CARTESIAN PRODUCTS OF LENS SPACES AND THE KUNNETH FORMULA

by

JAN FRANS VERSTER
B. Math., University of Waterloo, 1974

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE DEPARTMENT OF MATHEMATICS

We accept this thesis as conforming to
the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
May, 1976

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Department of Mathematics

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date April 30, 1976

Abstract

The graded cohomology groups of a cartesian product of two cellular spaces are expressible in terms of the cohomology groups of the factors. This relationship is given by the (split) short exact Künneth sequence.

However the multiplicative structure on the cohomology of a cartesian product can in general not be derived by solely referring to the Künneth formula.

In this thesis we explicitly exhibit the cup product structure on a cartesian product of two (standard) lens spaces.

This result is obtained by analyzing the Künneth sequence and by making use of the particular geometry of the spaces involved.

Contents

Acknowledgements

The author would like to thank Dr. René P. Held and Dr. Denis Sjerve for their many helpful suggestions and their patient readings of this thesis.

I would also like to thank the National Research Council and the University of British Columbia for their generous financial support over the previous two years.

1. Introduction

The Künneth formula relates the cohomology groups of a cartesian product of topological spaces to the cohomology groups of its factors. If X and Y are (finite) CW-complexes, then the Künneth formula can be written in the form of a short exact sequence of cellular cohomology groups (with integer coefficients) as follows:

$$0 \longrightarrow (H^*(X) \otimes H^*(Y))_q \xrightarrow{m} H^q(X \times Y) \longrightarrow \mathrm{Tor}(H^*(X), H^*(Y))_{q+1} \longrightarrow 0$$

where m denotes the cross product for cellular cohomology.

If X and Y are spaces with distinguished base points, then we get an analogous sequence for reduced cohomology:

$$0 \longrightarrow (\tilde{H}^*(X) \otimes \tilde{H}^*(Y))_q \xrightarrow{\tilde{m}} \tilde{H}^q(X \wedge Y) \longrightarrow \mathrm{Tor}(\tilde{H}(X), \tilde{H}(Y))_{q+1} \longrightarrow 0$$

where $X \wedge Y = X \times Y / X \vee Y$ denotes the smash product and $\tilde{m}$ is the cross product for reduced cellular cohomology.

Similar Künneth formulas have also been obtained for extraordinary cohomology theories h^* ; see e.g. [ATIYAH], [ADAMS]. In general the cohomology of $X \wedge Y$ and the cohomology of the factors X and Y are related by the so called Künneth spectral sequence involving higher Tor-terms:

$$E_2^{p,q} \cong \mathrm{Tor}_p^{\Lambda}(\tilde{h}^*X, \tilde{h}^*Y)_{p+q} \qquad E_\infty^{p,q} = \mathrm{Gr}_p\, \tilde{h}^q(X \wedge Y)$$

where $\Lambda = \tilde{h}^*S^0$.

Although the abelian group structure of $h^*(X\times Y)$ seems to be (at least theoretically) computable, there still remains the problem of finding the multiplicative structure of $h^*(X\times Y)$.

The main purpose of this thesis is to actually compute the cohomology ring of a cartesian product of lens spaces, i.e. $H^*(L_p^{2n-1} \times L_q^{2m-1})$.

According to the Künneth formula, $H^*(L_p^{2n-1} \times L_q^{2m-1})$ contains the "tensor product part" $H^*(L_p^{2n-1}) \otimes H^*(L_q^{2m-1})$ as a subring. The "Tor part", however, can only be identified with a subgroup of $H^*(L_p^{2n-1} \times L_q^{2m-1})$.

For a precise statement of the ring structure of $H^*(L_p^{2n-1} \times L_q^{2m-1})$, we refer to section 4.

We would like to point out some interesting features of the multiplicative structure of the ring $H^*(L_p^{2n-1} \times L_q^{2m-1})$. There are generators $\alpha,\beta,\tau \in H^*(L_p^{2n-1} \times L_q^{2m-1})$ (see section 4), where α and β are the generators carried by the factors of the tensor product $H^*(L_p^{2n-1}) \otimes H^*(L_q^{2m-1})$, and τ generates $H^3(L_p^{2n-1} \times L_q^{2m-1}) \cong \mathrm{Tor}(H^2(L_p^{2n-1}),H^2(L_q^{2m-1})), \cong Z_{(p,q)}$.

It turns out that τ^2 hits the "tensor product part", while $\alpha\tau$ and $\beta\tau$ live in the "Tor part". Hence the "tensor product part" can only be imbedded into $\tilde{H}^*(L_p^{2n-1} \times L_q^{2m-1})$ as a subring, not as an ideal, and the "Tor part" is not closed under multiplication. These two observations probably illustrate a typical feature of the cup product structure on a cartesian product of spaces.

