CARTESIAN PRODUCTS OF LENS SPACES

AND THE KUNNETH FORMULA

by

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Abstract

The graded cohomology groups of a cartesian product of two cellular spaces are expressible in terms of the cohomology groups of the factors. This relationship is given by the (split) short exact Künneth sequence.

However the multiplicative structure on the cohomology of a cartesian product can in general not be derived by solely referring to the Künneth formula.

In this thesis we explicitly exhibit the cup product structure on a cartesian product of two (standard) lens spaces.

This result is obtained by analyzing the Künneth sequence and by making use of the particular geometry of the spaces involved.
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1. Introduction

The Künneth formula relates the cohomology groups of a cartesian product of topological spaces to the cohomology groups of its factors. If $X$ and $Y$ are (finite) CW-complexes, then the Künneth formula can be written in the form of a short exact sequence of cellular cohomology groups (with integer coefficients) as follows:

$$0 \rightarrow (\tilde{H}^*(X) \otimes \tilde{H}^*(Y))_q^m \rightarrow \tilde{H}^q(X \times Y) \rightarrow \text{Tor} (\tilde{H}^*(X), \tilde{H}^*(Y))_{q+1} \rightarrow 0$$

where $m$ denotes the cross product for cellular cohomology.

If $X$ and $Y$ are spaces with distinguished base points, then we get an analogous sequence for reduced cohomology:

$$0 \rightarrow (\tilde{H}^*(X) \otimes \tilde{H}^*(Y))_q^m \rightarrow \tilde{H}^q(X \wedge Y) \rightarrow \text{Tor} (\tilde{H}(X), \tilde{H}(Y))_{q+1} \rightarrow 0$$

where $X \wedge Y = X \times Y / X \vee Y$ denotes the smash product and $\tilde{m}$ is the cross product for reduced cellular cohomology.

Similar Künneth formulas have also been obtained for extraordinary cohomology theories $\tilde{h}^*$; see e.g. [ATIYAH], [ADAMS]. In general the cohomology of $X \wedge Y$ and the cohomology of the factors $X$ and $Y$ are related by the so called Künneth spectral sequence involving higher Tor-terms:

$$E^{p,q}_2 = \text{Tor}^{\Lambda}_{p} (\tilde{h}^*X, \tilde{h}^*Y)_{p+q} \quad E^{p,q}_\infty = \text{Gr}_p \tilde{h}^q(X \wedge Y)$$

where $\Lambda = \tilde{h}^* S^0$. 
Although the abelian group structure of $h^*(X \times Y)$ seems to be (at least theoretically) computable, there still remains the problem of finding the multiplicative structure of $h^*(X \times Y)$.

The main purpose of this thesis is to actually compute the cohomology ring of a cartesian product of lens spaces, i.e.,

$$H^*(L^{2n-1}_p \times L^{2m-1}_q).$$

According to the Künneth formula, $H^*(L^{2n-1}_p \times L^{2m-1}_q)$ contains the "tensor product part" $H^*(L^{2n-1}_p) \otimes H^*(L^{2m-1}_q)$ as a subring. The "Tor part", however, can only be identified with a subgroup of $H^*(L^{2n-1}_p \times L^{2m-1}_q)$.

For a precise statement of the ring structure of $H^*(L^{2n-1}_p \times L^{2m-1}_q)$, we refer to section 4.

We would like to point out some interesting features of the multiplicative structure of the ring $H^*(L^{2n-1}_p \times L^{2m-1}_q)$. There are generators $\alpha, \beta, \tau \in H^*(L^{2n-1}_p \times L^{2m-1}_q)$ (see section 4), where $\alpha$ and $\beta$ are the generators carried by the factors of the tensor product $H^*(L^{2n-1}_p) \otimes H^*(L^{2m-1}_q)$, and $\tau$ generates $H^3(L^{2n-1}_p \times L^{2m-1}_q) \cong \text{Tor}(H^2(L^{2n-1}_p), H^2(L^{2m-1}_q)) \cong \mathbb{Z}(p,q)$.

It turns out that $\tau^2$ hits the "tensor product part", while $\alpha \tau$ and $\beta \tau$ live in the "Tor part". Hence the "tensor product part" can only be imbedded into $H^*(L^{2n-1}_p \times L^{2m-1}_q)$ as a subring, not as an ideal, and the "Tor part" is not closed under multiplication. These two observations probably illustrate a typical feature of the cup product structure on a cartesian product of spaces.
2. The Cohomology Ring of the Lens Space \( L_{2n-1}^p \)

