0 < \theta < 1$, $1 \leq q \leq \infty$,

$$(A_o, A_1)_{\theta, q; K} \subset (A_o, A_1)_{\theta, \infty; K}.$$ 

We also have

$$K(t, f) \leq ct^\theta ||f||_{\theta, q; K},$$
where \( c \) is a constant depending only on \( \theta \) and \( q \).

We now wish to discuss the theory of reiteration but first we need the following definitions (\( A \) will be an intermediate space of \( A_0 \) and \( A_1 \)).

**Definition 1.18**: \( A \) belongs to the class \( K(\theta; A_0, A_1) \) for \( 0 \leq \theta \leq 1 \) iff there exists a constant \( c_1 \) with \( K(t, f) \leq c_1 \cdot t^\theta \| f \|_A \) for any \( f \in A \).

**Definition 1.19**: \( A \) belongs to the class \( J(\theta; A_0, A_1) \) for \( 0 \leq \theta \leq 1 \) iff there exists a constant \( c_2 \) such that \( \| f \|_A \leq c_2 t^\theta J(t, f) \) for \( f \in A_0 \cap A_1 \).

**Definition 1.20**: \( A \) belongs to the class \( H(\theta; A_0, A_1), 0 \leq \theta \leq 1 \) iff \( A \) belongs to both \( K(\theta; A_0, A_1) \) and \( J(\theta; A_0, A_1) \).

These classes can be characterized in the following manner.

**Theorem 1.21**: An intermediate space \( A \) of \( A_0 \) and \( A_1 \) belongs to

(a) \( K(\theta; A_0, A_1), 0 \leq \theta \leq 1 \) iff \( A \subseteq (A_0, A_1)_{\theta, \infty}; K \).

(b) \( J(\theta; A_0, A_1), 0 \leq \theta \leq 1 \) iff \( (A_0, A_1)_{\theta, 1}; J \subseteq A \).

(c) \( H(\theta; A_0, A_1), 0 \leq \theta \leq 1 \) iff

\[
(A_0, A_1)_{\theta, 1}; J \subseteq A \subseteq (A_0, A_1)_{\theta, \infty}; K.
\]
As an immediate corollary we see

**Corollary 1.21.1**: For $0 < \theta < 1$, $1 \leq q \leq \infty$, $(A_0, A_1)_{\theta,q}, K \in \mathcal{H}(\theta; A_0, A_1)$ and $A_o \in \mathcal{H}(0; A_o, A_1)$, $A_1 \in \mathcal{H}(1, A_o, A_1)$.

We are now ready to state the theorem of reiteration.

**Theorem 1.22**: Let $A_{\theta_1}$ and $A_{\theta_2}$ be two intermediate spaces of $A_o$ and $A_1$ belonging to $\mathcal{H}(\theta_1; A_o, A_1)$ and $\mathcal{H}(\theta_2; A_o, A_1)$, $(0 \leq \theta_1 < \theta_2 \leq 1)$ respectively. Write $\theta = (1 - \theta')\theta_1 + \theta'\theta_2$, $(0 < \theta' < 1)$ and let $1 \leq q \leq \infty$. Then $(A_{\theta_1}, A_{\theta_2})_{\theta',q}, K' = (A_o, A_1)_{\theta,q}, K$ with equivalent norms.

By this time it should be apparent that the above characterization does not make it obvious what the intermediate spaces for given "endpoint spaces" are. In many cases the characterization is well-known but in others, it is still an open question and often, what would appear to be the logical choice is wrong (see corollary 3.9.1). However, before explicitly defining the intermediate spaces which we will later find of interest some standard results in interpolation theory should be stated. We begin with the following definitions.

**Definition 1.23**: If $(A_o, A_1)$ and $(B_o, B_1)$ are interpolation pairs in $A$ and $L$ respectively, $T(A, L)$ is the set of all linear maps $T$ from $A_o + A_1$ into $B_o + B_1$ such $T|_{A_1}$ is a bounded linear transformation of $A_1$ into $B_1$. We will denote $||T|_{A_1}||$ by $M_1$. 

Definition 1.24: If $A$ and $B$ are intermediate spaces of $A_0$, $A_1$ and $B_0$, $B_1$ respectively, we say that $A$ and $B$ have the interpolation property iff for every $T \in T(A, L)$, $T|A$ is a bounded linear transformation of $A$ into $B$.

The spaces $A$ and $B$ are then called interpolation spaces with respect to $(A_0, A_1)$ and $(B_0, B_1)$. In particular we say that $A$ and $B$ are interpolation spaces of type $\theta$ (where $0 \leq \theta < 1$) if

$$||T|A|| = M \leq C M_\alpha^{1-\theta} M_\beta$$

for all $T \in T(A, L)$ where $C \geq 1$ is independent of $T$. The inequality is said to be exact if $C = 1$.

Theorem 1.25: Suppose $(A_0, A_1)$ and $(B_0, B_1)$ are two interpolation pairs of $A$ and $L$ respectively. Then the spaces $(A_0, A_1)_{\theta,q;K}$ and $(B_0, B_1)_{\theta,q;K}$ for $0 < \theta < 1$, $1 \leq q < \infty$ and/or $0 < \theta \leq 1$, $q = \infty$ are interpolation spaces of $(A_0, A_1)$ and $(B_0, B_1)$ of type $\theta$ and

$$M \leq M_\alpha^{1-\theta} M_\beta \quad (T \in T(A, L)).$$

Theorem 1.26: The intermediate spaces $(A_0, A_1)_{\theta,q;J}$ and $(B_0, B_1)_{\theta,q;K}$ for $0 < \theta < 1$ and $1 \leq q < \infty$ are interpolation spaces of $(A_0, A_1)$ and $(B_0, B_1)$ of type $\theta$.

As mentioned before we can extend the intermediate spaces of the Lebesgue spaces $L^0$ and $L^1$ to a larger class of spaces, namely the Lorentz spaces, which contain the $L^p$ spaces as particular cases. We begin
Definition 1.27: Let \( f \) be a measurable function. Its non-increasing (or equimeasurable) rearrangement \( f^* \) is given for \( x > 0 \) by

\[
f^*(x) = \inf\{t : D_f(t) \leq x\},
\]

where \( D_f \) is the distribution function of \( f \) (see definition 1.3).

Definition 1.28: The Lorentz space \( L(p, q) \) is the space of all measurable functions \( f \) such that

\[
||f||_{p,q}^* = \left( \frac{q}{p} \int_0^\infty \left[ t^{1/p} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} < \infty,
\]

for \( 1 \leq p < \infty \), \( 1 \leq q < \infty \) and \( ||f||_{p,q}^* = \sup_{t>0} t^{1/p} f^*(t) < \infty \), when \( 1 \leq p < \infty \) and \( q = \infty \).

It should be mentioned that \( ||\cdot||_{p,q}^* \) is not always a norm since the triangle inequality may fail. We will introduce a norm on \( L(p, q) \) (which then provides an alternate definition to 1.28), but it is much easier to work with \( ||\cdot||_{p,q}^* \).

Since \( ||f^*||_p = ||f||_p \) and \( ||f||_{p,p}^* = ||f^*||_p \) it follows that \( L(p, p) = L^p \).

