Two studies in hydrodynamic stability
Interfacial instabilities and applications of bounding theory

by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
Doctor of Philosophy

in

The Faculty of Graduate Studies
(Mathematics)

The University of British Columbia
July 2006
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Abstract

Interfacial phenomena, viz. roll waves and elastic tremor, are considered in the first part, whereas in the second bounding theory is applied to double-diffusive convection and shear flows.

Roll waves are investigated using shallow-water equations with bottom drag and diffusivity. We explore the effect of bottom topography on linear stability of turbulent flow, followed by an investigation of the nonlinear dynamics. Low-amplitude topography and hydraulic jumps are found to destabilize turbulent roll waves, while higher amplitude topography stabilizes them. The nonlinear dynamics of these waves is explored with numerical and asymptotic solutions of the shallow-water equations. We find that trains of roll waves undergo coarsening dynamics, however coarsening does not continue indefinitely but becomes interrupted at intermediate scales, creating patterns with preferred wavelengths. We quantify the coarsening dynamics in terms of linear stability of steady roll-wave trains.

For elastic tremor, e.g. observed in musical reed instruments and vocal cords, a first principles explanation is given for the onset of these oscillations using linear stability theory. An analytical solution is built on the assumptions of thin-film flow and stiff elastic material and the criterion for the destabilization of natural elastic oscillations is derived. Acoustic excitation (e.g. organ pipes) is treated as an analogue, with compressibility playing the role of elasticity, with similar mechanism possibly at work.

In double diffusive convection, the flux of the unstably stratified species is bounded using the background method in the presence of opposite stratification of the other species. In order to incorporate a dependence of the bound on the stably stratified component, Joseph's (Stability of fluid motion, 1976, Springer-Verlag) energy stability analysis is extended. At large Rayleigh number, the bound is found to behave like $\frac{1}{T^2}$ for fixed ratio $\frac{R_S}{R_T}$, where $R_T$ and $R_S$ are the Rayleigh numbers of the unstably and
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The energy stability of plane Couette flow is improved for two dimensional perturbations. The energy is chosen from a family of norms so as to maximize the critical Reynolds number. An explicit relation for the critical Reynolds number is found in terms of the perturbation direction.
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Acknowledgments

First and foremost, my gratitude and appreciation to Neil Balmforth. His energy and enthusiasm are truly inspirational. I have also learnt a lot from Alison Rust, who has been a great colleague and collaborator, and I appreciate it. I am grateful to Eli Tziperman for suggesting the investigation of roll waves to me and to Bill Young, Louis Howard, Richard Kerswell and Charles Doering for helpful discussions.

The Geophysical Fluid Dynamics Summer Program at the Woods Hole Oceanographic Institution has provided me with many great opportunities, for which I am indebted. Similarly, the Department of Mathematics, MIT, has been a very warm host for part of this degree program. I am also grateful to Anette Hosoi for her support and interaction at the Hastapoulos lab, MIT.

An honorary mention goes to my parents, who always supported my education, sometimes in the face of severe hardships. It gives me great joy to see their struggle come to fruition. Shilpa has been a very accommodating accomplice in the creation of this thesis; the thesis in the current form was not possible without her involvement. The interaction with the members of the Complex Fluids Lab was a pleasure that I cannot emphasize more.

Parts of this thesis were supported by an NSF grant (ATM0222109) and an NSERC discovery grant.
Co-authorship statement

This research and writing of this thesis was supervised by Prof. Neil J. Balmforth. The experiments on elastic oscillations in chapter 3 were designed and performed by Dr. Alison C. Rust, but more importantly she has performed thorough experiments that helped in setting the guidelines for the theory. In chapter 4, Shilpa A. Ghadge and Atichart Kettapun were involved at a grass-root level in the calculations of the upper bounds with me.
Chapter 1

Introduction

When wind blows, the sea relinquishes its tranquility to the formation of waves. The gentle blowing by a flautist generates serene music that soothes the heart. The rotation of gigantic galaxies makes their arms swirl. These and other observations have dumbfounded humans since the beginning of mankind. Man aspires to interpret nature's divine plan, and has invented arts and science for that purpose. While the methods may differ, arts and sciences share this common goal in their unending pursuit of knowledge and understanding.

It is hard to pin-point exactly when the scientific approach started. It is fair to say that analytical ability has developed into life gradually through the process of evolution. Nevertheless, the current school of scientific thought is attributed to the names of Pythagoras, Plato, Aristotle and Archimedes, though they cannot be the first to have demonstrated analytical ability; the wheel, for example, was invented much before them. Standing on the shoulders of these and other giants, the foundations of the modern approach were set down by Newton through his laws of motion and the invention of calculus. Newton provided very handy tools to describe the nature around us using mathematics, and scientists have been very busy since then crafting explanations of curious phenomena. From a scientific perspective, the phenomena mentioned above, and countless others, are examples of hydrodynamic instabilities.

1.1. Hydrodynamic stability

The laws of physics that govern fluid flow, the Navier-Stokes equations, have been known since 1822 [83, 108]. The fluid velocity field in physical flows is a solution of the Navier-Stokes equations, or a close approximation thereof. Hence, in a predictive approach, solutions to these governing equations are sought. No general solution is known, but
some can be found in simple circumstances. When the forces that drive the flow are steady, the simplest solutions to look for are steady. However, the governing equations are nonlinear and sometimes have multiple solutions. It is then not clear a priori which of the solutions the flow will evolve towards. Moreover, the flow may not evolve towards any of the steady states at all, positing the existence of time-dependent solutions. Thus, the conditions under which the solution of Navier-Stokes equations can be realized need to be carefully demarcated.

An important criterion for a known solution to be physically realized is that it be stable, i.e. small disturbances to the flow, which are always present, do not disrupt the flow. Unfortunately, not all solutions to the equations are stable. Perturbations about some of the solutions may be sufficient to disrupt them. Such a disruption is termed an instability. The part of fluid dynamics that concerns itself with distinguishing stable flows from unstable ones is called hydrodynamic stability theory.

A canonical example in this field is thermal convection, in which a layer of fluid between two infinite parallel plates is heated from the bottom and cooled from the top. A simple family of solutions for this configuration is a static fluid with a linear temperature profile and heat is transported from the hot plate to the cooler one by pure conduction. This solution exists irrespective of the temperature difference, the distance between the plates or properties of the fluid. However, not every member of this family of solutions is stable. If, for the same fluid and geometry, the temperature difference exceeds a critical value, this static solution is unstable to buoyancy driven flow. In this case, hydrodynamic stability theory furnishes the criterion on the problem parameters for which the members of the static-solution family are stable. This is the typical situation in most problems, where the stability properties of a family of solutions are quantified through a flow parameter. The identification of the characteristic parameter that governs the stability and its critical value are the objects of interest in this theory.

A widely used technique in hydrodynamic stability theory is called linear stability analysis. In this analysis, the evolution equations for perturbations about a steady solution are derived. The Navier-Stokes equations and the perturbation evolution equations are nonlinear, which makes it difficult to obtain solutions. The nonlinearity is neglected in this analysis based on the assumption that the perturbations are infinitesimally small. This makes the evolution equations linear, which are much easier to solve.
by superposition on the modes of the evolution operator. If all the modes of the linear operator eventually decay, so will any small perturbation and the flow will return to the known steady solution. But stability is lost when even a single mode keeps growing with time. Thus the analysis boils down to determining whether any modes of the evolution operator show persistent growth.

Linearly unstable solutions cannot be realised. Linear theory predicts that flows diverge exponentially from the unstable solutions. But as the difference becomes finite, nonlinear effects can no longer be neglected. Nonlinearities usually cause a saturation of the exponential divergence and the flow evolves to a new steady, periodic or aperiodic flow. Typically, the resulting flow is spatially and/or temporally more complicated than the unstable solution it evolved from. Nonlinear techniques have to be used to describe the resulting flows. These techniques usually involve computational exercises, but considerable insight can be gained by analytical simplification before resorting to numerical methods. One such simplification is provided by a weakly nonlinear analysis, which goes one step beyond linear stability analysis. In this analysis, the nonlinear terms are neglected by assuming the perturbations to be small but only to the leading order in an asymptotic expansion. Nonlinearities make their appearance at higher orders and may cause saturation of the exponential growth predicted by linear analysis for unstable flows. This analysis is only valid near the linear stability boundary and when the new flow is not very different from the unstable solution. Nevertheless, it addresses the question of what happens to an unstable flow.

If the new flow is not too complex, its stability may in turn be studied. Typically the new flow may become unstable at yet higher values of the characteristic parameter and further increase in complexity. This sequence of instabilities may repeat many times, as is observed in the case of thermal convection, until the flow becomes hopelessly intractable. This description is adopted by fluid dynamists to explore the parameter space to differentiate between the possibility of different flows.

What is even more common amongst theoreticians is the neglect of certain degrees of freedom from the full Navier-Stokes equations. For example, solutions are often found assuming that the flow is one-dimensional, neglecting the other two directions. That simplifies the analysis considerably and simple solutions are more conveniently found. For these solutions to be observed as real flows, they must be stable to three-dimensional
perturbations. Thus, all degrees of freedom must be considered when assessing stability of a particular flow.

The linear and weakly nonlinear analyses previously described are restricted to flows with small perturbations about the equilibrium flow. It is possible that certain flows are stable to all small perturbations, but a finite perturbation can destabilize them. Plane Couette flow is the flow of a fluid sheared between two infinite parallel plates and provides a notorious example of such a “finite-amplitude” instability [111]. Very few analytical techniques are known to locate the stability boundaries and consequence of such instabilities. Energy stability analysis is one such technique, which, unlike linear analysis, does not ignore nonlinear effects. It works by constructing equations for the evolution of certain positive definite functionals of the flow perturbations, called “energies”, and identifying flow parameters for which these functionals decay monotonically to zero. Decay of these functionals signifies an equilibrium flow stable to perturbations of arbitrary shape and size.

Unlike linear stability theory, energy stability cannot predict instability. While it can prove that below the energy stability threshold of the characteristic parameter all perturbations must decay, it is silent about the stability of the flow above the threshold. Above this threshold, it does predict that the energy grows instantaneously for certain perturbations, but the long term prospects of this growth are left unclear. In fact, it may be possible to find another energy functional that decays monotonically, when the former energy is inconclusive. Thus, the monotonic decay of even a single energy functional is a sufficient condition for stability. The improvement of energy stability theory amounts to the search for monotonically decaying energy functionals that predicts as large a region of parameter space to be stable as possible.

In this thesis, we have applied this theoretical framework to some simple flow situations. The dissertation is divided into two parts. The first part deals with some interfacial instabilities, i.e. instabilities occurring because of the presence of an interface between two materials. Two kinds of interfacial instabilities are studied, viz. the roll wave instability for turbulent flows in chapter 2 and flow induced elastic oscillations in chapter 3. The second part presents some work on energy stability theory and its extensions. It also has two chapters, chapter 4 on bounding double diffusive convection and chapter 5 on the energy stability of Couette flow. The final chapter concludes the
Figure 1.1: Spillway from Llyn Brianne Dam, Wales [115]. For an idea of the scale, the width of the spillway is about 75 feet.
thesis by describing the salient results and suggesting future directions. The remainder of this chapter provides background and context for the problems addressed in this thesis.

1.2 Interfacial instabilities

1.2.1 Roll waves

The interfacial instabilities presented in this dissertation are, in one form or another, manifestations or modifications of shallow water waves known as roll waves. Roll waves are waves appearing on thin films of water down inclined slopes. These are the same waves appearing on sloping streets on rainy days. Their appearance can be justified in the hydrodynamic stability framework as an instability of the uniform flow of thin films to wavy perturbations. Three elements are involved in the formation of these waves. The first is a driving force, like downslope gravity, that provides the energy for the flow itself and for the formation of waves too. The second ingredient is a resisting force, like the friction with the inclined plane that opposes the acceleration that the driving force provides and brings the flow to a steady equilibrium. The third is a surface restoring force, like the gravitational force normal to the plane, which tries to flatten the surface of the thin film. The formation of these waves is observed at many different scales where the three elements are provided by different physical mechanisms. For sub-millimeter thin films of liquids with strong interfacial tension, surface tension may provide the dominant restoring force. On the millimeter scale, for example for the waves on windows observed on rainy days, surface tension and gravity forces may be equally important. As the thickness increases, gravity takes over the role of providing restoring mechanism, while the contribution from surface tension subsides. If the thickness of the film increases further, the flow becomes turbulent. These waves are also seen on turbulent flows up to a thickness of several feet, first reported by Cornish [32]. A picture showing these waves on a dam spillway is shown in figure 1.1. Moreover, such waves are also observed on flows of granular material down inclines [45]. The phenomenology is quite robust and is seen on all length scales.

Experiments with laminar films are much simpler to perform due to shorter length scales and availability of sophisticated methods to measure film thickness [1]. An almost
uniform laminar flow starts at the inlet. But as it proceeds downstream, wave features start to appear [63]. Small variations in the inlet flow rate grow into larger waves as they are carried by the flow. This is an instance of a *convective* instability [2, 75], where the growth of perturbations occurs as they are carried with the flow. If one focuses on a particular location on the incline, the amplitude of the perturbations is a constant there. This is in contrast to *absolute* instabilities, where the size of the perturbations at a given location grows with time. As these convectively unstable waves grow, they generally steepen and form sharp fronts, called bores or hydraulic jumps, that propagate downstream. The distance between these hydraulic jumps is quite small in the beginning [2]. But further downstream, the wavelength increases. Casual observation of the laminar waves appearing on a rainy street reveals similar behaviour. This increase in the wavelength is attributed to the different speeds with which the bores travel, causing them to collide with their neighbours and merge with them. Such a decrease in the number of waves, and an increase in the wavelength, by merging is called *coarsening*. On the other hand, a wave-splitting process that causes an increase in the wavelength by spawning new waves have also been reported. The apparently random behaviour of the waves can be partly rationalized in terms of these coarsening and spawning mechanisms [74]. For faster flows, another possibility is that the waves develop three-dimensional herringbone patterns still maintaining their periodicity [76]. As the flow speed increases further, the flow becomes turbulent and statistically uniform again. This uniform flow becomes unstable to turbulent roll waves at even higher flow speeds.

Modeling of these waves started with the linear stability by Yih [120] and Benjamin [6] and weakly nonlinear analyses by Benney [10] of uniform films using the Navier-Stokes equations. These analysis identified the Reynolds number, which compares the inertial forces in the flow with the viscous ones, as the characteristic parameter for instability. Difficulties related to the divergence of solution from Benney’s weakly nonlinear analysis [95] led Shkadov [105] to alternative simplifications of the problem. By assuming a parabolic velocity profile across the film thickness *ad hoc*, Shkadov derived one-dimensional evolution equations for film thickness and average velocity. This set of equation does predict a instability of the uniform film but at a slightly different value of the Reynolds number as compared to the linear stability results from Navier Stokes equations themselves. Unlike the weakly nonlinear analysis, the solution of this set
of equations remains bounded, even for large Reynolds number. Chang, Demekhin & Kopelevich [25] reduced this set of equations to a Kuramoto-Sivashinsky equation and have also demonstrated some subharmonic and sideband instabilities related to coarsening dynamics. Efforts were mostly concentrated in this direction [24, 25, 94], despite the empirical nature of this model. Rigorous mathematical justification was later provided by Ruyer-Quil & Manneville [101, 102, 103], who used a more general polynomial velocity profile and modified some coefficients in the equations for agreement with the long-wave linear stability criterion.

As for turbulent roll waves, experiments are scarce owing mainly to the long channel length required to generate them. Although observations were available as early as 1904 [31], careful experimenting and documentation of results were missing until as late as 1965. The experiment undertaken by Brock [18, 19] remains the only experiment undertaken for the study of turbulent roll waves. Similar to the laminar case, the turbulent flow also starts quite uniformly at the inlet, but a convective instability causes the small variations in flow rate to grow and form waves. As these turbulent waves grow, they develop turbulent hydraulic jumps, which show coarsening. The picture in figure 1.1 also shows an increased distance between waves downstream than upstream and even some three-dimensional instabilities.

Modeling of turbulent films began with Jeffreys [58] using the St. Venant equations. These equations are similar to the ones used by Shkadov. But no mathematically rigorous justification can be provided for them starting from the Navier-Stokes equations, like the one provided by Ruyer-Quil & Manneville [102] for the Shkadov equations. Nevertheless, Jeffreys succeeded in describing the linear instability and that justified their validity in the absence of a better alternative. His criterion was based on the Froude number, that measures the strength of inertial forces to gravitational ones. A family of steadily propagating, periodic nonlinear wave solutions of the St. Venant equations were constructed by Dressier [39] and Needham & Merkin [85]. But which of these profiles does the solution growing from random noise evolve into was left unanswered. Kranenburg [69] and Yu & Kevorkian [121] were the first to describe the evolution of these waves from small perturbations, and they discovered the coarsening dynamics via subharmonic instabilities. Chang, Demekhin & Kalsidin [24] even demonstrated the self-similar coarsening behaviour to longer and longer wavelengths.
In nature, flow of water is rarely over perfectly flat inclines; there is invariably some (possibly three-dimensional) variation of the bedform through sediment erosion. This variation of bedform can significantly change the nature of the roll-wave instability. On the other hand, in artificial water courses, a bottom topography is sometimes deliberately constructed, so as to suppress the formation of these waves and lead to a more predictable and controllable flow [81, 100]. The cause behind this suppression has not yet been investigated. In chapter 2, we fill in this gap by modifying the St. Venant equations to include a periodically varying bottom topography and present its effects on the linear dynamics.

On the nonlinear dynamics front, a simplified model derived from the St. Venant equations for small-amplitude bottom topography via a weakly nonlinear analysis, with multiple temporal and spatial scales, was analyzed. The focus of this analysis is the mechanism that selects the wavelength of these waves. Numerical solution of the weakly-nonlinear model, along with stability analysis of a periodic wavetrain to small perturbations (including subharmonic ones), is used to paint a coherent picture from the initial growth of small perturbations to form the bores, to the coarsening dynamics, and to the ultimate saturation to a stable wavelength.

The stable wavelengths that emerge from the theory can be easily tested against experiments. However, other than the experiment by Brock, no significant experiments are reported in the literature. Even in Brock's experiment, the channel was not long enough for a stable wavelength to emerge. In fact, any experiments that will attempt to determine the stable wavelengths evolving from naturally occurring random perturbations will be faced with the problem of constructing an infeasibly long channel. Instead, periodic waves can be forced at the inlet of the channel and their evolution can be followed downstream to determine whether the wavelength is stable or not, much in the same spirit of the theoretical analysis. This was exactly the plan that we undertook; the results from experimental observations of coarsening and wavelength saturation were derived and compared with theory.

1.2.2 Elastic oscillations

Instabilities similar to roll waves exist when the restoring force provided by gravity is replaced by an elastic boundary. This replacement provides us with the opportunity to
explore another important class of problems, commonly classified under fluid-structure interaction, a study of fluid flow in the vicinity of flexible bodies.

The interest in fluid-structure interaction originally stemmed from the aviation community. Understanding the response of the flexibility of aircraft structures, and the feedback provided by the resulting deformations, to the aerodynamics was of crucial importance to the progress of powered flight, fueled especially by the two World Wars [16].

An independent study of fluid-structure interaction resulted from Gray’s paradox [48]. A calculation by Sir James Gray on the energetics of dolphin swimming ignited interest in the search of “soft” surfaces, which can help in either maintaining a laminar boundary layer or reducing the turbulent drag on the surface. Whilst the resolution of Gray’s paradox involved better knowledge of the dolphin’s muscles, it attracted attention to the nevertheless useful question of whether compliance of surfaces can modify the hydrodynamic drag on them. Indeed, experiments by Kramer [66, 67, 68] provided further reinforcement for these possibilities. However, subsequent experiments failed to reproduce Kramer’s results unanimously. An explanation has emerged only recently [23, 47, 119], that although the usual transition to turbulence is delayed by the compliant surface, the elasticity provides mechanisms for the existence of other instabilities and pathways to turbulence.

Benjamin classified the possible fluid-structure instabilities into three classes [7, 46]. The first kind is that of instabilities of hydrodynamic modes, which exist in the presence of rigid structures, modified by the flexibility of the structure. An example of this kind, termed as class A, is the modification of Tollmien-Schlichting waves present in the case of the flow over an semi-infinite rigid walls, by making the wall compliant. The fluid dissipation stabilizes this kind of instability, whereas any dissipation in the elastic wall further destabilizes it. The second kind of instabilities are the elastic modes and waves, which are destabilized by the fluid flow. An example of this in the context of flow over a flat plate is travelling wave flutter, which is stabilized by dissipation in the elastic body but destabilized by non-conservative interactions in the fluid. This kind is named class B. The final kind is known as divergence or class C and its mechanism is akin to Kelvin-Helmholtz instability, albeit in the presence of an elastic body instead of a fluid on one side of the interface. Its mechanism constitutes a unidirectional (i.e. without
oscillations) flow of energy from the flow to the solid. Dissipation in either fluid or solid cannot stabilize this kind of instability. The explanation that emerged for flow over a flat plate was that, although the Tollmien-Schlichting waves are suppressed because of the fluid-structure interaction, in order to reduce drag, attention must be paid to also prevent the other two kinds. Although exemplified using flow over a flat plate, this threefold classification is quite general and serves well in the identification and understanding of instabilities in other flow situations as well. For instance, in aeroelasticity, structures are designed to avoid flutter and divergence instabilities.

Fluid-based instabilities have a critical Reynolds number, which has to be exceeded for the instability to set in. Typically, for common shear flows under consideration, this critical Reynolds number is found to be in the thousands. One the other hand, in the inviscid limit, the criteria for elastic-based instabilities to set in is that the characteristic fluid speed has to exceed a certain multiple of the elastic wave speed, i.e. the critical parameter is an elastic Mach number. For many engineering applications, this criterion corresponds to a Reynolds number of the order of thousands as well or higher. However, there are also well-known examples of flow-induced elastic oscillations at low Reynolds numbers. At zero Reynolds number, a class of instabilities is known [40, 70, 71, 72, 110], in which viscous stresses destabilize the elastic body into an oscillatory instability. Flows at small but non-zero Reynolds numbers in collapsible conduits, are also susceptible to elastic analogues of roll waves, destabilized by fluid inertia [5, 93].

All oscillatory instabilities mentioned earlier are convective, but there are absolute oscillatory instabilities as well. Finite collapsible conduits [14, 15, 59, 60] are subject to such absolute oscillatory instabilities and are well investigated experimentally. The Reynolds number is considered the critical parameter for this instability, rather than the elastic Mach number. These are, however, quite different from shear instabilities and more akin to flutter instabilities. Quite elaborate analytical studies of such systems, motivated by applications like musical reed instruments, speech generation and other physiological systems, using lumped parameter models (see figure 1.2) have also been carried out [13, 42, 44, 56]. While one and two-dimensional computational models [49] of some of these systems are carried out, a complete understanding from a fundamental fluid dynamical point of view is yet to be developed.

A close relative of the flow induced elastic oscillations is the excitation of an acoustic
Figure 1.2: Schematic setup for the lumped parameter models used to describe elastic oscillations in vocal cords. The vocal folds are made up of elastic tissue and are forced by air at the glottal pressure \((p_g)\) flowing between them at speed \(u_g\). In the lumped parameter model, the vocal folds are replaced by a mass-spring-dashpot system forced hydrodynamically. The glottal pressure is related to the glottal velocity through Bernoulli's principle, \(p_g + \rho u_g^2/2 = p_0=\) stagnation pressure. The glottal velocity is given by mass conservation, \(u_g x + \dot{x} L/2 = q\), \(q\) being the constant inlet volume flux. The displacement of the masses about its unforced position \(x_0\) is governed by \(m \ddot{x} + b \dot{x} + k(x-x_0) = p_g L d\). For a given \(q\), an equilibrium separation between the masses, \(H\), can be found by solving \(k(H-x_0) = Ld(p_0 - \rho q^2/2H^2)\). Linearizing about this equilibrium, the displacement perturbation \(y = x - H\) satisfies, \(m \ddot{y} + b \dot{y} + k y = \rho q^2 L dy/H^3 + \rho q L^2 \dot{d} y/2H^2\). Re-arranging slightly, the equation reads \(m \ddot{y} + (b - \rho q L^2 d/2H^2) \dot{y} + (k - \rho q^2 L d/3H) y = 0\). Divergence, or a class C, instability corresponds to the case when the restoring force provided by the spring is inadequate in face of its hydrodynamic counterpart; i.e. when \(\rho q^2 L d/H^3 > k\). In this case a steady instability ensues, which re-adjusts the equilibrium displacement. Flutter, or a class B, instability corresponds to the case when \(\rho q L^2 d/2H^2 > b\) while \(k > \rho q^2 L d/H^3\). This leads to oscillations responsible for phonation. This simple model captures the essential mechanism for flutter and divergence. It does not have the effect of fluid viscosity, nor is it capable of demonstrating the analogues of class A instabilities.
cavity, known as a Helmholtz resonator [117]. A classic example of a Helmholtz resonator is an air column (for example, a beverage bottle or an organ pipe) in which acoustic modes are set by blowing a narrow jet near one of its open end. This mechanism is commonly employed in wind instruments like flutes and recorders. In principle, the mechanism of this excitation involves setting up oscillations in compressible gas by the action of a fluid flow. The acoustic oscillations can be considered to be the analogues of elastic modes, where the role of elasticity is played by compressibility. Again, plenty of lumped parameter models, mostly originating from analogy with electrical circuits, have been put forward and analyzed [42, 44, 106], but a fundamental understanding from a fluid dynamical point of view is still under development.

Chapter 3 is devoted to filling the gaps in the lumped-parameter analysis from a fluid dynamical point of view by starting from first principles. We present an inertia-driven hydrodynamic linear destabilization of elastic modes, which is the analogue of the oscillatory instability seen in the lumped-parameter models. Unlike traveling wave flutter, the criterion for this instability is that a critical Reynolds number be exceeded, irrespective of the elastic wave speed. Dissipation in both the elastic solid and the fluid stabilize this instability. The geometry we have chosen corresponds to a flow through a narrow channel made in an elastic body. Although this flow configuration has been studied in the past, all those analyses particularly looked for convective instabilities in channels of infinite length. Such analyses are not particularly applicable to many physical systems like the vocal folds and musical instruments, where the channel length is finite and the instability seems to be absolute. By having a channel of finite length, we are forced to impose physically motivated boundary conditions at the inlet and exit of the channel. It is precisely these boundary conditions that are found to be the cause of the instability.

The conceptual relation between the absolute flow-induced elastic instability and the acoustic excitation of a cavity suggests a similar mechanism at work in the acoustic case. The mechanism suggested in the literature for the acoustic oscillations involves shear instability of the jet to sinuous perturbations [30, 33, 41, 106]. These perturbations are fed by the modes in the cavity and they in turn force the modes by periodically blowing in and out of the cavity. When the feedback is strong enough and with the right phase difference with the modes in the cavity, the oscillations become stronger. However, a
simply devised experiment demonstrates (see chapter 3, §3.6) that there is more to the
story and acoustic excitation mechanism needs further scrutiny. It is natural to exploit
the analogy with the elastic instability and modify the model to propose a mechanism
for the acoustic excitations.

1.3 Energy stability and its extensions

1.3.1 Bounds on double diffusive convection

In chapter 4, we have found bounds on double diffusive convection using the background
method. Double diffusive convection is convection in the presence of two species that
diffuse at different rates and affect the buoyancy force via the density diffusing at
different rates. Applications include phenomena in the ocean [57, 80, 84, 104], where
the two species are heat and salt, and in stars, where salt is replaced by a heavier
element like helium [107, 113, 114]. The bounding technique used is an extension of the
energy stability theory. In this section, we present the relation between the two and the
motivation for deriving the bound.

The rate of change of the energy functional in energy stability theory is typically
divided into terms responsible for generation and others causing a dissipation of energy.
One of these two kinds, say the generation terms, are proportional to the characteristic
parameter of the system. Energy stability analysis can be construed in a functional anal­
ysis framework as the search for the values of this parameter, for which the dissipation
always dominates the generation, irrespective of the shape and size of the perturbation.
Beyond the energy stability boundary, a similar functional analysis procedure can be
devised to provide some information about the possible flows. In particular, the time
average of statistically steady flows satisfy the same power balance in a mean sense,
which the neutrally energy-stable state satisfies. The difference between these two per­
spectives is that in the latter case, instead of restricting the value of the characteristic
parameter, this power balance can be thought of as constraining the shape and size of
the perturbations. This concept gained momentum and was put to use starting with
the intuition of Malkus [78].

Malkus made two hypotheses concerning thermal convection, motivated by observa-
tions. His first hypothesis was that, when multiple solutions to the governing equations
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are possible, the one physically realized is the one that maximizes the average heat transport. The second hypothesis was even stronger. It postulated that not only is the heat transport maximal amongst all the solutions of the governing equations, but also over a wider class of functions; viz. functions that satisfy the integral power balance equations. While there are no bases to these postulates, they motivated the only questions about turbulent flows that have been answered rigorously, namely whether the heat flux is bounded. The maximum found via Malkus's second hypothesis serves as an upper bound for true heat flux. Optimistically, the bound is hoped to capture the scaling law of the heat flux dependence on the thermal forcing.

This concept was made rigorous by Howard [54], who viewed the solutions to the governing equations as the limit of a sequence in which more and more integral constraints are imposed. Thus, according to him, the bound obtained via the power integrals can be improved and made to approach the true maximum over the solutions of the governing equations by incorporating more and more constraints. In his attempt, he showed that the power integrals themselves are sufficient to bound the heat flux and derived the first rough bound on the heat flux in thermal convection. To better the bound, he used test functions with a sinusoidal variation in the horizontal directions, thus leading to a single-wavenumber theory. It was possible that the true maximum was not reached by any of the test functions used. Thus, this theory could underestimate the upper bound and hence was flawed. This flaw was removed by Busse [20, 21] in his multi-wavenumber theory, in which he showed that more and more wavenumbers need to be included in the analysis to approach the bound as the convection becomes stronger. These concepts were also applied to various shear flows to bound the viscous dissipation.

Interest was revived again in the nineties, when Doering & Constantin [36, 37, 38] devised an alternate extension of the energy stability theory that used test functions and derived true upper bounds on those quantities with relative ease. Their method was called the “background method”, as it incorporated test functions called the background functions. This method was understood, improved and interpreted by the works of Nicodemus, Grossmann & Holthaus [86, 87, 88, 89], Kerswell [64] and Ierley & Worthing [55]. It turned out that the background method was the optimal dual of the variational formulation posed by Howard and thus corresponded to the same variational problem. This method has been used in many different contexts to quantify transport, mixing
One of the ongoing debates about convection in the scientific community is about the limit in which the diffusivities vanish. This is a singular limit of the governing equations and is directly relevant to many physical applications of convection. The dimensionless characteristic parameter for the context of convection is the Rayleigh number ($R_T$), which quantifies the strength of the buoyancy force as compared to viscous and diffusive effects. The resulting heat flux is quantified through the Nusselt number, which measures the enhancement in heat transport due to convection compared to the purely conductive value. The Nusselt number is supposed to behave like a power law, $R_T^\gamma$, as $R_T \to \infty$, where the precise value of $\gamma$ is the matter of the debate.

Experimentally, the value of $\gamma$ was measured to be about $1/3$ for $R_T \approx 10^8-10^9$. This exponent can be derived from a simple scaling analysis, if it is assumed that the small values of diffusivities do play a role in determining the heat flux. However, Kraichnan [65] postulated that the exponent could as well be equal to $1/2$, which can be interpreted as a consequence of the diffusivities not playing a dominant role in limiting the heat flux. The bounding theories predict an exponent of $1/2$ for the bound, and thus do not exclude this possibility. In fact, as more and more experimental evidence is coming to light, it is believed that the limit $R_T \to \infty$ has not been reached by the experiments with $R_T \approx 10^8-10^9$. Above these Rayleigh numbers, the exponent shows an upward trend, possibly towards $\gamma = 1/2$.

Fluxes measured in laboratory for double-diffusive convection also scale like $R_T^{1/3}$, where $R_T$ is interpreted as the Rayleigh number corresponding to the destabilizing species. Again the question of whether this is the ultimate regime, or does the exponent increase to $1/2$ if $R_T$ is increased further is left unanswered. There is a possibility that because of the added degrees of freedom (i.e. the second diffusing species), this scaling law is different from the one in single-species convection. If the bounding theories predict an exponent less than $1/2$, it can help us in eliminating that possibility. Bounding theories for double diffusive convection were attempted previously, but they were flawed in the methodology employed, in that they followed Howard in restricting their function space to allow only one horizontal wavenumber [109]. Moreover, some attempts at thermohaline convection did not look for the largest heat flux possible, but instead sought stationary fields that simultaneously maximized the salt flux as well [73].
remedy these fallacies in the previous attempts and find a true upper bound.

More interest in obtaining the bounds is derived from the perspective of predicting finite-amplitude perturbations. Double diffusive convection is prone to these transitions [116], where the linearly-stable static state can become nonlinearly unstable through the appearance of new solutions. These new solutions do enhance the transport, but contrary to Malkus's postulate, the solution need not evolve to them. The true maximum over all possible solutions of the governing equations, however, has to jump from the purely conductive value to the transport given by these new solutions.

If one is now to adopt Howard's vision of sequentially constraining the allowed function space using more and more constraints derived from the governing equations, a sufficient number of these constraints may be able to produce a jump in the bound, corresponding to the appearance of new nonlinear solutions. This can be indicative of finite-amplitude instabilities. Thus the bounding exercise can also be made to yield information about the bifurcations in a particular system. Double diffusive convection has a simple enough bounding theory, while still bearing rich dynamics, that it provides a suitable ground for testing these ideas.

### 1.3.2 Energy stability of Couette flow

Energy analysis can be used to show that for thermal convection, the nonlinear stability boundary corresponds to the onset of linear instability. However that is not true for many other flows, especially some shear flows. The characteristic parameter to investigate is the Reynolds number, which measures the strength of inertial effects compared with viscous ones. For example, plane Poiseuille flow is linearly stable for Reynolds numbers below 5772 but experiments and numerical computations show that the solution is disrupted for Reynolds number of about 1000. Similarly, plane Couette flow and pipe Poiseuille flow are linearly stable for all Reynolds numbers, but they too show a transition to turbulence at Reynolds numbers of about 400 and 2000 respectively.

In the context of plane Couette flow, the process of this transition to turbulence has been a subject of thorough experimental investigation [3, 29, 34, 35, 79, 98]. All perturbations are seen to decay monotonically below a Reynolds number of about 310, while above it certain "turbulent spots" appear to grow. Just above this Reynolds number, the spots eventually decay, but they seem to survive for a longer duration
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as the Reynolds number is increased. This survival time increases to infinity at a Reynolds number of about 370, above which a turbulent spot can persist on its own. The school of thought, starting with Lord Kelvin [111], that instabilities caused by finite amplitude perturbations are responsible for this transition has gained widespread acceptance; however the precise mechanism of this instability is still at large.