2.1 Definitions

The standard lens space, \( L_{2n-1}^p \), is defined to be the orbit space of the usual free action of the cyclic group of order \( p, \mathbb{Z}_p \), on the standard sphere \( S^{2n-1} = \{ z \in \mathbb{C}^n : |z| = 1 \} \). Recall that if \( \mathbb{Z}_p \) is presented by \( \{ t \in \mathbb{C} : t^p = 1 \} \), then the action is given by:

\[
(2.1) \quad t \cdot (z_0, \ldots, z_{n-1}) = (tz_0, \ldots, tz_{n-1})
\]

where \( (z_0, \ldots, z_{n-1}) \) is an \( n \)-tuple of complex numbers representing a point of \( S^{2n-1} \). Note that in the case \( p = 2 \), the \( \mathbb{Z}_2 \) action on \( S^n \) (\( n \) may be even in this case) is the antipodal action and the resulting orbit space is the real projective space \( \mathbb{R}P^n \).

In this section we give an explicit calculation of the cohomology groups of \( L_{2n-1}^p \). We first describe a cellular structure on \( S^{2n-1} \) which is equivariant with regard to the action of \( \mathbb{Z}_p \). The canonical projection onto the orbit space induces a cellular structure on \( L_{2n-1}^p \). From the corresponding cellular chain and cochain complex we will derive the additive structure of \( H^*(L_{2n-1}^p) \). To determine the multiplicative structure we use a barycentric subdivision of the original cellular structure, which is again compatible with the action of \( \mathbb{Z}_p \) on \( S^{2n-1} \).
2.2 The Additive Structure of $H^*(L^{2n-1}_p)$

The following is standard and follows the treatment in [DOLD]. Give $S^{2n-1}$ the following cellular structure:

Let $e^2_{2k} = \{(z_0, \ldots, z_{n-1}) \in S^{2n-1} : z_j = 0 \text{ for } j > k \text{ and } \arg(z_k) = r \frac{2\pi}{p} \text{ or } z_k = 0\}$

(2.2)

and $e^2_{2k+1} = \{(z_0, \ldots, z_{n-1}) \in S^{2n-1} : z_j = 0 \text{ for } j > k \text{ and } r \frac{2\pi}{p} \leq \arg(z_k) \leq (r+1)\frac{2\pi}{p} \text{ or } z_k = 0\}$

for $r = 0, \ldots, p-1$; $k = 0, \ldots, n-1$

be the 2k (resp. 2k+1) dimensional cells of $S^{2n-1}$. If $t \in \mathbb{Z}_p$, $t = \exp(2\pi i/p)$, then $t \cdot e^k_r = e^k_{r+1}$. Hence this cellular structure is compatible with the action of $\mathbb{Z}_p$ on $S^{2n-1}$.

If $D^{2k} = \{(z = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n : ||z|| \leq 1 \text{ and } z_j = 0, j \geq k\}$

and $D^{2k+1} = \{(z = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n : ||z|| \leq 1 \text{ and } z_j = 0, j \geq k\}$

and $z_k \in \mathbb{R}$} are the standard 2k (resp. 2k+1) dimensional closed discs, then we can give the cells (2.2) orientations by homeomorphisms

$f^k_r : D^k \rightarrow e^k_r :$

(2.3)

\[ f^2_{2k}(z) = (z_0, \ldots, z_{k-1}, g(z)\exp(2\pi ir/p), 0, \ldots, 0) \]

\[ f^2_{2k+1}(z) = (z_0, \ldots, z_{k-1}, g(z)\exp(((z_k/g(z)) + 2r+1)\pi i/p), 0, \ldots, 0) \]

where $g(z) = \sqrt{1 - \left(z_0^2 + \ldots + z_{k-1}^2\right)}$. Note that $f^2_{2k+1}$ can be well defined for those $z$ for which $g(z) = 0$.
With respect to these orientations the cellular boundaries are given by:

\[ \partial(e_r^{2k}) = \{(z_0, \ldots, z_{n-1}) \in e_r^{2k}: z_k = 0\} \]

\[ = \bigoplus_{i=0}^{p-1} e_r^{2k-1} \]

(2.4) \[ \partial(e_r^{2k+1}) = \{(z_0, \ldots, z_{n-1}) \in e_r^{2k+1}: \arg(z_k) = r \frac{2\pi}{p} \]

\[ \text{or } \arg(z_k) = (r+1) \frac{2\pi}{p} \]

\[ = e_{r+1}^{2k} - e_r^{2k} \]

where \( e_r^{2k} = e_0^{2k} \). Let \( \pi: S^{2n-1} \longrightarrow \mathbb{L}_p^{2n-1} \) be the canonical projection. Denote the image of \( e_r^{k} \) by \( \pi(e_r^{k}) = e^k \), where \( r = 0, \ldots, p - 1 \). Passing to the cellular chain complex \( C_\ast(\mathbb{L}_p^{2n-1}) \) of \( \mathbb{L}_p^{2n-1} \) we have one generator in dim \( k \) denoted by \( e_r^{k} \) for \( k = 0, 1, \ldots, 2n - 1 \).
The chain complex $C_\ast(L_p^{2n-1})$ comes with the following boundary homomorphisms:

\[ \partial(e^{2k}) = \partial(\pi(e^{2k})) = \pi(\partial(e^{2k})) = \pi(\sum_{i=0}^{p-1} e_i^{2k-1}) = pe^{2k-1} \quad 0 < k < n - 1 \]

Moreover, we get $\partial(e^{2k+1}) = 0$ for $0 < k < n - 1$, and $\partial(e^0) = 0$.