2. The Cohomology Ring of the Lens Space L_p^{2n-1}

2.1 Definitions

The standard lens space, L_p^{2n-1}, is defined to be the orbit space of the usual free action of the cyclic group of order p, $\mathbb{Z}_p$, on the standard sphere $S^{2n-1} = \{z \in \mathbb{C}^n: ||z|| = 1\}$. Recall that if $\mathbb{Z}_p$ is presented by $\{t \in \mathbb{C}: t^p = 1\}$, then the action is given by:

$$t \cdot (z_0,\ldots,z_{n-1}) = (tz_0,\ldots,tz_{n-1}) \tag{2.1}$$

where $(z_0,\ldots,z_{n-1})$ is an n-tuple of complex numbers representing a point of S^{2n-1}. Note that in the case $p = 2$, the $\mathbb{Z}_2$ action on S^n (n may be even in this case) is the antipodal action and the resulting orbit space is the real projective space RP^n.

In this section we give an explicit calculation of the cohomology groups of L_p^{2n-1}. We first describe a cellular structure on S^{2n-1} which is equivariant with regard to the action of $\mathbb{Z}_p$. The canonical projection onto the orbit space induces a cellular structure on L_p^{2n-1}. From the corresponding cellular chain and cochain complex we will derive the additive structure of $H^*(L_p^{2n-1})$. To determine the multiplicative structure we use a barycentric subdivision of the original cellular structure, which is again compatible with the action of $\mathbb{Z}_p$ on S^{2n-1}.

2.2 The Additive Structure of $H^*(L_p^{2n-1})$

The following is standard and follows the treatment in [DOLD]. Give S^{2n-1} the following cellular structure:

$$\text{Let } e_r^{2k} = \{(z_0,\ldots,z_{n-1}) \in S^{2n-1}: z_j = 0 \text{ for } j > k \text{ and } \arg(z_k) = r\frac{2\pi}{p} \text{ or } z_k = 0\}$$

$$\text{and } e_r^{2k+1} = \{(z_0,\ldots,z_{n-1}) \in S^{2n-1}: z_j = 0 \text{ for } j > k \text{ and } r\frac{2\pi}{p} \leq \arg(z_k) \leq (r+1)\frac{2\pi}{p} \text{ or } z_r = 0\} \tag{2.2}$$

for $r = 0,\ldots,p-1$; $k = 0,\ldots,n-1$

be the $2k$ (resp. $2k+1$) dimensional cells of S^{2n-1}. If $t \in \mathbb{Z}_p$, $t = \exp(2\pi i/p)$, then $t\cdot e_r^k = e_{r+1}^k$. Hence this cellular structure is compatible with the action of $\mathbb{Z}_p$ on S^{2n-1}.

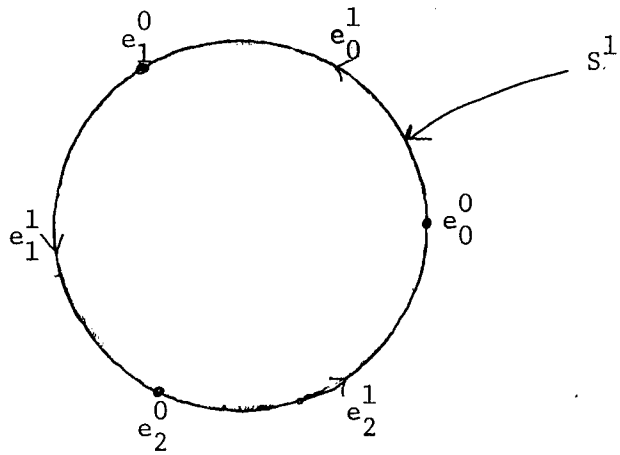
If $D^{2k} = \{z = (z_0,\ldots,z_{n-1}) \in \mathbb{C}^n: ||z|| \leq 1 \text{ and } z_j = 0,\ j \geq k\}$ and $D^{2k+1} = \{z = (z_0,\ldots,z_{n-1}) \in \mathbb{C}^n: ||z|| \leq 1 \text{ and } z_j = 0,\ j > k$ and $z_k \in R\}$ are the standard $2k$ (resp. $2k+1$) dimensional closed discs, then we can give the cells (2.2) orientations by homeomorphisms $f_r^k: D^k \to e_r^k$:

$$f_r^{2k}(z) = (z_0,\ldots,z_{k-1}, g(z)\exp(2\pi ir/p),0,\ldots,0)$$

$$f_r^{2k+1}(z) = (z_0,\ldots,z_{k-1}, g(z)\exp(((z_k/g(z)) + 2r+1)\pi i/p),0,\ldots,0) \tag{2.3}$$

where $g(z) = \sqrt{1 - (z_0^2 + \ldots + z_{k-1}^2)}$. Note that f_r^{2k+1} can be well defined for those z for which $g(z) = 0$.