To explain this transition to turbulence, energy stability analysis was first crudely attempted by Reynolds [99] for plane Couette flow. This technique was later refined by Orr [90], who showed that perturbations independent of the spanwise direction, irrespective of their size, must decay if the Reynolds number is below 177.22. Orr’s perturbations were independent of the spanwise direction. Joseph [61] then extended this analysis to arbitrary three-dimensional perturbations and found that energy stability is lost at a lower Reynolds number of 82.65 for perturbations independent of the streamwise direction. A similar situation is posed for plane Poiseuille flow where the critical Reynolds number for energy stability is 99.21 [20], again for perturbations independent of the streamwise direction. For pipe Poiseuille flow, the energy stability critical value is 81.49 [62]. Thus energy stability technique grossly underestimates the stability boundary for shear flows, whereas linear stability analysis overestimates it.

Several approaches are taken by investigators towards determining the finite-amplitude instability mechanism. One approach is to find nearest non-trivial steady (or steadily propagating) states of the governing equations. Determination of such solutions is an exclusively numerical exercise. It started with Nagata [82], who used a continuation technique starting from the known solutions of Taylor-Couette flow. Continuation was then carried from thermal convection [28] and plane Poiseuille flow [27] solutions to provide other solutions. This approach asserts the possibility that the domain of attraction of the simple shear flow is bounded and provides upper bounds on the size of perturbations required to cause transition. The way a critical perturbation size can be determined this way is still unclear.

Another direction followed is that of a secondary instability of the time-dependent solution of the equations linearized about the shear flow [8, 9, 12, 91, 92]. Since the shear flow is linearly stable, the solution decays to the equilibrium. From a random perturbation to the flow, the slowest decaying mode survives for the longest time. By Squire’s theorem, this slowest decaying mode is two-dimensional; it is independent of
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the spanwise direction in case of planar flows. In the process, flow is modified enough to introduce an inflection point in the shear profile, leading to linear instability in the inviscid limit as shown by Rayleigh's theory. These secondary instabilities are postulated to give rise to transition to turbulence.

The phenomenon of transient growth is the most popular proposition towards an explanation of the finite-amplitude instability. In the linear limit, although all small perturbations eventually decay, their energy may transiently grow by several orders of magnitude before decaying. Initially this growth was attributed to repeated eigenvalues [11, 50, 52] but it was later realised that this condition is not necessary. The cause of this growth can be found in the fact that the advection operator is not self-adjoint, due to which the linear eigenvectors are not orthogonal to each other. In fact they are very much aligned with each other [17, 22, 43, 51, 53, 96] at large Reynolds numbers, and consequently the transient growth is proportionally large. It is believed that transient growth may cause perturbations to grow enough for nonlinearities to start playing a significant role and cause a transition [4, 112, 118].

The way energy stability is applied to the case of convection, provides some insight into understanding the role of transient growth. Any positive definite quantity qualifies as an energy for this analysis. For thermal convection, a family of these functionals is defined and the criteria for each one of them to decay monotonically is found. The union of all these criteria in the parameter space gives the region that is energy stable. A poorly chosen energy may grow transiently well inside the energy stability region, however that growth must be followed by an eventual decay to zero.

Double-diffusive convection is more akin to shear flows in the manner that linear instabilities can be subcritical and finite amplitude instabilities are possible. For double-diffusive convection a larger family of energies, as compared to thermal convection, need to be analysed to find the energy stability region (see chapter 4, §4.3). In this case it is also found that a transient growth of the energy causes an underestimate of the energy stability boundary. The cause of this transient growth can be traced back to the fact that the linear operator for double-diffusive convection is not self-adjoint either. However, by suitably modifying the energy, a better energy stability criterion can be deduced. This better energy can be interpreted as being defined as the norm of the solution vector in a transformed coordinate system, where the eigenvectors are less aligned with each other.
In chapter 5, we define a similar family of energies for plane Couette flow, with the goal of improving energy stability theory. In the process of extending the usual energy (see the book by Joseph [61]) to a family of energies, the analysis is restricted to perturbations independent of an arbitrary direction, i.e. oblique modes. These oblique modes are believed to play a central role in some of the proposed theories of transition to turbulence via transient growth [26, 77, 97]. The extension of energy stability lets us determine the fate of each oblique mode, if the variations are restricted only to the two dimensions describing the mode. This improved energy stability result can be elegantly derived entirely in terms of already known two-dimensional results. The energy stability criteria for these oblique modes can be improved in this fashion and a restricted parameter space needs to be examined for transition. Results from an analysis on similar lines carried out for plane Poiseuille flow are also presented at the end.

1.4 References


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Part I

Interfacial instabilities
Chapter 2

Dynamics of roll waves

2.1 Introduction

Roll waves are large-amplitude shock-like disturbances that develop on turbulent water flows. Detailed observations of these waves were first presented by Cornish [131], although earlier sightings have been reported and their renditions may even appear in old artistic prints [146]. Roll waves are common occurrences in man-made conduits such as aquaducts and spillways, and have been reproduced in laboratory flumes [126]. The inception of these waves signifies that variations in flow and water depth can become substantial, both of which contribute to practical difficulties for hydraulic engineers [146, 151]. Although most often encountered in artificial water courses, roll waves have also been seen in natural flows such as ice channels [128], and on gravity currents in the laboratory [129], ocean [154] and lakes [134]. Moreover, disturbances identified as the analogues of roll waves occur in a variety of other physical settings, such as in multiphase fluid [158], mudflow [133], granular layers [136], and in flow down collapsible tubes and elastic conduits (with applications to air and blood flow in physiology [148], and a model of volcanic tremor [140]).

Waves are also common occurrences in shallow, laminar fluid films flowing on street gutters and window panes on rainy days. These objects are rationalized as wavy instabilities of uniform films and are the laminar relatives of the turbulent roll waves, arising typically under conditions in which surface tension plays a prominent role. As the speed and thickness of the films increases, surface tension becomes less important, and “capillary roll waves” are transformed into “inertial roll waves”, which are relevant to some processes of mass and heat transfer in engineering. It is beyond this regime,

\[A\text{ version of this chapter has been published. Balmforth, N. J. & Mandre, S., Dynamics of roll waves (2004), J. Fluid Mech. 514:1-33}\]
Figure 2.1: The picture on the left shows a laboratory experiment in which roll waves appear on water flowing down an inclined channel. The fluid is about 7 mm deep and the channel is 10 cm wide and 18 m long; the flow speed is roughly 65 cm/sec. Time series of the free-surface displacements at four locations are plotted in the pictures on the right. In the upper, right-hand panel, small random perturbations at the inlet seed the growth of roll waves whose profiles develop downstream (the observing stations are 3 m, 6 m, 9 m and 12 m from the inlet and the signals are not contemporaneous). The lower right-hand picture shows a similar plot for an experiment in which a periodic train was generated by moving a paddle at the inlet; as that wavetrain develops downstream, the wave profiles become less periodic and there is a suggestion of subharmonic instability.
and the transition to turbulence, that one finds Cornish’s roll waves. An experiment illustrated in figure 2.1 shows these roll waves in the laboratory at a Reynolds number of about $10^4$ and Froude number of around 2-3.5.

A class of models that have been used to analyze roll waves are the shallow-water equations with bottom drag and internal viscous dissipation:

$$
\begin{align}
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} - g \cos \theta (\tan \theta - h - \zeta_x) &= -C_f f(u, h) + \frac{1}{h} \left( h \nu_{eff} u_x \right)_x \quad (2.1) \\
\frac{\partial h}{\partial t} + (hu)_x &= 0, \quad (2.2)
\end{align}
$$

where $t$ is time, $x$ is the downstream spatial coordinate, and $g$ is the gravitational acceleration. The dependent variables of this model are the depth-averaged water velocity, $u(x,t)$, and depth, $h(x,t)$ and subscripts with respect to $x$ and $t$ denote partial derivatives. The flow configuration is illustrated in figure 2.2, and consists of a Cartesian coordinate system aligned with an incline of overall slope, $\tan \theta$, with $\zeta$ representing any departure due to an uneven bottom. The bottom drag is $C_f f(u, h)$, where $C_f$ is a parameter, and the effective viscosity is $\nu_{eff}$. The parameter $\alpha$ is a geometrical factor meant to characterize the flow profile in the direction transverse to the incline.

The drag law and $\alpha$ vary according to the particular model chosen, and reflect to some degree the nature of the flow. For example, the St. Venant model, a popular model in hydraulic engineering, pertains to turbulent stream flow. In this instance, one expects that the flow profile is fairly blunt, with sharp turbulent boundary layers, and dimensional analysis suggests a form for the drag law (a crude closure for the turbulent stress from the bed):

$$
\alpha = 1, \quad f(u, h) = \frac{|u|}{h}. \quad (2.3)
$$

There are empirical estimates of the friction coefficient, $C_f$, in the drag term, which is often referred to as the Chézy formula.

For a laminar flow, the shallow-water model can be crudely justified by vertically averaging the mass and momentum balance equations, using a von Karman-Polhausen technique to evaluate the nonlinearities [153]. The flow can be approximated to be parabolic in the transverse direction giving

$$
\alpha = \frac{4}{5}, \quad f(u, h) = \frac{u}{h^2}. \quad (2.4)
$$

In this instance, $C_f$ and $\nu_t$ are both given by the kinematic viscosity of the fluid. For
In 1925, Jeffreys [138] used the St Venant equations (2.1)-(2.2) to provide the first theoretical discussion of roll waves. He analyzed the linear stability of flow over a flat plane (ζ = 0 in equation (2.1)), including the Chézy drag term and omitting the turbulent viscosity. His main result was an instability condition, \( F > 2 \), where \( F \) is the Froude number of the flow, defined by \( F = \frac{V}{\sqrt{gD}} \cos \theta \), with \( D \) and \( V \) being the characteristic fluid depth and speed respectively. Subsequently, Dressier [132] constructed finite-amplitude roll waves by piecing together smooth solutions separated by discontinuous shocks. The necessity of shocks in Dressler’s solutions arises because, like Jeffreys, he also neglected the turbulent viscosity, which leaves the equations hyperbolic and shocking. Needham & Merkin [147] later added the eddy diffusion term to regularize the discontinuous shocks. The nonlinear evolution of these waves to the steadily propagating profile has interested many researchers since [130, 137, 144, 149, 160].

Previous investigations have incorporated a variety of forms for the viscous dissipation term, all of them of the form \( \nu h^{-m} \partial_x (h^n u_x) \). Of these, only those with \( m = 1 \) conserve momentum and dissipate energy. Furthermore, if \( n = 1 \), \( \nu \) has the correct dimension of viscosity and the total viscous dissipation is weighted by the fluid depth. Thus we arrive at the term included in (2.1), as did Kranenburg, which we believe is the most plausible.

The study of laminar roll waves was initiated by Kapitza & Kaptiza [141] somewhat after Cornish and Jeffries. Subsequently, Benjamin [124], Yih [159] and Benney [125] determined the critical Reynolds number for the onset of instability and extended the theory into the nonlinear regime. These studies exploited long-wave expansions of the governing Navier-Stokes equations to make analytical progress, and which leads to nonlinear evolution equations that work well at low Reynolds numbers. However, it was later found that the solutions of those equations diverged at higher Reynolds number [150]. This led some authors [122, 153] to resort to the shallow-water model (2.1)-(2.2) to access such physical regimes.

The present study has two goals. First, we explore the effect of bottom topography on the inception and dynamics of roll waves (\( \zeta \) is a prescribed function). Bottom topography is normally ignored in considering turbulent roll waves. However, real water
courses are never completely flat, and roll waves have even been observed propagating
down sequences of steps [155]. Instabilities in laminar films flowing over wavy surfaces
have recently excited interest, both theoretically [127, 135, 152] and experimentally
[156], in view of the possibility that boundary roughness can promote mixing and heat
and mass transfer in industrial processes, or affect the transition to turbulence. Also,
in core-annular flow (a popular scenario in which to explore lubrication problems in the
pipelining industry [139]), there have been recent efforts to analyze the effect of periodic
corrugations in the tube wall [143, 157]. With this background in mind, we present a
study of the linear stability of turbulent flow with spatially periodic bottom topography.

Our second goal in this work is to give a relatively complete account of the nonlinear
dynamics of roll waves. To this end, we solve the shallow-water equations (2.1)-(2.2) nu-
merically, specializing to the turbulent case with (2.3), and complement that study with
an asymptotic theory valid near onset. The asymptotics furnish a reduced model that
encompasses as some special limits a variety of models derived previously for roll waves
[144, 160, 161]. The nonlinear dynamics captured by the reduced model also compares
well with that present in the full shallow-water system, and so offers a compact descrip-
tion of roll waves. We use the model to investigate the wavelength selection mechanism
for roll waves. It has been reported in previous work that roll-wave trains repeatedly
undergo a process of coarsening, wherein two waves approach one another and collide
to form a single object, thereby lengthening the spatial scale of the wave-pattern. It has
been incorrectly inferred numerically that this inverse-cascade phenomenon proceeds to
a final conclusion in which only one wave remains in the domain. Such a conclusion
Chapter 2. Dynamics of roll waves

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is an artifact of restricting analysis to "short" waves, in which limit the shallow water equations do exhibit a self-similar coarsening dynamics [130, 144]. We intend to account for the longer spatial scales via a long wave analysis thus allowing us to study the arrest of coarsening dynamics beyond the regime of short waves. The asymptotic model we derive indeed shows that coarsening does not always continue to the largest spatial scale, but becomes interrupted and roll-wave trains emerge over a range of selected wavelengths.

Coarsening dynamics was documented by Brock [126] in his experiments and is also clear in the experimental data of figure 2.1. This is indicative that these experiments are performed in the short-wave regime. Due to the lack of any experiments reporting the arrest of coarsening, we devised our own experiment to study the phenomenon. By generating periodic waves at the inlet of the channel, we force the flow to start out with longer spatial scales and thus directly probe if coarsening dynamics are universal for roll waves. This verification not only illuminates the mechanisms of pattern formation in flows down inclines but also helps us in validating the very mathematical model we have empirically assumed.

We start with non-dimensionalizing our governing equations in section §2.2. Next, in §2.3, we study the equilibrium flow profiles predicted by our model and follow it with a linear stability theory in §2.4. The asymptotic analysis is described in §2.5. We devote §2.6 to the study of the nonlinear dynamics of roll waves, mainly using the reduced model furnished by asymptotics and compare the predictions with observations from the experiments in §2.7. We summarize our results in §2.8. Overall, the study is focussed on the turbulent version of the problem (i.e. St. Venant with (2.3)). Some of the results carry over to the laminar problem (the Shkadov model with (2.4)). However, we highlight other results which do not (see Appendix B).

2.2 Mathematical formulation

We place (2.1)-(2.2) into a more submissive form by removing the dimensions from the variables and formulating some dimensionless groups: We set

\[ x = L \tilde{x}, \quad u = V \tilde{u}, \quad h = D \tilde{h}, \quad \zeta = D \tilde{\zeta} \quad \text{and} \quad t = (L/V) \tilde{t}, \]  

(2.5)
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where

\[ L = D \cot \phi, \quad C_f f(V, D) = g \sin \phi \quad \text{and} \quad V D = Q, \]  

(2.6)

which specifies \( D, L \) and \( V \) in terms of the slope, friction coefficient and water flux, \( Q \). We also assume that the dependence of the drag force on \( u \) and \( h \) is such that \( f(Vu, Dh) = f(V, D)f(u, h) \). After discarding the tilde decorations, the equations can be written in the form,

\[ F^2(u_t + auu_x) + h_x + \zeta_x = 1 - f(u, h) + \frac{\nu}{h}(hu_x)_x \]  

(2.7)

and

\[ h_t + (hu)_x = 0, \]

(2.8)

where,

\[ \nu = \frac{\nu V}{C_f L^2 f(V, D)}, \]  

(2.9)

is a dimensionless viscosity parameter, assumed constant. As demanded by the physical statement of the problem, that the flow is shallow, we typically take \( \nu \) to be small, so that the bottom drag dominates the internal viscous dissipation. In this situation, we expect that the precise form of the viscous term is not so important.

We impose periodic boundary conditions in \( x \). This introduces the domain length \( a \) as a third dimensionless parameter of the problem. As mentioned earlier, we also select topographic profiles for \( \zeta(x) \) that are periodic. For the equilibria, considered next, we fix the domain size to be the topographic wavelength, but when we consider evolving disturbances we allow the domain size to be different from that wavelength.

2.3 Equilibria

The steady flow solution, \( u = U(x) \) and \( h = H(x) \), to (2.7)-(2.8) satisfies

\[ F^2 a U U_x + H_x + \zeta_x = 1 - f(U, H) + \frac{\nu}{H}(HU_x)_x \quad \text{and} \quad HU = 1, \]  

(2.10)

since we have used the water flux \( Q \) to remove dimensions. For both drag laws in (2.3) and (2.4), \( f(U, H) = U^3 \). Also, by taking \( \tilde{F} = F \sqrt{\alpha} \) as a modified Froude number, we avoid a separate discussion of the effect of \( \alpha \).

By way of illustration, we consider a case with sinusoidal bottom topography:

\[ \zeta(x) = a \cos k_b x, \]  

(2.11)
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Figure 2.3: Viscous periodic equilibria for $\hat{F} = \sqrt{\alpha F} = 1.225$, $k_b = 2$ and $\nu = 0.04$, with varying $a$ (0.01, 0.1, 0.2, 0.3, 0.5, 0.75 and 1).

where $k_b$ is the wavenumber of the bottom topography and $a$ is its amplitude. Discussion on more general topographic profiles is included in §2.8. Some example equilibria are illustrated in figure 2.3. For a low amplitude topography, the response in the fluid depth appears much like $\zeta$, with a phase shift. As the amplitude increases, however, steep surface features appear. A similar trend was experimentally observed by Vlachogiannis & Bontozoglou [156] which they reported as a "resonance". We rationalize these features in terms of hydraulic jumps, based on the "inviscid" version of the problem (i.e. $\nu = 0$).

For $\nu = 0$, the equilibria equation simplifies and can be written in the form,

$$H_\eta = \left[ \frac{H^3(1 - f(1/H, H) - k_\eta \zeta_\eta)}{k_b(H^3 - \hat{F}^2)} \right]$$

where $\eta = k_b x$. All solutions to (2.12) reside on the $(\eta, H)$–phase plane; we require only those that are strictly periodic in $\eta$. Now, the extrema of $H(\eta)$ occur for $H = 1/(1 - k_\eta \zeta_\eta)^{1/3}$, whilst there is a singular point at $H = \hat{F}^{2/3}$. In general, $H(\eta)$ becomes vertical at the latter points, except if the numerator also vanishes there, in which case inviscid solutions may then pass through with finite gradient. Overall, the two curves, $H = 1/(1 - k_\eta \zeta_\eta)^{1/3}$ and $H = \hat{F}^{2/3}$, organize the geometry of the inviscid solutions on the $(\eta, H)$–phase plane. Four possible geometries emerge, and are illustrated in figure 2.4.

The two curves cross when $\hat{F}^2 = 1/(1 - k_\eta \zeta_\eta)$ somewhere on the $(\eta, H)$–plane. Thus,
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if the amplitude of the topography is defined so that \(-a \leq \zeta'(\eta) \leq a\), the curves cross when

\[ (1 + k_0 a)^{-1/2} < \tilde{F} < (1 - k_0 a)^{-1/2} \]  \hfill (2.13)

(if \(k_0 a > 1\), there is no upper bound on \(\tilde{F}\)). Outside this range, the inviscid system has smooth periodic solutions, and panels (a) and (d) of figure 2.4 illustrate the two possible cases.

When \(\tilde{F}\) falls into the range in (2.13), the two organizing curves cross, and the geometry of the phase plane becomes more complicated. For values of \(\tilde{F}\) adjacent to the two limiting values in (2.13), periodic inviscid solutions still persist and lie either entirely above or below \(H = \tilde{F}^{2/3}\) (panel (b)). We denote the ranges of Froude numbers over which the solutions persist by \((1 + k_0 a)^{-1/2} < \tilde{F} < F_1\) and \(F_2 < \tilde{F} < (1 - k_0 a)^{-1/2}\).

At the borders, \(F_1\) and \(F_2\), the inviscid periodic solutions terminate by colliding with a crossing point. Thereafter, in \(F_1 < \tilde{F} < F_2\), no periodic, continuous solution exists: all trajectories on the phase plane either diverge to \(H \to \infty\) or become singular at \(H = \tilde{F}^{2/3}\) (panel (c)).

Although there are no periodic inviscid solutions within the divergent range of Froude numbers, \(F_1 < \tilde{F} < F_2\), there are periodic, weakly viscous solutions that trace out inviscid trajectories for much of the period (see figure 2.4). The failure of the inviscid trajectories to connect is resolved by the weakly viscous solution passing through a hydraulic jump over a narrow viscous layer. The limiting inviscid jump conditions can be determined by integrating the conservative form of the governing equations across the discontinuity:

\[ U_+ H_+ = U_- H_- = 1 \quad \text{and} \quad \tilde{F}^2 U_+ + \frac{H_+^2}{2} = \tilde{F}^2 U_- + \frac{H_-^2}{2}, \]  \hfill (2.14)

where the subscripts + and − denote the values downstream and upstream respectively.

The jump region, \(F_1 < \tilde{F} < F_2\), is delimited by values of the Froude number at which an inviscid solution curve connects the rightmost crossing point to itself modulo one period. This curve is continuous, but contains a corner at the crossing point; see figure 2.5. The curves \(F_1\) and \(F_2\) are displayed on the \((\tilde{F}, k_0 a)\)-plane in figure 2.6. Hydraulic jumps form in the weakly viscous solutions in the region between these curves.

A departure from the classification shown in figure 2.4 occurs for Froude numbers
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Figure 2.4: Stationary flow profiles for $k_b = 5$, $a = 0.1$ and four values of $\hat{F} = F \sqrt{a}$. Light dotted curves show a variety of inviscid solutions ($\nu = 0$) to illustrate the flow on the phase plane ($\eta, H$). The thicker dots show a periodic viscous solution (with $\nu = 0.002$) Also included is the line, $H = \hat{F}^{2/3}$, and the curve, $H = (1 - \zeta_x)^{-1/3}$. In panels (b) and (c), with dashed lines, we further show the inviscid orbits that intersect the “crossing point”, $H = \hat{F}^{2/3} = (1 - \zeta_x)^{-1/3}$. 
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Figure 2.5: Limiting periodic inviscid solutions for $a = 0.1$, and (a) $k_b = 5$ and $\hat{F} \approx 1.311$ (b) $k_b = 10$ and $\hat{F} \approx 0.733$. The dots (which lies underneath the inviscid solution except near the corner at the rightmost crossing point) show the viscous counterparts for $\nu = 0.002$. The solid and dashed lines show $H = \hat{F}^{2/3}$ and $H = (1 - \zeta)_{-1/3}$.

Figure 2.6: The jump region on the $(\hat{F}, k_b a)$–plane. The solid lines show the limits, $F_1$ and $F_2$, for $k_b = 5$ and 10; the $F_2$ curve is also shown for $k_b = 2$. Shown by dotted lines are the borders (2.13) of the region in which the organizing curves $H = \hat{F}^{2/3}$ and $H = (1 - \zeta)_{-1/3}$ cross one another. The inset shows a magnification near $\hat{F} = 1$, and the curves $F = F_*(a)$ on which the inviscid solutions passing through both crossing points disappear.
near unity and low-amplitude topography. Here, the flow of the inviscid solutions on the \((\eta, H)\)-phase plane is sufficiently gently inclined to allow orbits to pass through both crossing points. This leads to a fifth type of equilibrium, as shown in figure 2.7. Although this solution is continuous, its gradient is not; again, there is a weakly viscous counterpart. As the amplitude of the topography increases, the flow on the phase plane steepens, and eventually the inviscid orbit disappears (see figure 2.7), to leave only viscous solutions with hydraulic jumps. This leads to another threshold, \(\hat{F} = F_*\), on the \((\hat{F}, k_b a)\)-plane, which connects the \(F_1\) and \(F_2\) curves across the region surrounding \(\hat{F} = 1\) (see the inset of figure 2.6).

### 2.4 Linear stability theory

We perform a linear stability analysis of the steady states described above to uncover how the bed structure affects the critical Froude number for the onset of roll waves. Let \(u = U(x) + u'(x, t)\) and \(h = H(x) + h'(x, t)\). After substituting these forms into the governing equations and linearizing in the perturbation amplitudes, we find the linear
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equations,

\begin{align*}
F^2 [u' t + a(Uu')_x] + h'_x &= -f_h h' - f_u u' + \nu u'_{xx} \quad (2.15) \\
h'_t + (U h' + H u')_x &= 0, \quad (2.16)
\end{align*}

where \( f_u = (\partial f / \partial u)_{U,H=H} \) and \( f_h = (\partial f / \partial h)_{U,H=H} \) denote the partial derivatives of the drag law, evaluated with the equilibrium solution.

Because of the spatial periodicity of the background state, a conventional stability analysis must proceed by way of Floquet, or Bloch, theory: We represent infinitesimal perturbations about the equilibria by a truncated Fourier series with a Bloch wavenumber, \( K \) (a Floquet multiplier), and growth rate, \( \sigma \):

\begin{align*}
&u' = \sum_{j=-N+1}^N u_j e^{ijk_b x + iK x + \sigma t}, \quad h' = \sum_{j=-N+1}^N h_j e^{ijk_b x + iK x + \sigma t}. \quad (2.17)
\end{align*}

We introduce these solutions into the governing equations and then linearize in the perturbation amplitudes, to find an algebraic eigenvalue equation for \( \sigma \). The system contains five parameters: the Froude number \( F \), the wavenumber of bottom topography \( k_b \), the amplitude of bottom topography \( a \), the Bloch wave number \( K \) and the diffusivity \( \nu \).

When the bottom is flat, the equilibrium is given by \( U = H = 1 \) and we avoid the Bloch decomposition by taking \( (u', h') \propto \exp(ikx) \). This leads to the dispersion relation,

\begin{equation}
\sigma = -ik \left( \frac{1 + a}{2} - \frac{f_u + \nu k^2}{2} \right) \pm \sqrt{\left( \frac{f_u + \nu k^2}{2} \right)^2 + \frac{(\alpha - 1)ik}{F^2} + \frac{ikf_h - k^2}{F^2}}. \quad (2.18)
\end{equation}

For long waves, the least stable root becomes

\begin{equation}
\sigma \sim -ik \left[ \frac{f_u}{f_h} - 1 \right] + \left[ \frac{F^2(f_u f_h (\alpha - 1) + f_h^2) - f_u^2}{f_u^2} \right] k^2 + ... \quad (2.19)
\end{equation}

which displays the instability condition,

\begin{equation}
F^2 > \frac{f_u^2}{f_u f_h (\alpha - 1) + f_h^2} \quad (2.20)
\end{equation}

For the turbulent case, \( f_u = 2 \) and \( f_h = -1 \), and so \( F > 2 \), as found by Jeffreys. For the laminar case, on the other hand, \( f_u = 1 \) and \( f_h(1,1) = -2 \), which gives \( F > \sqrt{5}/2 \).

We next provide a variety of numerical solutions to the linear stability problem for finite topography with the sinusoidal profile, \( \zeta = a \sin(k_b x) \), and using the St. Venant
Figure 2.8: Eigenvalues from numerical stability analysis and asymptotics for $\nu = 0.4$, $k_b = 10$, $a = 0.05$, and Froude numbers of 1.9, 2 and 2.1. The lines denote numerical calculations and the dots represents asymptotic theory (for $\nu \sim k_b^{-1}$; theory A §2.5.1). Panel (a) shows the growth rate, $\text{Re}(\sigma)$, and (b) the phase speed, $-\text{Im}(\sigma)/K$.

model ($f = u^2/h$ and $\alpha = 1$). In this instance, the Bloch wavenumber allows us to analyze the stability of wavenumbers which are not harmonics of $k_b$. We only need to consider

$$-\frac{k_b}{2} < K \leq \frac{k_b}{2};$$

values of $K$ outside this range do not give any additional information because the wavenumber combination, $k = jk_b + K$ for $j = 0, 1, 2, \ldots$, samples the full range.

The dependence of the growth rate on $K$ is illustrated in Fig. 2.8 for three Froude numbers straddling $F = 2$ and a low-amplitude topography. The case with larger Froude number is unstable for a band of waves with small wavenumber, and illustrates how the instability invariably has a long-wave character. This feature allows us to locate the boundaries of neutral stability by simply taking $K$ to be small (as done below).

A key detail of this stability problem is that low-amplitude topography is destabilizing. We observe this feature in figure 2.9, which shows the curve of neutral stability on the $(F,a)$–plane for fixed Bloch wavenumber, $K = 10^{-3}$, and three values of $\nu$, including $\nu = 0$. The curves bend to smaller $F$ on increasing $a$, indicating how the unstable region moves to smaller Froude number on introducing topography.

In this region of parameter space, we find that viscosity plays a dual role: As is clear from the classical result for a flat bottom, viscosity stabilizes roll waves of higher wavenumber. In conjunction with topography, however, viscosity can destabilize long waves, see figures 2.9 and 2.10. The second picture shows the depression of the $F = 2$
Figure 2.9: Stability boundaries on the \((a, F)\)-plane, near \((a, F) = (0, 2)\), for fixed Bloch wavenumber, \(K = 10^{-3}\), and three values of viscosity \((0, 0.01 \text{ and } 0.1)\). Also shown are the boundaries predicted by the two versions of asymptotic theory (theory \(A\) is used for \(\nu = 0.1\), and theory \(B\) (§2.5.2) for \(\nu = 0.01 \text{ and } 0\)).

Stability boundary on the \((\nu, F)\)-plane as the bottom topography is introduced. The boundary rebounds on increasing the viscosity further, and so the system is most unstable for an intermediate value of the viscosity (about 0.1 in the figures). These results expose some dependence on \(\nu\), which presumably also reflects the actual form of the viscous term. Nevertheless, the “inviscid” \(\nu \to 0\), results can also be read off the figures and are independent of that form. It is clear from figure 2.9 & 2.10 that the general trend is to destabilize turbulent roll waves.

Further from \((a, F) = (0, 2)\), a new form of instability appears that extends down to much smaller Froude number, see figure 2.11, panel (a). The growth rate increases dramatically in these unstable windows, as shown further in the second panel. In fact, for \(\nu = 0\), it appears as though the growth rate as a function of \(a\) becomes vertical, if not divergent (we have been unable to resolve precisely how the growth rate behaves, although a logarithmic dependence seems plausible). This singular behaviour coincides with the approach of the inviscid equilibrium to the limiting solution with \(F = F_2\). In other words, when the equilibrium forms a hydraulic jump, the growth rate of linear theory becomes singular (in gradient, and possibly even in value). The weakly viscous solutions show no such singular behavior, the jump being smoothed by viscosity, but the sharp peak in the growth rate remains, and shifts to larger \(a\) (figure 2.11). As a result,
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Figure 2.10: Stabilities boundaries on the \((\nu, F)\)-plane, near \(a = 0\), for fixed Bloch wavenumber, \(K = 10^{-3}\), and \(k_b = 10\). Also shown are the boundaries predicted by the two versions of asymptotic theory (labeled \(A\) and \(B\)).

the unstable windows fall close to the \(F_2\)-curve of a neighboring inviscid equilibrium; a selection of stability boundaries displaying this effect are illustrated in Figure 2.12. However, we have not found any comparable destabilization near the \(F_1\)-curve. In fact, near the \(F_1\)-curve, the growth rates appear to decrease suggesting that the hydraulic jump in this part of the parameter space is stabilizing.

Figure 2.12 also brings out another feature of the stability problem: for larger \(a\), the stability boundaries curve around and pass above \(F = 2\). Thus, large-amplitude topography is stabilizing.

2.4.1 An integral identity for inviscid flow

When \(\nu = 0\), an informative integral relation can be derived from the linear equations by multiplying (2.15) by \(2h'U - Hu'\) and (2.16) by \(2F^2Uu' - h'\), then integrating over \(x\) and adding the results:

\[
\frac{d}{dt} \left\langle F^2 H \left( u' - \frac{h'U^2}{2} \right)^2 + h'^2 \left( 1 - \frac{F^2U^2}{4H} \right) \right\rangle = - \left\langle U \left( 2u' - U^2 h' \right)^2 \right\rangle - \left\langle 3U \left( F^2u'^2 H + \frac{h'^2}{4} \right) \right\rangle \tag{2.22}
\]

where the angular brackets denote \(x\)-integrals.

For the flat bottom, \(U = H = 1\), and the left-hand side of this relation is the time derivative of a positive-definite integral provided \(F < 2\). The right-hand side, on the
Figure 2.11: Instability windows at smaller Froude number. Panel (a): Contours of constant growth rate ($\sigma$) for $\nu = 0.05$, $k_b = 10$, $K = 10^{-3}$. Thirty equally spaced contours (lighter lines) are plotted with the growth rate going from $1.14 \times 10^{-4}$ to $-4.28 \times 10^{-5}$. The darker line denotes the neutral stability curve and the dashed line shows the location of $F_2$ curve. Panel (b): Growth rates against $a$ for $F = 1.6$, $k_b = 5$, $K = 10^{-3}$ and four values of $\nu$. These sections cut through the window of instability at smaller Froude number. Also shown is the inviscid growth rate, which terminates as $F \to F_2$ (the vertical dotted line).
other hand, is negative definite. Thus, for such Froude numbers, the integral on the left must decay to zero. In other words, the system is linearly stable, and so (2.22) offers a short-cut to Jeffrey’s classical result.

Because of the integral involving $U_x$, a stability result is not so straightforward with topography, although (2.22) still proves useful. First, assume that this new integral is overwhelmed by the first term on the right of (2.22), so that the pair remain negative definite. This will be true for low-amplitude topography, away from the region in which hydraulic jumps form. Then stability is assured if $F > F_2$ or $F < 2\sqrt{H}/U$. That is, if the local Froude number is everywhere less than 2 (a natural generalization of Jeffrey’s condition).

Second, consider the case when the local Froude number condition is everywhere satisfied, so that stability is assured if the right-hand side of (2.22) is always negative. But on raising the amplitude of the topography, $U_x$ increases sharply as a hydraulic jump develops in the equilibrium flow. Provided $U_x < 0$ at that jump, the right-hand side of (2.22) can then no longer remain always negative, and allowing an instability to become possible. As illustrated in figure 2.4, the jump in $H$ is positive across the $F_2$ curve, so $U_x < 0$, and that feature is potentially destabilizing, as indicated in the stability analysis. Nonetheless, we have found no explanation for why the jump near $F_2$ is destabilizing but the one near $F_1$ is not.
2.5 Asymptotics

We complement the linear stability analysis with an analytical theory based on asymptotic expansion with multiple time and length scales. The theory is relevant near onset for low-amplitude, but rapidly varying topography, and proceeds in a similar fashion to that outlined by Yu & Kevorkian [160] and Kevorkian, Yu & Wang [142] for flat planes; topography is incorporated by adding a further, finer length scale. We offer two versions of the theory, suited to different asymptotic scalings of the viscosity parameter, $\nu$. We refer to the two versions as theories A and B.