We are now able to read off the homology groups:

\[ H_i(L_p^{2n-1}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, i = 2n - 1 \\ \mathbb{Z}/p & i \text{ odd, } 1 \leq i < 2n - 1 \\ 0 & \text{otherwise} \end{cases} \]

In passing to cohomology we choose a collection of generators for the cochain complex $C^\ast(L_p^{2n-1})$ dual to the ones used for the chain complex. Moreover, by abuse of notation we will use the same symbols for the duals of $e^k$; $k = 0, 1, \ldots, 2n - 1$.

The coboundaries of $C^\ast(L_p^{2n-1})$ are given by

\[ \delta(e^{2k})(e^{2k+1}) = e^{2k}(\delta e^{2k+1}) = e^{2k}(0) = 0 \]

and therefore $\delta(e^{2k}) = 0$, $0 \leq k \leq n - 1$. We further evaluate
\( \delta(e^{2k+1}) = p e^{2k+2}, \quad 0 \leq k < n - 1, \) and \( \delta(e^{2n-1}) = 0. \)

From this cochain complex we compute the cohomology groups of the lens space \( L_{2n-1}^p \):

\[
H^i(L_{2n-1}^p) = \begin{cases} 
\mathbb{Z} = \langle e^i \rangle & i = 0, \ i = 2n-1 \\
\mathbb{Z} / p = \langle e^i \rangle & i \text{ even}, \ 0 < i < 2n - 1 \\
0 & \text{otherwise}
\end{cases}
\]  

(2.8)

2.3 The Multiplicative Structure of \( H^*(L_{2n-1}^p; \mathbb{Z}) \)

In the previous section we computed the cohomology groups of \( L_{2n-1}^p \) from a free cochain complex which is of rank one in dimension \( i \), where \( i = 0, 1, \ldots, 2n - 1 \). To determine the multiplicative structure we will calculate cup products on the cochain level. To carry out this calculation we refine equivariantly (i.e. essentially barycentrically subdivide) the cellular structure in the previous section to get a simplicial structure for \( S^{2n-1} \). Calculating cup products in the corresponding cochain complex and relating this cochain complex with the one in the previous section, will give us the required cup products.

The additional vertices we introduce are given by the barycenters of the original cells, i.e. the images of the origin under the maps \( f_r^k \) in the previous section:
\[(2.9) \quad x_r^k = f_r^k(0) = \begin{cases} (0,\ldots,0,\exp(2\pi i/p),0,\ldots,0) & \in S^{2n-1} \quad k \text{ even} \\ (0,\ldots,0,\exp((2r+1)\pi i/p),0,\ldots,0) & \in S^{2n-1} \quad k \text{ odd} \end{cases} \]

for \( r = 0,\ldots,p-1; \ k = 0,\ldots,2n-1 \) and the non-zero entry occurs in the \( \left(\frac{k}{2}\right)+1 \)th component.

Let \( K_n \) be the simplicial complex with the above vertices and all simplices of the form \((x^i_{r_i}, x^j_{r_j}, \ldots, x^h_{r_h})\) where \( 0 \leq i < j < \ldots < h \leq 2n-1 \) and if \( x^2k_{r_{2k}} x^{2k+1}_{r_{2k+1}} \) occurs, then \( r_{2k} = r_{2k+1} \) or \( r_{2k} \equiv r_{2k+1} + 1 \)(mod \( p \)).

We claim that \(|K_n|\) is homeomorphic to \( S^{2n-1} \). We show this by induction on \( n \).

When \( n = 1 \), \( S^1 \) consists of the cells \( e^0_r \) and \( e^1_r \) for \( r = 0,\ldots,p-1 \). The barycentric subdivision of the cell \( e^1_r \) is \((\partial e^1_r)^* x^1_{r} = \{x^0_{r}, x^1_{r+1}\}^* x^1_{r} = (x^0_{r} x^1_{r}) \cup (x^0_{r+1} x^1_{r}) \). Hence \(|K_1|\) is homeomorphic to \( S^1 \).

Illustration \( n = 1, p = 3 \)
Assume inductively that $|K_{n-1}|$ is homeomorphic to $S^{2n-3}$.

