MAN AS PREDATOR: QUALITATIVE BEHAVIOUR OF A CONTINUOUS DETERMINISTIC MODEL OF A FISHERY SYSTEM

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ABSTRACT

A global portrait of the phase plane is obtained for any acceptable values of the parameters. 3 different structures of the phase plane are recovered. The first predicts an eventual collapse of the fishery. The second predicts an unstable limit cycle and an eventual stability of solutions which start inside the limit cycle. The last structure predicts 2 possible stable equilibria, one with high catch rate, and the other one with no catch. Each structure corresponds to a different domain in the parameter space. The boundaries of these domains are found by solving the relevant differential equation for a saddle-to-saddle separatrix in the phase plane. This procedure utilizes regular perturbation methods.

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INTRODUCTION

Commercial exploitation of animate resources is one of man's oldest occupations, already mentioned in Genesis. In recent years an accelerating decline in the productivity of important fisheries was observed. Clark [1] mentions the great whale fisheries, Grand Banks fisheries and the Peruvian anchovy fishery.

A model of the dynamics of animal population and human effort to harvest it is analysed in this work. The model is continuous and deterministic. For a non-harvested population it assumes logistic growth perturbed by predation. This implies that there are two possible equilibria for the population: a very low one and a high one. Harvesting may drive the population to the low equilibrium, where no harvesting is worthwhile.

For a harvested population we subtract the harvest from the natural growth, and obtain the equation for the dynamics of the population. For the human effort we assume that the rate of change in the effort is proportional to the net income. This reflects the fact that the hunted population is a common property. Everybody has free access to commercial hunting (or fishing) and therefore the total human effort increase is proportional to the total net income (negative net income and negative increase are not excluded). The net income is the difference between the total revenue (i.e. the harvest) and total cost.

The model is presented in Section I.

Section II is a preliminary discussion of the equations obtained in Section I. They are scaled and brought to the form

$$\frac{dx}{dt} = x\, g(x) - xy \ ,$$

$$\frac{dy}{dt} = ay(x - b) \ ,$$

$$g(x) = R(1 - \frac{x}{Q}) - \frac{x}{1 + x^2} \ .$$

The quantities x and y are proportional to the population density and the human effort respectively, and hence $x, y \geq 0$. $g(x)$ has three positive zeros x_i, $i = 1, 2, 3$. The possible equilibria of the system are $E_0 = (0,0)$; $E_i = (x_i, 0)$ $i = 1, 2, 3$, ; $E_4 = (b, g(b))$. E_1 is asymptotically stable, E_2 and E_3 are saddle points and E_4 is $\begin{matrix} \text{stable} \\ \text{unstable} \end{matrix}$ if $\frac{dg}{dx}(b) \begin{matrix} < 0 \\ > 0 \end{matrix}$. We denote by x_o the point where $\frac{dg}{dx}(x) = 0$, and then give a preliminary description of the motions in the phase plane. They are summarized in the following figure:

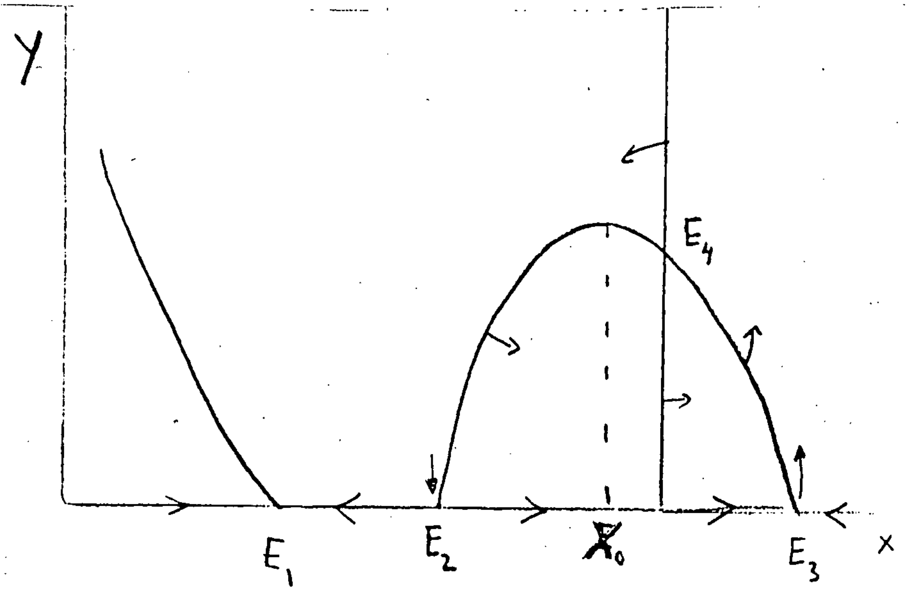

Figure 1. The equilibria in the phase plane.

Section II concludes with an indication of the main problem of the work: the completion of the phase plane portrait. The main clue for this is the information about T_2 and T_3 - the separatrix which goes to E_2 and the one which leaves E_3, respectively. In Section III these T_i are defined and the problem arises: for which value of b is a saddle-to-saddle separatrix obtained, i.e. when do the T_i intersect?

This b is found in Section IV. Approximate solutions for the saddle-to-saddle separatrix are obtained, and they imply a unique value of b (denoted $\hat{b}$). Of course, $\hat{b}$ depends on R, Q, a. This $\hat{b}$ is found with two alternative assumptions:

(i) $a \cdot Q^{3/2}$ is large, or

(ii) $a \cdot Q^{3/2}$ is small.

The interesting thing is that always $\hat{b} > x_0$. This enables us to draw only three distinct phase plane portraits and eliminate other possibilities. These portraits - as given in Section V - are:

(i) For $b < x_0$ all the trajectories converge to E_1.

(ii) For $x_0 < b < \hat{b}$ an asymptotically unstable limit cycle appears. Its existence is proved by the Poincaré-Bendixon theorem. All the solutions which start inside the limit cycle converge to E_4. Those which start outside the limit cycle converge to E_1.

(iii) For $b > \hat{b}$, E_4 is the attractor for orbits which start under T_2 while E_1 is the attractor for orbits which start above T_2.

From these 3 structurally different portraits we see that given an initial point $(x(0), y(0))$, the solution may or may not converge to E_1. What happens will depend upon the parameters. In certain cases, a modification c the parameters can avoid a collapse of the population (and the harvest).

The analytic approximations were accompanied and verified by a numerical scheme. It is described in the appendix.

I. THE MODEL

This model is a deterministic continuous model which describes an animal population subject to human harvest. A system of two coupled ordinary differential equations is introduced. One equation describes population changes while the other describes changes in human effort to catch the animals. The equations are simple and therefore one should not expect them to fit reality in every detail. They do not refer to any particular animal population, but mainly fish populations were in my mind during the work on this paper.

The basic equation for the dynamics of the population is

$$\frac{du}{d\tau} = NG(u) - H \ . \tag{I.1}$$

Where u is the population density, τ is time, NG is natural growth rate (i.e. growth rate of the population with the absence of human harvest) and H is the harvest. This equation has been employed in Clark and Munro [2] and Smith [9].

The equation for a population which is not subject to human catch is

$$\frac{du}{d\tau} = NG(u) = r_u u\left(1 - \frac{u}{K_u}\right) - \beta \frac{u^2}{\alpha^2 + u^2} \ . \tag{I.2}$$

The term $r_u\, u\left(1 - \frac{u}{K_u}\right)$ is the right hand side of the logistic equation, often used by biologists (cf. McNaughton-Wolf [3]). It describes a growth which is exponential initially and then decays due to the finiteness of the environment resources. K_u is the maximal possible population density. It is determined by factors such as limited food supply or space.

The term $-\beta \frac{u^2}{\alpha^2 + u^2}$ describes an effect on the growth rate, due to predation. This particular choice of the predation term represents a type III S-shaped functional response (cf. Holling [4]). According to Holling, the effect of predation saturates at fairly low population densities, i.e. there is an upper limit to the rate of mortality due to predation. This implies that $\frac{\alpha}{K_u}$ is small. Another feature of predation is a decrease in the effectiveness of predation at very low densities. This is attributed to searching and learning on the predator's part. Finally, we have to remark that $-\beta \frac{u^2}{\alpha^2 + u^2}$ is not the only way to represent a type III S-shaped response. This particular form is chosen because of mathematical convenience.

In order to incorporate the effect of human harvest we follow Clark-Munro [2] who give $H(E,u)$ - the human harvest - the form:

$$H(E,u) = \gamma\, u^{\gamma_1} E^{\gamma_2} , \qquad (I.3)$$

where E is the human effort and $\gamma_i \geq 0$. For reasons of convenience we set

$$\gamma_1 = \gamma_2 = 1 . \qquad (I.4)$$

Combining (I.1), (I.2), (I.3), and (I.4) we obtain

$$\frac{du}{d\tau} = r_u \; u(1 - \frac{u}{K_u}) - \beta \frac{u^2}{\alpha^2 + u^2} - \gamma UE . \qquad (I.5)$$

For the economic part we assume that

$$\frac{dE}{d\tau} = c \cdot Y , \tag{I.6}$$

where Y is the net income yield. Following Schaeffer [5] we assume that the total cost is proportional to the effort.

$$C = p \cdot E \quad . \tag{I.7}$$

This is a common assumption among economists (cf. Clark Munro [2]). In general, $p = p(E)$, but here p is assumed to be independent of E.