2.5.1 A first expansion; $\nu \sim \epsilon$ (theory A)

We take $\epsilon \equiv \kappa^{-1} \ll 1$ and $\zeta$ to be an $O(\epsilon)$ function of the coordinate, $\eta = x/\epsilon$, resolving the rapid topographic variation, $\zeta \rightarrow \epsilon A(\eta)$, where $A(\eta)$ describes the topographic profile. We introduce the multiple length and time scales, $(\eta, x)$ and $(t, \tau)$, where $\tau = \epsilon t$, giving

$$\partial_t \rightarrow \partial_t + \epsilon \partial_\eta \quad \text{and} \quad \partial_x \rightarrow \frac{1}{\epsilon} \partial_\eta + \partial_x,$$

and further set

$$\nu = \epsilon \nu_1 \quad \text{and} \quad F = F_0 + \epsilon F_1.$$  \hspace{1cm} (2.23)

We next expand the dependent variables in the sequences,

$$ u = 1 + \epsilon [U_1(\eta) + u_1(x, t, \tau)] + \epsilon^2 [U_2(\eta, x, t, \tau) + u_2(x, t, \tau)] + \epsilon^3 [U_3(\eta, x, t, \tau) + u_3(x, t, \tau)] + ...,$$

$$ h = 1 + \epsilon [H_1(\eta) + h_1(x, t, \tau)] + \epsilon^2 [H_2(\eta, x, t, \tau) + h_2(x, t, \tau)] + \epsilon^3 [H_3(\eta, x, t, \tau) + h_3(x, t, \tau)] + ... $$ \hspace{1cm} (2.25)

Here, $U_1$ and $H_1$ denote the fine-scale corrections due to the topography, whereas $u_1$ and $h_1$ represent the longer-scale, wave-like disturbance superposed on the equilibrium. To avoid any ambiguity in this splitting, we demand that $U_1$ and $H_1$ have zero spatial average. At higher order, we again make a separation into fine-scale and wave components, but the growing disturbance modifies the local flow on the fine scale and so, for example, $U_2$ and $H_2$ acquire an unsteady variation.

At leading order, we encounter the equations,

$$ F_0^2 U_{1\eta} + H_{1\eta} + A_\eta = \nu_1 U_{1\eta\eta}, \quad U_{1\eta} + H_{1\eta} = 0,$$  \hspace{1cm} (2.27)
and write the solution formally as

\[ U_1 = f(\eta) = -H_1, \quad (2.28) \]

where \( f(\eta) \) has zero spatial average. A convenient way to compute \( f(\eta) \) is via a Fourier series: Let

\[ A(\eta) = \sum_{j=1}^{\infty} A_j e^{i\eta} + \text{c.c.,} \quad (2.29) \]

where the coefficients \( A_j \) are prescribed (without loss of generality we may take \( A(\eta) \), i.e. \( \zeta \), to have zero spatial average). Then,

\[ f = \sum_{j=1}^{\infty} f_j e^{i\eta} + \text{c.c.,} \quad f_j = -\frac{A_j \exp(i\theta_j)}{\sqrt{(F_0^2 - 1)^2 + \nu_1^2 f_j^2}}, \quad \tan \theta_j = \frac{\nu_1 j}{F_0^2 - 1}. \quad (2.30) \]

At next order:

\[ F_0^2 U_{2\eta} + H_{2\eta} - \nu_1 U_{2\eta} = -F_0^2 (u_{1t} + u_{1x}) - h_{1x} - 2u_1 + h_1 - F_0^2 u_1 U_{1\eta} - F_0^2 U_1 U_{1\eta} \]
\[ -2U_1 + H_1 - 2F_0 U_1 U_{1\eta} + \nu_1 H_{1\eta} U_{1\eta}, \quad (2.31) \]
\[ U_{2\eta} + H_{2\eta} = -h_{1t} - h_{1x} - u_{1x} - (H_1 U_1)_{\eta} - (h_1 U_1 + H_1 u_1)_{\eta}. \quad (2.32) \]

We deal with these equations in two stages. First, we average over the fine length scale \( \eta \) to eliminate the corrections, \( U_2 \) and \( H_2 \). This generates our first set of evolution equations for the variables \( u_1 \) and \( h_1 \):

\[ F_0^2 (u_{1t} + u_{1x}) + h_{1x} + 2u_1 - h_1 = -\nu_1 U_{1\eta}^2, \quad (2.33) \]
\[ h_{1t} + h_{1x} + u_{1x} = 0. \quad (2.34) \]

This pair of equations has the characteristic coordinates, \( \xi = x - (1 + F^{-1}) t \) and \( \dot{\xi} = x - (1 - F^{-1}) t \). Moreover, along the characteristics, the solutions either grow or decay exponentially unless the undifferentiated terms in (2.33) cancel. To avoid such detrimental behavior, we demand that those terms vanish, which fixes

\[ h_1(\xi) = 2u_1(\xi) + \nu_1 U_{1\eta}^2 \quad (2.35) \]

and \( F_0 = 2 \), the familiar neutral stability condition. Hence, superposed on the fine-scale flow structure, there is a propagating disturbance characterized by the traveling-wave coordinate, \( \xi = x - 3t/2 \).
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Second, we decompose the fine-scale variation into two parts:

\[ U_2 = \tilde{U}_2(\eta) + \bar{U}_2(\eta)u_1(\xi, \tau) \quad \text{and} \quad H_2 = \bar{H}_2(\eta) + \bar{H}_2(\eta)u_1(\xi, \tau), \]  

(2.36)

with

\[
\begin{align*}
4\tilde{U}_{2\eta} + \bar{H}_{2\eta} - \nu_1\tilde{U}_{2\eta} &= -4ff_\eta - 3f - 4F_1f_\eta - \nu_1(f_\eta^2 - f_\eta^2), \\
\tilde{U}_{2\eta} + \bar{H}_{2\eta} &= 2ff_\eta - \nu_1f_\eta f_\eta^2, \\
4\tilde{U}_{2\eta} + \bar{H}_{2\eta} - \nu_1\tilde{U}_{2\eta} &= -4f_\eta, \\
\bar{U}_{2\eta} + \bar{H}_{2\eta} &= -f_\eta.
\end{align*}
\]

(2.37) - (2.40)

The solution \( \tilde{U}_2 \) and \( \bar{H}_2 \) represents a correction to the fine-scale flow structure, and is not needed for the evolution equation of the disturbance. The other component of the solution can again be determined by decomposition into Fourier series:

\[
\begin{align*}
\tilde{U}_2 &= \sum_{j=1}^{\infty} U_je^{ij\eta} + \text{c.c.} \\
\bar{H}_2 &= \sum_{j=1}^{\infty} \bar{H}_je^{ij\eta} + \text{c.c.,}
\end{align*}
\]

(2.41)

with

\[
\begin{align*}
\tilde{U}_j &= -\frac{3f_je^{ij\eta}}{\sqrt{9 + \nu_1^2j^2}} = \frac{3A_je^{2ij\eta}}{9 + \nu_1^2j^2}, \\
\bar{H}_j &= -\tilde{U}_j - f_j.
\end{align*}
\]

(2.42)

and \( \tan \theta_j = \nu_1j/3 \).

We proceed to one more order in \( \epsilon \), where the spatially averaged equations are

\[
\begin{align*}
h_{2\xi} - 2u_{2\xi} + 2u_2 - h_2 &= 2F_1u_{1\xi} - 4u_{1r} - 4u_1u_{1\xi} - (u_1 - h_1)^2 - 4\bar{U}_1^2 \\
&+ \nu_1 \left[ u_{1\xi\xi} + (H_{2\eta} - U_{2\eta})u_{1\eta} + (h_1 - U_{1\eta})\bar{U}_{1\eta}^2 \right] \\
\frac{1}{2}h_{2\xi} - u_{2\xi} &= h_{1r} + (u_1h_1)\xi.
\end{align*}
\]

(2.43) - (2.44)

Lastly, we eliminate the combination, \( 2u_2 - h_2 \), to arrive at the evolution equation of our first expansion:

\[
4u_{1r} + 3(u_1^2)\xi - 8u_{1r}\xi - 6(u_1^2)\xi + \nu_1(f_\eta^2 - 2\bar{U}_{2\eta}f_\eta)u_{1\xi} + 2(F_1 - \nu_1f_\eta^2)u_{1\xi\xi} + \nu_1u_{1\xi\xi} = 0.
\]

(2.45)

### 2.5.2 A second expansion; \( \nu \sim \epsilon^2 \) (theory B)

A distinctive feature of the expansion above is that if \( \nu_1 = 0 \), topographic effects disappear entirely. In other words, terms representing "inviscid" topographic effects must
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lie at higher order. To uncover these terms, we design a different expansion, with a smaller scaling for the viscosity. We sketch the alternative procedure: Again we take \( \varepsilon \equiv k_0^{-1} \ll 1 \) and \( \xi \to \varepsilon A(\eta) \). This time the slow timescale is even slower, \( \tau = \varepsilon^2 t \), and we pose

\[
\nu = \varepsilon^2 \nu_2 \quad \text{and} \quad F = F_0 + \varepsilon^2 F_2, \tag{2.46}
\]

and the asymptotic sequences,

\[
\begin{align*}
u &= 1 + \varepsilon U_1(\eta) + \varepsilon^2 [U_2(\eta) + u_2(x, t, \tau)] + \varepsilon^3 [U_3(\eta, x, t, \tau) + u_3(x, t, \tau)] + \ldots \\
h &= 1 + \varepsilon H_1(\eta) + \varepsilon^2 [H_2(\eta) + h_2(x, t, \tau)] + \varepsilon^3 [H_3(\eta, x, t, \tau) + h_3(x, t, \tau)] + \ldots
\end{align*}
\]

The corrections, \( U_1, H_1, U_2 \) and \( H_2 \), denote the fine-scale corrections due to the topography, whereas \( u_2 \) and \( h_2 \) now represent the growing disturbance.

The expansion proceeds much as before. A summary of the details is relegated to the Appendix A. The principal result is the amplitude equation,

\[
4u_{2r} + 3(u_2)^2 \xi - 8u_{2z} \xi - 6(u_2)^2 \xi \xi + 3(6U_1 + \nu_2 U_1^2)u_{2z} + 2(F_2 - 3U_1 - \nu_2 U_1^2)u_{2zz} + \nu_2 u_{2zzz} = 0, \tag{2.47}
\]

which explicitly contains the inviscid topography effects via the leading-order equilibrium correction, \( U_1 = -A/3 \).

### 2.5.3 Revisiting linear stability

We now revisit linear stability using the amplitude equations for the St. Venant model (equations (2.45) and (2.47)) by taking \( u_j = \nu e^{iK \xi + \lambda \tau} \), with \( j = 1 \) or \( 2 \), and linearizing in the perturbation amplitude \( \nu \):

\[
\lambda = \frac{K^2 q - iK (p - \nu_j K^2)}{4(1 - 2iK)}, \tag{2.48}
\]

where

\[
p = \begin{cases} \nu_1 (\overline{F_1^2} - 2\overline{U_{2n} F_n}) \\ 3(6U_1^2 + \nu_2 U_1^2) \end{cases} \quad q = \begin{cases} 2(F_1 - \nu_1 \overline{F_1^2}) \\ 2(F_2 - 3\overline{U_1^2} - \nu_2 \overline{U_1^2}) \end{cases}. \tag{2.49}
\]

The growth rate is

\[
\text{Re}(\lambda) = \frac{K^2(q + 2p - 2\nu_j K^2)}{4(1 + 4K^2)}, \tag{2.50}
\]

implying instability for \( q + 2p > 0 \). The neutral stability condition, \( q + 2p = 0 \), is written out fully as

\[
F_1 = 2\nu_1 \overline{U_{2n} F_n} \quad \text{or} \quad F_2 = -15\overline{U_1^2} - 2\nu_2 \overline{U_1^2}. \tag{2.51}
\]
In both cases the critical Froude number is reduced by the topography (the corrections \( F_1 \) and \( F_2 \) are negative). To see this for the first theory, we introduce the Fourier decompositions in (2.30) and (2.41), to find:

\[
F_1 = -36\nu \chi \sum_{j=1}^{\infty} \frac{j^2 |A_j|^2}{(9 + \nu^2 j^2)^2}.
\]

(2.52)

Thus small amplitude topography is destabilizing for any periodic profile.

On restoring the original variables, we find that the stability boundary near \((a, F) = (0, 2)\) is given by

\[
F - 2 = \begin{cases}
-36k_b^2 \nu \sum_{j=1}^{\infty} j^2 |\zeta_j|^2 (9 + \nu^2 j^2 k_b^2)^{-2} & \text{for } \nu \sim O(k_b^{-1}) \\
-(15\zeta_j^2 + 2\nu^2 \zeta_j^2)/9 & \text{for } \nu \sim O(k_b^{-2})
\end{cases}
\]

(2.53)

where \( A_j = k_b \zeta_j \) and \( \zeta_j \) denotes the unscaled Fourier mode amplitudes of \( \zeta(x) \). For the sinusoidal profile, \( \zeta = a \sin(k_b x) \), the mode amplitudes are \( \zeta_j = -ia\delta_{j1}/2 \). It follows that

\[
F - 2 = \begin{cases}
-9\nu k_b^2 a^2 (9 + \nu^2 k_b^2)^{-2} & \text{for } \nu \sim O(k_b^{-1}) \\
-(15 + 2\nu k_b^2) a^2 /18 & \text{for } \nu \sim O(k_b^{-2})
\end{cases}
\]

(2.54)

These predictions are compared with numerical solutions of the linear stability problem in figures 2.8–2.10. Both versions of the asymptotics are used in the comparison, choosing one or the other according to the size of \( \nu \). In figure 2.10 the stability boundary is shown over a range of \( \nu \); the numerical results span both ranges of the asymptotics, \( \nu \sim k_b^{-1} \) and \( k_b^{-2} \), and there is a distinctive cross over between the two asymptotic predictions for intermediate values of \( \nu \).

### 2.5.4 Canonical form

With periodic boundary conditions, the amplitude equation has the property that Galilean transformations cause a constant shift in \( u_j \). This allows us to place the amplitude equation into a canonical form by defining a new variable, \( \varphi = 3u_j/2 + C \), and introducing a coordinate transformation, \((\xi, \tau) \rightarrow (\xi', \tau) = (\xi + c\tau, \tau)\). We may then eliminate any correction to the background equilibrium profile using \( C \), and remove the term \( a\xi \xi \) by suitably selecting the frame speed \( c \). The result is our final amplitude equation,

\[
(1 - 2\partial_\xi)(\varphi_\tau + \varphi \varphi_\xi) + p \varphi_\xi + \mu \varphi_\xi_\xi = 0,
\]

(2.55)
which has the two parameters $p$ and $\mu = \nu_j/4$, and the unique equilibrium state, $\varphi = 0$. A third parameter is the domain size in which we solve the equation, $d$. If we scale time and amplitude, $\tau \to \tau/|p|$ and $\varphi \to |p|\varphi$, we may further set the parameter $p$ to $\pm 1$, leaving only $\mu$ and $d$ as parameters. Below we present some numerical solutions of the amplitude equation; we exploit this final scaling to put $p = 1$, focusing only on unstable flows.

The amplitude equation (2.55) is identical in form to reduced models derived by Yu & Kevorkian (1992) and Kevorkian, Yu & Wang (1995). An additional short-wave approximation leads to the modified Burgers equation derived by Kranenburg (1992), whilst a long-wave approximation gives a generalized Kuramoto-Sivashinsky equation, as considered by Yu, et al. (2000). In contrast to those two final models, (2.55) correctly describes both long and short waves (which can be verified by looking at linear stability - Mandre 2001). Yu, et al. (2000) and Kevorkian, et al. (1995) used a slightly different form for the diffusive term at the outset. Consequently, some of the coefficients in (2.55) differ from those of the corresponding amplitude equations of Yu et al. when compared in the appropriate limit. This reflects the extent to which the amplitude equation depends on the form of diffusion term.

### 2.6 Nonlinear roll-wave dynamics

In this section, we explore the nonlinear dynamics of roll waves, solving numerically both the shallow-water model (and, in particular, the St. Venant version) and the amplitude equation derived above. Related computations have been reported previously by Kranenburg (1992), Yu et al. (1995), Brook et al. (1999) and Chang et al. (2000), who ignored bottom topography and gave incomplete picture of the selection of wavelengths of nonlinear roll waves.

#### 2.6.1 St. Venant model

We numerically integrate the St. Venant model with sinusoidal topography, beginning from the initial conditions, $uh = 1$ and $h = 1$. A pseudo-spectral discretization in space and a fourth-order Runge-Kutta time-stepping scheme was used. A sample integration is shown in Fig. 2.13. In this run, the system falls into the eye of instability of §2.4,
Figure 2.13: A numerical solution of the St. Venant model with $F = 1.58$, $\nu = 0.05$, $a = 0.32$ and $k_t = 4$. The domain has size $5\pi$. Panel (a) shows $h(x,t)$, and (b) shows the flux, $hu$, as surfaces above the $(x,t)$-plane. The solution is “strobed” every 11 time units in order to remove most of the relatively fast propagation of the instability (and make the picture clearer).

and the domain contains ten wiggles of the background topography. The short-scale effect of the topography is evident in $h$, but is far less obvious in the flux, which makes $hu$ a convenient variable to visualize the instability. In Fig. 2.13, the instability grows from low amplitude and then saturates to create a steadily propagating nonlinear roll wave (modulo the periodic variation induced as the wave travels over the topography). Although the run in Fig. 2.13 lies in the eye of instability, similar results are obtained elsewhere in parameter space: Fig. 2.14 shows results from a run nearer the classical roll-wave regime.
Figure 2.14: Panel (a): the flux \((uh)\) associated with a nonlinear roll wave, computed from the St. Venant model (dots) and reconstructed from the amplitude equation (solid line), for \(\nu = 0.05, F = 2.05, a = 0.03\) and \(k_b = 4\). Panel (b) shows the corresponding evolution of the saturation measure, \(I(t)\), for the amplitude equation (solid line) and St. Venant model (dashed line).

We define a measure of the roll-wave amplitude by

\[
I^2(t) = \int_0^L (uh - \langle uh \rangle)^2 dx,
\]

where \(\langle uh \rangle\) denotes the spatial average of the flux. As illustrated in Fig. 2.14, this quantity can be used to monitor saturation. Fig. 2.15 shows the saturation amplitude as a function of the Froude number for \(a = 0.3, \nu = 0.05\) and \(k_b = 10\). This slice through the \((a, F)\)-parameter plane intersects the eye of instability at smaller Froude number as well as Jeffrey’s threshold. At each stability boundary, the saturation level declines smoothly to zero, and so we conclude that the bifurcation to instability is supercritical.

2.6.2 Amplitude equation

Figure 2.14 also includes a numerical solution of our amplitude equation for comparison with that from the St. Venant model. The numerical method employed a pseudospectral scheme in space and a Gear-type time-integrator. The asymptotic scalings have been used to match parameter settings and reconstruct the solution in terms of the original variables. Each of the computations begins with small perturbations about the equilibrium flow, although transients not captured by the asymptotic theory pre-
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Figure 2.15: Saturation amplitudes for the shallow-water equations (circles) for $k_b = 4$, $K = 0.2$, $\nu = 0.05$ and $a = 0.3$. The shaded region shows the range of linear instability of the steady background flow. Corresponding results from the amplitude equation (crosses) are also shown for comparison.

clude agreement over a brief initial period. To remove that transient and improve the comparison of the longer-time dynamics, we have offset the asymptotic solution in time. Figure 2.14 illustrates what appears to be the general result that the amplitude equation (2.55) reproduces the roll-wave dynamics of the St. Venant model (see also figure 2.15, which shows qualitative agreement in the saturation measure near $F = 2$, despite a relatively large topographic amplitude). We therefore focus on the amplitude equation in giving a fuller discussion of the roll-wave dynamics, thereby avoiding separate discussions of the problem with and without bottom topography.

Figure 2.16 shows the evolution of a typical roll wave pattern, and illustrates a key result found by previous authors – namely that roll waves coarsen: The simulation starts from an initial condition consisting of low-amplitude, rapidly varying perturbations about the uniform equilibrium state, $\phi = 0$. The instability grows and steepens into about eight non-identical roll waves. These waves propagate at different speeds, causing some of them to approach and collide. The colliding waves then merge into larger waves, a process that increases the length scale of the wave train. The collisions continually recur to create an inverse cascade that eventually leaves a pattern with the largest possible spatial scale, a single (periodic) roll wave. Coarsening has been observed in many different physical systems, and the dynamics seen in figure 2.16 seems, at first sight, like no exception.

Coarsening dynamics can be rationalized, in part, in terms of the subharmonic instability of trains of multiple roll waves. Specially engineered initial-value problems
Figure 2.16: Coarsening of roll waves predicted by the amplitude equation (2.55) for \( d = 20 \) and \( \mu = 0.05 \). The first panel shows \( \varphi(\xi, t) \) as a density on the \((t, \xi)\)-plane. The second and third panels show the amplitude measure \( \langle \varphi^2 \rangle \) (the spatial average of \( \varphi^2 \)) and final profile. The initial condition consisted of about eleven, low amplitude irregular oscillations.
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Figure 2.17: Coarsening dynamics as a subharmonic instability of steady roll-wave trains, for $\mu = 0.05$, and in a domain of length 10.

illustrate this notion quantitatively. For example, figure 2.17 shows the response of two periodic wave-trains to sub-harmonic perturbations: The two simulations begin with initial conditions dominated by wavenumber four and six, respectively, but also contain subharmonic wavenumbers with much lower amplitude. In each case, a train of waves appears that propagates steadily for a period. Somewhat later, the small subharmonic perturbations of the basic disturbance prompt collisions to trains with half the number of waves. Again, these trains persist for a period, but then final mergers occur to leave a single roll wave. We interpret the growth of subharmonic perturbations in the second experiment of figure 2.1 to be the analogue of this instability.

Despite the total coarsening evident in figures 2.16 and 2.17, we have also found that roll waves do not always complete an inverse cascade. Exploring a little, we find parameters for which periodic trains of multiple roll waves appear to be stable. Figure 2.18 shows such an example; a stable train of two roll waves emerges after a number
of mergers. Thus coarsening does not always continue to its final conclusion, but becomes interrupted at an intermediate scale. We have verified that this is also a feature of the St. Venant equations.

### 2.6.3 Linear stability of roll waves

The limitations of the coarsening dynamics can be better quantified with a linear stability analysis of periodic nonlinear roll-wave trains. Those steadily propagating solutions take the form \( \varphi = \Phi(\xi - ct) \), where \( c \) is the wave speed, and satisfy

\[
\frac{1}{2} (1 - 2d\xi) \left( \Phi - c \right)^2 \frac{\Phi}{\xi} + \Phi + \mu \Phi_{\xi\xi\xi} = 0 \tag{2.57}
\]

Auxiliary conditions on \( \Phi \) are periodicity, the choice of origin (equation (2.57) is translationally invariant) and the integral constraint,

\[
\int_0^L \Phi(s) ds = 0 \tag{2.58}
\]
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Figure 2.19: Top panel: Steadily propagating roll-wave solutions of the amplitude equation for $L = 4$ and $\mu = 0.04$ (dotted) and $\mu = 0$ (solid). The lower panel shows the real (solid) and imaginary (dashed) parts of an unstable eigenfunction with twice the spatial period as the basic roll wave. We use the integral of $\phi$ to display the eigenfunction because $\phi$ itself contains a delta function related to the movement of the shock for $\mu = 0$, or a large-amplitude spike for $\mu = 0.04$ which obscures the picture.

(which follows from the fact that this integral is a constant of motion for (2.55) and vanishes for the specified initial condition). This system can be solved numerically; a sample solution is shown in figure 2.19.

An analytical solution to (2.57) is possible in the inviscid case: After requiring regularity at the singular point, $\Phi = c$, we find

$$\Phi(\xi - c\tau) = A \exp \left( \frac{\xi - c\tau}{4} \right) + c - 2$$

where $A$ is a constant of integration. Because (2.59) is a monotonic function, a shock must be placed in the solution with jump condition,

$$c = \frac{\Phi_+ + \Phi_-}{2},$$

where subscripts denote the value of $\Phi$ upstream and downstream of the shock. After imposing the remaining auxiliary conditions, we find

$$A = 4/(e^{L/4} + 1) \quad \text{and} \quad c = 2 - \frac{16}{L} \tanh \left( \frac{L}{8} \right).$$

This solution is compared with the weakly viscous solution of (2.57) in figure 2.19.
To study the stability of the steady solutions, we introduce \( \varphi(\xi, \tau) = \Phi(\xi - c\tau) + \phi(\xi - c\tau)e^{\sigma \tau} \) into (2.55) and linearize in the perturbation amplitude \( \psi(\xi - c\tau) \):

\[
\sigma \psi - \frac{1}{2} \phi + [(\Phi - c) \phi]_\xi = \psi, \quad (2.62)
\]

\[
(2\partial_\xi - 1) \psi - \frac{1}{2} \phi - \mu \phi \xi \xi = 0, \quad (2.63)
\]

with \( \psi \) an auxiliary variable and \( \sigma \) the sought growth-rate. The solution proceeds by introducing another Bloch wavenumber, \( K \), to gauge stability with respect to perturbations with longer spatial scale than the steady wave train. Numerical computations then provide the growth rate, \( \text{Re}(\sigma) \), as a function of \( K \); an example eigenfunction of the weakly viscous solution shown in figure 2.19 is also displayed in a second panel of that picture.

In the inviscid problem, the stability theory is complicated by the shock, which, in general, shifts in space under any perturbation. The shifted shock contributes a delta function to the linear solution. We take this singular component into account using suitable jump conditions: Integrating (2.62) and (2.63) with \( \mu = 0 \) across the discontinuity, and allowing for an arbitrary delta-function of amplitude \( \Delta \) in \( \phi \), gives:

\[
\left( \sigma - \frac{1}{2} \right) \Delta + (\Phi - c) \phi = \psi \quad (2.64)
\]

\[
\psi^+ - \psi^- = \frac{1}{4} \Delta \quad (2.65)
\]

A boundary-layer analysis based on the weakly viscous stability problem provides exactly these relations, except as matching conditions across the boundary layer. The regularity condition, \( \psi = \sigma \phi \), must also be imposed at the singular point, \( \Phi = c \). Despite the lower-order of the linear stability problem, an analytical solution is not possible and we again solve the system numerically. Figure 2.19 once more compares inviscid and weakly viscous solutions.

Typical results for the dependence of the growth rate of the most unstable mode on wave spacing, \( L \), are shown in figure 2.20. Four values of the Bloch wave number are shown, corresponding to steady wave-trains with \( n = 1, 2, 3 \) and 4 waves, each a distance \( L \) apart, in a periodic domain of length, \( nL \). As we increase the wave spacing, there is a critical value beyond which periodic trains with multiple waves become stable. This stabilization of multi-wave trains applies to general values of \( K \) and \( \mu \), as illustrated by the neutral stability curves shown in figure 2.21. Thus, wave-trains with sufficiently
Figure 2.20: Linear stability results of roll-waves using the amplitude equation for \( \mu = 1 \) (top) and \( \mu = 0 \) (bottom). Growth rate is plotted against wave spacing \( (L) \) for perturbations having a Bloch wavenumber of \( K = 2\pi/nL \) (except for \( n = 1 \), where \( K = 0 \).)
Figure 2.21: Stability boundaries for nonlinear roll waves on the \((L, \mu)\)-plane. The first panel shows the stability curves for \(n = 2, 3\) and \(4\) (corresponding to roll-wave trains with \(n\) peaks in a periodic domain of size \(nL\)). The second panel shows the stability boundary for a single roll wave in much longer periodic domains.

Wide spacing become stable to subharmonic perturbations, removing any necessity for coarsening.

Figures 2.20 and 2.21 also illustrate that at yet larger wave spacing, a different instability appears which destabilizes a single roll wave in a periodic box \((n = 1)\). For these wavelengths, the nonlinear wave develops a long, flat tail resembling the unstable uniform flow. Hence, we interpret the large-\(L\) instability to result from perturbations growing on that plateau. We verify this character of the instability by solving the amplitude equation numerically, beginning from an initial condition close to the unstable nonlinear wave. Figure 2.22 illustrates how small disturbances grow and disrupt the original wave; eventually further peaks appear and four roll waves are present by the end of the computation, of which two are about to merge. Later still, the system converges to a steady train of three waves. In other words, trains with spacings that are too wide suffer wave-spawning instabilities that generate wavetrains with narrower separations.

The combination of the destabilization of trains of multiple waves at lower spacing and the wave-spawning instability at higher spacing provides a wavelength selection mechanism for nonlinear roll waves. We illustrate this selection mechanism further in figure 2.23, which shows the results of many initial value problems covering a range
Figure 2.22: A solution of the amplitude equation, beginning with an initial condition near an unstable roll wave. \((L = 62 \text{ and } \mu = 1)\). The dotted line in the final picture shows the initial condition.
of domain lengths, $d$. Each computation begins with a low-amplitude initial condition with relatively rapid and irregular spatial variation. The figure catalogues the final wave spacing and displays the range over which trains of a given wave separation are linearly stable. Also shown is the wavelength of the most unstable linear eigenmode of the uniform equilibrium, which typically outruns the other unstable modes to create a first nonlinear structure in the domain. At lower viscosities ($\mu$), the most unstable mode is too short to be stable, and the inception of the associated nonlinear wave is followed by coarsening until the wave separation falls into the stable range. As we raise $\mu$, however, the most unstable mode falls into the stable range, and the nonlinear wave-trains that appear first remain stable and show no coarsening. Thus, viscosity can arrest coarsening altogether.

### 2.7 Comparison with experiments

We verified the predictions of the long wave equation (2.57) using laboratory experiments. Water was poured down an channel, inclined to the horizontal at an angle of about $7^\circ$. The channel was 18 m long, 10 cm wide and 3 cm deep. At the end of the channel, water was collected and re-circulated using a small centrifugal pump and the flow rate was controlled using a valve. A flow rate of 20 liters/min was used which corresponds to a steady depth of 7 mm, speed of 65 cm/s and a Froude number of 2.5.

A video camera mounted above the channel recorded the propagation of these waves. Water was dyed red so that through the camera, deeper regions appeared darker. Thus color is a proxy for depth. By extracting columns of pixels from different frames of the recording, images similar to shown in figure 2.24 can be assembled. The figure shows the growth of small random perturbations to the water surface. Dark lines in the figure are crests, which are moving with a speed of 1 m/s. The speed of the waves from the weakly nonlinear theory is about the same. The wavelength that appears first is roughly 0.57 cm corresponding to a nondimensional wavenumber of about 0.63. If one is to believe that the random perturbations at the inlet do not have any preferred frequency, then one would expect the fastest growing mode to be observed downstream. Under this assumption the observed wavelength corresponds to a value of $\mu = 1$ from (2.57), whereas it suggests $\mu = 0.25$ from the St. Venant equations themselves. This
Figure 2.23: Final roll-wave spacings (crosses) in a suite of initial-value problems with varying domain size \( d \) and two values of \( \mu \). The shaded region shows where nonlinear wave trains are linearly stable. Also shown are the stability boundaries of the uniform flow (dashed line) and the fastest growing linear mode from that equilibrium (dotted line).
Figure 2.24: Roll waves appearing spontaneously on the flow on an incline. Color intensity shows perturbation from the mean, darker values representing deeper regions.

variance in the value of $\mu$ is expected as $F = 2.5$ may not be in the weakly nonlinear regime of the St. Venant equations, where (2.57) is valid.

Since in the analysis we have used periodic waveforms, more controlled experiments were performed to better correspond with the theory. Periodic waves were forced at the inlet of the channel at different frequencies. A small paddle attached to a pendulum carried out this forcing. The swinging of the pendulum caused the paddle to carve out periodic waveforms on the flow. The forcing frequency could be changed by changing the length of the pendulum. This allowed us to generate almost-periodic wavetrains.

To compare the steadily propagating roll wave profiles calculated from equation (2.57) with observed ones, we needed the instantaneous perturbation of the water surface at different locations in the channel. Measurement of the instantaneous height profile is extremely difficult (e.g. see [145]) and we resort instead to measuring a time series of the depth at a given location. The spatial dependence can then be inferred by assuming
Figure 2.25: Experimentally measured roll wave profiles (diamonds) compared with steadily propagating solutions of (2.57) for $\mu = 0.1$ (solid line) and $\mu = 0$ (dashed line). The flow rate was set to 25 liters/min and the average water depth was measured to be about 6 mm for an average speed of approximately 60 cm/s. This corresponds to a Froude number of 2.5. This profile is obtained from a time series measured at a distance of 6 m downstream of the wave-generating paddle.

That the waves are steadily propagating. The perturbation to the water surface was measured using a conductivity device similar to the ones used by Brock [126], which consists of a set of electrodes mounted above the water surface. As the water level rises, the electrode makes contact with the water surface and completes a circuit. By using 20 such electrodes at the same downstream location, but mounted at slightly different elevations compared to the mean, the instantaneous water level could be bracketed. The time series in figure 2.1 were obtained in this way.

The measured profile is compared with periodic solution of (2.57) in figure 2.25. The magnitudes and the shapes of the measured profiles seem to be predicted very well by (2.57). Another estimate for the parameter $\mu$ can be made using the "width" of the hydraulic jump as measured through the profile to be approximately 0.5. However, the measurement procedure used here is obtrusive to the flow, especially near sharp gradients, and hence the measurement of the width of hydraulic jump may not be accurate. Nevertheless, the measured profiles can be well approximated by using $\mu = 0 - 0.1$. All these estimates of $\mu$ suggest that the typical value is around 0 to 1.
Figure 2.26: Intensity data from an experiments with periodic inlet perturbations. The forcing frequency is 1.57 s\(^{-1}\), which corresponds to a wavelength of 0.68 m.
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We turn our attention towards experimental observation of the coarsening instability. The fate of the waves formed by periodically forcing them with different frequencies at the inlet recorded in the form of pictures similar to figure 2.24. Results for some sample forcing frequencies is shown in figure 2.26-2.28.

If the forcing frequency is sufficiently fast (e.g. figure 2.26), the distance between waves is short and coarsening instability disrupts the periodicity of the waves. As the wavelength is increased (e.g. figure 2.27), the periodic waves generated seem to be fairly robust. Small random perturbations do not disturb the flow very much, indicating stability for this wavelength. As the distance between the waves is increased further, as seen in figure 2.28, The periodic waves get disrupted again, but in this case via new waves spawning in between.

The results from all the forcing frequencies are summarized in figure 2.29. The critical ripple distance at which waves become stable to coarsening is much larger than the value predicted by the weakly nonlinear theory. The reason for the disagreement could be that the experimental flow is not in the weakly nonlinear regime any more. On the other hand, the transition to nucleation of new waves seems to be better predicted.

2.8 Discussion

In this article, we have investigated turbulent roll waves in flows down planes with topography. We combined numerical computations of both the linear and nonlinear problems with an asymptotic analysis in the vicinity of the onset of instability. The results paint a coherent picture of the roll-wave dynamics.

The addition of low-amplitude bottom topography tends to destabilize turbulent flows towards long-wave perturbations, depressing the stability boundary to smaller Froude number. At moderate topographic amplitudes, an eye of instability also appears at much smaller Froude number, a feature connected to the development of topographically induced hydraulic jumps in the background equilibrium flow (at least for the St. Venant model). At larger amplitudes, the topography appears to be stabilizing, and the onset of roll waves occurs at higher Froude numbers than expected for a flat bottom. This is consistent with observations of hydraulic engineers, who traditionally have exploited structure in the bed to eliminate roll waves in artificial water conduits, albeit
Figure 2.27: Intensity data for a forcing frequency of 0.86 s⁻¹ with a wavelength of 1.2 m.
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Figure 2.28: Intensity data for a forcing frequency of $0.2 \, s^{-1}$ and a wavelength of 5.9 m.