Let $K'_1$ be the simplicial complex obtained from $K_1$ by replacing $x^0_r$ with $x^{2n-2}_r$ and $x^1_r$ with $x^{2n-1}_r$. Then

$$|K_n| = |K_{n-1}| * |K'_1| = S^{2n-3} * S^1 = S^{2n-1}$$

as required. Since the homeomorphism $|K_1| = S^1$ and the homeomorphism $S^k * S^l = S^{k+l}$ (assuming suitable placement) can be given by projecting from the origin, we deduce that the homeomorphism $|K_n| = S^{2n-1}$ can be given by projecting from the origin.

If $t \in \mathbb{Z}_p$, $t = \exp(2\pi i/p)$, the action (2.1) gives $t^* x^k_r = x^{k+1}_r$, where $r+1$ is assumed to be mod $p$. Hence the subdivision is compatible with the action of $\mathbb{Z}_p$.

Under the projection $\pi: S^{2n-1} \rightarrow L^{2n-1}_p$, we denote the image of each simplex by:

$$\pi((x^i_{r_1} \ldots x^j_{r_j})) = [x^i_{r_1} \ldots x^j_{r_j}]$$

Let $C'_*(L^{2n-1}_p)$ be the chain complex of the associated simplicial structure on $L^{2n-1}_p$, and let $C'*(L^{2n-1}_p)$ be the corresponding cochain complex. Let $i_*: C_*(L^{2n-1}_p) \rightarrow C'_*(L^{2n-1}_p)$ be the chain complex homomorphism induced by the inclusion. The map $i_*$ is determined by:

$$i_*(e^r) = \sum [x^0_{i_1} x^1_{i_2} \ldots x^r_{i_r}] + \sum [x^0_{i_1} x^1_{i_2} \ldots x^r_{i_r}]$$

where the above two sums are to be taken over all $i_2, \ldots, i_r$ such that $i_{2k} = i_{2k+1}$ or $i_{2k} = i_{2k+1} + 1 \pmod{p}$ and $0 \leq i_k < p - 1$. 

Let $c^s$ be the following cochain in $C^\ast(L_p^{2n-1})$ for $0 \leq s \leq 2n - 1$:

$$c^s([x_{r_0} \cdots x_{r_s}]) = \begin{cases} 1 & r_{\ell+1} \equiv r_\ell - 1 \pmod{p} \quad 0 \leq \ell < s \\ 0 & \text{otherwise} \end{cases}$$

Under the induced surjection $i^\ast: C^\ast(L_p^{2n-1}) \longrightarrow C^\ast(L_p^{2n-1})$ we claim that $i^\ast(c^2) = e^s$. We verify this by a direct calculation:

$$i^\ast(c^s)(e^s) = c^s(i^\ast(e^s)) = c^s([x_{0}^2 x_{0}^p \cdots x_{p-r+1}]) = 1$$

Hence $i^\ast(c^s) = e^s$. To determine the multiplicative structure of $H^\ast(L_p^{2n-1})$ we proceed as follows. We calculate the cup products $c^r \smile c^s$ using the Alexander-Whitney diagonal approximation. This yields:

$$c^r \smile c^s([x_j^0 \cdots x_j^{r+s}]) = c^r([x_j^0 \cdots x_j^{r}])c^s([x_j^{i_r} \cdots x_j^{i_{r+s}}])$$

$$= \begin{cases} 1 & j_{\ell+1} \equiv j_{\ell} - 1 \pmod{p} \quad 0 \leq \ell < r \lor r \leq \ell < r + s \\ 0 & \text{otherwise} \end{cases}$$

$$= c^{r+s}([x_j^0 \cdots x_j^{r+s}])$$

and therefore $c^r \smile c^s = c^{r+s}$. Thus in $C^\ast(L_p^{2n-1})$ we have

$$e^r \smile e^s = e^{r+s}, \quad 0 \leq r + s \leq 2n - 1.$$
(2.14) \[ H^*(L^{2n-1}; \mathbb{Z}) \cong \mathbb{Z}[a]/(pa,a^n) \oplus <g> \]

where the generator \( a \) is in dimension 2, and \( g \) is in dimension \( 2n - 1 \). The top dimensional generator \( g \) generates an infinite cyclic group, i.e. \(<g> \cong \mathbb{Z} \).
3. The Cohomology Ring of Cartesian Products of Standard Lens Spaces.

Recall that the cohomology rings of the lens spaces \( L^p_{2n-1} \) and \( L^q_{2m-1} \) have the following structures respectively, (see 2.14):

\[
H^*(L^p_{2n-1}) = \mathbb{Z}[a]/(pa, a^{2n}) \oplus <g>
\]
\[
H^*(L^q_{2m-1}) = \mathbb{Z}[b]/(qb, b^{2m}) \oplus <h>
\]  

(3.1)

where \( \dim a = \dim b = 2, \dim g = 2n - 1, \dim h = 2m - 1 \) and \( <g> = <h> = \mathbb{Z} \).

