THE ESTIMATION OF A CHARACTERISTIC FUNCTION AND ITS DERIVATIVES

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ABSTRACT

In this thesis, we discuss the problem of estimating a characteristic function and its derivatives. We obtain estimates which are consistent and asympototically normal, and uniformly consistent with probability one.

The methods employed here are similar to the methods used in estimating a probability density function and its derivatives (see [7], [9] for references).

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0. INTRODUCTION.

Suppose given a sequence of independent identically distributed (iid) random variables $X_1, X_2, \ldots, X_n, \ldots$ with a common characteristic function $\phi(t)$.

The problem of estimating a characteristic function is interesting for many reasons. One possible application is to determine the components of a corresponding mixture distribution function (see [4]). In this thesis, we construct an estimate $\phi_n(t)$ of $\phi(t)$, which is based on the random sample $X_1, X_2, \ldots, X_n$, such that $\phi_n(t)$ will have some nice asymtotic properties, and converges uniformly to $\phi(t)$ with probability one.

In addition, if $E|X|^{2q}$ is finite for some integer $q > 0$, we are able to obtain an estimate $\phi_n^{(p)}(t)$, the estimate of the p-th derivative of $\phi(t)$ for $0 < p \leq q$, such that $\phi_n^{(p)}(t)$ will have the asymptotic properties parallel to those of $\phi_n(t)$. Furthermore, if $\sup_n E_n|X|^q = M$ for some constant M, then for $0 < p < q$, we are able to show that $\phi_n^{(p)}(t)$ converges uniformly to $\phi^{(p)}(t)$ with probability one .

I. THE ESTIMATE $\phi_n(t)$ OF $\phi(t)$.

Let $X_1, X_2, \ldots, X_n$ be iid as the random variable X whose distribution $F(x) = P[X \leq x]$ is absolutely continuous. That is

$$F(x) = \int_{-\infty}^{x} f(u)\, du$$

with the density $f(x)$. Let $\phi(t)$ be the characteristic function of $F(x)$. Then each estimate $\phi_n(t)$ of $\phi(t)$, based on the empirical data, will be in the form of

$$\phi_n(t) = \int_{-\infty}^{\infty} e^{itx} f_n(x)\, dx \,, \tag{1.1}$$

where

$$f_n(x) = \frac{1}{nh} \sum_{j=1}^{n} k\left(\frac{x - X_j}{h} \right) ,$$

is the kernel estimate of $f(x)$ as given in [8], and $h = h(n)$ is a function of the number n which converges to zero as $n \to \infty$; $k(y)$ is some symmetric density function, such that the moments of all orders exist. (1.1) can also be expressed as

$$\begin{aligned} \phi_n(t) &= \frac{1}{nh} \sum_{j=1}^{n} \int_{-\infty}^{\infty} e^{itx}\, k\left(\frac{x - X_j}{h} \right) dx \\ &= \left(\frac{1}{n} \sum_{j=1}^{n} e^{itX_j} \right)\left(\int_{-\infty}^{\infty} e^{ithy}\, k(y)\, dy \right) \\ &= \left(\frac{1}{n} \sum_{j=1}^{n} e^{itX_j} \right) \psi(th) \,, \end{aligned} \tag{1.2}$$

where

$$\psi(th) = \int_{-\infty}^{\infty} e^{ithy}\, k(y)\, dy \quad . \tag{1.3}$$

Since $k(y)$ is some symmetric density function, then by definition, $\psi(th)$ is a characteristic function, and is real and even.

I-1. Asymptotic Unbiasedness of $\phi_n(t)$.

$\phi_n(t)$ would be unbiased if $E\phi_n(t)$ were equal to $\phi(t)$. But

$$\begin{aligned} E\phi_n(t) &= \psi(th) \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \\ &= \psi(th)\phi(t) \ , \end{aligned} \tag{1.4}$$

and $\psi(th)$ equals one only when $(th) = 0$, however as $n \to \infty$, $(th) \to 0$, so $\psi(th) \to \psi(0) = 1$. It follows immediately that

$$\lim_{n\to\infty} E\phi_n(t) = \phi(t) \lim_{n\to\infty} \psi(th) = \phi(t)\psi(0) = \phi(t) \ .$$

Hence, $\phi_n(t)$ is asymptotically unbiased as $n \to \infty$ and $h \to 0$.

I-2. Quadratic Consistency.

The mean square error of $\phi_n(t)$ converges to zero as $n \to \infty$, and $h \to 0$. In notation,

$$E|\phi_n(t) - \phi(t)|^2 \to 0 \quad \text{as} \quad n \to \infty \ , \ h \to 0 \ . \tag{1.5}$$

or (1.5) can be rewritten as

$$E|\phi_n(t) - \phi(t)|^2 = E[\mathrm{Re}(\phi_n(t) - \phi(t))]^2 + E[\mathrm{Im}(\phi_n(t) - \phi(t))]^2 \tag{1.6}$$
$$= \mathrm{Var}[\mathrm{Re}\phi_n(t)] + [b(\mathrm{Re}\phi_n(t))]^2 + \mathrm{Var}[\mathrm{Im}\phi_n(t)] + [b(\mathrm{Im}\phi_n(t))]^2$$
$$= \mathrm{Var}[\phi_n(t)] + |b[\phi_n(t)]|^2 ,$$

where $\mathrm{Var}[\phi_n(t)]$ is the variance of $\phi_n(t)$, and $b[\phi_n(t)] = E\phi_n(t) - \phi(t)$ is the bias of $\phi_n(t)$. The quadratic consistency can be shown if $\mathrm{Var}[\phi_n(t)]$ and $|b[\phi_n(t)]|^2$ vanish as $n \to \infty$ and $h \to 0$. But $|b[\phi_n(t)]|^2 = |E\phi_n(t) - \phi(t)|^2 = |\psi(th) - 1|^2|\phi(t)|^2 \to 0$ as $n \to \infty$ and $\psi(th) \to \psi(0) = 1$. It remains to show that the variance of $\phi_n(t)$ vanishes as $n \to \infty$. As we know any complex random variable, say $Z = U + iV$, is such that its expectation and variance can be put in the following forms :

$$E[Z] = E[U] + iE[V] ,$$

$$\mathrm{Var}[Z] = \mathrm{Var}[U] + \mathrm{Var}[V] ,$$

where U and V are the real and the imaginary parts of Z.

Observe that

$$\phi_n(t) = \left(\frac{1}{n}\sum_{j=1}^{n} e^{itX_j}\right)\psi(th) = \left(\frac{1}{n}\sum_{j=1}^{n} (\cos tX_j + i \sin tX_j)\right)\psi(th) . \tag{1.7}$$

Hence

$$\mathrm{Var}[\phi_n(t)] = \frac{[\psi(th)]^2}{n}\left\{ \mathrm{Var}(\cos tX) + \mathrm{Var}(\sin tX) \right\} . \tag{1.8}$$

Since $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, it follows that

$$E\{\cos^2 tX\} = \frac{1}{2}\{1 + E(\cos 2tX)\} = \frac{1}{2}\{1 + \mathrm{Re}\phi(2t)\} ,$$

and

$$E\{\sin^2 tX\} = \frac{1}{2}\{1 - E(\cos 2tX)\} = \frac{1}{2}\{1 - \mathrm{Re}\phi(2t)\} ;$$

and since $\sin\theta \cos\theta = \frac{1}{2}\sin 2\theta$, it follows that,

$$E\{\sin tX \cos tX\} = \frac{1}{2}\{E \sin 2tX\} = \frac{1}{2}\mathrm{Im}\phi(2t) .$$

From above computations, one gets

$$\mathrm{Var}\{\cos tX\} = \frac{1}{2}\{1 + \mathrm{Re}\phi(2t)\} - \{\mathrm{Re}\phi(t)\}^2 ,$$

$$\mathrm{Var}\{\sin tX\} = \frac{1}{2}\{1 - \mathrm{Re}\phi(2t)\} - \{\mathrm{Im}\phi(t)\}^2 , \tag{1.9}$$

$$\mathrm{Cov}\{\cos tX, \sin tX\} = \frac{1}{2}\mathrm{Im}\phi(2t) - [\mathrm{Re}\phi(t)][\mathrm{Im}\phi(t)] .$$

It follows immediately that (1.8) can be replaced by

$$\begin{aligned} \mathrm{Var}[\phi_n(t)] &= \frac{[\psi(th)]^2}{n}\left\{ 1 - [\mathrm{Re}\phi(t)]^2 - [\mathrm{Im}\phi(t)]^2 \right\} \\ &= \frac{[\psi(th)]^2}{n}\left\{ 1 - |\phi(t)|^2 \right\} . \end{aligned} \tag{1.10}$$

Since $\psi(th) \to 1$, as $n \to \infty$, it follows from (1.10) that $\mathrm{Var}[\phi_n(t)] \to 0$ as $n \to \infty$.