Figure 2.29: Experimental observations are compared with those predicted from weakly nonlinear theory (shaded region). The circles correspond to cases where coarsening was observed, the plus signs denote cases where periodicity was not disrupted, whereas the crosses denote observations of nucleation.
usually in the direction transverse to the flow [146, 151].

We have also found that the reduced model furnished by asymptotic theory re­
produces the nonlinear dynamics of roll waves. The model indicates that roll waves
proceed through an inverse cascade due to coarsening by wave mergers, as found previ­
ously [130, 144, 160]. This phenomenon was also observed in the experiments conducted
by Brock [126]. However, the cascade does not continue to the longest spatial scale, but
instead becomes interrupted over intermediate wavelengths. Moreover, wave-trains with
longer scale are unstable to wave-nucleation events. Thus, roll-wave trains emerge with
a range of selected spatial scales.

Although our results for low-amplitude topograpy are quite general, the discussion
of instabilities caused by the hydraulic jump has surrounded a sinusoidal topographic
profile and one may wonder how the results differ when the bed is more complicated. To
answer this question we have made further explorations of the linear stability problem
with a less regular form for $\zeta$. In particular, we have tested the linear stability of equi­
libria flowing over “random topography”. Here, $\zeta$ is constructed using a Fourier series
representation; the coefficients of the series are chosen randomly from a normal distri­
bution whose mean and standard deviations depend on the order of the Fourier mode.
In this way, the topography conforms to a specific spectral distribution, as sometimes
used in descriptions of the ocean’s floor [123]. An example is shown in figure 2.30, which
displays the realization of $\zeta$, an inviscid equilibrium solution on the $(\eta, H)$—phase plane,
and inviscid and weakly viscous stability boundaries on the $(F, \alpha)$—plane. The overall
conclusions are much the same as for the sinusoidal case: the inclusion of topography
lowers the stability boundary below $F = 2$, and there is a close association with the
formation of hydraulic jumps in the equilibrium.

We close by remarking on the application of our results. We have considered shallow­
water equations with drag and viscosity, focusing mostly on the St. Venant parameteri­
ization for turbulent flows and briefly on the Shkadov model for laminar flows. We
found that introduction of small, periodic, but otherwise arbitrary, topography desta­
bilizes turbulent roll waves but stabilizes the laminar ones. For both kind of flows, the
formation of a hydraulic jump in the equilibrium can further destabilize the flow (at
least near the $F_2$—curve, if not near $F = F_1$), a feature that may play a role in other
physical settings. For example, carefully fabricated periodic ribbing in the elastic wall
Figure 2.30: A computation with "random" topography: Panel (a) shows the realization of the topography and its derivative, constructed as follows: $\zeta$ is built from a Fourier series in which real and imaginary parts of the amplitude, $\zeta_n$, are drawn randomly from normal probability distributions with zero mean and standard deviation, $(n^2 + 16)^{-5/4}$, for $n = 1, 2, \ldots, 32$, and then a reality condition is imposed. In panel (b), we show the inviscid equilibrium for $a = 3$, $k_b = 10$ and $F = 1.9$, together with the organizing curves, $H^3 = F^2$ and $(1 - \zeta_x)H^3 = 1$; the solution is about to form a hydraulic jump (and is marked by a star in panel (c)). Panel (c) shows the (shaded) instability region on the $(a, F)$–plane for $\nu = 0$; to the right of this region, the periodic equilibria cease to exist, and weakly viscous solutions develop hydraulic jumps. Also indicated are the viscous stability boundaries for $\nu = 0.25$ and 0.5; viscous equilibria are unstable above this curve. The dashed lines show the corresponding stability boundaries predicted by asymptotics (with theory A for $\nu = 0.25$ and 0.5 and theory B for $\nu = 0$).
of a conduit may promote instability in the related physiological and engineering problems. In contrast, our results on the nonlinear dynamics of roll waves are more general; whatever the underlying physical setting, this model should equally apply.

2.9 References


Chapter 2. Dynamics of roll waves


Chapter 3

Flow induced elastic oscillations

3.1 Introduction

Steadily forced flows interacting with elastic structures can spontaneously induce time-periodic oscillations. A commonly observed instance of such oscillations is the fluttering of a flag [162, 190]. In the pulp and paper industry, such oscillations are important to thin-film coating and paper production processes [167, 197, 198]. The disastrous Tacoma Narrows bridge collapse in 1940 and many others are also thought to be due to aeroelastic oscillations excited by a strong wind [183]. The flutter of an airplane wing or any of its other parts is yet another example where fluid-structure interaction can have severe consequences [192]. A brief review of some of these and other examples from engineering can be found in the articles by Shubov [193, 194].

In this chapter we study the oscillations excited by a fluid flow through a narrow channel interacting with an elastic structure. An example demonstrating this phenomenon is shown in figure 3.1. A through-cut is made in a freshly set block of gelatin and air is passed through it. The gelatin block starts to vibrate. Similar vibrations are seen in physiological systems, where these oscillations manifest themselves as audible acoustic signals [177]. Perhaps the most commonly experienced example of such oscillations is speech. Air flowing through the vocal cords causes them to vibrate producing sound. The dynamics of this process is of interest to the physiological community as well as computer scientists interested in speech synthesis. Lumped parameter models, pioneered by Ishizaka & Flanagan [180], have become popular to describe speech generation, but more sophisticated one and two-dimensional models [179, 199] have also
been solved numerically to understand the phenomenon. Other notable examples are
the sounds made by blood flowing through partially open arteries. These sounds are
called Korotkoff sounds and are routinely used by physicians in the measurement of
blood pressure [165, 170, 173, 184]. One of the contending theories is that these sounds
generated are due to an instability of the steady flow.

Another motivation for studying flow induced elastic oscillations comes from what
geologists term as “volcanic tremor”. It is a sustained $\sim 1$ Hz seismic signal measured
near volcanic sites, sometimes lasting for as long as months. The signal itself is some­
times very harmonic and its spectrum has sharp peaks, although at other times it is
broadband and noisy. A clear explanation of this tremor remains elusive, although sev­
eral theories have been proposed [181]. One of these theories postulates that tremor is
cau sed by magma or magmatic fluids flowing through cracks in rocks. Lumped param­
ter models, similar in principle to those used for phonation, were employed by Julian
[181], even the validity of this mechanism as a candidate is questionable [163]. A more
careful analysis is needed to verify the feasibility of such models to explain volcanic
tremor.

There is also a considerable amount of interest in understanding the excitation mech­
anism of wind-driven musical instruments. Fluid-structure interaction is an important
factor for instruments involving reeds, e.g. clarinet, saxophone, etc. Understanding
their mechanism is crucial to computationally synthesizing realistic music. The current
state of research is a set of lumped parameter models [175, 176], in which a detailed
modeling of the fluid dynamics is missing. An analogous problem is the excitation of
acoustic modes in flutes and organ pipes by an air jet, referred to as air-reed instruments,
similar to the sound made by blowing over beverage bottles. The role of elasticity in
this problem is played by the compressibility of the resonating air column. In this case
as well, lumped parameter models to explain the excitation exist however an accurate
modelling of the jet from first principles is required [168, 174, 176].

These oscillations can be rationalized as a case of oscillatory instability of a steady
equilibrium flow. We investigate one such mechanism for a linear instability, the one that
excites the natural modes of elastic oscillations. In the absence of an externally driven
flow and any significant damping, an initial disturbance causes the elastic structure to
exhibit time-dependent oscillations. For example, when a tuning fork is struck, the
prongs of the fork start to vibrate. These oscillations eventually decay because energy is lost due to radiation of sound to the surroundings, viscosity of surrounding fluid and any damping present in the elastic medium. However, if the fluid is now forced to flow by an external agency, it can exert additional hydrodynamic forces on the elastic structure and provide an energy source to the elastic oscillations. This can cause the elastic oscillations to grow, constituting an instability mechanism.

This mechanism is central to the lumped-parameter analysis used for modelling phonation and musical instruments, and though questionable, a promising candidate for explaining volcanic tremor. However, certain assumptions about the flow or the elastic structure had to be made ad hoc in the lumped-parameter models. Moreover, a lot of detail was used in their construction to achieve quantitative accuracy [174, 176, 180, 181]. This obfuscated the underlying physical mechanism for exciting the oscillations. The equations had to be numerically analyzed to reveal the oscillatory instability of the steady state and that left the underlying mechanism unclear to intuition.

Motivated by these shortcomings, we present an account of the fluid and solid mechanics from first principles with the objective of isolating and demonstrating the underlying instability mechanism. The mechanism involving lumped parameter models alluded to in figure 1.2 was only uncovered to us as a result of the present analysis. This mechanism provides a unified approach to explaining the elastic oscillations seen in the various examples. As a specific example for demonstrating the instability and the accompanying analysis, we consider a fluid flowing through a channel of finite length, with the channel walls made up of a block of rectangular elastic material (the details are provided in §3.2). This conceptual setup is motivated by and similar to the experiments with vibrations of the gelatin block depicted in figure 3.1. The elastic deformation is modelled by a Hookean elastic law, while the Navier-Stokes equations govern the fluid flow. Thus, this model is qualitatively and quantitatively faithful, albeit more complicated to analyse than lumped parameter models.

Of course, our aspiration of uncovering the instability mechanism analytically is not possible for the problem in its full generality. We have to appeal to certain features of the setup that simplify the mathematics and allow us to make progress. The most important assumption we make is that the channel is long and narrow. This allows us to exploit certain models which are rigorously derived as approximations of the Navier-
Figure 3.1: Details of the experiment on elastic oscillations in a gelatin block. A schematic setup of the experiment involving tremor of a gelatin block is shown in the upper panel. The base of the block is 9" x 9" and it was 3" high. Compressed air is forced from the bottom to top through a knife-cut in the block (dimensions 2" perpendicular to the plane of the paper in the top panel). As a critical flow rate is exceeded, the block starts to vibrate at a frequency of about 70 Hz. The microphone located over the block records the sound generated by these vibrations. The amplitude of the signal recorded is plotted in the lower panel as a function of the air flow rate.
Stokes equations for the flows of thin films. This derivation is briefly outlined in §3.3. As a result of these simplifications, the channel can be treated as one-dimensional with the only unknown flow quantities being the local channel width and the flow rate.

A second considerable simplification comes from the assumption that the elastic structure is stiff as compared to the stresses in the fluid. This assumption renders the hydrodynamic forces weak in comparison to the elastic stresses. As a result, the dominant motion of the elastic structure is decoupled from the flow and can be explained in terms of its natural modes of oscillations. This description, to leading order is true irrespective of the precise details of the elastic structure. Be it an elastic beam, a stretched membrane or an extended elastic body, it possesses a set of elastic modes which determines its dynamics. As shown in §3.4, an appropriately constructed asymptotic expansion then allows us to study the action of hydrodynamic forces on the normal modes of a the elastic structure. These two assumptions allow a unified treatment for all the previously mentioned examples and many more.

When the elastic structure is not very stiff, its motion is coupled with the flow. Such a situation can not be studied analytically in general. Insight can still be gained through a computational solution with a simpler elastic structure. In §3.5, we explore such a solution for the special case of the channel walls being formed by a stretched membrane.

Finally, we exploit the analogy between the elastic and acoustic oscillations to develop a theory for the latter. The excitation of acoustic oscillations has long been attributed to the sinuous instability of an inviscid jet. This mechanism is reviewed in detail in §3.6 and a simple experiment is devised to show that further investigation into the mechanism is required. The theory we develop is crude owing to the lack of a rigorous but simple model for the jet, unlike the model for thin films. The other feature that obscures the analogy is the absence of a clearly defined interface between the air jet and the air in the resonant cavity. Nevertheless, an ad hoc model is proposed based on the similarity with the elastic model, that acts as a proof of concept of the analogy and serves as a stepping stone to further experimental and theoretical analysis.

With this picture in mind, we start with the mathematical formulation and non-dimensionalization of the governing equations.
Chapter 3. Flow induced elastic oscillations

3.2 Mathematical formulation and non-dimensionalization

A fluid of density $\rho$ and kinematic viscosity $\nu$ is flowing through a channel of average width $2H$ and length $L$ (see figure 3.2). The flow is represented by the fluid velocity, $u = (u, w)$ and a dynamic pressure $p$. The channel wall is located by the function $z = h(x, t)$ and it separates the fluid from an linear elastic material. The displacement in this elastic medium, denoted by $\xi = (\xi, \eta)$, is represented in a Lagrangian frame using the $(X, Z)$-coordinate system. The gradient operator acting on these displacements is denoted with a subscript 'X'.

The displacement field is governed by the momentum balance law,

$$
\rho_s \xi_{tt} = \nabla_X \cdot \tau_e + \nabla_X \cdot \tau_v, \tag{3.1}
$$

where $\rho_s$ is the density of the solid, $\tau_e$ is an elastic stress and $\tau_v$ is a viscous stress. The elastic stress is given by the Hookean law

$$
\tau_e = \Lambda_e I (\nabla_X \cdot \xi) + \mu_e (\nabla_X \xi + \nabla_X \xi^T), \tag{3.2}
$$

Here $\Lambda_e$ and $\mu_e$ are the Lamé constants for the elastic material. They are related to the Young’s modulus and the Poisson ratio as

$$
Poisson \text{ ratio} = \frac{\Lambda_e}{2(\Lambda_e + \mu_e)}, \quad \text{Young’s modulus} = \frac{\mu_e(3\Lambda_e + 2\mu_e)}{\Lambda_e + \mu_e}. \tag{3.3}
$$
The viscous stress is given by the Newtonian constitutive relation

$$\tau_v = \Lambda_v I (\nabla \cdot \xi_t) + \mu_v d (\nabla \times \xi_t + \nabla \times \xi_t^T),$$  \hspace{1cm} (3.4)

where $\Lambda_v$ and $\mu_v$ are the coefficients of bulk and shear viscosities respectively. The total stress will be denoted by $\tau = \tau_v + \tau_e$.

The fluid in the channel is governed by the mass and momentum conservation equations

$$u_t + uu_x + wz + \frac{p_z}{\rho} = \nu \nabla^2 u, \hspace{1cm} (3.5)$$

$$w_t + uw_x + wz + \frac{p_z}{\rho} = \nu \nabla^2 w, \hspace{1cm} (3.6)$$

$$u_x + w_z = 0. \hspace{1cm} (3.7)$$

These equations are accompanied by a set of boundary conditions and interface matching conditions. On the boundary of the solid we either have a no-displacement condition ($\xi = \eta = 0$) or the stress-free condition ($\tau \cdot n = 0$, where $n$ is the normal to the boundary). At the channel inlet, the fluid velocity specified. The fluid exit boundary condition is specified later in this chapter. The fluid and solid satisfy continuity of stress and velocity at the interface, which behaves as a material boundary; i.e.,

$$\xi_t = u, \hspace{1cm} (3.8)$$

$$\tau_{xz} = \frac{1}{\sqrt{1 + h_x^2}} (-p + 2\rho u w_z - \rho u h_x (u_x + w_z)) \hspace{1cm} (3.9)$$

$$\tau_{xz} = \frac{1}{\sqrt{1 + h_x^2}} ((p - 2\rho u w_z) h_x + \rho u (u_x + w_z)) \hspace{1cm} (3.10)$$

where the elastic displacements are evaluated at $(X,0)$ and the fluid velocities and pressures at the Eulerian counterpart $(x = X + \xi, z = h(x))$.

For the fluid equations, we are heading towards a thin film approximation with $H \ll L$. We exploit the contrast in the length scales in $x$ and $z$ by rescaling

$$z \rightarrow zH, \hspace{1cm} x \rightarrow xL, \hspace{1cm} u \rightarrow uU, \hspace{1cm} w \rightarrow \frac{H w U}{L}, \hspace{1cm} p \rightarrow \frac{\rho u U L}{H^2},$$  \hspace{1cm} (3.11)
scales does not exist \textit{a priori} and we non-dimensionalize its governing equations using

\[ \xi \rightarrow \xi H, \quad t \rightarrow tL\sqrt{\frac{\rho_s}{\mu_e}}, \quad (x, z) \rightarrow (x, z) L, \quad \tau_e \rightarrow \tau_e \frac{\mu_e H}{L}, \quad \tau_v = \frac{\mu_e H}{L} \tau_v. \]

This leads to seven dimensionless parameters; viz.

\[ \epsilon = \frac{H}{L}, \quad c^2 = \frac{\mu_e}{U^2 \rho_s}, \quad \lambda_e = \frac{\lambda_e}{\mu_e}, \quad R = \frac{U H}{\nu}, \quad \delta = \frac{\rho \nu U L^2}{\mu_e H^3}, \]

\[ \lambda_v = \frac{\lambda_v}{\mu_e L} \sqrt{\frac{\mu_e}{\rho_s}} \quad \text{and} \quad \mu_v = \frac{\mu_v d}{\mu_e L} \sqrt{\frac{\mu_e}{\rho_s}}. \]

The parameter \( \epsilon \) can be identified as an aspect ratio, \( c \) is a non-dimensional elastic wave speed, \( \lambda_e \) is a ratio of the two Lamé constants, \( R \) is the Reynolds number, \( \delta \) compares the elastic stiffness to viscous stresses and \( \lambda_v \) and \( \mu_v \) are the non-dimensional viscosities. They appear in the governing equations as

\[ \epsilon R (u_t + \epsilon u_x + \epsilon w_z) + p_x = u_{zz} + \epsilon^2 u_{xx} \]

\[ \epsilon^3 R (u_t + \epsilon u_x + \epsilon w_z) + p_x = \epsilon^2 w_{zz} + \epsilon^4 w_{xx} \]

\[ u_z + w_z = 0 \]

\[ \xi_{tt} = \nabla_x \cdot \tau_e + \nabla_x \cdot \tau_v \]

and in the interface boundary conditions as

\[ u = \epsilon c \xi_t, \quad w = c n_t, \]

\[ \tau_{zz} = \frac{\delta}{\sqrt{1 + \epsilon^2 h_z^2}} \left( -p + 2 \epsilon^2 u_z - \epsilon^2 h_x (u_z + w_z) \right) \]

\[ \tau_{xz} = \frac{\delta \epsilon}{\sqrt{1 + \epsilon^2 h_z^2}} \left( (p - 2 \epsilon^2 u_x) h_z + u_z + w_z \right). \]

All this is, of course, accompanied by the dimensionless versions of the homogeneous elastic conditions on the remaining boundaries, which introduces the aspect ratio of the elastic block as yet another dimensionless parameter, and the fluid inlet and exit conditions.

### 3.3 An averaged model

The two-dimensional Navier-Stokes equations (3.15-3.17) are cumbersome to solve. In any case, the solution can only be obtained numerically, which is a pathway we would like
to avoid. Relief comes from the fact that not all the terms in these equations are equally important. For our system of interest, some of them are negligible in magnitude and secondary in significance. A look at some typical values of the dimensionless parameters sheds some light into the relative magnitudes of various terms. Treating the gelatin-block experiments as a benchmark, the values of the parameters from the experiments can be considered representative of the situations that exhibit such instabilities.

The channel in the experiments is about 1 mm wide and 10 cm long. The density of gelatin is about the same as that of water, 1 gm/cc, and its Lamé constants are \( \mu_e \sim 2 \times 10^4 \) Pa and \( \Lambda_e \sim 10^9 \) Pa. Simple observations of the decay rate of natural oscillations of the gelatin block reveal a time scale of about one second, which helps to estimate the values of the viscous damping coefficients for the elastic material. The typical air speeds required for instability are about 30 cm/s. This set corresponds to the values of the dimensionless parameters \( \epsilon \sim 0.01, R \sim 15, \lambda_c \sim 10, \delta \sim 5 \times 10^4, \sigma \sim 6 \times 10^{-3}, \mu_v = 0.03 \) (the large value of \( \lambda_c \) makes the gelatin block almost incompressible and hence the value of \( \lambda_e \) is irrelevant).

Now we focus on the particular limit that the channel aspect ratio is narrow, i.e. \( \epsilon \rightarrow 0 \). In the limit of an \( R \sim \mathcal{O}(1) \), this limit gives the popular lubrication theory approximation, where inertia is negligible. We, however, scale \( R \) such that \( \epsilon R \sim \mathcal{O}(1) \), thus making some of the inertial effects important. In fact, as we will see later, these very inertial terms will be responsible for the instability and should not be ignored.

In this limit, the fluid equations take the simple form

\[
\begin{align*}
  p_z &= 0, \quad (3.22) \\
  u_x + w_z &= 0, \quad (3.23) \\
  \epsilon R(cu_x + wu_x + wu_z) + p_x &= u_{zz}, \quad (3.24)
\end{align*}
\]

with the interface conditions

\[
\begin{align*}
  h &= \eta, \quad u = 0, \quad w = c\eta_t, \quad (3.25) \\
  \tau_{xz} &= -\delta p \quad \text{and} \\
  \tau_{xx} &= 0. \quad (3.26)
\end{align*}
\]

Because the displacements in the elastic material are small (they are caused by the changes in \( h \) which are small), both the fluid and elastic variables are evaluated at the
same point in these conditions and the distinction between the Eulerian and Lagrangian frames is lost.

In order to bring these equations into an even more manageable form, we use an averaging technique used in the literature for the flow of thin films of fluids [187]. It involves assuming a polynomial structure for $u$ in $z$. For example, by using a parabolic profile for $u$

$$u = \frac{3}{2} \bar{u}(x, t) \left(1 - \frac{z^2}{h^2}\right),$$

(3.28)

and integrating (3.24) and (3.23) in $z$ from $-h$ to $h$ to eliminate any $z$-dependence yields

$$R_{eq} \left( c \partial_t + \alpha \frac{q}{h} q_x - \beta \frac{q^2}{h^2} h_x \right) = -h p_x - 3 \frac{q}{h^2},$$

(3.29)

$$c h_t + q_x = 0$$

(3.30)

where $R_{eq} = e^R/r$ is a rescaled Reynolds number, $q = \int_0^h u(0, z) \, dz$ is the volume half-flux and $\alpha = 12/5$, $\beta = 6/5$, and $r = 1$ are constants. The above equation, sometimes called the Shkadov equation [191], models thin film flows over inclined planes [166, 178] qualitatively well in spite of the ad hoc nature of the assumption. This assumption is accurate when $eR \ll 1$, as can be demonstrated by a lubrication theory analysis, but fails to be quantitatively successful when $eR \sim 1$. A better profile assumption can be arrived at guided by a long-wave expansion, which prompts that the quadratic profile be replaced by a sixth degree polynomial in $z$ [188, 189],

$$u(x, z, t) = \sum_{j=0}^{4} a_j(x, t) \left\{ \left(1 - \frac{z}{h}\right)^{j+1} - \frac{j+1}{j+2} \left(1 - \frac{z}{h}\right)^{j+2} \right\},$$

(3.31)

with the $a_j$ being arbitrary functions. Again eliminating $z$ dependence by performing various averages, the details of which can be found elsewhere [187, 188, 189], leads to (3.29) and (3.30) but with different values of the constants. Here we get $\alpha = 17/7$, $\beta = 9/7$, and $r = 5/6$ using (3.31).

This reduction has converted the influence of the fluid in the channel effectively into a time-dependent boundary condition for the elastic medium. In this process, as (3.29) and (3.30) are differential equations for $q$ and $p$, we will need some boundary conditions on them. These boundary conditions have to come from the specification of fluid velocities or pressures at the inlet/exit. We cannot guarantee that the inlet profile will be of the form given by (3.31). However, we hope that because of the
small channel aspect ratio the velocity profile quickly develops into one which can be well approximated by (3.31). Thus the specified inlet velocity can be converted to an equivalent inlet flow rate $q_{in}$. The volume flux coming in to the channel could be determined by the flow upstream of the channel and that will correspond to a fixed flux inlet condition, $q(x = -1/2) = 1$, since we have non-dimensionalized the variables using this flux.

If the flux is specified at the inlet of the channel, it can not be specified at the exit. The option there is to have a condition on the pressure. The fluid has some momentum when it comes out of the channel and can gain some pressure as it reaches a stagnant state. In fact, the most appropriate thing to do is to solve for the flow of the fluid outside of the channel, assuming that the channel exit acts like a mass and momentum source, and match the solution inside the channel with the outside at the exit. Since it is impossible to solve for every conceivable flow outside, we resort to a Bernoulli-like condition that still hopes to captures the essence of the physics,

$$p + R_{eq} \frac{\gamma q^2}{2h^2} = 0 \quad \text{at} \quad x = \frac{1}{2}. \quad (3.32)$$

Here $\gamma$ is a parameter that models the flow outside the channel. The factor of $R_{eq}$ comes in because the origin of the $q^2/h^2$ term is thought to be inertial, and pressure is non-dimensionalized using viscosity. In order to understand the possible values of $\gamma$, let us look at a conceptual model for the flow outside the channel shown in figure 3.3. In this model, the channel is extended beyond the exit. Its width is time independent.
but increases as given by \( h(x,t) = H_{\text{out}}(x) \). Then we can apply (3.29)-(3.30) to this situation and get \( q_x = 0 \) and

\[
p_{\text{out}} + R \left[ q_t \int_x^\infty \frac{1}{H_{\text{out}}(s)} \, ds + \beta \frac{q^2}{2h^2} \right] + 3q \int_x^\infty \frac{1}{H_{\text{out}}^3(s)} \, ds = 0 \quad \text{at } x = \frac{1}{2}. \tag{3.33}
\]

If we ignore the acceleration term (the one proportional to \( q_t \)) and the viscous drag (proportional to \( 3q \)) then the remaining boundary condition is equivalent to (3.32) with \( \gamma = \beta \). Other values of \( \gamma \) are also possible depending on the precise flow situation.

### 3.4 Asymptotic analysis for \( \delta \ll 1 \)

We move on to exploiting the second assumption for simplifying the analysis, that of a stiff elastic structure. In the example of the elastic block, this assumption is reflected in the parameter \( \delta \) being small. The motivation for studying this limit comes from the fact that numerically \( \delta = 0.006 \) in the gelatin experiments. Moreover, making this assumption helps in analytically continuing the solution. The oscillatory instability can be understood as destabilization of elastic modes of vibration and this assumption is an analytical tool to bring out this interpretation. Along with a stiff elastic structure, the viscous damping in the structure is assumed to be small to explicitly illustrate the competing effect of the hydrodynamic forces overcoming the viscous damping. In particular, we assume

\[
\mu_v = \delta \mu_{v1}, \quad \text{and} \quad \lambda_v = \delta \lambda_{v1}. \tag{3.34}
\]

#### 3.4.1 Linear stability analysis

In order to assess the possibility of oscillations, we perform a linear stability analysis of a steady flow given by \( q = h = 1 \) and \( p_x = -3 \) through the channel. Admittedly, this flow generates a pressure field which deforms the elastic material, rendering the channel non-uniform. However, we ignore this equilibrium deformation of the channel for analytical convenience and assume the steady \( \xi = 0 \). Such an assumption has been previously made in the literature [171] and been termed as the "equilibrium fiddle" [163]. The error resulting from this assumption is anyway of \( O(\delta) \), as that is the strength of the coupling between the fluid pressure and elastic stresses. Moreover, in the undeformed state the channel width is assumed to be uniform. Relaying this assumption may make
a difference to the stability characteristics, but we have ignored those for simplicity of analysis. As such, the channel need not be straight and can be weakly curved and the following analysis would still be valid.

To determine the stability of this equilibrium, we substitute

\[
\xi = 0 + \xi e^{i\omega t} \tag{3.35}
\]

\[
(p, q, h) = (-3x, 1, 1) + (\bar{p}, \bar{q}, \bar{h}) e^{i\omega t} \tag{3.36}
\]

into the governing equations and retain the linear terms to obtain the following eigenvalue problem (tildes dropped) for the complex frequency \( \omega \)

\[
\omega^2 \xi + \nabla_x \cdot \tau_e + \nabla_x \cdot \tau_v = 0, \tag{3.37}
\]

with the interface conditions

\[
\tau_{zz} = 0, \tag{3.38}
\]

\[
\tau_{zz} = -\delta p, \tag{3.39}
\]

\[
i\omega c h + q_z = 0, \tag{3.40}
\]

\[
R_{cq} (i\omega c q + \alpha q_x - \beta h_x) + p_x = -3q + 9h \tag{3.41}
\]

and suitable conditions on the remaining boundaries of the elastic body and channel inlet and exit.

Next we proceed to present solution of this eigenvalue problem via a perturbation expansion in \( \delta \). Let us expand

\[
\omega = \omega_0 + \delta \omega_1 + \cdots, \tag{3.42}
\]

\[
\xi = \xi_0 + \delta \xi_1 + \cdots, \tag{3.43}
\]

\[
\tau_e = \tau_{e0} + \delta \tau_{e1} + \cdots, \tag{3.44}
\]

\[
\tau_v = \delta \tau_{v1} + \cdots, \tag{3.45}
\]

\[
(p, q, h) = (p_0, q_0, h_0) + \delta (p_1, q_1, h_1) + \cdots, \tag{3.46}
\]

where

\[
\tau_{ej} = \lambda_e \nabla_x \cdot \xi_j I + (\nabla_x \xi_j + \nabla_x \xi_j^T), \text{ for } j = 1, 2, \ldots \tag{3.47}
\]

\[
\tau_{v1} = i\omega_0 \left( \lambda_{v1} \nabla_x \cdot \xi_0 I + \mu_{v1} (\nabla_x \xi_0 + \nabla_x \xi_0^T) \right). \tag{3.48}
\]
Figure 3.4: First four modes of elastic vibrations of an elastic block. The calculations were motivated by a rectangular block of gelatin with aspect ratio 0.5 and $\lambda_e = \infty$. The displacements are forced to be zero on the top and the left boundaries, while the right and bottom boundaries are stress free. The frequencies of the modes are 4.24 (top left), 6.67 (top right), 7.46 (bottom left) and 9.41 (bottom right).
Substituting these in (3.37)-(3.41) and collecting like powers of \( \delta \) yields to the leading order

\[
\omega_0^2 \epsilon_0 + \nabla_x \cdot \tau_e = 0, \tag{3.49}
\]

with stress free conditions \( \tau_{e0zz} = \tau_{e0zz} = 0 \) on the interface, \( Z = 0 \). This, along with the conditions on the other boundaries, gives the free oscillations of the elastic body. Let us focus on a particular mode with \( \omega_0 \) being its frequency and \( \epsilon_0 \) the normal-mode displacement corresponding to that frequency. Figure (3.4) shows some of the modes of vibrations of a two-dimensional rectangular block calculated using FreeFEM++, a partial differential equation solver using finite elements. Given the non-dissipative properties of the elastic constitutive law, \( \omega_0 \) and \( \epsilon_0 \) are bound to be real. This motion of the elastic solid causes the channel to open and close as governed by the shape of the normal mode and thus causes the fluid flow to be perturbed. This induces a pressure field in the fluid, which is really the agency through which the alterations in channel width are converted to fluid velocities. This perturbed pressure field in turn exerts a force on the elastic body, which may cause its energy to increase with time, indicating an instability, or decrease, denoting stability.

The perturbation analysis proceeds on exactly the same lines. Assuming that the normal mode can be computed for the specific geometric and material parameters of the elastic solid, the shape of the perturbed interface \( h_0 \) is known to the leading order. We can then use the leading orders of (3.40) and (3.41) to compute the flux and pressure fields. To relate the pressure field to the displacement, we define \( \phi \) and \( \psi \) to be the following integrals of \( h_0 \):

\[
\phi(x) = \int_{-\frac{1}{2}}^{x} h_0(s) \, ds \quad \text{and} \quad \psi(x) = \int_{\frac{1}{2}}^{x} \phi(s) \, ds. \tag{3.50}
\]

Notice that since \( h_0 \) is real, so are \( \phi \) and \( \psi \). The flux is related to the channel-width through (3.40), which gives

\[
q_0(x) = -i\omega_0 c \phi(x), \tag{3.51}
\]

where we have used the fixed flux condition at the inlet. As a matter of notation, subscript "in" refers to the inlet \( (x = -1/2) \) and "out" to refer to the exit \( (x = 1/2) \).
Finally, substituting $h_0$ and $q_0$ in (3.41) yields

$$p_0(x) = p_{\text{out}} + \{i\omega_0 c(3 + \omega_0 c R_{eq})\psi + (i\omega_0 c R_{eq} + 9)(\phi - \phi_{\text{out}}) + \beta R_{eq}(h_0 - h_{\text{out}}) + (3 + R_{eq} i\omega_0) i\omega_0 \psi\},$$

(3.52)

where $p_{\text{out}}$ is the exit pressure of the fluid, used as an arbitrary constant of integration. Using the Bernoulli-like condition, it can be evaluated to $p_{\text{out}} = R_{eq} \gamma (h_{\text{out}} + i\omega_0 c \phi_{\text{out}})$.

The frequency correction can be found at the next order of $\delta$ and its imaginary part will indicate the stability of the steady state. At $O(\delta)$, we have

$$\omega_0^2 \xi_1 + \nabla_x \cdot \tau_{e1} + \nabla_x \cdot \tau_{v1} = -2\omega_0 \omega_1 \xi_0,$$

(3.53)

with

$$\tau_{e122} = -\tau_{e122} \quad \text{and} \quad \tau_{e122} = -p_0 - \tau_{e122} \quad \text{at} \quad Z = 0$$

(3.54)

and the homogeneous conditions at the other boundaries. The correction in frequency, $\omega_1$, is the object of our interest here, which can be found by taking a dot product of (3.53) with $\xi_0$ and integrating over the whole elastic domain (denoted by $V$). The left hand side of this product can be simplified by multiple applications of divergence theorem to give

$$-2\omega_0 \omega_1 \langle|\xi_0|^2 \rangle = \langle \xi_0 \cdot (\omega_0^2 \xi_1 + \nabla_x \cdot \tau_{e1} + \nabla_x \cdot \tau_{v1}) \rangle$$

(3.55)

$$= \langle \nabla_0 h_0 - i\omega_0 \langle \lambda_{v1}(\nabla_x \cdot \xi_0)^2 + \mu_{v1} D_0 : D_0 \rangle \rangle$$

(3.56)

where

$$\bar{f}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f dx \quad \text{and}$$

(3.57)

$$\langle f(X, Z) \rangle = \int_V f dV,$$

(3.58)

$$D_0 = \nabla_x \xi_0 + \nabla_x \xi_0^T.$$

(3.59)

Only the imaginary part of $\omega_1$ contributes to the instability, the real part merely perturbs the frequency of free oscillations. The real part of the linear growth rate can then be written as

$$\Re(\omega) = \frac{\delta}{2\omega_0 \langle|\xi_0|^2 \rangle} \overline{3}(\bar{h}_0) \bar{h}_0 - \frac{\delta}{2} \frac{\langle \lambda_{v1}(\nabla_x \cdot \xi_0)^2 + \mu_{v1} D_0 : D_0 \rangle \rangle}{\langle|\xi_0|^2 \rangle} + \cdots,$$

(3.60)
which, if positive, means instability. Thus the whole analysis boils down to the sign of the integral $\Re(p_0)\delta$, which depends on the fluid inlet and exit boundary conditions. This integral can be interpreted as the work done by the fluid pressure on the elastic boundary, alluded in the mechanism mentioned earlier.

