ABSTRACT MODEL THEORY

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Abstract

We define a notion of logic that provides a general framework for the study of extensions of first-order predicate calculus. The concept of partial isomorphism and its relation to infinitary logics are examined. Results on the definability of ordinals establish the setting for our proof of Lindstrom's Theorem: this theorem gives conditions that characterize first-order logic. We then consider the analogues to the general case of the compactness and Lowenheim properties. For a wide class of logics it is shown that interesting connections exist between the analogues of these properties.

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INTRODUCTION

In the last two decades logicians have done considerable research on logics that extend the first-order predicate calculus. Bell and Slomson, for instance, devote the last two chapters of their textbook "Models and Ultraproducts" to the study of such extensions. The motivation for looking at these extensions stems from a desire to avoid certain shortcomings of first-order logic; in particular, as is well known, first-order logic is deficient in expressing many useful mathematical notions.

Any good introduction to the first-order predicate logic will list the following properties that this logic possesses:

(i) Compactness property,

(ii) Lowenheim-Skolem property.

A natural question arises: under what conditions does an extension also satisfy these properties? The answer is perhaps surprising: any logic possessing the above properties must be equivalent to first-order logic. This is a result of the Scandinavian logician P. Lindstrom [12]. Besides being interesting in itself, Lindstrom's result has stimulated the growth of a field of research, abstract model theory, which attempts to construct a model theoryffor general extensions of first-order logic.

Among the many extensions of first-order calculus there are two logics which have been the subject of especially close study. One, pioneered by Tarski, is obtained by allowing the formation of infinitely long sentences. The other, due to Mostowski, involves the use of generalized quantifiers.

Because these logics are proper extensions of first-order logic Lindstrom's result ensures the failure of either the compactness or the Lowenheim property. It turns out, however, that analogues of these properties exist for the above extensions. In addition, reduction techniques of Fuhrken [9] and Lopez-Escobar [13] can be used to show certain interesting connections between the analogues of these properties.

The purpose of this thesis is threefold. First, a general notion of logic is established that is adequate for formulating results in a wide framework, (Chapter One). Second, the machinery necessary to prove Lindstrom's result is developed, (Chapters Two and Three). Third, the analogues of the compactness and Lowenheim properties are examined in the general setting, (Chapter Four).

Our definition of a general logic and the proof of Lindstrom's result is based on the treatment of Jon Barwise in [1] . The material on general compactness and Lowenheim properties has its source in Flum 8 . In our presentation we have modified, reorganized and supplemented the material with examples and connecting links. Thus we condense Barwise's notion of a logic while emphasizing certain assumptions which prove to be very important. Complete proofs and results from the literature have been supplied in places. The constructions in the proofs of Chapter Three are given explicitly (for example, Theorems 3.1.9 and 3.2.3). The complete proof given of Theorem 4.3.1 provides the justification for the remark that follows that theorem. What were originally exercises and statements are also given proof (Lemmas 4.3.4,4.3.5). Lastly, the existence of the well-ordering number of $L(Q_\alpha)$ is nowhere explicitly expressed in the literature but falls out naturally from Fuhrken's reduction technique.

It is hoped that this thesis will be accessible to anyone familiar with first-order logic. The rudiments of cardinal and ordinal arithmetic are assumed; the material that is needed here may be found in Chapter 0 of the above mentioned book of Bell and Slomson. Finally, the terms 'set' and 'class' are understood in the sense of the Godel-Bernays system of set theory, a set being a class which is a member of another class.(see Mendelson [14] for more details).

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CHAPTER ONE

THE BARWISE NOTION OF A LOGIC

In this section we present the bare essentials of Barwise's treatment of logics. We have, for example, dispensed with his categorical approach which, while interesting, doesn't seem to be essential.

1.1 Some Preliminaries and Basic Definitions

Definition 1.1.1: A language L is a collection of relation, function, and constant symbols. L will always contain the symbol $\forall$, a unary relation symbol which denotes the domain of the universal quantifier. The use of $\forall$, although it is a somewhat artificial constraint, proves to be technically convenient.

Definition 1.1.2: A partial structure $\mathcal{M}$ for L is a function $\mathcal{M}$ with domain $\subseteq$ L such that:

(i) $M = \forall^{\mathcal{M}}$ is a set, called the universe of $\mathcal{M}$.

(ii) If $R \in L$ is a n-ary relation symbol then $R^{\mathcal{M}} \subseteq M^n$.

(iii) If $f \in L$ is a n-ary function symbol then $f^{\mathcal{M}}$ is a partial function from M^n to M .

(iv) If $c \in L$ is a constant symbol and $c^{\mathcal{M}}$ is defined then $c^{\mathcal{M}} \in M$.

If in (iii) and (iv) each $f^{\mathcal{M}}$ and each $c^{\mathcal{M}}$ is defined totally then $\mathcal{M}$ is an L-structure. An L-structure will sometimes be displayed as follows:

$$\mathcal{M} = \langle M, \underline{R}_1, \underline{R}_2, \ldots, \underline{f}_1, \underline{f}_2, \ldots, a_1, a_2, \ldots \rangle ,$$

where M is the universe of $\mathcal{M}$, $\underline{R}_1$ is the interpretation of R_1, $\underline{f}_1$ is the interpretation of f_1, etc. .

Definition 1.1.3 : Two partial structures M and N for a language L are isomorphic if there is a bijective mapping from the universe of M to the universe of N which preserves in a natural sense the relations and functions. We write $M \simeq N$ to indicate that M is isomorphic to N.

An expansion of a language L is any language K with $L \subseteq K$. An L-structure M is a reduct of a K-structure N, and N is an expansion of M, if M is $N\restriction_L$, the restriction of N to the language L.

Every language L hassa set of terms associated with it, defined inductively as follows: if c is a constant term of L then $\bar{c}$ is a term; if $t_1, \ldots, t_n$ are terms and f is a n-ary function symbol then $f(t_1,\ldots,t_n)$ is a term. Given a structure M for L every term t of L denotes an element t^M of M (the universe of M). If c is a constant symbol t^M is just c^M and if t is $f(t_1,\ldots,t_n)$ then $t^M = f^M(t_1^M,\ldots, t_n^M)$ where $t_1^M, \ldots, t_n^M$ are already assumed (inductively) to have been defined.

We now introduce the important concept of an interpretation(morphism).

Definition 1.1.4 : Let $r \geq 0$ be an integer. Let L, K be languages. An r-term interpretation or morphism α of L into K consists of r terms $t_1, \ldots, t_r$ of K and a mapping α from L into K satisfying :

(i) If $R \in L$ is an n-ary relation symbol then R^{α} is an $(n + r)$-ary relation symbol.

(ii) If $f \in L$ is an n-ary function symbol then f^{α} is an $(n + r)$-ary function symbol.

(iii) If $\alpha(x) = \forall$ then $x = \forall$.

We thus have that $\forall^{\alpha}$ is a $(1 + r)$-ary relation symbol, usually denoted by U or V. A 0-term interpretation α such that $\alpha(\forall) = \forall$ is called a <u>simple morphism</u>.

Now morphisms on languages induce operations on the corresponding structures. Let $\bar{\alpha} : L \to K$ be an r-morphism and let $\mathcal{M} = \langle M, \ldots, a_1, \ldots, a_r \rangle$ be a typical K-structure, where a_i is the interpretation in M of t_i. We wish to pull $\mathcal{M}$ back to a structure $\mathcal{M}^{-\alpha}$ for L. The universe of $\mathcal{M}^{-\alpha}$ is

$$N = \{ a \in M : (a, a_1, \ldots, a_r) \in U^{\mathcal{M}}, U = \alpha(\forall) \}$$

The interpretation of an n-ary relation symbol $R \in L$ is given by

$$(b_1, \ldots, b_n) \in R^{\mathcal{M}^{-\alpha}} \text{ iff } b_1, \ldots, b_n \in N \text{ and } (b_1, \ldots, b_n, a_1, \ldots, a_r) \in (R^{\alpha})^{\mathcal{M}} ;$$

The interpretation of an n-ary function symbol $f \in L$ is the restriction of $\underline{f}_{a_1,\ldots,a_r}$ to N^n where

$$\underline{f}_{a_1,\ldots,a_r}(b_1, \ldots, b_n) = (f^{\alpha})^{\mathcal{M}}(b_1, \ldots, b_n, a_1, \ldots, a_r)$$

Since N may not be closed under the functions $\underline{f}_{a_1,\ldots,a_r}$, $\mathcal{M}^{-\alpha}$ is only a partial L-structure. In many cases, however, $\mathcal{M}^{-\alpha}$ will be an L-structure and $\mathcal{M}$ is then said to be <u>α-invertible</u>.

There are two important examples of the above which are used repeatedly in future proofs.

<u>Example 1.1.5</u> : Let L be a language and V a unary relation symbol not in L and let $K = L(V)$. ($L(V)$ is the language obtained from L by adjoining the symbol V.) Let $\alpha : L \to K$ be the interpretation which sends $\forall$ to V but leaves the other symbols of L unchanged. A typical K-structure has the form $(\mathcal{M}, \underline{V})$ where $\mathcal{M}$ is an L-structure. The structure $(\mathcal{M}, \underline{V})$ is α-invertible iff the set V is closed under the various functions of $\mathcal{M}$, in which case

$\mathcal{M}^{-\alpha}$ is just the usual relativization of $\mathcal{M}$ to the set $\underline{V}$, $\mathcal{M}^{-\alpha} = \mathcal{M}^{(V)}$.

<u>Example 1.1.6</u> : Given a language L and a collection of L-structures $\{ \mathcal{M}_i : i \in I \}$ it is possible to code this set of structures into a single model for an expanded language K of L. For example, let L = { R } where R is a binary relation symbol and let K = { R', V, c } where R' is 3-ary, VVis binary and c is a constant symbol. Let $\alpha : L \to K$ be the 1-morphism which sends $\forall$ to V and R to R'. The term of α is just c. From the set of L-structures $\{ \mathcal{M}_i : i \in I \}$ we form the K-structure $\mathcal{M} = < M, \underline{V}, \underline{R}' >$ the indexed union of the $\mathcal{M}_i$, by defining

$$\underset{\sim}{M} = I \cup \bigcup_{i \in I} M_i \quad \text{where } M_i = \forall^{\mathcal{M}_i} ,$$

$$< a, i > \in \underline{V} \text{ if } i \in I \text{ and } a \in M_i ,$$

$$< a, b, i > \in \underline{VR}' \text{ if } i \in I \text{ and } < a, b > \in R^{\mathcal{M}_i} .$$

The interpretation of the constant symbol was deliberately left unspecified; if we denote by $(\mathcal{M}, j)$ the indexed union in which c receives the interpretation j then $\mathcal{M}_j = (\mathcal{M}, j)^{-\alpha}$, i.e. it is possible to recover each of the structures $\mathcal{M}_j$ from the indexed union $\mathcal{M}$.

One more preliminary definition is needed.

<u>Definition 1.1.7</u> : By a collection $\mathcal{C}$ of languages we will always mean a collection which satisfies the following two conditions:

(i) If L and K are in $\mathcal{C}$ then L $\cap$KK is in $\mathcal{C}$,

(ii) If L is in $\mathcal{C}$ and K is obtained from L by adding (finitely) many relation, function or constant symbols to L then K is in $\mathcal{C}$.

1.2. The Main Notion

Definition 1.2.1 : A logic $\mathcal{L}^*$ on a collection $\mathcal{C}$ of languages consists of two components, a syntax and a semantics. The syntax of $\mathcal{L}^*$ assigns to each L in $\mathcal{C}$ a class L^* of sentences. The elements of L^* are called $\mathcal{L}^*$-sentences. The syntax satisfies the following two properties:

(i) If $L \subseteq K$ then $L^* \subseteq K^*$.

(ii) (Occurrence Property): For every $\mathcal{L}^*$-sentence ϕ there is a smallest (under inclusion) language L_ϕ in $\mathcal{C}$ such that $\phi \in L^*_\phi$.

For each morphism α from L to K, the syntax also induces a map α^* from L^* to K^*. If ϕ is an L^*-sentence, $\alpha^*(\phi)$ will be denoted by ϕ^α. The semantics of $\mathcal{L}^*$ is a relation $\models$ such that if $M \models \phi$ then M is an L-structure for some L in $\mathcal{C}$ and $\phi \in L^*$. The semantics satisfies:

(iii) (Isomorphism Property): If $M \models \phi$ and $M \cong N$ then $N \models \phi$.

$M \models \phi$ is read as "M is a model of ϕ". The syntax and semantics of $\mathcal{L}^*$ fit together according to the final property:

(iv) (Translation Property): For every $\mathcal{L}^*$-sentence ϕ, every morphism $\alpha : L_\phi \to K$ and every K-structure M

$$M \models \phi^\alpha \quad \text{iff} \quad M \text{ is } \alpha\text{-invertible and } M^{-\alpha} \models \phi.$$

The next two examples provide some motivation for the requirement of the translation property.