We first find the additive cohomology structure of \( H^*(L^p_{2n-1} \times L^q_{2m-1}) \). For that purpose we assume that the graded cohomology groups \( H^*(L^p_{2n-1} \times L^q_{2m-1}) \) are derived from cochain complexes which have one generator in each dimension; see previous section. We let \( c \) (resp. \( d \)) denote the cochain generator in dimension 1 for \( L^p_{2n-1} \) (resp. \( L^q_{2m-1} \)). Note that \( c \) (resp. \( d \)) does not represent a cohomology class \( (\delta c = pc^2, \text{ resp. } \delta d = qd^2) \); however \( [c^2] = a \) (resp. \( [d^2] = b \)), \( [c^{2n-1}] = g \) (resp. \( [d^{2m-1}] = h \)) where the square brackets denote cohomology classes as usual.

Using the Künneth formula we deduce:

\[
H^j(L^p_{2n-1} \times L^q_{2m-1}) \cong \bigoplus_{r=0}^j H^r(L^p_{2n-1}) \otimes H^{j-r}(L^q_{2m-1})
\]

(3.2)

\[
\bigoplus_{r=0}^{j+1} \text{Tor}(H^r(L^p_{2n-1}), H^{j+1-r}(L^q_{2m-1}))
\]
Referring to (3.1) we deduce:

\[
H^{2k}(L_p^{2n-1} \times L_q^{2m-1}) = \bigoplus_{r=0}^{2k} H^r(L_p^{2n-1}) \otimes H^{2k-r}(L_q^{2m-1}) \\
= \bigoplus_{r=0}^{k} \langle a^r \times b^{k-r} \rangle \quad 0 \leq k < n + m - 1
\]

Let \( \alpha = a \times 1, \beta = 1 \times b \). Then

\[
H^{2k}(L_p^{2n-1} \times L_q^{2m-1}) = \bigoplus_{r=0}^{k} \langle a^r \beta^{k-r} \rangle \quad 0 \leq k < n + m - 1
\]

(3.4) \[\text{and } \langle a^r \beta^{k-r} \rangle = \begin{cases} z_p & r = k \\ z_q & r = 0 \\ 0 & r > n - 1 \text{ or } k - r > m - 1 \\ z_{(p,q)} & \text{otherwise} \end{cases}\]

Note that if \((p,q) = 1\), then \(a \beta = 0\). For \(2k = 2n + 2m - 2\) we have:

\[
H^{2n+2m-2}(L_p^{2n-1} \times L_q^{2m-1}) = H^{2n-1}(L_p^{2n-1}) \otimes H^{2m-1}(L_q^{2m-1}) \\
= \langle g \times h \rangle
\]

(3.5) \[= \langle \gamma \eta \rangle\]

\[= Z\]

where \(\gamma = g \times 1, \eta = 1 \times h\).
When $j$ is odd, say $j = 2k + 1$, the term in the direct summand for $H^j(L_p^{2n-1} \times L_q^{2m-1})$ not involving Tor is:

$$
\bigoplus_{r=0}^{2k+1} H^r(L_p^{2n-1}) \otimes H^{2k+1-r}(L_q^{2m-1})
$$

(3.6) \quad \cong H^{2(k-m)+2}(L_p^{2n-1}) \otimes H^{2m-1}(L_q^{2n-1}) \otimes H^{2n-1}(L_p^{2m-1}) \otimes H^{2(k-n)+2}(L_q^{2m-1})

$$
\cong {}_{\alpha}^{k-m+1} \eta \oplus {}_{\gamma}^{k-n+1}
$$

where each term occurs only if $m - 1 \leq k \leq m + n - 2$ or $n - 1 \leq k \leq m + n - 2$ respectively and,

$${}_{\alpha}^{k-m+1} \eta = \begin{cases} 
\mathbb{Z} & k = m - 1 \\
\mathbb{Z}_p & k > m - 1 
\end{cases}
$$

(3.7) \quad {}_{\gamma}^{k-n+1} \eta = \begin{cases} 
\mathbb{Z} & k = n - 1 \\
\mathbb{Z}_q & k > n - 1 
\end{cases}
$$