Theorem 1.29: Let \( f \in L(p, q_1) \) and select \( q_2 \geq q_1 \). Then

\[
||f||_{p, q_2}^* \leq ||f||_{p, q_1}^*
\]

and hence \( L(p, q_1) \subseteq L(p, q_2) \).
Definition 1.30: A subadditive operator $T$ is of restricted weak type $(p, q)$ if it satisfies the weak type $(p, q)$ conditions when restricted to characteristic functions of sets of finite measure.

Theorem 1.31: Let $T$ be a subadditive operator of restricted weak types $(r_0, p_0)$ and $(r_1, p_1)$ where $r_0 < r_1$ and $p_0 \neq p_1$. Then there exists $B = B(\theta)$ such that

$$||Tf||_{p,q}^* \leq B||f||_{r,q}^*$$

for all $f \in L(r, q)$, where $1 \leq q < \infty$ and \( \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \), \( \frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1} \) and $0 < \theta < 1$.

Theorem 1.5 is a special case of the last theorem but theorem 1.31 is sharper as it concerns a larger class of spaces.

Application of this to the Fourier transform yields a stronger form of the Hausdorff-Young inequality which will be of later interest.

Corollary 1.31.1: Let $f \in L^p(\mathbb{R}^n), 1 < p \leq 2$. Then $\hat{f} \in L(p', p)$ (where \( \frac{1}{p'} + \frac{1}{p} = 1 \)) and there exists a constant $B$ dependent only on $p$ such that

$$||f||_{p',p}^* \leq B||f||_p.$$

Definition 1.32: The average function of $f^*$ is

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(x) \, dx.$$
We now define the norm

\[ \|f\|_{p, q} = \left( \frac{\int_0^\infty \left[ t^{1/p} f^{**}(t) \right]^q \frac{dt}{t} \right)^{1/q} \]

where 1 ≤ p < ∞, 1 ≤ q < ∞, and

\[ \|f\|_{p, q} = \sup_{t>0} t^{1/p} f^{**}(t) , \]

where 1 ≤ p ≤ ∞, q = ∞.

Many authors (see [3]) define the \( L(p, q) \) spaces in terms of this norm. However the introduction of \( f^{**} \) greatly increases the complexity of calculation for any given function \( f \). In any case the two definitions are equivalent as

**Theorem 1.33**: For \( f \in L(p, q) \), 1 < p ≤ ∞,

\[ \|f\|_{p, q}^* \leq \|f\|_{p, q} \leq \frac{p}{p-1} \|f\|_{p, q}^* , \]

Furthermore

**Theorem 1.34**: For 1 < p ≤ ∞, 1 ≤ q ≤ ∞, \( L(p, q) \) with the norm

\[ \|\cdot\|_{p, q} \] is a Banach space.

We also have the following

**Theorem 1.35**: The spaces \( L(p, q) \) (for 1 < p < ∞, 1 ≤ q < ∞ and/or 1 ≤ p ≤ ∞, q = ∞) are intermediate spaces of \( L^1 \) and \( L^\infty \). They are
equal to the spaces \((L^1, L^\infty)\) and \(||f||_{p,q;K} = ||f||_{1-\frac{1}{p},q;K} \).

Additionally for \(1 < p < \infty, 1 \leq q \leq \infty\), the bounded functional on \((L^1, L^\infty)\) defined by \(||t^{1/p}f^*(t)||_{L^*_q} \) satisfies

\[ ||t^{1/p}f^*(t)||_{L^*_q} \leq ||f||_{1-\frac{1}{p},q;K} \leq \frac{p}{p-1} ||t^{1/p}f^*(t)||_{L^*_q} . \]

**Theorem 1.36**: For \(1 \leq p_0 < p_1 \leq \infty, 1 \leq q \leq \infty\), the spaces \(L(p, q)\) are the intermediate spaces \((L^{p_0}, L^{p_1})_{\theta,q;K}\) of \(L^{p_0}\) and \(L^{p_1}\), where

\[ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1 . \]

**Proof**: This follows from 1.35 by reiteration.

**Definition 1.37**: Let \(\omega\) be a measurable function. Then we define the weighted \(L^p\) space \(L^p_\omega\) to be the class of all functions \(f\) such that the product \(f\omega\) belongs to \(L^p\) and we define \(||f||_{L^p_\omega} = ||f\omega||_p\).

We wish to discuss the intermediate spaces \(\left(\frac{L^p_\omega}{\omega}, \frac{L^p_\omega}{\omega_1}\right)_{\theta,p;K}\). It is known that the "diagonal spaces" \(\left(\frac{L^p_\omega}{\omega_0}, \frac{L^p_\omega}{\omega_1}\right)_{\theta,p;K}\) = \(L^p_\omega\), with an equivalent norm, where \(\omega = \omega_0^{1-\theta} \omega_1^\theta\).

We mention that Calderón in [4] characterized the intermediate spaces obtained by the complex method. They are \([L^p, L^p_\omega]_\theta = L^p_v\) where \(v(x) = \omega(x)^\theta\). As before however we are interested in the "real"
interpolation methods as they yield more general results. Due to the complexities of the arguments involved we quote from Gilbert [6], correcting a typographical error, but omitting the proof.

**Theorem 1.38**: If $1 \leq p \leq \infty$, and $0 < \theta < 1$, $1 \leq q \leq \infty$ and $r > 1$, the expression

$$
\left\{ \sum_{k=-\infty}^{\infty} \left( \int_{r^{k-1} < \omega(x) \leq r^k} |f(x)|^p \, d\mu \right)^{1/p} \right\}^{1/q}
$$

defines a norm on $(L^p, L^q)$.

More generally $(L^p, L^q)_{\theta, q; K}$ is the class of functions $f$ such that the above expression is finite where $\omega = \omega_{-1}^{-1} \omega_1$ and $d\mu = \omega_0 \, dt$.

An equivalent norm on the intermediate spaces is given by

$$
\left\{ \sum_{k=-\infty}^{\infty} \left( \int_{r^{k-1} < \omega(x) \leq r^k} |f(x)\omega_0(x)^{1-\theta} \omega_1(x)^{\theta}|^p \, dx \right)^{q/p} \right\}^{1/q}
$$

In the case that $p = q$ these reduce to weighted $L^p$ spaces. In the general case however, they are called mixed norm spaces. The reader is referred to Benedek and Panzone [2] for a detailed discussion.
CHAPTER TWO

LIPSCHITZ CLASSES ON TOTALLY DISCONNECTED GROUPS

Let $G$ be a Vilenkin group, that is, a compact, metrizable, 0-dimensional, abelian group. Then the dual group $X$ of $G$ is a countable discrete, abelian, torsion group. Vilenkin [11] proved

Theorem 2.1: There exists an increasing sequence $\{X_n\}$ of finite subgroups of $X$ and a sequence $\{\phi_n\}$ of characters in $X$ such that

(i) $X_0 = \{\chi_0\}$ where $\chi_0(x) = 1$ for any $x \in G$;

(ii) for $n \geq 1$, $X_n/X_{n-1}$ is of prime order $p_n$;

(iii) $X = \bigcup_{n=0}^{\infty} X_n$;

(iv) $\phi_n \in X_{n+1} \setminus X_n$ for all $n \geq 0$;

(v) $\phi_{n+1} \in X_n$ for every $n \geq 0$.

