AN EXTENSION OF THE KREIN-MILMAN THEOREM AND APPLICATIONS

by

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ABSTRACT

The Krein-Milman Theorem says that each compact, convex subset of a locally convex space is the closed convex hull of its extreme points. In the case of a separable Banach Space several collections of extreme points are known to be dense in the whole set of extreme points (e.g. the set of exposed points [5; theorem 4]; the set of denting points [8; remarks following definition44]). Consequently these sets can be used instead of the whole set of extreme points to generate compact convex sets. In this thesis we examine such a dense subset of extreme points in the context of less structured separable locally convex spaces. We also examine some applications of the resulting extended Krein-Milman Theorem.

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INTRODUCTION

This paper might well have been called the chipping Lemma and its applications rather than its present title, since the chipping lemma is the main tool in extending the Krein-Milman theorem. Breifly, given a separable, Hausdorff, locally convex space and a continuous pseudonorm p defined on it, the chipping lemma states that for any compact, convex set, a small (of p-diameter less than a given $\varepsilon > 0$) but non-empty set can be "chipped away" leaving intact the properties of compactness and convecity of the remainder.

In chapter 1 of this paper this result of I.Namioka and E. Asplund [2] is motivated and slightly generalized. The proof in [2] of the Ryll-Nardzewski fixed point theorem using the chipping lemma is reproduced, and some applications of it are examined to motivaten this line of research historically. Following Namioka in [1], the lemma is then applied to extend the Krein-Milman theorem. The method of proof here is the same as Namioka's, however the notation has been somewhat simplified, and the procedures motivated.

We begin chapter 2 of applications by examining specific locally convex hausdorff spaces to which the extended Krein-Milman theorem can be non-trivially applied, and as in [1; theorem 2.3] the result is reformulated in greater generality. Next, diverse applications are examined. For example the Krein-Milman theorem is extended in a different direction [1; theorem 3.6] -- in a Frechet space where second dual is quase-separable relative to the strong topology, the closed, bounded (not necessarily compact), convex subsets are the closed convex hull of their extreme points. This and other applications from [1; section 3] are simplified by the addition of numerous details to the proofs. We conclude with a slight generalization of the Ryll-Nardzeurki fixed point theorem [1; theorem 3.7].

Chapter 0, the introduction, provides ample background theory through which this thesis should be accessible to any student having a first functional analysis course.

Section 0

Preliminary Notations Definitions and Theorems

Notations:

Number Spaces

$\mathbb{N}$ denotes the set of natural numbers.

$\mathbb{R}$ denotes the set of feal numbers.

$\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{C}^n$ denotes the Cartesian product of C with itself n-times.

inf(X) is the infemum of a set X of real numbers.

(α,β) , $[\alpha,\beta]$ denotes the open, closed intervals from α to β , respectively.

Set Theory

AXB is the Cartesian product of the sets A and B .

$\prod_{i\in I} P_i$ is Cartesian product of the sets p_i $i \in I$.

$A \sim B$ is the set of elements which are in the set A but not in the set B .

comp(A) is the complement of the set A .

Topology

Cl(K) denotes the closure of the set K .

int(K) denotes the interior of the set K .

G_δ -sets are sets formed by taking countable intersections of open sets.

<u>Linear Algebra</u>

dim (X) denotes the dimension of vector space X .

Co(K) denotes the set of convex combinations of a subset K of some vector space.

rational convex combinations refers to convex combinations with rational coefficients.

<u>Functional Analysis</u>

For $(X,\mathcal{J})$ a topological vector space the scalar field of X will be presumed to be $\mathbb{R}$ unless otherwise stated. By a well known result [6; remarks preceding theorem 3.2], the theory herein presented applies to complex topological vector spaces as well.

A subset K of X will be described as $\mathcal{J}$-closed (open etc.) to indicatedthat K is closed (open etc.) in $(X,\mathcal{J})$. Similarly a continuous (lower semicontinuous, etc.) function on $(X,\mathcal{J})$ will be described as $\mathcal{J}$-continuous (lower $\mathcal{J}$-semicontinuous, etc.) on X .

<u>Other Topologies on X</u>

For $(X,\mathcal{J})$ a locally convex topological vector space $(X,\mathcal{J})^*$ (or X^* when $\mathcal{J}$ is understood) will denote the continuous dual of $(X,\mathcal{J})$.

The weak topology on $(X,\mathcal{J})$ wwill be denoted by $(X,\mathcal{J}_w)$.

Topologies on X*

<u>Topologies on X*</u>

$(X,\mathcal{J})^*$ with the weak-star topology will be denoted (X^*,w^*) . $(X,\mathcal{J})^*$ with the strong topology is denoted as (X^*,s) and is the topology of uniform convergence on bounded sets in $(X,\mathcal{J})$. That is: a local base for (X^*,s) is given by $\{B^o : B$ is bounded in $X\}$ where $B^o = \{f \in X^* : f(x) \leq 1$ for every $x \in B\}$ is the polar of B.

Other Derived Topologies

For $(X,\mathcal{J})$ a topological vector space and $K \subseteq X$, $(K,\mathcal{J})$ denotes K with the induced topology.

DEFINITIONS

Pseudo Norms

0.1 Definition: A pseudo-norm (called semi norms by some authors) p on a vector space X is a map $p: X \rightarrow [0,\infty)$ such that

$$\left\{\begin{array}{l} p(x+y) \leq p(x) + p(y) \\ p(\alpha x) = |\alpha|\, p(x) \end{array}\right\} \begin{array}{l} \text{for each } x,y \in X \\ \text{and } \alpha \in \mathbb{R} \end{array}.$$

0.2 Definition: Let $(X,\mathcal{J})$ be a topological vector space and p a pseudo-norm on X. P is lower $\mathcal{J}$-semicontinuous if and only if

$$\{x: p(x) \leq 1\} \text{ is } \mathcal{J}\text{-closed.}$$

0.3 Remark: A pseudo-norm p on a vector space X generates a topology on X in the same manner as does a norm. That is, a local base for the topology is given by $\{B_\varepsilon\}_{\varepsilon>0}$ where $B_\varepsilon = \{x: p(x) < \varepsilon\}$. Of course this topology is not in general hausdorff.

0.4 Notation: The topology derived from pseudo-norm p will be denoted $\mathcal{J}p$.

0.5 Definition: Let $\{p_\alpha\}_{\alpha\in I}$ be a set of pseudo-norms on a locally convex topological vector space $(X,\mathcal{J})$. $\{p_\alpha\}_{\alpha\in I}$ determines $\mathcal{J}$ means that a net $\{x_\delta\}$ will $\mathcal{J}$-converge to some x in X precisely when $\lim_{\delta} p_\alpha(x_\delta - x) = 0$ for every $\alpha \in I$.

<u>0.6 Definition</u>: Let A be a convex, balanced, absorbing set in a vector space X . The Minkowsky functional on A (denoted μ_A) is defined by $\mu_A(x) = \inf\{t > 0: t^{-1}x \in A\}$. (By [6: theorem 1.35] it is easily seen that μ_A is a pseudo-norm on X .

<u>0.7 Definition</u>: A pseudo metric d on a vector space X is a map $d: X \times X \to [0,\infty)$ such that:

$$d(x,x) = 0$$

$$d(x,y) = d(y,x)$$

and $d(x,y) \le d(x,z) + d(z,y)$ for each $x,y,z \in X$.

$d(x,y) = 0$ where $x \neq y$ may occur.

d generates a topology on X in the usual way.

<u>Properties of Sets</u>

<u>0.8 Definition</u>: A subset of a topological space is said to be <u>nowhere dense</u> if its closure has empty interior.

<u>0.9 Definition</u>: A subset of a topological space S is of <u>first category in S</u> if and only if it is a countable union of nowhere dense sets.

<u>0.10 Definition</u>: A subset of a topological space S is said to be of <u>second catogory in S</u> provided that it is not of first category in S .

PRELIMINARY THEOREMS

<u>Classical Results</u>

The first four well known theorems are stated (without proof) due to their importance in the results of this thesis.

0.11 Theorem: (Krein-Milman) [6; theorem 3.21]

Suppose X is a locally convex topological vector space. If $K \subseteq X$ is compact and convex, then K is the closed convex hull of its extreme points.

0.12 Theorem: [13; V.8.3 Lemma 5 Page 440]

Let Q be a compact set in a locally convex linear topological space X whose closed convex hull is compact. Then the only extreme points of $C\ell[C_{\sigma}(Q)]$ are points of Q .

0.13 Theorem: (Baire) [6; theorem 2.2]

If S is either

(a) a complete metric space, or

(b) a locally compact Hausdorff space,

then the intersection of every countable collection of dense open subsets of S is dense in S .

0.14 Theorem: (Markov-Kakutani) [13: V.10.5 Page 456 theorem 6]

Let K be a compact convex subset of a Linear topological space X . Let $\mathcal{F}$ be a commuting family of continuous linear mappings which map K into itself. Then there exists a point $p \in K$ such that $Tp = p$ for each $T \in \mathcal{F}$.