Example 1.2.2: Let ϕ be an $\mathcal{L}^*$-sentence and let K be any language such that $L = L_\phi \subseteq K$. If $\alpha : L_\phi \to K$ is the natural embedding map and M is a K-structure then $M^{-\alpha}$ is just the L-reduct of M, i.e. $M^{-\alpha} = M\restriction_L$. The translation property asserts that $M \models \phi^\alpha$ iff $M\restriction_L \quad \phi$. So the translation property implies that $\mathcal{L}^*$ has the above 'reduct property'.

Example 1.2.3: Let ϕ be an $\mathcal{L}^*$-sentence and let $L = L_\phi$ be the set of symbols occuring in ϕ. Let V be a unary relation symbol not in ϕ (i.e. not in L^*_ϕ), and let $\alpha : L \to L(V)$ be the relativization defined in Example 1.1.5. We write ϕ^V for ϕ^α. The translation property asserts that the L(V)-structure $(M, \underline{V}) \models \phi^V$ iff $\underline{V}$ is closed under the functions of M and $M^{(V)} \models \phi$. ϕ^V is called the relativization of ϕ to V, and the translation property implies that our logic has this relativization property.

In the following we stipulate a few additional assumptions on a logic. These assumptions are not a necessary part of a logic but are designed to ensure that any logic under consideration extends the classical first-order predicate calculus.

Observe first that any language L gives rise to a set of atomic sentences. If t_1 and t_2 are terms of L then $t_1 = t_2$ is an atomic sentence of L. ('=' denotes equality and is treated here as a logical symbol; it doesn't occur in the language L.) If R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms then $R(t_1,\ldots,t_n)$ is an atomic sentence of L.

Let L^* be a logic on the collection C of languages. We formulate three assumptions which we require L^* to satisfy:

<u>Assumption 1</u> : L^* contains all atomic sentences. This means that:

(i) if $t_1 = t_2$ is an atomic sentence of L then it is an element of L^*, and for each L-structure M

$$M \models t_1 = t_2 \text{ iff } t_1^M = t_2^M$$

(ii) if $R(t_1,\ldots,t_n)$ is an atomic sentence of L then it is an element of L^*, and for each L-structure M

$$M \models R(t_1,\ldots,t_n) \text{ iff } (t_1^M,\ldots,t_n^M) \in R^M .$$

<u>Assumption 2</u> : L^* contains conjunction and negation. Suppose L is a language of C and ϕ and ψ are any sentences of L^*.

Conjunction: There is a sentence χ of L^* such that if M is any L-structure then $M \models \chi$ iff $M \models \phi$ and $M \models \psi$. The sentence χ will be denoted by $\phi \wedge \psi$

Negation: There is a sentence χ of L^* such that if M is any L-structure then $M \models \chi$ iff not $M \models \phi$. χ will be denoted by $\sim\phi$.

Assumption 3 : L^* is closed under existential quantification. Again let L be any language in C and let ϕ be a sentence of L^* which may or may not contain the constant symbol c. Then there is a sentence χ of L^* such that if M is any L-structure:

$M \models \chi$ iff there is some constant a ε M such that (M, a) $\models \phi$

((M, a) is the L-structure differing at most from M in that c receives the interpretation a in the universe of M.) The sentence χ will be denoted by $\exists x\phi\phi(x)$.

For our purposes then a logic L^* as defined earlier satisfies the four properties of the definition and the three assumptions given above. Additional restrictions might be introduced. For instance, Barwise proposes a tentativee definition for finitary syntactic operations. This definition, however, is extremely general and may be too weak to be useful; we will not look at it further. At this point it is interesting to look at some examples of logics that are covered in the Barwise set-up.

Example 1.2.4 : $L_{\omega\omega}$ - the classical first-order predicate calculus.

Syntax: Given any language L the set $L_{\omega\omega} = L^*$ is formed inductively as follows:

(i) $L_{\omega\omega}$ contains all atomic sentences (see page)

(ii) If ϕ, ψ ε $L_{\omega\omega}$ then $\phi \wedge \psi$ ε $L_{\omega\omega}$

(iii) If $\phi \in L_{\omega\omega}$ then $\sim\phi \in L_{\omega\omega}$

(iv) If $\phi \in L_{\omega\omega}$ then $\exists x\phi(x) \in L_{\omega\omega}$.

$L_{\omega\omega}$ is sometimes referred to as the set of well-formed sentences.

Semantics: Given an L-structure M we inductively define the relation $M \models \phi$ (ϕ a sentence of $L_{\omega\omega}$) as follows:

(i) If ϕ is an atomic sentence the relation $M \models \phi$ is exactly as given in the statement of assumption 1 above

(ii) $M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$

(iii) $M \models \sim\phi$ iff not $M \models \phi$

(iv) If ϕ is a sentence which may or may not contain the constant symbol c then $M \models \exists x\phi(x)$ iff there is some $a \in M$ such that $(M, a) \models \phi(c)$ where c receives the interpretation a in (M, a) .

(We wrote $\phi(c)$ for ϕ to indicate the special role of the constant symbol c.)

Note: In our formulation of a logic we have avoided the usual concept of a free variable. This is because a free individual variable of a language can be identified with a constant symbol of a larger language. More precisely, given a language L form the language $L(\bar{c})$ by adjoining the constant symbol c to L. If M is an L-structure we are free to interpret c arbitrarily in the universe of M to get an L(c)-structure (M, a). In this manner, c can be thought of as a free variable of L. And similar remarks apply in the case of (higher-order) logics which allow relation or predicate variables.

Example 1.2.5 : L^{II} - second-order logic. Informally, this is an extension of classical first-order logic in which quantification over predicate variables is allowed.

Syntax: We add a new logical symbol ε (the membership relation) to $L_{\omega\omega}$.

New atomic sentences $t \varepsilon U$ are allowed, where t is a term and U is a unary relation symbol. If L is a language L^* is obtained from L by adding the following formation rule to the syntax of $L_{\omega\omega}$:

(v) If $\phi \varepsilon L^*$ then $\exists X\phi \varepsilon L^*$.

Semantics : ε is interpreted in any L-structure M as set-membership.

Add the following semantic rule to $L_{\omega\omega}$:

(v) If ϕ is any sentence of L^* which may or may not contain the unary relation symbol U then

$$M \models \exists X\phi \text{ iff there is some } \underline{U} \subseteq M \text{ such that } (M, \underline{U}) \models \phi(U).$$

Here (M, $\underline{U}$) is the structure differing at most from M in that U receives the interpretation $\underline{U}$; we write $\phi(U)$ for ϕ to indicate the special role of the symbol U.

Example 1.2.6 : $L(Q_\alpha)$ - logics with the quantifier Q_α. These logics are obtained from $L_{\omega\omega}$ by adding the quantifier Q_α, where αiis an ordinal. They are but one example of the many different logics that result from employing generalized quantifiers.

Syntax: Add the following rule to the syntax of $L_{\omega\omega}$:

(v) If $\phi \varepsilon L^*$ then $Q_\alpha x\phi(x) \varepsilon L^*$.

Semantics: Add the following rule to the semantics of $L_{\omega\omega}$:

(v) If $\phi(c)$ is a sentence of L^* then

$$M \models Q_\alpha x\phi(x) \text{ iff cardinality}(\{a \varepsilon M : (M, a) \models \phi(c)\}) \geq \omega_\alpha .$$

(ω_α is the α cardinal in the sequence of infinite cardinals)

Intuitively, $Q_\alpha x\phi(x)$ holds in M if there are ω_α objects a such that (M, a) $\models \phi(c)$. One important specific example is $L(Q_0)$: in this logic, $M \models Q_0 x\phi(x)$ iff there are infinitely many $a \varepsilon M$ such that (M, a) $\models \phi(c)$.

Infinitary logics are important extensions of the predicate calculus and have been the object of close study in recent years. The next logics we will consider are examples of these infinitary logics; they will be examined in greater detail in Chapter II.

<u>Example 1.2.7 :</u> $L_{\infty\omega}$ - logic with infinite conjunctions and disjunctions.

Syntax: Add the following rule to the syntax of $L_{\omega\omega}$:

(v) If Φ is a set of sentences of L^* then $\bigwedge \Phi$ and $\bigvee \Phi$ are sentences of L^*.

Semantics: Add the following rule to the semantics of $L_{\omega\omega}$:

(v) $M \models \bigwedge\Phi$ iff $M \models \phi$ for <u>all</u> $\phi \in \Phi$ and

$M \models \bigvee\Phi$ iff $M \models \phi$ for <u>some</u> $\phi \in \Phi$.

<u>Example 1.2.8</u>: $L_{\alpha\omega}$, where α is an infinite cardinal. This logic has the same syntax and semantics as $L_{\infty\omega}$ except for the following restriction: the set Φ of the previous example must have cardinality less than α. Thus for $\alpha = \omega_1$ we get the logic $L_{\omega_1\omega}$ in which only the countable conjunction and disjunction of sets of sentences is allowed.

CHAPTER TWO

PARTIAL ISOMORPHISMS AND SCOTT SENTENCES

In this chapter we give some definitions and state some results from the infinitary logics described at the end of the last chapter. A more detailed account including proofs of the results may be found in [2].

Let M be an L-structure. A partial substructure M_0 of M is a subset M_0 of the universe of M together with the restrictions of the relations and functions to M_0 (so some of the functions may be partial). Given L-structures M and N, a partial morphism from M to N is just an isomorphism $F : M_0 \simeq N_0$ for partial substructures of M and N respectively.

Definition 2.1 : Let I be a set of partial morphisms from M to N. We say I has the back and forth property if for every $F \in I$ and $a \in M$ (respectively $b \in N$) there is a $G \in I$ with $F \subseteq G$ and $a \in \text{domain}(G)$ (respectively $b \in \text{range}(G)$). If there exists a set I with the back and forth property then M and N are said to be partially isomorphic, written $I : M \simeq_p N$ or simply $M \simeq_p N$.

Example 2.2 : Let $M = \langle M, <^M \rangle$ and $N = \langle N, <^N \rangle$ be dense linear orderings without endpoints. For example M and N could be the reals and the rationals respectively with the natural ordering. Let the set I consist of all maps f such that f is an isomorphism from a finite subordering of M onto a finite subordering of N. Then it is straightforward to check that I has the back and forth property and so $I : M \simeq_p N$.

The notion of partial isomorphism can be viewed as a 'weak' form of isomorphism. The next result shows that in certain cases the two notions are equivalent.

<u>Theorem 2.3</u> : If M and N are <u>countable</u> L-structures then $M \simeq N$ iff there exists a set I such that $I : M \simeq_p N$.

<u>Proof:</u> If $f : M \simeq N$ let $I = \{ f \}$ and clearly $I : M \simeq_p N$. To prove the converse we give a special case which is actually a classical result. The general case is then proved in a similar manner. The special case is where M, N and I are as given in Example 2.2 above with the added stipulation that M and N are countable. Since M and N are both countable let $M = \{ a_1, a_2, a_3, \ldots \}$ and $N = \{ b_1, b_2, b_3, \ldots \}$. The set I is used to construct an isomorphism f from $M = \langle M, <^M \rangle$ onto $N = \langle N, <^N \rangle$. In fact a sequence $f_1 \subseteq f_2 \subseteq f_3 \subseteq \ldots$ is constructed such that $f = \bigcup_n f_n$ is the desired isomorphism. Let $f_1 = \{ \langle a_1, b_1 \rangle \}$ be the function in I which maps a_1 to b_1. We now proceed inductively :

f_{2n} = some function $g \in I$ with $f_{2n-1} \subseteq g$ and $a_n \in \text{domain}(g)$

f_{2n+1} = some function $g \in I$ with $f_{2n} \subseteq g$ and $b_n \in \text{range}(g)$.

Then $f = \bigcup_n f_n$ has domain all of M, range all of N and preserves the relations $<^M$, $<^N$. Hence f is an isomorphism.

Recall the definition of $L_{\infty\omega}$ given in Example 1.2.7. Two L-structures M and N are said to be $L_{\infty\omega}$-elementarily equivalent , written $M \equiv_{L_{\infty\omega}} N$, if the same set of $L_{\infty\omega}$-sentences hold in both structures. The following theorem is fundamental in that it provides a connection between the algebraic concept of partial isomorphism and the logical concept of $L_{\infty\omega}$-elementary equivalence.

Theorem 2.4 : Given L-structures M and N the following are equivalent:

(i) $M \equiv_{L_{\infty\omega}} N$

(ii) There is an $I : M \simeq_p N$.

One effect of this theorem is to make clear the transitivity of the relation $\simeq_p$ because of the fairly obvious transitivity of $L_{\infty\omega}$-elementary equivalence.

Later results require a refinement of the above theorem. To this end a precise measure of the complexity of an $L_{\infty\omega}$-sentence is needed. The next definition gives one such measure.