Substituting (3.52) in (3.60) and performing some integrations by parts, we get

$$
\Re(\omega) = \frac{\delta}{2 \langle |\xi_0|^2 \rangle} \left( k_0^{-2} R_{eq} (\gamma - \frac{\alpha}{2}) - 3 \delta^2 \right) - \frac{\delta}{2} \frac{\lambda u_1 (\nabla \cdot \xi_0)^2 + \mu u_1 D_0 : D_0}{\langle |\xi_0|^2 \rangle} + \ldots
$$

(3.61)

The first term on the right hand side of (3.61) represents the contribution of hydrodynamic forces, while the second term arises because of the viscous damping in the elastic material. In the absence of any fluid, the second term is responsible for decaying of the elastic normal modes.

Even if the fluid in the channel is flowing, the steady solution is stable at $R_{eq} = R = 0$. The growth rate has a linear dependence on $R$. If the exit pressure is assumed to be fixed ($\gamma = 0$), then the coefficient of this linear dependence is negative and the growth rate becomes more negative with an increase in $R$. However, if $\gamma > \alpha/2$, this coefficient can be positive for a sufficiently large value of $R$. The critical value of $R$ in this case is given by

$$
(R_{eq})_{crit} = 6 \frac{\delta^2}{k_0^{-2} (2\gamma - \alpha)} + 2 \frac{\lambda u_1 (\nabla \cdot \xi_0)^2 + \mu u_1 D_0 : D_0}{k_0^{-2} (2\gamma - \alpha)}.
$$

(3.62)

The dependence of this instability on the nature of the boundary condition can be crudely rationalized as follows. The Bernoulli-like condition at the exit relates the pressure to the flow velocity there. In particular, an increase in the flow speed at the exit corresponds to a decrease in the pressure there. If at a given instant the channel volume is expanding (contracting), then the exit flux will be less (more) than the fixed value of the inlet flux due to conservation of volume. That leads to an increased (decreased) pressure at the exit, which contributes to an increase (decrease) in the pressure throughout the channel by some amount. If this increase (decrease) in pressure overcomes the pressure change due to an decreased (increased) wall drag and acceleration then it further helps in pushing the channel open (close). This does positive work on the vibrating structure and thus increases the vibrational mechanical energy and assists the instability. Thus we identify this instability to be due to the pressure-velocity dependence at the channel exit through the Bernoulli-like boundary condition.
Figure 3.5: Critical $R_{eq}$ for modes of vibrations of the elastic block in figure 3.4 as described by (3.63). Top panel corresponds to $\mu_1 = 0$ with a lowest $R_{eq}$ of 4.1 for a frequency of 22.69 and bottom panel corresponds to $\mu_1 = 10^{-4}$ giving a lowest $R_{eq}$ of 12.46 for a frequency of 14.82.

3.4.2 Mode selection

The question we are trying to answer is which modes will be observed in an experiment. Such a question is very difficult to answer in general and, at the least, will require the effect of nonlinearities to be included. We try to answer a simplified version of the question of which mode will be observed as we gradually increase the flow rate in the channel from zero.

For each of these modes, (3.62) gives a critical Reynolds number for instability. Since the Reynolds number is proportional to the flow rate, the mode with the lowest critical Reynolds number will become unstable for the smallest flow rate. Modes vibrating at higher frequencies typically have a finer spatial structure associated with them. The
lowest critical Reynolds number does not necessarily belong to the fundamental mode. In fact, qualitative dependence of the critical Reynolds number on the mode structure can be predicted from (3.62) by looking at the contribution of various physical processes.

The first term in (3.62), which represents the contribution from hydrodynamic forces, is proportional \( \phi^2 \). Comparing the modes with 25 lowest frequencies, the 21st mode with a frequency of 22.69 has the lowest contribution to this term. A simple qualitative analysis using sinusoidal functions for \( h_0 \) reveals that even lower values of this term are possible for modes with finer spatial scales. The cause can be traced to the fact that contribution from hydrodynamic forces is proportional to the integrals of \( h_0 \), which is related to the displacement fields of the mode. Thus, hydrodynamic forces seem to favour high frequency modes to be destabilized at lower Reynolds numbers.

Exactly the opposite behaviour is seen for the second term, which represents the effect of viscous damping, in (3.62). This term is expected to grow with finer spatial scales as derivatives of functions are involved. Moreover, its contribution is positive definite. For the case of the gelatin, the incompressibility condition can be used to reduce (3.62) to

\[
(R_{eq})_{\text{crit}} = 6 \frac{\phi^2}{h_0 (2\gamma - \alpha)} + 2 \frac{\mu_{v1}\omega^2}{h_0} \frac{\langle |\xi_0|^2 \rangle}{(2\gamma - \alpha)},
\]

(3.63)

in which the contribution from viscous damping can be seen to increase quadratically with the vibration frequency. Thus, it can be concluded that viscous damping in the elastic structure inhibits the excitation of high-frequency modes.

As a result of the opposing influence from the hydrodynamic forces and viscous damping forces, the mode with the lowest Reynolds number depends on the viscous damping coefficients \( \mu_{v1} \) and \( \lambda_{v1} \). If these coefficients are too small, then high frequency modes are favoured. On the other hand, if these coefficients are large then they inhibit modes with finer spatial structure and shift the favour towards the fundamental mode. Results from (3.63) using two representative values of \( \mu_{v1} \) shown in figure 3.5 brings out this behaviour.
3.5 Vibration of a membrane

The asymptotic analysis is quite general. Based solely on the shape of the normal modes, the result of their interaction with the flow can be determined. However, it is limited to the asymptotic regime $\delta \ll 1$, $R_{eq} \sim O(1)$. As the flow speed becomes faster, viscous forces become less and less important as compared to inertial ones and an inertial scaling for pressure needs to be used. Thus a different asymptotic regime is reached when $\delta \ll 1$ but $\delta R_{eq} \sim O(1)$. The fluid flow can no longer be neglected to leading order and the analysis can not be carried out in its full generality. Hence we focus our attention on flow between stretched membranes. This special case serves not only to demonstrate the general applicability of the analysis in the former asymptotic regime but also allows us to explore beyond it and into the latter one. The elasticity equation for a stretched membrane (in the limit of small displacements) can be written as

$$h_{tt} = h_{xx} + \delta p,$$  \hspace{1cm} (3.64)

where $c^2 = TH^2/\rho m Q^2$ is the dimensionless wave speed, $\delta = \rho \nu Q / TH c^3$, $T$ is the dimensional membrane tension per unit length, $Q$ is the dimensional flow rate and $\rho m$ is the membrane surface density. The membrane equation will be solved along with some elastic boundary conditions. On each of the boundaries, there is a choice for the boundary conditions. The edge of the membrane can either be held at a fixed position, prompting a constant value of $h$ at the boundary, or the displacement of the edge is freely determined by the condition that no net external force acts on the edge normal to the membrane. This corresponds to the free boundary condition $h_x = 0$ at the edge. There are other choices as well, specifically a linear relation between the displacement and the gradient of $h$. We will only deal with one of them, namely

$$h = 1 \quad \text{at} \quad x = -\frac{1}{2} \quad \text{and} \quad x = \frac{1}{2}$$  \hspace{1cm} (3.65)

These equations are coupled with the fluid equations (3.29) and (3.30).

The equilibrium flow field and membrane deflection denoted by $h = H(x)$ and $p =$
Chapter 3. Flow induced elastic oscillations

Figure 3.6: Equilibrium displacement for the fixed pressure fluid condition with \( h \) fixed at inlet and exit. Four values of \( \delta=2 \) (top left), 1 (top right), 0.5 (bottom left), 0.1 (bottom right) are shown and each panel contains solution of (3.68) for eleven equally spaced values of \( R_{eq} \) from 0 (bottommost) to 10 (topmost).

\[ P(x) \text{ satisfy,} \]
\[ \delta P + H_{xz} = 0 \]  \hspace{1cm} (3.66)
\[ q_e = 0 \quad \rightarrow \quad q(x) \equiv 1 \]  \hspace{1cm} (3.67)
\[ H_{xxx} = \frac{\delta}{H^3} (3 - R_{eq} \beta H_x) . \]  \hspace{1cm} (3.68)

Since volume flux is non-dimensionalized using the equilibrium value, it is fixed at unity throughout the entire domain for a steady state. This can be thought of as the application of the inlet fluid boundary conditions. The fluid exit boundary condition has to be imposed separately.

Figure 3.6 and 3.7 show the equilibrium displacements of the membrane and pressure in the fluid. The pressure falls below the ambient value, sometimes to the extent that the channel width is decreased. This is a consequence of the exit boundary condition,
Figure 3.7: Steady state fluid pressure for parameters corresponding to figure 3.6.
which requires that the exit pressure be smaller than the ambient at non-zero Reynolds numbers. If this reduction in exit pressure is sufficient so as to overcome the pressure gradient due to viscous drag, then the pressure in the whole channel will remain lower than the ambient and thus cause the channel width to be smaller than unity.

The stability of the steady state to small perturbations is studied using linear stability analysis. For that, we write \( h = H + \delta h \), \( q = 1 + \delta q \), and \( p = P + \delta p \), where the variable with hats are small and \( \sigma \) is the complex growth rate. Ignoring quadratic and higher terms (and dropping the hats), the perturbations satisfy

\[
\sigma^2 \hat{h} - \hat{h}_{xx} + \delta \hat{p} = 0, \\
\sigma \hat{h} + \hat{q}_x = 0, \\
R_{eq} \left( \sigma \hat{q} + \frac{\hat{q}_x}{H} - \frac{\beta}{H^2} \hat{h}_{xx} - 2\beta \frac{\hat{q}H_x}{H^3} + 3\beta \frac{H_x \hat{h}}{H^5} \right) = \hat{h} - 3 \frac{\hat{q}}{H^2} + 9 \frac{\hat{h}}{H^4}. 
\]

These equations are to be solved with boundary conditions, for which there are eight choices. The channel width at the entrance and the exit, \( x = \pm \frac{1}{2} \) is fixed, so \( \hat{h} = 0 \). For the fluid boundary conditions there the flux is specified at the inlet, leading to \( \hat{q} = 0 \). At the other end, \( x = \frac{1}{2} \), for the fluid we impose the Bernoulli-like condition giving

\[
\hat{p} + R_{eq} \gamma \left( \frac{\hat{q}}{H^2} - \frac{\hat{h}}{H^3} \right) = 0. 
\]

The growth rate \( \Re(\sigma) \) calculated numerically for \( c = 1 \) from these equations is plotted in figure 3.8. Other values of \( c \) yield qualitatively similar results. For small \( \delta \) there is a critical \( R_{eq} \), as also determined by a asymptotic analysis, analogous to (3.62), beyond which the flow becomes unstable. But for finite values of \( \delta \), increasing \( R_{eq} \) beyond yet another threshold makes the flow stable again.

### 3.6 Acoustic oscillations in a cylinder

As an innovative application of this analysis, we present the acoustic instability of the air in a cavity excited by a flowing jet. Here the role of elasticity is played by compressibility of the air column. It is a common experience that acoustic vibrations of the air inside beverage bottles can be sustained by blowing over them. In the scientific literature, the beverage bottle is known as a Helmholtz resonator [196]. Woodwind instruments like
Figure 3.8: Growth rate as a function of $R_{eq}$ and $\delta$ for $c = 1$. Solid lines are contours of constant growth rate for twelve equally spaced values from $5 \times 10^{-4}$ (bottommost) to $-5 \times 10^{-3}$. (topmost right corner). The dashed line shows the result from asymptotic analysis for small $\delta$. As the flow rate is increased from zero for a fixed set of material and geometric properties, both $\delta$ and $R_{eq}$ increase proportionally. This is depicted by the dash-dotted line. There are two thresholds in flow rate corresponding to the two intersections of this line with the neutral stability curve (the thick solid line corresponding to a growth rate of zero).
Figure 3.9: Schematic setup for the two-bottle experiment. Two plastic bottles acting as resonators are placed face-to-face and a jet of air is blown through the gap in between to excite acoustic oscillations. Microphones placed at the bottom of each bottle transduce the pressure perturbation to electric signals which were simultaneously recorded using a computer (not shown). Experiments were also performed in the same setting but with a single bottle.

The flute, organ pipes, panpipes, etc. employ this relatively less understood mechanism for excitation of acoustic modes of vibrations.

A simply devised experimental setup to demonstrate this phenomenon (see figure 3.6) consists of a jet flowing past the mouth of a plastic bottle. A microphone attached to the bottom of the bottle recorded pressure perturbations with the help of a computer.

The results shown in figure 3.10 plot the amplitude of the pressure perturbation as a function of the blowing velocity. The excited frequency was 176 Hz. It can be readily seen that the generation of sound occurs in a small velocity window of 2–3 m/s and then for velocities higher than about 5–8 m/s. It has been accepted in the scientific literature that the sound generated is due to an instability resulting from the interaction of the jet with the cavity. A popular mis-conception amongst the non-experts about this mechanism is that the resonating cavity acts like a linear system and the sound heard is just a linear response of the cavity to random perturbations in the jet. While this concept is correct, it is not responsible for the note produced in the cavity, as is demonstrated by experiments. Current theories contend that the acoustic oscillations force perturbations on the jet, which grow due to a shear instability.
The grown perturbations then feed energy back to the resonating cavity to sustain the oscillations [168, 169, 174]. While lumped parameter models have been put forward, much from the analogy with electronic circuits, a clear understanding of the feedback from a fluid dynamics perspective is lacking.

One possible mechanism for the feedback from the jet to the resonating cavity was presented by Elder [174] (see an excellent review by Smith & Mercer [195]), who unified the works of Cremer & Ising [169] with that of Coltman [168]. According to this mechanism, the oscillating acoustic pressure in the resonator imposes a periodic forcing on the jet, causing it to meander. Thus for fraction of the period of the oscillations, the jet is directed inside the mouth hole, while for the remaining duration it deviates away from it. If the phase difference between the oscillations in the cavity and the fluctuations in the jet at the edge of the mouth hole are in proper relation, this constitutes a constructive periodic forcing back on the air column. Borrowing from the shear instability of a jet [164, 172, 185], the phase relation at the edge of the mouth hole is believed to depend on the jet velocity and the lip-edge distance. According to this theory, the driving jet speed and the lip-edge distance has to be carefully adjusted depending on the note to be played. A note can be played by blowing hard enough, bringing the lips closer to the edge, or a combination of both. Stroboscopic photographs have visualized the jet via smoke streaks, which seem to agree with this theory.

To experimentally verify the meander of the jet, we devised an improved experiment involving two identical resonators placed face-to-face excited by an air stream. If it is believed that the sinuous perturbations to the jet cause excitation in the resonators, then while the jet is directed away from one bottle, it is forcing the other. Hence the acoustic signal in the two resonators must be 180° out of phase. The pressure perturbation in the two bottles was simultaneously recorded on a computer using two microphones placed at the bottom of the bottles. The results for representative points are shown in figure 3.11. At the onset of the first instability, this figure shows that the pressure perturbations in the two bottles are in phase with each other. As the jet velocity is increased further, the signals abruptly become 180° out of phase until the instability disappears. For the second instability with even larger jet speeds, the two signals were always out of phase. While it is plausible that for most jet speeds, sinuous perturbations of the jet excite the resonance, the existence of this small window of jet
speeds where the perturbations in the two bottles are in phase is contradictory to this mechanism and an alternative explanation must be found for that case.

To this end, we present an alternative theory for this jet induced excitation, which is consistent with all these observations. This theory is different in that it does not require the shear instability of the jet. Instead it is much similar in spirit to the previous asymptotic analysis and explores how pressure variations from the Bernoulli principle in the jet affect the energy balance of the acoustic mode set up in the resonant cavity. For that we consider a rectangular cavity open at one end as shown in figure 3.12. The dimensions of the cavity are $L \times d$ and a jet of width $b$ is forced on its open end, while the other end is closed. In reality, the acoustic cavity and the jet are made up of the same fluid and there is no interface between the two. But we conceptually separate the flow in the jet from the cavity by an imaginary interface.

Let us directly write the equations for the linear stability analysis of the steady equilibrium corresponding to no density perturbations and no motion in the resonator.
Figure 3.11: The acoustic signal as a function of time from the two microphones in the two-bottle experiment for jet velocities corresponding to point A (top) and point B (bottom) (see figure 3.10 for the locations of A and B). The dashed line is from Mic2, whilst the solid line shows data recorded from Mic1 (shown in figure 3.6).
Figure 3.12: Schematic setup for acoustic excitation of a rectangular cavity by an air jet.
and a constant flux $\dot{q}$ in the jet. The cavity is modeled so as to be able to sustain
acoustic modes and the jet forms a boundary condition to it. The equations governing
the density perturbations and fluid velocity of the air in the resonating cavity are then
given by the linearized inviscid Navier-Stokes equations

$$\hat{\rho} = c^2 \rho_x = 0, \quad \frac{\partial}{\partial t} \rho_x + c^2 \rho_z = 0 \quad \text{and} \quad \rho_t + \hat{\rho}(u_x + w_z) = 0,$$

where $\rho$ is the density fluctuation about the ambient density of $\rho$, $c$ is the speed of sound
and $(u, w)$ is the velocity. The relation between pressure $(p)$ and density fluctuations is
given by the isentropic equation of state

$$p = c^2 \rho.$$

For boundary conditions, we impose a zero normal velocity at the walls of the cavity.
At the open end, the cavity interacts with the jet. No theory equivalent to (3.29)-(3.30)
is known for a jet. Instead of deriving such a theory from scratch, we write similar
equations $\textit{ad hoc}$, with the exception that the coefficients $\alpha$, $\beta$ and $r$ are not known and
will be treated as parameters. This theory also adds more parameters as will be seen.
The motion in the cavity mainly causes air to entrain into the jet. In the absence of
any variations in the jet width, which we assume for simplicity, conservation of mass
dictates

$$b \rho_t + \hat{\rho} (q_X + w) = 0 \quad \text{at } z = 0$$

Similarly, a phenomenological momentum conservation equation can be written as

$$\hat{\rho} \left( q_t + \frac{\alpha}{b} \hat{q} q_x \right) + bc^2 \rho_X = -\frac{\dot{f}}{b^2} q \quad \text{at } z = 0,$$

$\alpha$ is a coefficient that models the velocity profile of the jet across its width and $\dot{f}$ is
a friction factor to crudely model the drag force on the jet. The boundary conditions
at $z = 0$, themselves require boundary conditions and we will use $q = 0$ at $X = 0$
and $bc^2 \rho + \gamma \hat{\rho} U q = 0$ at $X = d$, based on fixed inlet flux at the inlet and Bernoulli-like
condition at the exit respectively. The dimensionless parameter $\gamma$ models the exit
boundary condition as before.
These equations can be non-dimensionalized using

\[(u, w) \rightarrow V(u, w), \quad (x, z) \rightarrow (dx, Lz), \quad t \rightarrow \frac{L}{c} t, \quad \rho \rightarrow \frac{\dot{\rho}}{\dot{c}} \rho, \quad q \rightarrow VdU.\]  

(3.79)

Notice that since the equations are linear, all dependent variables are scaled to be proportional to an arbitrary velocity \(V\), which must be small enough so that all the nonlinear terms can be neglected. The non-dimensional equations in the resonator then become

\[
\begin{align*}
\frac{L}{d} u_t + \rho_x &= 0, \\
\rho_t + \rho_z &= 0, \\
\rho_t + \frac{L}{d} u_x + w_z &= 0
\end{align*}
\]

(3.80)  
(3.81)  
(3.82)

with the boundary conditions at the open end \(z = 0\),

\[
\begin{align*}
\frac{b}{L} \rho_t + q_x + w &= 0, \\
\rho_x + \delta (q_t + \tau \alpha q_x + f q) &= 0
\end{align*}
\]

(3.83)  
(3.84)

where the dimensionless numbers are defined as,

\[
\delta = \frac{d^2}{bL}, \quad \tau = \frac{\dot{q}L}{cdb} \quad \text{and} \quad f = \frac{\dot{f}L}{b^2 \rho c}.
\]

(3.85)

At the inlet of the jet \(q = 0\), while at the exit, we get \(\rho + \delta \tau \gamma q = 0\). At the closed end of the resonator, \(w = 0\).

The number \(\delta\) is a geometric parameter, \(\tau\) can be interpreted as the ratio of the acoustic time scale to the advective time scale of the jet, whereas \(f\) is a non-dimensional drag coefficient for the jet. Most wind instruments are slender, corresponding to a small aspect ratio \(d/L\). Moreover, the jet thickness is typically smaller than its length, but not by a lot. Mathematically, we consider the limit \(\delta \ll 1\) and treat \(\delta\) perturbatively, while \(b/d, \tau\) and \(f\) are assumed to be \(O(1)\). That allows us to ignore \(\rho_t b/L\) in (3.83). The first few modes of the case we are considering here are independent of \(x\), hence in the resonator cavity we set the \(x\) derivatives to zero, despite them being multiplied by the large number \(L/d\) and \(u = 0\).
Chapter 3. Flow induced elastic oscillations

Assuming that \((\rho, w, q) = (\tilde{\rho}, \tilde{w}, \tilde{q})e^{i\omega t}\), we expand

\[
\tilde{\rho} = \rho_0 + \delta \rho_1 + \cdots, \\
\tilde{w} = \tilde{w}_0 + \delta \tilde{w}_1 + \cdots, \\
\tilde{q} = \tilde{q}_0 + \delta \tilde{q}_1 + \cdots \quad \text{and} \\
\omega = \omega_0 + \delta \omega_1 + \cdots.
\]

(3.86)

(3.87)

(3.88)

At the leading order, after dropping the tildes, we get

\[
iw_0w_0 + \rho_{0z} = 0, \\
iw_0\rho_0 + w_{0z} = 0
\]

(3.90)

(3.91)

with momentum conservation in the jet leading to \(\rho_{0z} = 0 \rightarrow \rho_0 = 0\) at \(z = 0\). Here we have used the boundary condition at the exit of the jet, \(\rho_0 = 0\). The solutions of these equations correspond to undamped normal modes of the resonator. We can pick any one of the following for further analysis.

\[
w_0 = A \cos \left(\frac{(2n+1)\pi x}{2}\right), \quad \rho_0 = -iA \sin \left(\frac{(2n+1)\pi x}{2}\right), \quad \omega_0 = \frac{(2n+1)\pi}{2}.
\]

(3.92)

Mass conservation of the jet now gives the perturbation in the jet speed due to the entrainment from the resonant cavity. Solving, \(q_{0x} = -A\) with \(q_0 = 0\) at \(x = 0\), gives \(q_0 = -Ax\).

We are now ready to go to the next order in \(\delta\). At this order, we get

\[
iw_0w_1 + i\omega_1w_0 + \rho_{1z} = 0, \\
iw_0\rho_1 + i\omega_1\rho_0 + w_{1z} = 0.
\]

(3.93)

(3.94)

The jet momentum conservation at \(z = 0\) gives

\[
\rho_{1x} = (i\omega_0 A x + \tau \alpha A + f A x),
\]

(3.95)

with \(\rho_1 - \tau \gamma A = 0\) at \(x = 1\). This can be solved for \(\rho_1\) as

\[
\rho_1 = \frac{i\omega_0 A}{2} (x^2 - 1) + \tau \alpha A (x - 1) + \frac{f A}{2} (x^2 - 1) + \tau \gamma A.
\]

(3.96)

The correction in the frequency can be found by multiplying (3.93) and (3.94) by the complex conjugates (denoted by stars) of \(w_0\) and \(\rho_0\), respectively and integrating the
sum from \( z = 0 \) to \( 1 \). This yields

\[
\omega_1 = \int_0^1 \frac{\rho_1 \omega_0^*|_{z=0}}{<|\omega_0|^2 + |\rho_0|^2>} dx,
\]

which is the analogue of (3.60). Substituting the expression for \( \rho_1 \) from (3.96) gives value of \( \omega_1 \). The imaginary part of \( \omega_1 \) gives the exponential growth rate with which the mode may grow or decay. Since \( \omega_0 \) is real and

\[
i \omega_1 = \tau \left( \gamma - \frac{\alpha}{2} \right) - \frac{f}{3} - \frac{i \omega_0}{3},
\]

the real part of the growth rate will be

\[
\Re(i \omega) \sim \delta \left[ \tau \left( \gamma - \frac{\alpha}{2} \right) - \frac{f}{3} \right].
\]

The linear growth rate for the modes can be positive if \( \gamma > \alpha/2 \). The condition for instability is

\[
\tau > \tau_{\text{crit}} = \frac{2f}{3(2\gamma - \alpha)}.
\]

The analysis can be given a physical interpretation. As the air in the cavity is executing its normal mode motion, for half the cycle air leaves the cavity and is entrained into the jet. Since a fixed volume is coming in the jet, this increases the jet velocity at the exit. Due to the Bernoulli-like boundary condition, an increase in velocity causes a decrease in the fluid pressure there. This change in pressure is transmitted throughout the jet. But a negative pressure gradient is required to accelerate the entrained fluid. When the decrease in pressure due to the exit boundary condition is stronger than increase due to overcoming viscous drag and accelerating entrained fluid, positive work is done on the motion of the air in the cavity in this half of the cycle. This constitutes a constructive feedback giving rise to an instability. The criterion can be written in terms of \( \tau \), which increases by either blowing harder or decreasing the lip to mouth distance. Thus the flow can be made unstable that way, which is consistent with observations.

### 3.7 Summary and conclusion

In the presence of very stiff elastic structures, due to contrast of the advective time scales with that of propagation of the elastic waves, it may be tempting to assume
that the structure quickly adjusts itself to an equilibrium configuration dictated by the fluid forces and filter out elastic oscillations. One of the implications of our analysis is that such filtering of the elastic waves may not be entirely appropriate. The flow can be forced at the time scale of the elastic structure by the natural modes of oscillation themselves. The response of the flow to these oscillations can in turn feed back energy to sustain them.

The cause of the instabilities investigated is the Bernoulli-like boundary condition at the exit. This conclusion is derived from the asymptotic analysis that reveals the mechanism of the instability. The precise exit boundary condition for the channel is still a matter of discussion. The most accurate thing to do will be to solve for the flow of the fluid once it has left the channel exit and couple it to the flow in the channel. Such an exercise will be cumbersome to say the least, if not impossible. Moreover, it is also unnecessary because it will further complicate the analysis and may obscure the instability mechanism.

We have parameterized the exit boundary condition through the parameter $\gamma$. Having a free parameter in the analysis, say to be fixed later to fit experimental data, seems somewhat dissatisfactory. However, it can also be argued that identifying the form of the boundary conditions that lead to an instability is an achievement of this analysis. As mentioned earlier, the precise value of $\gamma$ depends on the flow of fluid outside the channel exit and it will be unreasonable to solve for every such kind of conceivable flow. Moreover, we have solved for one of the possible flows to estimate the appropriate boundary condition and the corresponding value of $\gamma$.

Since the only restriction for the instability mechanism is that the elastic structure should be capable of normal modes of vibrations, a host of experiments can be devised to test the theory. For example, flow past membranes, beams, rectangular elastic blocks and Helmholtz resonators are all candidates for further experiments. The value of $\gamma$ can then be estimated from critical Reynolds number for instability, as it is the only unknown parameter in the theory. A comparison of the value of $\gamma$ obtained from different experimental setups can then help to check the validity of the boundary condition itself.

We have not found any instability in the limit of zero Reynolds number, which is relevant for volcanic tremor. However, we have limited our analysis to situations in which the equilibrium channel width is uniform. It is suggested that a non-uniform
channel width can modify the results and lead to an oscillatory instability even at zero Reynolds numbers [182]. We have not ventured into such an analysis. Most physically motivated problems, though, have a varying channel width at equilibrium. The vocal folds, for example, consist of a very narrow elastic constriction through which air has to pass. Korotkoff sounds are generated when the blood artery is just beginning to open, thus again forming a constriction. It raises the question of whether the narrow constriction geometry is optimal for exciting these oscillations, which perhaps nature has exploited through the evolution of vocal cords.

Our experiments with acoustic excitation have revealed an instability that does not depend on the shear-driven meandering of the jet. Instead of a sinuous mode, which the meandering instability mechanism predicts, a varicose mode is excited. However, as the flow rate is increased further, an abrupt but reproducible transition to the sinuous mode is observed. A possible explanation is that for a viscous jet, a critical Reynolds number needs to be exceeded for the sinuous shear instability to set it. Below that Reynolds number, the sinuous excitation mechanism is not operative. If the critical Reynolds number for the varicose mode is lower than that required for the sinuous mode, the varicose mode will be observed first. However, as the flow rate is increased, the sinuous mode becomes unstable and, possibly through nonlinear interactions, annihilates the varicose mode and establishes itself. Currently, this is just a hypothesis but experiments with Helmholtz resonators of different frequencies can help to bring out the mechanism of this abrupt mode transition.

In the general framework, when the flow is not restricted to such narrow geometries, a similar mechanism can be operative. We have exploited some approximations suitable for our flow geometry, and in principle, similar analysis could uncover instabilities in flow past flags, aircraft structures like wings, fuselages and fins, and bridges, in which there is considerable interest.

Moreover, we have also limited the analysis to the linear regime. Preliminary experiments with flow through gelatin blocks [186] and past membranes have exhibited a rich array of dynamical behaviour like multiple equilibria and period doubling. The nonlinear dynamics of these systems pose a very interesting problem, with the potential of having a single canonical model to explain each of these systems.
3.8 References


Part II

Energy stability
Chapter 4

Bounds on double diffusive convection

4.1 Introduction

Mathematical models describing physical flows often have multiple possible solutions that prove difficult to find due to the complex nature of the basic equations. Worse still, such flows are often turbulent, which precludes computing some of the physically relevant solutions owing to the inability to resolve the finest scales. In this situation, it is helpful to search for other, more indirect approaches to the problem that may assist in understanding crucial characteristics of the actual flow. One such approach is upper bound theory, wherein one avoids the search for actual solutions, but places bounds on some of their average properties. Malkus [213] was the first to propose this kind of idea in the context of thermal convection (the Rayleigh-Benard problem), and Howard [206] subsequently set the theory on a firm mathematical basis and devised techniques to calculate the bound. Whilst the governing partial differential equations (PDEs) themselves are abandoned, the method retains two integrals relations, or “power integrals”, derived from them, which is the crux of how the approach is far simpler than direct computations.

Making use of clever inequalities, Howard deduced rigorous but rough bounds on the heat flux that scaled like $R_T^{1/2}$, where $R_T$ is the Rayleigh number. Howard also computed bounds using test functions with a single horizontal wavenumber, which leads to the alternative scaling, $R_T^{3/8}$, for large $R_T$. The single wavenumber bound is only valid if other functions do not lead to a higher value of the heat flux, which Busse [203] later

\footnote{A version of this chapter has been accepted for publication. Balmforth, N. J., Ghadge, S. A., Kettapun, A. & Mandre, S., Bounds on double diffusive convection, J. Fluid Mech., accepted.}
showed to be the case at sufficiently high Rayleigh number. He also generalized Howard’s approach using an elaborate procedure to account for more than one wavenumber, and recovered the $R_{T}^{1/2}$ scaling in the infinite Rayleigh number limit.

Howard’s method pivots about a decomposition of all the physical fields into horizontal averages and fluctuations about them. In an alternative approach, the Constantin-Doering-Hopf (CDH) background method [205], the variables are also decomposed, but this time exploiting arbitrary “background” fields. Integral identities similar to the ones used in Howard’s method are constructed, and the choice for the background is dictated by constraints similar to those obtained in energy stability theory [210]. Work by Nicodemus, Grossmann & Holthaus [216] and Kerswell [211] has proved that, although the Howard-Busse and the CDH methods are seemingly different, they are nothing but complementary or optimal dual of each other, and ultimately lead to the same result.

In the current article, we use the background method to place bounds on double-diffusive convection (i.e. convection resulting from the dependence of buoyancy on two properties that diffuse at different rates). Such systems are often termed as thermohaline referring to their most common occurrences in oceans and other large water bodies, with salt and heat playing the roles of the two components. Interesting dynamics ensues when the two components affect the density stratification in opposite senses, and convection may occur even when the total density gradient is gravitationally stable [201, 228]. For example, near the polar ice caps, melting of ice releases fresh but cold water near the surface, a situation prone to oscillatory double-diffusive (ODD) convection [208, 215]. ODD convection is also thought to occur in meddies (vortices of warm, salty water commonly observed in East Atlantic emanating from the Mediterranean, [218]). In an astrophysical context, ODD is believed to operate in the interiors of older stars where the two components are entropy and the elements produced by thermonuclear reactions [220]; mixing by ODD convection may replenish the reactive core of the star with fresh fuel and thus affect its evolution.

The opposite case, in which salt stratification is destabilizing but heat is stabilizing, is susceptible to the formation of salt fingers. Stern [221] proposed that enhanced fluxes resulting from these fingers are instrumental in forming the staircase-like salinity profiles observed in laboratory experiments and the open ocean. The articles by Merryfield [214] and Schmitt [219] provide recent review on this subject. Analogues of salt fingers
have also been suggested to arise in some astrophysical situations, where the role of salt is played by locally overabundant heavier elements such as helium [226, 227]. In all these applications quantifying the degree of mixing generated by thermohaline processes is paramount, which highlights the importance of characterizing the flux laws in double diffusive convection, especially in the turbulent regime. Whilst the desired characterization of the flux laws remains elusive to analysis, at least at present, we follow Malkus's vision and calculate an upper bound on the flux of the unstably stratified species.

For double-diffusion, this bounding exercise has two key novelties over the Rayleigh-Bénard problem. First, at the onset of convection, double-diffusive systems show a richer array of dynamics than purely thermal systems: In Rayleigh-Bénard convection, when the system first becomes convectively unstable, a branch of steady convection solutions bifurcates supercritically from the motionless state; that is, there is a smooth onset to steady overturning. This simple scenario does not carry over to the double-diffusive case: as one raises $R_T$ to drive the system into convection, the linear instability can take the form of either steady overturning or oscillatory convection. Furthermore, the steady bifurcation can become subcritical, which implies the existence of multiple, finite-amplitude solutions at lower Rayleigh number that must, in turn, appear in saddle-node bifurcations at yet lower $R_T$. The existence of multiple solutions demands that the conductive state be subject to finite amplitude instability, even if it is linearly stable. All such dynamics must become embedded in the upper bound, which may even jump discontinuously at the saddle node bifurcations. This raises the interesting question of whether the upper bound machinery can be used to detect and characterize finite-amplitude instability and saddle-node bifurcations.

Second, the bounding exercise also has some interesting mathematical twists. We first show that the upper bound obtained on the flux of the unstably stratified component in the absence of the other component also serves as a bound in the presence of that stabilizing field. In fact, this is the result that comes out when we extend the background method in a straightforward way to doubly-diffusive convection. Whilst this result is heuristically expected, since the stratification of the stable component can only diminish the flux, it fails to provide a dependence on both components. In previous attempts, Lindberg [212] and Straus [223] used variants of the single-wavenumber approach to bound the ODD and salt-fingering cases, respectively. In order to obtain a
non-trivial dependence of the bound on the salt flux, Lindberg maximized the heat and salt fluxes simultaneously. Not only is there no reason to expect a single wavenumber, there is also no justification for assuming the fluxes to be maximal simultaneously. The procedure yielded bounds that scaled like $R_{T}^{3/8}$, where $R_{T}$ now denotes the Rayleigh number of the unstably stratified component. Straus exploited the large difference between the diffusivities of heat and salt to solve the heat equation asymptotically, thereby building the full effects of the stabilizing component into the bounding formulation automatically. But like Lindberg's, Straus's bound also scales with the $3/8$ power of $R_{T}$, and again reflects the inadequacy of a single wavenumber. In the current work, we identify one more integral constraint on double-diffusive convection, which Joseph [209, 210] has shown to be crucial in energy stability analysis. By augmenting the upper bound analysis with this integral constraint, and avoiding the use of a single wavenumber, we capture the effect of both components and construct a true bound.