<u>Illustration</u> $n = 1,\ p = 3$



With respect to these orientations the cellular boundaries are given by:

$$\partial(e_r^{2k}) = \{(z_0,\ldots,z_{n-1}) \in e_r^{2k} : z_k = 0\}$$
$$= \sum_{i=0}^{p-1} e_i^{2k-1}$$

(2.4) $$\partial(e_r^{2k+1}) = \{(z_0,\ldots,z_{n-1}) \in e_r^{2k+1} : \arg(z_k) = r\,\frac{2\pi}{p} \text{ or } \arg(z_k) = (r+1)\frac{2\pi}{p}\}$$
$$= e_{r+1}^{2k} - e_r^{2k}$$

where $e_p^{2k} = e_0^{2k}$. Let $\pi : S^{2n-1} \longrightarrow L_p^{2n-1}$ be the canonical projection. Denote the image of e_r^k by $\pi(e_r^k) = e^k$, where $r = 0,\ldots,p-1$. Passing to the cellular chain complex $C_*(L_p^{2n-1})$ of L_p^{2n-1} we have one generator in dim k denoted by e^k for $k = 0,1,\ldots,2n-1$.

The chain complex $C_*(L_p^{2n-1})$ comes with the following boundary homomorphisms:

$$\begin{aligned} \partial(e^{2k}) &= \partial(\pi(e_0^{2k})) \\ &= \pi(\partial(e_0^{2k})) \\ &= \pi(\sum_{i=0}^{p-1} e_i^{2k-1}) \\ &= pe^{2k-1} \qquad 0 < k \leq n-1 \end{aligned} \tag{2.5}$$

Moreover we get $\partial(e^{2k+1}) = 0$ for $0 \leq k \leq n-1$, and $\partial(e^0) = 0$.

We are now able to read off the homology groups:

$$H_i(L_p^{2n-1};\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \quad i = 2n-1 \\ \mathbb{Z}_p & i \text{ odd}, \quad 1 \leq i < 2n-1 \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

In passing to cohomology we choose a collection of generators for the cochain complex $C^*(L_p^{2n-1})$ dual to the ones used for the chain complex. Moreover by abuse of notation we will use the same symbols for the duals of e^k; $k = 0,1,\ldots,2n-1$.

The coboundaries of $C^*(L_p^{2n-1})$ are given by

$$\begin{aligned} \delta(e^{2k})(e^{2k+1}) &= e^{2k}(\delta e^{2k+1}) \\ &= e^{2k}(0) = 0 \end{aligned} \tag{2.7}$$

and therefore $\delta(e^{2k}) = 0$, $0 \leq k \leq n-1$. We further evaluate

$\delta(e^{2k+1}) = p\, e^{2k+2}, \quad 0 \leq k < n-1, \quad$ and $\quad \delta(e^{2n-1}) = 0$.

From this cochain complex we compute the cohomology groups of the lens space L_p^{2n-1} :

$$(2.8) \qquad H^i(L_p^{2n-1}) = \begin{cases} \mathbb{Z} = \langle e^i \rangle & i = 0, \quad i = 2n-1 \\ \mathbb{Z}_p = \langle e^i \rangle & i \text{ even}, \quad 0 < i < 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

2.3 The Multiplicative Structure of $H^*(L_p^{2n-1};\mathbb{Z})$

In the previous section we computed the cohomology groups of L_p^{2n-1} from a free cochain complex which is of rank one in dimension i, where $i = 0,1,\ldots,2n-1$. To determine the multiplicative structure we will calculate cup products on the cochain level. To carry out this calculation we refine equivariantly (i.e. essentially barycentrically subdivide) the cellular structure in the previous section to get a simplicial structure for S^{2n-1} . Calculating cup products in the corresponding cochain complex and relating this cochain complex with the one in the previous section, will give us the required cup products.

The additional vertices we introduce are given by the barycenters of the original cells, i.e. the images of the origin under the maps f_r^k in the previous section:

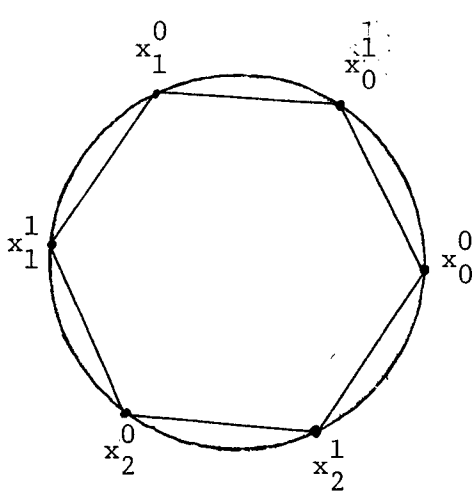
$$(2.9)\qquad x_r^k = f_r^k(0) = \begin{cases} (0,\ldots,0,\exp(2r\pi i/p),0,\ldots,0) \in S^{2n-1} & k \text{ even} \\ (0,\ldots,0,\exp((2r+1)\pi i/p),0,\ldots,0) \in S^{2n-1} & k \text{ odd} \end{cases}$$

for $r = 0,\ldots,p-1$; $k = 0,\ldots,2n-1$ and the non-zero entry occurs in the $([\frac{k}{2}]+1)$th component.

Let K_n be the simplicial complex with the above vertices and all simplices of the form $(x_{r_i}^i x_{r_j}^j \ldots x_{r_h}^h)$ where $0 \leq i < j < \ldots < h \leq 2n-1$ and if $x_{r_{2k}}^{2k} x_{r_{2k+1}}^{2k+1}$ occurs, then $r_{2k} = r_{2k+1}$ or $r_{2k} \equiv r_{2k+1} + 1 (\mathrm{mod}\ p)$. We claim that $|K_n|$ is homeomorphic to S^{2n-1}. We show this by induction on n.