Meanwhile we know that, for any given real random variable Y with absolutely continuous distribution, $E[Y^2] > \{E[Y]\}^2$ when $\mathrm{Var}[Y] > 0$. Similarly, when $t \neq 0$, and $\mathrm{Var}[\cos tX] > 0$,

$\mathrm{Var}[\sin tX] > 0$, one gets

$$|\phi(t)|^2 = [E(\cos tX)]^2 + [E(\sin tX)]^2$$

$$< E[\cos^2 tX] + E[\sin^2 tX] \equiv 1 .$$

Then it follows that

$$\frac{1}{n} > \mathrm{Var}[\phi_n(t)] > 0 . \tag{1.11}$$

From above, we know that the variance of $\phi_n(t)$ satisfies $n\,\mathrm{Var}[\phi_n(t)] \to \{1 - |\phi(t)|^2\}$, for $t \neq 0$ as $n \to \infty$; but for $t = 0$, $\phi_n(0) \equiv 1$ and $\mathrm{Var}[\phi_n(0)] \equiv 0$.

I-3. Asymptotic Normality of $\phi_n(t)$.

From (1.9) one sees that $\mathrm{Var}[\mathrm{Re}\phi_n(t)] \neq \mathrm{Var}[\mathrm{Im}\phi_n(t)]$, and $\mathrm{Cov}[\mathrm{Re}\phi_n(t), \mathrm{Im}\phi_n(t)] \neq 0$. It follows that $\phi_n(t)$ is not distributed according to the special kind of the univariate complex normal distribution of R.A. Wooding [11]. However, since $|\phi_n(t)| \leq 1$, $|\mathrm{Re}\phi_n(t)| \leq 1$ and $|\mathrm{Im}\phi_n(t)| \leq 1$, one may expect that $\mathrm{Re}\phi_n(t)$ and $\mathrm{Im}\phi_n(t)$ will be asymptotically bivariately normally distributed.

Consider $\phi_n(t)$ as an average of $\theta_{n1}, \theta_{n2}, \ldots, \theta_{nn}$ iid complex random variables with common distribution. In notation, one writes

$$\phi_n(t) = \frac{1}{n} \sum_{j=1}^{n} \theta_{nj} , \tag{1.12}$$

where $\theta_{nj} = \theta_j = \psi(th)e^{itX_j} = \psi(th)\{\cos tX_j + i \sin tX_j\}$, the univariate complex random variable, is considered as a bivariate random vector in R^2. One form of the k-variate central limit theorem is stated in S.S. Wilks [10], as follows : " Suppose $(X_{1j}, X_{2j}, \ldots, X_{kj}; j = 1, 2, \ldots, n)$ is a sample of size n from a k-variate distribution having finite means μ_i, $i = 1, 2, \ldots, k$, and (positive definite) covariance matrix $||\sigma_{im}||$, $i, m = 1, 2, \ldots, k.$, ..., then $(\overline{X}_1, \ldots, \overline{X}_k) \sim N\left(\{\mu_i\}, \left|\left|\frac{\sigma_{im}}{n}\right|\right|\right)$." We apply this result to the case $k = 2$ with sample means,

$$\mathrm{Re}\phi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{Re}\theta_{nj} = \frac{1}{n}\sum_{j=1}^{n} [\psi(th)\cos tX_j] ,$$

and

$$\mathrm{Im}\phi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{Im}\theta_{nj} = \frac{1}{n}\sum_{j=1}^{n} [\psi(th)\sin tX_j]$$

Let

$$\psi(th)\mu_1 = E[\psi(th)\cos tX] = \psi(th)\mathrm{Re}\phi(t) ,$$

$$\psi(th)\mu_2 = E[\psi(th)\sin tX] = \psi(th)\mathrm{Im}\phi(t) ,$$

be the expected means, which are both bounded by one respectively. Let

$$[\psi(th)]^2\sigma_{11} = \mathrm{Var}[\psi(th)\cos tX]$$

$$= [\psi(th)]^2 \left\{ \frac{1}{2}[1 + \mathrm{Re}\phi(2t)] - [\mathrm{Re}\phi(t)]^2 \right\} ,$$

$$[\psi(th)]^2\sigma_{22} = \mathrm{Var}[\psi(th)\sin tX]$$

$$= [\psi(th)]^2 \left\{ \tfrac{1}{2}[1 - \mathrm{Re}\phi(2t)] - [\mathrm{Im}\phi(t)]^2 \right\},$$

$$[\psi(th)]^2\sigma_{21} = \mathrm{Cov}[\psi(th)\cos tX,\ \psi(th)\sin tX]$$

$$= [\psi(th)]^2 \left\{ \tfrac{1}{2}\, \mathrm{Im}\phi(2t) - [\mathrm{Re}\phi(t)][\mathrm{Im}\phi(t)] \right\}$$

$$= [\psi(th)]^2\sigma_{12}$$

be the variances and the covariance of $[\mathrm{Re}\phi_n(t)]$ and $[\mathrm{Im}\phi_n(t)]$. It is clear that all of σ_{11}, σ_{22} and σ_{21}, σ_{12} are bounded at most by 2, and that $\det\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} > 0$. The mean vectors and the covariance matrix will have limits

$$(\psi(th)\mu_1,\ \psi(th)\mu_2) \to (\mu_1, \mu_2),$$

and

$$\frac{[\psi(th)]^2}{n} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \sim \frac{1}{n} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix},$$

as $n \to \infty$, $\psi(th) \to 1$.

It follows that $\left(\mathrm{Re}\phi_n(t),\ \mathrm{Im}\phi_n(t)\right) \sim N\left(\{\psi(th)\mu_i\},\ \left|\left|\frac{\psi^2(th)\sigma_{im}}{n}\right|\right|\right)$ for $i, m = 1, 2$. Since $\psi(th)$ is just a number for any constant t, and $\psi(th) \to 1$ as $n \to \infty$, so $\phi_n(t)$ is asymptotically normal.

I-4. Limits of the Bias and the Mean Square Error.

(a) The bias of $\phi_n(t)$ satisfies

$$|b[\phi_n(t)]| = |\psi(th) - 1||\phi(t)| \ . \tag{1.13}$$

This expression shows that the bias of $\phi_n(t)$ depends on the properties of $\psi(th)$ which are in turn based on the function $k(y)$ and h. Since $\psi(th)$ is real and even, $\{\psi(th) - 1\}$ is always real and negative for $t \neq 0$, and zero as $n \to \infty$ and $h \to 0$. Suppose $k(y)$ has moments of all orders, then the odd moments of $k(y)$ are zero, since $k(y)$ is symmetrically distributed. Hence, for any t held constant, and n sufficiently large, we can put $\psi(th)$ in the following form :

$$\psi(th) = \sum_{m=0}^{k} \frac{\alpha_m}{m!} (ith)^m + o(th)^k \ , \qquad \text{as} \quad (th) \to 0 \ , \tag{1.14}$$

where α_m is the moment of order m of $k(y)$, and is assumed to be finite. If α_2 and α_4 are finite and also non-zero, (1.14) can be written as

$$\psi(th) = 1 - \frac{\alpha_2}{2} (th)^2 + o(th)^2 \ , \ \ldots \qquad \text{as} \quad h \to 0 \ , \tag{1.15}$$

since t is constant, $(th) \to 0$ implies $h \to 0$. It follows that

$$|b[\phi_n(t)]| = \frac{\alpha_2}{2} (th)^2 |\phi(t)| + o(th)^2, \qquad \text{as} \quad (th) \to 0 \ , \tag{1.16}$$

or as $(th) \to 0$,

$$|b[\phi_n(t)]| \Big/ (th)^2 \to \frac{\alpha_2}{2} |\phi(t)| \ .$$

The expression (1.16) shows that for any real $t \neq 0$, the bias of $\phi_n(t)$, $|b[\phi_n(t)]| \to 0$ at the same rate as $h^2 \to 0$. In fact, the expression (1.16) can still be true, even when t is not held constant, but only required to increase slowly enough that (th) stays small or approaches zero as $n \to \infty$.

But even if t increases so fast that (th) becomes large or approaches infinity as $n \to \infty$, the bias of $\phi_n(t)$ still vanishes as $n \to \infty$ and $|t| \to \infty$. Given any $\varepsilon > 0$, there exists a T_ε so large that $|\phi(t)| < \frac{\varepsilon}{2}$, and $|b[\phi_n(t)]| = |\psi(th) - 1||\phi(t)| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$, whenever $|t| \geq T_\varepsilon$. Hence it follows that $\lim_{n\to\infty}|b[\phi_n(t)]| = 0$ uniformly for all real t.

(b) Limits for the mean square error.