Definition 2.5 : Let ϕ be an $L_{\infty\omega}$-sentence. The quantifier rank of ϕ, written $\mathrm{qr}(\phi)$, is defined inductively as follows:

(i) If ϕ is an atomic sentence in which no function symbols occur then $\mathrm{qr}(\phi) = 0$.

(ii) Suppose ϕ is an atomic sentence of the form $t_1 = t_2$ in which function symbols occur. Let n be the number of occurrences of function symbols in t_1 or t_2. Then $\mathrm{qr}(\phi) = n-1$.

(iii) Suppose ϕ is an atomic sentence of the form $R(t_1,\ldots,t_k)$ in which function symbols occur. Let n be the number of occurrences of function symbols in any of the t_i. Then $\mathrm{qr}(\phi) = n$.

(iv) $\mathrm{qr}(\sim\phi) = \mathrm{qr}(\phi)$, $\mathrm{qr}(\forall x\phi(x)) = \mathrm{qr}(\exists x\phi(x)) = \mathrm{qr}(\phi(c))$,
$\mathrm{qr}(\bigwedge\Phi) = \mathrm{qr}(\bigvee\Phi) = \sup\{\mathrm{qr}(\phi) : \phi \in \Phi\}$.

If α is an ordinal we write $M \equiv_{L^{\alpha}_{\infty\omega}} N$ to indicate that the same set of sentences of quantifier rank $\leq \alpha$ hold in both M and N.

Theorem 2.6 : Given L-structures M and N and an ordinal α the following are equivalent:

(i) $M \equiv_{L_{\infty\omega}} N$.

(ii) There is a sequence $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_\beta \supseteq \ldots \supseteq I_\alpha$ where, for each $\beta \leq \alpha$, I_β is a non-empty set of partial morphisms between M and N such that if $\beta+1 \leq \alpha$ and $F \in I_{\beta+1}$ then for each $a \in M$ (resp. $b \in N$) there is a $G \in I_\beta$ with F G and $a \in \text{domain}(G)$ (resp. $b \in \text{range}(G)$).

The sequence $\{ I_\beta \}_{\beta\leq\alpha}$ gives an approximation to a partial isomorphism in the case where $M \equiv_{L^\alpha_{\infty\omega}} N$ although not necessarily $M \equiv_{L_{\infty\omega}} N$.

Let L be a language and ϕ a sentence of L^*. In the subsequent definition the following notation is used: $\exists\ \phi(^c_x)$ denotes the sentence of L^* such that for any L-structure M

$M \models \exists x\phi(^c_x)$ iff $(M, a) \models \phi(c)$ where a is the interpretation of c in M .

Also, if the logic is $L_{\infty\omega}$ then $L_{\infty\omega}$ denotes the set L^* of sentences of L.

Definition 2.7 : Let L be a language. For each ordinal α, each L-structure M and each sequence $a = a_1,\ldots,a_n \in M$ we define a sentence $\sigma^\alpha_{(M,a)} \in L(c_1,\ldots,c_n)_{\infty\omega}$. This is called a Scott Sentence. The definition proceeds by induction on α:

$$\sigma^0_{(M,a)} = \bigwedge\{ \psi : \psi \text{ is an atomic or negated atomic sentence of } L(c_1,\ldots,c_n),\ qr(\psi) = 0, \text{ and } (M,a_1,\ldots,a_n) \models \psi \} .$$

For any α, $\sigma^{\alpha+1}_{(M,a)}$ is the conjunction of the following sentences:

$$\sigma^{\alpha}_{(M,a)}$$

$$\bigwedge_{b\in M} \exists x_{n+1}\, \sigma^{\alpha}_{(M,a_1,\ldots,a_n,b)}\left(\frac{c_{n+1}}{x_{n+1}}\right)$$

$$\forall x_{n+1} \bigvee_{b\in M} \sigma^{\alpha}_{(M,a_1,\ldots,a_n,b)}\left(\frac{c_{n+1}}{x_{n+1}}\right)$$

For limit ordinals λ,
For limit ordinals λ, $\sigma^{\lambda}_{(M,a)}$ is the conjunction $_{\alpha<\lambda}\ \sigma^{\alpha}_{(M,a)}$. We write σ^{α}_{M} for $\sigma^{\alpha}_{(M,a)}$ when a is the empty sequence.

Informally, the Scott sentence $\sigma^{\alpha}_{(M,a)}$ gathers up all the information about (M,a) that is contained in those sentences of quantifier rank $\leq \alpha$ which hold in (M,a). The next theorem makes this remark more precise.

<u>Theorem 2.8</u> : Let M and N be L-structures and let $a = a_1,\ldots,a_n$, $b = b_1,\ldots,b_n$ be sequences in M and N respectively. Then

$$(N,b) \models \sigma^{\alpha}_{(M,a)} \quad \text{iff} \quad (M,a) \equiv_{L^{\alpha}_{\infty\omega}} (N,b) \quad .$$

A special case of this theorem is when $n = 0$ and a,b are the empty sequences. In this case $N \models \sigma^{\alpha}_{M}$ iff $M \equiv_{L^{\alpha}_{\infty\omega}} N$. Hence Theorem 2.8 gives a nice criterion in terms of Scott sentences for $L^{\alpha}_{\infty\omega}$-elementary equivalence.

<u>Definition 2.9</u> : The sequence of beth cardinals is defined as follows:

$$\beth_0 = 0,\ \beth_{\alpha+1} = 2^{\beth_\alpha} \quad ;$$

$$\beth_\beta = \sup_{\alpha<\beta} \beth_\alpha \quad \text{if } \beta \text{ is a limit ordinal.}$$

We will be particularly interested in those cardinals α for which $\beth_\alpha = \alpha$. One important example of this type of cardinal is ω. We call cardinals such that $\beth_\alpha = \alpha$ fixed point beth cardinals.

The last theorem of this chapter provides additional information about Scott sentences in the case where cardinality(L) $\leq \kappa$ for some fixed point beth cardinal κ. The theorem is also crucial in the proof of the main result of Chapter Three.

Theorem 2.10 : Suppose κ is a cardinal such that $\kappa = \beth_\kappa$ and L is a language with cardinality(L) $\leq \kappa$. Then for each $\alpha < \kappa$ and each positive integer n:

(i) If M is any L-structure the sentence $\sigma^{\alpha}_{(M,a)}$ is in $L(c_1,\ldots,c_n)_{\kappa\omega}$.

(ii) There are less than κ sentences of the form $\sigma^{\alpha}_{(M,a)}$ as M ranges over all L-structures.

CHAPTER THREE

DEFINABILITY OF STRUCTURES AND LINDSTROM'S THEOREM

3.1 Definability of Structures

In this section we establish some results on the definability of structures and, in particular, the definability of ordinals in a logic. These results are important because the definability of ordinals gives some measure of the 'strength' of a logic. This becomes more clear in the material of 3.2 where the results are needed in some of the key proofs.

Throughout this section L^* is a logic on a collection C of languages.

Definition 3.1.1 : Structures M and N are similar if domain(M) = domain(N) = L. If K is a class of similar structures for a language L then K is L^*-definable if there is a sentence $\phi \in L^*$ such that $M \in K$ iff $M \models \phi$. So if $L^* = L_{\omega\omega}$ then K is L^*-definable iff K is an elementary class (i.e. the class of models of some first-order sentence.)

We now give the notion of Σ_1^1-definability. This is the notion of definability which will be used most often.

Definition 3.1.2 : Let K be a class of L-structures. K is said to be Σ_1^1-definable if (i) there is a language K and a 0-morphism $\alpha : L \to K$ which is the idenity on L except possibly that $\alpha(\forall) \neq \forall$ and (ii) there is a sentence ϕ in K^* such that each model M of ϕ is α-invertible and

$$K = \{M^{-\alpha} : M \models \phi\} .$$

The following remarks should help clarify what it means for a set of structures K of a language L to be Σ_1^1-definable. Suppose L is contained in a larger language K and let K(V) be the language obtained from K by adjoining the new unary relation symbol V. We denote a typical K(V)-structure by (M, $\underline{V}$) where M is a K-structure and $\underline{V}$ is the interpretation of V in the universe of M. Let ϕ be a sentence of K(V)* with the following property: if (M, $\underline{V}$) $\models \phi$ then $(M^{(V)})\restriction_L$ is a full L-structure. ($(M^{(V)})\restriction_L$ is the structure obtained by first relativizing M to $\underline{V}$ to get a (partial) K-structure and then taking the reduct of this structure to the language L.) Then a class K of L-structures is Σ_1^1-definable iff there exist a K, V and a sentence ϕ as above such that K consists of all and only those structures of the form $(M^{(V)})\restriction_L$ for $(M, \underline{V}) \models \phi$.

Clearly any definable class is Σ_1^1-definable. Also, the class of reducts of a definable class will be Σ_1^1-definable.

<u>Definition 3.1.3</u>: (i) A structure M is definable (Σ_1^1-definable) in L^* if $K = \{ N : N$ is similar to M and $N \simeq M \}$ is definable (Σ_1^1-definable).

(ii) A class A of ordinals is definable (Σ_1^1-definable) if the class of structures $K = \{ M : M \simeq \langle \alpha, < \rangle$ for some $\alpha \in A\}$ is definable (Σ_1^1-definable).

<u>Theorem 3.1.4</u> : The following are equivalent:

(i) The collection of finite sets is Σ_1^1-definable in L^*.

(ii) $\langle \omega, < \rangle$ is Σ_1^1-definable in L^*.

(iii) There is a class of ordinals A Σ_1^1-definable in L^* which contains an infinite ordinal.

<u>Proof</u>:

((i) implies (ii)): Let ϕ be the Σ_1^1-definition of the collection K of finite sets. We suppose ϕ is a sentence of the language $L = L_\phi$. Thus a unary relation symbol occurs in ϕ and the finite set $M \in K$ iff $M = U^{N}$ for some N such that $N \models \phi$. We may, by the isomorphism property of Definition 1.2.1, assume that M is an initial segment of the integers. For each $n < \omega$ there is an $N_n \models \phi$ with $|U^{N_n}| \geq n$. If the language $L = \{\forall, U, R_1, \ldots, f_1, \ldots\}$ then consider the language $K = \{\forall, V, U', R_1', \ldots, f_1', \ldots\}$ and the 1-morphism with constant $d \in K$ such that

$$\begin{aligned} \forall &\longmapsto V \\ U &\longmapsto U' \\ R_1 &\longmapsto R_1' \quad \ldots\text{etc.} \end{aligned}$$

We write $\phi'(d)$ for ϕ^{α}. We take the indexed union N_ω of $\{ N_n : n < \omega \}$ to form a K-structure (see Example 1.1.6). The universe of N_ω will be the set $\bigcup_{n<\omega} N_n \cup \omega$. Expand K to the language $K_1 = K(W, <)$ where W is a unary and $<$ a binary relation symbol. Let ψ be the conjunction of the following two K_1^* sentences: " $<$ is an infinite linear ordering of W "

$$\forall x[W(x) \to \exists y(\phi'(y) \wedge \forall z(z < x \to U'(z,y)))] .$$

We show two things: (a) that ψ has a model and (b) if $(N, \underline{W}, \prec)$ is a model of ψ then $\langle \underline{W}, \prec \rangle \simeq \langle \omega, < \rangle$.

(a) Consider the structure $(N_\omega, \omega, <)$ for $K(W, <)$ (here $<$ in $(N_\omega, \omega, <)$ is the natural ordering on ω). If $a \in \omega^{N_\omega}$ then it follows that $a \in U^{N_n}$ for some n. Let n be the interpretation of y in N_ω . Clearly if $b < a$ then $b \in U^{N_n}$, i.e. $(U')^{N_\omega}(b,n)$ holds. Therefore the sentence ψ holds in $(N_\omega, \omega, <)$.

(b) Let $(N, \underline{W}, \prec)$ be a model of ψ. To show $\langle \underline{W}, \prec \rangle \simeq \langle \omega, < \rangle$ it is only necessary to show that each $a \in \underline{W}$ has a finite number of predecessors. If $b \in N$ is such that $(N, b) \models \phi'(d)$ then all the predecessors of a are in $X = \{ z : \langle z, b\rangle \in (U')^{N} \}$. But then $M = (N, b)^{-\alpha}$ is a model of ϕ' with $U^M = X$ and so X is finite. This finishes ((i) implies (ii)).

((ii) implies (iii)) : This is trivial. Let A = { ω } and the result follows from the definitions.

((iii) implies (i)) : Let A be a class of ordinals Σ_1^1-definable in L^* and let ψ be the Σ_1^1-definitioniof A. So ψ contains the unary W and the binary < and $M \models \psi$ implies $< W^M, <^M >$ is well-ordered. Add a unary relation symbol to the language $L = L_\psi$ to form $L_\psi(U)$ and and let $\phi \in L_\psi(U)^*$ be the conjunction of ψ and $\forall x(U(x) \to W(x))$

$$\forall x \forall y(U(x) \wedge y < x \to U(y))$$

$$\forall x[U(x) \wedge \exists y(y < x) \to \exists y(y < x \wedge \forall z \sim(y < z \wedge z < x))]$$

Then a set M is finite iff $M = U^N$ for some N ϕ so ϕ is an Σ_1^1-definition of the collection of finite sets.