When $n = 1$, S^1 consists of the cells e_r^0 and e_r^1 for $r = 0,\ldots,p-1$. The barycentric subdivision of the cell e_r^1 is $(\partial e_r^1)*x_r^1 = \{x_r^0, x_{r+1}^0\}*x_r^1 = (x_r^0 x_r^1) \cup (x_{r+1}^0 x_r^1)$. Hence $|K_1|$ is homeomorphic to S^1.

<u>Illustration</u> $n = 1$, $p = 3$



Assume inductively that $|K_{n-1}|$ is homeomorphic to S^{2n-3}. Let K_1' be the simplicial complex obtained from K_1 by replacing x_r^0 with x_r^{2n-2} and x_r^1 with x_r^{2n-1}. Then $|K_n| \cong |K_{n-1}|*|K_1'| \cong S^{2n-3}*S^1 \cong S^{2n-1}$ as required. Since the homeomorphism $|K_1| \cong S^1$ and the homeomorphism $S^k*S^\ell \cong S^{k+\ell+1}$ (assuming suitable placement) can be given by projecting from the origin, we deduce that the homeomorphism $|K_n| \cong S^{2n-1}$ can be given by projecting from the origin.

If $t \in \mathbb{Z}_p$, $t = \exp(2\pi i/p)$, the action (2.1) gives $t\cdot x_r^k = x_{r+1}^k$, where $r + 1$ is assumed to be mod p. Hence the subdivision is compatible with the action of $\mathbb{Z}_p$.

Under the projection $\pi\colon S^{2n-1} \longrightarrow L_p^{2n-1}$, we denote the image of each simplex by:

$$\pi((x_{r_i}^i \ \dots \ x_{r_j}^j)) = [x_{r_i}^i \ \dots \ x_{r_j}^j] \tag{2.10}$$

Let $C_*'(L_p^{2n-1})$ be the chain complex of the associated simplicial structure on L_p^{2n-1}, and let $C'^*(L_p^{2n-1})$ be the corresponding cochain complex. Let $i_*\colon C_*(L_p^{2n-1}) \longrightarrow C_*'(L_p^{2n-1})$ be the chain complex homomorphism induced by the inclusion. The map i_* is determined by:

$$i_*(e^r) = \sum[x_0^0 x_0^1 x_{i_2}^2 \ \dots \ x_{i_r}^r] + \sum[x_1^0 x_0^1 x_{i_2}^2 \ \dots \ x_{i_r}^r] \tag{2.11}$$

where the above two sums are to be taken over all $i_2,\dots,i_r$ such that $i_{2k} = i_{2k+1}$ or $i_{2k} \equiv i_{2k+1} + 1 \pmod p$ and $0 \le i_k \le p - 1$.

Let c^s be the following cochain in $C'^*(L_p^{2n-1})$ for $0 \leq s \leq 2n-1$:

$$(2.12) \qquad c^s([x_{r_0}^{i_0} \cdots x_{r_s}^{i_s}]) = \begin{cases} 1 & r_{\ell+1} \equiv r_\ell - 1 \pmod p \quad 0 \leq \ell < s \\ 0 & \text{otherwise} \end{cases}$$

Under the induced surjection $i^*: C'^*(L_p^{2n-1}) \longrightarrow C^*(L_p^{2n-1})$ we claim that $i^*(c^2) = e^s$. We verify this by a direct calculation:

$$i^*(c^s)(e^s) = c^s(i_*(e^s)) = c^s([x_1^0 x_0^1 x_{p-1}^2 \cdots x_{p-r+1}^r]) = 1$$

Hence $i^*(c^s) = e^s$. To determine the multiplicative structure of $H^*(L_p^{2n-1})$ we proceed as follows. We calculate the cup products $c^r \smile c^s$ using the Alexander-Whitney diagonal approximation. This yields:

$$\begin{aligned} &(c^r \smile c^w)([x_{j_0}^{i_0} \cdots x_{j_{r+s}}^{i_{r+s}}]) \\ &= c^r([x_{j_0}^{i_0} \cdots x_{j_r}^{i_r}])c^s([x_{j_r}^{i_r} \cdots x_{j_{r+s}}^{i_{r+s}}]) \\ (2.13) \qquad &= \begin{cases} 1 & j_{\ell+1} \equiv j_\ell - 1 \pmod p \quad 0 \leq \ell < r \text{ or } r \leq \ell < r+s \\ 0 & \text{otherwise} \end{cases} \\ &= c^{r+s}([x_{j_0}^{i_0} \cdots x_{j_{r+s}}^{i_{r+s}}]) \end{aligned}$$

and therefore $c^r \smile c^s = c^{r+s}$. Thus in $C^*(L_p^{2n-1})$ we have $e^r \smile e^s = e^{r+s}$, $0 \leq r+s \leq 2n-1$.