The mean square error satisfies

$$
\begin{aligned}
(1.17) \qquad Q &= E|\phi_n(t) - \phi(t)|^2 \\
&= \mathrm{Var}[\phi_n(t)] + |b[\phi_n(t)]| \\
&= \frac{1}{n}\{\psi(th)\}^2\{1 - |\phi(t)|^2\} + \{\psi(th) - 1\}^2|\phi(t)|^2
\end{aligned}
$$

When t is held constant, one can, with the proper choice of $\psi(th)$, minimize the mean square error. Setting $\frac{\partial Q}{\partial \psi} = 0$, that is

$$\frac{2}{n}\psi(th)\{1 - |\phi(t)|^2\} + 2\{\psi(th) - 1\}|\phi(t)|^2 = 0 \quad .$$

One easily gets

$$\psi_{min}(th) = \frac{|\phi(t)|^2}{\frac{1}{n} + (1 - \frac{1}{n})|\phi(t)|^2} .$$

Formally, just looking at the last equation, one would say that $\psi_{min}(th)$ is not defined when $|\phi(t)|^2 = \frac{1}{1 - n}$. However, $\psi_{min}(th) = |\phi(t)|^2$ when $n = 1$. Since $\frac{1}{1 - n} < 0$ for $n = 2, 3, \ldots,$ while $0 \leq |\phi(t)|^2 \leq 1$, the exceptional case with $|\phi(t)|^2 = \frac{1}{1 - n}$ can not arise when $n > 1$. Thus we see that $\psi_{min}(th)$ is well defined for every whole number n. It is not difficult to check that $\psi_{min}(th)$ is indeed a characteristic function (see [3]). Then

$$1 - \psi_{min}(th) = \frac{1 - |\phi(t)|^2}{1 + (n - 1)|\phi(t)|^2} .$$

For the fixed $t \neq 0$ and n, a minimized mean square error, Q_{min}, can be obtained. It is

$$Q_{min} = \frac{1}{n} \{\psi_{min}(th)\}^2\{1 - |\phi(t)|^2\} + \{\psi_{min}(th) - 1\}^2|\phi(t)|^2$$

$$= \frac{|\phi(t)|^2\{1 - |\phi(t)|^2\}}{1 + (n - 1)|\phi(t)|^2} \doteq \frac{\{1 - |\phi(t)|^2\}}{n} \quad \text{as} \quad n \to \infty .$$

I-5. Uniform Consistency of $\phi_n(t)$ with Probability One.

To prove this uniform consistency of $\phi_n(t)$, it is enough to show that

$$\underset{n\to\infty}{\text{limit}} \ \underset{-\infty<t<\infty}{\text{supremum}} |\phi_n(t) - \phi(t)| = 0$$

with probability one.

An approximation $F_n(x)$ of $F(x)$, based on the empirical data, is given in [5]. It is

$$F_n(x) = \int_{-\infty}^{x} f_n(u)\, du \quad ,$$

where $f_n(x)$ is the kernel estimate of $f(x)$ as given in [8]. Since $k(y)$ is assumed to be a density function, $F_n(x)$ is some distribution function; in fact, $F_n(x)$ is absolutely continuous.

Assume $f(x) = F'(x)$ and $f_n(x) = F_n'(x)$ are defined for every x. Then from E.A. Nadaraya [5], it follows that if $f(x)$ is uniformly continuous, if the sequence h is such that the series $\sum e^{-\gamma n h^2}$ is finite for every positive value of γ, and if $k(y)$ is a function of bounded variation over $(-\infty, \infty)$, then

$$\underset{n\to\infty}{\text{limit}}\ \underset{-\infty<x<\infty}{\text{supremum}} \left| f_n(x) - f(x) \right| = 0$$

with probability one. Hence $\underset{n\to\infty}{\text{limit}}\ f_n(x) = f(x)$ for every x .

The sequence $F_n(x)$ of the absolutely continuous distributions converges uniformly to $F(x)$ for all x as $n \to \infty$ and $h \to 0$ (see [4]). Hence, the sequence $\{\phi_n(t)\}$ of the corresponding characteristic functions converges to $\phi(t)$ for every t. That is, $\underset{n\to\infty}{\text{limit}}\ \phi_n(t) = \phi(t)$ for every t. Since $F(x)$ is absolutely continuous, we know that $\underset{|t|\to\infty}{\text{limit}}\ \phi(t) = 0$. Similarly, we have $\underset{|t|\to\infty}{\text{limit}}\ \phi_n(t) = 0$ for every n. Moreover, for any real t ,

$$|\phi_n(t) - \phi(t)| = \left|\int_{-\infty}^{\infty} e^{itx} f_n(x)\, dx - \int_{-\infty}^{\infty} e^{itx} f(x)\, dx\right|$$

$$\leq \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx \quad .$$

Since the last expression is independent of t, it follows that

$$\text{(1.18)} \qquad \underset{-\infty<t<\infty}{\text{supremum}} |\phi_n(t) - \phi(t)| \leq \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx \quad .$$

To see the uniform consistency of $\phi_n(t)$, we only have to show that (1.18) $\to 0$ as $n \to \infty$ and $h \to 0$. Let

$$\{f - f_n\}^+(x) = \max\{0, f(x) - f_n(x)\} \quad ,$$

and

$$\{f - f_n\}^-(x) = -\min\{0, f(x) - f_n(x)\} \quad .$$

Then

$$\{f(x) - f_n(x)\} = \{f(x) - f_n(x)\}^+ - \{f(x) - f_n(x)\}^- \; ,$$

(1.19)

$$|f(x) - f_n(x)| = |f_n(x) - f(x)|$$

$$= \{f(x) - f_n(x)\}^+ + \{f(x) - f_n(x)\}^- \; .$$

Since $f(x)$ and $f_n(x)$ are densities, we have

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f_n(x)\, dx = 1$$

for every n. It follows that

$$\int_{-\infty}^{\infty} \{f(x) - f_n(x)\}\, dx = 0$$

$$= \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^+ dx - \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^- dx ,$$

and

$$(1.20) \qquad \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^+\, dx = \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^-\, dx \quad .$$

Since $\{f(x) - f_n(x)\}^+ \leq f(x)$, and $\underset{n\to\infty}{\text{limit}}\, f_n(x) = f(x)$ for every x, then by the Lebesque Dominated Convergence Theorem, one gets

$$(1.21) \qquad \underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^+ dx = \int_{-\infty}^{\infty} \underset{n\to\infty}{\text{limit}}\, \{f(x) - f_n(x)\}^+ dx = 0 \ .$$

From (1.20),

$$(1.22) \qquad \underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^- dx = \underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^+ dx = 0 \ .$$

Substituting (1.19) in (1.18), we have

$$\underset{-\infty<t<\infty}{\text{supremum}} |\phi_n(t) - \phi(t)| \leq \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^+ dx + \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^- dx \ .$$

From (1.21) and (1.22), we obtain finally that

$$0 \leq \underset{n\to\infty}{\text{limit}}\ \underset{-\infty<t<\infty}{\text{supremum}} |\phi_n(t) - \phi(t)|$$

$$\leq \underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^+ dx + \underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} \{f(x) - f_n(x)\}^- dx = 0 \ .$$

This shows that the convergence is uniform with respect to t as $n \to \infty$. From [5], we know $P\left\{ \underset{n\to\infty}{\text{limit}}\ \underset{-\infty<x<\infty}{\text{supremum}} |f_n(x) - f(x)| = 0 \right\} = 1$. This indicates that the probability of getting exceptional random sequences

$X_1(\omega)$, $X_2(\omega)$, ..., $X_n(\omega)$, ... is zero. These exceptional sequences are the only ones for which it may not be true that

$$\underset{n\to\infty}{\text{limit}}\ \underset{-\infty<t<\infty}{\text{supremum}} |\phi_n(t) - \phi(t)| = 0 .$$

Hence $\phi_n(t) \to \phi(t)$ uniformly with probability one.

We obtain the following theorems.

Theorem 1.5.1. Suppose $F(x)$ is absolutely continuous, and $f(x)$ is uniformly continuous. If $f_n(x)$ converges uniformly to $f(x)$ with probability one, then $\phi_n(t)$ converges uniformly to $\phi(t)$, the characteristic function of $F(x)$, with probability one.

Theorem 1.5.2. Suppose $\phi(t)$ and $\psi(th)$ are absolutely integrable over $(-\infty, \infty)$, and for any $\delta > 0$, $k(y)$ satisfies the Lipschitz condition of order α $(0 < \alpha \leq 1)$, and that $\phi_n(t)$ converges uniformly to $\phi(t)$ with probability one. Then $f_n(x)$ converges uniformly to $f(x)$, the uniform continuous density of $F(x)$, with probability one.

Proof : Since $\phi(t)$ is absolutely integrable over $(-\infty, \infty)$, and $F(x)$ is absolutely continuous, then by the Inversion Theorem, the density

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt ,$$

where $f(x)$ is continuous everywhere. In fact, it is uniformly continuous. To see this observe that we have, for any $A > 0$, $c > 0$,

$$(1.23) \qquad |f(x+c) - f(x)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |e^{-itc} - 1||\phi(t)| \, dt$$

$$\leq \frac{1}{\pi} \left\{ \int_{|t| \leq A} |\sin \tfrac{tc}{2}||\phi(t) \, dt + \int_{|t|>A} |\sin \tfrac{tc}{2}||\phi(t) \, dt \right\} .$$

For any $\varepsilon > 0$, we may choose c sufficiently small, that

$$\frac{1}{\pi} \int_{|t| \leq A} |\sin \tfrac{tc}{2}||\phi(t)| dt \leq \frac{Ac}{2\pi} \int_{|t| \leq A} |\phi(t)| dt < \frac{\varepsilon}{2} \ ;$$

and choose A sufficiently large, that

$$\frac{1}{\pi} \int_{|t|>A} |\sin \tfrac{tc}{2}||\phi(t)| dt \leq \frac{1}{\pi} \int_{|t|>A} |\phi(t)| dt < \frac{\varepsilon}{2} \ .$$

Hence $f(x)$ is continuous, and since the bounds above are independent of x, so it is also uniformly continuous.