To complete the calculation of $H^{2k+1}(L_p^{2n-1} \times L_q^{2m-1})$ we need only find an element in $H^{2k+1}(L_p^{2n-1} \times L_q^{2m-1})$ generating a subgroup isomorphic to $\text{Tor}(H^{2r}(L_p^{2n-1}), H^{2k+2-2r}(L_q^{2m-1}))$ under the "K"unneth isomorphism" (3.2). Since that isomorphism depends on the splitting chosen for the K"unneth exact sequence, we have to go back to a proof of the K"unneth formula in order to get a hold on the choice of the splitting involved.
If $X$ and $Y$ are finite CW-complexes, then the following is a split short exact sequence:

\[
(3.8) \quad 0 \to (H^*(X) \otimes H^*(Y))_q \xrightarrow{m} H^q(X \times Y) \to \text{Tor}(H^*(X), H^*(Y))_{q+1} \to 0
\]

**Proof.** (Spanier)

Let $C = C^*(X)$ and $C' = C^*(Y)$. Let $Z'$ and $B'$ be the complexes defined by $(Z')^q = Z^q(Y)$ and $(B')^q = B^{q+1}(Y)$, and where the boundary maps are all trivial. From the short exact sequence

\[
(3.9) \quad 0 \to Z' \to C' \to B' \to 0
\]

we get the following short exact sequence

\[
(3.10) \quad 0 \to C \otimes Z' \to C \otimes C' \to C \otimes B' \to 0
\]

since $C$ is free. From this we obtain an exact cohomology sequence

\[
(3.11) \quad \ldots \to H^q(C \otimes Z') \to H^q(C \otimes C') \xrightarrow{(1\otimes \delta)^*} H^q(C \otimes B') \xrightarrow{\partial^*} H^{q+1}(C \otimes Z') \to \ldots
\]

Observe that $C \otimes Z' \cong \bigoplus_j C^j$, where $(C^j)^q = C^{q-j}(X) \otimes Z^j(Y)$ and $C \otimes B' \cong \bigoplus_j C^j$ where $(C^j)^q = C^{q-j}(X) \otimes B^{j+1}(Y)$. Since $Z^j(Y)$ and $B^j(Y)$ are free, it follows from the Universal Coefficient Theorem, that

\[
(3.12) \quad H^q(C \otimes Z') \cong \bigoplus_j H^q(C^j) \cong \bigoplus_j H^i(X) \otimes Z^j(Y)
\]

and

\[
H^q(C \otimes B') \cong \bigoplus_j H^q(C^j) \cong \bigoplus_j H^i(X) \otimes B^{j+1}(Y)
\]
Under these isomorphisms, the map \( \varphi^* \) corresponds to the homomorphism \((-1)^i \otimes \nu_j \), where \( \nu_j \) is the inclusion map \( \nu_j : B^j(Y) \rightarrow Z^j(Y) \). Therefore there is a short exact sequence

\[
(3.13) \quad 0 \rightarrow \bigoplus_{i+j=q} \text{coker}(-1)^i \otimes \nu_j \rightarrow H^q(C \otimes C') \rightarrow \bigoplus_{i+j=q+1} \text{ker}(-1)^i \otimes \nu_j \rightarrow 0
\]

Tensoring the following short exact sequence with \( H^i(X) \)

\[
(3.14) \quad 0 \rightarrow B^j(Y) \xrightarrow{(-1)^i \otimes \nu_j} Z^j(Y) \rightarrow H^j(Y) \rightarrow 0
\]

gives (since \( Z^j(Y) \) is free):

\[
(3.15) \quad 0 \rightarrow \text{Tor}(H^i(X),H^j(Y)) \rightarrow H^i(X) \otimes B^j(Y) \xrightarrow{(-1)^i \otimes \nu_j} H^i(X) \otimes Z^j(Y) \rightarrow H^i(X) \otimes H^j(Y) \rightarrow 0
\]

and hence:

\[
\text{coker}((-1)^i \otimes \nu_j) \cong H^i(X) \otimes H^j(Y)
\]

(3.16)

and \( \text{ker}((-1)^i \otimes \nu_j) \cong \text{Tor}(H^i(X),H^j(Y)) \)

Using the Eilenberg-Zilber equivalence we identify \( H^q(C^*(X) \otimes C^*(Y)) \) with \( H^q(X \times Y) \) in (3.13) and thus get the Künneth formula. This completes the proof.

In the following lemma we characterize elements in \( H^*(X \times Y) \) "living in the Tor-part", i.e. elements in \( H^*(X \times Y) \) which are pre-images of elements in \( \text{Tor}(H^*(X),H^*(Y))_{q+1} \) under the map (3.8)
Lemma

Let \( u \) (resp. \( v \)) be a cocycle representing a class of order \( s \) in \( H^i(X) \) (resp. \( H^j(Y) \)), and \( u' \) (resp. \( v' \)) be a cochain such that \( \delta u' = su \) (resp. \( \delta v' = sv \)). Then under the notation of the above theorem

\[
(3.17) \quad [u' \otimes v - (-1)^i u \otimes v']
\]

is a class of order \( s \) in \( H^{i+j-1}(X \times Y) \) such that

\[
(3.18) \quad (1 \otimes \delta)[u' \otimes v - (-1)^i u \otimes v'] = (-1)^{i+1}[u] \otimes \delta v' \in H^{i}(X) \otimes B^j(Y)
\]

is a class of order \( s \) in \( \ker[(-1)^i \otimes j] \cong \text{Tor}(H^i(X), H^j(Y)) \).