CATEGORY THEOREMS

0.15 Definition: Any space which satisfies the conclusion of Baire's Theorem is called a Baire Space.

<u>0.16 Remark</u>: Every Baire space S is of second category in itself. Indeed, the assumption that $S = \bigcup_{i=1}^{\infty} C_i$ where each C_i is nowhere dense means that $S = \bigcup_{i=1}^{\infty} C\ell(C_i)$ where $\mathrm{int}[C\ell(C_i)] = \emptyset$ for every C_i . Clearly $\mathrm{comp}(C\ell(C_i))$ is open dense in S and thus $\bigcap_{i=1}^{\infty} \mathrm{comp}\,(C\ell(C_i))$ is dense in S , hence non-empty. Consequently $\bigcup_{i=1}^{\infty} C\ell(C_i) \neq S$ and the assumption is contradicted.

To deal with lower $\mathcal{J}$-semicontinuous pseudo-norms on a topological vector space $(X,\mathcal{J})$, and the subsequent $\mathcal{J}$-closed subsets we restate part of Baire's theorem for closed sets.

<u>0.17 Corollary</u>: [1; lemma 1.1]

Let X be a compact Hausdorff space. Let $\{C_i\}_{i\in\mathbb{N}}$ be a countable collection of closed subsets of X such that $X = \bigcup_{i=1}^{\infty} C_i$. Then $\bigcup_{i=1}^{\infty} \mathrm{int}(C_i)$ is dense in X .

<u>Proof</u>: Let $U \neq \emptyset$ be an arbitrery open set in X . Then $(U,\mathcal{J})$ is a locally compact Hausdorff space and $U = \bigcup_{i=1}^{\infty} U \cap C_i$ where each $U \cap C_i$ is closed in $(U,\mathcal{J})$. But $(U,\mathcal{J})$ is a Baire space and hence second category in itself (remark 0.15) thus $U \cap C_i$ has non-empty interior for some C_i and since U was arbitrarily chosen, $\bigcup_{i=1}^{\infty} C_i$ is dense in $(X,\mathcal{J})$.

<u>EVALUATION MAPS</u>

<u>0.18 Definition</u>: Let E be a topological vector space, and E^* its dual. For every $x \in X$ define $F_x\colon E^* \to \mathbb{R}$ by $F_x(f) = f(x)$. Then $F_x \in (E^*,s)^*$ and the map $I\colon E \to (E^*,s)^*$ defined by $I(x) = F_x$ called the <u>evaluation map</u> on E is one-to-one.

The following theorem and its proof are adopted from [10; 33.2 page 346] and [10; 36.5 page 373].

<u>0.19 Theorem:</u> For E a Frechet Space, the evaluation map $I: E \to [(E^*,s)^*,s]$ is continuous.

<u>Proof:</u> Let V be a 0-neighbourhood in $[(E^*,s)^*,s]$. Since $\{B^o: B$ is strongly bounded in $E^*\}$ forms a local base for $[(E^*,s)^*,s]$ we can find $B^o \subseteq V$, B strongly open in E^*.

$I^{-1}(B^o) = \{x: |f(x)| \leq 1$ for every $f \in B\}$ is absorbing in E. Indeed if $y \in E$ then by the continuity of scalar multiplication $\{y\}$ is bounded in E and so $\{y\}^o$ is a strong 0-neighbourhood in E^*. Slince B is strongly bounded in E^* there exists $m > 0$ such that $B \subseteq m\{y\}^o = m\{f: |f(y)| \leq 1\} = \{f: |f(y)| \leq m\}$. Thus $y \in mI^{-1}(B^o)$.

y being arbitrarily chosen, we get that $E = \bigcup_{m \in \mathbb{N}} mI^{-1}(B^o)$ and $mI^{-1}(B^o)$ is closed in E for every m. But E is Frechet, hence a Baire Space, hence second category in itself, (0.15 remark).

Thus $mI^{-1}(B^o)$ has non-empty interior for some m, and so $I^{-1}(B^o)$ has non-empty interior. But 0 is internal to $I^{-1}(B^o)$ in the algebraic sense (i.e. each line passing through 0 has some segment in $I^{-1}(B^o)$) and hence by a well-known theorem 0 is interior to $I^{-1}(B^o)$ (topologically). This proves that I is continuous.

<u>0.20 Corollary:</u> $I: E \to I(E)$ is a homeomorphism from the Frechet space E onto its image $I[E]$ in $[(E^*,s)^*,s]$.

<u>w*-compactness</u>

As an easy consequence of the Banach-Alaoglu theorem

[6; theorem 3.15 page 66] we state the following proposition.

<u>0.21 Proposition</u>: Let E be a Banach space. Let $K \subseteq E^*$ be norm bounded. Then $K_1 = w^* - C\ell(K)$ is w^*-compact.

Chapter 1

THE CHIPPING LEMMA AND THE KREIN-MILMAN EXTENSION THEOREM

In this chapter, an extension of the Krein-Milman theorem is obtained by means of the Chipping Lemma (I. Namioka [1]). A proof of the Ryll-Nardzewski fixed point theorem (I. Namioka and E. Asplund [2]), to which the chipping lemma was originally applied, is presented, and various applications of the Ryll-Nardzewski fixed point theorem are sketched to provide some motivation for this area of research.

1.1 Definition: An affine map T from a convex set K into itself is a map which satisfies $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ for every $x,y \in K$ and every $\alpha,\beta \geq 0$, $\alpha + \beta = 1$.

A subset Q of K is T-invariant if $T(Q) \subseteq Q$.

For $\mathcal{S}$ a collection of affine maps, Q is $\mathcal{S}$-invariant if and only if Q is T-invariant for each $T \in \mathcal{S}$.

1.2 Definition: A collection $\mathcal{S}$ of affine maps from K into K is a semigroup if it is closed with respect to composition of mappings. A semigroup $\mathcal{S}$ of affine maps is finitely generated if all members of $\mathcal{S}$ are compositions of a fixed finite subcollection of $\mathcal{S}$.

1.3 Definition: Let $(E,\mathcal{J})$ be a locally convex topological vector space. Let $Q \subseteq E$ and let $\mathcal{S}$ be a semigroup of affine maps such that Q is $\mathcal{S}$-invariant. $\mathcal{S}$ is $\mathcal{J}$-noncontracting on Q if for each distinct pair $x,y \in Q$, $Q \notin \mathcal{J} - C\ell(\{Tx - Ty: T \in \mathcal{S}\})$.

1.4 Proposition: Let $Q \subseteq (X,\mathcal{J})$, and $\mathcal{S}$ a semigroup of affine maps from Q into Q . Then $\mathcal{S}$ is $\mathcal{J}$-noncontracting if and only if for every distinct $x,y \in Q$ there exists a $\mathcal{J}$-continuous pseudo-norm p on Q such that

$$\inf_{T \in \mathcal{S}} \{p(Tx - Ty)\} > 0 \quad .$$

Proof: $\mathcal{S}$ is $\mathcal{J}$-noncontracting. Thus for every distinct $x,y \in Q$ there exists a balanced, convex, absorbing 0-neighbourhood $V \subseteq X$ such that $Tx - Ty \notin V$ for every $T \in \mathcal{S}$. The Minkowsky functional μ_V is the required continuous pseudo-norm.

Conversely assume that for every distinct $x,y \in Q$ there is a $\mathcal{J}$-continuous pseudo-norm p such that $\delta = \inf_{T \in \mathcal{S}}\{p(Tx - Ty)\} > 0$. Let $V = \{x \in Q: p(x) < \delta\}$. Then V is a 0-neighbourhood in X, and $Tx - Ty \notin V$ for every $T \in \mathcal{S}$.

1.5 Theorem: (Ryll-Nardzewski) [I.Namioka, E. Asplund; 2]

Let $(E; \mathcal{J})$ be a locally convex hausdorff topological vector space. Let $Q \subseteq E$ be non-empty, convex and weakly compact. Let $\mathcal{S}$ be a $\mathcal{J}$-noncontracting semigroup of weakly continuous affine maps of Q into Q . Then there is a point $z \in Q$ such that $Tz = z$ for all $T \in \mathcal{S}$. (That is, z is a common fixed point of $\mathcal{S}$ on Q).

Before presenting the proof of this theorem, we examine its principal application to the existance of a left invariant mean on W(G) - the set of weakly almost periodic functions from a locally compact group G into $\mathbb{C}$ (F. Greenleaf [12; chapter 3]).

1.6 Definition: Let G be a locally compact group. B(G) is the space of all bounded complex-valued functions on G equipped with the supnorm

$\|f\|_\infty$. CB(G) is the subspace of continuous functions.

<u>1.7 Definition</u>: Let G be a locally compact group. Let $f \in CB(G)$. The left orbit of f is defined by $LO(f) = \{_x f: x \in G\}$, where $_x f(y) = f(x^{-1}g)$, for every $g \in G$.

<u>1.8 Definition</u>: Let G be a locally compact group. $f \in CB(G)$ is weakly almost periodic if and only if $LO(f)$ is relatively weakly compact in CB(G) . (That is the weak closure of LO(f) is weakly compact in CB(G) .) The space of all such funtions is denoted by W(G) .