The total production has already been given by (I.3), (I.4) and when we combine this with (I.7) and substitute in (I.6) we obtain

$$\frac{dE}{d\tau} = c\ \gamma E(u - \frac{p}{\gamma}) \ . \tag{I.8}$$

Combining (I.5) and (I.8) we get the system

$$\frac{du}{d\tau} = r_u u(1 - \frac{u}{K_u}) - \beta \frac{u^2}{\alpha^2 + u^2} - \gamma E u \ , \tag{I.9}$$

$$\frac{dE}{d\tau} = c\ \gamma\ E(u - \frac{p}{\gamma}) \ .$$

The main aim of this work is to study system (I.9).

II. PRELIMINARIES

This section is devoted to a simplification of the problem and to the derivation of some straight forward results. Simplification is done by scaling the variables and bringing (I.9) to the form

$$x' = x\,g(x) - xy\ ,$$
$$y' = ay(x - b)\ ,$$

where $g(x) = R(1 - \frac{x}{Q}) - \frac{x}{1 + x^2}$. Zeros and a maximum of $g(x)$ are found and then used in a discussion on the equilibria and their asymptotic stability. The section concludes with a preliminary description of the motions in the first quadrant of the phase plane. The following sections complete this description.

II.1 Scaling

We introduce the following parameters and quantities:

$$R = \frac{\alpha r_u}{\beta}\ ,$$

$$Q = \frac{K_u}{\alpha}\ ,$$

$$a = \frac{c}{\beta} \cdot \gamma \cdot \alpha^2\ ,$$

$$b = \frac{p}{\gamma\alpha}\ ,$$

$$x = \frac{u}{\alpha} ,$$

$$y = \frac{\gamma\alpha}{\beta} E ,$$

$$t = \frac{\beta}{\alpha} \tau .$$

In terms of these we have

$$\begin{aligned} x' &= x\, g(x) - xy , \\ y' &= ay(x - b) , \end{aligned} \qquad \text{(II.1.1)}$$

where $' = \frac{d}{dt}$; $g(x) = R(1 - \frac{x}{Q}) - \frac{x}{1 + x^2}$.

Since α is the density where predation saturation occurs and K_u is the total capacity, $Q = \frac{K_u}{\alpha}$ is large. This will be used later.

II.2 g(x) : its zeros and maximum

$g(x)$ may have either one or three zeros. We are interested only in the latter case. This implies certain restrictions on R and Q . They are discussed in the Appendix.

We employ algebra to obtain:

$$g(x) = R(1 - \frac{x}{Q}) - \frac{x}{1 + x^2} = - \frac{R}{Q(1 + x^2)} P(x) ,$$

where

$$P(x) = x^3 - Qx^2 + (1 + \frac{Q}{R})x - Q .$$

To evaluate $x_i (i=1,2,3)$ we set

$P(x) = 0$.

Hence $x_{1,2}^2 - \frac{1}{R} x_{1,2} + 1 = \frac{1}{Q} (x_{1,2} + x_{1,2}^3)$

Therefore for $0 < \varepsilon_0 \leq R < \frac{1}{2}$ we have*:

$$x_1 = \frac{\frac{1}{R} + \sqrt{\frac{1}{R^2} - 4}}{2} + 0(Q^{-1}) ,$$

$$x_2 = \frac{-\frac{1}{R} + \sqrt{\frac{1}{R^2} - 4}}{2} + 0(Q^{-1}) .$$

Since $x_3 = -Q + \frac{Q}{R \cdot x_3} + \frac{1}{x_3} - \frac{Q}{x_3^2} = 0$, we obtain

$$x_3 = Q - \frac{1}{R} + 0(Q^{-1}) .$$

An approximate value of x_0 is found as follows:

$$\frac{dg}{dx} = -\frac{R}{Q} + \frac{x^2 - 1}{(1 + x^2)^2} , \text{ and}$$

$$\frac{dg}{dx}(x_0) = 0 .$$

Hence $x_0^2 = \dfrac{\frac{Q}{R} - 2 + \frac{Q}{R}\sqrt{1 - \frac{8R}{Q}}}{2}$.

Thus $x_0 = \sqrt{\frac{Q}{R}} - \frac{3}{2}\sqrt{\frac{R}{Q}} + 0(Q^{-1})$.

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*In general:

$$(x - x_1)(x - x_2) = \varepsilon ; \quad x_1 \neq x_2$$

$$\Rightarrow \quad x = x_1 + \frac{\varepsilon}{x - x_2} = x_1 + \frac{\varepsilon}{x_1 + \frac{\varepsilon}{x - x_2} - x_2} = x_1 + \frac{\varepsilon}{x_1 - x_2} + 0(\varepsilon^2) ,$$

II.3 Equilibria and their asymptotic stability

From the equations:

$$x' = x\,g(x) - xy\ ,$$
$$y' = ay(x - b)\ ,$$

we see that the equilibria in the first quadrant are:

$$E_0 = (0,0)\ ,$$
$$E_1 = (x_1,0)\ ,$$
$$E_2 = (x_2,0)\ ,$$
$$E_3 = (x_3,0)\ ,$$
$$E_4 = (b,g(b))\ .$$

E_4 occurs only if $0 \le b \le x_1$ or $x_2 \le b \le x_3$. The restriction $b > x_3$ means that even if the biomass were at its maximal possible value, it would not be worthwhile to make an effort to harvest it. The restriction $b < x_2$ means that it is worthwhile to make a harvesting effort in densities below x_2 - the collapse threshold. Neither case is realistic. We shall concentrate only on the case $x_2 < b < x_3$.

In order to compute the asymptotic stability of the equilibria, we first compute the variational matrix of the system.

$$M(x,y;b) = \begin{pmatrix} x\,g_x(x) + g(x) - y & -x \\ ay & a(x - b) \end{pmatrix} .$$

Let M_i be $M(x,y;b)$ evaluated at E_i . Then we obtain

$$M_0 = \begin{pmatrix} R & 0 \\ 0 & -ab \end{pmatrix},$$

$$M_1 = \begin{pmatrix} x_1 \cdot g_x(x_1) & -x_1 \\ 0 & a(x_1 - b) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} x_2 \cdot g_x(x_2) & -x_2 \\ 0 & a(x_2 - b) \end{pmatrix},$$

$$M_3 = \begin{pmatrix} x_3 \cdot g_x(x_3) & -x_3 \\ 0 & a(x_3 - b) \end{pmatrix},$$

$$M_4 = \begin{pmatrix} b\ g_x(b) & -b \\ a\ g(b) & 0 \end{pmatrix}.$$

From the assumptions on $g(x)$ and on b $(x_2 < b < x_3)$ we can draw the following conclusions:

(1) E_0 is a saddle point which attracts in the y direction and repels in the x direction.

(2) E_1 is a stable equilibrium (because $g_x(x_1) < 0$; $x_1 < b$)

(3) E_2 is a saddle point (because $g_x(x_2) > 0$; $x_2 < b$) that repels in the x direction.

(4) E_3 is a saddle point (because $g_x(x_3) < 0$; $x_3 > b$) which attracts in the x direction.

(5) E_4 is asymptotically stable (unstable) according to:

If $\frac{d}{dx} g(b) < 0$, then E_4 is asymptotically stable.

If $\frac{d}{dx} g(b) > 0$, then E_4 is asymptotically unstable.

Graphically, this means that if $b > x_0$ then E_4 is asymptotically stable; if $b < x_0$ then E_4 is asymptotically unstable.

II.4 Phase plane description

We consider again the system

$$x' = x\, g(x) - xy \; ,$$
$$y' = ay(x - b) \; .$$

Along the axes:

Along the y axis $x' = 0$; $y' < 0$.

Along the x axis $y' = 0$; $\operatorname{sgn}(x') = \operatorname{sgn} g(x)$.

In the interior of the first quadrant we have 5 distinct regions to discuss. Considering $\frac{dy}{dx} = \frac{ay(x-b)}{x\,g(x) - xy}$ we can immediately tell the direction of the motion in each case.

1. When $x < x_1$ and $y < g(x)$ then $\frac{dy}{dx} < 0$.
2. When $x_2 < x < b$ and $y < g(x)$ then $\frac{dy}{dx} < 0$.
3. When $b < x < x_3$ and $y < g(x)$ then $\frac{dy}{dx} > 0$.
4. When $x < b$ and $y > g(x)$ then $\frac{dy}{dx} > 0$.
5. When $x > b$ and $y > g(x)$ then $\frac{dy}{dx} < 0$.

At the boundaries of these regions we have

(1) $y = g(x)$ implies $x'=0$; $y' > 0$ if $x > b$ and $y' < 0$ if $x < b$.