Next we will enumerate the elements of $X$ by means of the $\phi_n$.

Set $m_0 = 1$ and define $m_n = \prod_{i=1}^{n} p_i$. Now, if $k \geq 1$ and if $k = \sum_{i=0}^{s} a_im_i$ with $0 \leq a_i < p_{i+1}$ for $0 \leq i \leq s$ define

$$\chi_k = \phi_0^{a_0} \cdots \phi_s^{a_s}.$$

Then we can write $X_n = \{\chi_i : 0 \leq i < m_n\}$. Next let

$$G_n = \{ x \in G : \chi_k(x) = 1, \text{ all } \chi_k \in X_n \}.$$
be the annihilator of $X_n$. It is clear that

$$G = G_0 \supset G_1 \supset \ldots \supset \bigcap_{n=0}^{\infty} G_n = \{0\}$$

and that the $G_n$'s form a fundamental system of neighbourhoods of zero in $G$. Furthermore, for each $n \geq 0$ there exists $x_n \in G_n \setminus G_{n+1}$ such that

$$\chi_{m_n}(x_n) = \exp \left( \frac{2\pi i}{p_{n+1}} \right)$$

and each $x \in G$ can be uniquely represented in the form $x = \sum_{i=0}^{\infty} b_i x_i$ where $0 \leq b_i < p_{i+1}$ for every $i > 0$.

Additionally

$$G_n = \{ x \in G : x = \sum_{i=0}^{\infty} b_i x_i, b_0 = \ldots = b_{n-1} = 0 \}.$$

Consequently each coset of $G_n$ in $G$ can be represented by $z + G_n$ where

$$z = \sum_{i=0}^{n-1} b_i x_i$$

for some choice of $0 \leq b_i < p_{i+1}$. We shall order these $z$ lexicographically and denote them by $z_{q,n}$ where $0 \leq q < m_n$.

If $dx$ is the normalized Haar measure on $G$, the Fourier series of $f \in L^1(G)$ is the series

$$S[f](x) = \sum_{n=0}^{\infty} \hat{f}(n) \chi_n(x)$$

where

$$\hat{f}(n) = \int_G f(t) \chi_n(t) \, dt$$

The partial sums are given by
\[ S_k(f, x) = \sum_{n=0}^{k-1} f(n) \chi_n(x) = \int_G f(x-t)D_k(t) \, dt \]

where \( D_k(t) = \sum_{n=0}^{k-1} \chi_n(t) \) is the Dirichlet kernel of order \( k \). The Dirichlet kernels have the property that

\[
D_m(x) = \begin{cases} 
0 & \text{if } x \notin G_n, \\
m_n & \text{if } x \in G_n;
\end{cases}
\]

(see Vilenkin [11]).

For the remainder of this discussion \( G \) and \( X \) will be as described above with the further restriction that \( \sup_{n} p_n = p < \infty \).

(The usual example of a group of this type is given by \( G = \prod_{n=1}^{\infty} \mathbb{Z}(p_n) \), where \( \{p_n\} \) is a bounded sequence of prime numbers (not necessarily distinct). Then \( x_n = (b_0, b_1, \ldots, b_n, \ldots) \) where \( b_i = 0 \) if \( i \neq n \) and \( b_n = 1 \), and if \( x \in G \) is arbitrary, \( f_n(x) = \exp(2\pi ib_n/p_{n+1}) \). In the case that \( p_n = 2 \) for every \( n \), \( G \) is merely \( 2^\omega \) and the elements of the character group \( X \) are the Walsh functions.)

We need the following definitions:

**Definition 2.2:** For \( \beta > 0 \) we define \( A(\beta) \) to be the set of all integrable functions \( f \) on \( G \) such that
\[ \sum_{n=0}^{\infty} |\hat{f}(n)|^\beta < \infty \quad \text{for} \quad \beta < \infty, \]
\[ \sup_{n} |\hat{f}(n)| < \infty \quad \text{for} \quad \beta = \infty; \]

i.e. we say \( f \in A(\beta) \) iff \( \hat{f} \in \ell^\beta \).

**Definition 2.3:** Let \( 1 \leq p \leq \infty, \quad k \in \mathbb{N} \). Then for \( f \in L^p(G) \) we define the integrated modulus of continuity of order \( k \) to be

\[ \omega_p(f, k) = \sup \{ ||T_y f - f||_p : y \in G_k \} , \]

where \( T_y f(x) = f(x + y) \).

**Definition 2.4:** If \( 0 < \alpha \leq 1, \quad 1 \leq p \leq \infty, \quad 1 \leq q \leq \infty \), then we define \( \text{Lip}(\alpha, q; L^p) \) to be the set of all \( f \in L^p \) such that \( \{m_k^{\alpha} \omega_p(f, k)\}_k \in \ell^q \) and we define a norm on this space by

\[ ||f||_{\text{Lip}(\alpha, q; L^p)} = ||f||_p + \left( \sum_{k=0}^{\infty} (m_k^{\alpha} \omega_p(f, k))^q \right)^{1/q} \]

with the appropriate modification if \( q = \infty \).

Some authors (for example Vilenkin [11] and Onneweer [7, 8]) consider a space \( \text{Lip} \alpha \) which in our notation is \( \text{Lip}(\alpha, \infty; L^\infty) \). The classes \( \text{Lip}(\alpha, q; L^\infty) \) are used to deal with Fourier multipliers in [5] and [12].

It is clear that for \( 1 \leq p \leq \infty, \quad 1 \leq q, s \leq \infty \), and \( 0 < \alpha_1 < \alpha_2 \leq 1 \), we have \( \text{Lip}(\alpha_2, q; L^p) \subseteq \text{Lip}(\alpha_1, s; L^p) \). Furthermore, if \( 0 < \alpha \leq 1, \quad 1 \leq q \leq \infty \) and \( 1 \leq p_0 \leq p_1 \leq \infty \) then
Lip(α, q; L^P) \subset Lip(α, q; L^P) \text{ and lastly, if } 0 < α \leq 1, \ 1 \leq p \leq \infty, \text{ and } 1 \leq q_0 \leq q_1 < \infty \text{ then } Lip(α, q_0; L^P) \subset Lip(α, q_1; L^P).

Definition 2.5: If f is a function on \( G \) and \( H \subset G \), then we set

\[ \text{osc}(f, H) = \sup \{ |f(x) - f(y)| : x, y \in H \} \text{.} \]

The following theorem is due to Onneweer [7] and is of interest to us mainly because of inequalities (1) and (2) in the proof.

Theorem 2.6: If \( f \in L^r(G) \), \( 1 \leq r \leq 2 \) and if

\[ \sum_{k=0}^{\infty} \left( \sum_{q=0}^{m_k-1} \text{osc}(f, z_{q,k}+C_k))^r \right)^{1/r} < \infty, \]

then \( f \in A(1) \).