This theorem tells us that $< \omega, < >$ is <u>not</u> Σ_1^1-definable in $L_{\omega\omega}$. If it were then there would be an $L_{\omega\omega}$ sentence ϕ containing the unary symbol U that Σ_1^1-defines the collection of finite sets. Let ϕ_n be the sentence:

$$\phi \wedge ((\exists v_1)\cdots(\exists v_n)(v_1 \neq v_2 \wedge \cdots \wedge v_1 \neq v_n \wedge v_2 \neq v_3 \wedge \cdots \wedge v_2 \neq v_n \wedge \cdots \wedge v_{n-1} \neq v_n \wedge$$
$$\wedge U(v_1) \wedge \cdots \wedge U(v_n))$$

Then the set of sentences $\{ \phi_n : n \lneqq \omega \}$ would have no model even though every finite subset has a model. And this contradicts the well known compactness property of $L_{\omega\omega}$.

In many of the other examples of logics given in Chapter One $< \omega \ < >$ is definable hence Σ_1^1-definable.

<u>Example 3.1.5</u> : L --second order logic. Consider the following sentences:

(i) " < is a total ordering"

(ii) $\forall X \exists y \forall x(y \in X \wedge x \in X \to y \leq x)$

(iii) $\forall x \exists y(x < y)$

(i), (ii) and (iii) say intuitively that < is a well ordering with no largest element.

(iv) $\forall x[\exists y(y < x) \rightarrow \exists y(y < x \wedge \forall z \sim (y < z < x))]$

(iv) says intuitively that each element has an immediate predecessor in the ordering <. Let ϕ be the conjunction of these four sentences. Then $\langle A, < \rangle$ is a model of ϕ iff $\langle A, < \rangle \simeq \langle \omega, < \rangle$ so ϕ is the L definition of $\langle \omega, < \rangle$.

Example 3.1.6 : $L(Q_0)$ - logic with the quantifier ' there exists infinitely many'. Let ϕ be the conjunction of the sentences:

" < is a total ordering", $\forall x(\sim Q_0 y(y < x))$, $Q_0 x(x = x)$.

Then $\langle A, < \rangle$ is a model of ϕ iff A is infinite, < well orders A and there are only finitely many predecessors of any $a \in A$. Consequently $\langle A, < \rangle$ is isomorphic to $\langle \omega, < \rangle$ and ϕ is the $L(Q_0)$ definition of $\langle \omega, < \rangle$.

Example 3.1.7 : For the definability of $\langle \omega, < \rangle$ in the logic $L_{\omega_1\omega}$ see 4.1.

Definition 3.1.8 : An ordinal α is L^*-accessible if $\alpha \in A$ for some class A of ordinals that is Σ^1_1-definable in L^*.

In $L_{\omega\omega}$ the definable and accessible ordinals coincide: they are just the finite ordinals. Clearly every finite ordinal is definable and therefore accessible. By the previous theorem if an infinite ordinal were $L_{\omega\omega}$-accessible then $\langle \omega, < \rangle$ would be Σ^1_1-definable; this, as was noted above, is not the case.

The following theorem is an important result for obtaining new L^*-accessible ordinals.

Theorem 3.1.9.: Let A be a class of ordinals which is Σ^1_1-definable in L^*. Then

(i) the class of ordinals $A' = \{ \alpha : \alpha \leq \beta$ for some $\beta \in A\}$ is

Σ^1_1-definable in L^*.

(ii) if A is a set and $\beta = \sup\{ \alpha : \alpha\alpha \in A\}$ then β is L^*-accessible.

Proof of (i) : Let ϕ be the Σ^1_1-definition ofAA. Thus ϕ contains a unary symbol U and a binary symbol <. $\alpha \in A$ iff $< \alpha, < > \simeq (U^N, <^N)$ for some N such that $N \models \phi$. Let U', <' be new unary and binary relation symbols respectively and let $\psi \in L_\phi(U',<')^*$ be the conjunction of the following two sentences: $\forall x(U'(x) \to U(x))$

" <' is the reduction of < to U' "

Then if $\gamma \leq \alpha$ for some $\alpha \in A$ then $\lessgtr \gamma, < > \simeq (U'^M, <^M)$ for some M such that $M \models \psi$. (M is obtained by modifying the above structure N.)

Proof of (ii) : Because A is a set $\beta = \sup\{\alpha: \alpha \in A\}$ is indeed an ordinal. By part (i) we may assume that $A = \beta = \{ \alpha : \alpha < \beta \}$. Let ϕ be the Σ^1_1-definition of A (as in part (i)). If $L_\phi = \{\forall , U, <,\ldots\}$ let $K = \{ \forall, V, U', R,\ldots\}$ and let $\alpha : L_\phi \to K$ be a 1-morphism with constant d such that

$$\begin{aligned} \forall &\longmapsto V \\ U\ U_U &\longmapsto U' \\ < &\longmapsto R \quad \ldots\text{etc.} \end{aligned}$$

Write $<\phi'(d)$ for ϕ^α. Expand K to form the language $K(W, \prec)$ (where W is unary and $\prec$ binary). Let θ be the conjunction of the following two $K(W, \prec)^*$ sentences; " $\prec$ is a linear ordering of W "

$$\forall x[W(x) \to \exists y(\phi(y) \wedge \forall z(z \prec x \to R(z,x,y)))]$$

As in the proof of the previous theorem we must show two things:

(a) $M \models \theta$ implies $\lessgtr W^M, \prec^M \gtrless$ is well-ordered and

(b) There exists a $M \models \theta$ such that $\lessgtr W^M, \prec^M \gtrless \simeq < \beta, < >$.

(a) Suppose $M = (N, W^M, \prec^M)$ where N is an K-structure. Assume that $M \models \theta\theta$and $< W^M, \prec^M \gtrless$ is not well ordered. Then there exists an infinite descending chain $a_1 \overset{M}{\succ} a_2 \overset{M}{\succ} a_3 \overset{M}{\succ} \ldots$ where each $a_i \in \mathcal{W}^M$. Let a be the interpretation of y in N such that $\phi(a)$ and $b \overset{M}{\prec} a_1$ implies $R^N(b,a_1,a)$. This implies $b < a_1$ in $N^{-\alpha}$. Letting $b = a_2, a_3, \ldots$ etc. we get a chain $a_1 > a_2 > a_3 > \ldots$ in $N^{-\alpha}$. Since < well orders $N^{-\alpha}$ this is a contradiction. Hence $< \mathcal{W}^M, \overset{M}{\prec} >$ is well ordered.

(b) We must find M such that $M \models \theta$ and $< \mathcal{W}^M, \overset{M}{\prec} > \simeq < \beta, < >$. For each $\alpha < \beta$ there is an N_α such that $N_\alpha \models \phi$ and $< U^{N_\alpha}, <^{N_\alpha} > \simeq <\alpha, < >$. Let N_β be the indexed union of the $\{ N_\alpha : \alpha \lneq \beta \}$. Then $(N_\beta, \beta, <)$ is a model of θ. For if $\alpha \in \beta$ (i.e. $\alpha < \beta$) let $\alpha+1$ be the interpretation of y in N_β. Then $\gamma < \alpha$ implies $\gamma < \alpha$ in $N_{\alpha+1}$ so that $R^{N_\beta}(\gamma,\alpha,\alpha+1)$.

We close this section with some definitions and examples.

<u>Definition 3.1.10</u>: The logic L^* is bounded (respectively bounded by β) if every Σ^1_1-definable class A of ordinals is bounded (respectively bounded by β) . <u>Note</u>: bounded here means strictly bounded.

<u>Definition 3.1.11</u> : If the logic L^* is bounded by some β then wo(L^*), the well ordering number of L^*, is the least such β.

<u>Example 3.1.12</u> : As was noted earlier the only sets of ordinals Σ^1_1-definable in $L_{\omega\omega}$ must consist of finitely many finite ordinals. Hence $L_{\omega\omega}$ is bounded by ω; in fact, wo($L_{\omega\omega}$) = ω .

<u>Example 3.1.13</u> : The logic L^{II} (second-order logic) is <u>not</u> bounded and therefore wo(L^{II}) is not defined. This is because the class A of <u>all</u> ordinals is definable hence Σ^1_1-definable by the sentence ϕ which is the conjunction of: " < is a total linear ordering "

$$\forall X\, [\exists y (y \in X \wedge \forall x(x \in X \to y \leq x)) .$$

3.2 Lindstrom's Theorem

The main result of this section is Lindstrom's Theorem. This theorem outlines conditions that characterize first-order logic in the sense that any logic that satisfies these condition must be equivalent to first-order logic (the meaning of 'equivalent' will be made precise). Actually a more general result is established that specifies conditions which characterize logics in terms of the hierarchy $L_{\kappa\omega}$, where κ is a certain type of infinite cardinal.

Let L^* be a logic on a collection C of languages. In this section we stipulate another assumption that a logic will be required to satisfy in addition to those given in Chapter One:

Assumption 4 : Let ϕ be any L^* sentence. Then the set L_ϕ is finite - only a finite number of symbols occur in ϕ.

The motivation for Assumption 4 will become apparent in the proofs of the results of this sectin.

Let L^* be a logic on C and let L be a language of C. If M and N are two L-structures then M and N are said to be elementarily equivalent with respect to L^*, written $M \equiv_* N$, if the same set of L^* sentences is true in both structures.

Definition 3.2.1 : The logic L^* has the Karp proprty if for all languages L in C, and all L-structures M and N, if $M \cong_p N$ then $M \equiv_* N$ i.e. partial isomorphism implies elementary equivalence.(see Chapter Two for an account of partial isomorphisms).

Definition 3.2.2 : The logic L^* has the Lowenheim property if every L^*-sentence ϕ which has a model has a model of power $\leq \aleph_0$.

<u>Theorem 3.2.3</u> : If L^* has the Lowenheim property then it also has the Karp property.

<u>Proof</u>: Suppose L^* doesn't have the Karp property and let L be a language with structures M_0 and M_1 such that $M_0 \simeq_p M_1$ but for some $\phi \in L^*$

$$M_0 \models \phi \text{ and } M_1 \models \sim\phi \quad .$$

We assume $L = L_\phi$ and hence is finite (by Assumption 4). Enrich L to form the language $K = L(U,W,E,p)$ where U and W are unary relation symbols, E is a binary relation symbol and p is a binary (pairing) function symbol. We write $<x,y>$ for $p(x,y)$ and $<x_1,\ldots,x_n,x_{n+1}>$ for $p(<x_1,\ldots x_n>,x_{n+1})$. Let $\psi \in L(U,W,E,p)^*$ be the conjunction of the following sentences:

(1) ϕ^U and $(\sim\phi)^W$ (see Example 1.2.3)

(2) $\forall xyu\,[E(x,y) \wedge U(u) \rightarrow \exists w(W(w) \wedge E(<x,u>,<y,w>))]$.

(3) $\forall xyw\,[E(x,y) \wedge W(w) \rightarrow \exists u(U(u) \wedge E(<x,u>,<y.w>))]$.

(4) For each n-ary relation symbol R the sentence:

$$\forall x_1\ldots x_n y_1\ldots y_n\,[E(<x_1,\ldots x_n>,<y_1,\ldots y_n>) \rightarrow \bigwedge_{i=1}^{n}(U(x_i) \leftrightarrow W(y_i)) \wedge R(x_1,\ldots x_n) \leftrightarrow R(y_1,\ldots y_n)] \quad .$$

(5) For each n-ary function symbol f the sentence:

$$\forall x_1\ldots x_n y_1\ldots y_n\,[E(<x_1,\ldots,x_n,x_{n+1}>,<y_1,\ldots,y_n,y_{n+1}>) \rightarrow \bigwedge_{i=1}^{n+1}(U(x_i) \leftrightarrow W(y_i)) \wedge f(x_1,\ldots,x_n) = x_{n+1} \leftrightarrow f(y_1,\ldots,y_n)=y_{n+1}]$$

Note that because L_ϕ is finite only finitely many sentences arise in (4) and (5). Thus by our assumptions on the logic L^* the conjunction ψ is indeed an L^*-sentence. We show that ψ has a model N. The universe of N will be the set: $N = M_0 \cup M_1$ where M_0 is the universe of M_0 and M_1 is the universe of M_1 .