(2) $x = b$ implies $y'=0$; $x' > 0$ if $y < g(x)$ and $x' < 0$ if $y > g(x)$

The motions in the phase plane can be illustrated by the following figure:

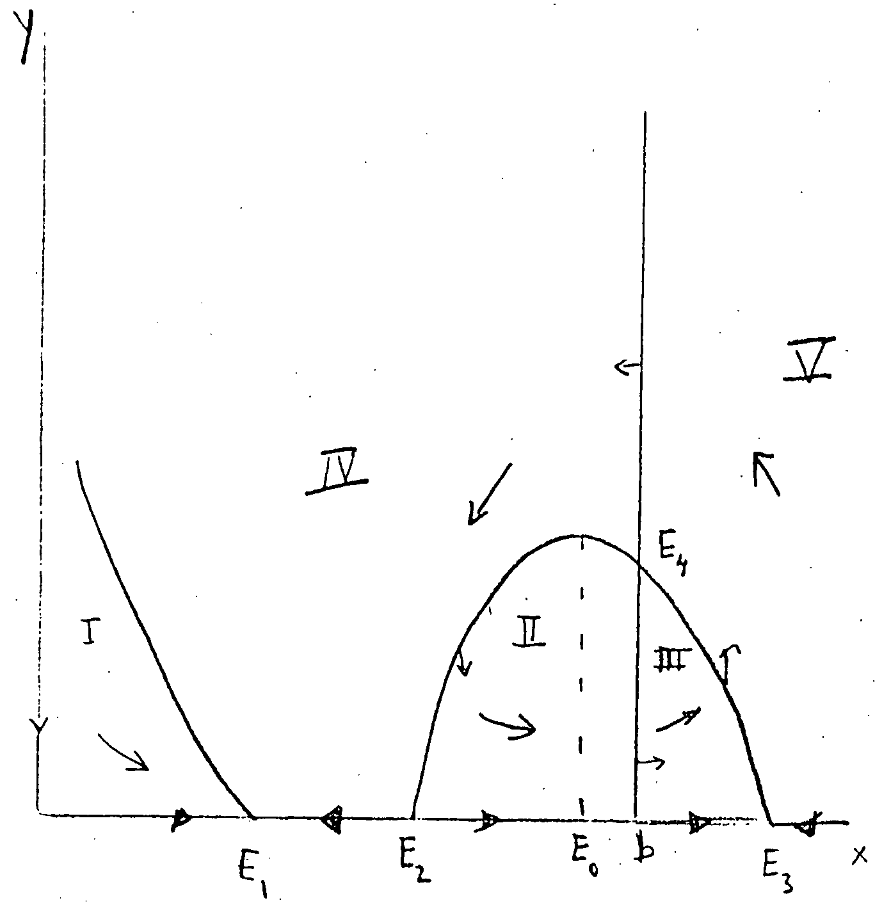

Figure 2. A preliminary description of the motions in the phase plane.

At this stage we are ready to indicate the main problem of the work - the global picture of the phase plane. Given initial x and y, we would like to know where the solution of (II.1.1) goes as $t \to \infty$. Clearly, when E_4 is asymptotically stable there are two possible answers - E_1 and E_4. But even when E_4 is asymptotically unstable there is no reason to believe that all the solutions converge to E_1. The answer depends not only upon the initial data but also on the particular value of the parameters. In the next sections the asymptotic behaviour of the solutions will be discussed in the various domains of the parameter space.

III. THE TRAJECTORIES AFFILIATED WITH THE SADDLE POINTS

A possible way to deal with the problem raised at the end of the last section is a division of the first quadrant into domains bounded by solution trajectories. A solution which starts in such a domain is destined to remain there because two solutions cannot intersect. For the same reason there are only two possible boundaries of this type: periodic solutions and orbits which connect critical points. E_1, E_2 and E_3 are already known to be connected by one orbit, namely the x axis. There exists another unique orbit which goes to E_2 and another one which goes from E_3. Motivated by this, one is led to investigate properties of these trajectories.

Since the right hand side of the system (II.1.1) is twice continuously differentiable we may use the following theorem, which is a slightly modified version of a theorem given by Coddington-Levinson [6].

Theorem. Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix},$$

for which the following conditions hold:

(i) (x_0,y_0) is a saddle point.

(ii) f_1, $f_2 \in C^2$ in the neighbourhood of (x_0,y_0).

Then there exist exactly two orbits tending to (x_0,y_0) as $t \to \infty$ The angle between these two orbits is 180°, and any orbit starting sufficiently near either of these orbits in the neighbourhood of (x_0,y_0)

tends away from them as $t \to \infty$.

A corollary of the theorem is that if (i), (ii) hold, there exist exactly two orbits tending to (x_0, y_0) as $t \to -\infty$. The angle between the orbits is 180°, and any orbit starting sufficiently close to either of these orbits and to (x_0, y_0) tend away from them as $t \to -\infty$.

At $E_2 = (x_2, 0)$ we already know that the latter orbits lie on the x-axis. We define T_2 as the orbit of the former type which lies above the x-axis.

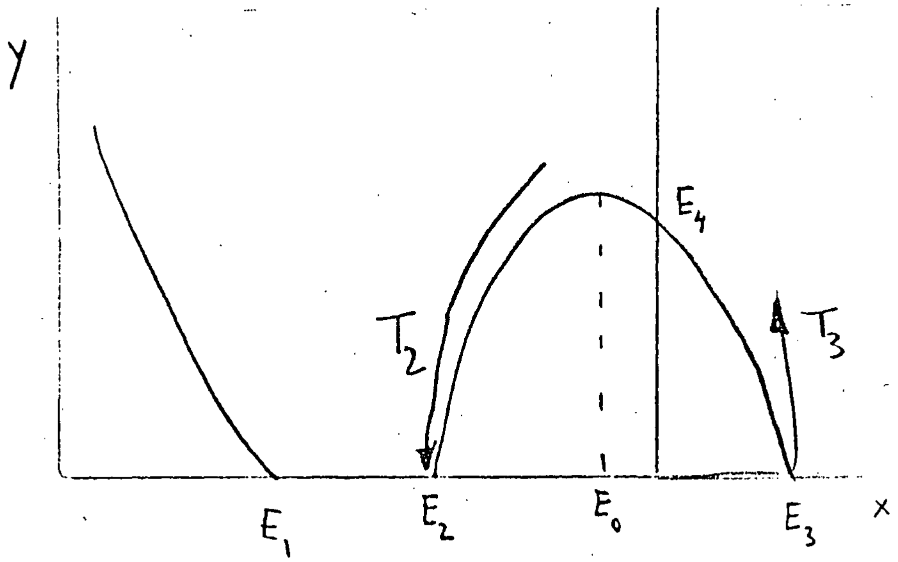

Figure 3. T_2 and T_3

T_2 must lie above $g(x)$ near E_2 . Indeed, otherwise $x' > 0$ along T_2 near E_2, and $x > x_2$ on T_2 for T_2 under $g(x)$ imply that T_2 does not tend to E_2 . Therefore T_2 lies in region 4 near E_2 .

At E_3 we define T_3 - the trajectory leaving E_3 . In a similar way to T_2, we can show that T_3 lies above $g(x)$ and T_3 lies in region 5 near E_3.

T_2 lies in region 4 near E_2. Therefore, either it crosses the line $x = b$ or it converges to E_4 as $t \to -\infty$, or both happen (if T_2 spirals around E_4 in a converging fashion).

T_3 lies in region 5 near E_3. Hence, either it crosses the line $x = b$ or it converges to E_4 as $t \to \infty$ or both.

If both T_2 and T_3 cross $x = b$ we say $T_2 > T_3$ if T_2 crosses "higher" than T_3 and $T_3 > T_2$ in the opposite case

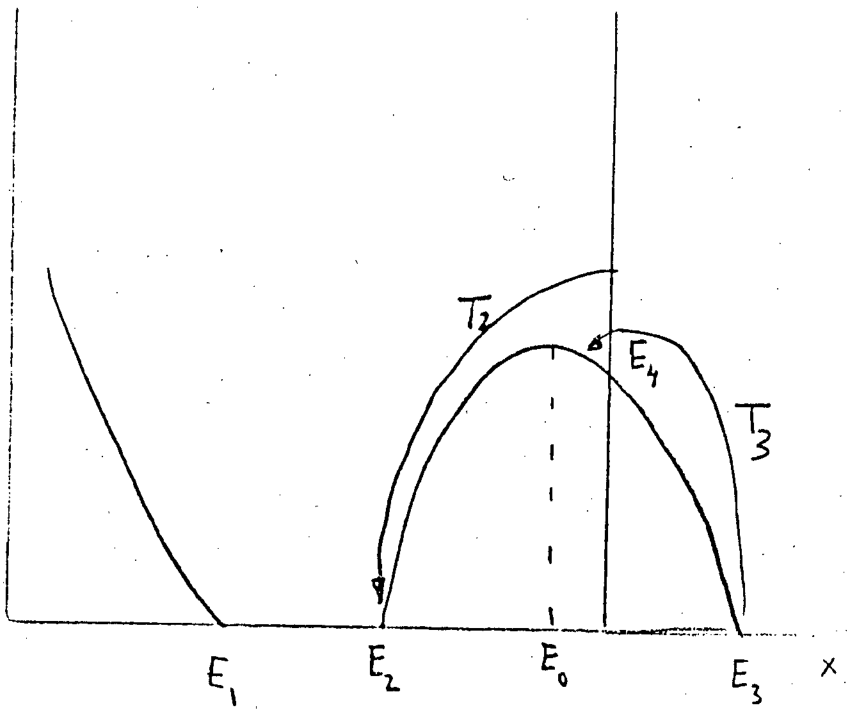

Figure 4. $T_2 > T_3$

If one T_{i_0} does not cross the line $x = b$, then $T_j > T_{i_0}$. At least one of them must cross $x = b$. Otherwise both converge to E_4, but a simple non saddle equilibrium cannot attract and repel at the same time.