Proof: For every \( k \geq 0 \) the Fourier series \( f(x + x_k) - f(x) \) is

\[ \sum_{n=0}^{\infty} \hat{f}(n)(\chi_n(x_k) - 1)\chi_n(x) \text{.} \]

For \( m_k \leq n < m_{k+1} \), we have \( n = \sum_{i=0}^{k} a_i m_i \) with \( 0 \leq a_i < p_{i+1} \) for all \( i \) and \( a_k \neq 0 \). Now as \( x_k \in C_k \), it follows that \( \chi_n(x_k) = \exp(2\pi i a_k/p_{k+1}) \) implying \( |\chi_n(x_k) - 1| > \pi p_{k+1}^{-1} > \pi p^{-1} \). Hence the Hausdorff-Young inequality implies that if \( k \geq 0 \) and \( s \) is the index conjugate to \( r \), then
(1) \[
\frac{1}{p} \left\{ \sum_{n=m_k}^{m_k+1-1} \left| \hat{f}(n) \right|^s \right\}^{1/s} \\
\leq \left\{ \sum_{n=m_k}^{m_k+1-1} \left| \hat{f}(n) \right|^s \chi_n(x_k) - 1 \right\}^{1/s} \\
\leq \left\{ \int_G |f(x + x_k) - f(x)|^r \, dx \right\}^{1/r} \\
= \left\{ \sum_{q=0}^{m_k-1} \int_{G_k} |f(x + x_k - z_q, k) - f(x - z_q, k)|^r \, dx \right\}^{1/r} \\
\leq \left\{ \sum_{q=0}^{m_k-1} (osc(f, -z_q, k + G_k))^r \right\}^{1/r}.
\]

An application of Hőlder's inequality yields

(2) \[
\frac{1}{p} \left\{ \sum_{n=m_k}^{m_k+1-1} \left| \hat{f}(n) \right|^s \right\}^{1/s} \\
\leq (m_{k+1} - m_k)^{1/r} \left\{ \sum_{n=m_k}^{m_k+1-1} \left| \hat{f}(n) \right|^s \right\}^{1/s} \\
\leq (m_{k+1} - m_k)^{1/r} \frac{p}{\pi} \frac{1}{m_k} \left( \sum_{q=0}^{m_k-1} (osc(f, -z_q, k + G_k))^r \right)^{1/r}.
\]

Hence

\[
\sum_{n=1}^{\infty} \left| \hat{f}(n) \right| \leq C \sum_{k=0}^{\infty} \left( \sum_{q=0}^{m_k-1} (osc(f, z_q, k + G_k))^r \right)^{1/r}
\]

which is finite, i.e. \( f \in A(1) \).
The following theorem is also due to Onneweer [8].

**Theorem 2.7 :** Let \( 1 \leq p \leq 2 \) and \( 0 < \beta \leq p' \). If \( f \in L^p(G) \) and if

\[
\sum_{k=0}^{\infty} m_k^{(p' - \beta)/p'} (\omega_p(f, k)) < \infty ,
\]

then \( f \in A(\beta) \).

Restated in our notation this becomes

**Theorem 2.7':** Let \( 1 \leq p \leq 2 \) and \( 1 \leq \beta \leq p' \). If \( f \in \text{Lip} \left( \frac{1}{\beta} - \frac{1}{p} , \beta; L^p \right) \), then \( f \in A(\beta) \).

**Proof :** First let \( 1 < p' \leq 2 \). Then by inequality (1), for all \( k \geq 0 \)

\[
(3) \quad \left( \sum_{n=m_k}^{m_{k+1}-1} |\hat{f}(n)|^{p'} \right)^{1/p'} \leq C \left( \int_G |f(x + x_k) - f(x)|^p \, dx \right)^{1/p} \leq C \omega_p(f, k) .
\]

Hölder's inequality then yields

\[
\sum_{n=m_k}^{m_{k+1}-1} \left| \hat{f}(n) \right|^{\beta} \leq \left( \sum_{n=m_k}^{m_{k+1}-1} |\hat{f}(n)|^{p'} \right)^{\beta/p'} \left( \sum_{n=m_k}^{m_{k+1}-1} 1 \right)^{(p' - \beta)/p'}
\]
\[ \leq C(p(f, k)) (m_k + 1 - m_k) (p' - \beta)/p' \].

Hence

\[ \sum_{n=1}^{\infty} |\hat{f}(n)|^\beta \leq C \sum_{k=0}^{\infty} (m_{k+1} - m_k) (p' - \beta)/p' (\omega_p(f, k))^\beta \]

\[ \leq C \sum_{k=0}^{\infty} m_k (p' - \beta)/p' (\omega_p(f, k))^\beta < \infty ; \]

i.e. \( f \in A(\beta) \).

If \( p = 1 \) then \( p' = \infty \) and the above proof holds with the appropriate modifications.

Setting \( \beta = 1 \) in theorem 2.7' yields

**Corollary 2.7.1**: Let \( 1 \leq p \leq 2 \). If \( f \in \text{Lip}(\frac{1}{p}, 1; L^p) \) then \( f \in A(1) \).

**Corollary 2.7.2**: Let \( 0 < \alpha < 1, \ 1 \leq p \leq 2, \) and \( \beta > p/(\alpha p + p - 1) \). If \( f \in \text{Lip}(\alpha, \infty; L^p) \) then \( f \in A(\beta) \).

**Proof**: Note that our assumptions imply \( \alpha > \frac{p' - \beta}{\beta p'} \). Hence

\( \text{Lip}(\alpha, \infty; L^p) \subseteq \text{Lip}(\frac{p' - \beta}{\beta p'}, \beta; L^{p'}) \). The result follows by 2.7'.

Choose \( \beta = 1 \) in corollary 2.7.2 and obtain

**Corollary 2.7.3**: Let \( 1 \leq p \leq 2 \) and \( p^{-1} < \alpha < 1 \). If \( f \in \text{Lip}(\alpha, \infty; L^p) \) then \( f \in A(1) \).
In [8] Onneweer shows that the last two corollaries are best possible as there does exist an \( f \in \text{Lip}(\alpha, \infty; L^p) \) which does not belong to \( A(p/(\alpha p + p - 1)) \), \((1 < p \leq 2, 0 < \alpha \leq 1)\). This will be discussed in detail in chapter 3.

**Theorem 2.8**: If \( 1 \leq p \leq 2, 0 < \alpha < 1 \) and \( f \in \text{Lip}(\alpha, 1; L^p) \) then for all \( \beta \geq \frac{1}{p} - \alpha \), \( \sum_{n=1}^{\infty} n^{-\beta} |\hat{f}(n)| < \infty \).

**Proof**: An application of Hölder's inequality yields

\[
\sum_{n=m_k}^{m_{k+1}-1} |\hat{f}(n)| \leq (m_{k+1})^{1/p} \left( \sum_{n=m_k}^{m_{k+1}-1} |\hat{f}(n)|^{p'} \right)^{1/p'} \leq C(m_{k+1})^{1/p} \omega_p(f, k)
\]

by (3). Then we have

\[
\sum_{n=m_k}^{m_{k+1}-1} n^{-\beta} |\hat{f}(n)| \leq C_1 (m_{k+1})^{1-\beta} \omega_p(f, k)
\]

\[
\leq C_2 m_k^{1-\beta} \omega_p(f, k).
\]

Now, in the fundamental case where \( \beta = 1/p - \alpha \), we have

\[
\sum_{n=1}^{\infty} n^{-\beta} |\hat{f}(n)| \leq C_2 \sum m_k^{\alpha} \omega_p(f, k) < \infty
\]

as \( f \in \text{Lip}(\alpha, 1; L^p) \).
If \( \beta > \frac{1}{p} - \alpha \) then

\[
\sum_{n=1}^{\infty} n^{-\beta} |\hat{f}(n)| \leq \sum_{n=1}^{\infty} n^{-\frac{1}{p}} |\hat{f}(n)| < \infty.
\]

As a corollary to this we obtain a result due to Onneweer [7].