Similarly, $\psi(th)$ is absolutely integrable over $(-\infty, \infty)$, so

$$k(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ithy} \, \psi(th) \, d(th) \quad ,$$

or

$$k\left(\frac{x - X_j}{h} \right) = \frac{h}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left\{ e^{itX_j} \, \psi(th) \right\} dt \ .$$

It follows that

$$(1.24) \qquad f_n(x) = \frac{1}{nh} \sum_{j=1}^{n} k\left(\frac{x - X_j}{h} \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left\{ \frac{1}{n} \sum_{j=1}^{n} e^{itX_j} \, \psi(th) \right\} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \, \phi_n(t) \, dt \quad .$$

Since $f(x) = F'(x)$ and $f_n(x) = F'_n(x)$ are assumed to be defined everywhere, for any $\Delta > 0$, we may let

$$\omega(x, \Delta) = \begin{cases} \dfrac{F(x + \Delta) - F(x - \Delta)}{2\Delta}, & \text{if } \Delta \neq 0, \\[2ex] f(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt, & \text{if } \Delta = 0; \end{cases}$$

and similarly,

$$\omega_n(x, \Delta) = \begin{cases} \dfrac{F_n(x + \Delta) - F_n(x - \Delta)}{2\Delta}, & \text{if } \Delta \neq 0, \\[2ex] f_n(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{-itx} \phi_n(t) dt, & \text{if } \Delta = 0 . \end{cases}$$

Now, for any real x, we can write

$$(1.25) \qquad |f_n(x) - f(x)|$$

$$\leq |f_n(x) - \omega_n(x, \Delta)| + |\omega_n(x, \Delta) - \omega(x, \Delta)| + |\omega(x, \Delta) - f(x)| .$$

By the Mean Value Theorem, $\omega(x, \Delta) = f(\xi)$, $\omega_n(x, \Delta) = f_n(\xi)$ with $x - \Delta \leq \xi \leq x + \Delta$. There is strict inequality if $\Delta \neq 0$. Consider the first term on the right side of (1.25), for $\Delta \neq 0$,

$$(1.26) \qquad |f_n(x) - \omega_n(x, \Delta)| = |f_n(x) - f_n(\xi)|$$

$$\leq \frac{1}{nh} \sum_{j=1}^{n} \left| k\left(\frac{x - X_j}{h}\right) - k\left(\frac{\xi - X_j}{h}\right) \right| .$$

By hypothesis $k(y) \in \mathrm{Lip}(\alpha)$, $0 < \alpha \leq 1$, that is, for $\delta > 0$,

$$\underset{\substack{-\infty<y<\infty \\ |y'-y|\leq\delta}}{\text{supremum}} |k(y') - k(y)| \leq L\delta^{\alpha} ,$$

for some constant L. Now, we have $y = \frac{x - X_j}{h}$. Then

$$\left| \frac{x - X_j}{h} - \frac{\xi - X_j}{h} \right| = \left| \frac{x - \xi}{h} \right| \leq \frac{\Delta}{h} .$$

Let $\delta = \frac{\Delta}{h}$. Then (1.26) is not greater than

$$\frac{1}{nh} \sum_{j=1}^{n} \underset{-\infty<\frac{x-X_j}{h}<\infty}{\text{supremum}} \left| k\left(\frac{x - X_j}{h} \right) - k\left(\frac{\xi - X_j}{h} \right) \right|$$

$$\leq \frac{1}{nh} \sum_{j=1}^{n} L\left(\frac{\Delta}{h} \right)^{\alpha} = L\left(\frac{\Delta^{\alpha}}{h^{1+\alpha}} \right) .$$

Let $\delta^{\alpha} = o(h)$ as $n \to \infty$, it is clear that $L\left(\frac{\Delta^{\alpha}}{h^{1+\alpha}} \right) \to 0$ as $n \to \infty$.

For any $\varepsilon > 0$, there exists an integer $N_1 > 0$ such that, for $n > N_1$, one has, for $\Delta \neq 0$, $|f_n(x) - \omega_n(x, \Delta)| < \varepsilon/3$. We assume that $\phi_n(t) \to \phi(t)$ uniformly for all t with probability one as $n \to \infty$. By the continuity theorem, it follows that $F_n(x) \to F(x)$ uniformly as $n \to \infty$. Given $\varepsilon > 0$, there exists an integer $N_2 > 0$ not dependent on x such that $|F_n(x) - F(x)| < \varepsilon/6$ whenever $n > N_2$. One takes $N = \max\{N_1, N_2\}$, and for $n > N$, one has, for $\Delta \neq 0$ that

$$|\omega_n(x, \Delta) - \omega(x, \Delta)|$$

$$= \left| \frac{F_n(x + \Delta) - F_n(x - \Delta)}{2\Delta} - \frac{F(x + \Delta) - F(x - \Delta)}{2\Delta} \right|$$

$$= \left| \frac{F_n(x+\Delta) - F(x+\Delta)}{2\Delta} - \frac{F_n(x-\Delta) - F(x-\Delta)}{2\Delta} \right|$$

$$\leq \frac{1}{2\Delta} \left\{ |F_n(x+\Delta) - F(x+\Delta)| + |F_n(x-\Delta) - F(x-\Delta)| \right\}$$

$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3} .$$

For the last term of (1.25) $|\omega(x, \Delta) - f(x)| < \varepsilon/3$, when Δ is chosen small enough. Hence (1.25) is $|f_n(x) - f(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$, and $f_n(x) \to f(x)$ as $n \to \infty$. We obtain that

$$\phi_n(t) \to \phi(t) \implies f_n(x) \to f(x) \quad \text{as} \quad n \to \infty .$$

Thus $\underset{n\to\infty}{\text{limit}}\, f_n(x) = f(x)$. It follows that

$$\underset{n\to\infty}{\text{limit}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_n(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt$$

for every x. It certainly holds for $x = 0$, so

$$\underset{n\to\infty}{\text{limit}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_n(t)\, dt = \int_{-\infty}^{\infty} \phi(t)\, dt ,$$

or

$$\underset{n\to\infty}{\text{limit}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\phi_n(t) - \phi(t)\}\, dt = 0 . \tag{1.27}$$

Since $\phi_n(t)$ and $\phi(t)$ are both complex, from (1.27) one gets

$$\underset{n\to\infty}{\text{limit}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re}\{\phi_n(t) - \phi(t)\}\, dt = 0 ,$$

$$\underset{n\to\infty}{\text{limit}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Im}\{\phi_n(t) - \phi(t)\}\, dt = 0 .$$

For any $\varepsilon > 0$, and $n \geq N$, some constant N,

$$\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\{\phi_n(t) - \phi(t)\}\, dt \right| < \frac{\varepsilon}{2} ,$$

$$\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Im}\{\phi_n(t) - \phi(t)\}\, dt \right| < \frac{\varepsilon}{2} .$$

As the result of Riemann-Lebesque's Lemma, we know $\lim_{|x|\to\infty} f(x) = 0$.

For any real x ,

$$|f_n(x) - f(x)| = \left|\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\{\phi_n(t) - \phi(t)\}\, dt \right|$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi_n(t) - \phi(t)|\, dt$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathrm{Re}\{\phi_n(t) - \phi(t)\}|\, dt$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathrm{Im}\{\phi_n(t) - \phi(t)\}|\, dt .$$

The last expression is independent of x , so

(1.28) $$\sup_{-\infty<x<\infty} |f_n(x) - f(x)|$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathrm{Re}\{\phi_n(t) - \phi(t)\}|\, dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathrm{Im}\{\phi_n(t) - \phi(t)\}|\, dt .$$

Let

(1.29)
$$R_n^+(t) = \{\mathrm{Re}[\phi(t) - \phi_n(t)]\}^+ , \quad R_n^-(t) = \{\mathrm{Re}[\phi(t) - \phi_n(t)]\}^- ;$$

$$I_n^+(t) = \{\mathrm{Im}[\phi(t) - \phi_n(t)]\}^+ , \quad I_n^-(t) = \{\mathrm{Im}[\phi(t) - \phi_n(t)]\}^- .$$

It follows immediately that

$$
(1.30)\qquad
\begin{aligned}
\mathrm{Re}[\phi(t) - \phi_n(t)] &= R_n^+(t) - R_n^-(t) , \\
|\mathrm{Re}[\phi(t) - \phi_n(t)]| &= R_n^+(t) + R_n^-(t) , \\
\mathrm{Im}[\phi(t) - \phi_n(t)] &= I_n^+(t) - I_n^-(t) , \\
|\mathrm{Im}[\phi(t) - \phi_n(t)]| &= I_n^+(t) + I_n^-(t) .
\end{aligned}
$$

From (1.29) and (1.30), one has, for any $\varepsilon > 0$ and $n \geq N$, some constant N, that

$$
\begin{aligned}
&\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}[\phi(t) - \phi_n(t)]\, dt \right| \\
&\qquad = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} R_n^+(t)\, dt - \frac{1}{2\pi} \int_{-\infty}^{\infty} R_n^-(t)\, dt \right| < \frac{\varepsilon}{2}
\end{aligned}
$$

or

$$
0 < \frac{1}{2\pi} \int_{-\infty}^{\infty} R_n^-(t)\, dt < \frac{1}{2\pi} \int_{-\infty}^{\infty} R_n^+(t)\, dt + \frac{\varepsilon}{2} .
$$