There is another possibility yet: both T_2 and T_3 cross $x = b$ at the same point. Then - by the uniqueness of the initial value problem - they are identical. $T_2 = T_3$, and this is a saddle-to-saddle separatrix. This trajectory connects E_2 and E_3, so it can serve us in the way described at the beginning of this section. Hence the motivation to find the value of b for which such a separatrix is obtained. From here on we shall refer to this as the solution for b, and denote it by $\hat{b}$.

IV. A SADDLE TO SADDLE SEPARATRIX

IV.1 Introduction

In this section we shall concentrate on the equation

$$\frac{dy}{dx} = \frac{ay(x - b)}{x\ g(x;Q) - xy} \quad . \tag{IV.1.1}$$

We shall find the b for which exists a solution y which satisfies

$$y(x_i) = 0 \quad i = 2,\ 3 \tag{IV.1.1}$$

i.e., there is a saddle-to-saddle separatrix.

Since (IV.1.1) is too complicated to integrate exactly, we shall use a perturbation method. The crucial parameter was discovered in two steps:

(i) Clearly, the larger a is, the larger $\left|\frac{dy}{dx}\right|$ is. Assuming "large a" we obtained an approximate solution and then compared it with numerical results. It appeared that the approximation was good also for fairly small values of a such as 10^{-2}.

(ii) Observing that $x_0 = \sqrt{\frac{Q}{R}}$ is where structural changes occur* we scaled:

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*In fact, it will be shown later that a Hopf bifurcation takes place near $b = x_0$.

$$u = \frac{x}{\sqrt{Q}} \; ,$$
$$B = \frac{b}{\sqrt{Q}} \; ^{**} . \qquad \text{(IV.1.3)}$$

Now the new form of (IV.1.1) is:

$$\frac{dY}{du} = \frac{a\sqrt{Q}\,(u - B) \cdot Y}{u(G - Y)} \; . \qquad \text{(IV.1.4)}$$

Here $Y(u) = y\left(\frac{x}{\sqrt{Q}}\right)$; $G(u;Q) = R(1 - \frac{u}{\sqrt{Q}}) - \frac{u\sqrt{Q}}{1 + Qu^2}$.

We tried a parameter of the form $P = aQ^m$ and the following expansions for the saddle-to-saddle separatrix and $\hat{B}$:

For small P ,

$$Y^{(s)}(u;Q) = Y_0^{(s)}(u;Q) + PY_1^{(s)}(u;Q) + P^2Y_2^{(s)}(u;Q) + \dots ,$$
$$\hat{B}^{(s)} = B_0^{(s)}(Q) + PB_1^{(s)}(Q) + P^2B_2^{(s)}(Q) + \dots \; . \qquad \text{(IV.1.5)}$$

For large P ,

$$Y^{(\ell)}(u;Q) = Y_0^{(\ell)}(u;Q) + P^{-1}Y_1^{(\ell)}(u;Q) + P^{-2}Y_2^{(\ell)}(u;Q) + \dots ,$$
$$\hat{B}^{(\ell)} = B_0^{(\ell)}(Q) + P^{-1}B_1^{(\ell)}(Q) + P^{-2}B_2^{(\ell)}(Q) + \dots \; . \qquad \text{(IV.1.6)}$$

**Following this, $u_j = \frac{x_j}{\sqrt{Q}}$ $j = 0, 1, 2, 3$; $\hat{B} = \frac{\hat{b}}{\sqrt{Q}}$

$B_0^{(s)}(Q)$ appears to be $O(1)$ as $Q \to \infty$ and m will be chosen so that also $B_1^{(s)} = O(1)$ as $Q \to \infty$. This will indicate that (IV.1.5) is asymptotic. It appears that $m = 3/2$ is the appropriate m.

IV.2 An approximate solution for small P

We substitute (IV.1.5) in (IV.1.4) and obtain:

$$\frac{dY_0^{(s)}}{du} + P\,\frac{dY_1^{(s)}}{du} + P^2\,\frac{dY_2^{(s)}}{du} + \ldots \tag{IV.2.1}$$
$$= \frac{a\,Q^{\frac{1}{2}}(Y_0^{(s)} + PY_1^{(s)} + P^2Y_2^{(s)} + \ldots)(u - B_2^{(s)} - PB_1^{(s)} - P^2B_2^{(s)} - \ldots}{u(G(u) - Y_0^{(s)} - PY_1^{(s)} - P^2Y_2^{(s)} - \ldots)}$$

At this point we have to decide if $aQ^{\frac{1}{2}}$ is "small" or "large". This is necessary in order to simplify the equation. Certainly, if $m \geq \frac{1}{2}$, $aQ^{\frac{1}{2}}$ is small. We assume this, and will show later that $m = 3/2$.

Assuming that $aQ^{\frac{1}{2}}$ is small we can balance the two sides of (IV.2.1) by:

(i) setting $Y_0^{(s)}(u;Q) = 0$, or

(ii) setting $Y_0^{(s)}(u;Q) = G(u;Q)$.

We know that $Y^{(s)}(u;Q)$ lies above $G(u;Q)$, so we reject (i) and set

$$Y_0^{(s)}(u;Q) = G(u;Q)\,. \tag{IV.2.2}$$

Now the equation has the form:

$$\frac{dG}{du} + P\,\frac{dY_1^{(s)}}{du} + P^2\,\frac{dY_2^{(s)}}{du} + \ldots \tag{IV.2.3}$$
$$= -\,\frac{Q^{\frac{1}{2}-m}(G + PY_1^{(s)} + P^2Y_2^{(s)} + \ldots)(u - B_0^{(s)} - PB_1^{(s)} - \ldots)}{u(Y_1^{(s)} + PY_2^{(2)} + P^2Y_3^{(s)} + \ldots)}$$

Formally, we expand the right hand side:

$$-\frac{Q^{\frac{1}{2}-m}\,(G + PY_1^{(s)} + P^2Y_2^{(s)} + \ldots)(u - B_0^{(s)} - PB_1^{(s)} - P^2B_2^{(s)} - \ldots)}{u(Y_1^{(s)} + PY_2^{(s)} + P^2Y_3^{(s)} + \ldots)} =$$

$$= -\frac{Q^{\frac{1}{2}-m}(G + PY_1^{(s)} + P^2Y_2^{(s)} + \ldots)(u - B_0^{(s)} - PB_1^{(s)} - P^2B_2^{(s)} - \ldots)}{u\,Y_1^{(s)}\left(1 + P\,\frac{Y_2^{(s)}}{Y_1^{(s)}} + P^2\,\frac{Y_3^{(s)}}{Y_1^{(s)}} + \ldots\right)} =$$

$$= -\frac{Q^{\frac{1}{2}-m}}{u\,Y_1}\,(G + PY_1^{(s)} + P^2Y_2^{(s)} + \ldots)(u - B_0^{(s)} - PB_1^{(s)} - P^2B_2^{(s)} - \ldots)$$

$$\left(1 - P\,\frac{Y_2^{(s)}}{Y_1^{(s)}} - P^2\left(\frac{Y_3^{(s)}}{Y_1^{(s)}} - \left(\frac{Y_2^{(s)}}{Y_1^{(s)}}\right)^2\right) + \ldots\right) =$$

$$= -\frac{Q^{\frac{1}{2}-m}G(u - B_0^{(s)})}{u\cdot Y_1^{(s)}} - P\,\frac{Q^{\frac{1}{2}-m}}{u\,Y_1^{(s)}}\left[Y_1^{(s)}(u - B_0^{(s)}) - B_1^{(s)}G - \frac{Y_2^{(s)}}{Y_1^{(s)}}\,G(u - B_0^{(s)})\right] -$$

$$- P^2\,\frac{Q^{\frac{1}{2}-m}}{u\,Y_1^{(s)}}\left(Y_2^{(s)}(u - B_0^{(s)}) - Y_1^{(s)}B_1^{(s)} - B_2^{(s)}G - \frac{Y_2^{(s)}}{Y_1^{(s)}}\,(Y_1^{(s)}(u - B_0^{(s)}) - B_1^{(s)}G) -\right.$$

$$\left. - \left(\frac{Y_3^{(s)}}{Y_1^{(s)}} - \left(\frac{Y_2^{(s)}}{Y_1^{(s)}}\right)^2\right) G(u - B_0)\right) + \ldots$$

This expansion is valid only if $\frac{Y_2}{Y_1}$ is bounded.

Now we substitute this in (IV.2.3) and formally equate the coefficients of the respective powers of P.