<u>1.9 Definition</u>: A linear functional m on W(G) is a <u>mean</u> if

$$m(\bar{f}) = \overline{mf} \qquad \text{for all } f \in W(G) .$$

$\bar{f}$ denotes the conjugate function to f .

and $\inf_{x \in G}\{f(x)\} \leq m(f) \leq \sup_{x \in G} \{f(x)\}$ for all real valued $f \in W(G)$ f W(G)

If furthermore $m(_x f) = m(f)$ for each $x \in G$, and f in W(G) then m is said to be a <u>left invariant mean</u> on W(G) .

<u>1.10 Theorem</u>: [12; pages 38-40]

Let G be a locally compact group. Then W(G) has a left invariant mean.

<u>Sketch of Proof</u>: Let Q(f) be the weakly closed conves hull of LO(f) , where $f \in W(G)$. Then Q(f) is non-empty convex and weakly compact. Define $L_x : Q(f) \to Q(f)$ by $L_x(h) = x^{-1}h$ for $x \in G$. Then L_x is an affine map. Also $S = \{L_x: x \in G\}$ is norm-noncontracting. Indeed, if $f_1 \neq f_2$ then $\|f_1 - f_2\| > 0$, and hence $0 \notin C\ell\{L_x f_1 - L_x f_2\}_{x \in G}$

$= C\ell\{L_x(f_1 - f_2)\}_{x \in G}$ since $\inf_{x \in G} \|L_x(f_1 - f_2)\| = \inf_{x \in G} \|f_1 - f_2\| > 0$. Since S is a semigroup of weakly continuous maps which are norm non-contracting, the Ryll-Nardzewski fixed point theorem yeilds some $h_f \in Q(f)$ such that $L_x(h_f) = h_f$ for every $x \in G$. Then h_f is a constant function on G , since $h_f(xg) = h_f(g)$ for all $x, g \in G$. Hence for $g = x^{-1}$ one gets $h_f(e) = h_f(g^{-1})$ for all $g \in G$ so h_f takes the constant value $h_f(e)$ on G .

A detailed proof that; h_f is the unique fixed point of S in $Q(f)$; that the map $m\colon W(G) \to \mathbb{C}$ which assigns to each $f \in W(G)$ the value of the constant function h_f , is linear; and that $\inf_{x \in G} f(x) \leq m(f) \leq \sup_{x \in G} f(x)$ for all real valued $f \in W(f)$, is shown in [12; page 39-42] for details). Given then that m is a mean, it is clearly left invariant since $LO(f) = LO(xf)$ for all $x \in G$, thus the unique fixed point of $LO(f)$ coincides with that of $LO(f_x)$.

We now present I. Namioka's and E. Asplunds proof of the Ryll-Nardzewski fixed point theorem. [2‖2] (Theorem 1.5)

<u>Proof</u>: It suffices to prove the result for S a finitely generated set of affine maps. Indeed, the assumption that S has no common fixed point, but that each finite subset of S does, leads to a contradiction as follows:

Since S has no common fixed point $x \in Q$, $T_x x \neq x$ for some $T \in S$. That is $Q = \bigcup_{T \in S}\{x\colon\ Tx - x \neq 0\}$. Now $\{x\colon\ T_x x - x \neq \emptyset\} = \text{comp}(\{\colon\ Tx - x = 0\})$ is weakly open for each $T \in S$, since T is weakly continuous. Q is weakly compact, thus $Q = \bigcup_{i=1}^{n}\{x\colon\ T_{ix}x - x \neq 0\}$ for finitely many $T_i \in S$. This says that $\{T_1, \ldots, T_n\}$ has no common fixed point, which is a contradiction. Thus we assume that $\{T_1, \ldots, T_m\}$ is a finite generating set for S .

Consider $T_o = \frac{T_1 + \dots + T_m}{m}$

Q is convex, therefore T_o: $Q \to Q$. Also T_o is weakly continuous and affine. Thus the Markov-Kakutani (0.14) applies to T_o , and there exists a fixed point x_o of T_o .

We show that x_o is the required fixed point for S .

Assume not:

Without loss of generality we can assume that x_o is a fixed point of no $T_i \in S$ ((We simply discard those for which $T_i(x_o) = x_o$ and work with the remaining $\mathcal{J}$-noncontracting subsemigroup.)

Since S is $\mathcal{J}$-noncontracting, by proposition 1.3, there is a $\mathcal{J}$-continuous pseudo-norm p and an $\varepsilon > 0$ such that:

(1) $$p(TT_i(x_o) - T(x_o)) > \varepsilon_o \text{ for every } T_i \in S , i = 1 \dots m .$$

Let $K = \mathcal{J} - C\ell[C_o(\{Tx_{xo}: T \in S\})]$. K is weakly compact since K is a subset of the weakly compact set Q , and K is weakly closed since it is $\mathcal{J}$-closed and convex. Also K is $\mathcal{J}$-separable since the rational convex combinations of the countable collection $\{Tx_{xo}: T \in S\}$ (S is finitely generated) is a countable $\mathcal{J}$-dense subset of K .

If we further assume now that there exists a closed, convex $C \subsetneq K$ such that $C \neq K$, but that the p-diam$(K \setminus C) \leq \varepsilon_o$ (Chipping lemma) then a contradiction can be achieved as follows:

Let C be the above postulated subset of K . Now there is some $S \in S$ such that $Sx_o \in K \setminus C$ since $K \setminus C$ is open in K .

$$Sx_o = ST_o(x_o) = \frac{ST_1x_o + ST_2x_o + \dots + ST_mx_o}{m}$$

C is convex, thus $ST_i x_o \in K \setminus C$ for sôme $i = 1, \ldots, m$. Hence $p(ST_i x_o - S_x) \leq$ p-diam$(K \setminus C) \leq \varepsilon_o$ which contradicts (1).

It remains to prove the chipping lemma.

1.11 Chipping lemma: (I. Namioka, E. Asplund [2])

Let $(E, \mathcal{J})$ be a locally convex, hausdorff topological vector space. Let $K \subsetneq E$ be non-empty, weakly compact, convex, and such that K is contained in some $\mathcal{J}$-separable set in E . Then for every $\varepsilon > 0$ there is a $\mathcal{J}$-closed convex $C \subseteq K$ such that $C \neq K$, and the p-diam$(K \setminus C) \leq \varepsilon$.

Remark: In [2] K is taken to be $\mathcal{J}$-separable.

(outline of proof): The method of the proof consists in taking for some $u \in \text{ext}(K)$, convex combinations. $C_{\hat{r}} = \{\lambda u_i + (1 - \lambda)u: \ 0 < \hat{r} < 1, \lambda \in [\hat{r}, 1], u_i \in \text{ext}(K) \setminus \{u\}\}$. Then $u \notin C_{\hat{r}}$ since $\lambda \neq 0$ and u is an extreme point of K . As $\hat{r}$ tends towards 0 , $C_{\hat{r}}$ tends towards K . Eventually some $C_{\hat{r}}$ is chosen as C .

Of course the set C so derived does not conform to the requirements of this lemma, since it is neither closed nor convex.

The procedure which we followiis to find a weakly open set W such that p-diam$(W) \leq \frac{\varepsilon}{2}$ and such that W contains an extreme point of K .

δ Convex combination $C_{\hat{r}}$ of the form

$$C_{\hat{r}} = \{\lambda x_1 + (1 - \lambda)x_2: \ 0 < \hat{r} < 1 \ , \lambda \in [\hat{r}, 1], \ x_1 \in \mathcal{J}\text{-}C\ell[C_o(D \setminus W)], \ x_2 \in \mathcal{J}\text{-}C\ell[C_o(D \cap N)]\}$$

where D is the weak closure of $\text{ext}(K)$ will satisfy this lemma for some sufficiently small $\hat{r}$.

(proof of chipping lemma): Let $S = \{x: \ p(x) \leq \frac{\varepsilon}{4}\}$ / S is convex.

(1) Since p is $\mathcal{J}$-continuous, S is $\mathcal{J}$-closed. Thus S is weakly closed.

Next,

(2) since p is $\mathcal{J}$-continuous countably many translates of S cover K. This is true since K is contained in a $\mathcal{J}$-separable set, and $\mathcal{J}\text{-int}(S) \neq \emptyset$ Since S is weakly closed, the translates cover $D = \mathcal{J}_w\, C\ell[\text{ext}(K)]$.

But D is a weakly closed subset of the weakly compact set K . Therefore D is weakly compact. Thus $(D, \mathcal{J}_w)$ is a Baire Space, and hence second category in itself. (remark 0.15). Therefore there is a $k \in K$ such that $\mathcal{J}_w - \text{int}(k + S) \cap D \neq \emptyset$.

Clearly then $\text{ext}(K) \cap W \neq \emptyset$ where $W = \mathcal{J}_w\text{-int}(k + S)$. Let $u \in \text{ext}(K) \cap W$. Let

$$C_r = \{\lambda x_1 + (1 - \lambda)x_2 :\ 0 < r < 1,\ \lambda \in [r,1],\ x_1 \in \mathcal{J}\text{-}C\ell[C_o(D \setminus W)],$$
$$x_2 \in \mathcal{J}\text{-}C\ell[C_o(D \cap W)]\} \ .$$

We show that C_r is $\mathcal{J}$-closed, convex, that $C_r \neq K$, and that C_r can be made p-arbitrarily small by choosing r sufficiently small.