Similarly for the imaginary part, one gets

$$
0 < \frac{1}{2\pi} \int_{-\infty}^{\infty} I_n^-(t)\, dt < \frac{1}{2\pi} \int_{-\infty}^{\infty} I_n^+(t)\, dt + \frac{\varepsilon}{2} ,
$$

hence (1.28) can be put into the form

$$\underset{-\infty<x<\infty}{\text{supremum}} \left| f_n(x) - f(x) \right|$$

$$\leq \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} R_n^+(t)\, dt + \int_{-\infty}^{\infty} R_n^-(t)\, dt + \int_{-\infty}^{\infty} I_n^+(t)\, dt + \int_{-\infty}^{\infty} I_n^-(t)\, dt \right\}$$

$$< \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} R_n^+(t)\, dt + \int_{-\infty}^{\infty} I_n^+(t)\, dt \right\} + \varepsilon \ ,$$

where $R_n^+(t) \leq \mathrm{Re}\phi(t)$ and $I_n^+(t) \leq \mathrm{Im}\phi(t)$ for all t , By the Lebesque Dominated Convergence Theorem

$$\int_{-\infty}^{\infty} R_n^+(t)\, dt \to 0 \quad \text{and} \quad \int_{-\infty}^{\infty} I_n^+(t)\, dt \to 0$$

as $n \to \infty$. Hence

$$\underset{n\to\infty}{\text{limit}}\ \underset{-\infty<x<\infty}{\text{supremum}} \left| f_n(x) - f(x) \right| = 0 \ ,$$

and this convergence is with probability one.

II. THE ESTIMATE $\phi_n^{(p)}(t)$ OF $\phi^{(p)}(t)$.

In this section, we assume that $E|X|^{2q}$ is finite for some positive integer q. If r is any positive integer less than or equal to $2q$, then $E|X|^r$ exists and is also finite, the characteristic function is r times differentiable. Let $0 < p \leq q$, then the p-th derivative of $\phi(t)$, denoted by $\phi^{(p)}(t)$, can be put into the following form :

$$\phi^{(p)}(t) = \frac{d^p}{dt^p}\phi(t) = \int_{-\infty}^{\infty} (ix)^p e^{itx} f(x)\, dx .$$

Naturally, we may choose the estimate of $\phi^{(p)}(t)$ as

$$\phi_n^{(p)} = \int_{-\infty}^{\infty} (ix)^p e^{itx} f_n(x)\, dx \tag{2.1}$$

where $f_n(x) = \frac{1}{nh}\sum_{j=1}^{n} k\left(\frac{x - X_j}{h}\right)$ is the kernel estimate given in [8]; $k(y)$ is some symmetric density which is assumed to have moments of all orders. This ensures that $\psi(th)$ is any number of times differentiable. Hence, since $\phi_n(t) = \psi(th)\left(\frac{1}{n}\sum_{j=1}^{n} e^{itX_j}\right)$, (2.1) can be also replaced by the following expression

$$\begin{aligned}
\phi_n^{(p)}(t) &= \frac{1}{nh}\sum_{j=1}^{n}\int_{-\infty}^{\infty} (ix)^p e^{itx} k\left(\frac{x - X_j}{h}\right) dx \\
&\doteq \frac{1}{n}\sum_{j=1}^{n}\left\{\sum_{\ell=0}^{p}\binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)\,(iX_j)^{\ell}\, e^{itX_j}\right\} \\
&= \frac{1}{n}\sum_{j=1}^{n}\left\{\sum_{\ell=0}^{p-1}\binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)\,(iX_j)^{\ell}\, e^{itX_j}\right\} \\
&\qquad + \frac{1}{n}\sum_{j=1}^{n}\psi(th)\,(iX_j)^{p}\, e^{itX_j} ,
\end{aligned} \tag{2.2}$$

where $0 < p \le q$, and

$$\psi^{(p-\ell)}(th) = \int_{-\infty}^{\infty} (iy)^{p-\ell}\, e^{ithy}\, k(y)\, dy .$$

For any real t,

$$\psi^{(p-\ell)}(th) \to \psi^{(p-\ell)}(0) = \begin{cases} 0 & \text{if } (p-\ell) \text{ is odd,} \\ \text{finite} & \text{if } (p-\ell) \text{ is even,} \end{cases}$$

as $n \to \infty$, $h \to 0$.

It follows that, as $n \to \infty$ and $h \to 0$,

$$(2.3) \quad \begin{cases} \frac{1}{n}\sum_{j=1}^{n} \left\{ h^p \psi^{(p)}(0) + \binom{p}{2} h^{p-2}\psi^{(p-2)}(0)(iX_j)^2 + \cdots \right. \\ \qquad \left. + \binom{p}{p-2} h^2 \psi^{(2)}(0)(iX_j)^{p-2} \right\} e^{itX_j} \\ \qquad + \frac{1}{n}\sum_{j=1}^{n} (iX_j)^p\, e^{itX_j}, \quad \text{when } p \text{ is even;} \\ \frac{1}{n}\sum_{j=1}^{n} \left\{ h^{p-1}\psi^{(p-1)}(0)(iX_j) + \binom{p}{3} h^{p-3}\psi^{(p-3)}(0)(iX_j)^3 + \cdots \right. \\ \qquad \left. + \binom{p}{p-2} h^2 \psi^{(2)}(0)(iX_j)^{p-2} \right\} e^{itX_j} \\ \qquad + \frac{1}{n}\sum_{j=1}^{n} (iX_j)^p\, e^{itX_j}, \quad \text{when } p \text{ is odd.} \end{cases}$$

It doesn't matter whether p is even or odd, one can see that the first partial sum in (2.2) becomes small and approaches zero as $n \to \infty$ and $h \to 0$.

However, the last term in (2.2) $\rightarrow \frac{1}{n}\sum_{j=1}^{n}(iX_j)^p e^{itX_j}$ as $n \rightarrow \infty$.

Hence, when n is very large, one can simply approximate (2.2) by $\left\{\frac{1}{n}\sum_{j=1}^{n}\psi(th)(iX_j)^p e^{itX_j}\right\}$, which approaches to $\frac{1}{n}\sum_{j=1}^{n}(iX_j)^p e^{itX_j}$ as $n \rightarrow \infty$ and $\psi(th) \rightarrow 1$.

II-1. Assymptotic Unbiasedness.

Let $\tilde{\phi}_n^{(p)}(t) = \frac{1}{n}\sum_{j=1}^{n}\psi(th)(iX_j)^p e^{itX_j}$. Since the expectations of all terms in (2.2) are finite, the term involving $h^{p-\ell}$ goes to zero at the same rate as $h^{p-\ell}$ does. The term of h^o or 1 is the one involved in calculating $E[\phi_n^{(p)}(t)]$, $Var[\phi_n^{(p)}(t)]$ and etc. $\tilde{\phi}_n^{(p)}(t)$ is the term of h^o in (2.2). For the asymptotic properties, one may just study those of $\tilde{\phi}_n^{(p)}(t)$; $\phi_n^{(p)}(t)$ should have the same asymptotic properties as $\tilde{\phi}_n^{(p)}(t)$, since

$$(2.4) \qquad E\left\{\phi_n^{(p)}(t) - \phi^{(p)}(t)\right\} = E\left\{\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)\right\} + E\left\{\tilde{\phi}_n^{(p)}(t) - \phi^{(p)}(t)\right\}$$

where $E\left\{\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)\right\} = \sum_{\ell=0}^{p-1}\binom{p}{\ell} h^{p-\ell}\,\psi^{(p-\ell)}(th)\,\phi^{(\ell)}(t)$.

Since $n \rightarrow \infty$, $h \rightarrow 0$, and $|\phi^{(\ell)}(t)| \leq E|X|^\ell$ is finite, and since $\psi^{(p-\ell)}(th) \rightarrow \psi^{(p-\ell)}(0) \leq |\psi^{(p-\ell)}(0)|$ is finite, for $\ell = 0, 1, \ldots, p$, it follows immediately that $E\left\{\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)\right\} \rightarrow 0$ as $n \rightarrow \infty$.

$$\text{(2.5)}\qquad E\left\{\tilde{\phi}_n^{(p)}(t) - \phi^{(p)}(t)\right\} = E\left\{\frac{1}{n}\sum_{j=1}^{n} \psi(th)(iX_j)^p\, e^{itX_j} - \phi^{(p)}(t)\right\}$$

$$= \left\{\ \psi(th) - 1\ \right\} \phi^{(p)}(t) \ \rightarrow\ 0\ ,$$

since, as $n \rightarrow \infty$, $\psi(th) \rightarrow \psi(0) = 1$ (in fact $\{\psi(th) - 1\} = 0(h^2))$. Hence $E\left\{\tilde{\phi}_n^{(p)}(t)\right\} \rightarrow \phi^{(p)}(t)$ as $n \rightarrow \infty$. From (2.5), we know that $\phi_n^{(p)}(t)$ is also asymptotically unbiased.

II-2. Quadratic Consistency.