Corollary 2.8.1: If \( f \in \text{Lip}(\alpha, \infty; L^\infty) \) for \( 0 < \alpha \leq 1 \) then, for all \( \beta > \frac{1}{2} - \alpha \), \( \sum_{n=1}^{\infty} n^{-\beta} |\hat{f}(n)| < \infty \).

Proof: Let \( \beta > \frac{1}{2} - \alpha \) and set \( \alpha_1 = \frac{1}{2} - \beta \). Then

\[
\text{Lip}(\alpha, \infty; L^\infty) \subseteq \text{Lip}(\alpha, \infty; L^2) \subseteq \text{Lip}(\alpha_1, 1; L^2)
\]

Because \( \alpha_1 < \alpha \). The result follows by theorem 2.8.

Notice also that setting \( \alpha = 1/p \) in theorem 2.8 gives an alternate proof of corollary 2.7.1.

We next look at some results of a slightly different nature.

Theorem 2.9: Let \( 1 \leq p \leq 2 \) and \( 0 < \alpha \leq 1 \). If \( \{C_n\} \) is a sequence for which \( \sum_{n=1}^{\infty} |C_n|^p < \infty \), then there exists an \( f \in \text{Lip}(\alpha, \infty; L^p) \) such that \( \hat{f}(n) = C_n \) for all \( n \geq 1 \).

Proof: As \( \sum_{n=1}^{\infty} |n^{\alpha} C_n|^p < \infty \) then \( \sum_{n=1}^{\infty} |C_n|^p < \infty \). By the Hausdorff-Young theorem there exists an \( f \in L^p \) such that \( \hat{f}(n) = C_n \) for each \( n \geq 1 \). Next
choose any natural number \( q \) and any \( y \in G \). Now \( \chi_k(y) = 1 \) for \( 0 \leq k \leq m \) and hence

\[
T_y f(x) = f(x) - \sum_{k=m}^{\infty} \hat{f}(k)(\chi_k(y) - 1)\chi_k(x).
\]

Applying the Hausdorff-Young inequality yields

\[
| |T_y f - f| |_{L^p} \leq \left( \sum_{k=m}^{\infty} |\hat{f}(k)(\chi_k(y) - 1)|^p \right)^{1/p} \leq \left( 2^p \sum_{k=m}^{\infty} |\hat{f}(k)|^p \right)^{1/p} \leq 2 \left( \sum_{k=m}^{\infty} |\hat{f}(k)|^p \frac{k^{\alpha p}}{m^p} \right)^{1/p} \leq 2m^{-\alpha} \left( \sum_{k=m}^{\infty} |\hat{f}(k)|^p \right)^{1/p} \leq Cm^{-\alpha}.
\]

Thus \( f \in \text{Lip}(\alpha, \infty; L^p) \).

**Theorem 2.10**: Let \( 1 \leq p \leq 2 \) and \( 0 < \beta < \alpha \leq 1 \). Suppose that \( f \in \text{Lip}(\beta, p'; L^p) \). Then \( \sum |\hat{f}(n)n^\beta|^{p'} < \infty \).

**Proof**: By inequality (3) we have

\[
\sum_{n=m_k}^{m_{k+1}-1} |n^\beta \hat{f}(n)|^{p'} \leq C(m_k^\beta (f, k))^{p'}.
\]

Summing over \( k \) yields the desired result.
Corollary 2.10.1: Let $1 < p \leq 2$, $0 < \beta < \alpha < 1$. If $f \in \text{Lip}(\alpha, \infty; L^p)$ then 

$$\sum_{n=1}^{\infty} |\hat{f}(n)n^\beta|^{p'} < \infty.$$ 

Proof: If $\beta < \alpha$ then $\text{Lip}(\alpha, \infty; L^p)$ is a subspace of $\text{Lip}(\beta, p'; L^p)$. The result follows by Theorem 2.10.

We mention at this point that theorem 2.10 is best possible since there exists an $f \in \text{Lip}(\beta, q; L^p)$ for any $q > p'$ such that 

$$\sum_{n=1}^{\infty} |\hat{f}(n)n^\beta|^{p'} = \infty.$$ 

Corollary 2.10 is also best possible since there is an $f \in \text{Lip}(\alpha, \infty; L^p)$ such that 

$$\sum_{n=1}^{\infty} |\hat{f}(n)n^\alpha|^{p'} = \infty.$$ 

The reader is referred to chapter three for details.
CHAPTER THREE

APPLICATIONS OF INTERPOLATION THEORY

This chapter begins with a brief discussion of the intermediate spaces \((\text{Lip}(\alpha, r; L^p), \text{Lip}(\beta, s; L^p))_{\theta, q}\). After determining what these spaces are we examine the theorems of chapter 2 in the context of interpolation theory. In this manner we extend the results we already have. We also show that these new results are sharp and we give a counterexample in the interpolation theory of the spaces

\[
\left[\text{Lip}(\alpha, r; L^p_0), \text{Lip}(\beta, s; L^p_1)\right]_{\theta, q}.
\]

The following is a generalization of work of Fournier [5].

We begin by considering \((L^r, \text{Lip}(l, \infty; L^r))_{\theta, q}\) where \(1 < r < \infty\) and \(0 < \theta < 1\), \(1 \leq q \leq \infty\). We define

\[
\omega_r(f, 0) \quad \text{if } t \geq 1,
\]

\[
\omega_r(f, k) \quad \text{if } \frac{1}{m_k} \leq t < \frac{1}{m_{k-1}}
\]

It follows that \(f \in \text{Lip}(\alpha, q; L^r)\) iff \(f \in L^r\) and \(t^{-\alpha}\omega(f)(t) \in L^q\) and the \(L^q\) norm of \(t^{-\alpha}\omega(f)(t)\) is equivalent to the \(\text{Lip}(\alpha, q; L^r)\) norm of \(f\).

On the circle group \(T\) we have \(\omega(f)(s + t) \leq \omega(f)(s) + \omega(f)(t)\). However this does not hold on our totally disconnected group \(G\) (see [1]). Hence we introduce
\[
\omega^*(f)(t) = \sup_{s > t} \frac{\omega(f)(s)}{s}
\]

\(\omega^*\) has the property that

\[
\omega^*(f)(s + t) \leq \omega^*(f)(s) + \omega^*(f)(t).
\]

Replacing \(\omega\) by \(\omega^*\) gives the same spaces \(\text{Lip}(\alpha, q; L^r)\) with equivalent norms.

We proceed much as in [3]. In the real interpolation method, let \(A_0 = L^r, A_1 = \text{Lip}(1, \infty; L^r)\). Then we obtain

**Lemma 3.1**: If \(f \in L^r, 1 \leq r \leq \infty\), we have

(i) \(K(t, f) \leq (1 + p)\omega^*(f)(t) + \min(1, t)||f||_r\),

(ii) \(\omega^*(f)(t) \leq 2K(t, f)\),

(iii) \(\min(1, t)||f||_r \leq K(t, f)\).