Consider the jointly continuous map

$f_r : \mathcal{J}\text{-}C\ell[C_o(D \setminus W)] X \mathcal{J}\text{-}C\ell[C_o(D \cap W)] X [r,1] \rightarrow K$ defined by

$f_r(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2$. Clearly the image of this map in K is C_r , which is thus shown to be $\mathcal{J}$-closed since the domain is compact.

C_r is convex, since if α, β, γ $[r,1]$, $x_1, y_1 \in \mathcal{J}\text{-}C\ell[C_o(D \setminus W)]$, and $x_2, y_2 \in \mathcal{J}\text{-}C\ell[C_o(D \cap W)]$ then $\gamma(\alpha x_1 + (1 - \alpha)x_2) +$

$\gamma(\alpha x_1 + (1 - \alpha)x_2) + (1 - \gamma)(\beta y_1 + (1 - \beta)y_2) = \gamma r z_1 + (1 - r)z_2$ where

$\delta = \beta - \gamma\beta + \gamma\alpha \in [r,1]$ since $\gamma \leq \beta$

$\beta - \gamma\beta + \gamma\alpha \leq \beta - \gamma\beta + \gamma \leq \beta + \gamma(1 - \beta) \leq \beta + (1 - \beta) = 1$, and

$\beta - \gamma\beta + \gamma\alpha \geq \beta - \beta\gamma + \gamma r \geq \beta(1 - \gamma) + \gamma r \geq (1 - \gamma)r + \gamma r = r$.),

$$z_1 = \frac{\gamma\alpha}{\beta - \gamma\beta + \gamma\alpha} x_1 + \left(1 - \frac{\gamma\alpha}{\beta - \gamma\beta + \gamma\alpha}\right) y_1 \quad \text{and}$$

$$z_2 = \frac{\gamma - \gamma\alpha}{1 - \beta + \gamma\beta - \gamma\alpha} x_2 + \left(1 - \frac{\gamma - \gamma\alpha}{1 - \beta + \gamma\beta - \gamma\alpha}\right) y_2$$

and the coefficients of x_1 and x_2 can be shown to be in $[0,1]$ hence $z_1 \in \mathcal{J}\text{-}C\ell(C_o(D \setminus W))$ and $z_2 \in \mathcal{J}\text{-}C\ell(C_o(D \cap W))$.

u (the extreme point of K found in W) $\notin C_r$, since if $u \in C_r$, then u would be an extreme point of C_r . By theorem 0.12 this would imply $u \in D \setminus W$ contradicting $u \in W$. This shows $C_r \neq K$.

Finally we show that the p-diam of $C_{\hat{r}}$ is arbitrarily small for small $\hat{r}$. Consider $f_{\hat{r}}$ defined above with $\hat{r} = 0$. The image of f_o is $\mathcal{J}$-closed and convex and contains all of the extreme points of K , hence it equals K (Krein-Milman theorem). That is every $x \in K$ can be written $x = \lambda x + (1 - \lambda)x_2, \lambda \in [0,1], x_1 \quad \mathcal{J}\text{-}C\ell[C_o(D \setminus W)]$

$$x_2 \in \mathcal{J}\text{-}C\ell[C_o(D \cap W)] \ .$$

Consequently for any $y \in K \setminus C_{\hat{r}}$ $(\hat{r} \neq 0)$ $y = \lambda x_1 + (1 - \lambda)x_2$ $\lambda \in [0, \hat{r})$. Therefore $p(y - x_2) = \lambda p(x_1 - x_2) \leq \hat{r}d\ell$ where $C\hat{d} = $ p-diam(K) . But,

(3) p is $\mathcal{J}$-continuous, therefore $\{x: \ p(x) \leq 1\}$ is weakly open and K is weakly compact, hence covered by finitely many translates of $\{x: \ p(x) \leq 1\}$. That is $C\hat{d} = $ p-diam(K) $< \infty$.

now $x_2 \in \mathcal{J}\text{-}C\ell[C_o(D \cap W)]$ which has p-diam $\leq \frac{\varepsilon}{2}$ thus p-diam $(K \setminus C_r)$

$= \underset{y_1, y_2 \in K \setminus C_{\hat{r}}}{\text{supremum}} \{p(y_1 - y_2)\} \leq \underset{y_1, y_2 \in K \setminus C_{\hat{r}}}{\text{supremum}} \{[p(y_1 - x_2) + p(x_2 - x_2') +$

$+ p(x_2' - y_2)]\} \leq \frac{\varepsilon}{2} + 2\hat{r}d$ (where $x_2, x_2' \in \mathcal{J}\text{-}C\ell[C_o(D \cap W)]$)

$C = C_{\frac{\varepsilon}{4d}}$ satisfies the chipping lemma, and completes the proof of the Ryll-Nardzewski fixed point theorem.

1.12 Remark:

Definition: A subset K of a Banach space E is called dentable if for each $\varepsilon > 0$ there is a $u \in K$ such that $u \notin C\ell[C_o(K \setminus \{y: \|u - y\| \leq \varepsilon\})]$. $u \in K$ is a denting point of K if

$$u \notin C\ell[C_o(K \setminus \{y: \|u - y\| \leq \varepsilon\})] \text{ for each } \varepsilon > 0 .$$

Since the set W in the proof is defined as a translate of $\{x: p(x) \leq \frac{\varepsilon}{4}\}$ and since $u \notin C\ell[C_o(K \setminus W)]$ we might say that K is "p-dentable". Since the point $u \in W$ was dependent on ε we are not entitled to denote u aas a "p-denting point". (More on denting points in sequel).

A slight modification of the chipping Lemma leads into another paper by Namioka [1], where the lemma forms one of the two basic technical arguments.

The main thrust of Neighbourhoods of Extreme points [1] is towards an extension of the Krein-Milman theorem. Let K be a compact, convex subset of some hausdorff topological vector space $(E,\mathcal{J})$. This stronger version is achieved by determining a dense subset of special points of $ext(K)$. The closed convex hull of this subset is clearly, again K .

Since the pseudo-norm p of the chipping lemma is $\mathcal{J}$-continuous, the topology on $(K,\mathcal{J}_p)$ which p generates on K satisfies that each $\mathcal{J}_p$-open set contains a $\mathcal{J}$-open set. Thus, since E is hausdorff, the following property is readily seen to hold for each $x_o \in K$:

Every (K,J_p) neighbourhood of x_o contains a (K,J)-open set.

If however p is only lower J-semicontinuous, then for a given point $x_o \in K$, the above property is not guaranteed to hold. This can be more succinctly stated as: the identity map $i\colon (K,J) \to (K,J_p)$ may not be continuous at x_o. We show that the dense subset of special points of ext(K) refered to above are precisely the points of continuity of the identity map i which are in ext(K).

Since the chipping lemma is to be our main tool in proving this assertion, we broaden it to include lower J-semicontinuous pseudo-norms instead of just J-continuous ones. We compensate for this strengthened result by strengthening the separability condition on K.

<u>1.13 Proposition</u>: Let (E,J) be a locally convex, hausdorff topological vector space. Let p be a lower J-semicontinuous pseudo-norm on E. Let $K \subseteq E$ be convex, J-compact, and such that it is contained in some J_p-separable set. Then for each $\varepsilon > 0$ there is a J-compact, convex $C \subseteq K$ such that p-diam$(K \setminus C) \leq \varepsilon$, but $C \neq K$.

<u>proof</u>: To modify the chipping lemma we show that all of the steps justified by the J-continuity of p in the proof can be obtained with the present hypothesis. Since the J_w-compactness of K follows from the J-compactness of K, the result will follow.

The relevant steps have been numbered (1), (2) and (3) in the proof of Theorem 1.5.

(1) - S is J-closed since p is J-continuous. Since p in our present hypothesis is lower J-semicontinuous, $\frac{4}{\varepsilon}S = \{x\colon\ p(x) \leq 1\}$ is J-closed. Since K in our present hypothesis is [illegible]

[illegible] set we need only that J_p-int$(S) \neq \emptyset$. This [illegible]

(2) -- Since p is $\mathcal{J}$-continuous countably many translates of S cover K .--

Since K in our present hypothesis is contained in a $\mathcal{J}_p$-separable set we need only that $\mathcal{J}_p\text{-int}(S) \neq \emptyset$. This is clearly true since $S = \{x\colon\ p(x) \le \frac{\varepsilon}{4}\}$.

(3) -- $Cd = p\text{-diam}(K) < \infty$ since p is $\mathcal{J}$-continuous. --

Following is a proof of Cd's finiteness based on the lower $\mathcal{J}$-semicontinuity of p . The proof is modelled on the absorption theorem [3; page 91].