The mean square error of $\tilde{\phi}_n^{(p)}(t)$ can be written as

$$\text{(2.6)}\qquad E|\tilde{\phi}_n^{(p)}(t) - \phi^{(p)}(t)|^2 \ =\ \mathrm{Var}[\tilde{\phi}_n^{(p)}(t)] + |b[\tilde{\phi}_n^{(p)}(t)]|^2\ ,$$

where $|b[\phi_n^{(p)}(t)]| = |E\{\tilde{\phi}_n^{(p)}(t)\} - \phi^{(p)}(t)|$ and

$$\mathrm{Var}[\tilde{\phi}_n^{(p)}(t)] \ =\ \frac{\psi^2(th)}{n}\,\mathrm{Var}[(iX)^p e^{itX}]\ .$$

Since $E\left\{\tilde{\phi}_n^{(p)}(t) - \phi^{(p)}(t)\right\} \rightarrow 0$ as $n \rightarrow \infty$, as shown above,

$$|b[\phi_n^{(p)}(t)]| \ =\ |E\{\tilde{\phi}_n^{(p)}(t)\} - \phi^{(p)}(t)| \ \rightarrow\ 0 \ \text{ as } \ n \rightarrow \infty\ .$$

For $\mathrm{Var}[\tilde{\phi}_n^{(p)}(t)]$, we need to compute the variance of $\{(iX)^p\, e^{itX}\}$.

(a) When p is even,

$$\text{(2.7)}\qquad \begin{aligned} \mathrm{Re}\{(iX)^p\, e^{itX}\} &= (-1)^{p/2}\, X^p \cos tX\ , \\ \mathrm{Im}\{(iX)^p\, e^{itX}\} &= (-1)^{p/2}\, X^p \sin tX\ ; \end{aligned}$$

(b) When p is odd,

$$\text{Re}\{(iX)^p e^{itX}\} = (-1)^{\frac{p+1}{2}} X^p \sin tX ,$$

(2.8)

$$\text{Im}\{(iX)^p e^{itX}\} = (-1)^{\frac{p-1}{2}} X^p \cos tX .$$

Hence in case (a), where $p > 0$ is even,

$$\text{Re}\ \phi^{(p)}(t) = (-1)^{p/2}\ E\{X^p \cos tX\} ,$$

(2.9) $$\text{Im}\ \phi^{(p)}(t) = (-1)^{p/2}\ E\{X^p \sin tX\} ,$$

$$\text{Var}[(iX)^p e^{itX}] = \text{Var}[X^p \cos tX] + \text{Var}[X^p \sin tX] ;$$

in case (b), where p is odd,

$$\text{Re}\ \phi^{(p)}(t) = (-1)^{\frac{p+1}{2}}\ E\{X^p \sin tX\} ,$$

$$\text{Im}\ \phi^{(p)}(t) = (-1)^{\frac{p-1}{2}}\ E\{X^p \cos tX\} ,$$

$$\text{Var}[(iX)^p e^{itX}] = (-1)^{p+1}\ \text{Var}[X^p \sin tX] + (-1)^{p-1}\ \text{Var}[X^p \cos tX].$$

Since $(p + 1)$ and $(p - 1)$ are both even, if p is odd,

(2.10) $$\text{Var}[(iX)^p e^{itX}] = \text{Var}[X^p \sin tX] + \text{Var}[X^p \cos tX] .$$

From (2.9) and (2.10), one sees that the variance of $\{(iX)^p e^{itX}\}$ is the same in either cases (p even or odd). For $p \leq q$, we have

$$E\{X^{2p}\cos^2 tX\} = \frac{1}{2}\left\{ E(X)^{2p} + \frac{1}{2^{2p}} \operatorname{Re} \phi^{(2p)}(2t) \right\},$$

$$E\{X^{2p}\sin^2 tX\} = \frac{1}{2}\left\{ E(X)^{2p} - \frac{1}{2^{2p}} \operatorname{Re} \phi^{(2p)}(2t) \right\},$$

$$E\{X^{2p}\sin tX \cos tX\} = \frac{1}{2^{2p+1}} \operatorname{Im} \phi^{(2p)}(2t) .$$

Thus,

$$\operatorname{Var}[X^p \cos tX] = \frac{1}{2}\left\{ E(X)^{2p} + \frac{1}{2^{2p}} \operatorname{Re} \phi^{(2p)}(2t) \right\} - \left\{ \operatorname{Re} \phi^{(p)}(t) \right\}^2 ,$$

(2.11)
$$\operatorname{Var}[X^p \sin tX] = \frac{1}{2}\left\{ E(X)^{2p} - \frac{1}{2^{2p}} \operatorname{Re} \phi^{(2p)}(2t) \right\} - \left\{ \operatorname{Im} \phi^{(p)}(t) \right\}^2 ,$$

$$\operatorname{Cov}[X^p \cos tX,\ X^p \sin tX] = \frac{1}{2^{2p+1}} \operatorname{Im} \phi^{(2p)}(2t) - \operatorname{Re} \phi^{(p)}(t) \operatorname{Im} \phi^{(p)}(t) .$$

From the above computations, we obtain that

(2.12)
$$\operatorname{Var}[(iX)^p e^{itX}] = \left\{ E|X|^{2p} - |\phi^{(p)}(t)|^2 \right\},$$

where $E|X|^{2p}$ and $|\phi^{(p)}(t)|^2$ are both finite.

The variance of $\tilde{\phi}_n^{(p)}(t)$,

$$\frac{\psi^2(th)}{n} \operatorname{Var}[(iX)^p e^{itX}] = \frac{\psi^2(th)}{n} \left\{ E|X|^{2p} - |\phi^{(p)}(t)|^2 \right\}$$

approaches zero as $n \to \infty$ and $\psi(th) \to \psi(0) \equiv 1$. It follows that $E|\tilde{\phi}_n^{(p)}(t) - \phi^{(p)}(t)|^2 \to 0$ as $n \to \infty$, so $\tilde{\phi}_n^{(p)}(t)$ is quadractically consistent.

To see $\phi_n^{(p)}(t)$ is quadractically consistent, we need to show $E|\phi_n^{(p)}(t) - \phi^{(p)}(t)|^2 \to 0$ as $n \to \infty$. Now,

$$E|\phi_n^{(p)}(t) - \phi^{(p)}(t)|^2 \leq E|\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)|^2 + E|\tilde{\phi}_n^{(p)}(t) - \phi^{(p)}(t)|^2 , \tag{2.13}$$

and

$$E|\phi_n^{(p)}(t) - \tilde{\phi}^{(p)}(t)|^2 = \mathrm{Var}[\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)] + |E\{\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)\}|^2 .$$

Here we have

$$\{\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)\} = \frac{1}{n}\sum_{j=1}^{n}\left\{\sum_{\ell=0}^{p-1}\binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)(iX_j)^{\ell} e^{itX_j}\right\},$$

Then

$$E\{\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)\} = \sum_{\ell=0}^{p-1}\binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)\phi^{(\ell)}(t) = o(h) ,$$

$$|E\{\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)\}|^2 = o(h^2) , \tag{2.14}$$

as $n \to \infty$ and $h \to 0$.

For the variance of $[\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)]$, we have

$$\mathrm{Var}[\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)] = \frac{1}{n^2}\mathrm{Var}\left[\sum_{j=1}^{n}\sum_{\ell=0}^{p-1}\binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)(iX_j)^{\ell} \times e^{itX_j}\right]. \tag{2.15}$$

Since $X_1, X_2, \ldots, X_n$ are independent identically distributed, (2.14) can be written as

$$\mathrm{Var}[\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)] = \frac{1}{n}\,\mathrm{Var}\left[\sum_{\ell=0}^{p-1}\binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)(iX)^{\ell} e^{itX}\right]$$

$$= o(h^2) \qquad \text{as } n \to \infty\ ,\ h \to 0.$$

The exact calculations will be given in the following section II-3. As a result,

$$E|\phi_n^{(p)}(t) - \tilde{\phi}_n^{(p)}(t)|^2 = o(h^2) \qquad \text{as } n \to \infty\ ,\ h \to 0\ .$$

Hence

$$E|\tilde{\phi}_n^{(p)}(t) - \phi^{(p)}(t)|^2 = \frac{[\psi(th)]^2}{n}\left\{E|X|^{2p} - |\phi^{(p)}(t)|^2\right\} + o(h^2)$$

$$\to 0 \quad \text{as } n \to \infty \text{ and } h \to 0\ .$$

Clearly $\phi_n^{(p)}(t)$ is also quadractically consistent.

II-3. Asymptotic Normality of $\phi_n^{(p)}(t)$.