Proof

\[
\delta(u' \otimes v - (-1)^i u \otimes v')
\]

\[
= \delta u' \otimes v + (-1)^{i-1}u' \otimes \delta v - (-1)^i \delta u \otimes v' - (-1)^{2i} u \otimes \delta v'
\]

\[
= s(u \otimes v - u \otimes v) = 0
\]

Hence \( u' \otimes v - (-1)^i u \otimes v' \) is a cocycle.

\[
(-1)^i \otimes_{j+1} [(-1)^{i+1}[u] \otimes \delta v'] = -[u] \otimes \delta v'
\]

\[
= -[u] \otimes sv
\]

\[
= -[su] \otimes v = 0
\]

since \( v \in Z^j(Y) \). Suppose

\[
\delta(\sum_k \omega_k \otimes z_k) = \sum_k \delta \omega_k \otimes z_k = t[u \otimes v'] \in \tilde{C}^j.
\]
Since $\delta v'$ generates a direct summand of $B^{j+1}(Y)$ ($v$ is a class of order $s$), and $B^{j+1}(Y)$ is free, we can assume, using properties of the tensor product, that $z_k = \delta v'$ for all $k$. Hence

$$\sum_k \delta v_k \otimes z_k = (\delta \sum_k v_k) \otimes \delta v' = tu \otimes \delta v'.$$

Since $C^*(X)$ is free we can conclude $\delta \sum_k v_k = tu$ and hence that $s$ divides $t$.

Therefore $(-1)^{i+1}[u] \otimes \delta v'$ is an element of order $s$ in $\ker(-1)^{i+1} \otimes v'$. Observing that

$$\delta((-1)^{i-1}u' \otimes v') = s(u' \otimes v - (-1)^i u \otimes v')$$

we conclude that $[u' \otimes v - (-1)^i u \otimes v']$ is also a class of order $s$.

This completes the proof.

**Corollary**

If $X = L_{p}^{2n-1}$ and $Y = L_{q}^{2m-1}$, the generator of the subgroup of $H^{2k+1}(L_{p}^{2n-1} \times L_{q}^{2m-1})$ corresponding to $\text{Tor}(H^{2r}(L_{p}^{2n-1}), H^{2k+2-2r}(L_{q}^{2m-1}))$ under the isomorphism (3.2) is

$$[k_1 c^{2r-1} \otimes d^{2k+2-2r} - k_2 c^{2r} \otimes d^{2k+1-2r}]$$

$$= \alpha^{r-1} b^{k-r} [k_1 c \otimes d^2 - k_2 c^{2} \otimes d]$$

$$= \alpha^{r-1} b^{k-r} \tau$$

where $k_1 = \frac{q}{(p,q)}$, $k_2 = \frac{p}{(p,q)}$ and $\tau = [k_1 c \otimes d^2 - c^{2} \otimes d]$.
Remark

In general, a complete set of generators for $H^*(X \times Y)$ may be found in the above manner.

To complete the calculation of the ring structure we need only to determine the cup products involving $\tau$.

$$
\tau^2 = [(-1)^{2\cdot 1} k_1 c^2 \otimes d^4 - (-1)^{2\cdot 2} k_1 k_2 c^3 \otimes d^3
- (-1)^{1\cdot 1} k_1 k_2 c^3 \otimes d^3 + (-1)^{1\cdot 2} k_2 c^4 \otimes d^2]
= k_1^2 \alpha \beta^2 + k_2^2 \alpha^2 \beta
$$

(3.20)

where $k_1^2$ and $k_2^2$ are of course taken mod $(p,q)$.

$$
\alpha^{n-1} \tau = [c^{2n-2} \otimes 1] \tau = [k_1 c^{2n-1} \otimes d^2] = k_1 \gamma \beta.
$$

Similarly $\beta^{m-1} \tau = k_2 \alpha \eta$ and $\tau \gamma = \tau \eta = 0$. 