**Proof**: Write \(f = f_o + f_1\) with \(f_o \in A_0, f_1 \in A_1\). Then

\[
\omega^*(f)(t) \leq \omega^*(f_o) + \omega^*(f_1)
\]

\[
\leq 2||f_o||_r + t \sup_{s > t} \frac{\omega(f)(s)}{s}
\]

\[
\leq 2||f_o||_r + t||f_1||_{\text{Lip}(1, \infty; L^r)}
\]

since

\[
\sup_{s} \frac{\omega(f)(s)}{s} = \sup_k m_k \omega(f, k)
\]

In addition
\[ \min(1, t)||f||_r \leq \min(1, t)(||f_o||_r + ||f_1||_r) \]

\[ \leq ||f_o||_r + t||f_1||_{\text{Lip}(1, \infty; L^r)} \]

Passing to the infimum over all pairs \( f_o, f_1 \) with \( f_o + f_1 = f \), we obtain inequalities (ii) and (iii). To prove (i), if \( t \geq 1 \) write \( f = f_o + 0 \).

Then

\[ K(t, f) \leq ||f||_r + t||0||_{\text{Lip}(1, \infty; L^r)} = \min(1, t)||f||_r. \]

Now for \( t < 1 \), choose \( k \) such that \( m_k^{-1} \leq t \leq m_k^{-1} \). Let

\[ f_1(x) = \sum_{n=1}^{m_k} \hat{f}(n)\chi_n(x) \]

and let \( f_o(x) = f(x) - f_1(x) \). Then \( f_1 = f_k\delta_{m_k} \). As \( ||D_{m_k}||_1 = 1 \) and \( D_{m_k} = 0 \) off \( G_k \), it follows that \( ||f_o||_r \leq \omega_r(f, k) \), but

\[ \omega_r(f, k) = \omega(f)(t) \leq \omega^*(f)(t). \]

Also \( ||f_1||_r \leq ||f||_r \). Furthermore,

\[ \omega(f_1)(s) \]

is bounded above by \( \omega(f)(s) \) and \( ||f||_1 \omega(D_{m_k})(s) \). It follows from the second bound that \( \omega(f_1)(s) \) vanishes on \( 0 < s < m_k^{-1} \) because \( \omega(D_{m_k})(s) \) also vanishes there. Hence

\[ \sup_{s \geq m_k^{-1}} \frac{\omega(f_1)(s)}{s} \leq \sup_{s \geq m_k^{-1}} \frac{\omega(f)(s)}{s} \leq p \sup_{s \geq t} \frac{\omega(f)(s)}{s}. \]

Therefore
\[ K(t, f) \leq ||f_0||_r + t \left\{ ||f_1||_r + \sup_s \frac{\omega(f_1)(s)}{s} \right\} \]

\[ \leq \omega^*(f)(t) = t||f||_r + p \omega^*(f)(t) \]

\[ \leq (1 + p) \omega^*(f)(t) + \min(1, t)||f||_r , \]

which completes the proof of the inequalities.

Now (i) and (ii) imply that the map \( t \rightarrow t^\theta K(t, f) \) belongs to \( L^*_q \) iff \( t \rightarrow t^{-\theta} \omega^*(f)(t) \) belongs to \( L^*_q \). This, together with our earlier comment where we characterized membership in \( \text{Lip}(\alpha, q; L^P) \) in terms of membership of \( t^{-\alpha} \omega^*(f)(t) \) in \( L^*_q \), proves

**Lemma 3.2:** Let \( 1 \leq p \leq \infty, \ 1 \leq q \leq \infty, \ 0 < \theta < 1. \) Then

\[ (L^P, \text{Lip}(1, \infty; L^P))_{\theta,q} = \text{Lip}(\theta, q; L^P) . \]

Applying the theorem of reiteration (theorem 1.22) to lemma 3.2 yields

**Theorem 3.3:** Let \( 1 \leq p \leq \infty, \ 1 \leq r, s \leq \infty, \) and \( 0 < \alpha, \beta < 1. \) If \( 0 < \theta < 1, \ 1 \leq q \leq \infty \) and \( \alpha \neq \beta \) then the space

\[ (\text{Lip}(\alpha, r; L^P), \text{Lip}(\beta, s; L^P))_{\theta,q} \]

is \( \text{Lip}(\gamma, q; L^P) \) where \( \gamma = (1 - \theta)\alpha + \theta\beta . \)
Definition 3.4: $A(\beta, q)$ is defined to be the class of all integrable functions $f$ such that $\hat{f}$ belongs to $\ell(\beta, q)$ (where $\ell(\beta, q)$ is the discrete analogue of $L(\beta, q)$).

The reader should note that if $1 \leq \beta_1 < \beta_2 \leq \infty$, $1 \leq r \leq \infty$, then

$$A(\beta_1, r) \subset A(\beta_2, r);$$

also if $1 \leq \beta \leq \infty$, $1 \leq r \leq s \leq \infty$, then

$$A(\beta, r) \subset A(\beta, s).$$

Thus we realize that $A(1) = A(1, 1) \subset A(\beta, q)$ for any $1 \leq \beta \leq \infty$, $1 \leq q \leq \infty$. In addition, if $\alpha \neq \beta$, we have

$$(A(\alpha), A(\beta))_{\theta, q} = A((1-\theta)\alpha + \theta\beta, q),$$

with equivalent norms.

We next consider

Theorem 3.5: For $1 \leq p \leq 2$, $1 \leq q \leq \infty$ and $0 < \theta < 1$, define $\beta$ by

$$\frac{1}{\beta} = \frac{1-\theta}{p'} + \frac{\theta}{1}.$$ Then

$$\text{Lip}(\frac{\theta}{p'}, q; L^p) \subset A(\beta, q).$$

Proof: By 1.31.1 we have $L^p \subset A(p', p)$ and by setting $\alpha = 1/p$ in 2.8 we have

$$\text{Lip}(\frac{1}{p'}, 1; L^p) \subset A(1).$$

Interpolation yields
(L^p, \text{Lip}(\frac{1}{p}, 1; L^p))_{\theta, q} \subset (A(p', p), A(1))_{\theta, q} ,

i.e. \text{Lip}(\frac{\theta}{p}, q; L^p) \subset A(\beta, q) \quad \text{where} \quad \frac{1}{\beta} = \frac{1-\theta}{p'} + \frac{\theta}{1} .

Set \ q = \beta \ \text{in theorem 3.5 and obtain}

Corollary 3.5.1 : If \ 1 < p < 2, \ 0 < \theta < 1, \ \text{and} \ \beta \ \text{is given by}

\frac{1}{\beta} = \frac{1-\theta}{p'} + \frac{\theta}{1} , \ \text{then} \ \text{Lip}(\frac{\theta}{p'}, \beta; L^p) \subset A(\beta) .

If we set \ \theta = \frac{p}{p'} - \frac{p}{p'} \ \text{and restrict} \ \beta \ \text{to the interval} \ 0 < \beta < p' ,

we realize that this is exactly theorem 2.7' with the exception of the

endpoint \ \beta = p' .

Hence we see that given Hausdorff-Young and the endpoint result

2.8 (or 2.7.1), we get theorem 2.7' by interpolation. As a bonus, we get

additional results concerning inclusion in \ A(\beta, q) .