For P° we have

$$\frac{dG}{du} = -\frac{Q^{\frac{1}{2}-m}\ G(u - B_0^{(s)})}{u\ Y_1^{(s)}}\ ,$$

$$Y_1^{(s)}(u;Q) = -\frac{Q^{\frac{1}{2}-m} \cdot G(u;q) \cdot [u - B_0^{(s)}(Q)]}{u \cdot \frac{dG}{du}(u;Q)}\ . \qquad \text{(IV.2.4)}$$

This $Y_1^{(s)}(u;Q)$ is continuous except - maybe - at $u = u_0(Q)$, where $\frac{dG}{du}(u_0;Q) = 0$. To avoid a singularity there we set

$$B_0^{(s)} = u_0(Q)\ . \qquad \text{(IV.2.5)}$$

For P^1 we have

$$\frac{dY_1^{(s)}}{du} = \frac{-Q^{\frac{1}{2}-m}}{u\ Y_1} [Y_1^{(s)}(u - B_0^{(s)}) - B_1^{(s)}G - \frac{Y_2^{(s)}}{Y_1^{(s)}} G(u - B_0^{(s)})]\ .$$

Or: $$Y_2^{(s)} = \frac{(Y_1^{(s)})^2}{G} + \left(\frac{Q^{m-\frac{1}{2}} \cdot u \cdot (Y_1^{(s)})^2}{G} \cdot \frac{dY_1^{(s)}}{du} - B_1^{(s)}Y_1^{(s)}\right) \cdot \frac{1}{u - B_0^{(s)}}\ . \qquad \text{(IV.2.6)}$$

Now to avoid singularity we set

$$B_1^{(s)}(Q) = \frac{Q^{m-\frac{1}{2}} \cdot u_0 \cdot Y_1^{(s)}(u_0;Q)}{G(u_0;Q)} \cdot \frac{dY_1}{du}(u_0;Q)\ . \qquad \text{(IV.2.7)}$$

From (IV.2.6) we see that $Y_2^{(s)}(u) = H(u) \cdot Y_1^{(s)}(u)$ where $H(u)$ is a bounded function on $[u_2,u_3]$. Therefore $\frac{Y_2^{(s)}(u)}{Y_1^{(s)}(u)} = H(u)$ is bounded there. This indicates that the geometric expansion was valid.

To get a more explicit form of (IV.2.7) we use l'Hopital's rule to obtain $Y_1^{(s)}(u_0;Q)$ and $\frac{dY_1^{(s)}}{du}(u_0;Q)$ and substitute in (IV.2.7) to obtain:

$$B_1^{(s)} = - \frac{G(u_0;Q)[2\ G''(u_0;Q) + u_0\ G'''(u_0;Q)]}{2\ u_0^2\ (G''(u_0))^3} \cdot Q^{\frac{1}{2}-m} , \qquad \text{(IV.2.8)}$$

where $(\ )' = \frac{d}{du}$.

We recall that

$$u_0 = \frac{1}{\sqrt{R}} + o(1) ,$$

$$G(u_0;Q) = R + o(1) ,$$

$$G''(u_0;Q) = - \frac{2R^{3/2}}{Q^{\frac{1}{2}}} + O(Q^{-3/2}) ,$$

$$G'''(u_0;Q) = \frac{6R^2}{Q^{\frac{1}{2}}} + O(Q^{-3/2}) .$$

Therefore

$$B_1^{(s)}(Q) = \frac{Q^{3/2-m}}{8R} + o(1) .$$

This gives us two things

(i) m has to be $\frac{3}{2}$ in order that $B_1^{(s)}(Q) = O(1)$, $Q \to \infty$. This is required to make the expansion (IV.1.5) asymptotic.

(ii) $B_1^{(s)}(Q) > 0$, and therefore $B^{(s)}(Q) > u_0$ for large Q . This will serve us later.

The continuation of the expansion is similar to what was done for the $O(P)$ terms. For $O(P^n)$, $n > 1$ we have

$$Q \cdot u \cdot Y_1^{(s)} \frac{dY_{n-1}^{(s)}}{du} = \frac{Y_n^{(s)}}{Y_1^{(s)}} \cdot G \cdot [u - B_0^{(s)}] - B_{n-1} \cdot G + (u - B_0) \cdot$$

$$H_1(Y_0^{(s)}, \ldots, Y_{n-1}^{(s)}; \frac{Y_2^{(s)}}{Y_1^{(s)}} ; \frac{Y_3^{(s)}}{Y_1^{(s)}} - \frac{Y_{n-1}^{(s)}}{Y_1^{(s)}}) + H_2(Y_0, \ldots, Y_{n-1}, \frac{Y_2}{Y_1} , \ldots, \frac{Y_{n-1}}{Y_1}) ,$$

where H_1, H_2 are polynomials in $Y_0, \ldots, Y_{n-1}; \frac{Y_2}{Y_1}, \ldots, \frac{Y_{n-1}}{Y_1}$ which vanish at u_2, u_3. From this we calculate the expressions for $Y_n^{(s)}$, $B_n^{(s)}$. $Y_n^{(s)}$ will automatically satisfy $Y_n^{(s)}(u_i) = 0$, $0 = 2, 3$, and $B_{n-1}^{(s)}$ is determined so that $Y_n^{(s)}(u)$ has no singularities in $[u_2, u_3]$.

IV.3 An approximate solution for large P.

For large P we expect $y^{(\ell)}(u)$ to be large, so we scale it:

$$\text{Let } z(u) = \frac{y^{(\ell)}(u)}{P} . \tag{IV.3.1}$$

Then we assume the following expansions:

$$\begin{aligned} z(u;B^{(\ell)}) &= z_0(u;B_0^{(\ell)}) + P^{-1}Z_1(u;B_0^{(\ell)},B_1^{(\ell)}) \\ &\quad + P^{-2}Z_2(u;B_0^{(\ell)},B_1^{(\ell)},B_2^{(\ell)}) + \ldots , \\ B^{(\ell)} &= B_0^{(\ell)} + P^{-1}B_1^{(\ell)} + P^{-2}B_2^{(\ell)} + \ldots \end{aligned} \tag{IV.3.2}$$

We denote the solution which corresponds to T_i by $Z^{(i)}$. Substituting (IV.3.1), (IV.3.2), (IV.1.3) in (IV.1.1) yields:

$$\frac{dZ_0^{(i)}}{du} + P^{-1}\frac{dZ_1^{(i)}}{du} + \ldots = \frac{(Z_0^{(i)} + P^{-1}Z_1^{(i)} + \ldots)(u - B_0^{(\ell)} - P^{-1}B_1^{(\ell)} - \ldots)}{-Qu(Z_0^{(i)} + P^{-1}Z_1^{(i)} + \ldots - P^{-1}G)} \tag{IV.3.3}$$

Using the geometric series expansion we obtain:

$$\frac{dZ_0^{(i)}}{du} + P^{-1}\frac{dZ_1^{(i)}}{du} + \dots = \frac{(u - B_0^{(\ell)})}{-Qu}$$

$$+ \frac{P^{-1}}{Q}\left|\frac{B_1^{(\ell)}}{u} - \frac{G \cdot (u - B_0^{(\ell)})}{u \cdot Z_0^{(\ell)}(u)}\right| + \dots \qquad \text{(IV.3.4)}$$

This expansion is valid only if $\frac{G(u)}{Z_0^{(i)}(u)}$ is bounded. It will be shown later that $\frac{G}{Z_0^{(2)}}$ is bounded in $[u_2,B_0^{(\ell)}]$ and $\frac{G}{Z_0^{(3)}}$ is bounded in $[B_0^{(\ell)},u_3]$.

Along with (IV.3.4) we must satify

$$Z_j^{(i)}(u_i) = 0 \qquad j = 0, 1, \dots \quad i = 1, 2 . \qquad \text{(IV.3.5)}$$

Formally, we equate the coefficients of equal powers of P and solve term by term.

Hence

$$Z_0^{(i)}(u;B_0^{(\ell)}) = \frac{1}{Q}\,[B_0^{(i)} \ln \frac{u}{u_i} - (u - u_i)] \qquad i = 2, 3 \qquad \text{(IV.3.6)}$$

Now we can show that $\frac{G}{Z_0^{(2)}}$ is bounded in $[u_2,B_0^{(\ell)}]$. At $u = u_2$ we have $\lim_{u\to u_2}\frac{G(u)}{Z_0^{(2)}(u)} = \frac{G^1(u_z)}{\frac{u_2 - B_0^{(\ell)}}{-Qu}} < \infty$ (we assume $B_0^{(\ell)} > u_2$) .

Therefore $\frac{G(u)}{Z_0^{(2)}(u)}$ is bounded for $u \in [u_2,\delta]$ for some $\delta > 0$. $\frac{dZ_0^{(2)}}{du} > 0$ for $u \in [u_2,B_0^{(\ell)}]$ and $Z_0^{(2)}(u_2) = 0$ implies $Z_0^{(2)}(u) \geq \varepsilon$ for $u \in [\delta,B_0^{(\ell)}]$ and therefore $\frac{G(u)}{Z_0^{(2)}(u)}$ is bounded for $u \in [\delta,B_0^{(\ell)}]$. The final conclusion is that $\dfrac{G(u)}{Z_0^{(2)}(u)}$ is bounded for $u \varepsilon [u_2, B_0^{(\ell)}]$. In a similar way we can treat $Z_0^{(3)}(u)$ to show that $\dfrac{G(u)}{Z_0^{(3)}(u)}$ is bounded in $[B_0^{(\ell)}, u_3]$.