In conclusion we have:

$$H^*(L_p^{2n-1};\mathbb{Z}) \cong \mathbb{Z}[a]/(pa,a^n) \oplus \langle g\rangle \tag{2.14}$$

where the generator a is in dimension 2, and g is in dimension $2n - 1$. The top dimensional generator g generates an infinite cyclic group, i.e. $\langle g\rangle \cong \mathbb{Z}$.

3. The Cohomology Ring of Cartesian Products of Standard Lens Spaces.

Recall that the cohomology rings of the lens spaces L_p^{2n-1} and L_q^{2m-1} have the following structures respectively, (see 2.14):

$$(3.1)\qquad \begin{aligned} H^*(L_p^{2n-1}) &\cong \mathbf{Z}[a]/(pa,a^n) \oplus \langle g\rangle \\ H^*(L_q^{2m-1}) &\cong \mathbf{Z}[b]/(qb,b^m) \oplus \langle h\rangle \end{aligned}$$

where $\dim a = \dim b = 2$, $\dim g = 2n-1$, $\dim h = 2m-1$ and $\langle g\rangle \cong \langle h\rangle \cong \mathbf{Z}$.

We first find the additive cohomology structure of $H^*(L_p^{2n-1} \times L_q^{2m-1})$. For that purpose we assume that the graded cohomology groups $H^*(L_p^{2n-1} \times L_q^{2m-1})$ are derived from cochain complexes which have one generator in each dimension; see previous section. We let c(resp. d) denote the cochain generator in dimension 1 for L_p^{2n-1}(resp. L_q^{2m-1}) . Note that c(resp. d) does not represent a cohomology class ($\delta c = pc^2$, resp. $\delta d = qd^2$); however $[c^2] = a$ (resp. $[d^2] = b$), $[c^{2n-1}] = g$ (resp. $[d^{2m-1}] = h$) where the square brackets denote cohomology classes as usual.

Using the Künneth formula we deduce:

$$(3.2)\qquad \begin{aligned} H^j(L_p^{2n-1} \times L_q^{2m-1}) &\cong \bigoplus_{r=0}^{j} H^r(L_p^{2n-1}) \otimes H^{j-r}(L_q^{2m-1}) \\ &\quad \bigoplus_{r=0}^{j+1} \mathrm{Tor}(H^r(L_p^{2n-1}),H^{j+1-r}(L_q^{2m-1})) \end{aligned}$$

Referring to (3.1) we deduce:

$$H^{2k}(L_p^{2n-1} \times L_q^{2m-1}) \cong \bigoplus_{r=0}^{2k} H^r(L_p^{2n-1}) \otimes H^{2k-r}(L_q^{2m-1})$$

$$\cong \bigoplus_{r=0}^{k} \langle a^r \times b^{k-r} \rangle \qquad 0 \leq k < n + m - 1$$

Let $\alpha = a \times 1$, $\beta = 1 \times b$. Then

$$H^{2k}(L_p^{2n-1} \times L_q^{2m-1}) \cong \bigoplus_{r=0}^{k} \langle \alpha^r \beta^{k-r} \rangle \qquad 0 \leq k < n + m - 1$$

$$\text{(3.4)} \qquad \text{and} \quad \langle \alpha^r \beta^{k-r} \rangle \cong \begin{cases} \mathbb{Z}_p & r = k \\ \mathbb{Z}_q & r = 0 \\ 0 & r > n - 1 \text{ or } k - r > m - 1 \\ \mathbb{Z}_{(p,q)} & \text{otherwise} \end{cases}$$

Note that if $(p,q) = 1$, then $\alpha\beta = 0$. For $2k = 2n + 2m - 2$ we have:

$$\text{(3.5)} \qquad \begin{aligned} H^{2n+2m-2}(L_p^{2n-1} \times L_q^{2m-1}) &\cong H^{2n-1}(L_p^{2n-1}) \otimes H^{2m-1}(L_q^{2m-1}) \\ &= \langle g \times h \rangle \\ &= \langle \gamma\eta \rangle \\ &\cong \mathbb{Z} \end{aligned}$$

where $\gamma = g \times 1$, $\eta = 1 \times h$.

When j is odd, say $j = 2k + 1$, the term in the direct summand for $H^j(L_p^{2n-1} \times L_q^{2m-1})$ not involving Tor is:

$$\bigoplus_{r=0}^{2k+1} H^r(L_p^{2n-1}) \otimes H^{2k+1-r}(L_q^{2m-1})$$

$$\begin{aligned} (3.6) \qquad & \cong H^{2(k-m)+2}(L_p^{2n-1}) \otimes H^{2m-1}(L_q^{2m-1}) \oplus H^{2n-1}(L_p^{2n-1}) \otimes H^{2(k-n)+2}(L_q^{2m-1}) \\ & \cong \langle \alpha^{k-m+1}\eta \rangle \oplus \langle \gamma\beta^{k-n+1} \rangle \end{aligned}$$

where each term occurs only if $m - 1 \leq k \leq m + n - 2$ or $n - 1 \leq k \leq m + n - 2$ respectively and,

$$(3.7) \qquad \begin{aligned} \langle \alpha^{k-m+1}\eta \rangle &\cong \begin{cases} \mathbb{Z} & k = m - 1 \\ \mathbb{Z}_p & k > m - 1 \end{cases} \\ \langle \gamma\beta^{k-n+1} \rangle &\cong \begin{cases} \mathbb{Z} & k = n - 1 \\ \mathbb{Z}_q & k > n - 1 \end{cases} \end{aligned}$$