Consider $\tilde{\phi}_n^{(p)}(t)$ as the average of the independent identically distributed complex random variables, $Z_{n1}, Z_{n2}, \ldots, Z_{nn}$. That is $\tilde{\phi}_n^{(p)}(t) = \frac{1}{n}\sum_{j=1}^{n} Z_{nj}$, where $Z_{nj} = Z_j = (iX_j)^p\psi(th)e^{itX_j}$, for $j = 1, 2, \ldots, n$. Treat each complex random variable as a two dimensional random vector, such as

$$(-1)^{\frac{p}{2}} \left(X_j^p\, \psi(th) \cos tX_j,\ X_j^p\, \psi(th) \sin tX_j \right), \quad \text{if } p \text{ is even;}$$

and

$$(-1)^{\frac{p+1}{2}} \left(X_j^p\, \psi(th) \sin tX_j, -X_j^p\, \psi(th) \cos tX_j \right), \quad \text{if } p \text{ is odd.}$$

Hence, in the case, p is even,

$$\psi^2(th)\sigma_{11} = \psi^2(th)\ \mathrm{Var}[X^p \cos tX] \to \sigma_{11}\ ,$$

$$\psi^2(th)\sigma_{22} = \psi^2(th)\ \mathrm{Var}[X^p \sin tX] \to \sigma_{22}\ ,$$

$$\psi^2(th)\sigma_{12} = \psi^2(th)\ \mathrm{Cov}[X^p \cos tX,\ X^p \sin tX] \to \sigma_{12}\ , \tag{2.16}$$

$$\psi(th)\ \mu_1 = \psi(th)\ E\{X^p \cos tX\} \to \mu_1\ ,$$

$$\psi(th)\ \mu_2 = \psi(th)\ E\{X^p \sin tX\} \to \mu_2\ , \qquad \text{as} \quad n \to \infty\ .$$

In the case, p is odd,

$$\psi^2(th)\hat{\sigma}_{11} = \psi^2(th)\ \mathrm{Var}[X^p \sin tX] \to \sigma_{22}\ ,$$

$$\psi^2(th)\hat{\sigma}_{22} = \psi^2(th)\ \mathrm{Var}[X^p \cos tX] \to \sigma_{11}\ ,$$

$$\psi^2(th)\hat{\sigma}_{12} = (-1)\ \psi^2(th)\ \mathrm{Cov}[X^p \sin tX,\ X^p \cos tX] \to -\sigma_{21}\ , \tag{2.17}$$

$$\psi(th)\ \hat{\mu}_1 = (-1)^{\frac{p+1}{2}}\ \psi(th)\ E\{X^p \sin tX\} \to (-1)^{\frac{p+1}{2}}\ \mu_2\ ,$$

$$\psi(th)\ \hat{\mu}_2 = (-1)^{\frac{p-1}{2}}\ \psi(th)\ E\{X^p \cos tX\} \to (-1)^{\frac{p-1}{2}}\ \mu_1\ , \quad \text{as} \quad n \to \infty\ .$$

All $\mu_1, \mu_2, \sigma_{11}, \sigma_{22}$ and σ_{12} are finite. Then the covariance matrices,

$$\frac{\psi^2(th)}{n}\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix} \sim \frac{1}{n}\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}, \quad \text{when } p \text{ is even;}$$

in the case, when p is odd,

$$\frac{\psi^2(th)}{n}\begin{pmatrix}\hat\sigma_{11} & \hat\sigma_{12}\\ \hat\sigma_{21} & \hat\sigma_{22}\end{pmatrix} \sim \frac{1}{n}\begin{pmatrix}\sigma_{22} & -\sigma_{21}\\ -\sigma_{12} & \sigma_{11}\end{pmatrix} \qquad \text{as } n \to \infty .$$

Clearly $\tilde\phi_n^{(p)}(t)$ is asymptotically normal.

Similarly, we consider $\phi_n^{(p)}(t)$ as an average of the independent identically distributed complex variables, $\omega_{n1}, \omega_{n2}, \ldots, \omega_{nn}$, such as $\phi_n^{(p)}(t) = \frac{1}{n}\sum_{j=1}^{n}\omega_{nj}$, where $\omega_{nj} = \sum_{\ell=0}^{p}\binom{p}{\ell}h^{p-\ell}\psi^{(p-\ell)}(th)(iX_j)^{\ell}e^{itX_j}$ for $j = 1, 2, \ldots, n$. We want to show that $\phi_n^{(p)}(t)$ is also asymptotically normal. For each ω_{nj} is considered in a two-dimensional random vector, such as $(\text{Re}\,\omega_{nj}, \text{Im}\,\omega_{nj})$.

The expectations of $\text{Re}\,\phi_n^{(p)}(t)$ and $\text{Im}\,\phi_n^{(p)}(t)$, when p is even

$$\begin{aligned} E\left\{\text{Re}\,\phi_n^{(p)}(t)\right\} &= E\left\{\frac{1}{n}\sum_{j=1}^{n}\text{Re}\,\omega_{nj}\right\} = E\{\text{Re}\,\omega_n\} = \sum_{\ell=0}^{p} a_\ell\,\text{Re}\,\phi^{(\ell)}(t) ,\\ E\left\{\text{Im}\,\phi_n^{(p)}(t)\right\} &= E\left\{\frac{1}{n}\sum_{j=1}^{n}\text{Im}\,\omega_{nj}\right\} = E\{\text{Im}\,\omega_n\} = \sum_{\ell=0}^{p} a_\ell\,\text{Im}\,\phi^{(\ell)}(t) , \end{aligned} \tag{2.18}$$

where $a_\ell = \binom{p}{\ell}h^{p-\ell}\psi^{(p-\ell)}(th)$, and $a_p = \psi(th)$.

From (2.16), we can put

$$\sum_{\ell=0}^{p} a_\ell \text{ Re } \phi^{(\ell)}(t) = \psi(th)\text{Re } \phi^{(p)}(t) + \sum_{\ell=0}^{p-1} a_\ell \text{ Re } \phi^{(\ell)}(t) ,$$

and for $\sum_{\ell=0}^{p-1} a_\ell \text{ Re } \phi^{(\ell)}(t) = \sum_{\ell=0}^{p-1} \binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)\text{Re } \phi^{(\ell)}(t) = o(h)$

as $n \to \infty$. It follows immediately from (2.16) and (2.18) that, we have

$$E\left\{\text{Re } \phi_n^{(p)}(t)\right\} = E\left\{\text{Re } \tilde{\phi}_n^{(p)}(t)\right\} + o(h) ,$$

and

$$E\left\{\text{Im } \phi_n^{(p)}(t)\right\} = E\left\{\text{Im } \tilde{\phi}_n^{(p)}(t)\right\} + o(h) ,$$

as $n \to \infty$. The variances of $\text{Re } \phi_n^{(p)}(t)$ and $\text{Im } \phi_n^{(p)}(t)$ are

$$\text{Var}[\text{Re } \phi_n^{(p)}(t)] = \frac{1}{n} \text{Var}[\sum_{\ell=0}^{p} a_\ell U_\ell] ,$$

$$\text{Var}[\text{Im } \phi_n^{(p)}(t)] = \frac{1}{n} \text{Var}[\sum_{\ell=0}^{p} a_\ell V_\ell] ,$$

where $U_\ell = \text{Re}(iX)^\ell e^{itX}$, and $V_\ell = \text{Im}(iX)^\ell e^{itX}$ for $\ell = 0, 1, 2, \ldots, p$,

since $\left(\sum_{\ell=0}^{p} a_\ell U_\ell\right)^2 = \sum_{\ell=0}^{p} a_\ell^2 U_\ell^2 + 2 \sum\sum_{0\leq\ell<m\leq p} a_\ell a_m U_\ell U_m$. $\left(\sum_{\ell=0}^{p} a_\ell V_\ell\right)^2$ is

similar. It simply replaces U_ℓ, U_m by V_ℓ, V_m . Now

$$\text{Var}[\text{Re } \phi_n^{(p)}(t)] = \frac{1}{n} \left\{ \sum_{\ell=0}^{p} a_\ell^2 \text{ Var}[U_\ell] + 2 \sum\sum_{0\leq\ell<m\leq p} a_\ell a_m \text{ Cov}[U_\ell, U_m] \right\}$$

$$= \frac{1}{n} \left\{ a_p^2 \text{ Var}[U_p] + \sum_{\ell=0}^{p-1} a_\ell^2 \text{ Var}[U_\ell] \right.$$

$$\left. + 2 \sum\sum_{0\leq\ell<m\leq p} a_\ell a_m \text{ Cov}[U_\ell, U_m] \right\},$$

where $a_p^2 \operatorname{Var}[U_p] = \psi^2(th) \operatorname{Var}[X^p \cos tX]$, and

$$\sum_{\ell=0}^{p-1} a_\ell^2 \operatorname{Var}[U_\ell] = \sum_{\ell=0}^{p-1} \left[\binom{p}{\ell} h^{p-\ell}\psi^{(p-\ell)}(th)\right]^2 \operatorname{Var}[U_\ell] = o(h^2) \quad \text{as} \quad n \to \infty,$$

$$\sum_{0\leq\ell<m\leq p}\sum a_\ell a_m \operatorname{Cov}[U_\ell, U_m]$$

$$= \sum_{0\leq\ell<m\leq p}\sum \binom{p}{\ell}\binom{p}{m} h^{2p-(\ell+m)} \psi^{(p-\ell)}(th)\psi^{(p-m)}(th)\operatorname{Cov}[U_\ell, U_m]$$

$$= o(h) \qquad \text{as} \quad n \to \infty .$$

Finally, we have, as $n \to \infty$,

$$\operatorname{Var}[\operatorname{Re} \phi_n^{(p)}(t)] = \frac{1}{n}\left\{ \psi^2(th) \operatorname{Var}[X^p \cos tX] + o(h) \right\}$$

$$= \operatorname{Var}[\operatorname{Re} \tilde{\phi}_n^{(p)}(t)] + o(h) .$$

Similarly, the variance of $\operatorname{Im} \phi_n^{(p)}(t)$ is

$$\operatorname{Var}[\operatorname{Im} \phi_n^{(p)}(t)] = \operatorname{Var}[\operatorname{Im} \tilde{\phi}_n^{(p)}(t)] + o(h) .$$