Now to find $B_0^{(\ell)}$ we equate $Z_0^{(2)}(B_0^{(\ell)}; B_0^{(\ell)}) = Z_0^{(3)}(B_0^{(\ell)}; B_0^{(\ell)})$ and obtain the unique solution of this algebraic equation

$$B_0^{(\ell)} = \frac{u_3 - u_2}{\ln u_3 - \ln u_2} \tag{IV.3.7}$$

The next thing to be shown is that $B_0^{(\ell)} > u_0$. This was done numerically and for large Q we have

$$u_3 - u_2 = \sqrt{Q} + O(1), \qquad Q \to \infty$$

$$\ln \frac{u_3}{u_2} = \ln \frac{Q}{x_2} + O(1), \quad Q \to \infty, \text{ where } \quad x_2 = O(1)$$

Therefore $$B_0^{(\ell)} = \frac{\sqrt{Q}}{\ln \frac{Q}{x_2}} + O(1), \quad Q \to \infty$$

While $$u_0 = O(1) \quad , \qquad Q \to \infty$$

Hence $B_0^{(\ell)} > u_0$ for Q sufficiently large.

In order to continue this procedure one has to equate the higher order terms. In general, from (IV.3.4) we shall have:

$$\frac{dZ_n^{(i)}}{du} = \frac{B_n^{(\ell)}}{Qu} + \frac{H(G, Z_0^{(i)}, Z_1^{(i)}, \ldots, Z_{n-1}^{(i)}; B_0^{(\ell)}, \ldots, B_{n-1}^{(\ell)})}{QuZ_0(u)}$$

Where H is a polynomial such that $\dfrac{H}{Z_0}$ is continuous in $[u_2, u_3]$.

Since $Z_n^{(i)}(u_i) = 0 \quad i = 2, 3$ we shall have

$$Z_n^{(i)}(u) = \int_{u_i}^{u} \left[\frac{B_n^{(\ell)}}{Qs} + \frac{H(G(s), Z_0^{(i)}(s), \ldots, Z_{n-1}(s); B_0^{(\ell)}, \ldots, B_{n-1}^{(\ell)})}{Q\ sZ_0^{(\ell)}(s)} \right] ds \qquad i = 2, 3^*$$

Now we require $Z_n^{(2)}(v) = Z_n^{(3)}(v)$ for some $v \in [u_2, u_3]$ and obtain the expression for $B_n^{(\ell)}$ from

$$\frac{1}{Q} \int_{u_2}^{u_3} \left[\frac{B_n^{(\ell)}}{S} + \frac{H}{s \cdot Z_0(s)} \right] ds = 0$$

IV.4 Summary

Approximate solutions for the saddle-to-saddle separatrix were obtained in this section. One solution is based on the assumption that $aQ^{3/2}$ is a small parameter and the other assumes that $aQ^{3/2}$ is large. A unique value of b for which the existence of a saddle-to-saddle separatrix is possible was recovered in each case. In the former case this b has the expansion $\hat{b}^{(s)} = x_0 + Pb_1^{(s)} + \ldots$. The expression for $B_1^{(s)} = \frac{b_1^{(s)}}{\sqrt{Q}}$ is given in (IV.2.8). In the latter case we have $B^{(\ell)} = b_0^{(\ell)} + P^{-1}b_1^{(\ell)} + \ldots$ where $b_0^{(\ell)} = \frac{x_3 - x_2}{\ln x_3 - \ln x_2}$.

The important conclusion is that in both cases, $\hat{b} > x_0$. This will serve us as the main tool in the next section.

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*In fact at this stage we already know that $Z_j^{(2)}(u) = Z_j^{(3)}(u) \quad 0 \leq j < n$

V. THE GLOBAL PORTRAIT OF THE PHASE PLANE

At this stage we are ready to answer the question which was asked at the end of Section II. We use the ideas suggested at the beginning of Section III.

When $T_2 < T_3$, T_3 (which starts at E_3) must go to E_1. As a result we have a domain bounded by solution trajectories. The boundaries are the x-axis and T_3, and any solution which starts inside this domain cannot leave its boundaries.

When $T_2 > T_3$, T_2 serves as a part of the boundary, and similar conclusions will be drawn.

Throughout this section we assume $\hat{b} > x_0$. This has been shown for "large P" and "small P". For intermediate values of P it was verified by a numerical scheme. The details of the numerical scheme are given in the appendix.

<u>V.1 $T_3 > T_2$</u> In this case $x_2 < b \leq x_0$ or $x_0 < b < \hat{b}$.

For $x_2 < b < x_0$ there is only one asymptotically stable equilibrium in the phase plane, namely E_1. E_1 is on the boundary of K - the compact set bounded by T_3 and the x-axis. Therefore every solution (except those starting exactly at the equilibria or on T_2) will converge to E_1. This conclusion holds for solutions which start inside K or outside it in the first quadrant.

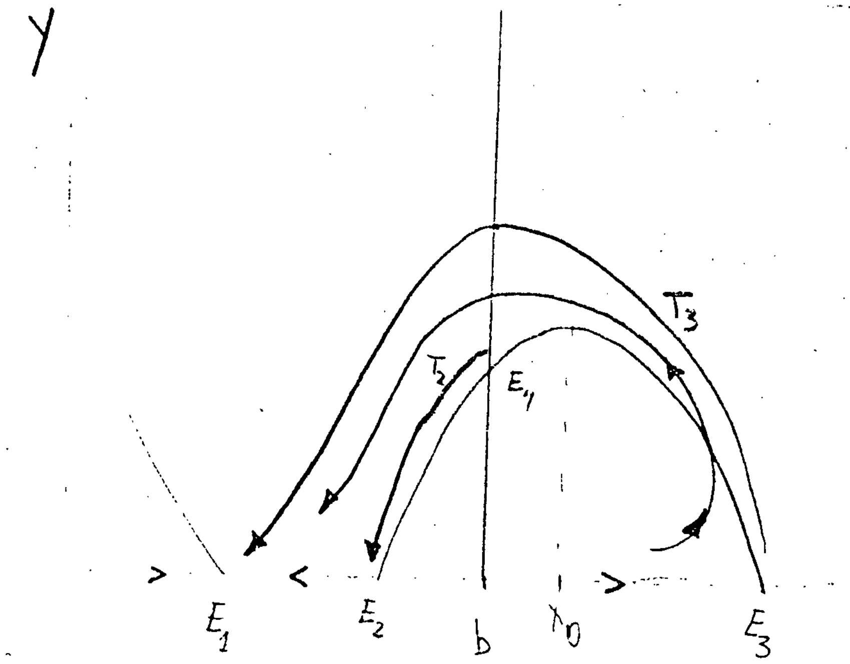

Figure 5. The phase plane portrait when $b < x_0$.

Another possiblity, still in case (1), is that $x_0 < b < \hat{b}$. Following the formulation of Coddington-Levinson [6] we define $L(T_2^-)$ as the negative limit set of T_2, and T_2^- as a negative semi-orbit of T_2 . *

Theorem. $L(T_2^-)$ is a limit cycle.

Proof. We use Poincaré-Benedixon theorem as given by Coddington-Levinson. There it refers to positive semi-orbits but it can be applied also to negative semi-orbits, which is our case.

$L(T_2^-) \subset K$ which is a bounded set. The singular points in K are E_i , $i = 1, 2, 3, 4$.

$E_1, E_4 \notin L(T_2^-)$ because they are asymptotically stable.

$E_2 \notin L(T_2^-)$ because there are only two trajectories which converge to E_3 as $t \to -\infty$, and they are on the x-axis.

$E_3 \notin L(T_2^-)$ because $E_3 = L(T_3^-)$ by definition of T_3 and there

*i.e. T_2^- is obtained by starting somewhere on T_2, and following the solution trajectory as time goes backwards.

is only one trajectory in the first quadrant which converges to E_3 as $t \to -\infty$.

The conclusion is that $L(T_2^-)$ contains only regular points and hence either

(i) T_2^- is a periodic orbit, or

(ii) $L(T_2^-)$ is a periodic orbit (a limit cycle).

(i) is excluded because if T_2^- is periodic then T_2 is periodic. But we know that $T_2 \to E_2$ as $t \to \infty$ and $T_2 \neq E_2$. Therefore T_2 is not periodic, and hence a limit cycle exists inside K .

E_4 is asymptotically stable so the limit cycle is asymptotically unstable. Every solution which starts in the domain bounded by the limit cycle will remain there, and converge to E_4 . Outside the closed set bounded by the limit cycle, all the solutions converge to E_1 .

The existence of a limit cycle for b near x_0 can be shown also by noticing that at $b = x_0$ a Hopf bifurcation takes place. We shall show this by using the Hopf bifurcation theorem as given by Howard-Koppel [7].

<u>Theorem</u>. For b near x_0 the system (II.1.1) has a one parameter family of solutions which lie in the neighbourhood of E_4, and there are no other periodic solutions wholly in this neighbourhood.

<u>Proof</u>. Near $b = x_0$ the eigenvalues at E_4 are $\lambda_1(b) \pm i\lambda_2(b)$ where

$$\lambda_1(b) = \frac{b\ g_x(b)}{2}$$

$$\lambda_2(b) = \frac{\sqrt{4\ ag(b) - b^2 g_x(b)}}{4}$$

At $b = x_0$, $g_x(b) = 0$ and therefore $\lambda_1(b) = 0$; $\lambda_2(b) \neq 0$.