Let $A = \{x\colon\ p(x) \le 1\}$. A is convex and it is $\mathcal{J}$-closed since p is lower $\mathcal{J}$-semicontinuous. Also, since p is defined on all of E , $E = \bigcup_{n\in\mathbb{N}} nA$, and consequently $K = \bigcup_{n\in\mathbb{N}} A \cap K$, where $nA \cap K$ is $\mathcal{J}$-closed for each $n \in \mathbb{N}$. But since $(K,\mathcal{J})$ is a Baire space, it is 2nd category in itself (remark 0.16) thus there exists an $N \in \mathbb{N}$ such that $\text{int}(NA \cap K) \neq \emptyset$. If $n \ge N$, then $\text{int}(nA \cap K) = \text{int}(\{x\colon\ p(x) \le n\} \cap K) \supseteq \text{int}(\{x\colon\ p(x) \le N\} \cap K) = \text{int}(NA \cap K) \neq \emptyset$.

Let U be a $\mathcal{J}$-open 0-neighbourhood and $y \in K$ such that $\emptyset \neq (y + U) \cap K \subseteq \text{int}(nA \cap K)$ for every $n \ge N$. Now E is a locally convex space and K is $\mathcal{J}$-bounded, therefore $K \setminus K$ is also $\mathcal{J}$-bounded. Hence we can find $b \in (0,1)$ such tha $b(K \setminus K) \subseteq U$, from which we get that $nA \supseteq (y + U) \cap K \supseteq [y + b(K \setminus K)] \cap K$.

K is convex, thus $bK + (1 - b)y \subseteq K$. But $y(1 - b) + bK = y + bK - by \subseteq y + b(K - K)$, therefore $y(1 - b) + bK \subseteq y + b(K \setminus K) \cap K \subseteq nA$ for every $n \ge N$.

Let $p(y) = s$. Then $p(\frac{1}{s}y) = 1$ so $\frac{1}{s}y \in A$ which is to say $y \in sA$. Therefore $-y(1 - b) \in sA$ since $1 - b < 1$. This shows that $rA \supseteq \frac{1}{2}[y(1 - b) + bK] - \frac{1}{2}y(1 - b) \supseteq \frac{1}{2}bK$ (A is convex) for each

$r > m = \max(s,N)$.

Thus $K \subseteq \frac{2m}{b} A$ ie $p\text{-diam}(K) \leq \frac{4m}{b} < \infty$.

We are now prepared to examine Namioka's extension of the Krein-Milman theorem [1; theorem 2.2]

1.14 Theorem: Let $(E,\mathcal{J})$ be a Hausdorff locally convex topological vector space. Let p be a lower $\mathcal{J}$-semicontinuous pseudo-norm on E . Let K be a $\mathcal{J}$-compact, convex subset of E such that K is $\mathcal{J}_p$-separable. Then the set of extreme points of K which are also points of continuity of the identity map $i: (K,\mathcal{J}) \to (K,\mathcal{J}_p)$ is a $\mathcal{J}$-dense G_δ set in $\text{ext}(K)$.

Remark: Namioka takes $(E,\mathcal{J}_p)$ separable.

Proof: Let Z be the set of points of continuity of the identity map $i: (K,\mathcal{J}) \to (K,\mathcal{J}_p)$. Then for $u \in K$, $u \in Z$ if and only if for each $\varepsilon > 0$ we can find a $(K,\mathcal{J})$-open neighbourhood of u of p-diam $\leq \varepsilon$.

Setting $\varepsilon = \frac{1}{n}$, and letting n increase through $\mathbb{N}$ we can reformulate this condition to $Z = \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}$ where $B_\varepsilon = \{u \in K:$ U is contained in a $(K,\mathcal{J})$-open set of p-diam $\leq \varepsilon\}$. Note that B_ε is the union of all open sets in $(K,\mathcal{J})$ of p-diameter $\leq \varepsilon$, hence it is open.

We must show that $Z \cap \text{ext}(K)$ is dense in $\text{ext}(K)$. This will be accomplished by showing:

(a) that $B_\varepsilon \cap \text{ext}(K)$ is dense in $\text{ext}(K)$ for each $\varepsilon > 0$,

and (b) $(\text{ext}(K),\mathcal{J})$ is a Baire space.

This will give us that $Z \cap \text{ext}(K) = B_{\frac{1}{n}} \cap \text{ext}(K)$ is a dense G_δ subset of $\text{ext}(K)$.

(a) Let W be an arbitrary open set in $(K,\mathcal{J})$ such that $W \cap \text{ext}(K)$

$\neq \emptyset$. By the chipping lemma we know that K contains a closed convex subset $C \neq K$. $\mathrm{Int}(K \setminus C)$ must contain an extreme point of K , since if $\mathrm{ext}(K) \subseteq C$ and C is closed and convex, then $K \subseteq C$ which cannot be. In other words, $\emptyset \neq \mathrm{ext}(K) \cap \mathrm{int}(K \setminus C) \subseteq B_\varepsilon \cap \mathrm{ext}(K)$. It remains to locate the set C so that it misses a part of W containing an extreme point of K . That is, $\emptyset \neq \mathrm{ext}(K) \cap \mathrm{int}(K \setminus C) \subseteq B_\varepsilon \cap W \cap \mathrm{ext}(K)$.

Let $S = \{x\colon\ p(x) \leq \frac{\varepsilon}{2}\}$. S is $\mathcal{J}$-closed, since p is lower $\mathcal{J}$-semicontinuous. Let $D = \mathcal{J}\text{-}C\ell[\mathrm{ext}(K)]$. D is compact since K is . Since K is $\mathcal{J}_p$-separable, a countable collection of translates of S will cover K and hence D .

That is, $D = \bigcup_{i=1}^{\infty} D \cap (x_i + S)$, $\{x_i\} \subseteq K$, and $D \cap (x_i + S)$ is closed for each $i \in \mathbb{N}$. By corollary 0.17 of the Baire Category theorem $\bigcup_{i=1}^{\infty} \mathrm{int}[D \cap (x_i + S)]$ is dense in D . It follows that $\mathrm{int}[D \cap (x_i + S)] \cap W \neq \emptyset$ for some $x_i \in K$, and clearly this intersection contains an extreme point of K .

But $\mathrm{int}[D \cap (x_i + S)] \subseteq B_\varepsilon$ for every $x_i \in K$. Therefore $\mathrm{ext}(K) \cap B_\varepsilon \cap W \neq \emptyset$.

(b) That $(\mathrm{ext}(K),\mathcal{J})$ is a Baire space is a theorem of Choquet's [4; page 355]. It is translated and included herein for the sake of completeness, and because of the interesting techniques of the proof. Explanatory comments are in italics.

1.15 Theorem: Let E be a locally convex separable space, C a compact, convex, subset of E, A the set of extreme points of C . Then A is a Baire Space.

proof: For all continuous linear functionals f on E , and for all real numbers α , let us denote by $U_{f,\alpha}$(resp $F_{f,\alpha}$) the set of $x \in C$ such that $f(x) < \alpha$ (resp. $f(x) \leq \alpha$) .

The strategy of the proof is to show that $F_{f,\alpha}$ *for which* $x \in U_{f,\alpha}$ *form a basic neighbourhood system for* x *if* $x \in A$.

Next given a sequence $\{V_n\}$ *of dense open sets in* A *and an arbitrary open* V *in* A *, we embed the* V_n*'s and* V *in appropriate sets in* C *. In particular* V *is embedded in* $U = U_{f_1,\alpha_1}$ *for some* $f_1 \in E^*$ *,* $\alpha_1 \in R$.

Next we find a sequence $\{(f_i, \alpha_i)\} \subseteq E^* X\ R$ *such that the* F_{f_i,α_i} *are descending compact, non-empty and such that* F_{f_n,α_n} *intersects with* V *and* V_{n+1} *. The non-empty intersection of the* $\{F_{f_i,\alpha_i}\}$ *will contain a member of each* V_n *, and of* V *, and the result follows.*

Let $x \in A$. We will first show that the set of $F_{f,\alpha}$ for which $x \in U_{f,\alpha}$ form a basic neighbourhood system of x . By the Hahn-Banach Theorem, the intersection of these $F_{f,\alpha}$ is $\{x\}$.

Indeed E^* *separates points on* E *, and thus if* $x,y \in F_{f,\alpha}$ *then* g,β *can be found such that* $y \notin F_{g,\beta}$.

Since the $F_{f,\alpha}$ are compact, it suffices to show that the family of the $F_{f,\alpha}$ for which $x \in U_{f,\alpha}$ is a descending net. Let F_1, f_2, α_1, α_2 be such that $x \in U_{f_1,\alpha_1} \cap U_{f_2,\alpha_2}$. Let C_1, C_2 be the complements of U_{f_1,α_1} , U_{f_2,α_2} in C . Since C_1 and C_2 are compact, the convex hull C_3 of $C_1 \cup C_2$ is compact and $x \notin C_3$ because x is an extreme point.

If $x \in C_3$ *then by theorem 0.12* $x \in C_1 \cup C_2$ *since* $C_1 \cup C_2$ *is compact and* $x \in A$ *. This contradicts* $x \in U_{f_1,\alpha_1} \cap U_{f_2,\alpha_2}$.