To complete the calculation of $H^{2k+1}(L_p^{2n-1} \times L_q^{2m-1})$ we need only find an element in $H^{2k+1}(L_p^{2n-1} \times L_q^{2m-1})$ generating a subgroup isomorphic to $\mathrm{Tor}(H^{2r}(L_p^{2n-1}), H^{2k+2-2r}(L_q^{2m-1}))$ under the "Künneth isomorphism" (3.2). Since that isomorphism depends on the splitting chosen for the Künneth exact sequence, we have to go back to a proof of the Künneth formula in order to get a hold on the choice of the splitting involved.

If X and Y are finite CW-complexes, then the following is a split short exact sequence:

$$(3.8) \qquad 0 \longrightarrow (H^*(X)\otimes H^*(Y))_q \xrightarrow{m} H^q(X\times Y) \longrightarrow \operatorname{Tor}(H^*(X),H^*(Y))_{q+1} \longrightarrow 0$$

<u>Proof</u>. (Spanier)

Let $C = C^*(X)$ and $C' = C^*(Y)$. Let Z' and B' be the complexes defined by $(Z')^q = Z^q(Y)$ and $(B')^q = B^{q+1}(Y)$, and where the boundary maps are all trivial. From the short exact sequence

$$(3.9) \qquad 0 \longrightarrow Z' \longrightarrow C' \longrightarrow B' \longrightarrow 0$$

we get the following short exact sequence

$$(3.10) \qquad 0 \longrightarrow C\otimes Z' \longrightarrow C\otimes C' \longrightarrow C\otimes B' \longrightarrow 0$$

since C is free. From this we obtain an exact cohomology sequence

$$(3.11) \qquad \ldots \longrightarrow H^q(C\otimes Z') \longrightarrow H^q(C\otimes C') \xrightarrow{(1\otimes\delta)^*} H^q(C\otimes B') \xrightarrow{\partial^*} H^{q+1}(C\otimes Z') \longrightarrow \ldots$$

Observe that $C\otimes Z' \cong \oplus\, C^j$, where $(C^j)^q = C^{q-j}(X)\otimes Z^j(Y)$ and $C\otimes B' \cong \oplus\, \bar{C}^j$ where $(\bar{C}^j)^q = C^{q-j}(X)\otimes B^{j+1}(Y)$. Since $Z^j(Y)$ and $B^j(Y)$ are free, it follows from the Universal Coefficient Theorem, that

$$(3.12) \qquad \begin{aligned} H^q(C\otimes Z') &\cong \bigoplus_j H^q(C^j) \cong \bigoplus_{i+j=q} H^i(X)\otimes Z^j(Y) \\ H^q(C\otimes B') &\cong \bigoplus_j H^q(\bar{C}^j) \cong \bigoplus_{i+j=q+1} H^i(X)\otimes B^j(Y) \end{aligned}$$

Under these isomorphisms, the map ∂^* corresponds to the homomorphism $(-1)^i \otimes \nu_j$, where ν_j is the inclusion map $\nu_j : B^j(Y) \longrightarrow Z^j(Y)$. Therefore there is a short exact sequence

$$(3.13) \qquad 0 \longrightarrow \bigoplus_{i+j=q} [\mathrm{coker}(-1^i)\otimes\nu_j] \longrightarrow H^q(C\otimes C') \longrightarrow \bigoplus_{i+j=q+1} [\ker(-1)^i\otimes\nu_j] \longrightarrow 0$$

Tensoring the following short exact sequence with $H^i(X)$

$$(3.14) \qquad 0 \longrightarrow B^j(Y) \xrightarrow{(-1)^i\nu_j} Z^j(Y) \longrightarrow H^j(Y) \longrightarrow 0$$

gives (since $Z^j(Y)$ is free):

$$(3.15) \qquad 0 \longrightarrow \mathrm{Tor}(H^i(X),H^j(Y)) \longrightarrow H^i(X)\otimes B^j(Y) \xrightarrow{(-1)^i\otimes\nu_j} H^i(X)\otimes Z^j(Y)$$

$$\longrightarrow H^i(X)\otimes H^j(Y) \longrightarrow 0$$

and hence:

$$(3.16) \qquad \begin{aligned} &\mathrm{coker}((-1)^i\otimes\nu_j) \cong H^i(X)\otimes H^j(Y) \\ \text{and} \quad &\ker((-1)^i\otimes\nu_j) \cong \mathrm{Tor}(H^i(X),H^j(Y)) \end{aligned}$$

Using the Eilenberg-Zilber equivalence we identify $H^q(C^*(X)\otimes C^*(Y))$ with $H^q(X\times Y)$ in (3.13) and thus get the Künneth formula. This completes the proof.