We also have $\left.\dfrac{d\lambda_1}{db}\right|_{b=x_0} = \dfrac{bg_{xx}(b)}{2} < 0$

Another condition which is automatically satisfied is that the other eigenvalues are bounded away from the imaginary axis.

We see that all the conditions of the theorem hold and hence the conclusion.

As we can see, a structural change takes place as b becomes bigger than x_0. One feature of this change is the appearance of a limit cycle. By numerical methods it was observed that its amplitude is small near $b = x_0$ and increases as b gets larger. This increase takes place until $T_2 = T_3$ and the limit cycle disappears.

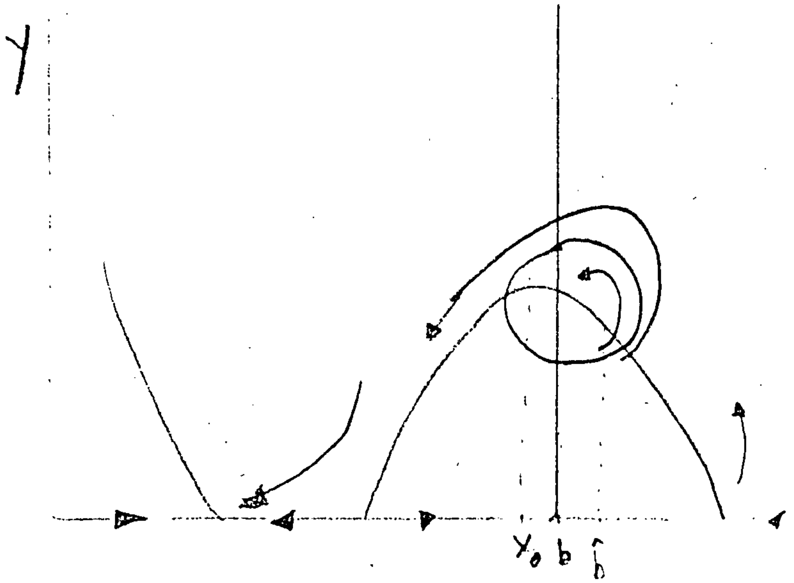

Figure 6. The phase plane portrait for $x_0 < b < \hat{b}$

V.2 $T_2 > T_3$

In this case T_2 divides the first quadrant to two connected sets, K_1 and K_2. K_1 is the set of points above and on the left of T_2 and K_2 is the set under T_2.

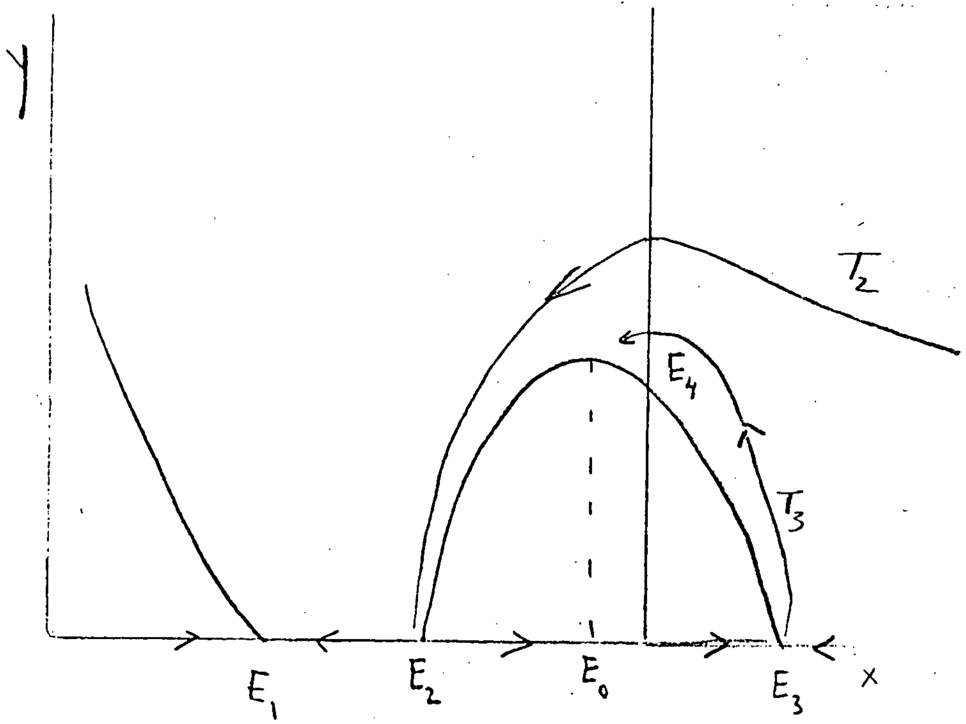

Figure 7. Division of the phase plane by T_2.

E_1 is the only asymptotically stable eqilibrium in $\overline{K_1}$ and E_4 is the only asymptotically stable equilibrium in $\overline{K_2}$. Therefore a solution which starts in K_1 will converge to E_1 and a solution which starts in K_2 will converge to E_4.

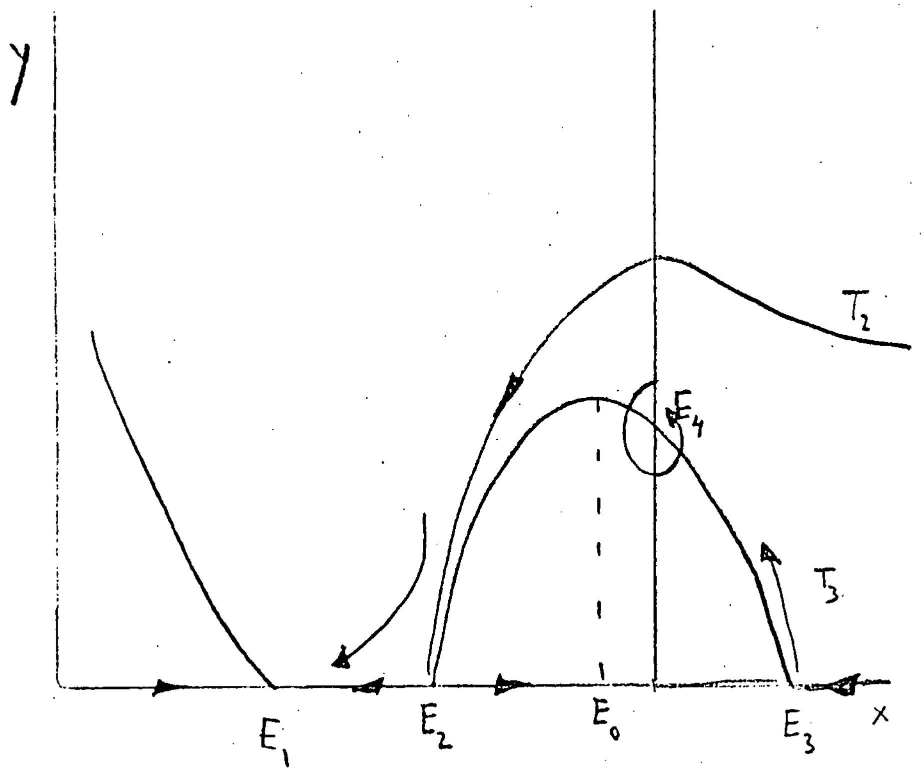

Figure 8. The phase plane portrait for $b > \hat{b}$.

Summary. We pointed at 3 structurally different portraits of the phase plane in the interior of the first quadrant.

The first portrait was obtained for $b < x_0$. Solutions which start at an equilibrium or on T_2 stay there, while all the other solutions converge to E_1 .

The second portrait was obtained for $b \in (x_0, \hat{b})$. An unstable limit cycle appears. Solutions which start on it, on T_2 or at an equilibrium stay there. Solutions which start in the interior of the domain bounded by the limit cycle converge to E_4 . All the other solutions converge to E_1 .

The last portrait was obtained for $b > \hat{b}$, i.e. when $T_2 > T_3$. Then solutions which start at an equilibrium or on T_2 , remain there. Solutions which start under T_2 converge to E_4, and all the other solutions converge to E_1 .

A fourth portrait that might have occurred would have E_4 as an asymptotically unstable equilibrium and an asymptotocally stable limit cycle surrounding E_4. It does not occur because $\hat{b} > x_0$.

The structural changes were derived from global considerations. However, at $b = x_0$ a Hopf bifurcation - a local phenomenon exists. Its significance is consistent with the portrait derived from a global considerations.

VI. INTERPRETATION

As we could see, there is a danger of driving the harvested population to E_1. Clearly, this is an undesirable equilibrium, an equilibrium of a graveyard: low animal density and no harvesting activity.

Another feature is the possibility of fluctuations both in the effort and in the population. With $b \in (x_0, \hat{b})$ these fluctuations may be undesirable but not fatal if they are inside the limit cycle. But if they are outside the limit cycle they end up at E_1 - a disaster.

If the situation is not too bad a collapse can be avoided by regulating the fishery, i.e. by controlling either a or b. The higher b is the smaller the attraction domain of E_1 is. Given $(x(0),y(0))$ we can determine b such that $(x(0),y(0))$ are in the interior of the limit cycle (in the second portrait) or under T_2 (in the third portrait), or we need $b > x_3$ to achieve one of those. Then a collapse is inevitable.

The main conclusion is that even from such a simple model it can be seen that nature is not always forgiving; there is a danger of depleting this type of resource.