Thus there exists a linear continuous funcitonal f on E , and a real number α such that $f(x) < \alpha$, and $f(y) \geq \alpha$ for $y \in C_3$.

separation theorem for convex sets

So $F_{f,\alpha}$ doesn't intersect C_1 or C_2 . That is $F_{f,\alpha} \subseteq U_{f_1,\alpha_1} \cap U_{f_2,\alpha_2}$ and $x \in U_{f,\alpha}$.

Let $\{V_n\}$ be a sequence of open dense subsets of A . It is required to prove that $\cap V_n$ is dense in A . That is it intersects every non-empty open subset V of A . Let $\{U_n\}$, U , be open subsets of C with U_n dense in C , such that $U_n \cap A = V_n$, $U \cap A = V$. One can assume the U_n's and V_n's are descending and that U is of the form U_{f_1,α_1} .

Let $V_n{}' = \bigcap_{i \leq m} V_i$ *and work with* $\{V_n{}'\}$ *instead of* $\{V_n\}$.

This being the case, we will prove the existence of $(f_1,\alpha_1),(f_2,\alpha_2)$, ... with the following properties: $F_{f_{n+1},\alpha_{n+1}} \subseteq U_{f_n,\alpha_n} \cap U_{n+1}$ and $U_{f_n,\alpha_n} \cap A \neq \emptyset$ for each $n \in \mathbb{N}$.

We already have (f_1,α_1) . Suppose we have found $(f_1,\alpha_1), \ldots (f_n,\alpha_n)$. There exists an $x \in U_{f_n,\alpha_n} \cap A \cap U_{n+1}$. Hence $U_{f_n,\alpha_n} \cap U_{n+1}$ is a neighbourhood of x in C , thus there is (f_{n+1},α_{n+1}) such that $x \in U_{f_{n+1},\alpha_{n+1}} \subseteq F_{f_{n+1},\alpha_{n+1}} \subseteq U_{f_n,\alpha_n} \cap U_{n+1}$

$\{U_{f,\alpha}\}$ *is a neighbourhood system of* x *in* C .

Since $x \in A$, we get that $U_{f_{n+1},\alpha_{n+1}} \cap A \neq \emptyset$ and we can continue inductively,

The F_{f_n,α_n} are descending, non-empty, compact. Therefore they have a non-empty intersection F . $F \subseteq U_{f_1,\alpha_1}$ and $F \subseteq \cap U_n$. Finally F

Thus there exists a linear continuous funcitonal f on E , and a real number α such that $f(x) < \alpha$, and $f(y) > \alpha$ for $y \in C_3$.

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So $F_{f,\alpha}$ doesn't intersect C_1 or C_2 . That is $F_{f,\alpha} \subseteq U_{f_1,\alpha_1} \cap U_{f_2,\alpha_2}$ and $x \in U_{f,\alpha}$.

Let $\{V_n\}$ be a sequence of open dense subsets of A . It is required to prove that $\cap V_n$ is dense in A . That is it intersects every non-empty open subset V of A . Let $\{U_n\}$, U , be open subsets of C with U_n dense in C , such that $U_n \cap A = V_n$, $U \cap A = V$. One can assume the U_n's and V_n's are descending and that U is of the form U_{f_1,α_1} .

Let $V_n' = \bigcap_{i \leq m} V_i$ *and work with* $\{V_n'\}$ *instead of* $\{V_n\}$.

This being the case, we will prove the existence of $(f_1,\alpha_1),(f_2,\alpha_2)$, ... with the following properties: $F_{f_{n+1},\alpha_{n+1}} \subseteq U_{f_n,\alpha_n} \cap U_{n+1}$ and $U_{f_n,\alpha_n} \cap A \neq \emptyset$ for each $n \in \mathbb{N}$.

We already have (f_1,α_1) . Suppose we have found $(f_1,\alpha_1), \ldots (f_n,\alpha_n)$. There exists an $x \in U_{f_n,\alpha_n} \cap A \cap U_{n+1}$. Hence $U_{f_n,\alpha_n} \cap U_{n+1}$ is a neighbourhood of x in C , thus there is (f_{n+1},α_{n+1}) such that $x \in U_{f_{n+1},\alpha_{n+1}} \subseteq F_{f_{n+1},\alpha_{n+1}} \subseteq U_{f_n,\alpha_n} \cap U_{n+1}$

$\{U_{f,\alpha}\}$ *is a neighbourhood system of* x *in* C .

Since $x \in A$, we get that $U_{f_{n+1},\alpha_{n+1}} \cap A \neq \emptyset$ and we can continue inductively,

The F_{f_n,α_n} are descending, non-empty, compact. Therefore they have a non-empty intersection F . $F \subseteq U_{f_1,\alpha_1}$ and $F \subseteq \cap U_n$. Finally F

is compact, convex, and its complement in C is convex.

A simple lemma proves that F contains at least one extreme point y of C. (In all F contains an extreme point x. If x is an extreme point of C, we're done. If not, let δ be a straight line through x and such that x is an interior point of $C \cap \delta$. Then one shows that one of the end points of $F \cap \delta$ is an extreme point of C).

The extreme point of C in $F \cap \delta$ can be found in a U_n :

$y \in A \cap U_n \cap V_n$ for each $n \in \mathbb{N}_3$, and $y \in A \cap U_{f_1,\alpha_1} = V$.

This concludes Choquet's theorem and theorem 1.14.

Chapter 2

APPLICATIONS

Many applications of the Krein-Milman extension theorem 1.14 occur by associating the lower $\mathcal{J}$-semicontinuous pseudo norm p on the locally convex space $(E,\mathcal{J})$ with another topology on E. We begin this section on applications by investigating these topologies on E and reformulating theorem 1.14 to facilitate the further applications.

The topology with which p can be most directly associated is a norm topology -- namely in the case that p is itself a norm. The problem with choosing $(E,\mathcal{J})$ a Banach space with norm p is that then p is $\mathcal{J}$-continuous, and hence the identity map $i: (E,\mathcal{J}_p) \to (E,\mathcal{J})$ is everywhere continuous making theorem 1.14 trivial.

Consider however $\mathcal{J}$ to be the weak topology on a normed space $(E,\mathcal{J}_p)$.

<u>2.1 Lemma</u>: In a normed space $(E,\mathcal{J}_p)$ the norm p is lower weakly semi-continuous. Furthermore if E is infinite dimensional then the norm p is not weakly continuous.

<u>Proof</u>: Let $S = \{x: p(x) \leq 1\}$

To show S is weakly closed consider a weakly convergent sequence $x_n \longrightarrow x$ for which each $x_n \in S$.

If $x \notin S$, then by the Hahn-Banach theorem, there is an $f \in E^*$ such that $f(x) > 1$ and $f(y) \leq 1$ for every $y \in S$. But $x_n \longrightarrow x$ weakly means precisely that $f(x_n) \to f(x)$ for each $f \in E^*$ which is a contradiction. Hence $x \in S$.

Next assuming that E is infinite dimensional, we show that p is not weakly continuous. $\{V = \{x: |f_i(x)| < r_i, 1 \leq i \leq n, f_i \in E^*\}\}$ is a local base for the weak topology on E . Thus every weak 0-neighbourhood contains a subspace of the form $N = \{x: f_i(x) = 0 , 1 \leq i \leq n\}$. But N is the null space of the map from E into $\mathbb{R}^n$ which takes an element $x \in E$ to $(f_1(x), f_2(x), \ldots, f_n(x)) \in \mathbb{R}^n$.

dim (E) $\leq n + \dim(N)$,, therefore $\dim(N) = \infty$.

This shows the N (hence V) is not p-bounded and p is not continuous.

This gives the first corollary to theorem 1.14.

2.2 Theorem: Let $(E,\mathcal{J})$ be a normed space, p its norm. Let $K \subseteq E$ be convex, weakly compact, and norm separable. Let Z be the points of continuity of the identity map $i:(K,\mathcal{J}_w) \rightarrow (K,\mathcal{J}_p)$. Then $Z \cap \text{ext}(K)$ is weakly dense in ext(K) . Hence $K = \mathcal{J}_w\text{-}C\ell[C_o(Z \cap \text{ext}(K)]$.

2.3 Remark: This result also follows from the work of Joram Lindenstrauss [5; theorem4].

Theorem: Every weakly compact, convex subset of a separable Banach space is the closed convex hull of its strongly exposedppoints, (since strongly exposed points are (a) points of continuity of the identity map $i: (K,\mathcal{J}_w) \rightarrow (K,\mathcal{J}_p)$ and (b) extreme points of K .

Definition: a point x in a convex subset K of a Banach-space E is a strongly exposed point of K if and only if there is an $f \in E^*$ such that

(i) $f(y) < f(x)$ for each $y \in K$, $y \neq x$.

and (ii) $f(x_n) \rightarrow f(x)$ implies $\|x_n - x\| \rightarrow 0$.

(a) x is a point of continuity of the identity map $(K,\mathcal{J}_W) \rightarrow (K,\mathcal{J}_p)$ means that $\|x_n - x\| \rightarrow 0$ whenever $x_n \rightarrow x$ weakly. Since $x_n \rightarrow x$ weakly is equivalent to $f(x_n) \rightarrow f(x)$ for every $f \in E^*$ we get that all strongly exposed points of K are points of continuity of the identity map i .