4.2 Mathematical formulation

As is traditionally done, we model our system by the Boussinesq equations:

\begin{align}
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{\nabla P}{\rho} + g(\alpha_{T} T - \alpha_{S} S) \hat{z} + \nu \nabla^2 \mathbf{u}, \\
T_t + \mathbf{u} \cdot \nabla T &= \kappa_{T} \nabla^2 T, \\
S_t + \mathbf{u} \cdot \nabla S &= \kappa_{S} \nabla^2 S \quad \text{and} \\
\nabla \cdot \mathbf{u} &= 0,
\end{align}

where $\mathbf{u}(\mathbf{x}, t)$ is an incompressible velocity field, and $T$ and $S$ represent two scalar components that affect the density of the fluid. We only deal with the situation in which the two components affect the density stratification in opposite senses; without loss of generality, we set $T$ to be unstably stratified and $S$ to be stably stratified. If the diffusivity of $T$ ($\kappa_{T}$) is larger than that of $S$ ($\kappa_{S}$) the system is susceptible to ODD convection. In the opposite case, the system is susceptible to "T-fingers" (because in this case $T$ is playing the role of salt). The other physical parameters are the acceleration due to gravity ($g$), coefficients of expansion due to variations in $S$ ($\alpha_{S}$) and $T$ ($\alpha_{T}$), and the kinematic viscosity ($\nu$).
We prescribe the values of $S$ and $T$ at the two boundaries, $z = 0$ and $z = H$:

\[ T(z = H) = T_{\text{top}}, \quad T(z = 0) = T_{\text{top}} + \Delta T, \]  
\[ S(z = H) = S_{\text{top}}, \quad S(z = 0) = S_{\text{top}} + \Delta S. \]  

(4.5)

(4.6)

For the velocity field, we either use the no-slip condition,

\[ u = 0, \]  

(4.7)

or stress-free conditions,

\[ w = 0, \quad u_z = v_z = 0. \]  

(4.8)

Both these cases are considered when calculating the bound. However, the linear stability and some nonlinear solutions that we present are computed using the shear stress-free conditions, mostly for computational convenience.

We place the equations in dimensionless form by rescaling,

\[ u \rightarrow \frac{\kappa T}{H} u, \quad T \rightarrow T_{\text{top}}, \quad S \rightarrow S_{\text{top}}, \quad x \rightarrow Hx, \]  
\[ t \rightarrow \frac{H^2}{\kappa T} t, \quad \text{and} \quad p - \rho g (\alpha_T T_{\text{top}} - \alpha_S S_{\text{top}}) z \rightarrow \frac{\rho c_T^2}{H^2} p. \]  

(4.9)

This gives rise to four dimensionless numbers,

\[ R_T = \frac{g \alpha T \Delta T H^3}{\nu \kappa T}, \quad R_S = \frac{g \alpha_S \Delta S H^3}{\nu \kappa S}, \quad \text{Pr} = \frac{\nu \kappa T}{\kappa S} \quad \text{and} \quad \beta = \frac{\kappa S}{\kappa T}. \]  

(4.10)

and the governing equations become

\[ \frac{1}{\text{Pr}} (u_t + u \nabla u + \nabla p) = (R_T T - \beta R_S S) \hat{z} + \nabla^2 u, \]  

(4.11)

\[ T_t + u \nabla T = \nabla^2 T, \]  

(4.12)

\[ S_t + u \nabla S = \beta \nabla^2 S, \]  

(4.13)

\[ \nabla \cdot u = 0. \]  

(4.14)

The accompanying boundary conditions are

\[ T(z = 1) = S(z = 1) = 0 \text{ and} \]  
\[ T(z = 0) = S(z = 0) = 1, \]  

(4.15)

plus (4.7) or (4.8) on $z = 0$ and $z = 1$ and periodicity in $x$ and $y$. 


Chapter 4. Bounds on double diffusive convection

4.3 Energy Stability

We start with the criteria for nonlinear stability of the purely conductive state of this system. A very brief account of this analysis was given by Joseph [210]. We elaborate and build upon Joseph’s results here in order to offer a more complete discussion and extract some important physical results; in doing so, we also emphasize the key connection with the bounding theory to follow.

4.3.1 Mathematical details

Consider $u = 0 + u(x,t)$, $T = T_0 + \theta(x,t)$, $S = S_0 + \sigma(x,t)$, and $p = P_0 + \Pi$, where $T_0 = S_0 = 1 - z$ and $P_0 = Pr(R_T - \beta R_S)(z - z^2/2)$ characterize the purely conductive solution of (4.11) - (4.14), and $u$, $\theta$, $\sigma$ and $\Pi$ are arbitrary perturbations. The perturbations satisfy

\begin{align}
\frac{1}{Pr} (u_t + u \cdot \nabla u + \nabla \Pi) &= (R_T \theta - \beta R_S \sigma) z + \nabla^2 u, \quad (4.16) \\
\theta_t + u \cdot \nabla \theta - w &= \nabla^2 \theta, \quad (4.17) \\
\sigma_t + u \cdot \nabla \sigma - w &= \beta \nabla^2 \sigma, \quad (4.18) \\
\nabla \cdot u &= 0 \quad (4.19)
\end{align}

The kinetic energy equation is constructed by taking the dot product of the momentum equation (4.16) with $u$ and integrating over the domain,

\begin{align}
\frac{1}{2Pr} \langle |u|^2 \rangle_t &= - \langle |\nabla u|^2 \rangle + R_T \langle \theta w \rangle - \beta R_S \langle \sigma w \rangle, \quad (4.20)
\end{align}

where

\begin{align}
\langle \cdots \rangle = \frac{1}{4L_x L_y} \int_{-L_y}^{L_y} \int_{-L_x}^{L_x} \int_{0}^{1} \cdots \, dz \, dx \, dy,
\end{align}

$2L_x$ and $2L_y$ are the periodicities in $x$ and $y$, respectively, and $|\nabla u|^2 = \nabla u : \nabla u^T$.

Similarly, by multiplying (4.17) and (4.18) with $\theta$ and $\sigma$, respectively, and integrating, we arrive at the following power integrals,

\begin{align}
\frac{1}{2} \langle \theta^2 \rangle_t &= - \langle |\nabla \theta|^2 \rangle + \langle \theta w \rangle, \quad (4.21) \\
\frac{1}{2} \langle \sigma^2 \rangle_t &= -\beta \langle |\nabla \sigma|^2 \rangle + \langle \sigma w \rangle. \quad (4.22)
\end{align}

While these integral equations are the obvious generalization of those used for the energy stability for thermal convection, there is a less obvious integral which is also crucial. It
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is constructed by multiplying (4.17) by $\sigma$ and adding it to the product of (4.18) with $\theta$ and integrating:

$$
\langle \theta \sigma \rangle_t = \langle (\theta + \sigma)w \rangle - (1 + \beta) \langle \nabla \theta \cdot \nabla \sigma \rangle. \tag{4.23}
$$

The optimal way in which to combine these integral equations so as to yield the best stability criterion is the essence of the analysis. Since we do not know the optimal combination a priori, we start with an arbitrary linear combinations of (4.20) - (4.23) and arrive at the generalized energy equation:

$$
E_t = - \langle |\nabla u|^2 \rangle - \lambda^2_T R_T \langle |\nabla \theta|^2 \rangle - \beta \lambda^2_S R_S \langle |\nabla \sigma|^2 \rangle + \lambda_T R_T b_T \langle \theta w \rangle
$$

$$
+ \frac{\sqrt{\beta} \lambda_S R_S b_T}{\alpha} \langle \sigma w \rangle - (1 + \beta) c \lambda_T \lambda_S \sqrt{R_T R_S} \langle \nabla \theta \cdot \nabla \sigma \rangle, \tag{4.24}
$$

where

$$
\alpha^2 = \frac{R_S}{R_T}, \tag{4.25}
$$

$$
E \equiv \frac{1}{2 \text{Pr}} \langle |u|^2 \rangle + \frac{\lambda^2_T R_T}{2} \langle \theta^2 \rangle + \frac{\lambda^2_S R_S}{2} \langle \sigma^2 \rangle + c \lambda_T \lambda_S \sqrt{R_T R_S} \langle \theta \sigma \rangle, \tag{4.26}
$$

$$
b_T \equiv \frac{1}{\lambda_T} + \lambda_T + \alpha \lambda_S, \tag{4.27}
$$

$$
b_S \equiv - \frac{\sqrt{\beta} \alpha}{\lambda_S} + \frac{\alpha}{\sqrt{\beta}} \lambda_S + \frac{c}{\sqrt{\beta}} \lambda_T. \tag{4.28}
$$

and $\lambda_T$, $\lambda_S$ and $c$ are the constants used to form the combination. When $R_S = 0$ the effect of $S$ disappears and we recover the result for thermal convection. The basic state is said to be "energy stable" when the energy-norm, $E$, of the perturbations is positive definite and decays monotonically (the right-hand side of (4.24) is negative definite) for all possible perturbations. It is straightforward to show that $E$ in (4.26) is positive definite when $|c| < 1$.

First, we demonstrate the inability of (4.20)-(4.22) to capture the stabilizing effect of $R_S$ when $c = 0$. In this case, the energy equation takes the form,

$$
E_t = R_T \left( 1 + \lambda^2_T \right) \langle \theta w \rangle + \alpha^2 R_T \left( -\beta + \lambda^2_S \right) \langle \sigma w \rangle
$$

$$
- R_T \left( \lambda^2_T \langle |\nabla \theta|^2 \rangle + \alpha^2 \beta \lambda^2_S \langle |\nabla \sigma|^2 \rangle \right) - \langle |\nabla u|^2 \rangle, \tag{4.29}
$$

and we refer to $E$ as the "regular" energy. The optimization problem of finding the critical $R_T$ leads to the criterion for stability,

$$
R_T < R_{Te} = \frac{4 R_c}{F(\lambda_T, \lambda_S)}, \tag{4.30}
$$
where $R_c$ is the critical Rayleigh number for the onset of thermal convection and

$$F(\lambda_T, \lambda_S) = \left(\frac{1}{\lambda_T} + \lambda_T\right)^2 + \left(-\frac{\beta}{\lambda_S} + \lambda_S\right)^2. \quad (4.31)$$

The choice of boundary conditions on velocity enter the consideration through the value of $R_c$. For no-slip, $R_c \approx 1707$, whereas for stress-free condition, we have $R_c \approx 657$. We now choose $\lambda_T$ and $\lambda_S$ to maximize the range of $R_T$ for which perturbations decay. That is, we look for the minimum value of the function $F(\lambda_T, \lambda_S)$, which occurs for $\lambda_S^2 = \beta$ and $\lambda_T = 1$, giving $R_T = R_c$, as stated earlier. To improve this stability condition we must take $c \neq 0$, thereby including (4.23).

Going back to (4.24), the terms involving $\nabla \theta$ and $\nabla \sigma$ are negative semi-definite only if $c \leq 2\sqrt{\beta}/(1 + \beta)$. We choose $c = 2\sqrt{\beta}/(1 + \beta)$ and then combine all three terms into $\langle|\nabla f|^2\rangle$, where

$$f \equiv \lambda_T \sqrt{R_T} \theta + \lambda_S \sqrt{\beta R_S} \sigma.$$

This leaves us with just two sign-indefinite terms, $\langle \theta w \rangle$ and $\langle \sigma w \rangle$. This pair can only be bounded if they can again be grouped together in the combination $f$, which prompts the constraint,

$$b_T = b_S. \quad (4.32)$$

The energy equation now takes the form,

$$E_t = R_T^{1/2} b_T \langle f w \rangle - \langle|\nabla f|^2\rangle - \langle|\nabla u|^2\rangle, \quad (4.33)$$

which is very similar to the one obtained for thermal convection, but with a modified thermal Rayleigh number and the field $f$ playing the role of temperature. Again finding the optimal perturbation, we conclude that the condition for energy stability is

$$R_T < R_T = \frac{4R_c}{b_T^2}. \quad (4.34)$$

We still have the freedom to choose one of either $\lambda_T$ or $\lambda_S$ so as to obtain the best possible stability criterion. This leads to the following minimization problem for the critical thermal Rayleigh number ($R_T$):

$$R_T = 4R_c \left(\min_{b_T = b_S} b_T (\lambda_T, \lambda_S)^2\right)^{-1}. \quad (4.35)$$
The details of the minimization are placed in Appendix C. The resulting stability condition can be encapsulated in the formulae,

\[
R_{TC} = \begin{cases} 
R_c + R_s, & \text{if } \alpha \leq \beta < 1 \text{ or } \beta \geq 1 > \alpha \\
\left(\sqrt{R_c(1-\beta^2)} + \beta \sqrt{R_s}\right)^2, & \text{if } \beta < \alpha < \beta^{-1} \\
\infty, & \text{if } \alpha \geq \beta^{-1} > 1 \text{ or } \alpha \geq 1 \text{ and } \beta \geq 1,
\end{cases}
\] (4.36)

which was derived previously by Joseph.

### 4.3.2 Interpretation for stress-free plates

Now we draw some conclusions from Joseph’s result and compare with linear stability theory, specifically for the case of stress-free plates. The linear stability theory is described by Veronis (1965) and can be summarized as follows: Linear instability can appear as either steady or oscillatory convection, the corresponding critical Rayleigh numbers being given by

\[
\text{Steady: } R_T = R_S + R_c, \quad \text{if } \alpha^2 \frac{\beta + Pr}{1 + Pr} < \beta < 1 \text{ or if } \beta > 1, \\
\text{Oscillatory: } R_T = (Pr + \beta) \left(\frac{\beta R_S}{1 + Pr} + \frac{R_c(1 + \beta)}{Pr}\right), \quad \text{if } \alpha^2 \frac{\beta + Pr}{1 + Pr} > \beta \text{ and } \beta < 1.
\] (4.37) (4.38)

In figure 4.1, the energy stability condition (4.36) is compared with the conditions for the onset of linear and nonlinear instability. First consider the fingering case \((\beta > 1)\), represented in the top left panel of figure 4.1, which is the same for all values of \(\beta\) and \(Pr\). In this case, the conductive state becomes linearly unstable to steady convection on the line (4.37), and is never unstable to oscillatory convection. The energy stability threshold agrees with linear onset everywhere proving that all perturbations, irrespective of their size, should decay below that line.

The ODD case \((\beta < 1)\) is rather more complicated, and the dynamics of the system depends on the detailed parameter settings. The top right and bottom panels of figure 4.1 show a representative case with \(\beta = 0.5\) and \(Pr = 2\); once we fix those parameters, the behaviour of the system is determined by where it falls on the \((R_S/R_c, R_T/R_c)\)-plane, and the range of possibilities is delimited by the four curves shown in the figure. For \(\alpha < \sqrt{\beta(1 + Pr)/(\beta + Pr)}\) (left of point B), steady convection appears on the line (4.37)
Figure 4.1: Stability boundaries on the \((R_S/R_c-R_T/R_c)\)-plane. The top left panel shows the fingering case \((\beta > 1)\), where the only linear instability of steady convection and generalized energy stability condition agree (topmost curve) for all \(\beta\) and \(Pr\). The curve below it shows the regular energy stability criterion. The top right panel shows the ODD case \((\beta = 0.5 \text{ and } Pr = 2)\) and to clarify the details, a magnified view is plotted in the lower panel. The topmost solid line corresponds to the onset of steady convection which is supercritical to the left of point A and subcritical to its right. The unstable branch bifurcating from the subcritical bifurcation turns around at a saddle-node bifurcation whose location is shown by the dashed-dotted line. The nonlinear solutions at the saddle node are calculated by expanding the variables in a truncated Fourier series. The dashed line shows the linear stability criterion for onset of oscillatory convection. The solid lines again show the generalized and regular energy stability conditions, respectively.
and is the only linear instability. For $\alpha < \beta$, or to the left of point A, the energy stability condition in (4.36) also agrees with steady onset. However, to the right of that point, the two conditions diverge from one another, indicating that energy stability is lost before the motionless state becomes linearly unstable to steady convection. The steady instability is further superceded by the onset of oscillatory convection for $\alpha > \sqrt{\beta(1 + Pr)/\beta + Pr}$, or to the right of point B in figure 4.1. Moreover, except at one special point (labelled D) where the two curves are tangential, energy stability never agrees with linear oscillatory instability. In other words, only over a limited parameter range does the loss of energy stability correspond to the onset of linear instability in contrast to thermal convection and the fingering case, where they always agree.

Part of the reason for the disagreement between the energy stability condition and linear onset arises because the steady bifurcation becomes subcritical at point A. To the right of this point, the subcritical instability leads to steady convection solutions even in the linearly stable regime. These steady convection solutions do not persist very far below the steady linear stability line because they turn around at a saddle-node bifurcation (Veronis 1965). The saddle node in the past has been located using a crude Galerkin truncation of the governing equations. To improve upon this, we have accurately computed the locus of that bifurcation numerically via Fourier expansion and a continuation algorithm. The locus is plotted in figure 4.1. In the region between this locus and the onset of steady convection, we are guaranteed that there are multiple steady solutions.

When multiple solutions exist, certain finite amplitude perturbations and the energy associated with them will not decay to zero but saturate to a finite value, reflecting a transition to one of the other solutions. As a result, the energy stability condition cannot agree with the onset of linear instability whenever there are multiple solutions. Indeed, we find that the energy stability condition is tangential to the saddle-node line at point A, suggesting that the saddle node is the cause of the loss of energy stability there. However, as $R_S$ increases, the saddle node line and energy stability condition diverge, indicating some other reason for the loss of energy stability. The saddle node line also crosses the threshold for the onset of oscillatory convection (point C in figure 4.1), whereupon oscillatory nonlinear solutions come into existence before the saddle node. Unlike steady convection, the onset of oscillatory convection is always supercritical.
(Veronis 1965). Nevertheless, the energy stability condition disagrees with oscillatory onset except at one point. This leaves us with a significant discrepancy between the energy stability condition and either the saddle node line or the oscillatory onset.

The discrepancy could arise from three possible sources, amongst which we are currently unable to distinguish: First, there could be other unidentified nonlinear solutions lying below the computed saddle-node line. To discriminate against such additional multiple equilibria, we would be forced into an intensive search of the solution space of the governing equations at each point on the parameter plane. Our original purpose was to avoid such a time-consuming, open-ended exercise at the outset, and hence we cannot say more on this particular score.

The second possibility is that the power integrals included in the energy stability formulation allow a wider class of trial functions than are solutions to the governing equations. Above the energy stability condition, the energy method indicates that there are trial functions for which the generalized energy grows in time, yet these may not, in fact, be real solutions. The cure is to better constraint the function space by, for example, adding more power integrals. A curious observation comes on exploring in more detail the point of intersection of the energy stability condition and the linear oscillatory onset. The former is independent of Prandtl number, but the latter is not. Yet, when one constructs the envelope of the oscillatory onset line for all possible Prandtl numbers, the energy stability condition is recovered exactly. This suspicious coincidence leaves one wondering whether the main problem is the lack of Prandtl-number dependence in the energy stability condition, which could be alleviated by building in extra constraints.

The final possibility is transient amplification. This is a purely linear mechanism wherein the energy norm chosen to determine stability grows initially for certain initial conditions. The growth can be attributed to the presence of non-orthogonal linear modes, even when each of these modes decays exponentially [200, 229]. In the thermohaline context, transient amplification has been invoked in studies of ocean circulation [204, 225], and with regard to possible transitions in the paleoclimate [202, 222]. As far as energy stability is concerned, a sub-optimal choice of the energy norm may lead to transient growth even in situations for which there is no finite-amplitude instability. Indeed, this is exactly what happens with the regular energy norm for $RT > R_T > R_c$. 


The remedy is to generalize the energy norm and curb transient amplification, leading to the improved energy stability condition. But even this generalized energy stability condition may not correspond to a finite-amplitude instability as all perturbations may still decay eventually beyond this condition. Whether a true finite-amplitude instability criterion can be derived from power-integral considerations remains an open question.

4.4 The Background Method

Energy stability rigorously predicts there to be no convective motion when $R_T < R_{TC}$, but the analysis is silent otherwise. More information can be gained by employing the background method to find a bound on a flow property like the average species transport over long times. We undertake this calculation in this section.

4.4.1 The general formulation

The average transport of $T$ is quantified by the Nusselt number (Nu), defined as

$$\text{Nu} = \lim_{t \to \infty} \frac{1}{4L_xL_yt} \int_0^L \int_{-L_y}^{L_y} \int_{-L_x}^{L_x} T(z = 1) \, dy \, dx \, dt.$$ (4.39)

A volume integration of (4.12) multiplied by $T$ puts the Nusselt number in a more usable form:

$$\text{Nu} = \langle |\nabla T|^2 \rangle,$$ (4.40)

where we now redefine $\langle \cdots \rangle$ to include a long time average:

$$\langle \cdots \rangle \equiv \lim_{t \to \infty} \frac{1}{4L_xL_yt} \int_0^t \int_{-L_y}^{L_y} \int_{-L_x}^{L_x} \int_0^1 \cdots \, dy \, dx \, dt.$$ (4.41)

The $T$ and $S$ fields are decomposed into backgrounds and fluctuations as

$$T(x, t) = 1 - z + \phi(x) + \theta(x, t), \quad S(x, t) = 1 - z + \psi(z) + \sigma(x, t),$$ (4.42)

where we term $\phi$ and $\psi$ as the backgrounds and $\theta$ and $\sigma$ as the fluctuations. With this selection, $\phi(z)$, $\theta(x, t)$, $\psi(z)$ and $\sigma(x, t)$ satisfy homogeneous boundary conditions. The decomposition is arbitrary at the moment but will be made unique as the analysis
proceeds. With the decomposition, we construct the power integrals:

\[ R_T (\theta w) - \beta R_S (\sigma w) - \langle |\nabla u|^2 \rangle = 0, \quad (4.43) \]
\[ \langle (1 - \phi')\theta w \rangle - \langle \phi' \theta_z \rangle - \langle |\nabla \theta|^2 \rangle = 0, \quad (4.44) \]
\[ \langle (1 - \psi')\sigma w \rangle - \beta \langle \psi' \sigma_z \rangle - \beta \langle |\nabla \sigma|^2 \rangle = 0, \quad (4.45) \]
\[ \langle (1 - \phi')\sigma w + (1 - \psi')\theta w \rangle - \langle \phi' \sigma_z \rangle - \beta \langle \psi' \theta_z \rangle - (1 + \beta) \langle \nabla \sigma \cdot \nabla \theta \rangle = 0, \quad (4.46) \]

where primes denote differentiation with respect to \( z \). To find a bound, we relax the condition that \( u, \theta \) and \( \sigma \) solve the governing PDEs, but require them to satisfy the above integral relations. As will be shown later, the inclusion of the equation (4.46) is crucial in obtaining the dependence of the bound on \( R_S \) in the same way it was needed for energy stability.

The method proceeds by writing a variational problem in which we maximize the Nusselt number subject to the integral constraints. Thus, we consider the Lagrangian,

\[ \mathcal{L}[u, \theta, \sigma] = 1 + \left\langle \phi'^2 \right\rangle + 2 \left\langle \phi' \theta_z \right\rangle + \langle |\nabla \theta|^2 \rangle + \sqrt{a} \langle \Pi(x) \nabla \cdot u \rangle + \lambda (R_T (\theta w) - \beta R_S (\sigma w) - \langle |\nabla u|^2 \rangle) \]
\[ + \lambda_2 (R_T \langle (1 - \phi')\theta w \rangle - \langle \phi' \theta_z \rangle - \langle |\nabla \theta|^2 \rangle) \]
\[ + \lambda_3 R_S \left[ \langle (1 - \psi')\sigma w \rangle - \beta \langle \psi' \sigma_z \rangle - \beta \langle |\nabla \sigma|^2 \rangle \right] \]
\[ + \lambda_4 \lambda_S \alpha R_T \left[ \langle (1 - \phi')\sigma w + (1 - \psi')\theta w \rangle - \phi' \sigma_z - \beta \psi' \theta_z - (1 + \beta) \langle \nabla \sigma \cdot \nabla \theta \rangle \right], \quad (4.47) \]

where \( a, \lambda_T, \lambda_S \) and \( c \) are constant Lagrange multipliers, and \( \Pi \) is a spatially dependent multiplier that enforces fluid incompressibility. The interested reader can easily verify that if \( c \) is chosen to be zero, thus avoiding the constraint (4.46), the best value for \( \lambda_S \) turns out to be \( \sqrt{\beta} \) and the problem reduces to that of thermal convection. That is, the effect of \( R_S \) disappears from the bound as in energy stability theory. We therefore retain \( c \), but resist making the same choice for \( c \) as in energy stability theory. Instead, we substitute \( c = 2q\sqrt{\beta}/(1 - e^2/(1 + \beta)) \), where \( q \) is a parameter (\( q = 1 \) corresponds to the choice of energy stability theory). For algebraic convenience, we further rescale the backgrounds and fluctuations as

\[ u \rightarrow \frac{1}{\sqrt{a}} u, \quad \theta \rightarrow \epsilon \theta, \quad \phi \rightarrow \epsilon \phi, \quad \sigma \rightarrow \eta \sigma, \quad \text{and} \quad \psi \rightarrow \eta \psi, \quad (4.48) \]
where \( \epsilon \equiv 1/(\lambda_T \sqrt{a R_T}) \) and \( \eta \equiv 1/(\lambda_S \sqrt{a R_S}) \). Then \( \mathcal{L}[u, \theta, \sigma] \) can be written as

\[
\mathcal{L}[u, \Theta] = 1 + \epsilon^2 \langle \theta'^2 \rangle - \langle |\nabla u|^2 \rangle - \left( \frac{\partial \Theta^T}{\partial x} P \Psi' \right) + R_T^{1/2} \langle (B_T \theta + B_S \sigma) w \rangle - \left( \frac{\partial \Theta^T}{\partial x_i} R \frac{\partial \Theta}{\partial x_i} \right) + \langle \Pi \nabla \cdot u \rangle,
\]

(4.49)

where

\[
\Theta = \begin{pmatrix} \theta \\ \phi \end{pmatrix}, \quad \Psi = \begin{pmatrix} \phi \\ \psi \end{pmatrix},
\]

(4.50)

\[
B_T = b_T - \left( \phi' + \frac{2q \psi' \sqrt{1 - \epsilon^2}}{1 + \beta} \right) \epsilon \lambda_T, \quad B_S = b_S - \left( \frac{\psi'}{\beta} + \frac{2q \phi' \sqrt{1 - \epsilon^2}}{1 + \beta} \right) \epsilon \lambda_T.
\]

(4.51)

\[
b_T = \frac{1}{\lambda_T} + \frac{\lambda_T}{1 + \beta} + \frac{\sqrt{\beta} \eta q \sqrt{1 - \epsilon^2} \lambda_S}{1 + \beta}, \quad b_S = -\frac{\sqrt{\beta} \eta q \sqrt{1 - \epsilon^2}}{1 + \beta} + \frac{\lambda_S}{\sqrt{\beta}} + \frac{2q \sqrt{1 - \epsilon^2} \lambda_T}{1 + \beta},
\]

(4.52)

\[
P = \begin{pmatrix} 1 - 2\epsilon^2 & \frac{2q \sqrt{1 - \epsilon^2}}{1 + \beta} \\ \frac{2q \sqrt{1 - \epsilon^2}}{1 + \beta} & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 - \epsilon^2 & q \sqrt{1 - \epsilon^2} \\ q \sqrt{1 - \epsilon^2} & 1 \end{pmatrix}
\]

(4.53)

and a summation is implied on the repeated index \( i = 1, 2, 3 \).

The first variation of \( \mathcal{L}[u, \Theta] \) demands that the optimal fields, denoted by the subscript \( \ast \), satisfy the Euler-Lagrange equations,

\[
\nabla \cdot u = 0, \quad 2 \nabla^2 u + R_T^{1/2} (B_T \theta_\ast + B_S \sigma_\ast) \hat{u} - \nabla \Pi = 0, \quad \text{and}
\]

(4.54)

\[
P \Psi'' + R_T^{1/2} w_\ast \begin{pmatrix} B_T \\ B_S \end{pmatrix} + 2 R \nabla^2 \Theta_\ast = 0.
\]

(4.55)

For the stationary fields to be maximizers, the second variation of \( \mathcal{L}[u, \Theta] \) requires

\[
\langle |\nabla \hat{u}|^2 \rangle + \left( \frac{\partial \hat{\Theta}^T}{\partial x_i} R \frac{\partial \hat{\Theta}}{\partial x_i} \right) - R_T^{1/2} \langle (B_T \hat{\theta} + B_S \hat{\sigma}) \hat{w} \rangle \geq 0,
\]

(4.56)

where the hat denotes deviations from the stationary fields. If we now set

\[
f \equiv \hat{\theta} \sqrt{1 - \epsilon^2} + q \hat{\sigma},
\]

then (4.56) can be expanded into

\[
\langle |\nabla \hat{u}|^2 \rangle + \langle |\nabla f|^2 \rangle + (1 - q^2) \langle |\nabla \hat{\sigma}|^2 \rangle - R_T^{1/2} \left( \frac{B_T}{\sqrt{1 - \epsilon^2}} \hat{f} \hat{w} + \left( B_S - \frac{q B_T}{\sqrt{1 - \epsilon^2}} \right) \hat{\sigma} \hat{w} \right) \geq 0.
\]

(4.57)
In order to ensure that the third term is not negative, we must choose $|q| \leq 1$.

The most general version of our variational problem is now to find the smallest possible value of the extremal Nusselt number, $\mathcal{L}[\mathbf{u}, \Theta]$, subject to the Euler-Lagrange equations (4.54)-(4.55) and condition (4.57). At our disposal in this optimization are the various Lagrange multipliers and the choices of the background fields. Plasting & Kerswell (2003) [217] have used a general formulation of this kind in bounding the Couette flow. Here, we proceed less ambitiously and consider a less optimal, but certainly more straightforward version of the problem.

4.4.2 Reduction to a more familiar formulation

The general variational formulation can be reduced to a more familiar form if we make two further assumptions. First, following Doering & Constantin [205], we simplify the solution of the Euler-Lagrange equations by taking $\mathbf{u} = 0$. Therefore, $\Theta = \Theta(z)$, with

$$\Theta' = -\frac{1}{2} R^{-1} P\Psi'.$$

Second, by analogy with energy stability theory, we impose the constraints,

$$q b_T = \sqrt{1 - \epsilon^2} b_S \quad \text{and} \quad q B_T = \sqrt{1 - \epsilon^2} B_S,$$

which have the advantage of eliminating the final term in (4.57), leaving

$$\langle |\nabla \mathbf{u}|^2 \rangle + \langle |\mathbf{f}|^2 \rangle + (1 - q^2) \langle |\nabla \delta|^2 \rangle - B_T \sqrt{\frac{R_T}{1 - \epsilon^2}} \langle \delta \mathbf{u} \rangle \geq 0. \quad (4.60)$$

The second relation in (4.59) also connects the two background fields to one another:

$$\Psi' = \frac{(\beta + 2\epsilon^2 - 1)\beta q \phi'}{(\beta + 1 - 2q^2\beta)\sqrt{1 - \epsilon^2}} \quad (4.61)$$

The extremal value of the heat flux, $\mathrm{Nu}_*$, can now be written in the form,

$$\mathrm{Nu}_* = \mathcal{L}[0, \Theta_*] = 1 + \langle \Psi' M \Psi' \rangle,$$  \quad (4.62)

where

$$M = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4} \mathbf{P}^T (R^{-1})^T \mathbf{P}, \quad (4.63)$$
and the positive-definiteness of $R^{-1}$ makes the bound, $\text{Nu}_*,$ bigger than or equal to unity. Note that (4.61) implies that $\left< \Psi^T M \Psi \right>$ can be written formally in terms of a parameter-dependent coefficient times $\left< \phi'^2 \right>$.

At this stage, the variational problem amounts to locating the smallest value of $\text{Nu}_*$ such that (4.60) holds. If we insist that $|q| < 1,$ then we may simply omit the term $\langle |\nabla \sigma|^2 \rangle$ leaving a formulation much like that explored for the Rayleigh-Bénard problem (with, once again, $f$ playing the role of temperature). The problem posed, however, is more complicated because of the richer structure of the coefficients in both the second-variation constraint (4.60) and the maximum Nusselt number (4.62).

Although any background field for which the second variation condition is satisfied will furnish a valid upper bound, some profiles may lead to a better bound than others. Hence, it is desirable to find that background which not only satisfies the second variation but also leads to the lowest bound. Such an exercise involves a nonlinear functional optimization problem. In the next subsection, we reduce this optimization problem to an algebraic one by using piece-wise linear background profiles. Before making this selection, however, we remark briefly on the choices in (4.59). These selections have the advantage of reducing the general variational formulation to something closer to the familiar, Rayleigh-Bénard problem. Better still, because they also coincide with the choices made in energy stability theory, the bound is guaranteed to reduce to the energy stability condition when $R_T < R_{TC}.$ Moreover, one can show that these selections are, in fact, the best possible choices if the background fields are piece-wise linear, as in our main computations. Nevertheless, for general backgrounds and above the energy stability threshold, we cannot judge the optimality of the selection, which exposes a flaw in the current theory; one possible consequence is mentioned later.

### 4.4.3 Piece-wise linear background fields

We now reformulate the variational problem in purely algebraic terms by introducing the piece-wise linear background fields,

$$
\Psi(z) = \begin{cases} 
-\left( \frac{1}{2\delta} - 1 \right) \Psi'_{in} z, & 0 \leq z \leq \delta, \\
\Psi'_{in}(z - \frac{1}{2}), & \delta \leq z \leq 1 - \delta \\
-\left( \frac{1}{2\delta} - 1 \right) \Psi'_{in}(z - 1), & 1 - \delta \leq z \leq 1,
\end{cases}
$$

(4.64)
where $\delta$ ($0 \leq \delta \leq 1/2$) is loosely referred to as the "boundary-layer thickness", and $\Psi_{in}'$ denotes the slopes of the two backgrounds in the interior region ($\delta < z < 1 - \delta$). Because of (4.61), the components of the latter are not independent of one another. The shapes of the background fields are illustrated in figure 4.2.

The next step is to make the sign-indeterminate term in (4.60) as small as possible. We achieve this by choosing $\Psi_{in}'$ so that $B_T = B_S = 0$ in the interior, which demands that

$$\Psi_{in}' = \frac{1}{\ell \lambda_T} S^{-1} \begin{pmatrix} b_T \\ b_S \end{pmatrix},$$

where

$$S \equiv \begin{pmatrix} 1 & 2q \sqrt{1 - \epsilon^2} \\ 2q \sqrt{1 - \epsilon^2} & \frac{1}{1 + \beta} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\beta} \end{pmatrix}.$$

We are then left with only boundary layer contributions to the sign-indeterminate term, but these hopefully remain controlled and small because $\hat{\Theta}$ and $\hat{u}$ vanish on the boundaries.