Appendix A: Possible Values of R and Q

$g(x)$ can have either one or three roots. The number of roots is a function of R and Q. To find the domain in the R-Q plain where $g(x)$ has 3 roots we solve simultaneously:

$$g(x) = 0 ,$$

$$\frac{dg}{dx} = 0 .$$

Then we obtain $R = R(x)$; $Q = Q(x)$. This is a parametric representation of the curves on which $g(x)$ has one double roots and one simple roots. At $\tilde{x}$ where $\frac{d^2g}{dx^2}(\tilde{x}) = 0$ we have one triple root. These curves are the boundary which confines the domain where $g(x)$ has 3 roots.

The equation $g(x) = 0$ implies $R(1 - \frac{x}{Q}) - \frac{x}{1 + x^2} = 0$.

The equation $\frac{d}{dx} g = 0$ implies $-\frac{R}{Q} - \frac{1 - x^2}{(1 + x^2)^2} = 0$.

Hence

$$R(x) = \frac{2x^3}{(1 + x^2)^2} \quad ; \quad Q(x) = \frac{2x^3}{x^2 - 1} .$$

By the nature of the problem R and Q are positive. The following properties can be derived from these forms:

$$x \to 1^+ \quad \text{implies} \quad Q \to \infty \quad \text{and} \quad R \to \tfrac{1}{2}^+ ,$$

$$x \to +\infty \quad \text{implies} \quad Q \to \infty \quad \text{and} \quad R \to 0^+ .$$

To find $\tilde{x}$ we consider:

$$\frac{dR}{dx} = \frac{2x^2(3 - x^2)}{(1 + x^2)^3} ; \quad \frac{dQ}{dx} = \frac{2x^2(x^2 - 3)}{(x^2 - 1)^2} \Longrightarrow \frac{dR}{dx} = \frac{dQ}{dx} = 0 \quad \text{at} \quad \tilde{x} = \sqrt{3}$$

$$Q(\sqrt{3}) = 3\sqrt{3} ; \quad R(\sqrt{3}) = \frac{3\sqrt{3}}{8} ; \quad \frac{dR}{dQ}(\sqrt{3}) = \lim_{x \to \sqrt{3}} \frac{\frac{dR}{dx}}{\frac{dQ}{dx}} = -\frac{1}{16}$$

From these features we can draw the following plot:

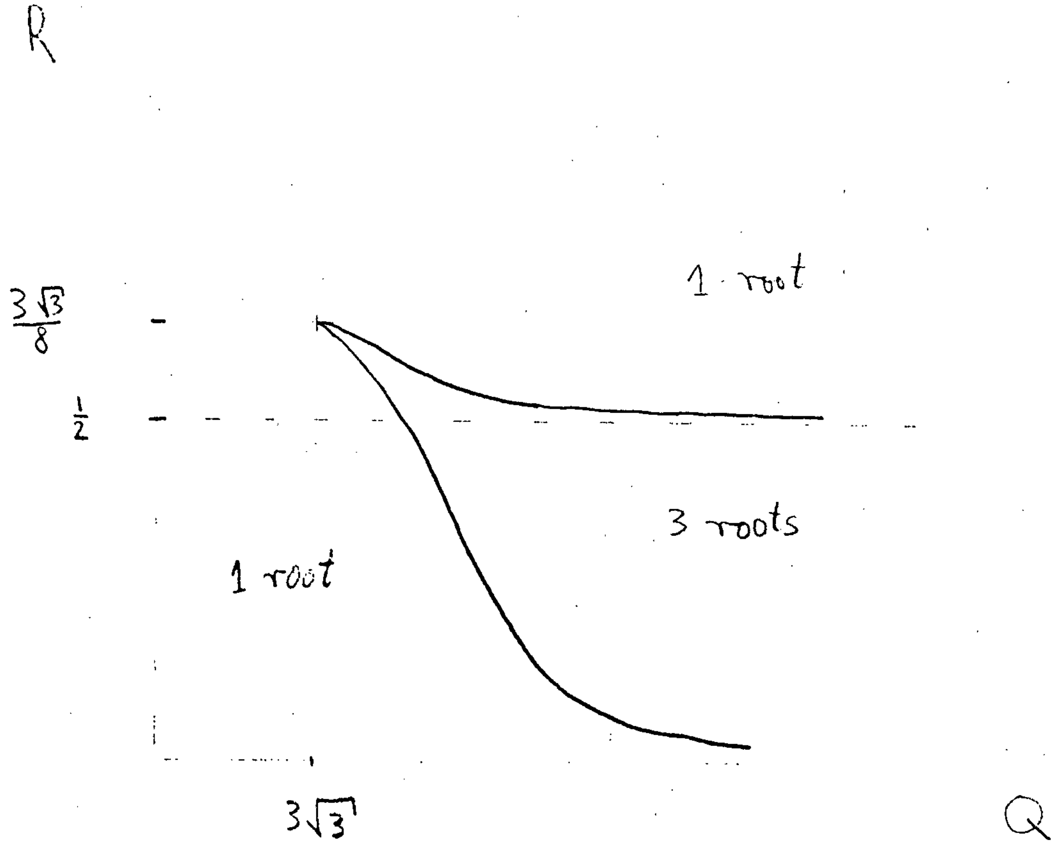

Figure 9. The number of roots of $g(x)$ as a function of various values for R and Q .

Appendix B: The Numerical Scheme

A numerical scheme was used in order to verify the results which utilized asymptotic techniques. The main part of the scheme was an integration routine which used a predictor-corrector formula of fourth order. The routine also measured a quantity proportional to the relative error (namely, $\left|\frac{\text{predicted value - corrected value}}{\text{corrected value}}\right|$), and changes the time step so that this quantity will be smaller than 10^{-3}. When the time step was decreased (or to start the solution) a fourth order Runge-Kutta method was used to generate the next three terms.

The system was integrated with initial (x,y) near E_2 or E_3, as the case required. Near E_i the slopes of T_i are $m_i = g'(x_i) - a\frac{x_i - b}{x_i}$ $(i = 2, 3)$. This observation was used to make the integration slightly faster. The initial y was 10^{-3} and the initial x was taken to be on the line which goes through E_i and has the slope m_i $(i = 2, 3)$.

To obtain T_2 the system (II.1.1) was integrated with time going backwards. This was done until the solution crossed the line $x = b$. The value of T_2 at $x = b$ was not necessarily given by the routine because the integration was done with respect to time (rather than integrating $\frac{dy}{dx} = \frac{ay(x - y)}{x\ g(x) - xy}$). Therefore, the routine took the value at the closest point to $x = b$, say at $x = x_B$. Incidentally, the value at x_B was either the last or the one next to the last value of T_2 to be computed.

The next step was integration of T_3. Now time went forwards and the routine stopped integrating when the solution crossed the line $x = x_B$. This time we were interested in the value of T_3 at $x = x_B$, and used a third order Lagrangian interpolation formula to obtain it.

At this point comparison could be made between T_2 and T_3. If y_i is the y-value of T_i at x_B, then $F = Y_2 - Y_3$ was the interesting quantity. As it was shown in the work, for any (R, Q, a) fixed we had $F = F(b)$. To find $\hat{b}$ we had to find the zero of F. This was done by the method of bisection until $|F| < 10^{-1}$. Then the method of false position was used until $|F| < 10^{-3}$. Obviously, each iteration involves an integration of T_2 and T_3.

The program was run for R values ranging from 0.014 to 0.5, Q values from 50 to 5000 and a from 10^{-5} to 10^{3}. For $P > 10$, the numerical solutions for $\hat{b}$ was not more than 5% away from the asymptotic "large P" approximation. For $P < 1$ the numerical solution was not more than 7% away from the asymptotic "small P" approximation. For P between 1 and 10 the results varied. Sometimes they were close to the "large P" approximation, sometimes to the "small P" approximation, sometimes to both and sometimes to none. Yet even in the latter case, the computed $\hat{b}$ was larger than x_0.

The formulas which were used in the scheme were based on Ralston [8].

	R	Q	$P = aQ^{3/2}$	computed $\hat{b}$	"small P" $\hat{b}$	"small P" relative error (%)	"large P" $\hat{b}$	"large P" relative error (%)
"small P" is good for P = 1.77	0.5	50	17.67	12.57	27.37	117	12.8	1.8
	0.5	50	1.77	11.06	11.6	4.8	12.8	15
	0.5	50	0.177	10.04	10.02	0.1	12.8	27
none is good for P = 1.77	0.3	50	17.67	15.87	37.08	133	16.25	2.3
	0.3	50	1.77	14.03	15.22	8.5	16.25	15
	0.3	50	0.117	13.045	13.035	0.08	16.25	15
both are good for P = 1.77	0.108	50	17.67	22.57	41.58	84	22.58	0.04
	0.108	50	1.77	22.35	23.48	5	22.58	1
	0.108	50	0.117	21.74	21.67	0.3	22.58	3.9
"large P" is good for P = 5.59	0.0389	500	55.90	155.80	2308.6	1381	155.99	0.1
	0.0389	500	5.59	150.78	332.91	120.8	155.99	3.5
	0.0389	500	0.56	131.21	135.35	3.2	155.99	18.9

Table: A sample of the computer results.

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