(b) That all strongly exposed points are extreme is clear, since if x is strongly exposed and $x = \lambda x_1 + (1 - \lambda)x_2$ for $\lambda \in [0,1]$, then $f(x) = f(\lambda x_1 + (1 - \lambda)x_2) = \lambda f(x_1) + (1 - \lambda)f(x_2)$ which can only occur if $x = x_1$ or x_2 ((If $x_1 \neq x$ then $f(x_1) < f(x)$)

Let E be a normed space. Then E* , its continuous dual is also a normed space. Analagously to lemma 2.1 we have that the norm on E* is lower w*-semicontinuous, and not w*-continuous. Thus:

2.4 Theorem: [1; theorem 3.2]

Let K be a norm separable, w*-compact convex subset of E* , where E is a normed space. Let Z be the set of points of continuity of the identity map i: (E*,w*) → (E*, norm) . Then $Z \cap \text{ext}(K)$ is w*-dense in ext(K), hence

$$K = w^*\text{-}C\ell[C_o(Z \cap \text{ext}(K))]$$

We abstract from the foregoing, the following

2.5 Theorem: Let E be a normed space, p its norm. Let $\mathcal{J}$ be a locally convex topology on E such that p is lower $\mathcal{J}$-semicontinuous. Let K be a $\mathcal{J}$-compact, convex subset of E , such that K is norm-separable. Then $Z \cap \text{ext}(K)$ is $\mathcal{J}$-dense in ext(K) , where Z is the set of points of continuity of the identity map i: $(K,\mathcal{J}) \rightarrow (K,\mathcal{J}_p)$.

The next obvious spaces to look at are locally convex pseudo-metrizable spaces.

If $(E,\mathcal{J}_1)$ is a locally convex pseudo-metrizable space, then an invariant pseudo-metric (d) can be chosen so that for

$A_n = \{x: \ d(x,o) \leq n\}$, $\{\mu_{A_n}\}_{n \in \mathbb{N}}$ is a family of pseudo-norms on E which determines $\mathcal{J}_1$ (see definition 0.5). A simple device extends theorem 1.14 to pseudo-metric spaces.

2.6 Theorem: [1; theorem 2.3].

Let $(E,\mathcal{J}_1)$ be a locally convex pseudo-metric space. Let $\mathcal{J}_2$ be another topology on E such that $(E,\mathcal{J}_2)$ is hausdorff. Let $\{p_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{J}_1$-continuous, lower $\mathcal{J}_2$-semicontinuous pseudo-norms on E which determines $\mathcal{J}_1$. Let K be a $\mathcal{J}_2$-compact, convex $\mathcal{J}_1$-separable subset of E . Let Z be the set of points of continuity of the identity map $i: (E,\mathcal{J}_2) \to (E,\mathcal{J}_1)$. Then $Z \cap \text{ext}(K)$ is a $\mathcal{J}_2$-dense G_δ subset of ext(K) . Hence $K = \mathcal{J}_2 - C\ell[C_o(Z \cap \text{ext}(K))]$.

Proof: Consider E with the topology generated by p_n , with $\mathcal{J}_2$ as a second topology on E , such that p_n is lower $\mathcal{J}_2$-semicontinuous. Let Z_n be the set of points of continuity of the identity map $i: (K,\mathcal{J}_2) \to (K,\mathcal{J}_{p_n})$. Then by theorem 1.14 $Z_n \cap \text{ext}(K)$ is a $\mathcal{J}_2$-dense G_δ subset of ext(K) . By theorem 1.15 of Choquet $(\text{ext}(K),\mathcal{J}_2)$ is a Baire space. Thus $Z \cap \text{ext}(K) = \bigcap_{n \in \mathbb{N}} Z_n \cap \text{ext}(K)$ is a $\mathcal{J}_2$-dense subset of ext(K) . Also this intersection is G_δ , since G_δ'ness is closed under countable intersections.

Included in the diverse applications which we cover of the foregoing theory are that: each bounded subset of a separable dual Banach space is dentable, and that each closed convex, bounded (not necessarily compact) subset of a Frechet space whose second dual is separable relative to its strong topology is the closed convex hull of its extreme points. We conclude with a slight generalization of the Ryll-Nardzewski fixed point theorem, also due to Namioka. [1; theorem 3.7]

The following lemma gives a slightly stronger version of theorem 2.4.

<u>2.7 Lemma</u>: Let E be a Banach space such that E^* is separable. Let $K \subseteq E^*$ be bounded, norm-closed and convex. Let $K_1 = w^*\text{-}C\ell(K)$. Then $K \cap \text{ext}(K_1)$, which is clearly contained in $\text{ext}(K)$ is w*-dense in $\text{ext}(K_1)$.

<u>Proof</u>: By proposition 0.20 we get that K_1 is w*-compact. Thus theorem 2.4 applies to $K_1 \subseteq (E^*,w^*)$. That is, $Z \cap \text{ext}(K_1)$ is w*-dense in $\text{ext}(K_1)$ where Z is the set of points of continuity of the identity map $i: (K,w^*) \to (K_1, \text{norm})$. We show that $Z \subseteq K$ which completes the proof.

Let $z \in Z$. K is w*-dense in K_1 , therefore we can find a net $\{x_\alpha\}$ on K which converges w^* to z . That is, for each w*-open neighbourhood U of z , there is an α_0 such that for each $\alpha > \alpha_0$, (α in the directed index set I) , $x_\alpha \in U$. But $z \in Z$ means that each ε-ball about z contains a w*-open neighbourhood U . Thus x_α tends to z in norm. Since K is norm closed, $z \in K$.

<u>2.8 Theorem</u>: [1; corollary 3.4].

Let E be a Banach space, such that E^* is separable. Then each norm closed, convex bounded subset of E^* is the norm closed convex hull of its extreme points.

<u>Proof</u>: Let $K_1 = w^*\text{-}C\ell(K)$ where K is norm closed, bounded and convex in E^* . $\text{Ext}(K_1) \neq \emptyset$, thus as in lemma 2.7 we get that $\emptyset \neq K \cap \text{ext}(K_1) \subseteq \text{ext}(K)$. We show that this is sufficient to prove that

$K = C\ell[C_o(ext(K))]$, following a proof by Richard Bourgin as presented by N. T. Peck in [7; lemma 1], (and in a written communication from I. Namioka).

Lemma 2.9: Let E be a locally convex space. Then every closed, bounded, convex subset of E has an extreme point if and only if every closed, bounded, convex subset of E is the closed convex hull of its extreme points.

Proof: Assume that the non-trivial of the implications is false. Then there is a closed, bounded, convex set $C \subseteq E$ such that $C_o = C\ell[C_o(ext(C))] \subsetneq C$.

Let $y \in C \setminus C_o$. Then by the separation theorem for convex sets [6; theorem 3.4, page 58], there is an $f \in E^*$ and $\beta \in \mathbb{R}$ such that $f(c) < \beta \leq f(y)$ for every $c \in C_o$. That is $K_1 = \{c \in C: f(c) \geq \beta\} \neq \emptyset$, and $K_1 \cap C_o = \emptyset$. Now $D = \{x \in C: f(x) = \beta\} \neq \emptyset$, and it is closed, bounded and convex. By our hypothesis, $ext(D) \neq \emptyset$ say $u \quad ext(D)$. Clearly $u \in ext(K_1)$.

Since $u \notin C_o \supseteq ext(C)$, $u = \lambda a + (1 - \lambda)b$ for some $\lambda \in (0,1)$ and $a,b \in C$. Since $u \in ext(K_1)$, one of $a,b \notin K_1$. Say $a \notin K_1$. But then $b \in K_1$, since if $b \notin K_1$, then $\beta = f(u) = \lambda f(a) + (1 - \lambda)f(b) < \lambda\beta + (1 - \lambda)\beta = \beta$ which cannot be.

Without loss of generality, we can let

$$b = a + t(u - a) \text{ , where } t = \sup\{\lambda \in \mathbb{R}: a + \lambda(u - a) \in C\} .$$

Indeed, since $t > 1$, $f(a + t(u - a)) = tf(u) - (t - 1)f(a) > (t - 1)\beta = \beta$, so $a + t(u - a) \in K_1$.

Now $b \notin C_o \supseteq ext(C)$, therefore there are $c_1, c_2 \in C$ such that

$b = 1/2(\hat{c}_1 + \hat{c}_2)$, and clearly we can find $c_1, c_2 \in K_1$.

Let $p_i = \frac{\varepsilon_i}{\delta + \varepsilon_i} a + \frac{\delta}{\delta + \varepsilon_i} c_i$ for $i = 1,2$

where $\delta = \beta - f(a) > 0$ and $\varepsilon_i = f(c_i) - \beta \geq 0$ for $i = 1,2$.

Then $p_1, p_2 \in C$ since C is convex and $a, c_1, c_2 \in C$. Note that $f(p_1) = f(p_2) = \beta$, hence $p_1, p_2 \in D$.