The inequality in (4.60) can now be written as

$$\langle |\nabla \tilde{u}|^2 \rangle + \langle |\nabla f|^2 \rangle + (1 - q^2) \langle |\nabla \hat{\omega}|^2 \rangle - \frac{b_T}{2\delta} \sqrt{\frac{R_T}{1 - \epsilon^2}} \langle \tilde{f} \hat{\omega} \rangle_{bl} \geq 0,$$

where

$$\langle \cdot \rangle_{bl} \equiv \lim_{t \to \infty} \frac{1}{4L_x L_y L_z} \int_0^t \int_{-L_y}^{L_y} \int_{-L_z}^{L_z} \left( \int_0^\delta \cdots \ dz + \int_{-\delta}^1 \cdots \ dz \right) \ dxdydt.$$
For convenience, we replace (4.67) by the constraint,
\[
\langle |\nabla \hat{u}|^2 \rangle_{bl} + \langle |\nabla f|^2 \rangle_{bl} - \frac{b_T}{2\delta} \sqrt{\frac{R_T}{1 - \epsilon^2}} \langle f \hat{u} \rangle_{bl} \geq 0.
\] (4.69)
which is sufficient for (4.67) to be satisfied, and depends on the integrals of \( \hat{u} \) and \( f \) only over the boundary layers. Hence the interior region can be omitted completely from the analysis, noting only that \( \hat{u} \) and \( f \) should be smooth there. The inequality can be cast as the variational problem,
\[
\frac{2\delta}{|b_T|} \sqrt{\frac{1 - \epsilon^2}{R_T}} \leq \max_{f, \hat{u}} \langle f \hat{u} \rangle_{bl} \quad \text{s.t.} \quad \langle |\nabla \hat{u}|^2 \rangle_{bl} + \langle |\nabla f|^2 \rangle_{bl} = 1, \quad \nabla \cdot \hat{u} = 0, (4.70)
\]
with \( f \) and \( \hat{u} \) vanishing at \( z = 0 \) and \( z = 1 \) and free at \( z = \delta \) and \( z = 1 - \delta \). The Euler-Lagrange equations corresponding to this maximization are identical to the linear stability equations obtained for thermal convection with a layer of height \( 2\delta \) and an equilibrium temperature gradient of unity. Thus, the results from thermal convection can be adapted using a suitable rescaling of the variables. Doing that, we obtain the following constraint on \( \delta \):
\[
\delta < \delta_{\max} = \frac{\sqrt{1 - \epsilon^2}}{|b_T|} \sqrt{\frac{R_T}{R_e}}. \] (4.71)

Finally, we simplify the bound on the Nusselt number:
\[
\text{Nu}_* = 1 + \left( \frac{1}{2\delta} - 1 \right) \Psi_{in}^T M \Psi_{in}'. \] (4.72)

Since we would like to obtain the smallest \( \text{Nu}_* \), we choose the biggest \( \delta \) allowed by (4.71), and arrive at
\[
\text{Nu}_* = 1 + \frac{b_T^2 \left[ 1 - \beta q^2 (2 - \beta) \right]}{4\epsilon^2 \lambda_T^2 (1 - \epsilon^2) (1 - q^2)} \left( \frac{1}{2\delta} - 1 \right), \] (4.73)
where
\[
\delta = \begin{cases} \delta_{\max}, & \delta_{\max} < \frac{1}{2}, \\ \frac{1}{2}, & \delta_{\max} \geq \frac{1}{2}. \end{cases} \] (4.74)

This leaves us with a choice of the constants \( \lambda_T, \lambda_S, \epsilon \) and \( q \), which are constrained by (4.59) and must be selected to minimize \( \text{Nu}_* \):
\[
\text{Nu}_{\text{max}} = 1 + \min_{\lambda_T, \lambda_S, \epsilon, q} \frac{b_T^2 \left[ 1 - \beta q^2 (2 - \beta) \right]}{4\epsilon^2 \lambda_T^2 (1 - \epsilon^2) (1 - q^2)} \left( \frac{1}{2\delta} - 1 \right), \] (4.75)
subject to \( q b_T = b_S \sqrt{1 - \epsilon^2}, -1 < q < 1 \) and \( 0 < \epsilon < 1 \). If \( \delta_{\max} \geq 1/2 \) for a suitable choice of the parameters, we set \( \delta = 1/2 \) and, consequently, \( \text{Nu}_{\text{max}} = 1 \). The condition for that to happen coincides with energy stability.
Chapter 4. Bounds on double diffusive convection

4.5 Results

The optimization in (4.75) to find the lowest upper bound on the Nusselt number is performed numerically. We made extensive use of the Matlab function fminsearch to serve the purpose. The results for the ODD convection and the fingering case are presented separately.

4.5.1 ODD convection

Figure 4.3 shows the typical behaviour of the bound for ODD convection using $\beta = 0.1$. The lower left panel demonstrates that the scaling of the bound is $R_T^{1/2}$ for fixed $\alpha$, as $R_T$ becomes large, which can be extracted from (4.75) simply by observing the limiting dependence, $\delta \sim R_T^{-1/2}$, in the constraint (4.71). The $1/2$ scaling mirrors the equivalent result in the Rayleigh-Bénard problem, and one might at first sight guess that little has been gained. In fact, much more information is included in the $\alpha-$dependent pre-factor to the scaling, which does not heed to asymptotic analysis and must be computed numerically. For example, an increase of $\alpha (R_S)$ at fixed $R_T$ lowers the bound, as can be seen in the lower right panel of the figure. The bound continues to decrease smoothly as $\alpha$ is increased, until this parameter reaches the threshold for energy stability, whereupon the bound discontinuously jumps to unity. Thus, the $\alpha-$dependence of the bound encapsulates the ability of the stabilizing component to turn off convection completely.

Although the optimization must in general be performed numerically, there is one particular limit in which we can make further progress: $\beta \ll 1$ (which is relevant to the oceanic application, where $\beta \approx 10^{-2}$). We begin by writing the bound as

$$\mathbf{N}_u - 1 = \frac{b_T^2 [1 - \beta q^2 (2 - \beta)]}{4 \epsilon^2 \lambda_T^2 (1 - \epsilon^2) (1 - q^2)} \left( \frac{1}{2 \delta} - 1 \right)$$

and find the values of $\lambda_T$, $q$ and $\epsilon$ so as to minimize the coefficient of $\sqrt{R_T / R_c}$. Guided by energy stability theory, we set $\alpha \beta \sim O(1)$. In this limit, the constraint (4.59) gives $\lambda_S = -\sqrt{\beta}$ and

$$b_T = \frac{1}{\lambda_T} + \lambda_T - 2 \alpha \beta \chi,$$
Figure 4.3: The bound on Nusselt number for ODD convection, shown as a density on the $(\alpha, R_T/R_c)$-plane for $\beta = 0.1$ (top panel). The solid lines are contours of constant $\text{Nu}_{\text{max}}$ for values of 70 (topmost), 60, 50, 40, 30, 20, 10 and 5 (last but one), and the lowermost solid line corresponds to the energy stability threshold $R_T = R_{Tc}$. The lower left panel plots the bound for $\alpha = 0$ (topmost solid), 1, 4, 7 (lowermost) as a function of $R_T/R_c$. The dotted line shows a $R_T^{1/2}$ scaling for comparison. In the lower right panel, the effect of $\alpha$ is shown for $R_T/R_c = 5$ (lowermost), 10, 50 and 100 (uppermost).
Figure 4.4: The coefficient of \( (R_T/R_c)^{1/2} \) in the bound for \( \beta \ll 1 \). The solid curve is the result of the analysis given in the text. The circles correspond to the data shown in figure 4.3 for \( R_T = 1000R_c \). The dashed line shows the asymptotic result for \( \alpha \beta \sim 1 \), \( C(\alpha \beta) \sim 27(1 - \alpha \beta)/4 \).

where \( \chi = q\sqrt{1 - \epsilon^2} \). We minimize (4.76) with respect to \( \lambda_T \) to obtain,

\[
\lambda_T = -2\alpha \beta \chi + \sqrt{4\alpha^2 \beta^2 \chi^2 + 5},
\]

which then leads to

\[
\begin{align*}
\text{Nu}_* - 1 &\leq \frac{\epsilon^2}{5^3} \left( -3\alpha \beta \chi + \sqrt{4\alpha^2 \beta^2 \chi^2 + 5} \right)^3 \left( 2\alpha \beta \chi + \sqrt{4\alpha^2 \beta^2 \chi^2 + 5} \right)^2 \left( \frac{R_T}{R_c} \right) \left( 1 - \epsilon^2 \right)^{1/2} \left( 1 - \chi^2 - \epsilon^2 \right) \\
\epsilon^2 &\approx \frac{7}{10} - \frac{3\chi^2}{10} + \left[ \frac{9}{100} (1 + \chi^2)^2 - \frac{\chi^2}{5} \right]^{1/2},
\end{align*}
\]

which leaves \( \text{Nu}_* \) as a function of only \( \chi \). The final minimization in \( \chi \) must be done numerically. The result is \( \text{Nu}_{\text{max}} = C(\alpha \beta)\sqrt{R_T/R_c} \), where the function \( C(\alpha \beta) \) is plotted in figure 4.4. At \( \alpha \beta = 0 \), the coefficient takes the value for thermal convection, \( C(0) = \sqrt{27}/4 \), and then decreases smoothly to zero as \( \alpha \beta \) approaches 1 (the energy stability condition for \( R_T/R_c \to \infty \)). Also included in the figure are the results of the full numerical optimization for \( \beta = 0.1 \) and \( R_T = 1000R_c \), which display quantitative agreement with the limiting solution.
4.5.2 T-fingers

The bound for $\beta = 10$ is plotted in figure 4.5. As is clear from this picture, the asymptotic behaviour of the bound is again $R_T^{1/2}$ for large $R_T$, and, once more, $\text{Nu}_{\text{max}}$ is discontinuous at the energy stability boundary. A closer look reveals a relatively weak dependence of the bound on $\alpha$. Indeed, the bound obtained for $\alpha = 0$ is a very good approximation to the bound for other values of $\alpha$. Figure 4.6 shows the dependence of the bound on $\alpha$ and $\beta$ for fixed $R_T = 1000\beta_c$, and illustrates again how $\text{Nu}_{\ast}$ is only weakly sensitive to $\alpha$ in the limit of large $\beta$. Thus, we infer that, with the constraints employed and the family of backgrounds chosen, the bound is not reduced on adding the stabilizing component in this limit. Perhaps Straus' asymptotic solution of the $S-$equation could be used to improve the situation.

4.5.3 Discontinuity in the bound

There are two obvious reasons why the bounds computed above could be discontinuous on the energy stability curve, neither of which are correct. First, a discontinuity can arise due to the appearance of new finite-amplitude solutions in a saddle-node bifurcation. Indeed, the loss of energy stability at the point where the saddle node first appears (see figure 4.1) suggests that a jump of this kind might well be present around these parameter settings. In this way, the bounding machinery could prove an effective tool in exploring the nonlinear dynamics of the system. Unfortunately, it turns out that the bound jumps discontinuously even in cases where there is no saddle-node and the energy stability condition coincides with linear onset (as for the fingering case). Moreover, no qualitative change occurs in the extent of the discontinuity when we approach parameter settings for which we know a saddle-node exists. Thus, the discontinuity observed in our computations does not appear to be caused primarily by the appearance of new nonlinear solutions.

The second reason why the bound could be discontinuous is that the background profiles change from being linear to piecewise linear on passing through the energy stability curve. In fact, for exactly this reason, discontinuities exist in bounds for thermal convection. As shown by Doering & Constantin [205], those discontinuities can be removed by using a smoother background profile near the energy stability threshold,
Figure 4.5: Shown in the top panel is the bound computed for $\beta = 10$ (T-fingers). The solid lines are contours of constant $\text{Nu}_{\text{max}}$ for values of 70 (top most), 60, 50, 40, 30, 20, 10, 5 (lowest but one) and the lowest solid line shows the energy stability threshold $R_T = R_{Tc}$. The lower left panel plots the bound for $\alpha = 0$ (topmost solid), 1, 4, 7 (lowermost) as a function of $R_T/R_c$. The dotted line shows $R_T^{1/2}$ for scaling. In the lower right panel, the effect of $\alpha$ is shown for $R_T/R_c = 100$ (topmost), 50, 10 and 5 (lower most).
which begs the question of whether we can smooth out the current discontinuity by similar means.

To address this question one can return to the formulation of the variational problem in §4.4.2. Near the energy stability threshold, it is possible to develop asymptotic solutions via perturbation theory without choosing a particular background. The final value of the bound depends on integrals of various functions that are related to the background fields, and one could, in principle, optimize the procedure to find the best bound. However, it becomes immediately clear on heading down this avenue that the bound always jumps discontinuous at the energy stability boundary, irrespective of the choice of background. The reason can be traced to the conditions in (4.59) which, in combination with the solution of the Euler-Lagrange equations in (4.58), lead to the optimal Nusselt number in (4.62). The trouble is that the matrix $R$ becomes singular at the energy threshold (where $q \to 1$ and $\epsilon \to 0$), and with the choice (4.59) already made, there is no way to adjust the background fields to ensure that $\Psi_*$ remains regular there. The result is that $Nu_*$ always converges to a finite value as $RT$ approaches $R_{Te}$.
from above. Given the failure of the perturbation expansion, it seems clear that the only possible way in which the discontinuity might be eliminated is by avoiding one of the two extra assumptions made at the beginning of §4.4.2 (namely \( u = 0 \) or (4.59)).

### 4.6 Discussion and open questions

In this work, we have bounded fluxes in double diffusive convection using the Constantin-Doering-Hopf background method. Of particular interest is the behaviour of the bound for large Rayleigh numbers, where we find the dependence, \( R^{1/2} \). This bound is different from empirical flux laws often quoted in the literature [224] which show \( Nu \sim R^{1/3} \). One reason for this discrepancy is that our bound may simply be too conservative and grossly overestimate the physically realized flux. Indeed, many examples of double-diffusive convection in the laboratory and ocean show the formulation of internal boundary layers (salt finger interfaces, diffusive steps), yet our optimal backgrounds only exhibit such sharp features next to the walls and do not capture whatever process is responsible.

However, as also true in Rayleigh-Bénard problem, it is not clear whether the observed flows have converged to the ultimate asymptotic state of double-diffusive convection. If that state is characterized by flux laws which do not depend explicitly on the molecular values of diffusivity and viscosity, a 1/2 scaling law must eventually emerge.

A main difficulty addressed in this article is to account for the effect of the stabilizing element on the bound. This effect disappears from the most straightforward implementation of the background method, as it does from regular energy stability theory. A similar problem is posed for geophysical and astrophysical systems in a rotating frame of reference, where there is no effect of rotation rate in standard energy stability theory and its extensions. The Prandtl number also plays no role in the bounding theory of thermal as well as double-diffusive convection. The fact that the theory does not depend on these parameters does not mean that the system is insensitive to them, but is merely a result of throwing away the governing PDEs and keeping only certain integral equations derived from them. Thus, the problem facing us is to add more integral constraints in order to incorporate the missing physics [207].

Here, we have identified and exploited a key constraint for doubly diffusive convection. The role of this constraint in energy stability theory is instructive, and amounts
to generalizing the definition of the energy function so that one can suppress transient amplification in the absence of finite-amplitude instability. The constraint, however, is far from sufficient in describing all the features of double-diffusive convection. In fact, the generalized energy stability threshold still seems to fall short of where we expect nonlinear solutions to come into existence. This leaves one suspicious that there may still be inconsequential transient amplification above threshold, and prompts the two key questions: Is it possible to differentiate between such transient growth and a true finite amplitude instability? Is it possible to improve energy analysis further so that the loss of energy stability always signifies a linear or nonlinear instability?

The bound we have derived is discontinuous along the energy stability boundary. Such jumps could reflect the appearance of additional finite-amplitude solutions in saddle-node bifurcations, an eventuality that certainly occurs for double-diffusive convection. Unfortunately, our numerical computations offer little evidence that this is the main cause of the discontinuity. The jump could also have been introduced because we have used piece-wise linear background fields. Forcing the backgrounds to be smooth removes any discontinuity of this kind in the Rayleigh-Bénard problem. For the current problem, however, the difficulty is far more insidious: one can establish for the simplified variational formulation in §4.4.2 that the bound remains discontinuous even for smooth backgrounds fields. The only remaining possibility for further progress in using the bounding machinery to detect saddle-node bifurcations and the like is to retain the more general version variational problem in §4.4.1.

Finally, the background method is geared towards extending energy stability theory to find the properties of the solution with the biggest norm. While this method has provided us with some useful insight, other modifications of energy stability theory must also be possible. In particular, it is conceivable that one may be able to incorporate thresholds on the norm of perturbations that decay to the trivial state, thus allowing one to extend the energy stability threshold for sufficiently “small” disturbances. Such a method could address important issues like the abrupt transition to turbulence in some shear flows. Double diffusive convection remains a rich testing ground for all such future developments.
4.7 References


Chapter 4. Bounds on double diffusive convection


Chapter 5

Energy stability of Couette flow

5.1 Introduction

Plain Couette flow is the flow of a viscous fluid confined between two infinite, parallel plates (see figure 5.1). The bottom plate is held stationary while the top plate moves with unit dimensionless speed. Mathematically, the flow is described by the Navier-Stokes equations as

\[ v_t + v \cdot \nabla v + \nabla p = \frac{1}{R} \nabla^2 v \quad \text{and} \]
\[ \nabla \cdot v = 0, \]

where \( v \) is the velocity field, \( p \) is the pressure and \( R \) is the Reynolds number based on the speed of the top plate and the gap-width between the plates. For small enough \( R \) the flow that is attained is the unidirectional flow with a uniform shear as shown in figure 5.1. However, experimentally the uniform shear flow graduates to a complicated

\[ u_0 \]
\[ q \]
\[ y \]
\[ x \]

Figure 5.1: The schematic setup for plane Couette flow.
three-dimensional time-dependent flow for larger \( R \). Unfortunately, this transition to turbulence cannot be explained through an linear instability of the basic flow because the basic flow is stable for all values of \( R \) as shown by Romanov [238]. That makes computation of solutions other than the uniform shear flow difficult. Flows in other geometries that in limiting cases converge to plain Couette flow are used to find such solutions (see [232, 233, 235]). Three-dimensional steady solutions of the system for values of \( R \) as low as 600 have been computed in this fashion. A similar situation is also posed for the plane Poiseuille flow where transition to turbulent states is observed for Reynolds numbers much below the onset of linear instability at \( R = 5772 \).

Several approaches are taken by researchers to explain this discrepancy, the most popular of which is the possibility of “transient growth” due to non-normal linear modes. This transient growth coupled with nonlinearities can give rise to finite amplitude instabilities, as was demonstrated for a number of toy models [230, 240]. Verification of this hypothesis for shear flows, however, remains elusive.

A disturbing fact about the business of transient growth is that the growth of a particular energy norm of the perturbations may not mean the absolute growth of perturbations themselves. To distinguish between the two consider a linear system given by \( \dot{q} = Aq \) where \( q \) is a two dimensional vector and \( A \) is a \( 2 \times 2 \) matrix with two non-orthogonal eigenvectors, \( u \) and \( v \). If both eigenvectors have eigenvalues with negative real parts then the solution decays exponentially, \( q = aue^{\lambda_u t} + bve^{\lambda_v t} \), where \( \lambda_u \) and \( \lambda_v \) are corresponding eigenvalues and \( a, b \) are arbitrary constants of integration which depend on initial conditions. It is well known that for some initial conditions, \( \|q\| \) grows initially before the eventual exponential decay. However, if we resolve \( q \) on the two eigenvectors as \( q = au + \beta v \), then the quantity \( \sqrt{\alpha^2 + \beta^2} \) decays monotonically to zero. This goes to show that transient growth of the energy depends on the coordinate system used to describe the problem. Alternatively, in a given coordinate system, even if the 2-norm grows transiently, in a transformed coordinate system a similarly defined 2-norm can decay monotonically.

In this article, we follow this line of thought using energy methods to describe the evolution of norm of perturbations in uniform shear flows [234, 239]. These methods are different from linear stability analysis in the sense that by proving monotonic decay of a norm (called the generalized energy) they predict global stability of a solution.
Using this technique for Couette flow, Orr [236] calculated the criteria for stability towards perturbations independent of the spanwise direction to be $R = 177.22$. Joseph [234] later showed that the flow is more vulnerable to energy instability in the form of perturbations independent of the streamwise direction where the stability boundary comes out to be $R = 82.65$. Above these critical values, there exist perturbations that grow initially and may either saturate to finite values signifying instability or decay eventually denoting stability. The motivation of this paper is the hope, that perhaps for the case of initial growth of the kinetic energy, another energy norm of the velocity perturbations that decays monotonically can be used to distinguish between instability and transient growth.

We are not able to prove monotonic decay of any norm other than the one given by the kinetic energy for three-dimensional perturbations. Instead, we find an generalized energy and a criteria for the stability of uniform shear flow to two-dimensional perturbations based on such an energy. We begin by formulating the generalized energy analysis in section §5.2. The stability calculation is carried out in §5.3 and we discuss the results in §5.4.

### 5.2 Generalized energy formulation

We will choose a coordinate system in which the $z$-axis is normal to the walls and the $x$-axis makes an angle of $\theta$ with the direction of motion of the top plate (see figure 5.1). The equilibrium velocity field is then given by

$$v = v_0 = U(z)(\cos \theta, \sin \theta, 0), \quad (5.3)$$

where $U(z) = z$ for the Couette flow. We begin by perturbing the uniform shear flow as

$$v = (U(z) \cos \theta + u(x,t), U(z) \sin \theta + v(x,t), w(x,t)), \quad (5.4)$$
where \( u(x,t), v(x,t) \) and \( w(x,t) \) are perturbations periodic in \( x \) and \( y \). The perturbed equations become

\[
\begin{align*}
    u_t + U(\cos \theta u_x + \sin \theta u_y) + u \nabla u + p_x &= \frac{1}{R} \nabla^2 u - w \cos \theta U_z, \\
v_t + U(\cos \theta v_x + \sin \theta v_y) + u \nabla v + p_y &= \frac{1}{R} \nabla^2 v - w \sin \theta U_z, \\
w_t + U(\cos \theta w_x + \sin \theta w_y) + u \nabla w + p_z &= \frac{1}{R} \nabla^2 w \quad \text{and} \\
    u_x + u_y + w_z &= 0.
\end{align*}
\]

To denote volume averages, we adopt the notation

\[
\int_{-L_x}^{L_x} \int_{-L_y}^{L_y} \int_0^1 f \, dz \, dy \, dx = 4L_x^2 L_y^2 \langle f \rangle
\]

where \( L_x \) and \( L_y \) are periodicities of the perturbations in the respective directions. The separate energy integral equations are evaluated as follows:

\[
\begin{align*}
    \left\langle \frac{u^2}{2} \right\rangle_t &= -\langle up_x \rangle - \frac{1}{R} \langle |\nabla u|^2 \rangle - \langle (U'uw) \cos \theta \rangle, \\
    \left\langle \frac{v^2}{2} \right\rangle_t &= -\langle vp_y \rangle - \frac{1}{R} \langle |\nabla v|^2 \rangle - \langle (U'vw) \sin \theta \rangle, \\
    \left\langle \frac{w^2}{2} \right\rangle_t &= -\langle wp_z \rangle - \frac{1}{R} \langle |\nabla w|^2 \rangle.
\end{align*}
\]

We combine these integrals by taking suitable linear combinations to give

\[
\frac{d}{dt} \left( \frac{u^2 + c^2 v^2 + b^2 w^2}{2} \right) = -\langle U'uw \rangle \cos \theta - c^2 \langle (U'vw) \sin \theta \rangle + \langle p(u_x + c^2 p_y + b^2 w_z) \rangle - \frac{1}{R} \langle |\nabla u|^2 + c^2 |\nabla v|^2 + b^2 |\nabla w|^2 \rangle, \quad (5.13)
\]

where \( b \) and \( c \) are positive constants. For brevity, let

\[
\begin{align*}
    \mathcal{E}[u,v,w] &= \left\langle \frac{u^2 + c^2 v^2 + b^2 w^2}{2} \right\rangle, \\
    \mathcal{D}[u,v,w] &= \langle |\nabla u|^2 + c^2 |\nabla v|^2 + b^2 |\nabla w|^2 \rangle \quad \text{and} \\
    \mathcal{G}[u,v,w,p] &= \langle U'uw \rangle \cos \theta + c^2 \langle (U'vw) \sin \theta \rangle + \langle p(u_x + c^2 v_y + b^2 w_z) \rangle.
\end{align*}
\]

Equation (5.13) can now be written as,

\[
\frac{d\mathcal{E}}{dt} = -\mathcal{G} - \frac{1}{R} \mathcal{D} \quad (5.17)
\]

the quadratic positive definite functionals \( \mathcal{E} \) and \( \mathcal{D} \) are called the generalized energy and dissipation terms. The non-definite term in the equation, \( \mathcal{G} \), is called the generation
Chapter 5. Energy stability of Couette flow

The stability result in this case is derived through the following variational problem:

\[
\frac{1}{R_c} = \min_{\{u, p \mid \nabla \cdot u = 0\}} \mathcal{G} \tag{5.18}
\]

s.t. \( \mathcal{D} = 1. \) \( \tag{5.19} \)

The solution of this optimization will give the critical Reynolds number, \( R_c \). Now equation (5.13) can be manipulated as

\[
\frac{d\mathcal{E}}{dt} = -\mathcal{G} \geq \frac{\mathcal{D}}{R} \leq \left( \frac{1}{R_c} - \frac{1}{R} \right) \mathcal{D} \tag{5.20}
\]

If \( R \) is smaller than \( R_c \), then for all perturbations, the dissipation term dominates over the generation term causing the energy, and consequently the perturbations, to decay monotonically. However, as we increase \( R \) above \( R_c \), there exist perturbations \((u, v, w)\) such that the generation term may dominate over the dissipation term leading to a potential instability. Note that, in this case all we can definitely say is that the energy does not decay monotonically anymore. It may experience a transient growth and a subsequent decay (as in the case of systems with non-normal modes) or it may grow and saturate to a finite value denoting instability.

The generation term contains a nasty average involving the instantaneous pressure field, which is intimately related to the velocity perturbations through the incompressibility constraint. We do not know how to bound the generation term as written in (5.16) to get a meaningful result.

Previously, a special case corresponding to \( b = c = 1 \) of this equation was first derived by Orr [236] in an attempt to improve the technique used by Reynolds [237]. The equation, called the Reynolds-Orr equation, causes the integral term involving pressure to drop out from the generation term owing to the continuity equation. This avoids further complications due to the dependence of the pressure on instantaneous velocity field. The result of considering this special case, \( R_J(\theta = \alpha) \), is shown in figure 5.2.

We follow a similar route by using continuity to eliminate the pressure term from the analysis. We limit ourselves to two dimensional perturbations, thereby avoiding the path taken by Reynolds, Orr and Joseph and still make use of incompressibility to eliminate the pressure term from the energy generation.
Specifically, we choose perturbations independent of \( y \) thus modifying the generation integral as

\[
\mathcal{G}[u,v,w,p] = \langle U'uw \rangle \cos \theta + c^2 \langle U'vw \rangle \sin \theta + \langle p(u_x + b^2 w_z) \rangle .
\]  

(5.21)

The gradients in the dissipation term also do not have any \( y \) derivatives and the continuity equation becomes

\[
u_x + w_z = 0
\]

(5.22)

Forcing \( b = 1 \) now eliminates the pressure integral from the generation term. Notice, however, that we still have the freedom of choosing \( c \).

### 5.3 Energy stability

The variational problem is solved by writing the Lagrangian,

\[
\mathcal{L} = \mathcal{G} - \frac{D - 1}{R_c} - \langle r(u_x + w_z) \rangle .
\]

(5.23)

The Euler-Lagrange conditions for stationarity are

\[
\frac{\delta \mathcal{L}}{\delta u} = U'w \cos \theta + r_x + \frac{2}{R_c} \nabla^2 u = 0, \quad (5.24)
\]

\[
\frac{\delta \mathcal{L}}{\delta v} = c^2 U'w \sin \theta + \frac{2c^2}{R_c} \nabla^2 v = 0 \quad (5.25)
\]

\[
\frac{\delta \mathcal{L}}{\delta w} = U'(u \cos \theta + c^2 v \sin \theta) + r_x + \frac{2}{R_c} \nabla^2 w = 0 \quad \text{and, } (5.26)
\]

\[
\frac{\delta \mathcal{L}}{\delta r} = -(u_x + w_z) = 0. \quad (5.27)
\]

Along with the boundary conditions

\[
u = v = w = 0 \text{ at } z = 0, 1,
\]

(5.28)

this is an eigenvalue problem for \( u, v, w, r \) and \( R_c \). The value of \( c \) is chosen so as to maximize the critical Reynolds number (this being the essence of generalized energy analysis).

We demonstrate a solution for the case of plane Couette flow \((U(z) = z)\) by eliminating \( u, v \) and \( r \) to obtain an equation for \( w \) as

\[
\nabla^6 w - \frac{c^2 R_c^2 \sin^2 \theta}{4} w_{xx} - R_c \cos \theta \nabla^2 w_{xx} = 0.
\]

(5.29)
Figure 5.2: \( R_J \) as a function of \( \alpha \) is shown on the left. On the right, the critical Reynolds number for 5 values of \( c \) are shown.

The boundary conditions on \( w \) are \( w = w_z = \nabla^4 w = 0 \) on \( z = 0,1 \).

This differential equation is a small modification of the one treated by Joseph [234] (obtained by putting \( c = 1 \)) and so is the solution. Using \( c = 1 \) the equation becomes

\[
\nabla^6 w - \frac{R_J^2 \sin^2 \alpha}{4} w_{xx} - R_J \cos \alpha \nabla^2 w_{xx} = 0.
\]

This equation permits a solution of the form

\[
w(x, z) = W(z)e^{ikx}
\]

thus getting rid of the \( x \)-dependence but adding a parameter \( k \), the horizontal wavenumber, to the problem. The critical Reynolds number obtained depends on this wavenumber. The most dangerous wavenumber is the one with the minimum value for \( R_J(\alpha) \).

The dependence of this critical value on \( \alpha \) is plotted in figure 5.2.

The problem with \( c \neq 1 \) can now be solved using the transformation

\[
R_c(\theta; c) = \frac{R_J(\alpha)}{\sqrt{c^2 \sin^2 \theta + \cos^2 \theta}}, \quad \text{where}
\]

\[
\tan \alpha = c \tan \theta.
\]

The panel on the right of figure 5.2 shows this critical Reynolds number for different values of \( c \).

Our goal here is to choose a value for \( c \) so as to maximize \( R_c(\theta; c) \) for each \( \theta \). By reducing the value of \( c \), it can be seen that the denominator of (5.32) decreases,
increasing the value of $R_c$. At the same time, $\alpha$ in (5.33) decreases causing an increase in $R_J(\alpha)$, which is the numerator of (5.32). This make the most favourable value of $c$ to be zero, in which limit the critical Reynolds number has a simple dependence on the angle $\theta$ given by

$$R_c(0; 0) = \frac{R_J(0)}{\cos \theta} \approx \frac{177.21}{\cos \theta}. \quad (5.34)$$

The critical Reynolds number for plane Poiseuille flow is also calculated similarly and is plotted in figure 5.3. The expression for critical Reynolds number is

$$R_c = \frac{175.18}{\cos \theta}, \quad (5.35)$$

where 175.18 is the Reynolds number obtained from the energy stability of plane Poiseuille flow [231]. In general, the two-dimensional criterion for stability is

$$R \leq \frac{R_{\text{energy}}(\theta = 0)}{\cos \theta}, \quad (5.36)$$

where $R_{\text{energy}}(\theta = 0)$ is the critical Reynolds number obtained from the energy stability for perturbations aligned with the equilibrium flow.

### 5.4 Discussion

An immediate consequence (5.34) is that plane Couette flow is stable towards all perturbations independent of streamwise coordinate. This can be seen by taking $\theta = \pi/2$
and seeing that $R_c$ becomes infinite. This means that, although the kinetic energy for perturbations may grow for Reynolds number above 82, as discovered by Joseph, this growth is only transient. By using a generalized energy, we have differentiated between transient growth and instability. This conclusion, though, must be taken with a pinch of salt because the addition of slight three-dimensionality to the perturbation revives the possibility of its growth. Thus, although we have demonstratedly proved that the critical Reynolds number to all perturbations is not the one predicted by Joseph, we have only marginally improved on its value in this analysis.

As a function of the angle $\theta$ made by the equilibrium flow with the perturbation direction, the critical Reynolds number is monotonic. Amongst all two-dimensional perturbations, the most critical is independent of the spanwise coordinate. The value of the critical Reynolds number in this case is 177, identical to the value calculated by Orr, which we were not able to improve.

The difficulties surfaced during the course of this analysis attract attention towards issues that do not seem to be widely expressed in the literature. In particular, although the nonlinearity in the advection term $(v \cdot \nabla v)$ is accepted to be a hindrance in successful analysis of this problem, there is little mention of incompressibility playing any role. The justification for incompressibility contributing to the difficulty comes from the observation that the generation term involving the pressure integral in (5.16) makes this analysis difficult in general. The pressure is dependent on the instantaneous velocity field and the pressure gradient term in the Navier-Stokes equations is really a nonlinear term in velocities. In fact, it was that term that forced us to focus only on two-dimensional perturbations. If one considers a hypothetical problem of a perfectly compressible fluid (one without the continuity equation (5.2) and the pressure variable set to zero), then it can be shown that plane Couette flow is unconditionally stable to all perturbations. This is indicative of a fundamental dependence of the instability mechanism on incompressibility.

Finally, the results have some implications on the numerical calculation on nonlinear states of plane Couette flow. Since perturbations, irrespective of their magnitude, decay to zero below the $c = 0$ curve in figure 5.2, the uniform shear flow is an unique solution to the problem in that regime. So far the two-dimensional solutions discovered by Cherhabili & Ehrenstein [232] are way above this curve and are independent of the
spanwise direction. If any oblique states are found, they will be below the $c = 0$ curve. This limits the parameter space to be explored numerically to find any such states.

5.5 References


Chapter 6

General conclusions and future directions

Individual chapters on the four problems dealt with in this thesis, have their own conclusions and discussions at their end. In this chapter, we provide a general view of what was achieved in this thesis. This chapter can be treated as an overview of the salient results obtained in the previous chapters. It is written for a reader who is not particularly interested in the intricate details of the derivations, but instead is more keen on applying the results to a higher level problem. Some information already discussed in the previous chapter is repeated with the intention of collecting all the results in a place and making them accessible to the reader.

The organization of this chapter is as follows. Discussion on the interfacial instabilities is divided into two subsections of §6.1. The first subsection deals with roll waves and the second with oscillatory elastic instabilities. The second part of this thesis on energy stability and its extensions is also similarly treated in §6.2, with the first subsection on bounding double diffusive convection and the second on the energy stability of Couette flow.

6.1 Interfacial instabilities

6.1.1 Roll waves

When a uniform, turbulent, thin film of water flows down an incline it may become unstable to wavy perturbations. One-dimensional shallow water equations with bottom drag and turbulent diffusivity, also known as the St. Venant equations, were used to model this flow as a representative from the family of such models for thin film flows.
in various regimes. Another member of this family are the Shkadov equations, used to describe flows of thin laminar films. The characteristic parameter that controls the instability is the Froude number, which can be interpreted as the ratio of the flow speed to the speed of shallow water gravity waves. When the Froude number just exceeds 2, perturbations with very large wavelength are destabilized. When the Froude number is above 2, the linear instability growth rate is maximum for a finite wavelength.