Let p^N be any injective map from N N into N and let $U^N = M_0$, $W^N = M_1$. Define E^N as follows: $E^N(\langle x_1,\ldots,x_n\rangle,\langle y_1,\ldots,y_n\rangle)$ iff $x_1,\ldots,x_n \in U^N$, $y_1,\ldots,y_n \in W^N$ and there is a partial morphism from M_0 to M_1 such that $x_1 \longmapsto y_1, \ldots, x_n \longmapsto y_n$. Define R^N as follows:

$R^N(x_1,\ldots,x_n)$ holds iff $x_1,\ldots,x_n \in M_0$ and $R^{M_0}(x_1,\ldots,x_n)$ or

$x_1,\ldots,x_n \in M_1$ and $R^{M_1}(x_1,\ldots,x_n)$.

Define f^N as follows:

if $x_1,\ldots,x_n \in M_0$ then $f^N(x_1,\ldots x_n) = f^{M_0}(x_1,\ldots,x_n)$

if $x_1,\ldots,x_n \in M_1$ then $f^N(x_1,\ldots,x_n) = f^{M_1}(x_1,\ldots,x_n)$

otherwise define f arbitrarily.

Since $M_0 \simeq_p M_1$ sentences (2)-(5) in the conjunction of ψ hold in N. And because $M_0 \models \phi$ and $M_1 \models \sim\phi$ ψ N is indeed a model of ψ. Assume now that the logic L^* has the Lowenheim property. Then there exists a model N_0 for ψ of cardinality $\leq \aleph_0$. Let $\alpha : L \to L(U,W.E,p)$ be the 0-morphism which is the idenity except that $\alpha(\forall) = U$. Let $\beta : L \to L(U,W,E,p)$ be the 0-morphism which is the idenity except that $\alpha(\forall) = W$. Then $N_0^{-\alpha}$ is a countable model of ϕ and $N_0^{-\beta}$ is a countable model of $\sim\phi$. Moreover, the conjunction of (2)-(5) in the formation of ψ ensure that $N_0^{-\alpha} \simeq_p N_1^{-\beta}$. By the classical result given in Chapter Two (Theorem 2.3) partially isomorphic <u>countable</u> structures are isomorphic. Hence $N_0^{-\alpha} \simeq N_1^{-\beta}$. Since $N_0^{-\alpha} \models \phi$ and $N_1^{-\beta} \models \sim\phi$ this is a contradiction. Thus L^* does have the Karp property.

This result tells us that a large class of logics have the Karp property. For instance, it is well known that $L_{\omega\omega}$ satisfies the Lowenheim property and it is shown in [4] that $L(Q_0)$ (see Example 1.2.6) does also.

By the above theorem both these logics have the Karp property. The converse of this theorem is false: as was noted in Chapter Two $L_{\infty\omega}$ has the Karp property; however, it doesn't satisfy the Lowenheim property.

The next definition gives a partial ordering on the class of all logics.

Definition 3.2.4 : Given logics L^* and $L^\#$ on the same collection C of languages we say that $L^\#$ is as strong as L^* and write $L^\# \geq L^*$ if for every L^*-sentence ϕ there is a $L^\#$-sentence ψ such that:

(i) $L_\psi \subseteq L_\phi$,i.e., every symbol occurring in ψ occurs in ϕ.

(ii) $M \models^* \phi$ iff $M \models^\# \psi$ for all L_ϕ-structures M.

We say $L^\#$ is stronger than L^*, and write $L^\# > L^*$, if $L^\# \geq L^*$ but not $L^* \geq L^\#$. If $L^* \geq L^\#$ and $L^\# \geq L^*$ we write $L^* \equiv L^\#$ and say that L^* and $L^\#$ are equivalent.

The assumptions listed in Chapter One ensure that $L_{\omega\omega} \leq L^*$ for all logics L^*. Also, it is clear that $L_{\alpha\omega} \leq L_{\infty\omega}$ for all infinite cardinals α.

The following theorem contains the key result of this chapter. It provides a more precise measure of the strength of certain logics.

Theorem 3.2.5 : Let κ be a cardinal such that $\kappa = \beth_\kappa$. (For the definition of $\beth_\kappa$ see Chapter Two.) If L^* has the Karp property and $\mathrm{wo}(L^*) \leq \kappa$ then the logic $L_{\kappa\omega}$ is as strong as L^*.

Proof: Assume $L_{\kappa\omega}$ is not as strong as L^* and suppose L^* has the Karp property. We show that κ is L^*-accessible; this contradicts the condition $\mathrm{wo}(L^*) \leq \kappa$. Because $L_{\kappa\omega}$ is as not as strong as L^* there is an L^*-sentence ϕ which does not have the same models as any sentence of $L_{\kappa\omega}$.

Claim: For each $\alpha < \kappa$ there are L_ϕ-structures M, N with $M \equiv_{L^\alpha_{\kappa\omega}} N$, $M \models^* \phi$ and $N \models^* \sim\phi$.

Verification: suppose the claim were false. Then there would be some $\alpha < \kappa$ such that $M \equiv_{L^{\alpha}_{\kappa\omega}} N$, $M \models^* \phi$ implies $N \models^* \phi$ for all L_ϕ-structures M and N. By Theorem 2.8 $M \equiv_{L^{\alpha}_{\kappa\omega}} N$ iff $N \models \sigma^{\alpha}_{M}$ (here $\models$ denotes the semantic implication relation with respect to the logic $L_{\infty\omega}$). Hence we would have $N \models \sigma^{\alpha}_{M}$, $M \models^* \phi$ implies $N \models^* \phi$ for all L_ϕ-structures M, N. Consider the sentence $\bigvee\{ \sigma^{\alpha}_{M} : M \models^* \phi \}$. By Theorem 2.10 it is a sentence of $L_{\kappa\omega}$ (this is where the assumption $\kappa = \beth_\kappa$ is used). If N is a model of this sentence then $N \models \sigma^{\alpha}_{M}$ for some L_ϕ-structure M such that $M \models^* \phi$. By the above comment this gives $N \models^* \phi$. Similarly, if $N \models^* \phi$ then (since $N \models \sigma^{\alpha}_{N}$) $N \models \bigvee\{ \sigma^{\alpha}_{M} : M \models^* \phi \}$. Thus if the above claim were false ϕ would have the same models as $\bigvee\{ \sigma^{\alpha}_{M} : M \models^* \phi \}$, a sentence of $L_{\kappa\omega}$, and this contradicts our assumption on ϕ. This establishes the claim.

Using Theorem 2.6 we can rephrase the claim as follows:

For each $\alpha < \kappa$ there are L_ϕ structures M and N such that

(i) $M \models^* \phi$, $N \models^* \sim\phi$

(*) (ii) There is a sequence $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_\beta \supseteq \ldots \supseteq I_\alpha$ where, for each $\beta \leq \alpha$, I_β is a nonempty set of isomorphisms between partial substructures of M and N. This sequence has the following property: if $\beta+1 \leq \alpha$ and $F \in I_{\beta+1}$ then for each $a \in M$ (respectively $b \in N$) there is a $G \in I_\beta$ with $F \subseteq G$ and $a \in \text{domain}(G)$ (respectively $b \in \text{range}(G)$).

We now proceed with the proof. The technique resembles that used in proving the previous theorem. Let K be the language obtained from L_ϕ by adding the following new symbols: U, W_0, W_1, (all unary), < (binary), p (a binary pairing function symbol) and E a 3-ary relation symbol. We construct a sentence ψ of K^* such that: (i) if $N \models^* \psi$ then $< U^N, <^N >$ is a well-ordering and (ii) for each $\alpha < \kappa$ there is a model $N \models^* \phi$ with $< U^N, <^N >$ of order type α. It will then follow from Theorem 3.1.9 of the previous section that κ is L^*-accessible. ψ is the conjunction of the following sentences:

(1) ϕ^{W_0} and $(\sim\phi)^{W_1}$

(2) " $<$ is a linear ordering of U "

(3) " p is a pairing function "

(4) " if E(c,x,y) and $c' < c$ then for every $a \in W_0$ there is a $b \in W_1$ and for every $b \in W_1$ there is an $a \in W_0$ such that $E(c',<x,a>,<y,b>)$.

(5) For each n-ary relation symbol R: " if $E(b,<x_1,\ldots,x_n>,<y_1,\ldots,y_n>)$ and U(b) then $\bigwedge_{i=1}^{n}(W_0(x_i) \wedge W_1(y_i))$ and $R(x_1,\ldots,x_n) \leftrightarrow R(y_1,\ldots,y_n)$ "

(6) Similar to (5) except for function symbols f.

The notation $<x_1,\ldots,x_n>$ is the same as was used in the proof of Theorem 3.2.3 . Our assumptions on a logic ensure that ψ is indeed an L^*-sentence. Now for each $\alpha < \kappa$ using condition (*) above it is easy to construct a model N of ψ such that $N \models^* \psi$ and $< U^N,<^N > \simeq < \alpha, < >$ (condition (*) is used in satisfying (4),(5) and (6)). The construction of N is similar to the construction given in Theorem 3.2.3. We need only show that $N \models^* \psi$ implies that $< U^N, <^N >$ is a well ordering. Suppose there was some $X \subseteq U^N$ with no least element. Using X we can define a set I of partial morphisms from $N^{(W_0)}$ to $N^{(W_1)}$. I consists of all partial morphisms F such that for some $b \in X$, F is given by:

$x_1 \longmapsto y_1,\ldots,x_n \longmapsto y_n$ where $E^N(b,<x_1,\ldots,x_n>,<y_1,\ldots,y_n>)$. Because X has no least element I has the back and forth property. Hence $N^{(W_0)}$ and $N^{(W_1)}$ are partially isomorphic, $I : N^{(W_0)} \simeq_p N^{(W_1)}$. Since L^* has the Karp property this gives $N^{(W_0)} \equiv_* N^{(W_1)}$. But $N^{(W_0)} \models^* \phi$ and $N^{(W_1)} \models^* \sim\phi$, this a contradiction. Thus $< U^N, <^N >$ is well-ordered. We have shown that κ is L^*-accessible and this completes the proof.

If we let $\kappa = \omega$ in the above theorem we obtain Lindstrom's Theorem which is actually a corollary to the above result.

Theorem 3.2.6 (Lindstrom): Let L^* be a logic on a collection C of languages. If L^* has the Lowenheim property and either of the properties stated below, then $L^* \equiv L_{\omega\omega}$.

(a) (Upward Lowenheim-Skolem) : If a sentence ϕ of L^* has an infinite model then it has an uncountable model.

(b) (Countable Compactness) : If T is a countable set of sentences of L^* for any L in C and if every finite subset of T has a model then T has a model.

Proof: Observe first that because of our assumptions on a logic L^*, L^* is always stronger than $L_{\omega\omega}$. Also, since L^* has the Lowenheim property by Theorem 3.2.4 it has the Karp property. We show that conditions (a) or (b) imply that $wo(L^*) \leq \omega$; by the previous theorem this gives $L_{\omega\omega} \geq L^*$.

It was shown in an argument in Section 3.1 that the Σ_1^1-definability of $< \omega, < >$ in L^* makes countable compactness fail. So (b) implies $wo(L^*) \leq \omega$. Assume now that (a) holds and suppose $< \omega, < >$ is Σ_1^1-definable by an L^*-sentence ϕ containing the unary U and binary <. Let ψ be an L^*-sentence which says that f is an injective function with range contained in U. Clearly the conjunction of ϕ and ψ have only countable models and this contradicts (a) . Hence $< \omega, < >$ is not Σ_1^1-definable in L^*.

CHAPTER FOUR

HANF NUMBERS AND WELL-ORDERING NUMBERS

In this chapter we continue our investigation of general logics. We will examine the analogues in the general setting of the familiar compactness and Lowenheim-Skolem theorems of the first-order predicate calculus.

We assume that a logic L^* on a collection C of languages is as defined in Chapter One; hence a logic satisfies the four properties and three assumptions given there. We will, however, not require that a logic satisfy Assumption Four stipulated at the beginning of Chapter Three. That is, we drop the requirement that an L^*-sentence ϕ must contain only finitely many symbols.

4.1 Definitions and Preliminaries

The first definition of this section gives a generalization of the Upward Lowenheim-Skolem property.

Definition 4.1.1 : The Hanf number of a logic, written $h(L^*)$, is the least cardinal α such that if an L^*-sentence ϕ has a model of cardinality $\geq \alpha$ then it has arbitrarily large models. More generally, $h_\lambda(L^*)$ is the least α such that if a set Σ of L^*-sentences with cardinality$(\Sigma) \leq \lambda$ has a model of cardinality $\geq \alpha$ then Σ has arbitrarily large models.

Note that in general the Hanf number of a logic need not exist (a later example shows this). Clearly $h_\lambda(L_{\omega\omega}) = \omega$ for all cardinals λ:

this is the content of the Upward Lowenheim-Skolem theorem for $L_{\omega\omega}$.

Recall the definition of $\mathrm{wo}(L^*)$ given in 3.1.11. $\mathrm{wo}(L^*)$, if it exists, is the least ordinal ζ which is an upper bound for any class A of ordinals Σ_1^1-definable in L^*. $\mathrm{wo}(L^*)$ is also called the well-ordering number of the logic L^*. The next definition provides a generalization of the well-ordering number to sets of sentences.