But $\hat{u} = \frac{\delta + \varepsilon_1}{2\delta + \varepsilon_1 + \varepsilon_2} p_1 + \frac{\delta + \varepsilon_2}{2\delta + \varepsilon_1 + \varepsilon_2} p_2$ which contradicts that $u \in \text{ext}(D)$, and the proof is complete.

We next refer back to denting points as defined in Remark 1.12 following the proof of the chipping lemma. We examine the problem posed by M. Rieffel [8; question 3] namely for which spaces are all bounded subsets dentable. Namioka gives a partial answer in [1; theorem 3.5].

<u>2.10 Lemma</u>: [1; remarks preceding theorem 3.5].

Let E be a Banach Space. Let $\mathcal{J}$ be a hausdorff, locally convex topology on E such that the norm is lower $\mathcal{J}$-semicontinuous. Let $K \subseteq E$ be such that $\mathcal{J}\text{-}C\ell[C_o(K)]$ is $\mathcal{J}$-compact, and K is norm-separable. Then K is dentable.

<u>Proof</u>: Let $K_1 = \mathcal{J}\text{-}C\ell[C_o(K)]$. By the chipping lemma - proposition 1.13, there exists a $\mathcal{J}$-closed, convex $C \subseteq K$, such that $C \neq K_1$, and the $\text{diam}(K_1 \setminus C) \leq \frac{\varepsilon}{2}$.

But clearly $K \setminus C \neq \emptyset$, since if $C \supseteq K$ then C will also contain the closed convex hull of K , namely K_1 . Let $x \in K \setminus C$. Then clearly $C \supseteq K \setminus B_\varepsilon(x)$ where $B_\varepsilon(x)$ is the closed ball of radius ε around x . Therefore $C \supseteq \mathcal{J}\text{-}C\ell[C_o(K \setminus B_\varepsilon(x))]$. That is $x \notin \mathcal{J}\text{-}C\ell[C_o(K \setminus B_\varepsilon(x))]$.

<u>2.11 Theorem</u>: [1; theorem 3.5].

Let E be a Banach space such the E^* is separable. Then each non-empty, norm-closed, convex, bounded subset of E^* contains a denting point. Hence each bounded subset of E^* is dentable.

<u>Proof</u>: Let K be a norm-closed, convex, bounded subset of E^* . Let $K_1 = w^*\text{-}C\ell(K)$, and let $u \in ext(K_1)$ be such that u has arbitrarily norm-small w*-neighbourhoods (theorem 2.4). As in the proof of lemma 2.7, $u \in ext(K)$. But then u is a denting point of K_1 . Indeed let $\varepsilon > 0$ and W a w*-neighbourhood of u such that $diam(W) \leq \varepsilon$. Then $u \notin K_1 \setminus W$ and since u is extreme, $u \notin w^*\text{-}C\ell[C_o(K_1 \setminus W)]$ $\supseteq w^*\text{-}C\ell[C_o(K_1 \setminus B_\varepsilon(u))] \supseteq$ norm closure of $C_o(K_1 \setminus B_\varepsilon(u))$, where again, $B_\varepsilon(u)$ is the closed ball of radius ε around u . But K is norm bounded hence so is K_1 , and K_1 is w*-closed. Thus K_1 is w*-compact, and by lemma 2.10 K is dentable.

We now prove another type of generalization of the Krein-Milman theorem. [1; theorem 3.6].

<u>2.12 Definition</u>: A topological vector space is called <u>quasi-separable</u> if each bounded subset is separable.

<u>2.13 Theorem</u>: Let E be a Frechet Space such that $(E^*,s)^*$ is quasi-separable with respect to the strong topology. Let $K \subseteq E$ be closed, bounded and convex. Then K is the closed, convex hull of its extreme points.

<u>Proof</u>: Let $I: E \to (E^*,s)^*$ be the evaluation map. Let

$K_1 = w^*\text{-}C\ell(I[K])$.

Consider the bipolar $(K^o)^o$ of K . $(K^o)^o = \{F \in (E^*,s)^*:$ $|F(f)| \leq 1$ for each $f \in E^*$ which satisfies $|f(x)| \leq 1$ for each $x \in K$.

Clearly $I[K] \subseteq (K^o)^o$.

But K^o is a neighbourhood of 0 in (E^*,s) thus $(K^o)^o$ is w*-compact in $(E^*,s)^*$ (Banach Alaoglu theorem). Hence $(K^o)^o$ is w*-closed and so $K_1 \subseteq (K^o)^o$. That is K_1 is a closed subset of a compact set in a Hausdorff space. Therefore K_1 is w*-compact.

Also, we get that K_1 is strongly bounded. Let V be a strong 0-neighbourhood in $(E^*,s)^*$. Since $\{B^o$: B is strongly bounded in $E^*\}$ is a local base for $[(E^*,s)^*,s]$, $V \supseteq B^o$ for some such B . Now I is continuous (theorem 0.19), thus $I^{-1}(B^o)$ is a 0-neighbourhood in E . K is bounded in E , therefore $K \subseteq nI^{-1}(B^o)$ for some sufficiently large $n \in \mathbb{N}$. That is $I[K] \subseteq nB^o$. But B^o is w*-closed in $(E^*,s)^*$, thus $K_1 \subseteq nB^o \subseteq nV$.

Let K' be the subspace of $(E^*,s)^*$ generated by K_1 . Since $(E^*,s)^*$ is quasi-separable, K' is strongly separable and metrizable with the induced topology. Thus theorem 2.6 applies to K_1 in (K',s) with the w*-topology as the second topology on K_1. So for Z = set of points of continuity of the identity map $i: (K_1,w^*) \to (K_1,s)$

(1) $Z \cap \text{ext}(K_1)$ is w*-dense in $\text{ext}(K_1)$.

Consider a net $\{x_\alpha\}$ in $I[K]$ which converges strongly to $F \in (E^*,s)^*$. I is a homeomorphism of E onto $(I[E],s)$, therefore $\{x_\alpha\}$ is a Cauchy net in E . But E is complete, so $\{x_\alpha\}$ converges to some $x \in E$. Since K is closed $x \in K$. Thus $F = I(x) \in I[K]$, and I[K] is strongly closed in $(E^*,s)^*$. This gives, as in the proof of lemma 2.7, that $Z \subseteq I[K]$. Thus $Z \cap \text{ext}(K_1) \subseteq \text{ext}(I[K]) = I[\text{ext}(K)]$. By (1) above, $w^*\text{-}C\ell[C_o(I[\text{ext}(K)])] = K_1 = w^*\text{-}C\ell(I[K])$. Inverting back through I , we get that $\text{weak-}C\ell[C_o(\text{ext}(K))] = K$.

We conclude with a slight generalization of the first theorem proved-The Ryll-Nordzewski Fixed point theorem.

<u>2.14 Theorem</u>: [1; theorem 3.7].

Let $(E,\mathcal{J})$ be a locally convex separable topological vector space. Let $\mathcal{J}_2$ be a second locally convex, hausdorff topology on E , such that $\mathcal{J}$ is determined by lower $\mathcal{J}_2$-semicontinuous pseudo-norms p_α on E . Let $Q \subseteq E$ be non-empty, convex and $\mathcal{J}_2$-compact. Let $\mathcal{S}$ be a semigroup of $\mathcal{J}_2$-continuous affine maps of Q into itself, such that $\mathcal{S}$ is $\mathcal{J}$-noncontracting on Q . Then $\mathcal{S}$ has a common fixed point in Q .

<u>outline of proof</u>: $\mathcal{S}$ is $\mathcal{J}$-noncontracting implies that for every distinct pair $x,y \in Q$, there is a $\mathcal{J}$-continuous pseudo-norm p on E such that $\inf_{T\in\mathcal{S}} \{p(Tx - Ty)\} > 0$ (prosition 1.4). Since $\mathcal{J}$ is determined by a set $\{p_\alpha\}$ of lower $\mathcal{J}_2$-semicontinuous pseudo-norms, for each distinct x,y p itself can be chosen to be lower $\mathcal{J}_2$-semicontinuous. Indeed a $\mathcal{J}_1$-0 neighbourhood B which is $\mathcal{J}_2$-closed can be found within the p-unit ball $(= \{x: p(x) \leq 1\})$. This is true since the p_α unit balls are a local base for $\mathcal{J}_1$ and each is $\mathcal{J}_2$-closed. μ_B is then $\mathcal{J}_1$-continuous, lower $\mathcal{J}_2$-semicontinuous, and since $p \leq \mu_B$, $\inf\{\mu_B(Tx - Ty): T \in \mathcal{S}\}$ $\geq \inf\{p(Tx - Ty): T \in \mathcal{S}\} > 0$.

If $\mathcal{J}_2 = \mathcal{J}_w$, theorem 2.14 becomes theorem 1.5 with additional hypothesis that $(E,\mathcal{J})$ is separable. In the present more general form, the added hypothesis is required since the chipping lemma to theorem 1.5

requires that the set $K = \mathcal{J}\text{-}C\ell[C_o\{T_{x_o} : T \in \mathcal{S}\})]$ be contained in a $\mathcal{J}$-separable set. Consequently this theorem can be proved by the same method as theorem 1.5 with proposition 1.14 replacing the chipping lemma 1.13.

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