In the following lemma we characterize elements in $H^*(X\times Y)$ "living in the Tor-part", i.e. elements in $H^*(X\times Y)$ which are pre-images of elements in $\mathrm{Tor}(H^*(X),H^*(Y))_{q+1}$ under the map (3.8)

Lemma

Let u(resp. v) be a cocycle representing a class of order s in $H^i(X)$(resp. $H^j(Y)$), and u'(resp. v') be a cochain such that $\delta u' = su$ (resp. $\delta v' = sv$) . Then under the notation of the above theorem

$$(3.17) \qquad [u'\otimes v - (-i)^i u\otimes v']$$

is a class of order s in $H^{i+j-1}(X\times Y)$ such that

$$(3.18) \quad (1\otimes\delta)^*[u'\otimes v - (-1)^i u\otimes v'] = (-1)^{i+1}[u]\otimes\delta v' \in H^i(X)\otimes B^j(Y)$$

is a class of order s in $\ker[(-1)^i\otimes\nu_j] \cong \mathrm{Tor}(H^i(X),H^j(Y))$.

Proof

$$\begin{aligned}
&\delta(u'\otimes v - (-1)^i u\otimes v') \\
&= \delta u'\otimes v + (-1)^{i-1}u'\otimes\delta v - (-1)^i\delta u\otimes v' - (-1)^{2i}u\otimes\delta v' \\
&= s(u\otimes v - u\otimes v) = 0
\end{aligned}$$

Hence $u'\otimes v - (-1)^i u\otimes v'$ is a cocycle.

$$\begin{aligned}
(-1)^i\otimes\nu_{j+1}[(-1)^{i+1}[u]\otimes\delta v'] &= -[u]\otimes\delta v' \\
&= -[u]\otimes sv \\
&= -[su]\otimes v = 0
\end{aligned}$$

since $v \in Z^j(Y)$. Suppose

$$\delta(\sum_k w_k\otimes z_k) = \sum_k \delta w_k\otimes z_k = t[u\otimes\delta v'] \in \bar{C}^j .$$

Since $\delta v'$ generates a direct summand of $B^{j+1}(Y)$ (v is a class of order s), and $B^{j+1}(Y)$ is free, we can assume, using properties of the tensor product, that $z_k = \delta v'$ for all k. Hence

$$\sum_k \delta w_k \otimes z_k = (\delta \sum_k w_k) \otimes \delta v' = tu \otimes \delta v' .$$

Since $C^*(X)$ is free we can conclude $\delta \sum_k w_k = tu$ and hence that s divides t.

Therefore $(-1)^{i+1}[u] \otimes \delta v'$ is an element of order s in $\ker(-1)^i \otimes \nu_{j+1}$. Observing that

$$\delta((-1)^{i-1} u' \otimes v') = s(u' \otimes v - (-1)^i u \otimes v')$$

we conclude that $[u' \otimes v - (-1)^i u \otimes v']$ is also a class of order s. This completes the proof.

<u>Corollary</u>

If $X = L_p^{2n-1}$ and $Y = L_q^{2m-1}$, the generator of the subgroup of $H^{2k+1}(L_p^{2n-1} \times L_q^{2m-1})$ corresponding to $\mathrm{Tor}(H^{2r}(L_p^{2n-1}), H^{2k+2-2r}(L_q^{2m-1})) \cong \mathbb{Z}_{(p,q)}$ under the isomorphism (3.2) is

$$\begin{aligned} &[k_1\, c^{2r-1} \otimes d^{2k+2-2r} - k_2\, c^{2r} \otimes d^{2k+1-2r}] \\ &= \alpha^{r-1} \beta^{k-r} [k_1\, c \otimes d^2 - k_2\, c^2 \otimes d] \\ &= \alpha^{r-1} \beta^{k-r} \tau \end{aligned}$$

where $k_1 = \frac{q}{(p,q)}$, $k_2 = \frac{p}{(p,q)}$ and $\tau = [k_1\, c \otimes d^2 - c^2 \otimes d]$.

Remark

In general, a complete set of generators for $H^*(X\times Y)$ may be found in the above manner.

To complete the calculation of the ring structure we need only to determine the cup products involving τ .

$$\begin{aligned}\tau^2 &= [(-1)^{2\cdot 1}\, k_1^2 c^2 \otimes d^4 - (-1)^{2\cdot 2}\, k_1 k_2 c^3 \otimes d^3 \\ &\quad - (-1)^{1\cdot 1}\, k_1 k_2 c^3 \otimes d^3 + (-1)^{1\cdot 2}\, k_2^2 c^4 \otimes d^2] \\ &= k_1^2\, \alpha\beta^2 + k_2^2\, \alpha^2\beta\end{aligned} \tag{3.20}$$

where k_1^2 and k_2^2 are of course taken mod (p,q) .

$$\begin{aligned}\alpha^{n-1}\tau &= [c^{2n-2}\otimes 1]\tau \\ &= [k_1 c^{2n-1}\otimes d^2] = k_1\gamma\beta\end{aligned}$$

Similarly $\beta^{m-1}\tau = k_2\alpha\eta$ and $\tau\gamma = \tau\eta = 0$.