**Linear stability with bottom topography**

In the presence of small, periodic bottom topography of small wavelength (such that the maximum perturbation in slope is $O(1)$), long waves are destabilized for Froude numbers even smaller than 2. An asymptotic expression for this new critical Froude number was found analytically (see §2.5). Numerically carried out linear stability analysis also shows moderate decrease in the critical Froude number for small amplitude topography. Exactly the opposite is observed for the Shkadov model; bottom topography stabilizes the uniform flow and larger Froude numbers are required for waves to grow. As the amplitude is increased further, the equilibrium flow develops hydraulic jumps. The existence of these hydraulic jumps is seen to destabilize the steady flow at as low Froude numbers as 1.4 (see figure 2.11 and §2.4.1 for turbulent waves and figure B.1 for laminar).

**Nonlinear dynamics**

The nonlinear asymptotic theory for small amplitude topography near onset is then used to study the nonlinear dynamics of these roll waves. This theory furnishes an amplitude equation for the evolution of roll waves. Switching the bottom topography off merely changes some of the coefficients in this equation, leaving the same canonical form (see (2.55)). The same equation was derived by Yu & Kevorkian [261] for flat inclines. Thus a unified treatment of the roll-wave dynamics with or without bottom topography is possible. A comparison of the solution of this evolution equation with that of the original St. Venant model shows good agreement. Similar amplitude equation derived from the Shkadov equations, differs from the one derived from the St. Venant equations only by the values of some of the coefficients.

The equation reduces to a modified Burger's equation derived by Kranenburg [253] on short length scales and to a generalized Kuramoto-Sivashinsky equation found by
Chapter 6. General conclusions and future directions

[262] for long turbulent waves. The evolution equation predicts that sinusoidal waves grow starting from random perturbations, but nonlinearities soon take over and lead to propagating bores. The fastest growing mode determines the approximate wavelength of roll waves that appear first. The waves then undergo a process of merging, which can be viewed as the manifestation of a subharmonic instability. That causes an increase in the wavelength. This behaviour is predicted by the modified Burger's equation, in the short wave regime. As the wavelength increases, the merging stops and a stable periodic wavetrain emerges. This corresponds to a stabilization of the subharmonic instability. These observations are corroborated by a linear stability analysis of a periodic wavetrain as seen in figure 2.23. The stability analysis and numerical solutions also show that very long wavelength wavetrains undergo a spawning instability where new waves are formed in between existing waves of a wavetrain.

Comparison of predicted wavelength with experiments

Experiments were performed to verify these predictions. Figure 2.29 shows a comparison of the experimental stability results with theoretical predictions using the amplitude equation. The flow corresponded to a Froude number of 2.5, which seems to be beyond the quantitative validity of the amplitude equation. However, there is qualitative agreement in the sense that all of coarsening, stable and spawning regimes are observed. The theory predicts that wavelengths longer than 7 (in terms of the horizontal length scale) are stable. Experimentally, for the Froude number studied, wavelengths above 20 are found to be stable. Spawning instability is predicted for wavelengths larger than about 50, and the corresponding experimental number lies in the range 40-60.

Criticism and future directions

This treatment suffers from many imperfections, all contributing to the possible disagreement between theoretical predictions and experimental observations. For example, the channel length used for experiments was finite. Thus, it is possible that flows for which coarsening was not observed may have displayed coarsening if the channel was long enough. An experiment with a longer channel can be performed to check the sensitivity of the results to the channel length.

A perturbation theory is used to derive an amplitude equation valid only near the
Chapter 6. General conclusions and future directions

theoretical onset of the instability. The question about how close to onset is close enough for the validity of this theory can only be answered experimentally. Our experiments may very well be beyond the asymptotic regime. This concern can be resolved by dealing directly with the St. Venant equations without simplification.

Finally, the St. Venant equations used are phenomenological in nature. While there is a good amount of thought gone into their structure, they are in no sense rigorous. A uniform velocity profile is assumed across the film thickness, the pressure is assumed to be hydrostatic and the bottom drag is parametrized using an empirical law. Currently, there is no remedy to this; direct numerical simulations of thin films of turbulent flows with a free interface over such long domains may be possible as the memory and speed of digital computers increase, but they are not possible at present. The Shkadov equations, on the other hand, are also ad hoc, but a proper thin film theory can be derived from first principles to lead to a set of equations very similar to the Shkadov equations [257, 258, 259]; the values of only some coefficients are changed.

6.1.2 Flow induced elastic oscillations

The flow of a fluid through a narrow channel made in an elastic substance can excite elastic oscillations. We have looked at the possibility that these oscillations are a perturbed version of the free elastic modes. Free elastic modes, that normally decay because of the dissipation in the elastic body as well as radiation of sound to the surrounding medium, can be made to grow if the flow through the channel is fast enough.

Mathematical model and simplifications

The fluid flow is assumed to be laminar, making a first-principles approach starting from the Navier-Stokes equations possible. The dimensionless parameters that enter the flow problem are the Reynolds number, the elastic Mach number and the channel aspect ratio. Certain simplifications result from the assumption of a long, narrow channel, the details of which can be found elsewhere in the literature [257, 258, 259]. We adopt this formulation, which retains the effect of viscosity and inertia. In the limit of a vanishing aspect ratio (very long and narrow) or vanishing Reynolds number, this model reduces to the well-known lubrication approximation. However, we find that inertial terms cannot be neglected as they provide the destabilizing mechanism for the elastic modes.
The fluid flow is coupled with the motion of an elastic structure that forms the walls of the channel. The displacements and the stresses at the interface of the two materials must match. This introduces a parameter that measures the stiffness of the elastic structure to the stresses in the fluid. Based on the assumption that the structure is stiff and the structure is almost non-dissipative, the analysis can be carried out without the precise knowledge of the geometrical and material details of the structure.

To demonstrate this, a block of elastic solid, through which the channel is carved, is considered as an example. A linear Hookean law models the elasticity and a viscosity is assumed to account for the dissipation. Three parameters enter this model. The first one is the ratio of the Lamé constants, which can be thought of as related to the Poisson ratio. The other two are the non-dimensionalized versions of the solid shear and bulk viscosities, assumed to be small. This model is only valid when the displacements in the solid are small, which is exactly the regime we are interested in.

To get an idea about the effect of finite stiffness of the elastic body, a simpler structure is used. A channel flow between two stretched membranes is considered. The simple model for the stretched membrane allows us to relax the assumption of a stiff structure.

**Instability mechanism**

A physical interpretation of the analysis exploiting the asymptotic limit of very stiff elastic walls is also made. To leading order, the fluid flow is too weak to influence the motion of the structure. The dissipation is also very small, so the structure exhibits undamped, natural modes of elastic oscillations. These oscillations open and close the channel at different locations, depending on the mode of oscillation that sets in, and pushes the fluid around. This induces minor variations in the fluid pressure, which force a feedback on the elastic mode. The feedback is considered positive when it increases the mechanical energy of the elastic modes by doing positive work on the structure. The condition for a positive feedback is that on average the fluid pressure should drop when the channel is closing and vice versa.

Positive feedback from the fluid essentially comes from the dependence of the fluid pressure on velocity similar to Bernoulli's principle. When the channel is closing, fluid is squeezed out and the flow velocity increases. Bernoulli's law then translates this
increase in velocity to a decrease in pressure, thus satisfying the condition for a positive feedback. Exactly the opposite happens when the channel is closing. Mathematical analysis shows that this Bernoulli pressure-velocity dependence needs to be imposed at the exit of the channel for this mechanism to materialize. Bernoulli principle is an inertial phenomenon, thus fluid inertia is found to be destabilizing the modes. On the other hand, viscous and dissipative effects provide a negative feedback, i.e. they remove energy from the elastic modes and can be considered stabilizing. An instability ensues when the inertial effects dominate over viscous and dissipative ones. Thus the characteristic parameter for an elastic mode to grow is the flow Reynolds number. The instability criterion is independent of the elastic Mach number; instability can set even in the limit of infinite elastic wave speed.

Mode selection

There are infinitely many modes of free oscillations possible for an elastic body. As the Reynolds number of the flow in increased, the first mode to be destabilized will be observed in practice. Thus modes are selected based on the their critical Reynolds number. Typically, modes with higher frequencies have a smaller scale spatial structure associated with them.

From numerical computation of the modes for a two-dimensional elastic block and the asymptotic analysis of the flow suggests the feedback from inertia is approximately the same as the mode frequency increases. On the other hand, the viscous feedback from pressure associated with the modes of higher frequency is weaker. Thus in the absence of any dissipation in the solid, modes of higher and higher frequencies will be destabilized before the modes of lower frequencies.

But the viscous dissipation in the elastic body also depends on the spatial scale and structure of the mode. In general, high frequencies and fine spatial scales corresponds to increased dissipation. Thus, the dissipation from the elastic body will inhibit the instability for high frequency modes. A balance between the two effects of dissipative effects in the fluid that favours high frequency modes and in the solid that favour the low frequency modes gives rise to an intermediate mode that has the lowest critical Reynolds number. The stretched membrane also shows a similar mode selection mechanism.
Acoustic excitation

A simple experiment was devised to show that the mechanisms proposed for excitations of acoustic modes in Helmholtz oscillators are incomplete, at best. The proposed mechanisms all hinge on an sinuous perturbation of the jet to drive the feedback mechanism, whereas a varicose mode was observed in our experiments. This gap can be partly filled in by extending the analysis developed earlier for destabilization of elastic modes towards acoustic excitation. The flow of a thin film has to be replaced by the flow of jet and the elasticity by the compressibility of air.

An ad hoc model, similar in spirit to the thin-film model, was used. The limit of a stiff elastic body corresponds to having the length of the acoustic cavity much longer than the mouth (as flutes, pan-pipes, organ pipes, recorders and even beverage bottles are usually designed). The instability mechanism is analogous to the one discussed for elastic instabilities.

Criticism and future directions

On the down side, the instability mechanism is hinged on the exit boundary condition which is a matter of controversy. In principle, the exit boundary condition depends on the what is “beyond” the channel exit. The fluid flow outside the channel is simply parametrized in terms of the Bernoulli-like boundary condition (3.32). In a sense, this identifies the basic element responsible for the instability to be the pressure-velocity dependence at the exit. But it will be much more satisfying to write a more general solution of the governing equations beyond the exit of the channel and then derive the boundary condition from it. One such attempt is made in the thesis, but the state of affairs is still far from satisfactory.

Experimentally, flow through a channel made in an elastic block is seen to show multiple states of oscillations. This can be rationalized as multiple modes being destabilized and selected based on a nonlinear criteria. The role of nonlinearities was completely ignored for this problem in this thesis. It may be of interest to indulge into a proper account of the nonlinear dynamics of the mode selection process.

For the explanation of the acoustic instability, the state of affairs is also far from complete. The mathematical model used by us serves the purpose of phenomenologically justifying the possibility of the varicose oscillations observed in experiments. But the
model is written down \textit{ad hoc}. The jet is assumed to be of constant thickness, the location of center of the jet was left unperturbed and physical effects like inertia and drag were parameterized empirically. This criticism is very much reminiscent of the criticism of the St. Venant equations. A long wave theory of the Bickley jet \cite{241} may be able to remedy the situation by linking the Navier-Stokes equations to this model.

The sinuous mechanism proposed for this instability is quite popular and has been supported by experimental measurements. But a mathematical explanation from a first principles perspective is still missing. Incorporating the position of the center of the jet as a variable in the long wave theory of the jet may be able provide an analytical handle on the sinuous instability mechanism.

Experiments conducted by us show an abrupt transition from a varicose mode to a sinuous mode. This transition is not understood at all. More experiments need to be performed to indentify the parameters on which this transition depends before an explanation can emerge.

From a more general perspective, there is a host of other flow situations, that may be susceptible to this kind of instability. The only necessity is that oscillatory normal modes interact with the flow of a thin film or jet. Sloshing instigated by interaction with a jet have been reported in the literature \cite{260}. Flows in different geometric configurations, like flow past a flag or an airplane wing, may also be susceptible to this kind of instability. The underlying mechanism is whether the fluid pressure provides positive feedback on the elastic oscillations.

\section{6.2 Energy stability and its extensions}

\subsection{6.2.1 Bounds on double diffusive convection}

Double diffusive convection can lead to a myriad of possibilities. The convection can be steady at onset, or it can be oscillatory. The system can be linearly unstable even if it is gravitationally stable. A transition to nontrivial state can happen despite the trivial solution being linearly stable. The approach taken to understand double diffusive phenomena is through functional analysis like energy stability theory and the background method of Doering \& Constantin \cite{245, 246, 247} to bound the species transport.
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Energy stability

A generalized energy stability analysis of double diffusive convection was carried out by Joseph [249, 250]. We have provided more details of this analysis and have more thoroughly interpreted the energy stability methodology and results.

For thermal convection, the energy stability condition agrees with the linear instability threshold. This property carries over to double diffusive convection, when the species with the stabilizing density gradient has the faster diffusivity of the two species. However, if the stabilizing species diffuses slower than the other, the energy stability criteria coincides with linear instability only when the stabilizing density gradient is relatively weak. As the stabilizing density gradient is made stronger, energy stability boundary departs from the linear instability threshold.

This disagreement between the energy stability condition and linear instability threshold can be attributed to three causes. The first possibility is the existence of steady, periodic or statistically steady nonlinear solutions below the linear instability threshold. Such solutions make the basic state non-unique and consequently, the energy may not decay to zero at all.

The second possibility is that the energy is not generalized enough to sufficiently constraint the function space. The perturbation that shows the growth of energy, under such circumstances, may not be a solution of the governing equations at all. In other words, a family of governing equations may lead to the same energy evolution equation. The energy stability result has to be valid for every member of this family. Appearance of a nonlinear state in a even a single member will correspond to loss of energy stability. This appearance could well be for a different set of governing equations and below the linear stability threshold of the governing equations of our interest.

The final cause for the loss of energy stability is the possibility of transient growth, which is a purely linear phenomena. The non-orthogonality of linear eigenvectors may cause the energy to grow transiently, even when individual eigenmodes decay exponentially. Energy stability theory has to honour such growth and consequently fail to provide a conclusive stability statement.
Bound on species flux

A piece-wise linear background profile was used to calculate the bound. The bounds calculated behave like $R_T^{1/2}$ for fixed $R_S/R_T$ as $R_T \to \infty$. This is very similar to thermal convection, where the bound shows a similar scaling. The prefactor to this scaling law depends on the stability number $(R_S/R_T)$ and the ratio of diffusivities ($\beta$).

By better accounting for the second variation analytically, which ensures that the extremum obtained is indeed a maximum, we have improved the prefactor to the scaling law. A comparison with thermal convection, which is a special case of double diffusive convection with a stability number of zero, shows this improvement over previous treatments by Nicodemus, Grossman & Holthaus [254]. Using piece-wise linear background profiles and some crude bounding methods, they found the prefactor to be $3\sqrt{3}/16$. The Rayleigh number at which the bound departed from unity was 64 for this calculation. The prefactor calculated by us is $3\sqrt{3}/2\sqrt{R_c}$. The bound departs from unity at the energy-stability critical Rayleigh number and it has an explicit dependence on the value. This improvement in the prefactor carries over to double diffusive convection as well.

As the stability number is increased above zero, the prefactor starts to diminish. It decreases continuously until the energy stability condition is reached, at which point the bound precipitously and discontinuously drops to zero. This discontinuity is pronounced for the salt-fingering case, where the continuous decrease in the bound is minimal. The exception is the limit of small $\beta$, for which the bound has no discontinuity at the energy stability threshold.

Discontinuity in bound and nonlinear dynamics

Appearance of nonlinear solutions in a saddle node can cause the maximum species transport to be discontinuous. A bound, sufficiently faithful to the dynamics, should capture this discontinuity. However, the bound can also be discontinuous for the reason that the background profiles we have chosen for the computation are non-smooth. In an attempt to resolve the difference between the two possibilities, we solved a restricted version of the optimization problem asymptotically just beyond energy stability condition. However, even using smooth profiles does not remove the discontinuity.

We can isolate several reasons for the discontinuous jump in the bound. Firstly,
in order to keep things tractable, we had assumed a relation between the Lagrange multipliers in our formulation, so as to reduce the second variation constraint to a version obtained in thermal convection. This may have led to a sub-optimal bound.

The second reason is similar to the failure of energy stability theory to predict saddle-node bifurcations. The integral constraints used may allow certain functions that are not solutions of the governing equations. Hence, a saddle-node bifurcation in the integral equations may not correspond to anything in the differential equations. The third possibility is the choice of the velocity background profile. Experimentally, a large scale circulation, called “thermal wind”, is observed for thermal convection. It has been suggested [247] that the inclusion of a non-zero velocity background profile may better represent the physics and thus, further reduce the bound. This reduction may remove the discontinuity.

**Criticism and future directions**

The energy stability theory and the bound derived has no dependence whatsoever on the Prandtl number. In a way, it can be argued that the bound derived is valid and can be applied without requiring the explicit knowledge of the Prandtl number. But it will be more desirable to properly account for the Prandtl number since the dynamics seem to be sensitive to it. For example, the linear stability condition depends on the Prandtl number. An immediate motivation to incorporate Prandtl number dependence is the observation that energy stability boundary corresponds to the envelope of linear stability curves for different Prandtl numbers.

The bound on the species flux behaves like $R_{1/2}$. Whether the bound reflects the behaviour of the maximum possible species flux is still questionable. Stricter scaling for the bound, with an exponent of 1/3, has been calculated for the special case of infinite Prandtl number [244] by imposing the momentum conservation point-wise rather than in an average sense for thermal convection. Can the bound derived for the double diffusive case be improved in any such limiting cases?

The discontinuity in the bound at the energy stability boundary can furnish more information about any saddle-node bifurcations occurring there. An even better treatment of the second variation, possibly numerically, can help in improving the bound and removing the discontinuity where the nonlinear solutions bifurcate continuously.
In one way or another, the way to improve energy stability theory and its derivatives is to constrain the function space to better mimic the solutions of the governing equations. The identification of these key constraints that furnish useful information about the problem, yet keep the problem tractable, is required.

6.2.2 Energy stability of Couette flow

The energy stability of Couette flow had been studied relatively scarcely. Only the works of Reynolds [256], Orr [255], Busse [242] and Joseph [251] come to mind. In related contexts, Howard [248] had suggested that the way to gain more and more information is by successively constraining the possibilities. In a short treatise, we follow Howard’s vision and derive a better energy stability boundary by incorporating more integral constraints from the governing equations. This amounts to defining a family of energies and choosing the one that gives the best stability boundary. The generalized energy so crafted leads to nonlinear Euler-Lagrange equations, which are difficult to solve. We have avoided the nonlinearity at the expense of restricting perturbations to two dimensions. By two-dimensional we mean that the perturbations are chosen to depend on the coordinate direction normal to the channel walls and an arbitrary direction parallel to them. The critical Reynolds number calculated from this analysis depends on the arbitrary direction that defines the perturbation.

Relation with previous work

For one particular member of this family of energies, the nonlinearity drops out owing to continuity. The solution of the Euler-Lagrange equations is then easily possible without any further restriction. This particular energy was considered by Joseph & Carmi [252] and Busse [242]. The solutions to the Euler-Lagrange equations turn out to be two-dimensional, without any such assumption a priori. For each two-dimensional perturbation, a critical Reynolds number is identified. The lowest critical Reynolds number turns out to be 82.65 for spanwise perturbations. Below this Reynolds number, only the trivial solution to the perturbation equations can exist.
Chapter 6. General conclusions and future directions

Generalized energy analysis

In the spirit of Howard's suggestion, we have incremented our knowledge beyond the Reynolds number of 82.65 using generalized energy analysis. We have found that no two-dimensional non-trivial solutions can exist below a Reynolds number of 177.22. More generally, the energy stability condition gives a Reynolds number for each direction that parameterizes the perturbation. The efforts to compute two-dimensional nonlinear states of Couette flow [243] may find this result useful. The Reynolds number of 177.22, first derived by Orr [255], is the lowest of the critical Reynolds numbers and occurs for streamwise perturbations.

Using the generalized energy, the critical Reynolds number for spanwise perturbations turns out to be infinity. A perturbation, initially independent of the streamwise direction, will remain independent of the direction as time evolves. According to generalized energy analysis, such a perturbation will always decay to zero. Joseph's three-dimensional energy stability theory, however, attributes the lowest critical Reynolds number to such perturbations. The generalized energy analysis definitely shows that spanwise perturbations are the least vulnerable.

Criticism and future work

Of course, the generalized analysis becomes invalid as soon as the slightest three-dimensionality is introduced. A nonlinear eigenvalue problem needs to be solved for the Euler-Lagrange equations to locate the energy stability boundary. The nonlinear eigenvalue, which is related to the critical Reynolds numbers, can be a function of the amplitude of the perturbation. It is believed that the Reynolds number for transition to turbulence depends on the amplitude of the perturbation. The three-dimensional calculations by Joseph & Carmi and Busse are rigorous but they are linear. The critical Reynolds number they lead to is independent of perturbation amplitude and consequently their relevance in identifying the physical processes that lead to transition away from the basic state is questionable. On the other hand, the solution of the nonlinear eigenvalue problem is also expected to give rise to a dependence of the critical Reynolds number on the amplitude. But it is not clear how such an eigenvalue problem should be solved. What is even less clear is the extent to which energies should be generalized to get a true representation of the effects of nonlinearities in transition to turbulence.
These are all avenues for future research.

6.3 References


Chapter 6. General conclusions and future directions


Appendix A

The second expansion

In the second expansion, we introduce
\[ \partial_t \rightarrow \partial_t + \epsilon^2 \partial_\tau, \quad \partial_x \rightarrow \frac{1}{\epsilon} \partial_\eta + \partial_x, \quad \nu = \epsilon^2 \nu_2 \quad \text{and} \quad F = F_0 + \epsilon^2 F_2, \quad (A.1) \]

and
\[ u = 1 + \epsilon U_1(\eta) + \epsilon^2 [U_2(\eta) + u_2(x, t, \tau)] + \epsilon^3 [U_3(\eta, x, t, \tau) + u_3(x, t, \tau)] + \ldots \]
\[ h = 1 + \epsilon H_1(\eta) + \epsilon^2 [H_2(\eta) + h_2(x, t, \tau)] + \epsilon^3 [H_3(\eta, x, t, \tau) + h_3(x, t, \tau)] + \ldots \]

At leading order:
\[ F_0^2 U_{1\eta} + H_{1\eta} + A_\eta = 0, \quad U_{1\eta} + H_{1\eta} = 0, \quad (A.2) \]

with solution,
\[ U_1 = -\frac{1}{F_0^2 - 1} A(\eta) \equiv -H_1. \quad (A.3) \]

At order \(\epsilon^2\), we find inconsequential equilibrium corrections. At order \(\epsilon^3\):
\[ F_0^2 U_{3\eta} + H_{3\eta} = \nu_2 U_{2\eta} - 2U_2 + H_2 - (U_1 - H_1)^2 + \nu_2 H_{1\eta} U_{1\eta} - F_0^2 (U_1 U_2)_\eta \]
\[ -2F_0 F_1 U_{1\eta} - F_0^2 (u_{2t} + u_{2x}) - h_{2x} - 2u_2 + h_2 - F_0^2 U_{1\eta} u_2, \quad (A.4) \]
\[ U_{3\eta} + H_{3\eta} = -(H_2 U_1 + H_1 U_2 + H_1 u_2 + h_2 U_1)_\eta - h_{2x} - u_{2x} - h_{2t}. \quad (A.5) \]

We average over the fine length scale \(\eta\) to eliminate the corrections, \(U_3\) and \(H_3\):
\[ F_0^2 (u_{2t} + u_{2x}) + h_{2x} + 2u_2 - h_2 = -4U_1^2 - \nu_2 U_{1\eta}^2, \quad h_{1t} + h_{1x} + u_{1x} = 0. \quad (A.6) \]

To avoid exponential growth along the characteristics we impose
\[ h_2(\xi) = 2u_2(\xi) + \nu_2 U_{1\eta}^2 + 4U_1^2 = 2u_2(\xi) + \nu_2 A_\eta^2 + 4A_1^2, \quad (A.7) \]

and \(F_0 = 2\). We decompose the fine-scale variation into two parts:
\[ U_3 = \tilde{U}_3(\eta) + \tilde{U}_3(\eta) u_2(\xi, \tau) \quad \text{and} \quad H_3 = \tilde{H}_3(\eta) + \tilde{H}_3(\eta) u_2(\xi, \tau). \quad (A.8) \]
Appendix A. The second expansion

The solution, $\mathcal{U}_3$ and $\mathcal{H}_3$, is not needed. The other component satisfies

$$4\dot{\mathcal{U}}_3 - \dot{\mathcal{H}}_3 = -4\mathcal{U}_1, \quad \ddot{\mathcal{U}}_3 - \ddot{\mathcal{H}}_3 = -\mathcal{U}_1.$$  \hfill (A.9)

That is, $\dot{\mathcal{H}}_3 = 0$ and $\dot{\mathcal{U}}_3 = -\mathcal{U}_1 = A/3$.

At orders $\epsilon^3$ and $\epsilon^4$, we arrive at equations for $\mathcal{H}_4$, $\mathcal{U}_4$, $\mathcal{H}_5$ and $\mathcal{U}_5$, which are not needed. We skip directly to the $\eta$–averaged equations at order $\epsilon^4$:

$$h_{4\xi} - 2u_{4\xi} + 2u_4 - h_4 = 2F_2u_{2\xi} - 4u_2 - 4u_2u_{2\xi} - (u_2 - h_2)^2 + 4\mathcal{U}_1(2h_2 - u_2)$$
$$+ \nu_2 U_{1\eta}^2 h_2 + \nu_2 u_{2\xi} - 4U_{1\xi} U_{3\xi} + 4U_{1\xi}(H_3 - U_3) - 4U_{1\xi}^3$$
$$+ \nu_2 U_{1\eta}(H_3 - U_3) + H_{2\eta} U_{2\eta} - 4U_{2\xi}(U_2 - 2H_2)$$
$$- (U_2 - H_2)^2 + \nu_2 U_{1\eta}^2 H_2 + (H_2 - U_2)U_{1\eta} U_1, \quad (A.10)$$

$$h_{4\xi} - 2u_{4\xi} = 4u_2 + 2(h_2 u_2) + 2(U_1 H_{3\xi} + U_{3\xi} H_1). \quad (A.11)$$

Lastly, we eliminate the combination, $2u_4 - h_4$, to arrive at (2.47).
Appendix B

The laminar model

Throughout this article, we have used the turbulent drag law (2.3) to provide a closure to equations (2.1) and (2.2). Here we provide linear stability and asymptotic results using the laminar law (2.4).

Results from the linear stability analysis for the laminar problem reveal a slightly different picture than for the turbulent counterpart. As seen in figure B, when topography is introduced, the critical Froude number is raised above $\sqrt{5/22}$, the critical Froude number for a flat bottom in this case. In this sense, topography is stabilizing. The figure also shows a sharp spike in the linear growth rate, similar to that seen in figure 2.11. This spike is close to the $F_2$-curve and is reminiscent of the instability induced by the hydraulic jump in the turbulent case, except that it now occurs above the counterpart of Jeffrey’s threshold.

We repeat the asymptotic analysis for the laminar model using scalings identical to the ones used for the turbulent problem. We provide here the final amplitude equations for both possible scalings. For “Theory A”,

\begin{equation}
\begin{aligned}
&u_{1\tau} - \frac{21}{22} u_{1\tau \xi} + \frac{3}{2} (u_1^2)_{\xi} - \frac{15}{22} (u_1^2)_{\xi \xi} + 2 \nu_1 (\tilde{f}^2_{\eta} - 2 \tilde{U}_{2\eta} f_{\eta}) u_{1\xi} \\
&\quad + \left( 2 \sqrt{\frac{22}{5}} F_1 - \nu_1 \tilde{f}^2_{\eta} \right) u_{1\xi \xi} + 2 \nu_1 u_{1\xi \xi \xi} = 0,
\end{aligned}
\end{equation}

(B.1)

where $f$ and $\tilde{U}_2$ satisfy

\begin{equation}
\begin{aligned}
&- \frac{9}{11} f - \nu_1 f_{\eta} + A = 0, \\
&- \frac{9}{11} \tilde{U}_2 - \nu_1 \tilde{U}_{2\eta} + \frac{15}{22} f = 0.
\end{aligned}
\end{equation}

(B.2)
Figure B.1: Contours of constant growth rate ($\sigma$) for $\nu = 0.02$, $k_b = 10$, $K = 10^{-3}$. Thirty equally spaced contours (dotted lines) are plotted with the growth rate going from $6.09 \times 10^{-5}$ to $-4.77 \times 10^{-7}$. The solid line denotes the neutral stability curve and the dashed line shows the location of $F_1$, $F_2$ and $F_*$-curve.

For "theory B",

$$u_{2r} - \frac{21}{22} U_{2r} - \frac{15}{22}(u_{1r})_x + \frac{3}{2}(u_{tr})_x + \left( f_2^2 - \frac{4}{3} \nu_2 f_2^2 \right) u_{2r}$$

$$+ \left( 2 \sqrt{\frac{22}{5}} f_2 - \frac{18}{11} J_2 - \frac{\nu_2}{2} J_2^2 \right) u_{2r} + 2 \nu_2 u_{2r} \xi = 0,$$  \(\text{B.4}\)

where $f(\eta) = -(11/9) A(\eta)$.

Linear stability theory applied to equations (B.1) and (B.4) provides the corrections to the critical Froude number:

$$F_1 = -\frac{31}{968} \sqrt{110} f_2 + \frac{21 \nu_1 \sqrt{110}}{242} f_2 U_{2r}$$  \(\text{Theory A, B.5}\)

$$F_2 = -\frac{15 \sqrt{110}}{968} f_2^2 + \frac{39 \sqrt{110}}{968} f_2^2$$  \(\text{Theory B, B.6}\)

The correction for theory A can be written as

$$F_1 = \sum_{j=1}^{\infty} \left[ \frac{315}{5324} \left( \frac{81}{121} + \nu_1^2 j^2 \right)^{-\frac{1}{2}} - \frac{31}{968} \right] \frac{|A_j|^2}{\frac{81}{121} + \nu_1^2 j^2},$$  \(\text{B.7}\)

where the expression in square parenthesis is positive for $j > 0$. Thus small-amplitude topography is stabilizing in both limits.

The amplitude equations (B.1) and (B.4) are similar to (2.45) and (2.47) although they cannot be conveniently factorized into the form (2.55). This failure can be tracked back to the fact that mass is advected at a different rate than momentum in the laminar...
model when the parameter $\alpha$ is not equal to unity. The change in the structure of the amplitude equation could conceivably affect the general character of the nonlinear roll-wave dynamics. However, we have not explored this in the current work.
Appendix C

Energy Stability

Starting with (4.35), we consider two cases.

Case 1: \( \beta < 1 \)

We substitute
\[
\lambda_T = k_T \sqrt{\frac{1 + \beta}{1 - \beta}} \quad \text{and} \quad \lambda_S = k_S \sqrt{\frac{\beta(1 + \beta)}{1 - \beta}}
\]
into the constraint (4.32), to obtain
\[
k_T - \frac{1}{k_T} = -\alpha(k_S - \frac{1}{k_S}).
\]
By letting \( A = k_S - 1/k_S \), (C.2) leads to the following relations:
\[
k_T = \frac{-\alpha A \pm \sqrt{\alpha^2 A^2 + 4}}{2} \quad \text{and} \quad k_S = \frac{A \pm \sqrt{A^2 + 4}}{2}.
\]
We would like to find the biggest \( R_T \) for nonlinear stability which satisfies (4.34). Therefore, we would like to minimize \(|b_T|\). By substituting (C.1) and (C.3) in (4.27), we see that the best choice to make \(|b_T|\) as small as possible is when the signs of second terms of \( k_T \) and \( k_S \) in (C.3) are different. Therefore,
\[
|b_T| = \frac{1}{\sqrt{1 - \beta^2}} \sqrt{\alpha^2 A^2 + 4 - \beta \alpha \sqrt{A^2 + 4}}.
\]

Case: \( \alpha \leq \beta \)

\( b_T \) attains the minimum when
\[
A^2 = \frac{4}{\alpha^2} \frac{\beta^2 - \alpha^2}{1 - \beta^2},
\]
which gives \(|b_T| = 2\sqrt{1 - \alpha^2}\). From (4.34) and the definition of \( \alpha \), we end up with
\[
R_T - R_S < R_c.
\]
Appendix C. Energy Stability

Case: $\beta \leq \alpha \leq 1/\beta$

$b_T$ attains the minimum when $A = 0$, which gives $b_T = 2(1 - \beta \alpha)/\sqrt{1 - \beta^2}$. In this case, we end up with

$$\sqrt{R_T} - \beta \sqrt{R_S} < \sqrt{1 - \beta^2} \sqrt{R_c}. \quad (C.6)$$

Case: $\alpha \geq 1/\beta$

In this case $|b_T| = 0$ because we may choose

$$A^2 = \frac{4(\beta^2 \alpha^2 - 1)}{1 - \beta^2}. \quad (C.7)$$

Therefore, the system is nonlinear stable for all values of $R_T$.

Case 2: $\beta > 1$

Here, we substitute

$$\lambda_T = k_T \sqrt{\frac{\beta + 1}{\beta - 1}} \quad \text{and} \quad \lambda_S = k_S \sqrt{\frac{\beta(\beta + 1)}{\beta - 1}} \quad (C.8)$$

into the constraint (4.32) to obtain

$$k_T + \frac{1}{k_T} = -\alpha (k_S + \frac{1}{k_S}). \quad (C.9)$$

By letting $A \equiv k_S + 1/k_S$, (C.9) leads to the following relations:

$$k_T = \frac{-\alpha A \pm \sqrt{\alpha^2 A^2 - 4}}{2} \quad \text{and} \quad k_S = \frac{A \pm \sqrt{A^2 - 4}}{2} \quad (C.10)$$

By substituting (C.8) and (C.10) in (4.27) and choosing different signs of second terms of $k_T$ and $k_S$ in (C.10), we obtain

$$|b_T| = \frac{1}{\sqrt{\beta^2 - 1}} \left| \beta \alpha \sqrt{A^2 - 4 - \sqrt{\alpha^2 A^2 - 4}} \right|. \quad (C.11)$$

Case: $\alpha < 1$

$|b_T|$ attains the minimum when

$$A^2 = \frac{4 \beta^2 - \alpha^2}{\alpha^2 (\beta^2 - 1)},$$
which gives \(|b_T| = 2\sqrt{1 - \alpha^2}\). We then obtain

\[
R_T (1 - \alpha^2) < R_c \tag{C.12}
\]

or

\[
R_T - R_S < R_c \tag{C.13}
\]

Case: \(\alpha \geq 1\)

By substituting

\[
A^2 = \frac{4 \beta^2 \alpha^2 - 1}{\alpha^2 \beta^2 - 1},
\]

in (C.11), we obtain \(|b_T| = 0\). It is straightforward to show that \(\alpha \geq 1\) is a sufficient and necessary condition for \(A^2 \geq 4\). Therefore, the system is nonlinearly stable for all values of \(R_T\).