Definition 4.1.2 : Let λ be a cardinal. Suppose ζ is an ordinal with the following property:

> If Σ is any set of $\leq \lambda$ L^*-sentences containing the binary relation symbol $\lessdot$ and M is a model in which $<^M$ well-orders its field in order-type $\geq \zeta$, then Σ has a model N in which $<^N$ is not well-ordered.

Then $\mathrm{wo}_\lambda(L^*)$, if it exists, is the least ordinal ζ with this property.

In the case $\lambda = 1$ this definition is equivalent to our original definition, i.e. $\mathrm{wo}_1(L^*) = \mathrm{wo}(L^*)$. In this chapter we will find it convenient to work with this new definition of the well-ordering number of a logic.

In Chapter Three the compactness property of $L_{\omega\omega}$ was used to show that $\mathrm{wo}(L_{\omega\omega}) = \omega$. In fact, the well-ordering number of a logic is closely connected with compactness properties. Some evidence for this is provided by the next theorem.

Theorem 4.1.3 : Let L^* be a logic on a collection C of languages and let L be a countable language in C. Then the following are equivalent:

(i) (Countable Compactness) If each finite subset of a countable set Σ of L^*-sentences has a model, then Σ has a model.

(ii) ($\mathrm{wo}_\omega(L^*) \leq \omega$) Assume L contains the binary symbol $\lessdot$ and let

Σ be a countable set of L^*-sentences. Suppose that for each positive integer n, Σ has a model M such that $<^M$ well-orders its field in order-type $\geq$ n. Then Σ has a model N such that $<^N$ is <u>not</u> a well-ordering.

<u>Proof</u>: ((i) implies (ii)): Consider the set $\Sigma \cup \{ c_{n+1} < c_n : n \in \omega \}$. By compactness and the assumptions in (ii) this set has a model M. The c_n^M give an infinite descending chain in M so $<^M$ is not a well-ordering.

((ii) implies (i)): Let $\phi_0, \phi_1, \ldots$ be countably many L^*-sentences and suppose that for all n, $\{\phi_0, \ldots, \phi_n\}$ has a model. Let U and < be new unary and binary relation symbols respectively. Let Σ be the set of sentences of $L(U,<)^*$ consisting of the following:

"U is f closed" for each function symbol f in L

"If the field of < has more than n elements then ϕ_n^U" for each positive integer n .

For each n, if M is a model of $\{\phi_0, \ldots, \phi_{n-1}\}$ we can construct a model M_n of Σ. The universe of M_n will be $M \cup \{0,1,\ldots,n-1\}$. The field of < in M_n will be $\{0,1,\ldots,n-1\}$ with the natural well-ordering. By (ii) there is a model N of Σ in which $<^N$ is not well-ordered. Thus the field of $<^N$ is infinite and so N is a model of ϕ_n^U for all n. Taking the U-reduct of N we get a model of $\{\phi_0, \phi_1, \ldots\}$.

Our aim laterrin this chapter will be to examine what relationship exists between the Hanf and well-ordering numbers of a logic L^*. The Upward Lowenheim-Skolem theorem for $L_{\omega\omega}$ is usally proved using compactness. We will discover that the size of the Hanf number of many general logics is determined by their well-ordering numbers. Indeed, our results may be viewed as a welcome extension of first-order model theory if the following two 'equations' are accepted:

Hanf number = generalization of Upward Lowenheim-Skolem property

well-ordering number = generalization of compactness property.

We will focus our attention on the logics $L_{\kappa^+\omega}$ and $L(Q_\alpha)$ introduced in Examples 1.2.8,1.2.6. (If κ is any cardinal then κ^+ denotes the next largest cardinal, eg. $\omega^+ = \omega_1$.) A small digression is necessary to provide for future reference certain results about these logics.

The next two theorems are Downward Lowenheim-Skolem results for the logics $L_{\kappa^+\omega}$ and $L(Q_\alpha)$;; proofs of the theorems may be found in [11] and [4] .

Theorem 4.1.4 : (Downward Lowenheim-Skolem Theorem for $L_{\kappa^+\omega}$) Assume that M is a model of the $L_{\kappa^+\omega}$-sentence ϕ. Let $M_0 \subseteq M$ and suppose $|M_0| + \kappa \leq \mu \leq |M|$. Then there is a structure N such that

$N \subset M$, $N \models \phi$, $M_0 \subseteq N$ and $|N| = \mu$.

Theorem 4.1.5 : (Downward Lowenheim-Skolem Theorem for $L(Q_\alpha)$) Let $\kappa = \omega_\alpha$. Assume M is a model of the set Σ of $L(Q_\alpha)$-sentences where $|\Sigma| \leq \kappa$. Let M_0 M and suppose $|M_0| + \kappa \leq \mu \leq |M|$. Then there is a structure N such that

$N \subset M$, $N \models \Sigma$, $M_0 \subseteq N$ and $|N| = \mu$.

We now show that well-orderings of the form $< \zeta, < >$ are definable in $L_{\kappa^+\omega}$ for any ordinal $\zeta < \kappa^+$. Suppose L is a language containing only the binary relation symbol < and the constant symbol c. For each $\zeta < \kappa^+$ we define sentences $\phi_\zeta(c)$ and ψ_ζ of $L_{\kappa^+\omega}$ such that :

(i) For all L-structures M and all $a \in M$

$(M,a) \models \phi_\zeta(c)$ iff $\langle \{b : b \in M,\ b <^M a\}, <^M \rangle \simeq \langle \zeta, < \rangle$

(ii) For all L-structures M, $M \models \psi_\zeta$ iff $M \simeq \langle \zeta, < \rangle$.

The definition of $\phi_\zeta(c)$ proceeds by recursion. Suppose $\phi_\eta(c)$ has been defined for all $\eta < \zeta$. Then $\phi_\zeta(c)$ is the sentence

$$\Big[\bigwedge_{\eta<\zeta}(\exists y)(y < c \wedge \phi_\eta(y))\Big] \bigwedge \Big[\forall y(y < c \rightarrow \bigvee_{\eta<\zeta}\phi_\eta(y))\Big]$$

The sentences ψ_ζ are then defined in terms of the $\phi_\eta(c)$:

$$\psi_\zeta = \Big[\bigwedge_{\eta<\zeta} \exists x \phi_\eta(x)\Big] \bigwedge \Big[\forall x(\bigvee_{\eta<\zeta}\phi_\eta(x))\Big]$$

A straightforward induction argument shows that (i) and (ii) above hold.

The sentences ψ_ζ tell us immediately that $\mathrm{wo}(L_{\kappa^+\omega}) \geq \kappa^+$. They also show that the Hanf number and well-ordering numbers need not exist for a given general logic. More specifically, since ψ_ζ is an $L_{\infty\omega}$-sentence for <u>all</u> ordinals ζ it is clear that $h(L_{\infty\omega})$, $\mathrm{wo}(L_{\infty\omega})$ are both undefined.

We close out this section with some observations which will be useful later.

(i) There is a sentence ϕ_0 of $L_{\kappa^+\omega}$ which only has models of cardinality κ.

(ii) If $\kappa = \omega_\alpha$ then there is a sentence ϕ_0 of $L(Q_\alpha)$ having only models of cardinality κ.

To verify (i) take ϕ_0 to be the sentence ψ_κ defined above. For (ii) take ϕ_0 to be the conjunction of the following $L(Q_\alpha)$-sentences:

"$<$ is a total ordering"

$Q_\alpha x(x = x)$

$\forall x \sim Q_\alpha y(y < x)$

These sentences also serve to show that $h(L_{\kappa^+\omega}) \geq \kappa^+$, $h(L(Q_\alpha)) \geq \kappa^+$.

4.2 Existence Theorems

The main result of this section employs an argument of Hanf ([10]) to show that the Hanf number of a large class of logics exists.

Assume L^* is a logic on a collection C of languages. Let L be a language in C and suppose that L^*, the class of sentences of L, is a set. For each $\Sigma \subseteq L^*$ with $|\Sigma| \leq \lambda$ (λ a cardinal) define κ_Σ by:

$$\kappa_\Sigma = \begin{cases} 0 \text{ if } \Sigma \text{ has arbitrarily large models} \\ \\ \sup\{ |M| : M \models \Sigma \} \text{ otherwise.} \end{cases}$$

Let $\kappa_\lambda^L = \sup\{ \kappa_\Sigma^+ : \Sigma \subseteq L^*, |\Sigma| \leq \lambda \}$. Because L^* is a set, an application of the set theoretic axiom of replacement shows that $\kappa_{\lambda\lambda}^L$ is a cardinal. The λ-Hanf number of the logic L^*, if it exists, will equal

$$\sup\{ \kappa_\lambda^L : \text{L a language in } C \}.$$

We now prove that for many logics this supremum does in fact exist.

Theorem 4.2.1 (Hanf) : Assume that a logic L^* on a collection C of languages satisfies the following properties:

(i) The number of symbols occuring in any L^*-sentence is bounded. i.e., there is a cardinal μ such that for each L^*-sentence ϕ $|L_\phi| \leq \mu$.

(ii) L^* is a set for all L in C.

Then $h_\lambda(L^*)$ exists for all λ.

Proof: We construct a language K_0 such that $\kappa_\lambda^L \leq \kappa_\lambda^{K_0}$ for each L in C. (We are assuming the cardinal λ is fixed). Intuitively K_0 will contain within it a copy of each of the languages L. More precisely, for each positive integer n the set K_0 contains $\mu \cdot \lambda$ n-ary predicate symbols and

$\mu\cdot\lambda$ n-ary function symbols. Assume now L is any language in C. If Σ is a set of $\leq \lambda$ L^*-sentences we show that $\kappa_\Sigma \leq \kappa_\lambda^{K_0}$. Let α be a 0-morphism from L to K_0 such that $\alpha : \forall \longmapsto \forall$. Define $\Sigma^\alpha = \{ \phi^\alpha : \phi \in \Sigma \}$. If M is any model of Σ it is easy to construct from M a K_0-structure N such that: (a) $N^{-\alpha} = M$ and (b) $N \models \Sigma^\alpha$. By (a) $|M| \doteq |N|$. Because K_0^* is a set, $\kappa_\lambda^{K_0}$ exists and $|M| = |N| \leq \kappa_\lambda^{K_0}$. Hence $\kappa_\Sigma \leq \kappa_\lambda^{K_0}$ and so it follows that $\kappa_\lambda^L = \sup\{ \kappa_\Sigma^+ : \Sigma \subseteq L^*, |\Sigma| \leq \lambda \} \leq \kappa_\lambda^{K_0}$. Since this is true for any language L in C, $h_\lambda(L^*) = \sup\{ \kappa_\lambda^L : \text{L in } C \}$ exists.

Example 4.2.2 : (Application to $L_{\kappa^+\omega}$, $L(Q_\alpha)$))Assume that the logics $L_{\kappa^+\omega}$ and $L(Q_\alpha)$ are defined on a collection C of languages and assume that $L_{\kappa^+\omega}$, L_{Q_α} are sets for each L in C ($L_{\kappa^+\omega}$,L_{Q_α} denote L^* for the logics $L_{\kappa^+\omega}$, $L(Q_\alpha)$ respectively). If ϕ and ψ are sentences of $L_{\kappa^+\omega}$ aand $L(Q_\alpha)$ respectively it is clear that

$$\text{cardinality}(\{\text{symbols occuring in } \phi\}) < \kappa^+$$

$$\text{cardinality}(\{\text{symbols occuring in } \psi\}) < \omega.$$

Thus these logics satisfy the hypotheses of the above theorem and $h_\lambda(L(Q_\alpha))$, $h_\lambda(L_{\kappa^+\omega})$ exist for all λ.

Remark: We noted earlier that the Hanf number of $L_{\infty\omega}$ doesn't exist. The above theorem might suggest that the reason for this is that an $L_{\infty\omega}$ can contain arbitrarily many symbols. This, however, is quite misleading. Let $L^*_{\infty\omega}$ be the logic obtained from $L_{\infty\omega}$ by adding the requirement that only finitely many symbols occur in each sentence of $L_{\infty\omega}$. Then all of the sentences ψ_ζ defined in the last section belong to $L^*_{\infty\hat{\omega}}$; consequently $h(L^*_{\infty\omega})$ is undefined. The failure of the existence of the Hanf number of $L_{\infty\omega}$ and

and $L^*_{\infty\omega}$ is actually an outcome of the following: given a language L the class $L_{\infty\omega}$ of sentences of L is <u>not</u> a set.

No general method similar to the above theorem is available for determining whether wo(L^*) exists and each logic must be investigated individually. Lopez-Escobar shows in 13 that $wo_\lambda(L_{\alpha\omega})$ exists for all cardinals λ and α. The existence of $wo_\lambda(L(Q_\alpha))$ will be a direct corollary of the reduction given in Example 4.3.7 of the next section. We assume in the future therefore that $wo_\lambda(L_{\kappa^+\omega\omega})$ and $wo_\lambda(L(Q_\alpha))$ both exist.