Theorem 3.6 : For \ 0 < \alpha < 1, \ 1 < p < 2, \ 1 < q < \infty \ \text{and} \ \beta > p/(\alpha p + p - 1)

we have \ \text{Lip}(\alpha, q; L^p) \subset A(\beta) .

Proof : As \ \alpha > \frac{1}{\beta} - \frac{1}{p'} , \ \text{the result follows by 2.7' .}

Theorem 3.6 \ can also be proven using 2.7.2 \ and a simple

interpolation argument.

Theorem 3.7 : If \ 1 < p < 2, \ 1 < q < \infty \ \text{and} \ p^{-1} < \beta < 1, \ \text{then any}

f \in \text{Lip}(\beta, q; L^p) \ \text{also belongs to} \ A(1) .
Proof: As $\beta > \frac{1}{p}$ we have $\text{Lip}(\beta, q; L^p) \subseteq \text{Lip}(\frac{1}{p}, 1; L^p) \subseteq A(1)$ by 2.7.1.

We wish to show that the last two results are, in some sense, best possible. (Onneweer did this in the special cases given by corollaries 2.7.2 and 2.7.3). We begin by defining $R_k(x) = D_{m_k}(x) - D_{m_{k-1}}(x)$ for $k \geq 1$. Now

$$R_k(x) = \begin{cases} m_k - m_{k-1} & \text{if } x \in G_k \\ -m_k & \text{if } x \in G_{k-1} \setminus G_k \\ 0 & \text{if } x \in G \setminus G_{k-1} \end{cases}$$

and for $p > 1$ we have

$$||R_k||_p^p = \int_{G_k} |R_k(x)|^p \, dx + \int_{G_{k-1} \setminus G_k} |R_k(x)|^p \, dx + \int_{G \setminus G_{k-1}} |R_k(x)|^p \, dx$$

$$= m_k^{-1}(m_k - m_{k-1})^p + (m_{k-1} - m_k^{-1})^p m_{k-1}^p$$

$$\leq m_k^{p-1} + m_{k-1}^{p-1}$$

$$\leq C m_k^{p-1}.$$ 

Next for $1 < p \leq 2$, $0 < \alpha \leq 1$, $\delta > 0$ define

$$h_{\alpha, p, \delta}(x) = \sum_{k=1}^{\infty} \frac{m_k^{-(\alpha+1-p)-1}}{k^\delta} R_k(x).$$
Clearly \( h_{\alpha,p,\delta} \in L^p \) as
\[
||h_{\alpha,p,\delta}||_p^p \leq \sum_{k=1}^{\infty} \left( \frac{m_k^{-(\alpha+1-p)}}{k^\delta} \right)^p ||R_k(x)||_p^p < \infty.
\]

Now for every \( n \), if \( y \in C_n \) and \( k \leq n \) then \( R_k(x+y) = R_k(x) \). Hence
\[
||T^h_{y_\alpha,p,\delta} - h_{\alpha,p,\delta}||_p \leq \sum_{k=n+1}^{\infty} \frac{m_k^{-(\alpha+1-p)}}{k^\delta} ||T^h_{y_k} - R_k||_p
\]
\[
\leq \sum_{k=n+1}^{\infty} \frac{m_k^{-(\alpha+1-p)}}{k^\delta} \left( ||T^h_{y_k}||_p + ||R_k||_p \right)
\]
\[
\leq C \sum_{k=n+1}^{\infty} \frac{m_k^{-(k+1-p)}}{k^\delta} m_k^{1-p-1}
\]
\[
= C \sum_{k=n+1}^{\infty} \frac{m_k^{-\alpha}}{k^\delta}
\]
\[
< \sum_{k=n}^{\infty} \frac{k^{-\alpha-\delta}}{n}.
\]

since \( \{m_n\} \) grows geometrically. Hence
\[
\sum_{k=1}^{\infty} (m_k^{\alpha}_{k^p} (f, k))^r \leq \sum_{k=1}^{\infty} (m_k^{\alpha-\alpha_k-\delta})^r
\]
\[
= \sum_{k=1}^{\infty} k^{-\delta r}.
\]
which is finite for $\delta r > 1$. If we put $\delta = 0$, then $w_p(f, n) \leq m_n^{-\alpha}$.

Consequently $\{m_n^{\alpha} w_p(f, n)\} \in \ell^\infty$ and $h_{\alpha, p, \delta} \in \text{Lip}(\alpha, \infty; L^p)$.

If we select $\delta > 0$, then, if $r > 1/\delta$, we have $h_{\alpha, p, \delta} \in \text{Lip}(\alpha, r; L^p)$.

Furthermore, it is clear that for $m_{k-1} \leq n \leq m_k$,

$$\hat{h}_{\alpha, p, \delta}(n) = \frac{m_k^{-(\alpha+1-p^{-1})}}{k^\delta}.$$  

Thus

$$\sum_{n=1}^{\infty} |\hat{h}_{\alpha, p, \delta}(n)|^{p/(ap+p-1)}$$

$$= \sum_{k=1}^{n} (m_k - m_{k-1}) \frac{m_k^{-(\alpha+1-p^{-1}) \cdot p/(ap+p-1)}}{k^\delta \cdot p/(ap+p-1)}$$

$$= \sum_{k=1}^{n} (m_k - m_{k-1}) \frac{m_k^{-1}}{k^\delta \cdot p/(ap+p-1)}$$

$$= \sum_{k=1}^{\infty} (1 - \frac{m_{k-1}}{m_k}) \frac{1}{k^\delta \cdot p/(ap+p-1)}$$

which clearly converges iff $\delta > (ap+p-1)/p$.

In general $h_{\alpha, p, \delta} \in A(\beta)$ for any $\beta > p/(ap + p - 1)$ and $h_{\alpha, p, \delta} \in A(p/(ap + p - 1))$ iff $\delta > (ap+p-1)/p$.

Thus we have
Theorem 3.8: If $1 < p < 2$, $0 < \alpha \leq 1$ and $r > p/(\alpha p + p - 1)$ and $\delta$ is chosen such that $p/(\alpha p+p-1) < \frac{1}{\delta} < r$ the function

$h_{\alpha,p,\delta} \in \text{Lip}(\alpha, r; L^p)$ but is not in $A(p/(\alpha p+p-1))$. In particular, for $r > 1$, the function $h_{\frac{1}{p},p,1} \in \text{Lip}(\frac{1}{p}, r; L^p)$ but its Fourier series

is not absolutely convergent.

Theorem 3.5 can also be seen to be sharp; that is, if $s < q$, then for suitable choice of $\alpha$, $p$, $\delta$, the function $h_{\alpha,p,\delta}$ belongs to $\text{Lip}(\alpha, q; L^p)$ but does not belong to $A(\beta, s)$.

We state the following theorem:

Theorem 3.9: Let $1 < r < 2$, $0 < \theta < 1$, $1 \leq q \leq \infty$. Then

$$(\text{Lip}(\frac{1}{2}, 1; L^2), \text{Lip}(\frac{1}{r}, 1; L^r))_{\theta,q} \subset A(1).$$

Proof: In corollary 2.7.1 we make the following substitutions:

$p = 2$, hence $\text{Lip}(\frac{1}{2}, 1; L^2) \subset A(1)$,

$p = r$, hence $\text{Lip}(\frac{1}{r}, 1; L^r) \subset A(1)$.