For the covariance of $\operatorname{Re} \phi_n^{(p)}(t)$ and $\operatorname{Im} \phi_n^{(p)}(t)$,

$$\operatorname{Cov}[\operatorname{Re} \phi_n^{(p)}(t), \operatorname{Im} \phi_n^{(p)}(t)]$$

$$= \frac{1}{n} \operatorname{Cov}\left[\sum_{\ell=0}^{p} a_\ell U_\ell, \sum_{\ell=0}^{p} a_\ell V_\ell \right]$$

$$= \frac{1}{n}\left\{ \sum_{\ell=0}^{p} a_\ell^2 \operatorname{Cov}[U_\ell, V_\ell] + 2 \sum_{0\leq\ell<m\leq p}\sum a_\ell a_m \operatorname{Cov}[U_\ell, V_m] \right\}$$

$$= \frac{1}{n}\Big\{ a_p^2 \operatorname{Cov}[U_p, V_p] + \sum_{\ell=0}^{p-1} a_\ell^2 \operatorname{Cov}[U_\ell, V_\ell]$$

$$+ 2 \sum_{0\leq\ell<m\leq p}\sum a_\ell a_m \operatorname{Cov}[U_\ell, V_m] \Big\} ,$$

where

$$a_p^2 \operatorname{Cov}[U_p, V_p] = \psi^2(th) \operatorname{Cov}[X^p \cos tX, X^p \sin tX] ,$$

$$\sum_{\ell=0}^{p-1} a_\ell^2 \operatorname{Cov}[U_\ell, V_\ell] = o(h^2) ,$$

$$\sum_{0\leq\ell<m\leq p}\sum a_\ell a_m \operatorname{Cov}[U_\ell, V_m] = o(h) \qquad \text{as} \quad n \to \infty .$$

It follows that, as $n \to \infty$,

$$\operatorname{Cov}[\operatorname{Re} \phi_n^{(p)}(t), \operatorname{Im} \phi_n^{(p)}(t)] = \operatorname{Cov}[\operatorname{Re} \tilde{\phi}_n^{(p)}(t), \operatorname{Im} \tilde{\phi}_n^{(p)}(t)] + o(h).$$

The covariance matrix for $\operatorname{Re} \phi_n^{(p)}(t)$ and $\operatorname{Im} \phi_n^{(p)}(t)$ is

$$\frac{\psi^2(th) + o(h)}{n} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \sim \frac{1}{n} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \qquad \text{as} \quad n \to \infty .$$

If p is odd, the covariance matrix is

$$\frac{\psi^2(th) + o(h)}{n} \begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{pmatrix} \sim \frac{1}{n} \begin{pmatrix} \sigma_{22} & -\sigma_{21} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix} \qquad \text{as} \quad n \to \infty .$$

Since $\tilde{\phi}_n^{(p)}(t)$ is asymptotically normal, so $\phi_n^{(p)}(t)$ is .

II-4. Uniform Consistency of $\phi_n^{(p)}(t)$ with Probability One.

The absolute moment of order q of $F(x)$ is defined as $E|X|^q = \int_{-\infty}^{\infty} |x|^q f(x)\, dx$, for $q > 0$ and is assumed to be finite. We approximate $E|X|^q$ by

(2.19)
$$E_n|X|^q = \int_{-\infty}^{\infty} |x|^q \, dF_n(x)$$
$$= \int_{-\infty}^{\infty} |x|^q f_n(x)\, dx .$$

Assume that $\sup_n E_n|X|^q = M < \infty$, for some constant M. Since $F_n(x) \to F(x)$ uniformly for all x as $n \to \infty$, then according to a Theorem 4.52 in Chung's Book [1], for each $r < q$,

$$\operatorname*{limit}_{n\to\infty} E_n|X|^r = E|X|^r \quad .$$

In this section, we choose only $p < q$, then

(2.20)
$$\operatorname*{limit}_{n\to\infty} E_n|X|^p = E|X|^p \quad ,$$

or

(2.21)
$$\operatorname*{limit}_{n\to\infty} \int_{-\infty}^{\infty} |x|^p \, dF_n(x) = \operatorname*{limit}_{n\to\infty} \int_{-\infty}^{\infty} |x|^p f_n(x)\, dx$$
$$= \int_{-\infty}^{\infty} |x|^p \, dF(x) = \int_{-\infty}^{\infty} |x|^p f(x)\, dx .$$

In order to see that $\phi_n^{(p)}(t) \to \phi^{(p)}(t)$ uniformly for all t with probability one as $n \to \infty$, we need to show that $|x|^p f_n(x) \to |x|^p f(x)$

uniformly for all x with probability one.

Let $E|X|^p = M_o$, and $E_n|X|^p = M_n$ for every n, then, $\lim_{n\to\infty} M_n = M_o$, for some non-zero constants M_o and M_n. Consider the density function

$$g(x) = \frac{|x|^p f(x)}{E|X|^p} = \frac{1}{M_o} |x|^p f(x) ,$$

and its estimate

$$g_n(x) = \frac{1}{M_n} |x|^p f_n(x) = \frac{1}{M_n}\left\{ \frac{1}{nh} \sum_{j=1}^{n} |x|^p k\left(\frac{x - X_j}{h} \right)\right\}.$$

They are continuous everywhere.

Let $|x|^p f(x)$ be uniformly continuous over $(-\infty, \infty)$, and $|x|^p k\left(\frac{x - X_j}{h}\right)$ be of bounded variation over $(-\infty, \infty)$. Clearly, $g(x)$ is uniformly continuous. We claim the following

Lemma 2.4.1. Suppose $|x|^p k\left(\frac{x - X_j}{h}\right)$ is of bounded variation over $(-\infty, \infty)$, and that $|x|^p f(x)$ is uniformly continuous, and that the series $\sum_{n=1}^{\infty} e^{-\gamma nh^2}$ converges for every positive value of γ, then

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} |g_n(x) - g(x)| = 0 , \tag{2.22}$$

with probability one.

Proof : With the above assumptions, the proof can be followed from a theorem of E.A. Nadaraya [6] .

Now, with $\lim_{n\to\infty} M_n = M_o$, one can easily obtain that

$$\text{(2.23)} \qquad \underset{n\to\infty}{\text{limit}} \ \underset{-\infty<x<\infty}{\text{supremum}} \ |x|^p |f_n(x) - f(x)| = 0 \ ,$$

with probability one.

Theorem 2.4.1. Suppose each $F_n(x)$ and $F(x)$ are absolutely continuous. Then, for $0 < p < q$, $\underset{n\to\infty}{\text{limit}} \ \underset{-\infty<x<\infty}{\text{supremum}} \ |x|^p |f_n(x) - f(x)| = 0$ with probability one implies $\underset{n\to\infty}{\text{limit}} \ \underset{-\infty<x<\infty}{\text{supremum}} \ |\phi_n^{(p)}(t) - \phi^{(p)}(t)| = 0$ with probability one.

Proof : For any real t, we have

$$|\phi_n^{(p)}(t) - \phi^{(p)}(t)| = \left|\int_{-\infty}^{\infty} e^{itx} |f_n(x) - f(x)| \, |x|^p \, dx\right|$$

$$\leq \int_{-\infty}^{\infty} |x|^p |f_n(x) - f(x)| \, dx \ ,$$

this expression on the right is independent of t, so

$$\text{(2.24)} \qquad \underset{-\infty<t<\infty}{\text{supremum}} \ |\phi_n^{(p)}(t) - \phi^{(p)}(t)| \leq \int_{-\infty}^{\infty} |x|^p |f_n(x) - f(x)| \, dx \ .$$

We use the same method as given in Section I. Let

$$D_n^+(x) = \left\{|x|^p \ (f(x) - f_n(x))\right\}^+ \ ,$$

$$D_n^-(x) = \left\{|x|^p \ (f(x) - f_n(x))\right\}^- \ .$$

Clearly, $D_n^+(x) \leq |x|^p f(x)$ for every x. By Lebesque's Dominated Convergence Theorem, we have

$$\underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} D_n^+(x)\, dx = 0 \;,$$

and by (2.21)

$$\underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} |x|^p\{f(x) - f_n(x)\}\, dx = 0$$

$$= \underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} \{D_n^+ - D_n^-\}(x)\, dx \;.$$

Hence
$$\underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} D_n^+(x)\, dx = \underset{n\to\infty}{\text{limit}} \int_{-\infty}^{\infty} D_n^-(x)\, dx = 0 \;.$$

Finally (2.24) can be put as

$$\underset{-\infty<x<\infty}{\text{supremum}} |\phi_n^{(p)}(t) - \phi^{(p)}(t)|$$

$$\leq \int_{-\infty}^{\infty} |x|^p |f_n(x) - f(x)|\, dx$$

$$\leq \int_{-\infty}^{\infty} D_n^+(x)\, dx + \int_{-\infty}^{\infty} D_n^-(x)\, dx \;\to\; 0 \quad \text{as} \quad n \to \infty \;.$$

This shows that the convergence is uniform for all t, and furthermore, it is with probability one as $n \to \infty$.

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