4.3 Relationship of Hanf and Well-ordering Numbers

In Chapter Two the beth cardinals were defined in terms of cardinal exponentiation. An extension of this idea gives rise to the <u>generalized</u> beth cardinals. For each cardinal κ and ordinal ζ the general beth cardinal $\beth^\kappa_\zeta$ is defined by recursion as follows:

$$\beth^\kappa_0 = \kappa, \quad \beth^\kappa_\zeta = \sup_{\eta<\zeta}(2^{\beth^\kappa_\eta}) .$$

If $\kappa = 0$ we get the usual beth cardinals: $\beth_\zeta = \beth^0_\zeta$. Generalized beth cardinals are of interest to us here because they provide a connecting link between the Hanf and well-ordering numbers of many logics. That is , for many logics L^* the following relation holds: $h_\lambda(L^*) = \beth^\kappa_{wo_\lambda(L^*)}$. For example, since $wo_\lambda(L_{\omega\omega}) = h_\lambda(L_{\omega\omega}) = \omega$, we get the relation

$$h_\lambda(L_{\omega\omega}) = \omega = \beth_\omega = \beth_{wo_\lambda(L_{\omega\omega})} .$$

The next theorem provides a partial result in this direction for more general logics.

Theorem 4.3.1 Let λ and κ be cardinals with $1 \leq \lambda \leq \kappa$ and κ infinite. Assume that the logic L^* satisfies the following properties:

(i) $wo_\lambda(L^*)$ exists.

(ii) There is an L^*-sentence ϕ_0 that has only models of power κ.

(iii) Let Σ be any set of $\leq \lambda$ L^*-sentences of some language L. If the L-structure $M \models \Sigma$ and $M_0 \subseteq M$ then there is an L-structure N such that: $N \models \Sigma$, $N \subset M$, $M_0 \subseteq N$ and $|N| \leq |M_0| + \kappa$. (This condition is a Downward Lowenheim-Skolem property.)

Then $h_\lambda(L^*) \geq \beth^{\kappa}_{wo_\lambda(L^*)}$.

Proof: An easy argument shows that $wo_\lambda(L^*)$ is a limit ordinal. Hence it is sufficient to show that for each $\zeta < wo_\lambda(L^*)$ there is a set Σ of $\leq \lambda$ many L^*-sentences that has a model of power $\beth^{\kappa}_{\zeta}$ but not arbitrarily large models. Since $\zeta < wo_\lambda(L^*)$ there is a language L containing the binary symbol $<$ and a set Σ of $\leq \lambda$ L^*-sentences such that:

(i) $M \models \Sigma$ for some L-structure M in which $<^M$ well-orders its field in order-type ζ.

(ii) If $M \models \Sigma$ then $<^M$ well-orders its field in order-type less than $wo_\lambda(L^*)$.

Let K be the language obtained from L by adjoining U and V (unary relations), E (binary relation), and g (unary function). The set Σ will consist of the following K^*-sentences:

ϕ_0^U

ϕ^V for $\phi \in \Sigma$

$\forall x(V(g(x)) \wedge$ " $g(x) \in$ field$(<)$ ")

$\forall x \forall y(Eyx \rightarrow g(y) < g(x))$

$\forall x(U(x) \rightarrow$ "$g(x)$ is the first element of $<$")

$$\forall x \forall y(\sim U(x) \wedge \sim U(y) \wedge \forall z(Ezx \leftrightarrow Ezy) \to x = y)$$

We now show two things: (a) $\mathcal{M} \models \Sigma$ implies $|M| \leq \beth^{\kappa}_{wo_\lambda(L^*)}$. In particular, Σ doesn't have arbitrarily models. (b) Σ has a model of power $\beth^{\kappa}_{\zeta}$.

(a) If $\mathcal{M} \models \Sigma$ then $\mathcal{M} \models \phi^{VV}$ for all $\phi \in \Sigma$. In the subset $V^{\mathcal{M}}$ of M, $<^{\mathcal{M}}$ well-orders its field in order-type $\eta < wo_\lambda(L^*)$. We assume η is a limit ordinal; the proof of (a) in the case where η is not a limit ordinal is essentially the same. Let $X_0 = U^{\mathcal{M}}$ and for $0 < \beta < \eta$ let $X_\beta = P(\bigcup_{\alpha<\beta} X_\alpha)$. (For any set A $P(A)$ denotes the power-set of A). Set $X = \bigcup_{\beta<\eta} X_\beta$. Clearly cardinality(X) is equal to $\beth^{\kappa}_{\eta}$. To prove (a) we define an injective map f from M to X. This will show that $|M| \leq \beth^{\kappa}_{\eta} \leq \beth^{\kappa}_{wo_\lambda(L^*)}$. The definition of f for a $\in$ M proceeds by induction on $g^{\mathcal{M}}(a)$. If $g^{\mathcal{M}}(a) = 0$ (i.e. $a \in U^{\mathcal{M}}$) $f(a) = a$. Assume f has been defined for all b such that $g^{\mathcal{M}}(b) <^{\mathcal{M}} \beta <^{\mathcal{M}} \eta$. If $g^{\mathcal{M}}(a) = \beta$ then $f(a) = \{f(b) : E^{\mathcal{M}}ba\} \in X_\beta$. The last sentence included in Σ ensures that f is injective and this proves (a).

(b) We construct a model $\mathcal{M}$ of Σ with $|M| = \beth^{\kappa}_{\zeta}$. Let $\mathcal{M}_1$ be a model of Σ in which $<^{\mathcal{M}_1}$ well-orders its field in order-type ζ. As in (a) we suppose ζ is a limit ordinal; the other case is treated similarly. Using hypothesis (iii) we may assume that $|\zeta| \leq |M_1| \leq |\zeta| + \kappa$. By hypothesis (ii) there is a model $\mathcal{M}_2$ of ϕ_0 with $|M_2| = \kappa$. Let $V^{\mathcal{M}} = M_1$ and $U^{\mathcal{M}} = M_2$. Define $X_0 = U^{\mathcal{M}}$ and if $\beta < \zeta$ define $X_\beta = P(\bigcup_{\alpha<\beta} X_\alpha)$. Set $X = \bigcup_{\beta<\zeta} X_\beta$. We assume that M_1 is embedded in X so that $M_1 \subseteq X$ (this is possible since $|M_1| \leq |\zeta| + \kappa \leq |X|$). The universe of $\mathcal{M}$ is then $M = X$. If $a \in M$ let β be the least ordinal such that $a \in X_\beta$. Then define $g^{\mathcal{M}}(a) = \beta$. Define $E^{\mathcal{M}}$ by $E^{\mathcal{M}}ab$ iff $g(a) < g(b)$. The structure $\mathcal{M}$ so constructed is a model of Σ and $|M| = \beth^{\kappa}_{\zeta}$. This proves (b).

Remark: Let L^* be a logic for which $h_\lambda(L^*)$ exists and assume L^* is bounded (see Definition 3.1.10). If L^* satisfies properties (ii) and (iii) of Theorem 4.3.1 then the above proof can easily be adapted to show that $wo_\lambda(L^*)$ exists for each $\lambda \leq \kappa$.

Example 4.3.2: (Application to $L_{\kappa^+\omega}$, $L(Q_\alpha)$ where $\kappa = \omega_\alpha$) The results at the end of section 4.1 show that the logics $L_{\kappa^+\omega}$ and $L(Q_\alpha)$ satisfy the hypotheses of the above theorem. Hence for each $\lambda \leq \kappa$

$$h_\lambda(L_{\kappa^+\omega}) \geq \beth^\kappa_{wo_\lambda(L_{\kappa^+\omega})}, \quad h_\lambda(L(Q_\alpha)) \geq \beth^\kappa_{wo_\lambda(L(Q_\alpha))} .$$

As mentioned earlier, we wish to obtain results of the form $h(L^*) = \beth^\kappa_{wo(L^*)}$. For logics satisfying the hypotheses of the above theorem we need only show inequalities of the form $h(L^*) \leq \beth^\kappa_{wo(L^*)}$. The logics introduced in the next definition, Ω-logics, are designed for this purpose. Ω-logics are an invention of the logician Flum and our treatment of them is taken from [8]. (Note: we have altered Flum's notation; he calls Ω-logics M-logics.) Essentially an Ω-logic is just $L_{\omega\omega}$ with semantic restrictions on the class of allowable structures. We will see that $L_{\kappa^+\omega}$ and $L(Q_\alpha)$ can be reduced in some sense to these Ω-logics; the results holding for the latter can then be applied to the former.

We assume throughout that all logics belong to some collection C of languages.

Definition 4.3.3: Let L_0 be a language and Ω a class of L_0-structures. Let U be a unary relation symbol not in L_0. Given any language L, an L-structure M is said to an Ω-model if the following two conditions hold:

(i) $L_0 \cup \{U\} \subseteq L$.

(ii) Let $\alpha : L_0 \to L$ be the identity map except that $\alpha : \forall \longmapsto U$. Then

M is α-invertible and $M^{-\alpha} \simeq N$ for some N in Ω.

Condition (ii) says simply that the L_0-reduct of U^M is a full L_0-structure isomorphic to some structure in Ω, i.e. each Ω-model contains within it a relativized reduct which lies in Ω. The Ω-logic $L(\Omega)$ is then defined as follows. If L is any language the set of sentences of L, written L_Ω, is given by:

$$L_\Omega = \begin{cases} \emptyset & \text{if } L_0 \cup \{U\} \not\subseteq L \\ L_{\omega\omega} & \text{if } L_0 \cup \{U\} \subseteq L \end{cases}$$

For each L-structure M and L_Ω-sentence ϕ the relation $\models_\Omega$ is given by:

$$M \models_\Omega \phi \quad \text{iff} \quad M \text{ is an } \Omega\text{-model and } M \models \phi .$$

Here $\models$ denotes the semantic entailment relation in the first-order logic $L_{\omega\omega}$ (this makes sense since any sentence of L_Ω is first-order).

Observe that $L(\Omega)$ is not a logic as defined in Chapter One. For example, the relativization property given in Example 1.2.3 fails. This, however, should cause no alarm: the Ω-logics can be viewed as devices for establishing results about more familiar, genuine, logics.

Lemma 4.3.4: Assume that all structures in Ω have the same infinite cardinality κ. Then

(a) $\forall x U(x)$ has only Ω-models of power κ.

(b) If Σ has an Ω-model and $|\Sigma| \leq \kappa$ then Σ has an Ω-model of power κ.

(c) $wo_\lambda(L(\Omega))$ exists for all cardinals λ.

Proof: (a) If M is an Ω-model then $|U^M| = \kappa$. Thus $M \models_\Omega \forall x U(x)$ implies that $|M| = |U^M| = \kappa$.

(b) Let M be an Ω-model of Σ. Using the Downward Lowenheim-Skolem theorem for $L_{\omega\omega}$, take an elementary substructure N of M such that $U^M \subseteq N$ and $|N| = \kappa$. N will be an Ω-model of Σ of power κ.

(c) By choosing the cardinal μ large enough and taking a sufficiently rich language it is possible to find an $L_{\mu\omega}$-sentence ϕ such that

$$M \models_{L_{\mu\omega}} \phi \quad \text{iff} \quad (M^{(U)}) \restriction \mathbb{L}_0 \simeq N \text{ for some } N \in \Omega \ .$$

Now suppose $wo_\lambda(L(\Omega))$ doesn't exist. For each ordinal ζ there will be some set Σ_ζ of $\leq \lambda$ $L(\Omega)$-sentences such that Σ_ζ has only Ω-models M with $<^M$ well-ordered and one with $<^M$ in order-type ζ. Form the $L_{\mu\omega}$-sentence :

$$\delta_\zeta = (\bigwedge \Sigma_\zeta) \wedge \phi.$$

Then the sentences δ_ζ imply that $wo(L_{\mu\omega})$ doesn't exist and this is a contradiction. Hence $wo_\lambda(L(\Omega))$ exists.

For the remainder of this chapter it is assumed that each structure in Ω has the same infinite cardinality κ.

<u>Lemma 4.3.5</u>: For each $\lambda \leq \kappa$, $\hbar_\lambda(L(\Omega)) \underset{=}{\gtrsim} \beth^{\kappa}_{wo_\lambda(L(\Omega))}$.

<u>Proof</u>: We modify the proof of Theorem 4.3.1 making changes necessitated by the failure of the relativization property for $L(\Omega)$. Let the set Σ be as in that proof and let V, E and g be new symbols. The set Σ consists of the following $KL_{\omega\omega}$-sentences of L(V,E,g):

$\forall x(U(x) \rightarrow V(x))$

For each $\phi \in \Sigma$ the first-order relativization of ϕ to V, ϕ^V

$\forall x(V(g(x)) \wedge$ "g(x) ε field(<)") ele[illegible] of

$\forall x(U(x) \rightarrow$ "g(x) is the first element of <")

$\forall x \forall y(Eyx \rightarrow g(y) < g(x))$

$\forall x \forall y(\sim U(x) \wedge \sim U(y) \wedge \forall z(Ezx \leftrightarrow Ezy) \rightarrow x = y)$

Applying the previous lemma, the rest of the proof proceeds in a similar fashion to the proof of 4.3.1.