Interpolating between these statements gives the desired result.

The obvious question to ask at this point is what exactly are the spaces $(\text{Lip}(\frac{1}{2}, 1; L^2), \text{Lip}(\frac{1}{r}, 1; L^r))_{\theta,q}$. In light of what we already known about the intermediate spaces $(L^p, L^1)_{\theta,q}$ and the spaces
(\text{Lip}(\alpha, r; \mathbb{L}^p), \text{Lip}(\beta, s; \mathbb{L}^p))_{\theta, q}$ it would seem reasonable to expect that

for $0 < \theta < 1$, $1 < q \leq \infty$, $(\text{Lip}(\frac{1}{2}, 1; \mathbb{L}^2), \text{Lip}(\frac{1}{r}, 1; \mathbb{L}^r))_{\theta, q}$ would be

the space $\text{Lip}(\frac{1}{p}, q; \text{L}(p, q))$ where $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r}$. However, this is not

the case. In fact, if we consider the case $q = p$ we see

**Corollary 3.9.1**: If $0 < \theta < 1$, $(\text{Lip}(\frac{1}{2}, 1; \mathbb{L}^2), \text{Lip}(\frac{1}{r}, 1; \mathbb{L}^r))_{\theta, p}$ does

not contain $\text{Lip}(\frac{1}{p}, p; \mathbb{L}^p)$ where $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r}$. In fact, if $1 < q \leq \infty$,

$\text{Lip}(\frac{1}{p}, q; \mathbb{L}^p)$ is not contained in the intermediate space in question.

**Proof**: Consider the function $h_{\alpha, p, \delta}$ already defined. For fixed $q > 1$, setting $\delta = 1$ produces a function in $\text{Lip}(\frac{1}{p}, q; \mathbb{L}^p)$ which is not in $A(1)$

and hence we see $\text{Lip}(\alpha, q; \mathbb{L}^p)$ is not even contained in the intermediate

space we desire.

As was mentioned in chapter 2, theorem 2.10 and corollary 2.10.1

are best possible. In fact

**Theorem 3.10**: Let $1 < p \leq 2$, $0 < \alpha < 1$, $r > p'$. Then there exists an

$f \in \text{Lip}(\alpha, r; \mathbb{L}^p)$ such that

$$
\sum_{n=1}^{\infty} |\hat{f}(n)^{\alpha}|^p' = \infty.
$$

**Proof**: Consider the function $h_{\alpha, p, \delta}$ already defined. If $\delta = 0$, it

belongs to $\text{Lip}(\alpha, \infty; \mathbb{L}^p)$. For any other $r > p'$ select $\delta > 0$ such that

$p' < \frac{1}{\delta} < r$. We then have $h_{\alpha, p, \delta} \in \text{Lip}(\alpha, r; \mathbb{L}^p)$. Furthermore, by (4),
we see

\[
\sum_{n=1}^{\infty} |\hat{h}_{\alpha,p,\delta(n)n^\alpha}|p' = \sum_{k=1}^{\infty} \sum_{n=m_k-1}^{m_k-1} |\hat{h}_{\alpha,p,\delta(n)n^\alpha}|p'
\]

\[
\geq \sum_{k=1}^{\infty} (m_k - m_{k-1}) \left[ \frac{m_k}{k} \right]^{(\alpha+1-p-1)/\alpha} p'
\]

\[
\geq C \sum_{k=1}^{\infty} \frac{m_k^{1-ap'-p'+p'/p+ap}}{k^{\delta p'}}
\]

\[
= C \sum_{k=1}^{\infty} \frac{1}{k^{\delta p'}} = \infty \quad \text{as} \quad \delta p' < 1.
\]

**Definition 3.11**: Let \(1 \leq p \leq \infty, \ 1 \leq q \leq \infty\). Then the mixed norm space \(\ell_{\alpha,q}^p\) is defined to be the space of all sequences \(\{c_n\}\) such that

\[
\left\{ \sum_{k=0}^{\infty} \left( \sum_{2^k \leq n < 2^{k+1}} |n^\alpha c_n|^p \right)^{q/p} \right\}^{1/q} < \infty,
\]

if \(1 \leq p, q < \infty\); with the appropriate modifications if \(p\) or \(q\) is infinite.

One can show that for integrable functions \(f\) on \(G\), \(\hat{f} \in \ell_{\alpha,q}^p\) iff

\[
\left\{ \sum_{k=0}^{\infty} \left( \sum_{n=m_k}^{m_{k+1}-1} |n^\alpha \hat{f}(n)|^p \right)^{q/p} \right\}^{1/q} < \infty
\]
and this produces an equivalent norm. Hence

**Theorem 3.12**: Let \( 1 \leq p \leq \infty, \ 0 < \alpha_0, \alpha_1 < \infty \). Then if \( 0 < \theta < 1, \ 1 \leq q \leq \infty \), the space

\[
\left( \ell^p_{\alpha_0}, \ell^p_{\alpha_1} \right)_{\theta, q} = \ell^p_{\beta, q},
\]

where \( \beta = (1 - \theta)\alpha_0 + \theta \alpha_1 \).

**Corollary 3.12**: Let \( 1 \leq p \leq \infty, \ 0 < \theta < 1, \ 1 \leq q \leq \infty \). Then

\[
\left( \ell^p_n, \ell^p_{n^2} \right)_{\theta, q} = \ell^p_{\theta, q}.
\]

**Proof**: \( \ell^p_n = \left( \ell^p_n, \ell^p_{n^2} \right)_{1/2, \infty} \). Now reiterate.

**Theorem 3.13**: For \( 1 \leq p \leq 2 \) and \( 0 < \theta < 1 \), if \( f \in \text{Lip}(\theta, q; L^p) \) then

\( \hat{f} \in \ell^p_\theta \).

**Proof**: By Hausdorff-Young, \( (L^p)^* \supset \ell^p_\theta \) and by inequality (3) we have \( \text{Lip}(1, \infty; L^p)^* \supset \ell^p_1 \). Interpolation yields the desired result.

As an immediate corollary, we obtain theorem 2.10.

We have already seen theorem 2.10 is sharp. An analogous argument to that used in theorem 3.10 will show that 3.13 is sharp; that is, if \( r > q \) and we select \( \delta \) in the interval \( \left( \frac{1}{r}, \frac{1}{q} \right) \), the function
h_{\theta,p,\delta} \in \text{Lip}(\theta, r; L^P) \text{ but } h_{\theta,p,\delta} \text{ does not belong to } \mathcal{L}_{\theta}^{p'}_{n,q}.

Finally

**Theorem 3.14**: If $1 < p \leq 2$, $0 < \theta < 1$, $1 \leq q < \infty$, then if $\{C_n\}$ is a sequence in $\mathcal{L}_{\theta}^p$, there exists an $f$ belonging to $\text{Lip}(\theta, q; L^{p'})$ such that $f(n) = C_n$ for each $n \geq 1$.

**Proof**: Interpolate between the statements

\[ \mathcal{L}_{n_0}^p \subset (L^{p'})^\wedge \]

and

\[ \mathcal{L}_{1}^p \subset \text{Lip}(1, \infty; L^{p'})^\wedge. \]
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