4. The Main Result

4.1 The Reduced Cohomology of $L_{p\times q}^{2n-1} \times L_{p\times q}^{2m-1}$

In the preceding section we have shown that $\tilde{H}^*(L_{p\times q}^{2n-1} \times L_{p\times q}^{2m-1})$ is a ring with 5 generators $\alpha, \beta, \tau, \gamma, \eta$, where $\dim \alpha = \dim \beta = 2$, $\dim \tau = 3$, $\dim \gamma = 2n - 1$, $\dim \eta = 2m - 1$, and which satisfy the following relations:

$$\alpha^n = \beta^m = \alpha \gamma = \beta \eta = \gamma^2 = \eta^2 = \tau \gamma = \tau \eta = 0$$

$$\tau^2 = \left(\frac{q}{p,q}\right)^2 \alpha \beta + \left(\frac{p}{p,q}\right)^2 \alpha \beta$$

$$\alpha^{n-1} \tau = \left(\frac{q}{p,q}\right) \gamma \beta$$

$$\beta^{m-1} \tau = \left(\frac{p}{p,q}\right) \alpha \eta$$

$\alpha^i \beta^j$ generates a cyclic subgroup of order $p$ if $1 \leq i < n$ and $j = 0$, of order $q$ if $i = 0, 1 \leq j < m$, and of order $(p,q)$ if $1 \leq i < n$ and $1 \leq j < n$. $\alpha^i \beta^j \tau$ generates a cyclic subgroup of order $(p,q)$ if $0 \leq i < n$ and $0 \leq j < n$. $\gamma, \eta, \gamma \eta$ each generate an infinite cyclic subgroup, $\alpha^i \eta$ generates a cyclic subgroup of order $p$ if $1 \leq i < n$. $\beta^i \gamma$ generates a cyclic subgroup of order $q$ if $1 \leq j < m$.

From the above relations it follows that if $n = 1$ (resp. $m = 1$) $\tau$ is not necessary as a generator since $\tau = \left(\frac{q}{p,q}\right) \gamma \beta$ (resp. $\tau = \left(\frac{p}{p,q}\right) \alpha \eta$). Also if $n = m = 1$, in which case $L_{p\times q}^{2} \times L_{p\times q}^{2} \cong S^1 \times S^1$, $\tau = 0$. 


4.2 Corollary \( H^*(RP^n \times RP^m) \)

If \( n \) and \( m \) are odd, say \( n = 2k - 1 \) and \( m = 2\ell - 1 \), the structure of \( H^*(RP^n \times RP^m) \) is the same as in the previous section. It is a ring with 5 generators \( \alpha, \beta, \tau, \gamma, \eta \), where \( \dim \alpha = \dim \beta = 2 \), \( \dim \tau = 3 \), \( \dim \gamma = n \) and \( \dim \eta = m \), and which satisfy the following relations:

\[
\frac{n+1}{2} \alpha = \frac{m+1}{2} \beta = \alpha \gamma = \beta \eta = \gamma^2 = \eta^2 = \tau \gamma = \tau \eta = 0
\]

\[
\tau^2 = \alpha^2 + \alpha \beta; \quad \frac{n-1}{2} \alpha \tau = \gamma \beta, \quad \frac{m-1}{2} \beta \tau = \alpha \eta
\]

and every non-zero product generates a subgroup isomorphic to \( \mathbb{Z}_2 \), except \( \gamma, \eta \), and \( \gamma \eta \) which generate subgroups isomorphic to \( \mathbb{Z} \).

If \( n \) is even and \( m \) is odd, say \( n = 2k \) and \( m = 2\ell - 1 \), \( H^*(RP^n \times RP^m) \) is a ring with 4 generators \( \alpha, \beta, \tau, \eta \) where \( \dim \alpha = \dim \beta = 2 \), \( \dim \tau = 3 \), and \( \dim \eta = m \), and which satisfy the following relations:

\[
\frac{n+1}{2} \alpha = \frac{m+1}{2} \beta = \beta \eta = \eta^2 = \tau \eta = 0
\]

\[
\tau^2 = \alpha^2 + \alpha \beta, \quad \frac{n}{2} \alpha \tau = 0, \quad \frac{m-1}{2} \beta \tau = \alpha \eta
\]

and every non-zero product generates a subgroup isomorphic to \( \mathbb{Z}_2 \), except \( \eta \) which generates a subgroup isomorphic to \( \mathbb{Z} \).

If \( n \) and \( m \) are even, say \( n = 2k \) and \( m = 2\ell \), \( H^*(RP^n \times RP^m) \) is a ring with 3 generators \( \alpha, \beta, \tau \) where \( \dim \alpha = \dim \beta = 2 \) and \( \dim \tau = 3 \), and which satisfy the following relations:
\[
\begin{align*}
\alpha^{n+1} &= \beta^{m+1} = 0, \\
\tau^2 &= \alpha\beta^2 + \alpha^2\beta
\end{align*}
\]

(4.4) \[
\begin{align*}
\frac{n}{2} \tau &= 0, \\
\frac{m}{2} \tau &= 0
\end{align*}
\]

and every non-zero product generates a subgroup which is isomorphic to \( \mathbb{Z}_2 \).
Bibliography