The next very important theorem provides the raison d'être for Ω-logics.

<u>Theorem 4.3.6</u>: For each $\lambda \leq \kappa$ $h_\lambda(L(\Omega)) = \beth^\kappa_{wo_\lambda}(L(\Omega))$.

<u>Proof</u>: This theorem is quite deep and the proof is rather complicated. Since we are mainly interested in applications of this result we give a very brief sketch proof. Details may be found in [8]

By Lemma 4.3.5 it is only necessary to show that $h_\lambda(L(\Omega)) \leq \beth^\kappa_{wo_\lambda}(L(\Omega))$. So let Σ be a set of $\leq \lambda \leq \kappa$ sentences with an Ω-model M of power $\geq \beth^\kappa_{wo_\lambda}(L(\Omega))$; the idea is to construct arbitrarily large models of Σ. From the Ω-model M a set-theoretic partition result of Erdos and Rado and the well-ordering number of $L(\Omega)$ are used to construct an Ω-model N of Σ containing an infinite set X of elements indiscernible with respect to U. This means that any two finite sequences in X satisfy the same formulas with parameters in U. An Ω-model S of Σ of arbitrarily large cardinality is then formed by throwing into the universe of N $\mu > |N|$ new elements which behave as the elements in X.

The next two examples are applications of Theorem 4.3.6 to the logics $L(Q_\alpha)$ and $L_{\kappa^+\omega}$. In each case a reduction is defined which allows us to transfer the results for Ω-logics to the above logics.

<u>Example 4.3.7</u>: (Application to $L(Q_\alpha)$). Assume that $\kappa = \omega_\alpha$. Let $L_0 = \{<\}$ and let Ω consist of all the models of the sentence ψ_κ defined at the end of

4.1. Thus the structure $\mathcal{M} = \langle M, < \rangle \in \Omega$ iff

(i) $|M| = \kappa$.

(ii) $<$ is a linear ordering of M.

(iii) If $a \in M$ then the set $\{b : b < a\}$ of predecessors of a has cardinality $< \kappa$.

Fix the unary relation symbol U. With L_0, Ω and U thus specified we get an Ω-logic $L(\Omega)$. In the following discussion it is assumed that $L_0 \cup \{U\} \subseteq L$ for any language L under consideration. This assumption can always be justified by expanding L (if necessary) to include $L_0 \cup \{U\}$.

Suppose L is a language and ϕ is a sentence of L_{Q_α}. Form the language K by adjoining the relation symbols V (unary) and F (ternary) so that $K = L(V,F)$. Let $\alpha : L \to K$ be the 0-morphism which is the identity except that $\forall \longmapsto V$. We find a sentence $\hat{\phi}$ of $L_{\omega\omega}$ such that:

(*) Models of $\phi = \{\mathcal{M}^{-\alpha} : \mathcal{M} \models_\Omega \hat{\phi}\}$

That is, an L-structure $\mathcal{N} \models_{L(Q_\alpha)} \phi$ iff $\mathcal{N}$ is of the form $(\mathcal{M}^{(V)}) \upharpoonright L$ for some K-structure $\mathcal{M}$ with $\mathcal{M} \models_\Omega \hat{\phi}$. Result (*) thus gives the desired reduction of the $L(Q_\alpha)$-sentence ϕ to the $L(\Omega)$-sentence $\hat{\phi}$.

First we outline the idea behind this reduction. We wish to replace a sentence of the form $Q_\alpha x\phi(x)$ by the first-order sentence:

"There is a 1-1 map from a subset of $\{x : V(x)$ and $\phi(x)$ holds$\}$ onto the set U."

$\sim Q_\alpha \phi(x)$ is replaced by the first-order sentence:

"There is a 1-1 map from $\{x : V(x)$ and $\phi(x)$ holds$\}$ onto a proper initial segment of U."

The 1-1 maps referred to belong in a collection which is indexed using the ternary relation F. That is, for each x an injective map g_x from a subset of V to U is given such that: $g_x(y) = z$ iff $\langle x, y, z\rangle \in F$.

The precise details are now provided. To each sentence ψ of $L(Q_\alpha)$ is associated by recursion a sentence ψ^* of $L_{\omega\omega}$ as follows:

(i) $\psi^* = \psi$ if ψ is atomic.

(ii) $(\sim\psi)^* = \sim(\psi^*)$, $(\psi_1 \vee \psi_2)^* = \psi_1^* \vee \psi_2^*$.

(iii) $(\exists x\psi(x))^* = \exists x(V(x) \wedge \psi^*)$.

(iv) $(Q_\alpha x\psi(x))^* = \exists y\forall z\,\exists x(V(x) \wedge \psi^* \wedge Fyxz)$.

If the sentence ψ is first-order then (i)-(iii) ensure that ψ^* is just the relativization of ψ to V. (iv) says that there is a map indexed by F from a subset of the set defined by ψ^* in V onto U.

Given a sentence ϕ of L_{Q_α} we now define a finite set Σ of first-order sentences. Σ consists of the following sentences:

(i) ϕ^*

(ii) The sentence "F is an indexed collection of 1-1 maps from a subset of V into U"

(iii) For each subformula of ϕ of the form $Q_\alpha x\psi(x)$ the universal closure of

$$\exists y\{\forall z\,\exists x[\psi^* \wedge V(x) \wedge F(y,x,z)] \vee \exists w\forall x\,\exists z[\psi^* \wedge V(x) \rightarrow z < w \wedge F(y,x,z)]\}$$

This sentence says that among the maps indexed by F there is one that either (a) maps a subset of the set defined by ψ^* in V onto U, or (b) maps the set defined by ψ^* in V onto a proper initial segment of U.

(iv) $\forall x(U(x) \vee V(x))$

Note that Σ is finite. Let $\hat{\phi}$ be the conjunction of the sentences in Σ. Then $\hat{\phi}$ is a sentence of $K_{\omega\omega}$. Fuhrken proves in [9] that $N \models_\Omega \hat{\phi}$ iff $(N^{(V)}) \upharpoonright L \models_{L(Q_\alpha)} \phi$ and hence establishes result (*).

<u>Remark</u>: The reduction result((*) confirms (among other things) the existence

of $wo_\lambda(L(Q_\alpha))$ via the existence of the well-ordering number of $L(\Omega)$. Actually we have more.

Claim: For all λ $h_\lambda(L(\Omega)) = h_\lambda(L(Q_\alpha))$ and $wo_\lambda(L(\Omega)) = wo_\lambda(L(Q_\alpha))$.

Since Ω is just the set of all models of the $L(Q_\alpha)$-sentence ψ_κ it is clear that $h_\lambda(L(\Omega)) \leq h_\lambda(L(Q_\alpha))$ and $wo_\lambda(L(\Omega)) \leq wo_\lambda(L(Q_\alpha))$. The above described reduction applied to any set of $\leq \lambda$ $L(Q_\alpha)$-sentences shows the reverse inequalities.

We are finally in a position to apply Theorem 4.3.6. Using this theorem for each $\lambda \leq \kappa$ from the above claim we get:

$$h_\lambda(L(Q_\alpha)) = h_\lambda(L(\Omega)) = \beth^\kappa_{wo_\lambda(L(\Omega))} = \beth^\kappa_{wo_\lambda(L(Q_\alpha))}.$$

In particular, taking $\lambda = 1$, if the $L(Q_\alpha)$-sentence ϕ has a model of cardinality $\geq \beth^\kappa_{wo(L(Q_\alpha))}$ then ϕ has arbitrarily large models.

Example 4.3.8: (Application to $L_{\kappa^+\omega}$). Let $L_0 = \{ c_\zeta : \zeta < \kappa \}$ and let $\Omega = \{ (\kappa, (\zeta)_{\zeta<\kappa}) \}$. Ω consists of one structure with universe κ (where κ is conceived as the set of all ordinals less than κ). If ϕ_0 is the $L_{\kappa^+\omega}$-sentence $\bigwedge_{\zeta<\eta<\kappa} \sim(c_\zeta = c_\eta) \wedge \forall x \bigvee_{\zeta<\kappa}(x = c_\zeta)$ then $N \models_{L_{\kappa^+\omega}} \phi_0$ iff $N \cong (\kappa, (\zeta)_{\zeta<\kappa})$ for any L_0-structure N. With Ω and L_0 specified we get an Ω-logic $L(\Omega)$. As in the previous example it is assumed that for any language L under consideration, $L_0 \cup \{U\} \subseteq L$.

Suppose now that L is a language and ϕ is a sentence of $L_{\kappa^+\omega}$. We find a languge K containing the unary relation symbol V and with $L_0 \cup \{U\} \subseteq K$ which has the following property:

(*) Let $\alpha : L \to K$ be the 0-morphism which is the identity except that $\forall \longmapsto V$. Then there is a set of $\leq \kappa$ sentences of $K_{\omega\omega}$

such that: Models of $\phi = \{M^{-\alpha}: M$ is an Ω-model and $M \models_{\Omega} \Sigma\}$. (*) says that an L-structure $N \models_{L_{\kappa^+\omega}} \phi$ iff N is of the form $(M^{(V)}) \restriction L$ for some M with $M \models_{\Omega} \Sigma$. Result (*) thus gives the desired reduction of the $L_{\kappa^+\omega}$-sentence ϕ to the set Σ of $L(\Omega)$-sentences.

The idea behind the reduction is to replace a disjunction of $\leq$ κ formulas by an existential quantification over the set U. First, to each sentence ψ of $L_{\kappa^+\omega}$ we associate by recursion an $L_{\omega\omega}$-sentence ψ^*:

(i) $\psi^* = \psi$ if ψ is atomic.

(ii) $(\sim\psi)^* = \sim(\psi^*)$, $(\psi_1 \vee \psi_2)^* = \psi_1^* \vee \psi_2^*$.

(iii) $(\exists x\psi)^* = \exists x(V(x) \wedge \psi^*)$.

The purpose of (i)-(iii) is to relativize ψ to the set V. Suppose now that $\psi(x)$ is a formula of the form $\bigvee_{\zeta<\kappa} \psi_\zeta(x)$ with 1 free variable. (The generalization to n free variables is straightforward; We consider formulas with finitely many free variables because we will only be dealing with subformulas of sentences of $L_{\kappa^+\omega}$.) Let P_ψ be a new binary relation symbol. Then $(\psi(x))^*$ is:

(iv) $\exists y(U(y) \wedge P_\psi yx)$.

For ψ as in (iv) let D_ψ be the set $D_\psi = \{\forall x(P_\psi c_\zeta x \leftrightarrow \psi_\zeta^*(x)): \zeta<\kappa\}$.

Given a sentence ϕ of $L_{\kappa^+\omega}$ let K be the language consisting of (a) $L \cup \{V\}$ and (b) all the predicates P_ψ for any subformula ψ of ϕ which is an infinite disjunction. Then Σ is the following set of sentences of $K_{\omega\omega}$: $\{\phi^*\} \cup \bigcup_\psi D_\psi \cup \{\forall x(V(\bar{x}) \vee U(x))\}$ where the middle union is taken over all subformulas ψ of ϕ which are infinite disjunctions. Clearly $|\Sigma| \leq \kappa$ and an induction argument shows that an Ω-model $M \models_{\Omega} \Sigma$ iff $(M^{(V)}) \restriction L \models_{L_{\kappa^+\omega}} \phi$. This establishes result (*).

Result (*) and the fact that Ω is the set of models of the $L_{\kappa^+\omega}$-

sentence ϕ_0 prove the following two equalities:

$$h(L_{\kappa^+\omega}) = h_\kappa(L(\Omega)),\ wo(L_{\kappa^+\omega}) = wo_\kappa(L(\Omega)).$$

Applying Theorem 4.3.6 we then get:

$$h(L_{\kappa^+\omega}) = h_\kappa(L(\Omega)) = \beth^\kappa_{wo_\kappa(L(\Omega))} = \beth^\kappa_{wo(L_{\kappa^+\omega})} .$$

We noted earlier that wo
We noted earlier that $wo(L_{\kappa^+\omega}) \geq \kappa^+$. Thus $\kappa + wo(L_{\kappa^+\omega}) = wo(L_{\kappa^+\omega})$. A small argument shows from this: $\beth^\kappa_{wo(L_{\kappa^+\omega})} = \beth_{wo(L_{\kappa^+\omega})}$. Hence we obtain the final result $h(L_{\kappa^+\omega}) = \beth_{wo(L_{\kappa^+\omega})}$ which is an Upward Lowenheim-Skolem result for the logic $L_{\kappa^+\omega}$.

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