4. The Main Result

4.1 The Reduced Cohomology of $L_p^{2n-1} \times L_q^{2m-1}$

In the preceding section we have shown that $\tilde{H}^*(L_p^{2n-1} \times L_q^{2m-1})$ is a ring with 5 generators $\alpha,\beta,\tau,\gamma,\eta$, where $\dim \alpha = \dim \beta = 2$, $\dim \tau = 3$, $\dim \gamma = 2n - 1$, $\dim \eta = 2m - 1$, and which satisfy the following relations:

$$\alpha^n = \beta^m = \alpha\gamma = \beta\eta = \gamma^2 = \eta^2 = \tau\gamma = \tau\eta = 0$$

$$\tau^2 = \left(\frac{q}{(p,q)}\right)^2\alpha\beta^2 + \left(\frac{p}{(p,q)}\right)^2\alpha^2\beta \tag{4.1}$$

$$\alpha^{n-1}\tau = \left(\frac{q}{(p,q)}\right)\gamma\beta$$

$$\beta^{m-1}\tau = \left(\frac{p}{(p,q)}\right)\alpha\eta$$

$\alpha^i\beta^j$ generates a cyclic subgroup of order p if $1 \leq i < n$ and $j = 0$, of order q if $i = 0$, $1 \leq j < m$, and of order (p,q) if $1 \leq i < n$ and $1 \leq j < n$. $\alpha^i\beta^j\tau$ generates a cyclic subgroup of order (p,q) if $0 \leq i < n$ and $0 \leq j < n$. $\gamma,\eta,\gamma\eta$ each generate an infinite cyclic subgroup, $\alpha^i\eta$ generates a cyclic subgroup of order p if $1 \leq i < n$. $\beta^i\gamma$ generates a cyclic subgroup of order q if $1 \leq j < m$.

From the above relations it follows that if $n = 1$ (resp. $m = 1$) τ is not necessary as a generator since $\tau = \left(\frac{q}{(p,q)}\right)\gamma\beta$ (resp. $\tau = \left(\frac{p}{(p,q)}\right)\alpha\eta$). Also if $n = m = 1$, in which case $L_p' \times L_q' \cong S^1 \times S^1$, $\tau = 0$.

4.2 Corollary $\tilde{H}^*(RP^n \times RP^m)$

If n and m are odd, say $n = 2k - 1$ and $m = 2\ell - 1$, the structure of $\tilde{H}^*(RP^n \times RP^m)$ is the same as in the previous section. It is a ring with 5 generators $\alpha,\beta,\tau,\gamma,\eta$, where $\dim \alpha = \dim \beta = 2$, $\dim \tau = 3$, $\dim \gamma = n$ and $\dim \eta = m$, and which satisfy the following relations:

$$(4.2)\qquad \begin{aligned} &\alpha^{\frac{n+1}{2}} = \beta^{\frac{m+1}{2}} = \alpha\gamma = \beta\eta = \gamma^2 = \eta^2 = \tau\gamma = \tau\eta = 0 \\ &\tau^2 = \alpha\beta^2 + \alpha^2\beta; \quad \alpha^{\frac{n-1}{2}}\tau = \gamma\beta, \quad \beta^{\frac{m-1}{2}}\tau = \alpha\eta \end{aligned}$$

and every non-zero product generates a subgroup isomorphic to $\mathbb{Z}_2$, except γ,η, and $\gamma\eta$ which generate subgroups isomorphic to $\mathbb{Z}$.

If n is even and m is odd, say $n = 2k$ and $m = 2\ell - 1$, $\tilde{H}^*(RP^n \times RP^m)$ is a ring with 4 generators α,β,τ,η where $\dim \alpha = \dim \beta = 2$, $\dim \tau = 3$, and $\dim \eta = m$, and which satisfy the following relations:

$$(4.3)\qquad \begin{aligned} &\alpha^{\frac{n}{2}+1} = \beta^{\frac{m+1}{2}} = \beta\eta = \eta^2 = \tau\eta = 0 \\ &\tau^2 = \alpha\beta^2 + \alpha^2\beta, \quad \alpha^{\frac{n}{2}}\tau = 0, \quad \beta^{\frac{m-1}{2}}\tau = \alpha\eta \end{aligned}$$

and every non-zero product generates a subgroup isomorphic to $\mathbb{Z}_2$, except η which generates a subgroup isomorphic to $\mathbb{Z}$.

If n and m are even, say $n = 2k$ and $m = 2\ell$, $\tilde{H}^*(RP^n \times RP^m)$ is a ring with 3 generators α,β,τ where $\dim \alpha = \dim \beta = 2$ and $\dim \tau = 3$, and which satisfy the following relations:

$$\alpha^{\frac{n}{2}+1} = \beta^{\frac{m}{2}+1} = 0, \quad \tau^2 = \alpha\beta^2 + \alpha^2\beta$$

(4.4)

$$\alpha^{\frac{n}{2}}\tau = 0, \qquad \beta^{\frac{m}{2}}\tau = 0$$

and every non-zero product generates a subgroup which is isomorphic to $\mathbb{Z}_2$.

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